Sequence Risk Analysis

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Abstract

Preparing for retirement consists of two phases. The investment phase is during which someone is working and depositing money into a retirement portfolio. The withdrawal phase occurs once the person retires and can begin to withdraw money from their retirement portfolio. Based on previous analysis from Larry Frank, John Mitchel, David Blanchett, and Andrew Clare; and a previous MQP; our group decided to further explore and quantify sequence risk. Sequence risk is the risk associated with differing returns, where the order of those returns is uncertain, and multiple deposits or withdrawals occur. After creating simulations of return streams for both the investment and withdrawal phases, we worked to quantify the risk for investors to identify the risk they would face.

1 Introduction

Life is about the journey, not the destination. However, it is easy to get swept up in the adventure of it all and forget the expenses that begin to pile on. Whether it be for a home or apartment, student loans, or even a car payment, personal finances are something most people will never be able to ignore. Every day, 10,000 Americans reach age 65, the typical age for retirement. There is no one-size-fits-all answer for how much money an individual needs at the time of retirement due to the various risks associated with retirement.

Some challenges include:

- Lack of diversification in an individual's investment portfolio
- Too many defaults in the portfolio
- Insufficient funds in the portfolio

These challenges can be mitigated through careful planning and investment strategies. There is another challenge that retirees cannot reduce: sequence risk. In this paper, we will be focusing on how sequence risk affects a retirement account (assuming all other risks will be addressed separately). Throughout this paper when we refer to sequence risk we are talking about the following:

Sequence Risk: the risk associated with the final account balance attributable to the permutation of returns

Frank, Mitchell, and Blanchett claim "the first decade of retirement is the most crucial one in determining whether your retirement plan will be successful." The value of a retirement account can be significantly impacted by sequence risk. Even if an individual could correctly predict their future average rates of return, the uncertainty associated with the order of returns leads to risk. Sequence risk speaks to the uncertainty of returns; note that it does not always have a negative effect on an account, but instead could lead to major benefits.

Ultimately, to understand sequence risk, an individual must understand the effect of their returns and what happens to their money in an account. When talking about retirement accounts, it is important for an individual to know their goals. How much money do they want to save? How often do they want to make investments? What is their investment horizon? Do they want to exhaust their account by their time of death, or do they hope to have a certain amount of money “left over” in their account for their family? Having a goal in mind does not necessarily mitigate sequence risk but can help someone plan for the potential effects and determine possible withdrawal tactics.

2 Investment Period: Saving for Retirement

During the years leading up to retirement, individuals invest money into a variety of accounts to create their retirement portfolio. Ideally this portfolio is diversified to mitigate the risk associated with each specific investment. Individuals may invest their money into different types of retirement accounts to further diversify their retirement portfolio.

Potential types of retirement accounts include:

- 401(k) accounts: A benefit is provided by an individual's employer.
- Individual Retirement Accounts (IRA): An account an individual may set up, without the help of an employer, to prepare for their retirement.
- 403(b) accounts: An account specific for school professionals or civilian faculty, which function similarly to that of a 401(k).

No matter what type of account an individual invests in all are subjected to varying returns impacting their final balances. The riskiness an individual experience stems from the inability to forecast the final balance in their retirement account.

Sequence risk is a great concern for individuals who are looking to retire. If the retiree experiences negative effects of sequence risk, it can make it hard for them to recover their money. An individual could also experience positive effects of sequence risk and that could result in them earning far more than expected.

2.1 Investing: When is There No Sequence Risk?

Examples 1 and 2 show when sequence risk is not present. The lack of sequence risk is shown by the final account balances being the same for all of the presented scenarios within an example. The two major occurrences in which sequence risk is not present is when there is only one deposit and/or when the returns throughout a period is constant.

Example 1 demonstrates how sequence risk is **not** a concern when there is a single deposit, and the value of the account is left to grow untouched.

Example 2 shows how sequence risk is also **not** a concern when the returns throughout a period are constant.

Both of these examples are unrealistic in terms of actually modeling the way an individual saves for retirement, but they do provide a useful look at how and why sequence risk arises.

Example 1: Investing: Simple Demonstration

This example shows what happens to an individual's account after making one initial investment of $1,000 and letting it grows with interest for three years. The six different scenarios are all the permutations of the returns -10%, 10% and 20%.

Table 1. Grow single 1,000 deposit with returns –10%, 10%, and 20%

Scenario				Account Balance			
	Return Year 1	Return Year 2	Return Year 3	Time 0	Time 1	Time 2	Time 3
1	-10%	10%	20%	1,000	900	990	1,188
2	-10%	20%	10%	1,000	900	1,080	1,188
3	10%	-10%	20%	1,000	1,100	990	1,188
4	10%	20%	-10%	1,000	1,100	1,320	1,188
5	20%	-10%	10%	1,000	1,200	1,080	1,188
6	20%	10%	-10%	1,000	1,200	1,320	1,188

To perform the calculation for scenario one, with returns -10%, 10%, and 20%, the returns are all multiplied by the investment during the corresponding year.

1) Given scenario one, the initial deposit is $1,000.

$$Initial\ Deposit: 1{,}000$$

2) This deposit is brought forward to year one using the return of -10%.

$$Time\ 1:\ 1{,}000 \times\ (1 + (-0.1)) = 900$$

3) The time 1 account balance receives a 10% return.

$$Time\ 2: 900 \times\ (1 + (0.1)) = 990$$

4) The time 2 account balance receives a return of 20%.

$$Final\ Balance\ at\ Time\ 3: 900 \times\ (1 + (0.2)) = \$1{,}188$$

This same calculation occurs for each of the six permutations.

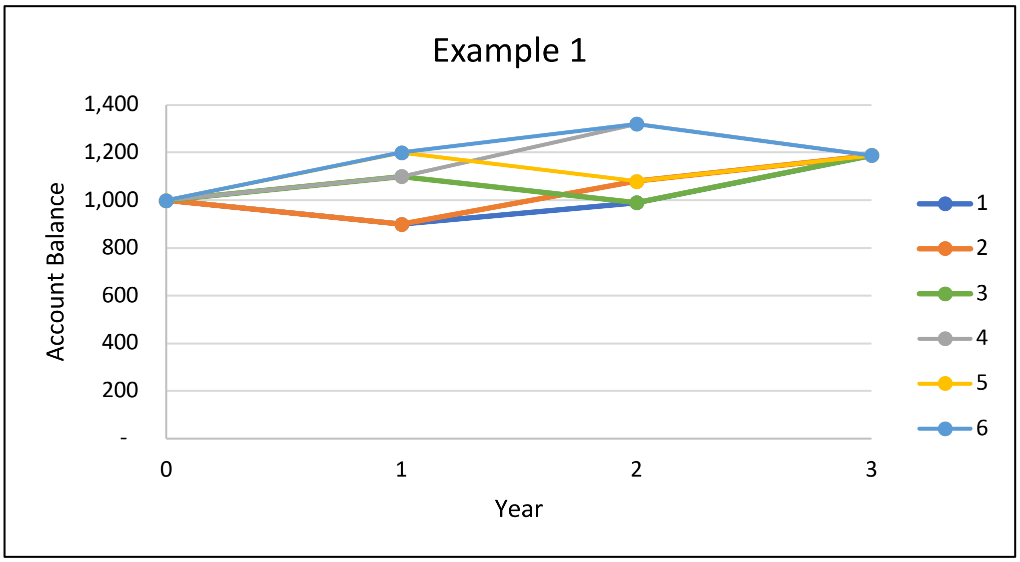


Figure 1. Grow single 1,000 deposit with returns –10%, 10%, and 20%

Result: The biggest takeaway from Example 1 is that the final balance at the end of the three-year period is the same for each scenario at $1,188. Although at the end of each intermediate period the final account balances are different, once all three returns are applied to the initial investment of $1,000 the outcome is the same.

This example shows the sequence of returns does **<u>not</u>** cause a risk for a single deposit. Since there was only an initial deposit, these accounts were left alone to grow without disruption. Therefore, the order of the returns does not change the final account balance since there were no other deposits or withdrawals to the account.

Example 2: Investing: Constant 6% Return

In this example, three annual deposits of $1,000 will be made. A return of 6% is applied for each of the three years.

Table 2. Deposit $1,000 each period for 3 years with returns 6%, 6%, and 6%

Scenario			**Account Balance**					
Return Year 1	**Return Year 2**	**Return Year 3**	**Initial +$1,000 Deposit**	**Time 1 Before Deposit**	**Time 1 After +$1000 Deposit**	**Time 2 Before Deposit**	**Time 2 after +$1,000 Deposit**	**Final Balance after Deposits**
6%	6%	6%	1,000	1,060	2,060	2,184	3,184	3,375

To perform the calculations for investing for Scenario one, complete the following steps:

1) Given scenario one, the initial deposit is $1,000.

$$Initial\ Deposit: 1{,}000$$

2) This deposit is brought forward to the end of year one using the return 6%.

$$Time\ 1\ Before\ Deposit: 1{,}000 \times (1 + 0.06) = 1{,}060$$

3) At the end of the first year, $1,000 is deposited into the account.

$$Time\ 1\ After\ Deposit: 1{,}060 + 1{,}000 = 2{,}060$$

4) This amount is then brought forward to the end of year two using the return of 6%.

$$Time\ 2\ Before\ Deposit: 2{,}060 \times (1 + 0.06) = 2{,}184$$

5) At the end of the second year, $1,000 more is deposited into the account.

$$Time\ 2\ After\ Deposit: 2{,}184 + 1{,}000 = 3{,}184$$

6) This value is then brought forward to the end of year 3 using the 6% return.

$$Final\ Balance\ at\ the\ End\ of\ the\ Period: 3{,}184 \times (1 + 0.06) = \$3{,}375$$

Result: This example demonstrated how sequence risk is **not** a concern when the return is constant. If the return is constant for the entire time period, there will only be one unique outcome given all potential permutations. There is no added risk from sequence risk since there is no variation in the account at the end of the period.

Sequence Risk is present when there are:

1. Non-constant returns (the usual situation)
2. Unknown permutations of those returns
3. Multiple cash flows in or out of the account

2.2 Investing: When is Sequence Risk Present?

In this paper there are many examples of the investment and withdrawal patterns for retirement accounts with varying returns. For each of these examples, all possible sequences of the returns, or the permutations, are shown in the tables. For a trial with three different returns, there are 3!, or 6 possible permutations. These permutations were generated in Microsoft Excel. Each of the tables has the most favorable outcome first, and the least favorable outcome last. This favorability is based on the order of the returns for each permutation calculation. For more information on the permutation calculations, reference *Appendix A*. Example 1 and Example 2 showed scenarios when there is **no** sequence risk during an investment period.

Sequence risk is a concern when money is deposited or withdrawn more often than just once, and when returns vary each period. Examples 3 through 5 show a range of examples with varying positive and negative returns. Since there is no guaranteed order of these returns, a risk is presented because each of the scenarios could have different final balances. Sequence risk occurs in varying degrees depending on the range of return values.

- Example 3 demonstrates how final balances will vary slightly when all returns are within 2% of each other.
- Example 4 demonstrates an example with positive and negative returns, all within 10% of each other. Finally,
- Example 5 demonstrates drastically changing positive and negative returns that are within 30% of each other.

Each of the Examples 3, 4, and 5 occur over a three-year period with three returns. The process of multiple deposits as shown in Example 2 are repeated for Examples 3 through 5.

Example 3: Investing: Positive Returns Ranging by 3%

This example is for three years where annual deposits of $1,000 will be made at the beginning of each year. The order of returns, 1%, 2%, and 3% varies in each scenario.

Table 3. Deposit $1,000 each period for 3 years with returns 1%, 2%, and 3%

Scenario				**Account Balance**						
	Return Year 1	**Return Year 2**	**Return Year 3**	**Initial +$1,000 Deposit**	**Time 1 Before Deposit**	**Time 1 After +$1000 Deposit**	**Time 2 Before Deposit**	**Time 2 After +$1,000 Deposit**	**Final Balance After Deposits**	**IRR**
1	1%	2%	3%	1,000	1,010	2,010	2,050	3,050	3,141.71	2.326%
2	1%	3%	2%	1,000	1,010	2,010	2,070	3,070	3,131.71	2.164%
3	2%	1%	3%	1,000	1,020	2,020	2,040	3,040	3,131.41	2.159%
4	2%	3%	1%	1,000	1,020	2,020	2,081	3,081	3,111.41	1.834%
5	3%	1%	2%	1,000	1,030	2,030	2,050	3,050	3,111.31	1.833%
6	3%	2%	1%	1,000	1,030	2,030	2,071	3,071	3,101.31	1.670%

To perform the calculations for investing for scenario 1, complete the following steps.

1) Given scenario one, the initial deposit is $1,000.

$$Initial\ Deposit: 1{,}000$$

2) This deposit is brought forward to the end of year one using the return 1%.

$$Time\ 1\ Before\ Deposit: 1{,}000(1 + 0.01)\ =\ 1{,}010$$

3) At the end of the first year, $1,000 is deposited into the account.

$$Time\ 1\ After\ Deposit: 1{,}010 + 1{,}000\ =\ 2{,}010$$

4) This amount is then brought forward to the end of year two using the return 2%.

$$Time\ 2\ Before\ Deposit: 2{,}010(1 + 0.02) = 2{,}050$$

5) At the end of the second year, $1,000 more is deposited into the account.

$$Time\ 2\ After\ Deposit: 2{,}050 + 1{,}000\ =\ 3{,}050$$

6) This value is then brought forward to the end of year 3 using the return 3%.

$$Final\ Balance\ at\ the\ End\ of\ the\ Period: 3{,}050(1 + 0.03)\ =\ \$3{,}141.71$$

The balance $3,142 is the ending balance for this example given the returns in the order 1%, 2%, and 3%. The same process is completed for each of the 6 scenarios for the example.

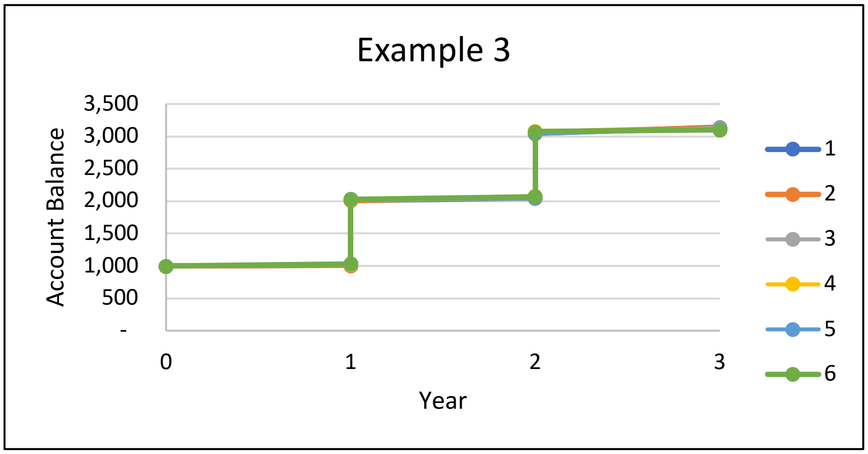


Figure 2. Deposit $1,000 each period for 3 years with returns 1%, 2%, and 3%

Table 4: Final Balance an IRR of Example 3

Scenario	Return Year 1	Return Year 2	Return Year 3	Final Balance	IRR
1	1%	2%	3%	3,141.71	2.326%
2	1%	3%	2%	3,131.71	2.164%
3	2%	1%	3%	3,131.41	2.159%
4	2%	3%	1%	3,111.41	1.834%
5	3%	1%	2%	3,111.31	1.833%
6	3%	2%	1%	3,101.31	1.670%

Result: Each scenario's final balance at the end of the period is different, demonstrating sequence risk on a small scale. Figure 2 shows the growth for three of the scenarios over the three-year period. Scenario 1 has returns going from lowest to highest and ends with the best outcome final account balance which occurs because the larger returns occur later in the period when the account value is higher. In scenario 6, returns 3%, 2%, then 1% has the worst outcome since the greatest returns occur at the beginning of the scenario when the account balance is not very large. To compare the results for each scenario, the Internal Rate of Return (IRR) was calculated. The IRR for the best case is 2.326% and the IRR for the worst scenario is 1.670%. The difference in IRR percentage for this example is 0.656%. The small difference in the IRR values between each scenario is consistent with the small difference in returns over this short, three-year period.

Example 4: Investing: Positive and Negative Returns Ranging by 7%

This example is for three years where three annual deposits of $1,000 will be made at the beginning of each year. The order of the returns, -2%, 2%, and 5% varies in each scenario.

Table 5. Deposit $1,000 each period for 3 years with returns -2%, 2%, and 5%

Scenario				**Account Balance**						
	Return Year 1	**Return Year 2**	**Return Year 3**	**Initial +$1,000 Deposit**	**Time 1 Before Deposit**	**Time 1 After +$1000 Deposit**	**Time 2 Before Deposit**	**Time 2 After +$1,000 Deposit**	**Final Balance After Deposits**	**IRR**
1	-2%	2%	5%	1,000	980	1,980	2,020	3,020	3,171	2.80%
2	-2%	5%	2%	1,000	980	1,980	2,079	3,079	3,141	2.31%
3	2%	-2%	5%	1,000	1,020	2,020	1,980	2,980	3,129	2.12%
4	2%	5%	-2%	1,000	1,020	2,020	2,121	3,121	3,059	0.98%
5	5%	-2%	2%	1,000	1,050	2,050	2,009	3,009	3,069	1.14%
6	5%	2%	-2%	1,000	1,050	2,050	2,091	3,091	3,029	0.48%

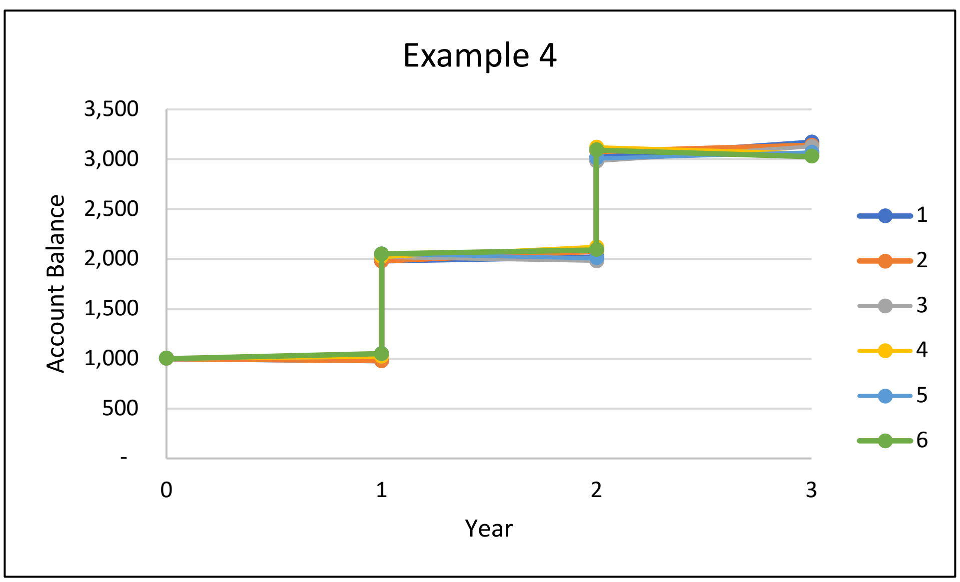


Figure 3. Deposit $1,000 each period for 3 years with returns -2%, 2%, and 5%

Results: The final balances at the end of the period are different for each scenario showing sequence risk in a short period with a small range of returns. As shown in the **Final Balance After Deposits** column, scenario 1 has the best outcome, due to the fact the returns are in ascending order. The first return is negative, which is the most desirable position for a negative return since the value in the account for the first year is at the lowest value. Furthermore, the highest return, 5% is the last return in the sequence which is also the most desirable since the value of the account is at a significantly greater amount at the end of year two compared to any of the other years. Scenario 6 has the worst outcome since the returns occur from largest to smallest, with the last return being a negative value. This example shows how negative returns cause a greater difference for final balances between the scenarios.

The IRR for the best case is 2.80% and the IRR for the worst scenario is 0.480%. The difference in IRR percentage for this example is 2.32%. This difference in IRR is greater than Example 3 since the returns have a greater range, which is consistent with the slightly larger, but still relatively small differences in final account values.

Example 5: Investing: Positive and Negative Returns Ranging by 30%

This example is for three years where three annual deposits of $1,000 will be made at the beginning of each year. The order of returns, -10%, 10%, and 20% varies in each scenario.

