Dimensions of Matrix Subalgebras

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Abstract

An *algebra over a field* (simply "algebra" for short) is an algebraic structure consisting of a vector space augmented with a vector multiplication operation such that vector addition and vector multiplication form a ring and scalar multiplication commutes with vector multiplication. Algebras are well behaved and have notions of dimension, basis, subalgebras, algebra ideals, algebra homomorphisms, and quotient algebras largely analogous to those of vector spaces or rings.

Algebras occur often in mathematics, for example the set of all $n \times n$ matrices valued in a field k form an algebra $M_n(k)$. We investigate the subalgebras of $M_n(k)$ and in particular which integers occur as dimensions of subalgebras and the number of dimensions of subalgebras for a given $M_n(k)$. We give a description of the dimensions of simple, nilpotent, and semisimple matrix subalgebras along with several sequences that represent various properties of matrix subalgebras.

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Chapter 1

Introduction

1.1 Vector Spaces and Algebras

Recall that a vector space V over a field k is an abelian group together with a binary operation $k \times V \to V$ called scalar multiplication that obeys certain axioms. If we write V additively we can express the vector space axioms as:

$\forall a, b \in k \quad u, v \in V$

1. $a(bv) = (ab)v$
2. $a(u + v) = au + av$
3. $(a + b)v = av + bv$
4. $1_k v = v$

Vector spaces are natural structures to work with due to their ubiquity and convenient properties. Often when we work with a vector space we find that the underlying concept, be it a vector space of functions or a vector space of matrices, also carries a sensible ring structure–we can multiply functions and we certainly take matrix products. We note that in both cases the vector space structure and the ring structure seem compatible in some sense, we know that the addition operations align and that the scalar product interacts with the ring product in a sensible way. We call these vector space / ring objects *algebras*, or more properly, *algebras over a field.*

Definition 1.1. An (associative) *algebra* over a field k is a k-vector space A equipped with an additional binary operation $\times : A \times A \to A$ such that $\times$ is associative, left and right distributive, and we have $(xa)(yb) = (xy)ab$ for all $x, y \in k$ and $a, b \in A$.

We will also refer to algebras over a field k as k-algebras, following the convention for vector spaces.

Algebras are very well-behaved structures and have sensible notions of subalgebras, ideals, homomorphisms, and quotient algebras.

Definition 1.2. A *subalgebra* of an algebra A is a subset B of A that is closed under addition, multiplication, and scalar products. In this case we write $B \leqslant A$.

Viewing $\mathbb{C}$ as a two-dimensional $\mathbb{R}$ algebra, we have $\mathbb{R} \leqslant \mathbb{C}$. In fact an algebra A over a field k will always have $k \leqslant A$, and we can use this property to give an alternative definition of an algebra.

Definition 1.3. An algebra A over a field k is a ring A together with a non-zero homomorphism $\phi : k \to A$ such that $\phi(k)$ is in the center of A.

In this definition scalar multiplication is defined by ring multiplication after applying ϕ.

Definition 1.4. An algebra *homomorphism* is a map $\phi : A \to B$ between algebras A and B over a common field k such that $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)$, $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$, and $\phi(ca_1) = c\phi(a_1)$ for all $a_1, a_2 \in A$ and $c \in k$. If ϕ is bijective then we say A and B are *isomorphic* and that ϕ is an *isomorphism*.

Definition 1.5. Recall that a subset S of a ring R is an *ideal* if S is closed under addition and "captures" multiplication from R, i.e., that

$$sr \in S \text{ and } rs \in S \quad \forall\, r \in R,\ s \in S.$$

We can generalize this to subsets that capture multiplication from one side only. We call $L \subseteq R$ a *left ideal* if L is closed under addition and $rl \in L \quad \forall\, r \in R,\ l \in L$, a *right ideal* is defined in the corresponding way. A normal ideal is both left and right, or *two-sided*.

Example 1.6. For $n \in \mathbb{N}$ we have that $n\mathbb{Z} \lhd \mathbb{Z}$, it is easy to see that $n\mathbb{Z}$ is closed under addition and the product of a multiple of n