Table 6. Deposit $1,000 each period for 3 years with returns -10%, 10%, and 20%

Scenario				**Account Balance**						
	Return Year 1	**Return Year 2**	**Return Year 3**	**Initial +$1,000 Deposit**	**Time 1 Before Deposit**	**Time 1 After +$1000 Deposit**	**Time 2 Before Deposit**	**Time 2 After +$1,000 Deposit**	**Final Balance After Deposits**	**IRR**
1	-10%	10%	20%	1,000	900	1,900	2,090	3,090	3,708	10.97%
2	-10%	20%	10%	1,000	900	1,900	2,280	3,280	3,608	9.25%
3	10%	-10%	20%	1,000	1,100	2,100	1,890	2,890	3,468	7.43%
4	10%	20%	-10%	1,000	1,100	2,100	2,520	3,520	3,168	2.75%
5	20%	-10%	10%	1,000	1,200	2,200	1,980	2,980	3,278	4.50%
6	20%	10%	-10%	1,000	1,200	2,200	2,420	3,420	3,078	1.29%

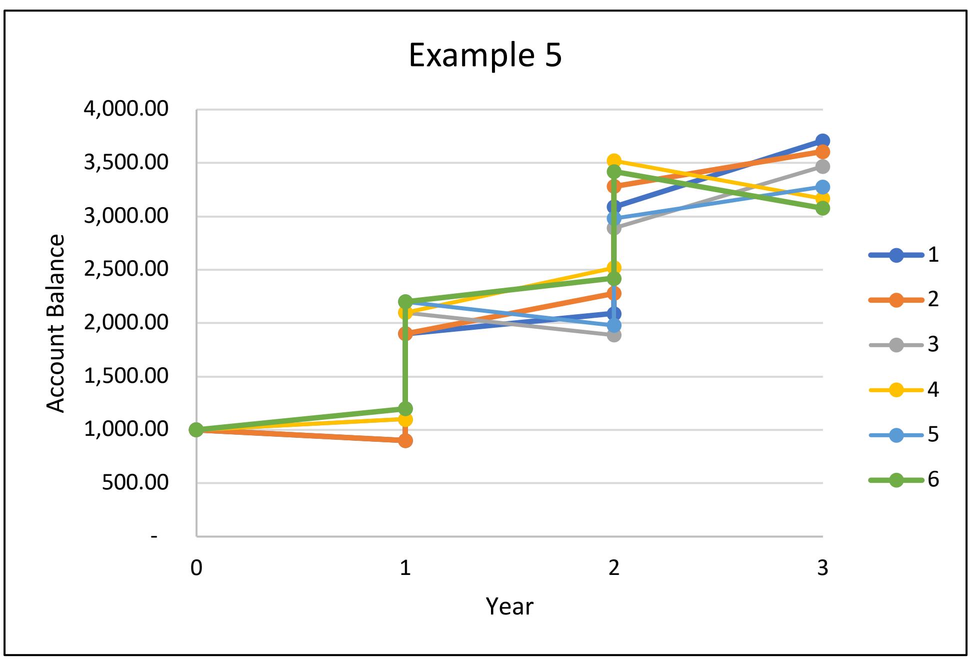


Figure 4. Deposit $1,000 each period for 3 years with returns -10%, 10%, and 20%

Results: The final balances at the end of the period are different for each scenario and more demonstrably than the previous examples.

Scenario 1 in this example has the best outcome for final balance, with $3,708. since the returns occur in the order from smallest to largest.

Scenario 6 has the worst outcome, with a final balance of $3,078. Since the returns occur from largest to smallest, with the last return being a negative value.

The IRR for the best case is 10.97% and the IRR for the worst scenario is 1.29%. The difference in IRR percentage for this example is 9.68%. This difference in IRR is greater than Example 3 and Example 4 since the since the range is the greatest we have explored, which is once again consistent with the greater differences between final balances.

2.3 Long-Term Sequence Risk Investment Trials

To understand sequence risk on a more realistic scale, 30-year trials were completed using historic S&P 500 returns and various deposit strategies. The main goal of these trials was to use realistic data to understand how sequence risk works on a larger timeline. The historic S&P 500 returns were the yearly returns from 1926 to 2019. Four additional returns were added to the set, so the returns had more potential variation than the historic values. When determining the additional returns, it was essential that the averages did not change from the initial historic values. The historic returns used in the trials can be found in *Appendix B*. Once the returns were adjusted, two Python programs were written to choose the 30 returns randomly with no duplication. With these 30 returns, 100,000 permutations were generated pseudo randomly and the best- and worst-case scenarios were added in to total 100,002 permutations to be used in each strategy. For more information regarding this permutation selection process, reference the Code Description document. These 100,002 permutations were saved and used for each of the given strategies. Since the permutations of the returns were constant for all the deposit strategies, we could compare between the trials.

A second Python code was focused on performing investment calculations, similar to those done in Examples 1 through 5, with an amount being deposited each year but with a timeline of 30 years. There were four specific strategies employed with multiple sub-strategies, resulting in 10 different combinations, as follows:

Table 7. Investing for Retirement Strategies

Strategy	Strategy Description
1.1	Deposit $100 each year
1.2	Deposit $1,000 each year
1.3	Deposit $5,000 each year
2.1	Deposit $5,000 initially, then increases the deposit by a constant 3% each year
2.2	Deposit $5,000 initially, and increase the deposit by a constant 5% each year
2.3	Deposit $5,000 initially, and increase the deposit by a constant 10% each year
3.1	Deposit $5,000 initially, and increase the deposit by 1%, 2%, 3%… each year
3.2	Deposit $5,000 initially, and increase the deposit by 5%, 8%,11%... every 5 years
4.1	Deposit $1,000 initially, and increase the deposit by $50 each year
4.2	Deposit $1,000 initially, and increase the deposit by $100 each year

Each strategy used the same 25,000 permutations of returns, but the deposit patterns of each differed. The key takeaway from the investment trials was to see the spread of final balance amounts when different depositing strategies were employed. By analyzing these values within each strategy as well as between strategies, we were able to see how the risk of investing enough money for retirement can vary greatly depending on the deposit amounts even when the returns stay the same. Strategies 1 and 4 are the same concept with deposits of a constant amount each year. Strategy 2 and 3 are meant to mimic a potential raise or change in salary of a person saving for retirement. Table 8 shows the minimum, maximum, and average final account balance value for each strategy. The lowest value in each category (minimum, maximum, and average account balances) are shown in orange and the highest value in each category are shown in green.

Table 8. Descriptive Statistics of Investing Strategies

Strategy	Maximum Account Balance	Minimum Account Balance	Average Account Balance
1.1	71,456	4,696	14,634
1.2	714,563	46,962	146,342
1.3	3,572,813	234,811	731,716
2.1	4,978,898	279,805	983,366
2.2	6,292,623	324,012	1,227,967
2.3	11,863,879	535,142	2,339,231
3.1	11,578,475	74,8052	2,814,149
3.2	10,355,491	569,512	2,268,942
4.1	3,953,532	246,776	689,431
4.2	4,334,251	258,741	865,344

The strategy with the lowest overall return was *Strategy 1.1* which involved depositing $100 each year. *Strategy 1.1* accounted for the lowest values for the minimum, maximum, and average account balance due to the small deposit amounts.

The strategy with the highest return was *Strategy 2.3* which involved incrementing deposit amounts by 10% each year. Each account balance was accompanied by an IRR to show return on the deposit amounts.

The IRRs values are dependent on both the account balance and the deposits occurring each year. To determine the IRR value, a Python function was used to assess the account balance and deposits. By analyzing the amount of each deposit and looking at that in comparison to the final account, the IRRs in Table 9 were found.

Table 9. Average IRR of Investing Strategies

Strategy	Average IRR
1.1	8.691%
1.2	8.691%
1.3	8.691%
2.1	8.710%
2.2	8.721%
2.3	8.734%
3.1	8.687%
3.2	8.722%
4.1	8.737%
4.2	8.702%

Unlike the account balances in Table 8, *Strategy 1.1* did not have the lowest return in comparison to the amount deposited. Strategies in category 1 all achieved the same IRR value since the deposit amounts were scaled values with varying deposit amounts of 100, 1,000, and 5,000.

The strategy with the lowest return on investment was S*trategy 3.1*, adding a 5000 initial deposit and increasing the deposit amount by 1% each year, with a return of 8.687%. The strategy with the highest return was *Strategy 2.3* with a return on 8.734%. The IRR values ranges from 8.687% to 8.734%.

From the IRR values, we were able to compare the returns for each strategy and see how differing deposit amounts impact the final return to the investor.

2.3.1 Strategy Comparisons

In addition to the overall values for each strategy, the 100,002 permutations were compared to determine where risk is more common. Histograms and descriptive statistics were run for each of the strategies and then compared. The risk is evident by looking at the spread of potential final account balances for the 100,002 permutations.

Investing Strategy 1.1: Depositing $100 Each Year

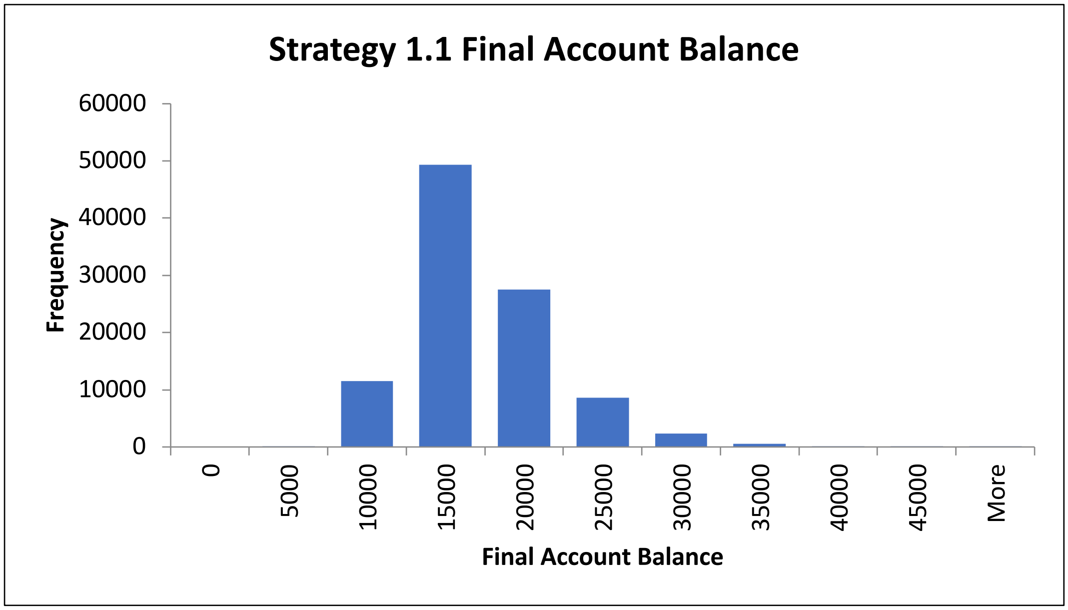


Figure 5. Histogram of Strategy 1.1 Final Balances

Figure 5 shows the account balances for *Strategy 1.1* which entailed depositing 100 each year for 30 years. Almost 50,000 of the permutations fell within the $10,000- $15,000 bin. This strategy presented a skewness of 1.16 for final account balances which is caused by the differing orders of returns. **Skewness** is a measure of how much a random variable's probability density function deviates from the normal distribution. A skewness value of zero means the distribution is symmetric. A negative skewness value means the distribution has a longer left tail, and positive skewness means the distribution has a longer right tail. Figure 5 shows the long right tail showing the positive skewness of *Strategy 1.1*. The range of values is only $66,760 which is much less than the other strategies and caused by the smaller deposit amounts.

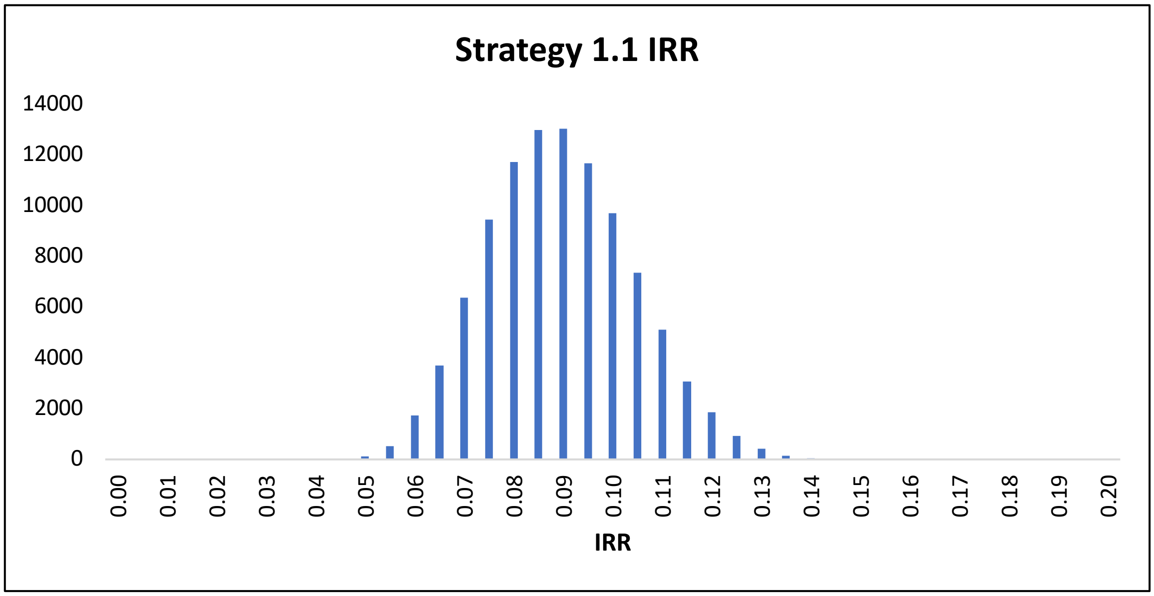


Figure 6. Histogram of Strategy 1.1's IRRs

Figure 6 shows the IRR values for *Strategy 1.1*. The distribution of IRRs was more widespread. The range of values is 13.97% with nearly half of the values falling between 7.5% and 9.5%. The skewness of the IRRs is 0.2308. This difference in skewness values is due to the changes in graphical representation between Figure 5 and Figure 6.

Investing Strategy 3.2: Depositing $5,000 the First Year, and Increasing Deposit Every 5 Years

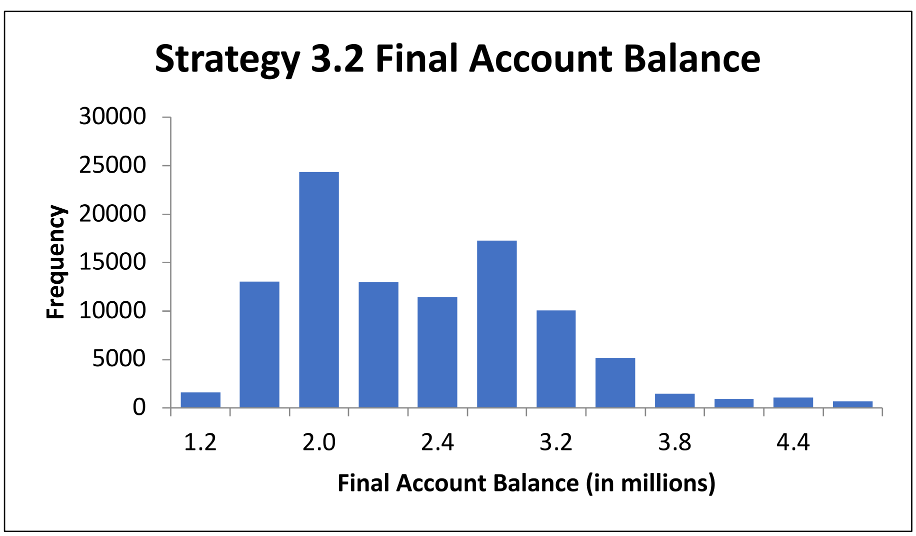


Figure 7. Strategy 3.2's Final Account Balances

Strategy 3.2, depositing an increasing amount beginning with $5,000 and increasing the amount by 3% every 5 years beginning with a 5% increase, is shown by Figure 7 and Figure 8. Figure 7 shows the final account balances for *Strategy 3.2*. The account balances for this strategy are far higher than *Strategy 1.1*. The maximum balance for *Strategy 1.1* is almost $500,000 less than the minimum balance for *Strategy 3.2*. The skewness of Figure 8 is 0.9066 showing the values are more symmetric than those seen in *Strategy 1.1.* Additionally, the range for this strategy is $9,785,979 which is drastically larger than the values seen previously.

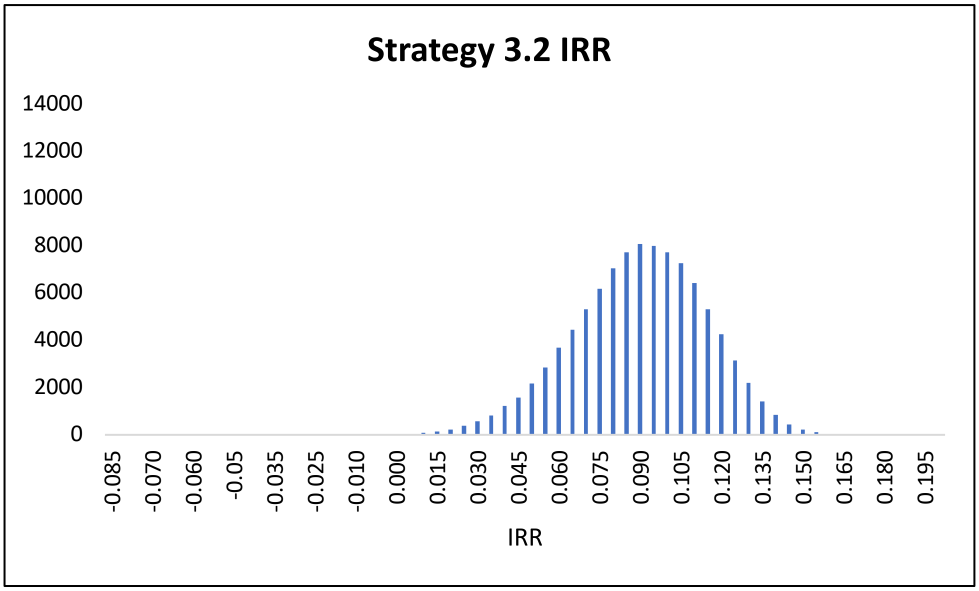


Figure 8. Histogram of Strategy 3.2's IRRs

The IRR values for *Strategy 3.2* showed the change in skewness for the dataset. The skewness for the IRR is –0.02724 (an indication of the distribution skewing to the left). Given the IRRs are graphed using the same x axis scale, it is easier to compare the IRR distribution between strategies. *Strategy 3.2* has 8,408 values within the "more" bucket for IRR showing the return on investment is much higher for this increased investment amount. The range of values for IRR is also more widespread with a value of 27.48% since the maximum IRR is 19.4% and the minimum return is -8.08%.

The rest of the strategies' graphs for IRR and final account balances can be found in the Investing and Withdrawing Excel Workbook. Sequence risk was shown by the comparison of all 10 strategies. Keeping the returns and time periods constant while comparing deposit amounts and IRR, the risk of depositing in the market were shown. If smaller deposit amounts are used, then the risk of the investment is not as far reaching in comparison to larger, increasing deposit amounts which vary greatly in their return amounts. The 30-year trials offered a more realistic look at what happens when a person deposits money into their retirement account.

3 Withdrawal Period: Retirement Spending

One common goal for retirees is to ensure their retirement accounts have enough money to last for the rest of their lives. With this common goal in mind, there is a differentiation between each retiree based on their individual needs in terms of withdrawing from their accounts. Each retiree and family lives different lifestyles with different expenses which leads to larger or smaller withdrawal amounts for everyone. Not having sufficient money in an account before someone dies means they might have to go back to work or go through the process of retrenchment where they must reevaluate their spending habits ("Probability-of-Failure-Based Decision rules to Manage Sequence risk in retirement"). It is also possible to put trust money aside to be left to family members. In this case, there is a specific end balance that should be left in the account when the individual dies.

One of the many risks during the withdrawal phase of retirement is poor investment returns. During the withdrawal phase one's account continues to earn interest. Depending on the returns their account experiences it can impact the amount of money they can withdraw from their account. These below average returns can greatly change the end value in an account and cause someone to run out of money sooner than they wished if returns are below expected. Below are three examples showing this idea where the tables reveal the highest and lowest remaining values. Similar to the investing examples the biggest takeaway is that depending on the order that the returns come in, someone will end up with varying amounts of money left in their accounts when they die. Due to the risk presented by the varying markets and returns, at the time of an individual's death, their account may not have their desired final value if they do not work to mitigate the sequence risk.

Similar to the investing period, sequence risk is not always present. In Example 1, we showed that sequence risk is not present when only one deposit occurs. The only ways to completely avoid sequence risk during the withdrawal period is to exhaust the account the first day of retirement, or to never make any withdrawals at all! Since this is not how retirement accounts are used, sequence risk during the withdrawal period is always a risk. One valid strategy to mitigate sequence risk in the withdrawal period would be to liquidate the account the first day of retirement and purchase a life annuity, to shift the risk from the individual to the insurance company. Example 8 shows how when returns are all the same, there is no risk associated with their order, like Example 2.

Example 8: Withdrawal period: Initial Demonstration

This example shows what happens when an individual withdraws $1,000 at the end of each year for three years with returns being the same at 6%. The initial balance for this example starts at $3,000.

Table 10. Withdraw $1,000 each period for 3 years with returns 6%, 6%, and 6%

Scenarios			**Account Balance**				
Return Year 1	**Return Year 2**	**Return Year 3**	**Initial Balance**	**Year 1 Balance**	**Year 2 Balance**	**Final Balance ($)**	**Final Balance (%)**
6%	6%	6%	3,000	2,180	1,311	389	12.97%

To perform the calculations for the withdrawals for scenario one, complete the following steps.

1) Given scenario one, the initial balance of $3,000.

$$Initial\ Balance: 3{,}000$$

2) This deposit is brought forward to the end of year one using the return 6%.

$$Year\ 1\ Balance: 3{,}000(1+0.06) = 3{,}180$$

3) At the end of the first year, $1,000 is withdrawn from the account.

$$Year\ 1\ Withdrawal: 3{,}180 - 1{,}000 = 2{,}180$$

4) This amount is then brought forward to the end of year two using the return 6%.

$$Year\ 2\ Balance: 2{,}180(1+0.06) = 2{,}311$$

5) At the end of the second year, $1,000 more is withdrawn from the account.

$$Year\ 2\ Withdrawal: 2{,}311 - 1{,}000 = 1{,}311$$

6) This value is then brought forward to the end of year 3 using the return 6%.

$$Year\ 3\ Balance: 1{,}311(1+0.06) = 1{,}389$$

7) At the end of the third year, $1,000 more is withdrawn from the account.

$$Final\ Balance\ (Year\ 3\ Withdrawal): 1{,}389 - 1{,}000 = 389$$

8) To find the percent of the balance left, take the final balance divided by initial balance.

$$Percentage: 389/3000 = \mathbf{12.97}\%$$

These 8 steps will be completed for each of the 6 scenarios but with the respective order of returns for the given scenario.

Result: Similar to Example 2, investing with the same returns, this example shows there is no risk associated with changing the order of the returns since they are the same value.

Sequence Risk is present when there are:

1. Non-constant returns (the usual situation)
2. Unknown permutations of those returns
3. Multiple cash flows in or out of the account

3.1 Withdrawal Period: When is Sequence Risk Present?

Similar to the investment examples, withdrawal examples were completed. For a trial with three different returns, there are 3!, or 6 possible permutations. These permutations are the same as those presented in the investment examples. Each of the tables is ordered from the lowest final account balance to the highest account balance. Since the returns are ordered from lowest to greatest value, the permutations were calculated in that same order. For more information on the permutation calculations, reference *Appendix A*.

Sequence risk is a concern when money is withdrawn and returns vary each period. Examples 9 through 11 show a range of examples with varying returns similar to those in Examples 3 through 5. The risk associated with these scenarios is present since the order of returns is not certain. Sequence risk occurs in varying degrees depending on the proximity of return values.

Example 9 demonstrates how final balances will vary slightly when all returns are within 2% of each other.

Example 10 demonstrates an example with positive and negative returns, which are all within 10% of each other.

Finally, Example 11 demonstrates drastically changing positive and negative returns that are within 30% of each other.

Each of the Examples 9,10, and 11 occur over a three-year period with three returns. The process of withdrawing money for Examples 9-11 is the same as shown in Example 8.

Example 9: Withdrawal Period: Positive Returns Ranging by 2%

This example is for three years where the beginning value of the account is $3,000 and at the end of each year there is a withdrawal of $1,000. The order of the returns, 1%, 2%, and 3% vary in each scenario. The first scenario has the returns in ascending order and the sixth scenario has the returns in descending order.

Table 11. Withdraw $1,000 each period for 3 years with returns 1%, 2%, and 3%

	Scenario			**Account Balance**				
	Return Year 1	**Return Year 2**	**Return Year 3**	**Initial Balance**	**Year 1 Balance**	**Year 2 Balance**	**Final Balance ($)**	**Final Balance (%)**
1	1%	2%	3%	3,000	2030	1071	102.72	3.42%
2	1%	3%	2%	3,000	2030	1091	112.72	3.76%
3	2%	1%	3%	3,000	2060	1081	113.02	3.77%
4	2%	3%	1%	3,000	2060	1122	133.02	4.43%
5	3%	1%	2%	3,000	2090	1111	133.12	4.44%
6	3%	2%	1%	3,000	2090	1132	143.12	4.77%

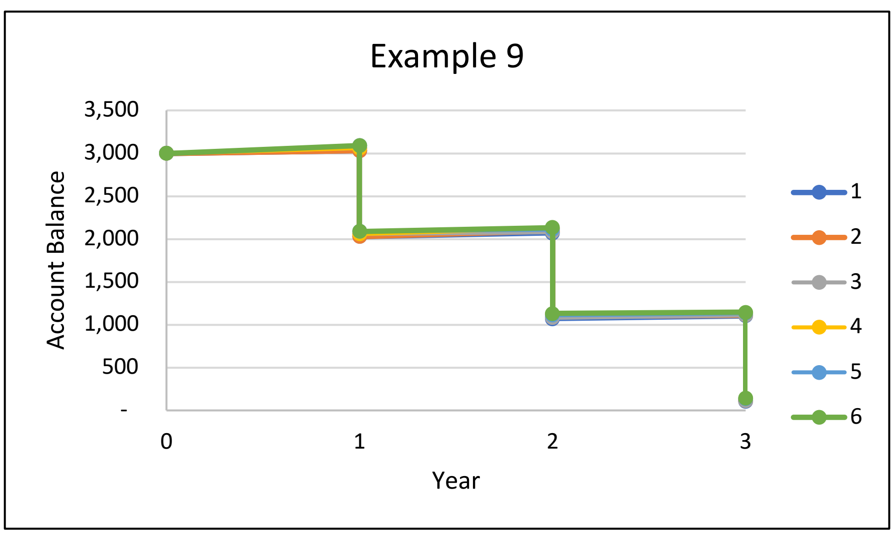


Figure 9. Withdraw $1,000 each period for 3 years with returns 1%, 2%, and 3%

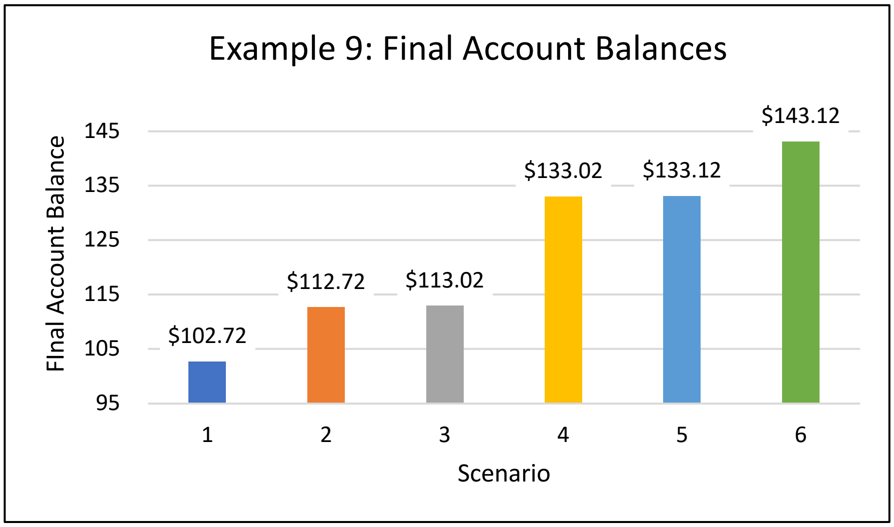


Figure 10. Final Account Balances

Results: Figure 10 demonstrates sequence risk on a small scale by showing the final balances for all of the scenarios over the three-year period. For someone who hopes to drain their retirement fund, scenario one has the best outcome since the final balances are in order from highest to lowest. This occurs because the larger returns occur later in the period when the account value is higher. In scenario six the returns are ordered from highest to lowest which produces the largest final account value. Since the greatest returns occur at the beginning of the scenario when the account balance is larger the withdrawals do not impact the account as greatly. In this example, the highest and lowest final balances only differ by 1.35% of the initial balance. Since the returns are close together and the scenario only occurs over 3 years, this percent difference is quite low.

Example 10: Withdrawal Period: Positive and Negative Returns Ranging by 7%

This example is for three years where the beginning value of the account is $3,000 and at the end of each year there is a withdrawal of $1,000. The order of the returns, -2%, 2%, and 5% vary in each scenario. The first scenario has the returns in ascending order and the sixth scenario has the returns in descending order. This example features a negative return.

Table 12. Withdraw $1,000 each period for 3 years with returns -2%, 2%, and 5%

Scenario				Account Balance				
	Return Year 1	Return Year 2	Return Year 3	Initial Balance	Year 1 Balance	Year 2 Balance	Final Balance ($)	Final Balance (%)
1	-2%	2%	5%	3,000	1,940	979	27.74	0.92%
2	-2%	5%	2%	3,000	1,940	1,037	57.74	1.92%
3	2%	-2%	5%	3,000	2,060	1,019	69.74	2.32%
4	2%	5%	-2%	3,000	2,060	1,163	139.74	4.66%
5	5%	-2%	2%	3,000	2,150	1,107	129.14	4.30%
6	5%	2%	-2%	3,000	2,150	1,193	169.14	5.64%

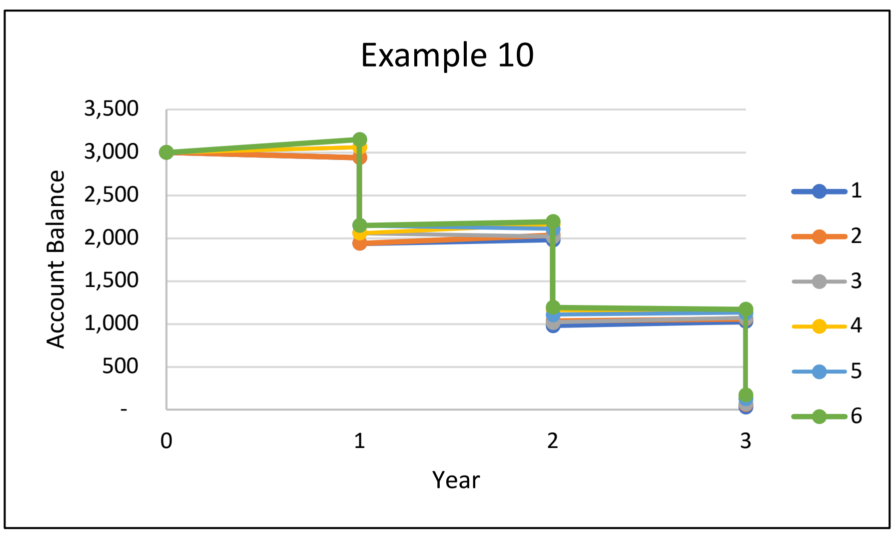


Figure 11. Withdraw $1,000 each period for 3 years with returns -2%, 2%, and 5%

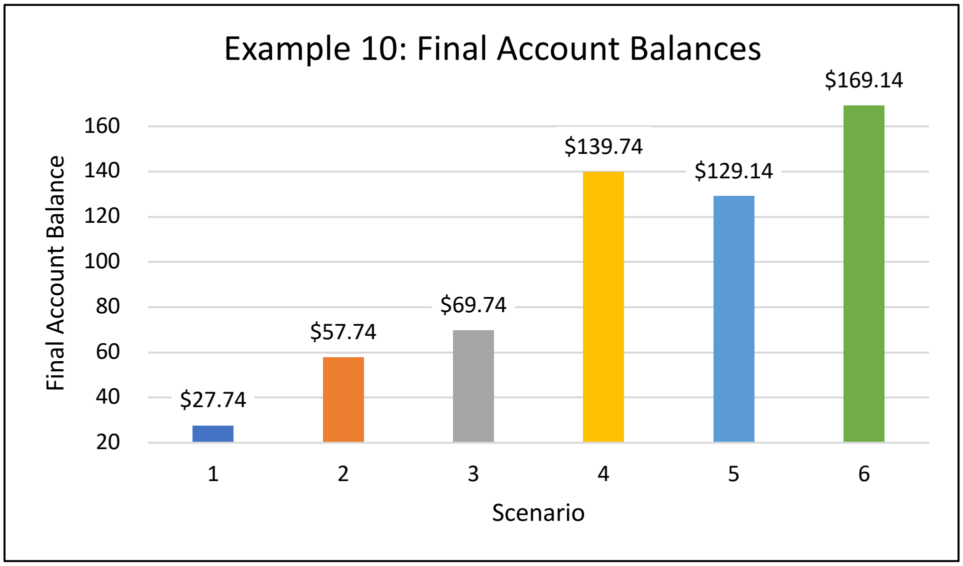


Figure 12 Final Account Balances

Results: Each scenario's final balance at the end of the period is different. Like Example 9, for someone who hopes to drain their retirement fund, scenario one has the best outcome since the returns are in order from highest to lowest, resulting in the lowest final balance. In scenario six the returns are order highest to lowest which causes the account to have the highest final balance since the greatest returns occur at the beginning of the scenario when the account balance is larger. In this example, the highest and lowest final balances only differ by 4.72% of the initial balance since the returns are close together and the scenario occurs over 3 years.

Example 11: Withdrawal Period: Positive and Negative Returns Ranging by 30%

This example is for three years where the beginning value of the account is $3,000 and at the end of each year there is a withdrawal of $1,000. The order of the returns, -10%, 10%, and 20% vary in each scenario. The first scenario has the returns in ascending order and the sixth scenario has the returns in descending order. This example features a negative return, and the rates vary more drastically that previous examples.

Table 13. Withdraw $1,000 each period for 3 years with returns -10%, 10%, and 20%

Scenario				Account Balance				
	Return Year 1	**Return Year 2**	**Return Year 3**	**Initial Balance**	**Year 1 Balance**	**Year 2 Balance**	**Final Balance ($)**	**Final Balance (%)**
1	-10%	10%	20%	3,000	1,700	870	44	1.47%
2	-10%	20%	10%	3,000	1,700	1,040	144	4.80%
3	10%	-10%	20%	3,000	2,300	1,070	284	9.47%
4	10%	20%	-10%	3,000	2,300	1,760	584	19.47%
5	20%	-10%	10%	3,000	2,600	1,340	474	15.80%
6	20%	10%	-10%	3,000	2,600	1,860	674	22.47%

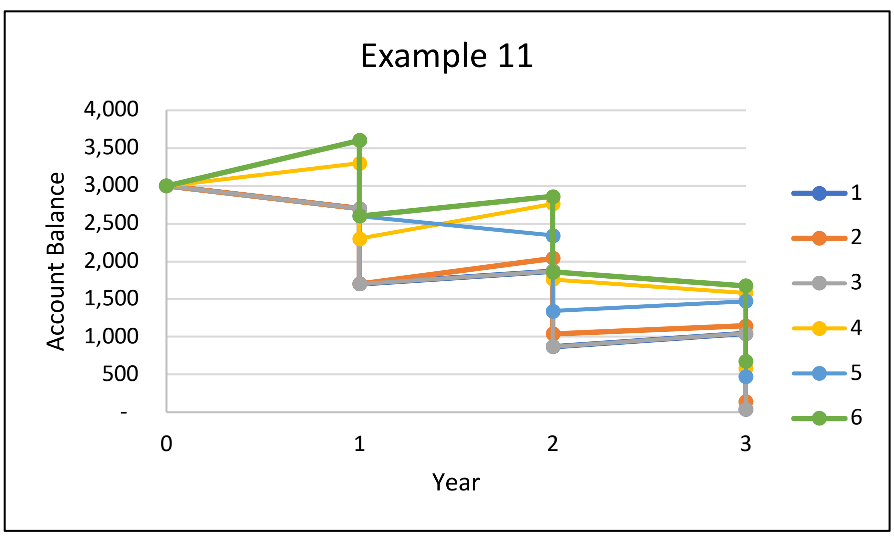


Figure 13. Withdraw $1,000 each period for 3 years with returns -10%, 10%, and 20%

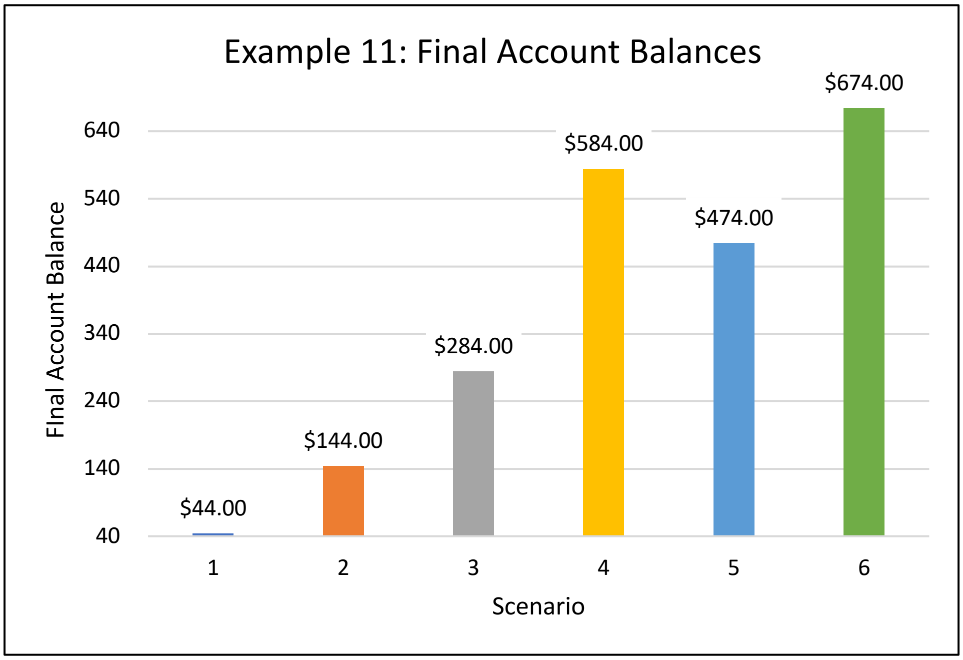


Figure 14. Final Account Balances

Results: Each scenario's final balance at the end of the period is different. Figure 14 is an example of sequence risk on a small scale. Figure 13 shows the account balances for three of the scenarios over the three-year period. Once again, for someone who hopes to drain their retirement fund, scenario one has the best outcome and scenario six has the worst outcome. In this example, the highest and lowest final balances differ by 21% of the initial balance. Since the returns in this example have a greater range than the returns in the previous two examples, it makes sense that the final balance's also have a greater variance in this example.

3.2 Long-Term Sequence Risk Withdrawal Trials

One major variable affecting sequence risk during the withdrawal period is the time, or the future expected lifetime of an individual. To better understand sequence risk on a more realistic scale during the same 30-year trials, that were completed for the investing period were completed for the withdrawal period. The same 100,000 permutations that were generated pseudo randomly and the best- and worst-case scenario returns were used to compose the same 100,002 permutations to be used for each strategy.

A third Python code was focused on performing withdrawal calculations like those done in Examples 8 through 13 with an amount being withdrawn each year. There was 1 general strategy that had 6 sub strategies. The strategies are as follows:

Table 14. Withdrawal Strategies

Strategy	Strategy Description
1.1	Withdraw 12.5% of initial account balance each year
1.2	Withdraw 10% of initial account balance each year
1.3	Withdraw 6.25% of initial account balance each year
1.4	Withdraw 5% of the initial account balance each year
1.5	Withdraw 4% of the initial account balance each year
1.6	Withdraw 3% of the initial account balance each year

The key information from these trials was the ending account value at 30 years, or when the account was exhausted. By analyzing these values within each strategy as well as between strategies, we were able to see the risk of exhausting one's retirement fund before the period is over.

Table 15 shows the number of times an account failed, meaning that the balance hit zero before the end of the period.

Table 15. Failure Counts of Withdrawal Strategies

Strategy	Number of Failures	Percent of Account Failures
1.1	85,519	85.52%
1.2	62,115	62.11%
1.3	68,377	68.38%
1.4	3,779	3.78%
1.5	666	0.67%
1.6	34	0.03%

Table 16 shows the minimum time of failure, maximum time of failure, and the average time of failure given the account failed. Furthermore, the average was calculated by taking the average of the time of failures for all trials, divided by the number of failures the strategy faced.

Table 16. Description of Time of Failure for Withdrawal Strategies

Strategy	Minimum Time of Failure	Maximum Time of Failure	Average Time of Failure
1.1	4	29	13
1.2	6	29	16
1.3	7	29	18
1.4	8	29	19
1.5	10	29	20
1.6	12	29	20

Strategy 1.5, acts in a similar manner to *Strategy 1.4*, but it is just a bit riskier. We can also see that *Strategy 1.1* has the lowest success rate by a significant value.

Withdrawal Strategy 1.1: Withdrawing $10,000 Each Year

Strategy 1.1 is the riskiest strategy in this group. With only a 13.59% success rate, it is safe to say that most accounts were exhausted before the 30 years was over. Figure 15 shows the distribution of the time that accounts failed. On the x-axis, the year is displayed, on the y-axis the number of failed accounts divided by the total number of accounts. Note that this distribution's maximum value is 0.8641 since 86.41% of all of the accounts were exhausted before the end of the period.

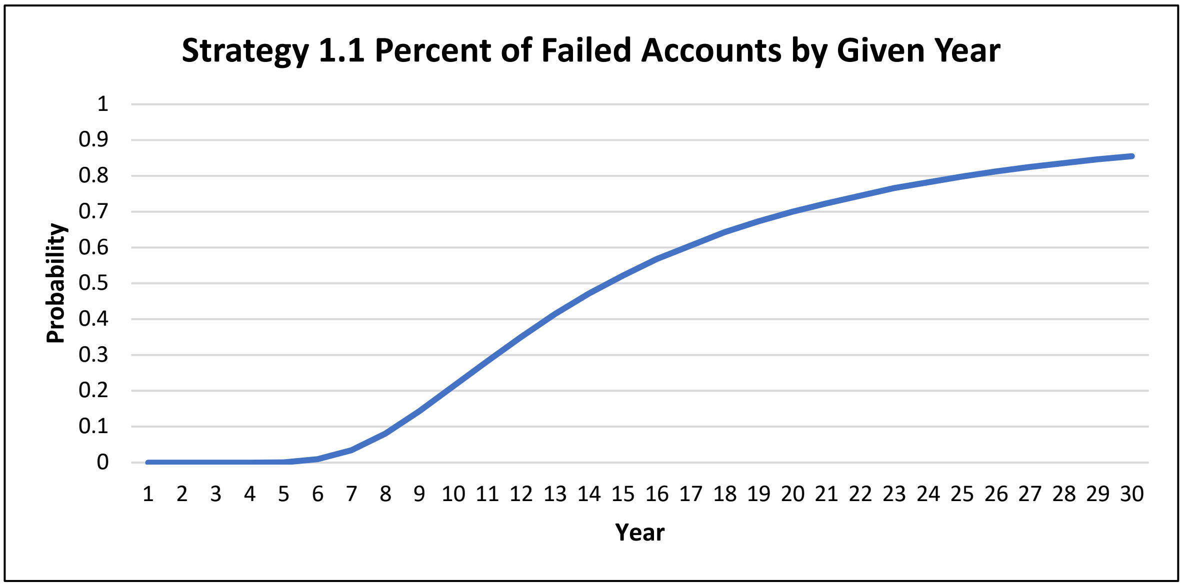


Figure 15. Strategy 1.1's Percent of Failed Accounts per Year

Figure 15 shows how drastically the failures occur throughout *Strategy 1.1*. It is important to note that all of the data labels are rounded to the hundredths place for ease of viewing. Because of this rounding, it might seem as though no accounts failed until time three, but that is not true. By time two, 385 accounts failed, but since the total number of failures is so large (86,410) only 0.45% of the failed accounts have already failed.

Withdrawal Strategy 1.6: Withdrawing 3% of the Initial Balance Each Year

As previously said, *Strategy 1.6* had the highest success rates of accounts that survived to the end of the period. Figure 16 shows the cumulative probability of the time of failure for this strategy.

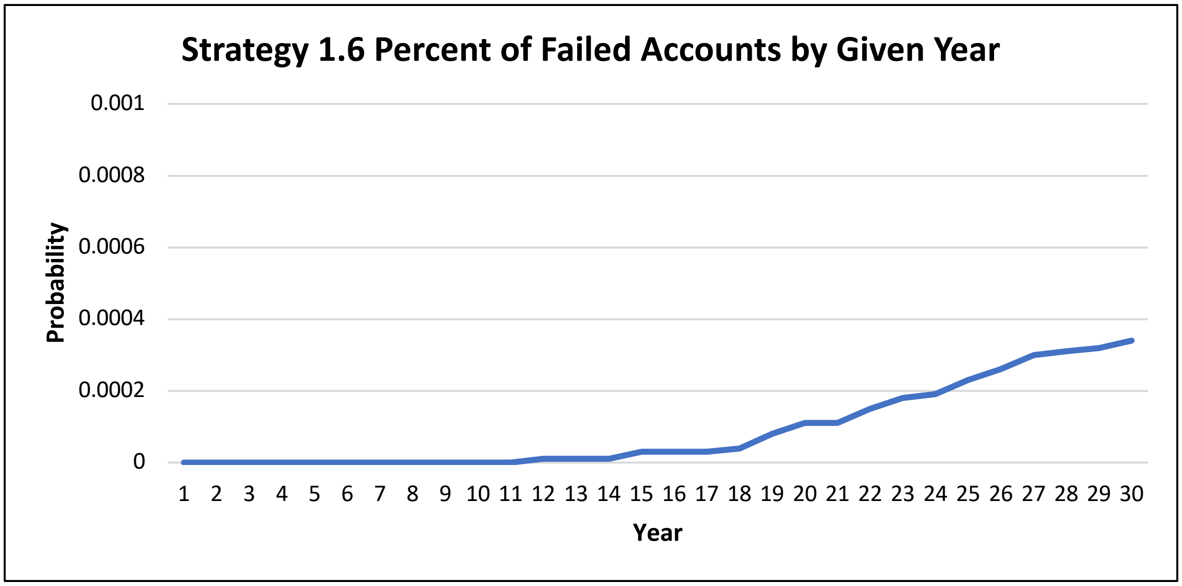


Figure 16. Strategy 2.3's Probability of Failure

When comparing Figure 15 to Figure 16, *Strategy 1.6* had significantly less failures. It is important to note that the y-axis for each of these figures differ since *Strategy 1.1* has a rate of failure of 85.52% and *Strategy 1.6* has a rate of failure of 0.03%. These failures are a big reason as to why *Strategy 1.6* distribution looks a little more rigid. It appears to look more like a straight line instead of the curve observed for *Strategy 1.1.* As shown in the figure, all the failures occurred between time 12 and time 29. For more analysis on each of the strategies, reference the Investing and Withdrawing Excel Workbook.

Overall, when looking at real life examples of the withdrawal period, the main concern is not exhausting the fund before the period is over. For example, if someone retires at 65 and exhausts their retirement fund by 75, they will be without any form of income until they die. On average, someone living in the United States likes to 79, so for the next four years this person who exhausted their account will constantly be short on money. It is also important to note that everyone has a different opinion on what the best way to plan for the withdrawal period of their retirement fund. For some people, the goal may be to have enough money to live, and have the account as close to zero as possible by their time of death. Other people might consider leaving a bequest to their loved ones, so they might want to leave a predetermined amount in their retirement fund by the time that they die so that they can help to support their loved ones. Whichever way is your ideal to the withdrawal period of a retirement fund, it is inevitable that you will experience sequence risk. But once again, sequence risk does not always affect accounts negatively, sometimes sequence risk can cause an account to end up much higher than the expected, but it is always important to expect the worst and hope for the best.

3.3 Perfect Withdrawal Amount: If You Could See into the Future

One theoretical way sequence risk could be eliminated would be to use the perfect withdrawal rate. **The Perfect Withdrawal Rate (PWR)** is the constant annual rate of withdrawal from a portfolio that will ultimately leave the portfolio with a balance of zero at the end of a given timeframe. A common rule for the perfect withdrawal rate is to follow the rule of withdrawing 4% of your account's initial balance each year, adjusted for inflation. Bengen created the 4% rule of thumb in 1994 when he compared a retirement fund to a sample of historical stock and bond returns (Clare, Andrew et al.)

In contrast, the **Perfect Withdrawal Amount (PWA)** is the constant annual withdrawal amount that can be taken from an account to leave the balance at the desired amount at the end of a given timeframe. These two strategies are important to analyze and assess as one ages. It is important to ensure that someone has had the ability to live their life to the fullest as well as the ability to leave money behind to one's beneficiaries, if desired. It is important to recognize that these rates of withdrawal are not foolproof. In examples like these, the equations and answers may seem trivial, but in real life, with the unpredictable returns in the market, there is no easy way to make predictions with complete certainty. Furthermore, theoretically someone will be able to withdraw the PWA, but if they live 15 years longer than projected, they will not have enough money in their accounts for the last 15 years of their life. As many always say, hindsight is 20/20. While looking back on past accounts, finding the PWA or PWR is a rather simple calculation. But since no one will know the exact returns for the rest of their life, there is no easy way to calculate the PWA before the withdrawal period. In general, if an account holder knows the initial balance and the returns the account will face during a desired period, the PWA can be calculated like it is in Example 14.

Example 12: Three Year Perfect Withdrawal Amount

This example has an account value beginning at $5,000 with the goal of exhausting the account by the end of the three-year period. So, looking at scenario one, someone currently has $5,000 in their account and in the next 3 years they want to exhaust their account given the returns their account will face are -10%, 5% and 10%, in that order. To be able to find the PWA we have to know when the account holder will be withdrawing from their account. In this example, withdrawals are made at the end of each year. We can begin to solve for the PWA by assign the variable X to the amount that will be withdrawn. Since the withdrawal is made at the end of each year, the initial account value will be subjected to the growth factor of 0.90 (or 1 + (-10%)). Affect the initial value is multiplied by the year one growth factor, the PWA of X can be withdrawn from the account. This pattern of multiplying by the growth rate and then withdrawing the PWA will continue for throughout the period. The calculation for scenario one can be represented in the following way:

$$EOY1\ Balance\text{: } 5{,}000(0.90) - X$$

$$EOY2\ Balance\text{: } (1.05)(EOY1) - X = 5{,}000(0.90)(1.05) - (1.05)X - X$$

$$EOY3\ Balance\text{: } (1.10)(EOY2) - X = 5{,}000(0.90)(1.05)(1.10) - (1.05)(1.10)X - (1.10)X - X$$

$$Since\ the\ account\ should\ be\ exhausted\ by\ EOY3\text{:}$$

$$0 = 5{,}000(0.90)(1.05)(1.10) - (1.05)(1.10)X - (1.10)X - X$$

$$5{,}000(0.90)(1.05)(1.10) = (1.05)(1.10)X + (1.10)X + X$$

$$5{,}197.50 = (3.255)X$$

$$\boldsymbol{X = 1{,}596.77}$$

For scenario one, the amount to withdraw each year should be $1596.77 to ensure the account is exhausted by the end of year 3. This same calculation process can be followed to receive all perfect withdrawal amounts. Refer to *Appendix C* for more calculations and derivations of the Perfect Withdrawal Amount formula and how the equation was founded.

Table 17. Perfect Withdrawal Amount Permutations of Returns

Scenario				Account Balance
	Return Year 1	**Return Year 2**	**Return Year 3**	**Perfect Withdrawal Amount**
1	-10%	5%	10%	$1,598
2	-10%	10%	5%	$1,622
3	5%	-10%	10%	$1,682
4	5%	10%	-10%	$1,798
5	10%	-10%	5%	$1,735
6	10%	5%	-10%	$1,827

To see how the perfect withdrawal amount formula works, we will look at scenario 6 from the above example. We will assume the initial balance of $5,000, the returns of 10%, 5% and -10% in that order, and that $1,827 will be withdrawn each year. The goal is for the account to be exhausted by the end of the three-year period.

The calculation is as follows:

$$EOY1\ Balance: 5{,}000(1.10) - 1827 \ = \ 3{,}673$$

$$EOY2\ Balance: (1.05)(EOY1) - 1{,}827 = 3{,}673(1.05) - 1{,}827 \ = \ 2{,}030$$

$$EOY3 Balance: (1.10)(EOY2) - 1{,}827 = 2{,}030(0.90) - 1{,}827 \ = \ 0$$

$Since\ the\ account\ is\ exhausted\ by\ EOY3, \$1{,}827\ is\ the\ perfect\ withdrawl\ amount\ of\ scenario\ 6.$

Our group was then able to derive a generalized formula for the PWA by conducting various examples, which can be found in *Appendix C*. These various examples included longer time periods and greater variation in the returns. The generalized PWA formula was derived as:

$$B\left(\prod_{i=1}^{n} G_i\right) \ = \ X\left(\left(\sum_{i=2}^{n}\left(\prod_{j=i}^{n} G_j\right)\right) + 1\right)$$

Equation 1. Perfect Withdrawal Amount Formula

The variables in Equation 1 above are as follows:

B = Beginning Account Balance

G_i = Growth Rate at Time i

n = Length of Time Period

X = Perfect Withdrawal Amount for the Period

4 Developing a Sequence Risk Score

Our group developed a measure for sequence risk from our analysis of various simulations and calculations performed on different sets of returns (that is, different permutations of the same set of returns). Here is the formula we developed to assign a score to any premutation:

$$Sequence\ Risk\ Score\ = \frac{Range\ \times \sqrt{SemiVariance}}{(Geometric\ Mean)^2}$$

Equation 2. Sequence Risk Score Formula

Range: Maximum Return – Minimum Return

Semi-Variance: The variance of the returns below the mean

Geometric Mean: The average of the returns

Each component of the score above contributes to its ability to differentiate between different permutations:

- The range is indicative of the spread of the returns. The larger the range of returns, the more potential risk that is present.
- The square root of semi-variance, also known as the semi-deviation, incorporates the component of risk for when a return is under-performing the average.
- Lastly, dividing by the geometric mean squared eliminates the units within the equation and scales the values of the score.

Since the score calculation is dimensionless, this measure can easily be compared between different sets of returns. The score calculation is best used as a comparison measure, like the Sharpe Ratio, which is discussed in Section 4.3.2. For instance, if Set 1 has a sequence risk score of 2.0 and Set 2 has a sequence risk score of 10.0, then Set 1 has a lower sequence risk than Set 2, so Set 1 would be preferred since there is less risk associated with the returns. Another important characteristic of our risk score calculation is that a set of returns with no variance, meaning the returns are the same each year, has a sequence risk score of 0.0, which was proven in the above sections. Overall, this score took trial and error and research to decide upon. In the next section, we describe our team's approach to developing this risk score.

4.1 Initial Research

Many studies have analyzed PWR, PWA, and ways to quantify sequence risk. This analytical work is often completed using models, and many times probability of failure (POF) to show the exposure to sequence risk.

According to Frank, Mitchell, and Blanchett the POF, along with a 3D model, can be used to best determine what would be the perfect withdrawal rate for a retiree. In their study, they determined that if your probability of failure is equal to or about 30% it would be in the best interest of the retiree to decrease their withdrawal amount slightly. This idea is meant to help someone adjust their rates to keep them on track with their finances.

Another model that was used to analyze simulations is called the Cyclically Adjusted Price to Earnings Ratio (CAPE Ratio). In *Reducing Sequence Risk Using Trend Following and the CAPE Ratio*, this model was used in combination with trend following to minimize sequence risk for a retirement fund. Trend following is the idea of investing in an asset when the rates are in an uptrend and then switching the asset to cash once the rate is below its typical average. The conclusion from this study was that the CAPE Ratio could be used as a predictive power to create a much better retirement experience.

Instead of looking for ways to mitigate sequence risk, our project was looking to quantify sequence risk in a variety of different investment situations. Before we developed a way to quantify sequence risk, our group reflected on what we learned from our earlier exploration.

Through the analysis of our investing and withdrawing scenarios we learned:

1. The order of returns matter.
2. The greater the spread of returns, the greater the sequence.

The longer the period, the greater the sequence risk. All of these points should be considered in our risk measure.

4.2 Developing Initial Sets of Returns and Calculations

The first step in developing a score for sequence risk was to create sets of returns and compare each set's summary statistics. We first created 10 sets of returns for a period of 10 years. We used scenarios where we felt varying degrees of sequence risk would affect the final balances, based on our intuitive understanding of sequence risk. The various returns were chosen in an attempt to have clear differences in the expected sequence risk scores. One set had the same rates for all 10 years which gave us a baseline score of zero when performing our calculations. Each set used the same group of 25,000 permutations as the previous examples.

Table 18 shows the 10 set of returns we initially used when performing sequence risk score calculations.

Table 18. Rates of Returns for Initial Risk Score Calculations

Initial Returns										
	Year 1	**Year 2**	**Year 3**	**Year 4**	**Year 5**	**Year 6**	**Year 7**	**Year 8**	**Year 9**	**Year 10**
Set 1	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%
Set 2	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%	7.0%	8.0%	9.0%	10.0%
Set 3	-20.0%	-15.0%	-10.0%	-5.0%	5.0%	10.0%	15.0%	20.0%	25.0%	30.0%
Set 4	-20.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%
Set 5	2.0%	4.0%	6.0%	8.0%	10.0%	12.0%	14.0%	16.0%	18.0%	20.0%
Set 6	-2.0%	-2.0%	-2.0%	-2.0%	-1.0%	-1.0%	-1.0%	-1.0%	6.0%	10.0%
Set 7	-20.0%	2.0%	2.0%	2.0%	2.0%	2.0%	2.0%	2.0%	2.0%	30.0%
Set 8	2.0%	3.0%	4.0%	5.0%	6.0%	7.0%	10.0%	11.0%	12.0%	13.0%
Set 9	1.0%	1.0%	2.0%	2.0%	3.0%	3.0%	4.0%	4.0%	5.0%	5.0%
Set 10	5.0%	5.0%	5.0%	6.0%	6.0%	6.0%	7.0%	7.0%	8.0%	20.0%

The next step in the process was looking to see if we could illustrate sequence risk in a graphical manner. The goal of having graphical representations of the returns was to order the graphs from low to high sequence risk by visually analyzing them. However, once we generated the histograms for the 10 sets of returns using the 25,000 permutations we felt as though we could only categorize the sets by low, medium, and high as it was difficult to determine which set had more sequence risk to another. Example 13 shows a table of four sets that were used when determining the sequence risk as well as each of the sets respective histograms that were categorized in the low, medium, and high categories.

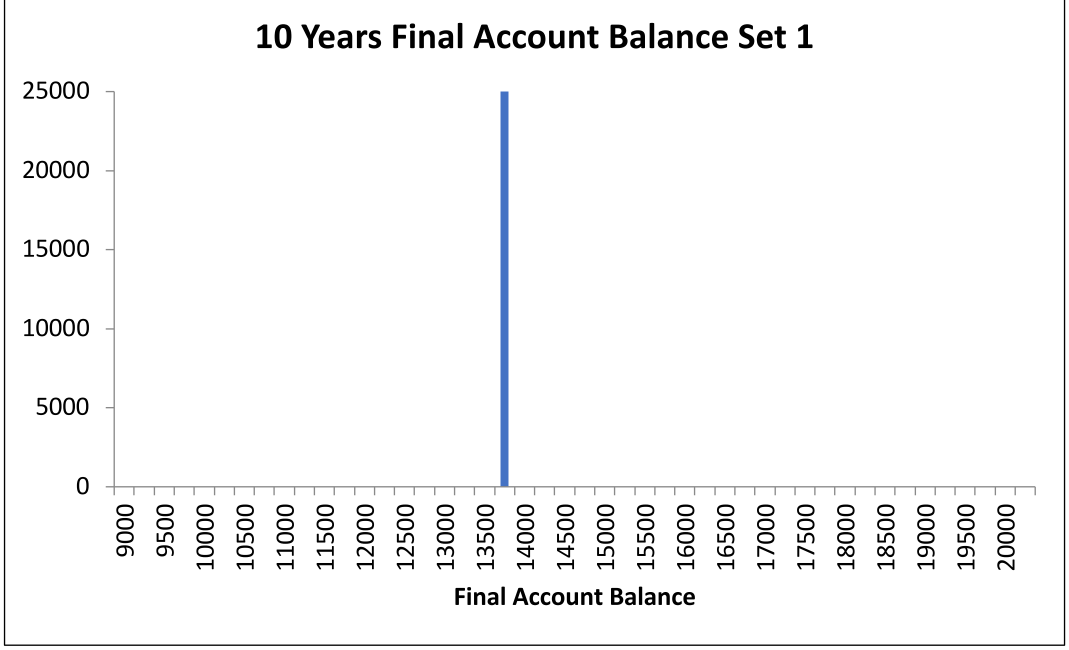
Example 13: Establishing Histograms with No, Low, Medium, and High Sequence Risk

In the following table the returns for sets 1, 2, 3, and 5 are shown. The color scale shows the level of sequence risk each of these sets has. Green represents no sequence risk, yellow represents low sequence risk, orange represents medium sequence risk and red represents high sequence risk. These categories of sequence risk were determined by comparing the histograms of the final account balances of 25,000 permutations of these rates. These histograms can be seen in Figure 17.

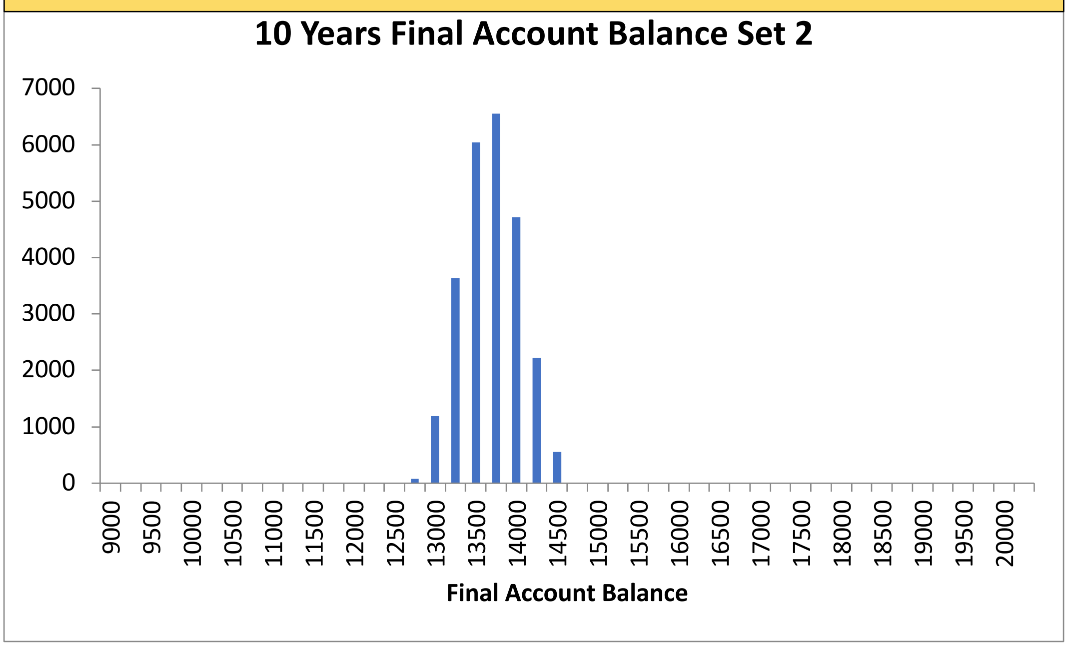
Table 19. Categorized Sets of Returns of Initial Sets

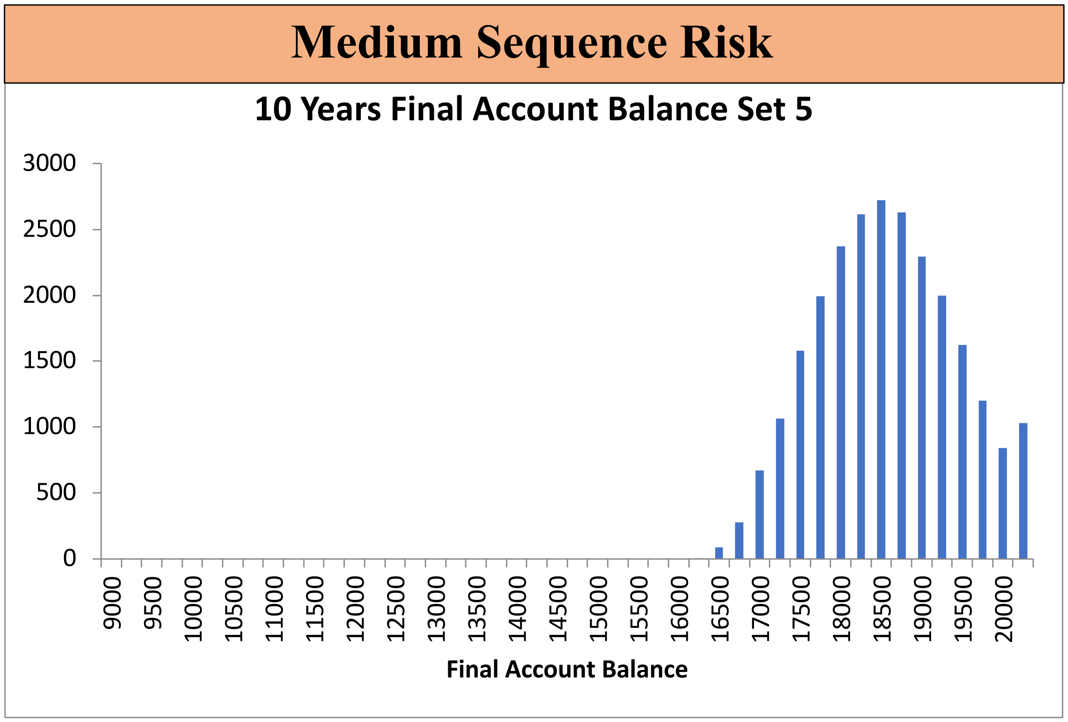
Initial Returns										
	Year 1	**Year 2**	**Year 3**	**Year 4**	**Year 5**	**Year 6**	**Year 7**	**Year 8**	**Year 9**	**Year 10**
Set 1	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%	5.5%
Set 2	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%	7.0%	8.0%	9.0%	10.0%
Set 5	2.0%	4.0%	6.0%	8.0%	10.0%	12.0%	14.0%	16.0%	18.0%	20.0%
Set 3	-20.0%	-15.0%	-10.0%	-5.0%	5.0%	10.0%	15.0%	20.0%	25.0%	30.0%

No Sequence Risk



Low Sequence Risk





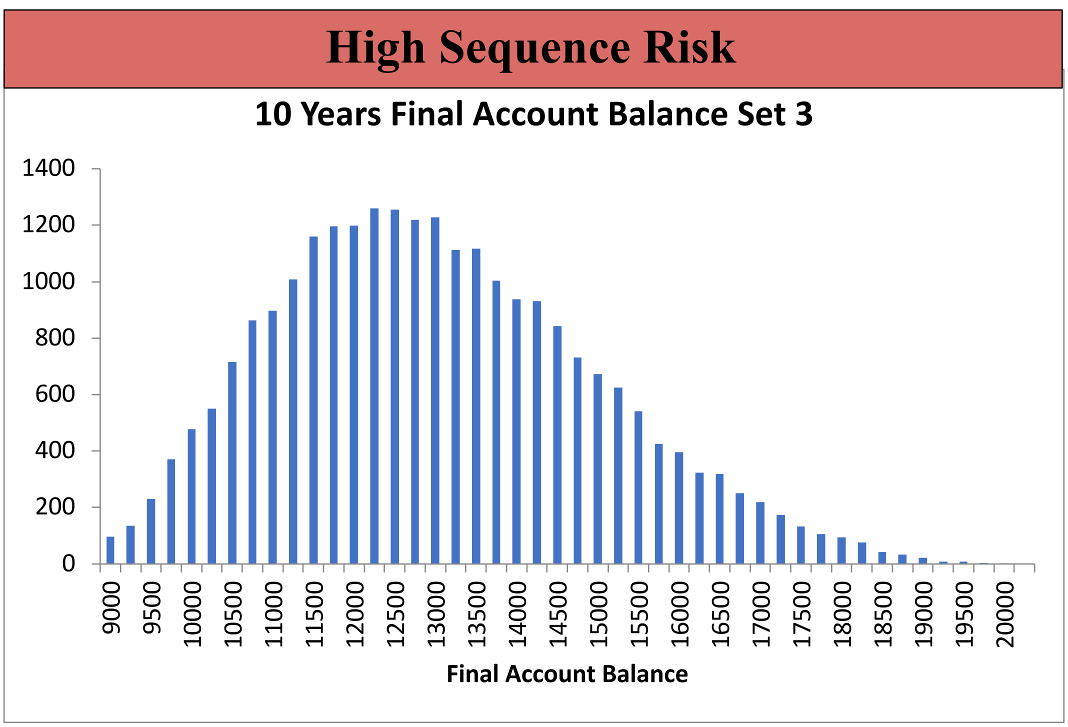


Figure 17. Categorized Histograms of Initial Sets

The ranking of the histograms was based on the visual representation. The gauge for rankings was based on the spread of the distribution, leading it to be a very subjective measure. Set 1 faces no sequence risk since the returns were constant over the 10-year period. As described in Example 1, since all of the returns are the same, there is no sequence risk present and all of the final account balances are the same. All of the Final balances are $13,583.50. The set with low sequence risk is Set 2. Set 2 has a larger range of final balances leading to more sequence risk than Set 1. The final balances are all between the values $12,750 and $14,750. This range is slightly higher than Set 1, at around $2,000, which leads to more potential risk. Set 5 has medium sequence risk since the frequencies are slightly lower for the range of values. Additionally, all of the final balances for this graph are between $16,111.51 and $21,102.09. The range of Set 5 is nearly $5,000 which leads to even more riskiness than what is faces by other examples. The set ranked as a high sequence risk is Set 3. The range of Set 3 is almost $12,000 which is over double the range seen in the medium sequence risk ranking. The values for Set 3 range from $8,364.90 to $19,949.64. The variability of final balances is what causes the highest amount of sequence risk. These four sets of returns have obvious differences between the ranking values, but that is not always the case.

The difficulty that arose when trying to categorize the histograms was regarding gauging the difference between the low and medium categories. Example 14 shows Set 2 and Set 6 with their corresponding graphs that demonstrated this difficulty. Set 6 was categorized as medium sequence risk and Set 2 was categorized as low. Notice the difficulty in pin-pointing the sequence risk from the sets of returns given their similar distributions, but different upper and lower bounds.

Example 14: Histograms Showing the Difficulty Distinguishing Between Low and Medium

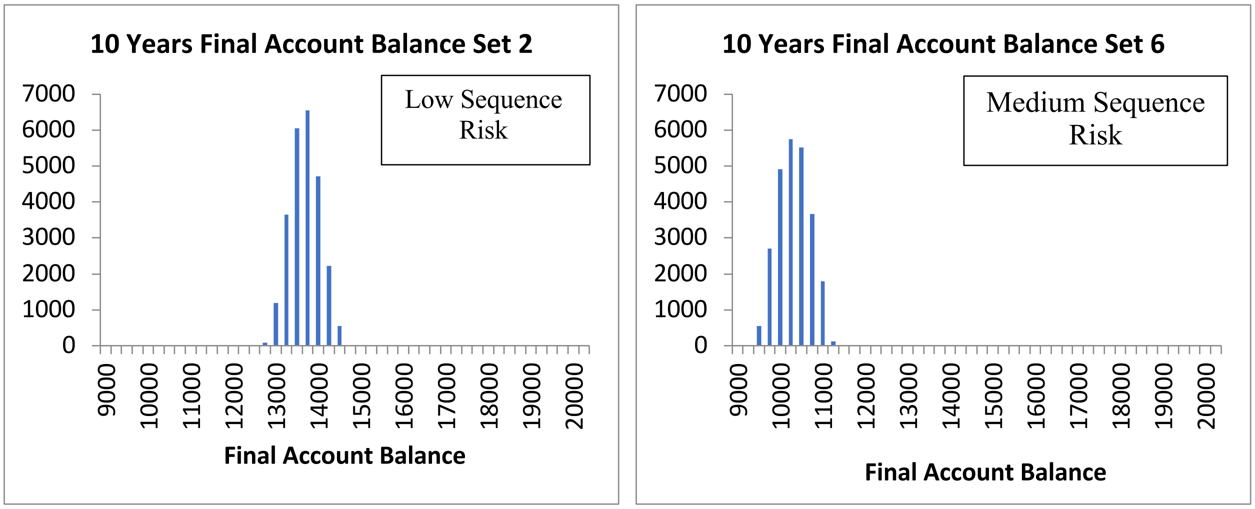


Figure 18. Distinguishing Sequence Risk Between Histograms

We determined from these sets of returns that the low, medium, and high categories would still be a good visual check for developing a sequence risk score moving forward. We can identify clear differences between low and high sequence risk but sets with medium sequence risk are less obvious. It is important to note, that these categorizations do not consider the final balances of the accounts, but instead focus on the spread of the final balances. We planned to continue to generate the histograms and compare them to see if the various score calculations corresponded to the low, medium, and high categories. The results from these histograms indicated that a score is necessary to be able to determine the differences between sets of returns and their respective sequence risks.

Along with the histograms, for each of our 10 sets of 25,000 permutations we generated summary statistics that included the range, standard deviation, variance, and other typical statistical calculations. These summary statistics were used to provide statistics on the 10 returns within each set as well as between sets. Initially, we combined these different measures to see if there were any number that started to look promising. Table 20 shows notable summary statistics for the 10 sets of returns. The rest of the summary statistics can be found in the Initial Data Excel Workbook.

Table 20. Summary Statistics of the Initial Sets of Returns

Summary Statistics					
	Mean	**Geometric Mean**	**Range**	**Standard Deviation**	**Variance**
Set 1	0.0550	0.0550	0.0000	0.0000	0.0000
Set 2	0.0550	0.0453	0.0900	0.0287	0.0008
Set 3	0.0550	0.1327	0.5000	0.1650	0.0272
Set 4	0.0250	0.0574	0.2500	0.0750	0.0056
Set 5	0.1100	0.0906	0.1800	0.0574	0.0033
Set 6	0.0040	0.0199	0.1200	0.0393	0.0015
Set 7	0.0260	0.0330	0.5000	0.1124	0.0126
Set 8	0.0730	0.0622	0.1100	0.0374	0.0014
Set 9	0.0300	0.0261	0.0400	0.0141	0.0002
Set 10	0.0750	0.0680	0.1500	0.0427	0.0018

In addition to ranking the final account balances, we computed summary statistics for the initial sets of returns. The goal of these summary statistics was to see if any of the calculations matched the rankings seen by the visual ranking process completed in Example 13. Additionally, we hoped to determine if the specific sets ranked in similar areas across the statistics. The first statistic calculated was the mean. Set 5 represented the highest mean value at 0.11 while Set 6 represented the lowest mean value at 0.004. The mean was the only statistic that has Set 5 in the maximum ranking. Set 6 had the lowest mean as well as the lowest geometric mean. The set with the highest geometric mean was Set 3. Set 3 ranked the highest across the other categories of range, standard deviation, and variance. Set 7 ranked as the highest range value which is equal to Set 3's range at 0.5. Set 1 ranked at the lowest values for range, standard deviation, and variance, which is a trend we hoped to see based on the visual rankings of the sets. The inconsistencies across the summary statistics made it difficult to create a sequence risk score. Our initial summary statistic calculations helped to develop different elements of a potential sequence risk score.

We learned was that we needed a way to keep consistency within the sets of returns. To increase consistency, we developed a new set of 10 rates, similar to our first set, but where the geometric mean of the returns was the same for each set of returns.

4.3 Sets of Returns and Calculations Holding the Geometric Mean Constant

Our next step was generating sets of returns for 10, 20 and 30 years and then running each of the sets through our simulator for 25,000 permutations. We began by developing ten sets of returns for 10 years, each with the same geometric mean return. If the geometric mean was impacting the results of our balances, then we would be potentially misinterpreting those impacts as sequence risk. To select the returns for our 20 and 30 years sets we used the set of returns from the 10-year trials twice and three times, respectively. Table 21 shows the 10-year sets of returns each with a geometric mean of 2.81%.

Table 21. Sets of Returns with a Constant Geometric Mean

Returns with a Geometric Mean of 0.281										
	Year 1	**Year 2**	**Year 3**	**Year 4**	**Year 5**	**Year 6**	**Year 7**	**Year 8**	**Year 9**	**Year 10**
Set 0	2.81%	2.81%	2.81%	2.81%	2.81%	2.81%	2.81%	2.81%	2.81%	2.81%
Set 1	-20.00%	-10.00%	-5.00%	2.00%	8.50%	9.75%	10.75%	11.75%	12.75%	13.75%
Set 2	-12.00%	-5.00%	-3.00%	-2.00%	6.00%	7.00%	8.00%	9.00%	11.00%	12.00%
Set 3	-16.00%	-14.00%	-12.50%	-11.00%	10.00%	13.75%	15.50%	16.50%	17.50%	18.50%
Set 4	-40.00%	-20.00%	-1.00%	5.00%	7.15%	10.00%	15.00%	20.00%	25.00%	30.00%
Set 5	-5.00%	-4.00%	-3.00%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	21.40%
Set 6	-8.30%	-3.50%	-1.50%	1.00%	4.00%	5.00%	6.00%	8.00%	9.00%	10.00%
Set 7	-10.00%	-4.00%	-0.50%	1.00%	3.00%	5.00%	6.00%	9.00%	10.00%	10.50%
Set 8	-9.50%	-4.00%	-1.75%	-1.15%	5.25%	6.25%	7.25%	8.25%	9.25%	10.25%
Set 9	-5.00%	-2.00%	-1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	10.00%
Set 10	-11.00%	-5.50%	-1.00%	2.75%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%

Table 22. Summary Statistics for the Constant Geometric Mean Sets

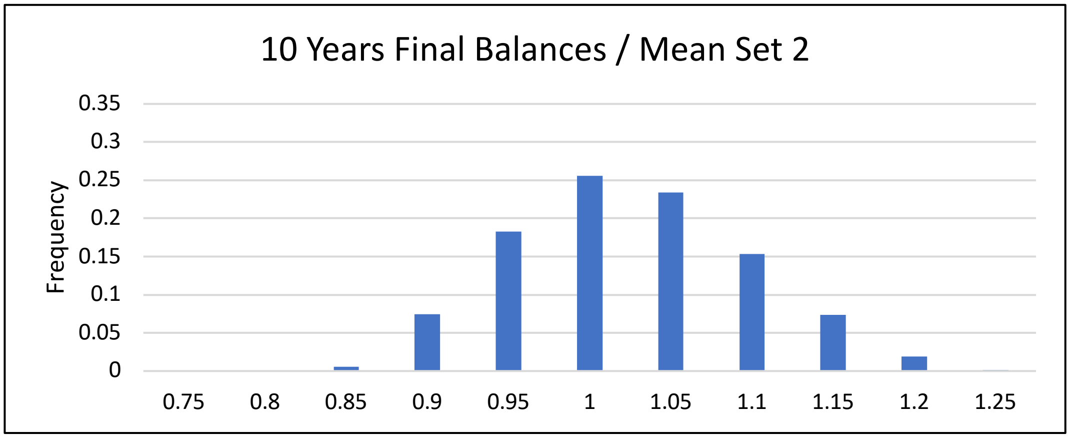
Summary Statistics					
	Mean	**Geometric Mean**	**Range**	**Standard Deviation**	**Variance**
Set 0	0.0281	0.0281	0.0000	0.0000	0.0000
Set 1	0.0343	0.0281	0.3375	0.1088	0.0118
Set 2	0.0310	0.0281	0.2400	0.0762	0.0058
Set 3	0.0383	0.0281	0.3450	0.1426	0.0203
Set 4	0.0512	0.0281	0.7000	0.2016	0.0407
Set 5	0.0304	0.0281	0.2640	0.0712	0.0051
Set 6	0.0297	0.0281	0.1830	0.0564	0.0032
Set 7	0.0300	0.0281	0.2050	0.0623	0.0039
Set 8	0.0301	0.0281	0.1975	0.0631	0.0040
Set 9	0.0290	0.0281	0.1500	0.0430	0.0018
Set 10	0.0303	0.0281	0.2100	0.0651	0.0042

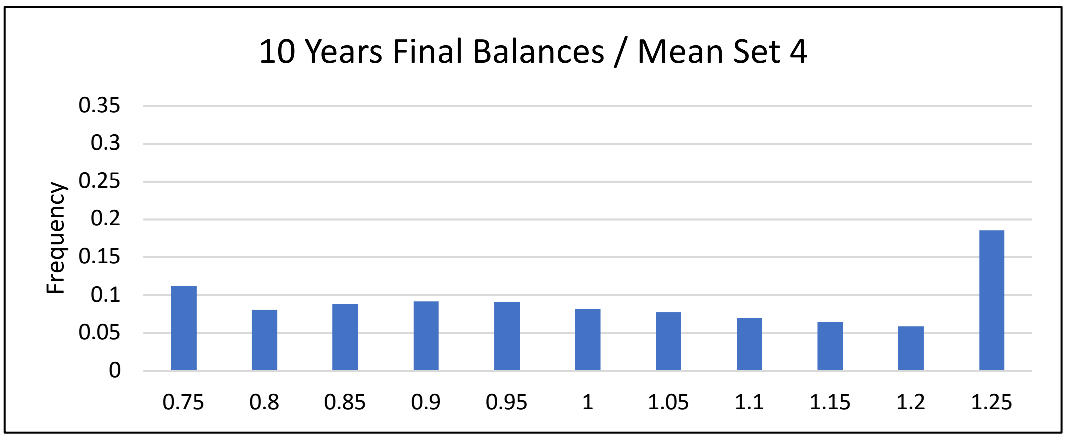
Similar to the initial return values, we computed summary statistics across the sets of returns. The main difference between this set of returns and the initial set of returns is that the geometric mean was held constant across the sets of returns to minimize the outside effects from the geometric mean. The values for the means had Set 4 with the highest value at 0.0512 while Set 9 had a mean value of 0.0290. The summary statistic with similar rating is Standard Deviation. Set 4 also ranked the highest at 0.0700, and set 9 had the lowest standard deviation of 0.1500. Set 4 ranked the highest in the statistics, and Set 9 had ranked the lowest in the statistics, not including Set 0, which was the baseline set to include no sequence risk. While, there seems to be a trend across the sets made, it is difficult to determine one specific metric using summary statistics. The summary statistics were helpful to determine potential calculations using a combination of multiple calculations.

4.3.1 Graphing the Sets of Returns of the 25,000 Permutations using Same Geometric Mean

Similar to the initial set of calculations, our group looked to graph the various histograms for the 10, 20, and 30 years. Examples 15, 16 and 17 show histograms for Set 2, Set 4, and Set 6 for each of the 10, 20 and 30 years respectively. These three sets were chosen to represent one graph from each sequence risk category, low, medium, and high. In addition to the geometric mean being constant, the portrayal of the graphs changed for these sets of returns. Instead of using frequency in terms of the total number of trials as the y-axis, we decided to graph the values with the frequency being the percent of the 25,000 permutations in the specific x-axis bin. Additionally, the x-axis now represents “final account balance divided by the mean of the final account balances from the set of returns”. The format of these axis allowed us to visually see if something is above or below the mean based off the value of 1 being the middle of the distribution.

Example 15: 10 Years Trials Showing the Final Balance / Mean





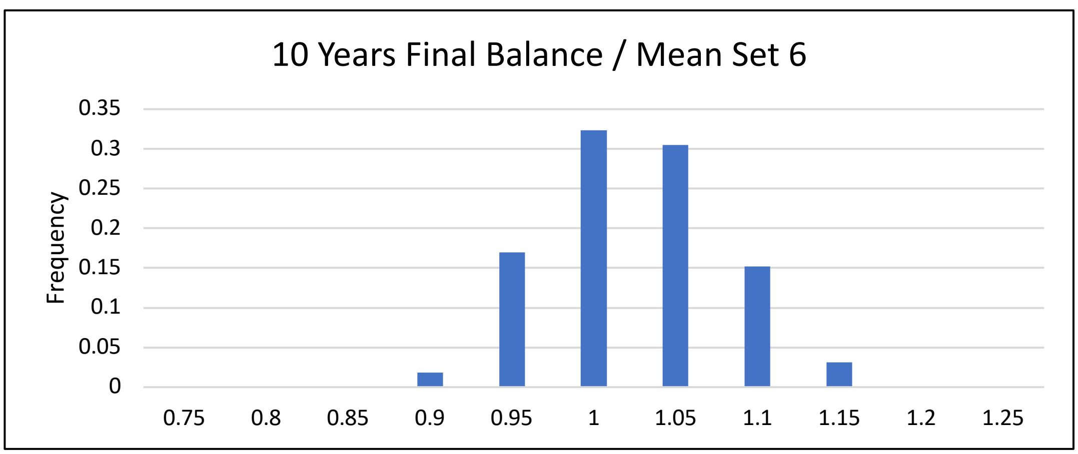


Figure 19. 10 Year Trials Set 2, 4, 6 Histograms

Based on the histograms for the 10-year trials, the ranking for the three sets is:

Low: Set 6

Medium: Set 2

High: Set 4

The ranking of score is based on the distribution across the x-axis. Set 4 is distributed with most values above or at 1 with the maximum frequency being a little more than 0.30 for one of the categories. Set 2, has a slightly higher range of values with the maximum frequency around 0.25. Set 4 is ranked as a high since the set of returns has the highest range of x-axis values with a lower frequency on the y-axis. The maximum frequency is between 0.15 and 0.20 for the highest x-axis bins. The ranking of these three graphs is in relation to these three specific sets. If another set of returns was compared, the ranking on the individual sets might vary. The same ranking process was completed for the 20-year and 30- year sets of returns.

Example 16: 20 Year Trials Showing Final Balance / Mean

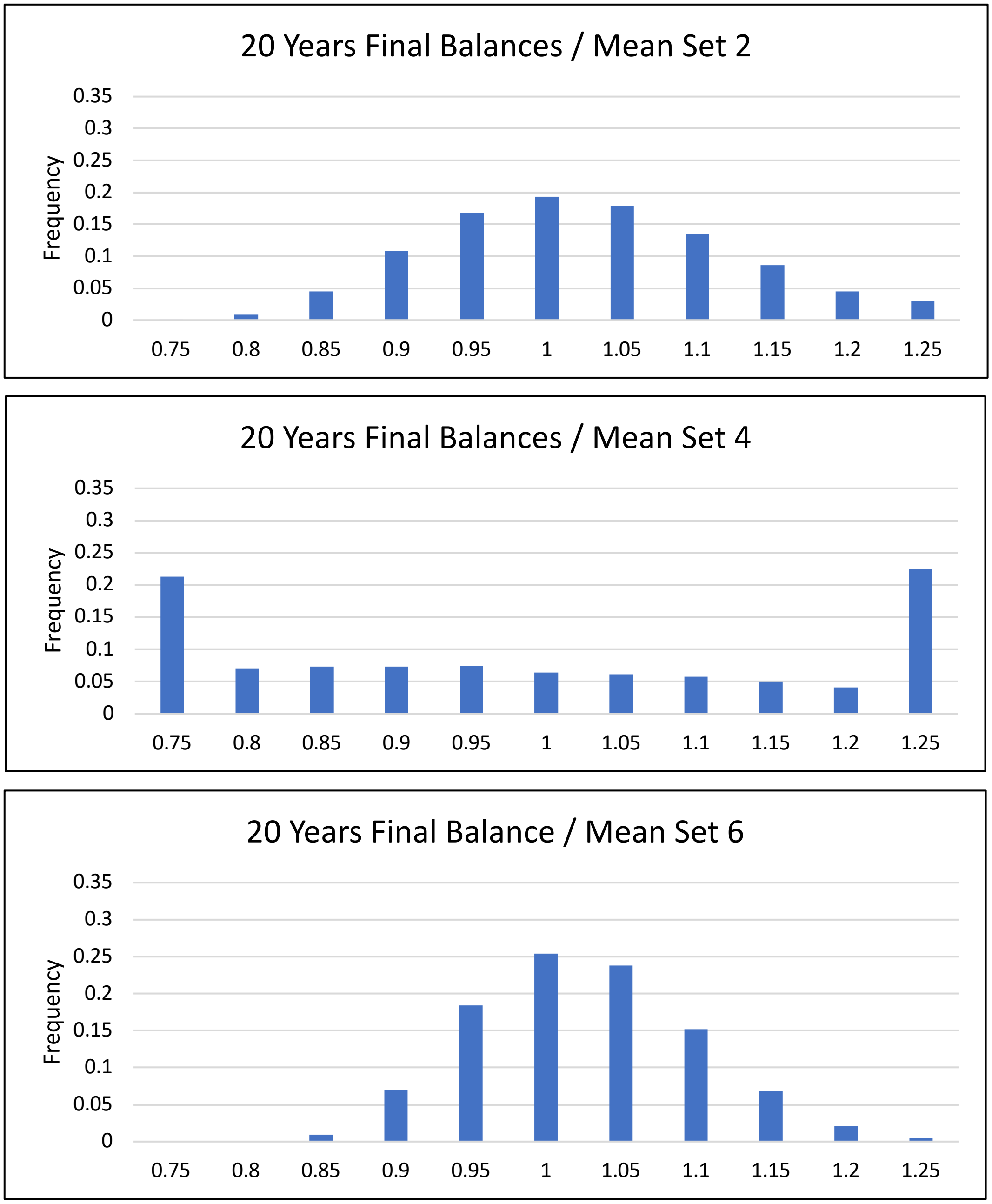


Figure 20. 20 Year Trials Set 2, 4, 6 Histograms

Based on the histograms for the 20-year trials, the ranking for the three sets is:

Low: Set 6

Medium: Set 2

High: Set 4

The ranking of score is based on the distribution across the x-axis. Set 6 is distributed with most values above or at 1 with the maximum frequency being almost around 0.25 for two categories. Set 2, has a slightly higher range of values with the maximum frequency being between 0.15 and 0.20 in three categories. Set 4 is ranked as a high since the set of returns has the highest range of x-axis values with a lower frequency on the y-axis. The maximum frequency is over 0.20 for the two highest x-axis bins but at opposite ends of the graph. It is important to note, the ranking of these sets of returns is relative to the specific three sets being shown. If another set were pulled out, the ranking of Set 2 might change to be low since for example Set 1 has a higher sequence risk.

Example 17: 30 Years Trials Showing Final Balance / Mean

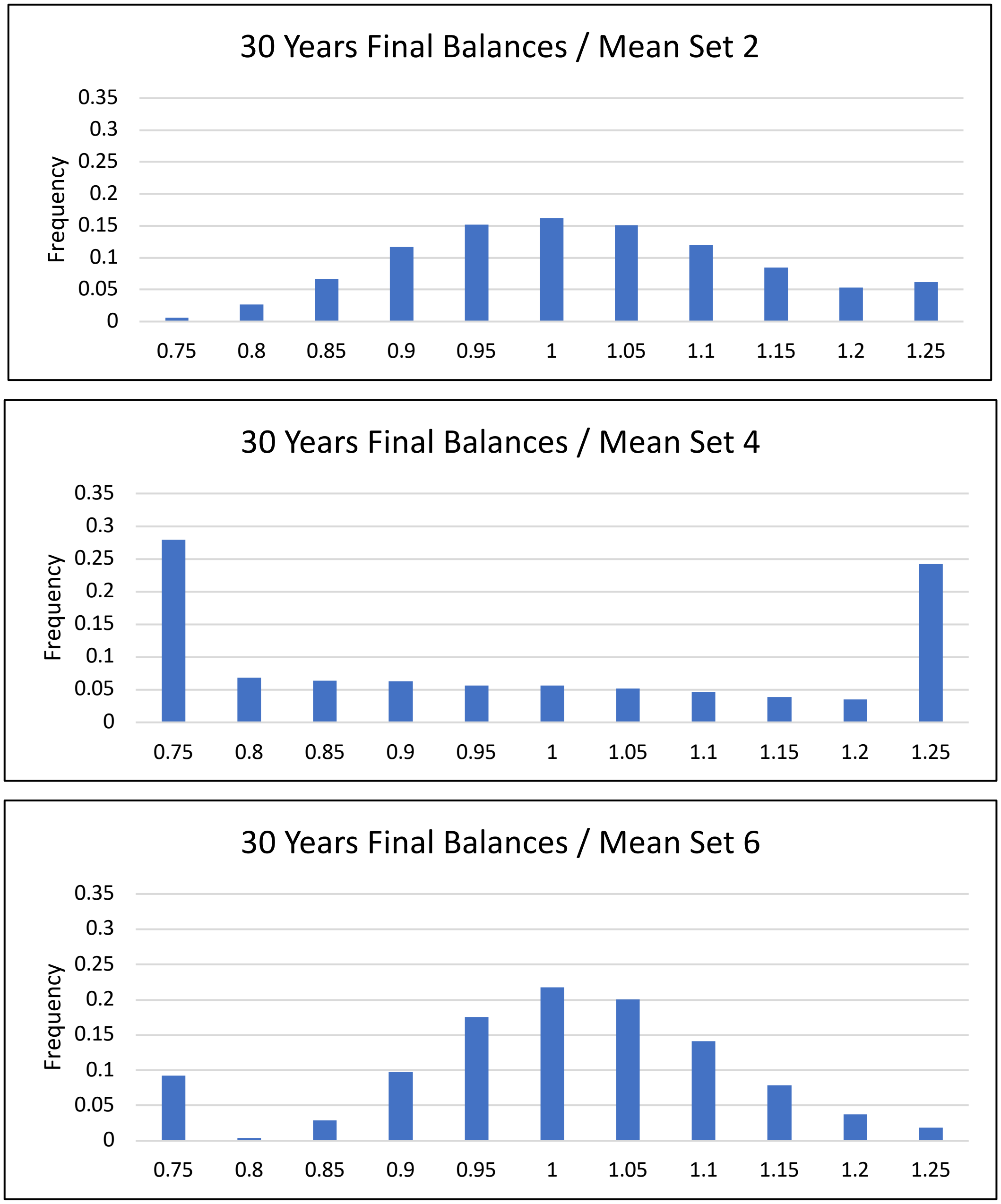


Figure 21. 30 Year Trials Set 2, 4, 6 Histograms

Based on the histograms for the 30-year trials, the ranking for the three sets is:

Low: Set 6

Medium: Set 2

High: Set 4

The ranking of score is based on the distribution across the x-axis. Set 6 is distributed with most values above or at 1 with the maximum frequency being between 0.20 and 0.25 for one of the categories. Set 2, has a slightly higher range of values with the maximum frequency being around 0.15. Set 4 is ranked as a high since the set of returns has the highest range of x-axis values with a lower frequency on the y-axis. The maximum frequency is around 0.25 but on both ends of the graph. The differences between the graphs for the 30-years is more difficult to determine than the graphs with fewer years since Set 2 and Set 6 follow more similar distributions. However, we did find that each of the time periods, 10, 20, and 30 years had the same sets of returns that were in the low, medium, and high categories. Intuitively, this makes sense because we are using the same returns in the 20 and 30 years but repeating the 10-year returns twice or three times as needed.

4.3.2 Sharpe Ratio and Inverse Sharpe Ratio Using Same Geometric Mean

Our group chose to begin with these summary statistics for each of the sets since these were the summary statistics produced in Excel's Data Analysis add-on for each of the trials that were run in our Python code. When working towards a formula to quantify sequence risk, we realized that these measures would not be enough. We then began researching other statistical measures, risk measures and coherent risk measures. For instance, as shown in Table 23, we analyzed the Sharpe Ratio of each of the sets.

Sharpe Ratio is a measure of return used to compare the performance of portfolios. In our case, the portfolio would have the already known 10 returns from our various sets. The Sharpe Ratio formula for a portfolio is:

$$Sharpe\ Ratio = \frac{Average\ Return - Risk\ Free\ Return}{Standard\ Deviation\ of\ Returns}$$

Equation 3. Sharpe Ratio Equation

For our sets of returns, since there was no risk-free return, the Sharpe Ratio was the averages of the returns divided by the standard deviation of returns. The Sharpe Ratio for each of the sets was calculated as followed:

$$Sharpe\ Ratio(Set\ 1) = \frac{Mean(Set\ 1)}{Standard\ Deviation(Set\ 1)} = \frac{0.0343}{0.1088} = 0.3147$$

Table 23. Sharpe Ratio of the Constant Geometric Mean Sets of Returns

Summary Statistics	
	Sharpe Ratio
Set 0	Undefined
Set 1	0.3147
Set 2	0.4067
Set 3	0.2683
Set 4	0.2537
Set 5	0.4271
Set 6	0.5265
Set 7	0.4813
Set 8	0.4772
Set 9	0.6744
Set 10	0.4649

A Sharpe Ratio greater than 1.00 indicates that the volatility associated with the returns is low, which would be a would be considered a good investment in terms of sequence risk. One important factor is that the Sharpe Ratio is best used in comparison rather than as a stand-alone number. In the case of our sets from Table 23, all of the sets had a ratio below 1.00. Since Sharpe Ratio is comparable between portfolios, given that all of the share ratios are below 1.00, the measure of 1.00 being a sound investment might need to be adjusted for the specific set of returns. An adjusted ratio of 0.50 might be a more worthwhile measure.

The Sharpe Ratio taught us about investments and how it could be applicable to sequence risk. One theory of portfolio analysis that was realized by the Sharpe Ratio analysis was that the ratios or scores can be comparative measures. When working towards our final sequence risk score, we thought that a comparative measure could be very beneficial. In some cases, a set might have the highest sequence risk score, but when compared to a different population of sets, the same set might no longer have the highest risk score, meaning that a measure of comparison is subjective to the investment options presented.

Additionally, the Sharpe Ratio lead us into another calculation that seemed to be a good measure for our sets which was performing the inverse of the Sharpe Ratio. The equation for the inverse of the Sharpe Ratio is:

$$Inverse\ of\ the\ Sharpe\ Ratio: \frac{Standard\ Deviation}{Average\ Return}$$

Equation 4. Inverse Sharpe Ratio Equation

Since the best available return is zero, the inverse of the Sharpe Ratio is equal to the coefficient of variation for a set of returns. Table X shows the results from calculating the inverse of the Sharpe Ratio.

The inverse of the Sharpe Ratio, or the coefficient of variation, for the sets were calculated in excel. For example, Set 1 was calculated both of the following ways to ensure accuracy:

$$Inverse\ of\ the\ \ Sharpe\ Ratio(Set\ 1) = \frac{1}{Sharpe\ Ratio(Set\ 1)} = \frac{1}{0.3147} = 3.1773$$

$$Coefficient\ of\ Variation(Set\ 1)\ = \frac{Standard\ Deviation(Set\ 1)}{Average\ Return(Set\ 1)} = \frac{0.1088}{0.0343} =\ 3.1773$$

Table 24. Inverse of the Sharpe Ratio Calculations

Inverse of the Sharpe Ratio	
Set 0	0.0000
Set 1	3.1773
Set 2	2.4586
Set 3	3.7277
Set 4	3.9419
Set 5	2.3412
Set 6	1.8992
Set 7	2.0777
Set 8	2.0957
Set 9	1.4828
Set 10	2.1508

The coefficient of variation measures the volatility of a portfolio in relation to the expected overall return. A smaller coefficient of variation leads to a better risk-return trade-off. Through exploring the inverse Sharpe Ratio/coefficient of variation, we learned that Set 4 would be considered to have the most sequence risk with a score of 3.9419 whereas set 9 has the lowest amount with 1.4828. We felt that this calculation was the most promising as the values coincided with the histogram ranking from Example 15.

4.3.3 Incorporating Time into Sets of Returns and Calculations

One thing that seemed important to our team when investigating a score calculation was to incorporate the aspect of time. We believed that a 10-year period and a 20-year period would have different amounts of sequence risk. When our group researched this concept more and started to construct formulas, we found that we might not need to incorporate time into an equation as directly as we first thought. Ultimately, we came to believe that the IRR could be a good measure of time for sequence risk. The assumption was that a lower IRR would mean that there could be worse returns towards the end of the investing period whereas a higher IRR would mean the investments are experiencing more favorable returns at the end.

With this thought in mind our group was looking to perform various calculations with the IRR values associated with each of the 25,000 permutations for each set. We decided to investigate taking the average final balance for the 25,000 permutations and compare it to a scenario 1,000 each year using the average IRR as the return value. Initially, we performed these calculations to find the difference between the two values. Example 18, shows the results of these calculations for the 10-year sets.

Example 18: Comparing the Average Final Balance to Investing Using the Average IRR

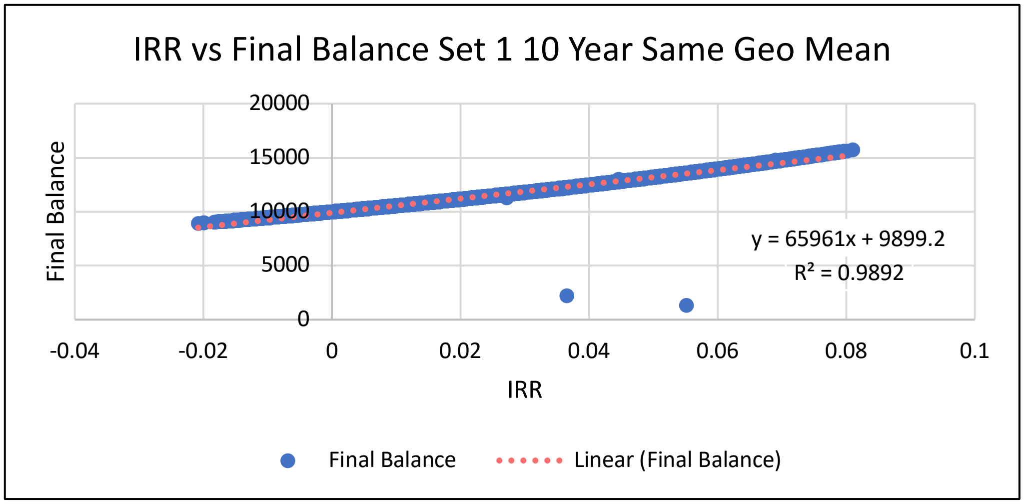
Table 25. Average Final Balance vs. Average IRR Investing

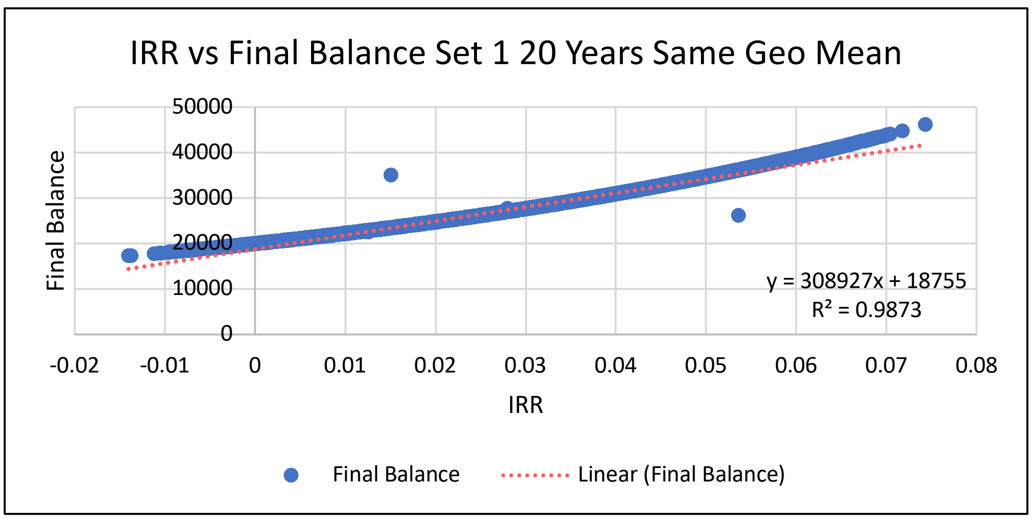
	Time 10	Average Final Balance	Difference
Set 0	11,683	11,683	0
Set 1	11,748	11,818	70
Set 2	11,714	11,747	33
Set 3	11,788	11,902	114
Set 4	11,937	12,223	286
Set 5	11,704	11,728	24
Set 6	11,702	11,720	18
Set 7	11,702	11,724	22
Set 8	11,707	11,729	22
Set 9	11,707	11,729	22
Set 10	11,708	11,732	24

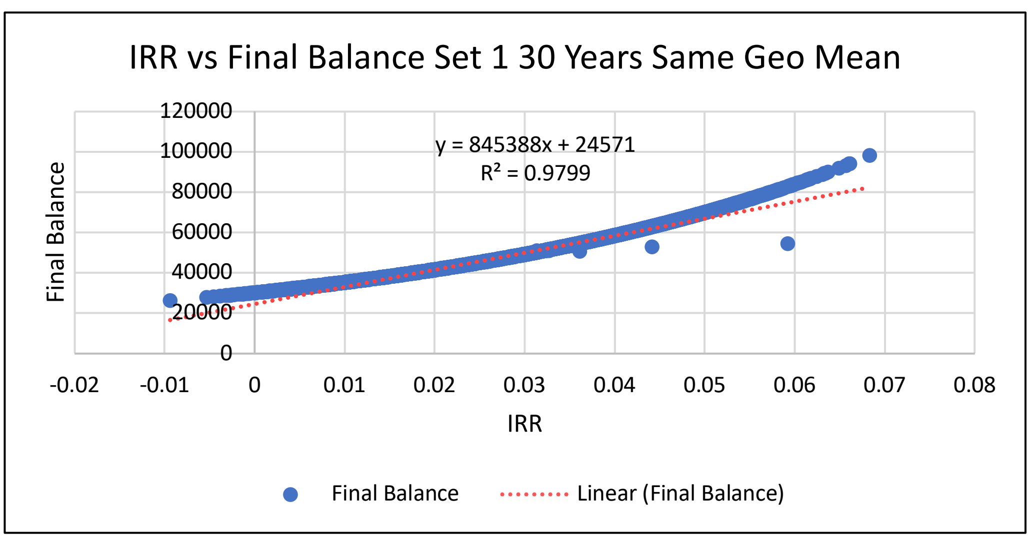
The difference column was calculated by subtracting the Time 10 value from the Average Final Balance. Notice how all the average final balances are greater than the time 10 value. Through this analysis we realized that our results implied positive convexity which is good for our sets of returns. Additionally, the data behaved how we expected it to be. We ran similar calculations for the 20 and 30 years as well as introduced a 50-year set to see the values for a longer period. The calculations for the 20,30-, and 50-year trials can be seen in the Constant Geometric Mean Excel Workbook.

In addition to performing these calculations, we graphed the values against each other. Our prediction was that the graphs would curve, due to the convexity, at either end. The straight line tells us that the final balances have a one-to-one relationship with the IRR values for the 25,000 permutations. Example 19 shows the four different graphs for each of the different scenarios of 10, 20, 30, and 50 years. As time increases, the convexity of the graphs increases. A linear trendline was graphed as a baseline to show the changing convexity for the different trials.

Example 19: Graphing Final Balances Against their IRR Values







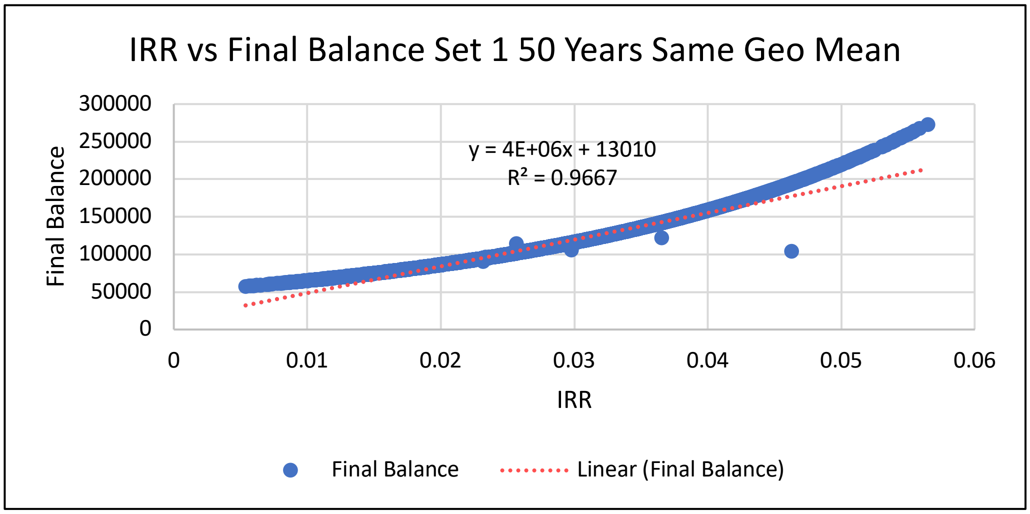


Figure 22. Set 1 IRR vs Final Balance for 10, 20, 30 and 50 Years

With each of these graphs we looked to compare their corresponding R-squared values. Each year had an R-squared value that was between 0.96 and 0.99. As expected, the R-squared value was the highest for the 10-year period at 0.9892 and the lowest at the 50-year period of 0.9667.

Continuing with the IRR calculations, our group investigated the various percentiles associated with the IRR and final account balances. To perform the percentile calculations, we looked at the 68^{th}, 75^{th}, 95^{th}, and 99.7^{th} percentiles. These calculations were performed similar to the other IRR calculations. Example 20 shows the results for the 10 years 68^{th} and 95^{th} percentiles. The results for the other percentiles and periods can be found in the Initial Data Excel Workbook.

Example 20: 68th and 95th Percentiles for the 10-Year Period

Table 26. 68th Percentile Calculation

10 Year 68th Percentile					
	IRR	**Time 10**	**Final Balance**	**Difference**	**% of Final Balances Less than Time 10**
Set 0	2.81%	11,683.40	11,683.40	0.00	0.00%
Set 1	0.75%	10,420.17	10,431.88	11.71	44.58%
Set 2	1.40%	10,802.23	10,808.35	6.12	44.28%
Set 3	0.26%	10,143.99	10,165.57	21.57	43.55%
Set 4	-1.09%	9,417.80	9,454.64	36.84	44.76%
Set 5	1.54%	10,886.66	10,890.60	3.94	45.43%
Set 6	1.76%	11,023.50	11,026.92	3.42	44.15%
Set 7	1.64%	10,950.77	10,954.77	4.00	44.06%
Set 8	1.64%	10,949.31	10,953.29	3.98	44.21%
Set 9	1.64%	10,949.31	10,953.42	4.11	44.20%
Set 10	1.58%	10,914.37	10,918.87	4.50	44.43%

The 68th percentile for the 10-year period showed slightly lower variation than the average values from Example 18. The highest difference between the final balance was in set 4 with a value of 36.84. Additionally, the differences had consistently lower values ranging from 36.84 to 3.42. An additional consideration for the percentile calculations was the percent of final balances less than the time 10 balance, represented in the last column of the table. The range in percentages was from 43.55% to 45.43%. The highest percent corresponds to the lowest difference, and the lowest percent corresponds to the highest difference.

Table 27. 95th Percentile Calculation

10 Year 95th Percentile					
	IRR	**Time 10**	**Final Balance**	**Difference**	**% of Final Balances Less than Time 10**
Set 0	2.81%	11,683.40	11,683.40	0.00	0.00%
Set 1	-0.69%	9,627.08	9,629.44	-2.36	44.16%
Set 2	0.38%	10,208.75	10,210.21	-1.46	41.76%
Set 3	-1.74%	9,090.02	9,095.04	-5.02	42.40%
Set 4	-3.74%	8,155.59	8,162.41	-6.82	41.60%
Set 5	0.73%	10,408.81	10,409.77	-0.95	41.20%
Set 6	1.01%	10,574.75	10,575.57	-0.81	42.80%
Set 7	0.83%	10,465.72	10,466.70	-0.98	42.16%
Set 8	0.80%	10,448.66	10,449.70	-1.04	41.04%
Set 9	0.80%	10,448.66	10,449.70	-1.04	40.96%
Set 10	0.74%	10,417.01	10,419.46	-2.45	43.92%

The 95th percentile for the 10-year period showed variation similar to the average values from Example 19 but continued to decrease. All sets had positive differences between the Time 10 and final balances where the highest difference had a value of 6.82 in set 5. Note that this highest difference is a different set compared to example 19. The values of the percentage of final balances less than time 10 had a slightly smaller range between 40.88% to 42.16%. This smaller range corresponds to the difference values all being closer together.

4.3.4 Researching a Measurement for the Spread of a Distribution

Skewness

Other calculations were also performed on each of these returns. Like our initial thoughts we found the mean, standard deviation, skewness, and kurtosis as well as many others. When comparing these calculations to the previous ones we realized that we were using the incorrect formula for skewness in excel. We learned that the function we were supposed to use was Skew.p which measures the degree of asymmetry of a distribution around the mean but based on a population. Before our skewness calculations were looking to use Skew.s which is for a sample. Below is the formula that excel uses to calculate the skewness for a population.

$$v = \frac{1}{N}\sum_{i=1}^{N}\frac{(x_i - \bar{x})^3}{\sigma}$$

Equation 5. Population Skewness Equation

Examples 21 and 22 show updated calculations, one that incorporate our newly calculated skewness and the other one looking into Coefficient of Variation.

Example 21: Skewness Calculation

Table 28. (Range x Variance) + Skewness Calculation

$(\boldsymbol{Range} \times \boldsymbol{Variance}) + \boldsymbol{Skewness}$			
	10 Years	**20 Years**	**30 Years**
Set 0	-1.0000	-1.0000	-1.0000
Set 1	-0.9744	-0.9744	-0.9744
Set 2	-0.6151	-0.6151	-0.6151
Set 3	-0.3554	-0.3554	-0.3554
Set 4	-0.9503	-0.9503	-0.9503
Set 5	1.4228	1.4228	1.4228
Set 6	-0.5945	-0.5945	-0.5945
Set 7	-0.6496	-0.6496	-0.6496
Set 8	-0.6344	-0.6344	-0.6344
Set 9	-0.2557	-0.2557	-0.2557
Set 10	-0.9689	-0.9689	-0.9689

Our group began to test different combinations of the summary statistics to see if we could create an intuitive formula that provided insight into the sequence risk that was present in a specific set of rates. One of the first combinations of statistics was the range times the variance plus the skewness of the set. Once this calculation was completed for each set in 10, 20 and 30 years, we applied a color-coded conditional formatting to the table. The color-coding helped to visualize the distribution of the results throughout each trial. When compared to the ranking of histograms in the Constant Geometric Mean Excel Workbook, our group realized this calculation was not the best match.

One key finding was the importance of units in a sequence risk score. For instance, Variance has units squared so when multiplied by Range, which has units, we are left with units cubed. Adding the units cubed to skewness, which has no units, results in something that cannot be done. Moving forward, we felt it would be best to have no units for a sequence risk score moving forward which meant us making sure the units either cancel or have none in future calculations.

Example 22: Inverse of the Sharpe Ratio into a Calculation

Table 29. (Range x Coefficient of Variation)/Geometric Mean Calculation

$\dfrac{Range \times Coefficient\ of\ Variation}{Geometric\ Mean}$			
	10 Years	**20 Years**	**30 Years**
Set 0	0.0000	0.0000	0.0000
Set 1	38.2275	38.2275	38.2275
Set 2	21.0109	21.0109	21.0109
Set 3	45.8323	45.8323	45.8323
Set 4	98.3393	98.3393	98.3393
Set 5	22.0105	22.0105	22.0105
Set 6	12.3612	12.3612	12.3612
Set 7	15.1805	15.1805	15.1805
Set 8	14.7230	14.7230	14.7230
Set 9	7.9168	7.9168	7.9168
Set 10	16.0727	16.0727	16.0727

We also wanted to use the inverse of the Sharpe Ratio, or the coefficient of variation in a calculation. Since the coefficient of variation is a measure of volatility, we thought that maybe also using the spread of the returns, the range, and the geometric mean could be insightful. Since the geometric mean for each of these sets is the same, we thought dividing by the geometric mean would be best for comparison. When the calculations were complete, we used the same conditional formatting strategy to compare the results with the histograms. While we did like that *Set 0*, which has no sequence risk was calculated to be 0, the rest of the calculations did not align as nicely. For example, *Set 4* has the highest solution for this calculation, and *Set 4* does have the most sequence risk in this group of sets, yet we felt as though we could find a more consistent calculation.

Kurtosis

Another way to measure the shape of the distribution is kurtosis. Here kurtosis measures how heavy the tails differ from the normal distribution. Below is a formula that explains how kurtosis is calculated.

$$Kurtosis = \frac{\sum_{i=1}^{N}(x_i - \bar{x})^4}{(N-1)\sigma^4}$$

$$\boldsymbol{where}:$$

$$N = total\ number\ of\ observations\ in\ the\ sample$$

$$\bar{x} = \text{mean of distribution}$$

$$x_i = \text{observed value}$$

$$\sigma = standard\ deviation\ of\ the\ distribution$$

Equation 6. Kurtosis Equation

In addition to understanding how it is calculated it is important to understand that there are three different types of kurtosis depending on the value you receive. To get the value for these types of kurtosis you must calculate the excess kurtosis which is simply subtracting the result by three. The three different types of kurtosis are as follows:

Mesokurtic: This is the Normal distribution where kurtosis is 3.

Leptokurtic: Kurtosis has a value more than 3 and the distribution has a higher peak as well as flatter tails

Platykurtic: Kurtosis has a value less than 3 and the distribution has a lower peak as well as a thinner tail

For our distributions we expected many of our distributions to be close to the mesokurtic result. We believe by running 25,000 permutations we be able to get close to a normal distribution. Table X, shows the various results we calculated for using the different sets of returns associated with 10 years by generating the 25,000 permutations. The results include both the kurtosis and excess kurtosis calculations.

Table 30. Kurtosis and Excess Kurtosis Calculations

10 Years		
	Kurtosis	**Excess Kurtosis**
Set 0	22737.2013	22734.2013
Set 1	-0.1787	-3.1787
Set 2	-0.4433	-3.4433
Set 3	-0.1170	-3.1170
Set 4	-0.3859	-3.3859
Set 5	-0.7542	-3.7542
Set 6	-0.4909	-3.4909
Set 7	-0.5279	-3.5279
Set 8	-0.4576	-3.4576
Set 9	-0.4575	-3.4575
Set 10	-0.5513	-3.5513

In this group of sets you can see that all kurtosis and excess kurtosis are negative values. For example, *Set 5* has the lowest excess kurtosis value of -3.7542 and *Set 3* has the highest excess kurtosis value of -3.1170, not including *Set 0*. This means between all ten sets there is only a difference in excess kurtosis of about 0.6 which is not big. Additionally, since the excess kurtosis is below three all the distributions are platykurtic, so we are not observing any differences there. This lack of differences led us to believe kurtosis will not help us in our analysis of sequence risk and that there are better measures to use in our score.

4.3.5 Investigating a Measurement for Downside Risk

Semi-Variance

The next set of calculations investigated used semi-variance to measure downside risk. The appeal to using semi-variance is because it classifies risks very intuitively. If someone is doing better than the average return, they most likely will not care about their performance however if they are doing worse than the average return, they would care. Semi-variance takes that idea into account. It is the measure of the variance of the data that is below the mean which means it is the average of the squared deviations of the values less than the mean. Typically, it is used to estimate the potential downside risk of an investment portfolio. Below is a formula explaining how to calculate the semi-variance of a portfolio.

$$SemiVariance = \frac{1}{n} \times \sum_{r_t < Average}^{n} (Average - r_t)^2$$

$$\boldsymbol{where}:$$

$$n = total\ number\ of\ observations\ below\ the\ mean$$

$$r_t = \text{observed value}$$

$$Average = the\ mean\ or\ target\ value\ of\ the\ dataset$$

Equation 7. Semi-variance Equation

Example 23: Calculation 3 with Semi-Variance

Table 31. (SemiVariance/ variance) x Sharpe Ratio Calculation

$\frac{SemiVariance}{Variance} \times Sharpe\ Ratio$			
	10 Years	**20 Years**	**30 Years**
Set 0	0.0000	0.0000	0.0000
Set 1	0.4954	0.4954	0.4954
Set 2	0.5901	0.5901	0.5901
Set 3	0.3500	0.3500	0.3500
Set 4	0.4923	0.4923	0.4923
Set 5	0.2445	0.2445	0.2445
Set 6	0.7661	0.7661	0.7661
Set 7	0.6953	0.6953	0.6953
Set 8	0.7029	0.7029	0.7029
Set 9	0.9053	0.9053	0.9053
Set 10	0.7532	0.7532	0.7532

Two of the risk measures that seemed promising where semi-variance and Sharpe Ratio, so we looked to use both to see if it could be a good measure of sequence risk together. Our thought was to take the ratio of semi-variance and variance and then multiple by the Sharpe Ratio. Since semi-variance measures the dispersion of observations below the mean and variance is the measure of the dispersion around the mean, we thought that the downside risk divided by volatility could provide valuable information about the distribution. When this ratio was multiplied by the Sharpe Ratio, another measure of volatility, the outcome was not as promising as expected. Once again, the conditional formatting did not match the expected rankings of our histograms, so we had more formulas we needed to test.

Example 24: Calculation 4 with Semi-Deviation

Table 32. (SemiDeviation x Coefficient of Variation)/Geometric Mean Calculation

$\frac{SemiDeviation \times Coefficients\ of\ Variation}{Geometric\ Mean}$			
	10 Years	**20 Years**	**30 Years**
Set 0	0.0000	0.0000	0.0000
Set 1	15.4643	15.4643	15.4643
Set 2	8.0370	8.0370	8.0370
Set 3	21.6360	21.6360	21.6360
Set 4	39.4605	39.4605	39.4605
Set 5	4.4893	4.4893	4.4893
Set 6	4.5960	4.5960	4.5960
Set 7	5.5476	5.5476	5.5476
Set 8	5.7073	5.7073	5.7073
Set 9	2.6295	2.6295	2.6295
Set 10	6.3380	6.3380	6.3380

Two of the measures that our group found to be insightful to measure sequence risk was semi-deviation, or the square root of semi-variance and coefficient of variation. Since semi-deviation measures the downside risk and coefficient of variation is a measure of volatility, we thought using both measures in an equation could be helpful. It turned out that this combination was not insightful for measuring sequence risk, but our group knew that we wanted to incorporate semi-deviation to account for the downside risk in future calculations.

Value at Risk and Tail-Value at Risk

We decided to investigate two other risk measures in addition to semi-variance. For example, tail-value-at-risk (TVaR) is a risk measure that was adapted from the value-at-risk (VaR) measure. VaR is a measure that quantifies the level of financial risk in each scenario in a timeframe. The VaR measure has many faults and limitations such as not being a coherent risk measure since it does not satisfy the subadditivity property, simply meaning, VaR is not able to model diversification. Mathematicians realized that this non-coherent risk measure could be improved by accounting for the shape of a tail, since many tails go beyond the VaR measure's threshold, and the coherent risk measure TVaR became the solution. Ultimately TVaR uses the average value of the expected losses above a threshold to provide the expected loss, given that the loss exceeds a specified percentile.

Using the 11 sets of returns with the same geometric mean, we were able to calculate the TVaR of each set using the final balances in the account at the end of the period. We were able to apply the empirical formula for TVaR to the final account balances of each accumulation scenario. Simply, the empirical TVaR formula for the 95th percentile is the average of the lowest 5% of values. Once computing the TVaR for each set of rates, there did not seem to be a way to easily interpret the risk of each set. Since the TVaR considers the deposit amounts of an investment due to evaluating the final account balances, we did not see this as an ideal measure since we were focused more on knowing just the returns of an investment. Due to this dead end, our group decided to leave the TVaR calculations alone. We recognize that TVaR is a powerful coherent risk measure, but for this specific problem we wanted to continue to work towards deriving a different, more meaningful score.

5 Reaching a Conclusion to Final Score Calculation

Through our analysis of downside risk the most promising calculations performed were using semi-variance. We looked to incorporate other measures with it to see if it generated a good measure for a score. Once again, the final calculation we reached to was as follows:

$$Sequence\ Risk\ Score\ = \frac{Range\ \times \sqrt{SemiVariance}}{(Geometric\ Mean)^2}$$

Equation 8. Final Sequence Risk Score Formula

Range: Maximum Return – Minimum Return

Semi-Variance: The variance of the returns below the mean

Geometric Mean: The average of the returns

Something that was important to our group when working to quantifying sequence risk was to find an intuitive formula. The biggest appeal is that the partial derivatives of range and semi-variance increase sequence risk whereas the second derivative of geometric mean square decreases sequence risk. In Example 25, a table of the different scores for each of the 10-, 20-, and 30-year rates of returns is shown. We then compared the histograms of the three sets with the lowest, medium, and largest values for the 10 year returns to see if they visually fit in the group as well in this example. Note that the returns that are used in these examples are the same as the ones mentioned in the earlier sections.

Example 25: Applying our Proposed Sequence Risk Score to our Returns

Table 33. Final Sequence Risk Score of the Sets with a Constant Geometric Mean

Final Score			
	10 Years	**20 Years**	**30 Years**
Set 0	0.0000	0.0000	0.0000
Set 1	58.5574	58.5574	58.5574
Set 2	27.9358	27.9358	27.9358
Set 3	71.3629	71.3629	71.3629
Set 4	249.7284	249.7284	249.7284
Set 5	18.0278	18.0278	18.0278
Set 6	15.7499	15.7499	15.7499
Set 7	19.5093	19.5093	19.5093
Set 8	19.1327	19.1327	19.1327
Set 9	9.4684	9.4684	9.4684
Set 10	22.0210	22.0210	22.0210

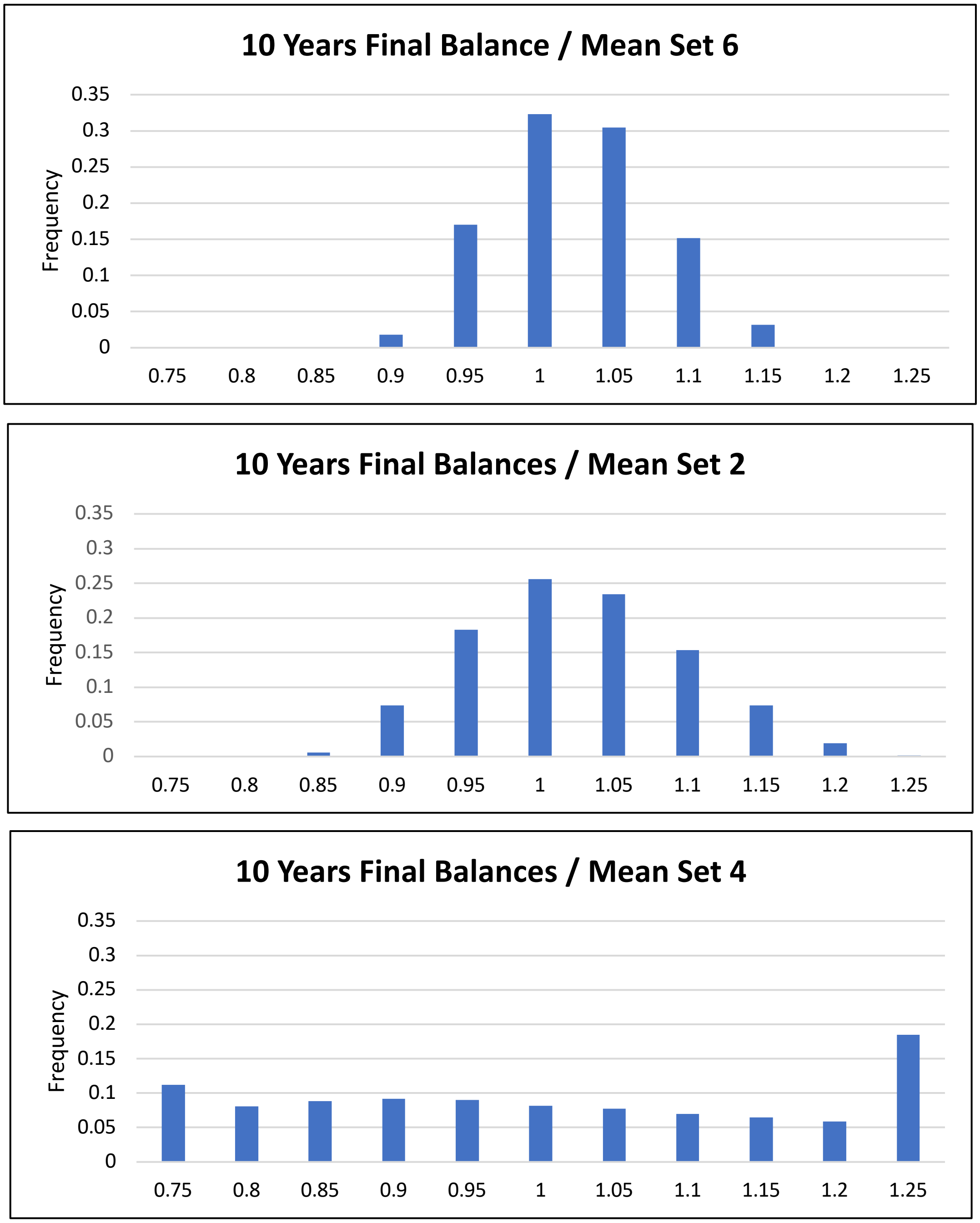


Figure 23. 10 Year Trials Set 2, 4, 6 Histograms

One of the first things that stood out with performing this calculation in the table above was that it effectively divided the various sets into low (Green), medium (Yellow Orange), and high (Red) sequence risk similar to our histogram results we discussed earlier. Additionally, we can see from the table Set 0 has a score of 0 which is what we wanted as the returns for the set stay the same. As for the other sets we see from the table Set 5, Set 7, and Set 8 have low sequence risk whereas Set 2, Set 7, Set 10 have medium and Set 1, Set 2, and Set 4 have high. Additionally, we were able to compare these groupings to our histograms where are eyes can visually see a similar result. The histograms go in low, medium, high order which are the same sets of Set 2, Set 4, and Set 6 from the table. For this specific group of sets, we also started to see if our sequence risk score could be a range of numbers where potentially a score of 1-20 would be considered low risk, 20-25 would be medium and anything above 25 would be high.

In addition to creating the table and observing the resulting scores our group came up with the pros and cons associated with using this type of calculation. It is also important to keep in mind we developed a score using hindsight calculations and that to use it in the real world we would have to know the returns we are receiving (or to at least have a projection of those rates). The pros and cons list are below:

Pros	**Cons**
• No negative values are possible • No units • Matches if a person were to eyeball sequence risk from a histogram • Looks at where risk is most important (i.e everything below the mean) • Partially accounts for time as the longer the period the more potential returns that could be below the mean as well as the range of returns could be greater	• Not the most intuitive calculation • Semi-variance does not work with the set of all the same returns (however we assumed it to be 0 for our purposes)

Through our pros and cons list as well as the way our to apply our score one step further and to determine to see if there was indeed a way to define the score based off a certain value our group used various rolling averages of the S&P 500 returns to see if the scores made sense there. We took the 10-, 20-, 30-, 40-, and 50-year rolling averages and provided scores. To see the list of all the scores, see the Sequence Risk Score Rolling Years Excel Workbook. We then ran some of the 10 year rolling averages of the S&P 500 returns to see if their histograms show a similar story to the calculation. Example 26 shows the histogram for the lowest sequence risk score as well as the highest and a table of the returns.

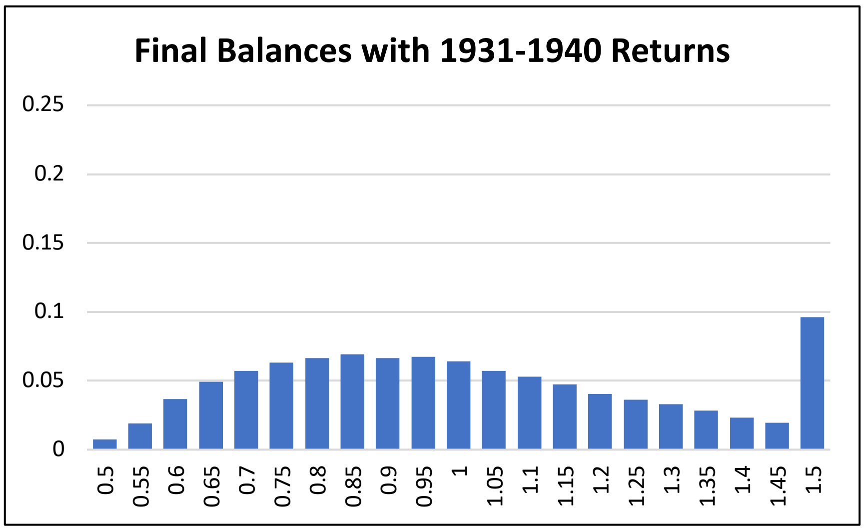
Example 26: Histograms of the Lowest and Highest Sequence Risk Score with 10 year Rolling Average of S&P 500 Returns

Table 34. 10 Year Rolling Average Returns

10 Year Rolling Average Returns										
	Year 1	**Year 2**	**Year 3**	**Year 4**	**Year 5**	**Year 6**	**Year 7**	**Year 8**	**Year 9**	**Year10**
1931-1940	-43.34%	-8.19%	53.99%	-1.44%	47.67%	33.92%	-35.03%	31.12%	-0.41%	-9.78%
1984-1993	6.27%	31.73%	18.67%	5.25%	16.61%	31.69%	-3.10%	30.47%	7.62%	10.08%

1931- 1940 Sequence Risk Score: **741.75**

1984-1993 Sequence Risk Score**: 1.67**



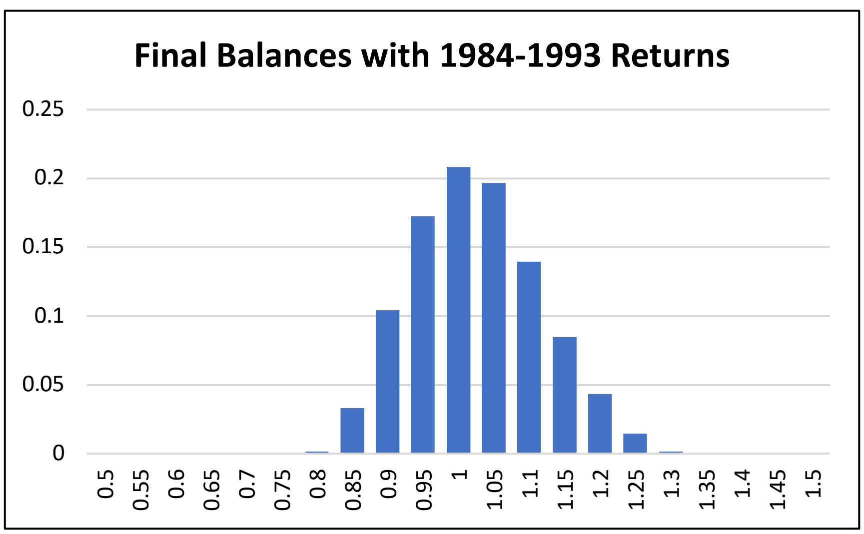


Figure 24. Maximum and Minimum Sequence Risk from S&P 500 Returns in a 10 Year Period

An initial observation from the set of returns is that many of the returns from 1931 to 1940 were negative whereas from 1984 to 1993 had all positive returns. Intuitively that lead 1931 to 1940 to have a sequence risk score that was exceedingly high at 18.35 and from 1984 to 1993 there was a low score of 1.94. Through the histograms we can also see the differences in their distributions and how again 1931 to 1940 was significantly more sequence risk than from 1984 to 1993. The graph of the 1931 to 1940 returns deviates more from the mean in comparison to the other graphs which can be expected. Lastly, by doing these calculations we learned that are score worked well with the S&P 500 rates.

Although our proposed measure seems to fit well with the examples we ran as well as the S&P 500 returns there are more ideas that still need to be explored. If our group had more time, we would begin to look at the various graphs of the sequence risk score to see if there is a notable trend in the years in which there is high sequence risk versus low. The goal here would be to see if there would be a way to predict future sequence risk scores for returns that will not be known.

Another idea that we would look to explore is finding a way to connect the results from our simulation work to the set of returns themselves. By performing a regression analysis on the two different results there is potential to connect the permutation values to the returns to find a better measure. Here we would look to incorporate more of the IRR measures as well as looking at other summary statistics where we could link the permutations and the sets of returns results together. We also believe it would be worth it in exploring the length of the period in a calculation and believe that IRR would be the best way to do that. We believe that drawing a link between the average IRR of the returns as well as the average IRR for the 25,000 permutations would be a good place to start when exploring this idea.

Lastly, we generated permutations randomly. We had our code randomly generate a list of 25,000 permutations and consistently use the same order making it pseudo random. An improvement can be made in the future to the process of determining these permutations. A way to improve the process would be to have a program choose which permutations from the list of permutations are used.

References

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Frank Sr, Larry R. and Mitchell, John B. and Blanchett, David, “Probability-of-Failure-Based Decision rules to Manage Sequence risk in retirement.” (2011).

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Appendix A

Permutations Function

A permutation is defined as the change the arrangement of a set. In the Examples in this paper, Sequence Risk is displayed by showing the various outcomes from each permutation of the list of returns. The number of permutations a set has is defined the factorial of the number of items. Many of the examples have 3 returns, meaning that there are 3! or 6 different sequences of those returns. Similarly, for the examples with 7 returns there are 7! or 5,040 different sequences of those returns.

Table. VBA Function Output

VBA ListPermut(num) Function Output			
Scenario	1st Number	2nd Number	3rd Number
1	1	2	3
2	1	3	2
3	2	1	3
4	2	3	1
5	3	1	2
6	3	2	1

Using Visual Basics for Applications (VBA) in Excel, all possible sequences were able to be found. The full code can be referenced in **Appendix A.2** below. The code produces the permutations of a number. So for the examples of 3 returns, the VBA code produces the different sequences of the numbers 1, 2 and 3. Then using Excel functions, each number was matched to a rate using an IF statement. The following tables are an example of what the VBA code produces for a value of 3 returns, and how the whole numbers were matched to the wanted returns, in this case that were -10%, 10% and 20%.

Table. Matching Returns

Excel Return Matching			
Scenario	**Return Year 1**	**Return Year 2**	**Return Year 3**
1	-10%	10%	20%
2	-10%	20%	10%
3	10%	-10%	20%
4	10%	20%	-10%
5	20%	-10%	10%
6	20%	10%	-10%

For a calculation with only 3 returns, finding the six sequences of the returns is not difficult to do manually. It because more of a roadblock when there are more returns. For example, 5! is 120, so creating 120 sequences manually becomes time consuming and there is a greater risk of error due to manual functions. This VBA code became extremely beneficial when examples of 7 years were conducted. The code produces all 5,040 permutations. Another road block was hit when longer examples were conducted. The VBA and Excel functions and code could only handle up to 9 permutations. So while this code was very beneficial for general understanding and explanations, a new method was needed to be able to conduct larger scaled examples.

Permutations VBA Code

```
Function ListPermut(num As Integer)
'Permutations without repetition
Dim c As Long, r As Long, p As Long
Dim rng() As Long, temp As Long, i As Long
Dim temp1 As Long, y() As Long, d As Long
'The Excel function Permut(num, num) displays the number of permutations
p = WorksheetFunction.Permut(num, num)
' Create array
ReDim rng(1 To p, 1 To num)
'Create first row in array (1, 2, 3, ...)
For c = 1 To num
 rng(1, c) = c
Next c
For r = 2 To p
' 1. Find the first smaller number rng(r-1, c-1)<rng(r-1,c)
 For c = num To 1 Step -1
  If rng(r - 1, c - 1) < rng(r - 1, c) Then
   temp = c - 1
   Exit For
  End If
 Next c
' Copy values from previous row
 For c = num To 1 Step -1
  rng(r, c) = rng(r - 1, c)
 Next c
' 2. Find a larger number than rng(r-1, temp) as far to the right as possible
 For c = num To 1 Step -1
   If rng(r - 1, c) > rng(r - 1, temp) Then
     temp1 = rng(r - 1, temp)
     rng(r, temp) = rng(r - 1, c)
     rng(r, c) = temp1
     ReDim y(num - temp)
     i = 0
     For d = temp + 1 To num
      y(i) = rng(r, d)
      i = i + 1
     Next d
     i = 0
     For d = num To temp + 1 Step -1
      rng(r, d) = y(i)
      i = i + 1
     Next d
     Exit For
   End If
  Next c
Next r
ListPermut = rng
End Function
```

Appendix B

S&P 500 Historic Returns							
Year	Return	Year	Return	Year	Return	Year	Return
"1922"	15.00%	1950	31.71%	1978	6.56%	2006	15.79%
"1923"	-10.00%	1951	24.02%	1979	18.44%	2007	5.49%
"1924"	-15.00%	1952	18.37%	1980	32.42%	2008	-37.00%
"1925"	44.00%	1953	-0.99%	1981	-4.91%	2009	26.46%
1926	11.62%	1954	52.62%	1982	21.55%	2010	15.06%
1927	37.49%	1955	31.56%	1983	22.56%	2011	2.11%
1928	43.61%	1956	6.56%	1984	6.27%	2012	16.00%
1929	-8.42%	1957	-10.78%	1985	31.73%	2013	32.39%
1930	-24.90%	1958	43.36%	1986	18.67%	2014	13.69%
1931	-43.34%	1959	11.96%	1987	5.25%	2015	1.38%
1932	-8.19%	1960	0.47%	1988	16.61%	2016	11.96%
1933	53.99%	1961	26.89%	1989	31.69%	2017	21.83%
1934	-1.44%	1962	-8.73%	1990	-3.10%	2018	-4.38%
1935	47.67%	1963	22.80%	1991	30.47%	2019	31.49%
1936	33.92%	1964	16.48%	1992	7.62%		
1937	-35.03%	1965	12.45%	1993	10.08%		
1938	31.12%	1966	-10.06%	1994	1.32%		
1939	-0.41%	1967	23.98%	1995	37.58%		
1940	-9.78%	1968	11.06%	1996	22.96%		
1941	-11.59%	1969	-8.50%	1997	33.36%		
1942	20.34%	1970	4.01%	1998	28.58%		
1943	25.90%	1971	14.31%	1999	21.04%		
1944	19.75%	1972	18.98%	2000	-9.10%		
1945	36.44%	1973	-14.66%	2001	-11.89%		
1946	-8.07%	1974	-26.47%	2002	-22.10%		
1947	5.71%	1975	37.20%	2003	28.68%		
1948	5.50%	1976	23.84%	2004	10.88%		
1949	18.79%	1977	-7.18%	2005	4.91%		

The yellow highlight denotes a non-historic return. Our group added four returns to create a more drastic risk.

Appendix C

The Perfect Withdrawal Amount is a calculation that can be computed when the returns are known. Often times, this calculation is computed as a historic computation, so the value of knowing this amount is only known after the return data is known. To find the best used formula for PWR, the group created a sample of 7 unique returns. In Example 1, the withdrawal amount X, which represents the PWA, was taken out at the end of each year. Example 2 is a simplified version of Example 1 which uses values instead of real numbers.

Example 1: Account value begins at \$7000, at the end of each year \$1000; returns vary.

Table. PWA Formula Derivation

YR	BOY	GR	W	EOY
1	7000	.98	X	7000(.98)−X
2	7000(.98)−X	.99	X	(.99)[7000(.98)−X] – X
3	7000(.98)(.99) − (.99)X - X	1.01	X	(1.01)[7000(.98)(.99) − (.99)X – X] – X
4	7000(.98)(.99)(1.01) − (.99)(1.01)X - (1.01)X − X	1.02	X	(1.02)[7000(.98)(.99)(1.01) − (.99)(1.01)X − (1.01)X – X] − X
5	7000(.98)(.99)(1.01)(1.02) − (.99)(1.01)(1.02)X − 1.01)(1.02)X − (1.02)X − X	1.03	X	(1.03)[7000(.98)(.99)(1.01)(1.02) − (.99)(1.01)(1.02)X − 1.01)(1.02)X − (1.02)X − X] − X
6	7000(.98)(.99)(1.01)(1.02)(1.03) − (.99)(1.01)(1.02)(1.03)X − (1.01)(1.02)(1.03)X − (1.02)(1.03)X − (1.03)X − X	1.04	X	(1.04)[7000(.98)(.99)(1.01)(1.02)(1.03) − (.99)(1.01)(1.02)(1.03)X − (1.01)(1.02)(1.03)X − (1.02)(1.03)X − (1.03)X − X] − X
7	7000(.98)(.99)(1.01)(1.02)(1.03)(1.04) − (.99)(1.01)(1.02)(1.03)(1.04)X − (1.01)(1.02)(1.03)(1.04)X − (1.02)(1.03)(1.04)X − (1.03)(1.04)X − (1.04)X − X	1.05	X	(1.05)[7000(.98)(.99)(1.01)(1.02)(1.03)(1.04) − (.99)(1.01)(1.02)(1.03)(1.04)X − (1.01)(1.02)(1.03)(1.04)X − (1.02)(1.03)(1.04)X − (1.03)(1.04)X − (1.04)X − X] − X

X = [7000(.98)(.99)(1.01)(1.02)(1.03)(1.04) (1.05)] / [(.99)(1.01)(1.02)(1.03)(1.04)(1.05) + (1.01)(1.02)(1.03)(1.04)(1.05) + (1.02)(1.03)(1.04)(1.05) + (1.03)(1.04)(1.05) + (1.04)(1.05) + (1.05) + 1]
X = 7869.38365\7.71988
X = 1019.36560

The formula for the PWA over seven years can be seen in equation one. The variable *B* is the value of the account at the beginning of the period. The variable G_i is the growth rate at time *i*. The variable of interest, or the PWR, is variable *X*.

$$B\left(\prod_{i=1}^{7} G_i\right) = X\left(\prod_{i=2}^{7} G_i + \prod_{i=3}^{7} G_i + \prod_{i=4}^{7} G_i + \prod_{i=5}^{7} G_i + \prod_{i=6}^{7} G_i + \prod_{i=7}^{7} G_i + 1)\right)$$

Equation. Perfect Withdrawal Amount

This equation can be generalized for any time period as:

$$B\left(\prod_{i=1}^{n} G_i\right) = X\left(\left(\sum_{i=2}^{n}\left(\prod_{j=i}^{n} G_j\right)\right) + 1\right)$$

Equation. General Perfect Withdrawal Amount

Example 2 is a simplified example. The variable *B* is the value of the account at the beginning of the period. The variable G_i is the growth rate at time *i*. The PWA is the variable X. Variable *A* represents the value desired at the end of the period. In this equation you may solve for the variable *X* to determine the amount that should be withdrawn to reach the desired amount.

Example 2:

Table. Derivation of Perfect Withdrawal Rate Formula

i	BOY_i	G_i	W	EOY_i
1	$B = BOY_1$	G_1	X	$B(G_1)$ - X
2	$BOY_2 = B(G_1)$ - X	G_2	X	$(G_2)(BOY_2)$ - X
3	$BOY_3 = (G_2)(BOY_2)$ - X	G_3	X	$(G_3)(BOY_3)$ - X
4	$BOY_4 = (G_3)(BOY_3)$ - X	G_4	X	$(G_4)(BOY_4)$ -X
5	$BOY_5 = (G_4)(BOY_4)$ -X	G_5	X	$(G_5)(BOY_5)$ - X
6	$BOY_6 = (G_5)(BOY_5)$ - X	G_6	X	$(G_6)(BOY_6)$ -X
7	$BOY_7 = (G_6)(BOY_6)$ -X	G_7	X	$(G_7)(BOY_7)$ – X

The equation for PWA for a seven-year period is very similar to that of the PWR equation. The only difference is the variable *A* which represents the value desired at the end of the period. In this equation you may solve for the variable *X* to determine the amount that should be withdrawn to reach the desired amount.

$$A = B\left(\prod_{i=1}^{7} G_i\right) - X\left(\prod_{i=2}^{7} G_i + \prod_{i=3}^{7} G_i + \prod_{i=4}^{7} G_i + \prod_{i=5}^{7} G_i + \prod_{i=6}^{7} G_i + \prod_{i=7}^{7} G_i + 1)\right)$$

Equation. Perfect Withdrawal Amount Without Exhausting Account

Appendix D

A common way to demonstrate sequence risk is through the development of various simulation work. For example, the Monte Carlo simulation, which can be applied in various ways, generates random values, and produces the probable outcomes (*The Dynamic Implications of Sequence Risk on a Distribution Portfolio*). With the method of the Monte Carlo simulation, we decided to explore various coding options to run a similar simulation.

Option 1: Python

Python is a coding language known for handling large datasets making it a popular language for simulation work. The main function used to generate a random list of a permutation of length n is called list(np.random.permutation(n)). This randomness function allows for no biases to enter the simulation and ensures that it is representative of a typical environment for returns in the case of retirement. During this simulation, the python code then matches the returns each year and then exports those values into Excel for calculations to be done.

Option 2: Excel & VBA

Visual Basic for Applications, or VBA, is another language used to create a list of permutations depending on the number of years in the simulation. Unfortunately, VBA can only handle a list up to 10!, or 3,628,800, entries in size before it fails and crashes. For this project, this size limitation may or may not be a problem depending on the number of years of investing and withdrawing in each simulation. In the examples below VBA was able to generate all 7!, or 5,040, permutations, which were matched to corresponding returns through a series of IF statements in Excel. Other calculations similar to those performed in the Python simulation were executed solving for the final account balance, ending balance when withdrawing money each year, and the corresponding IRR's for the time-period.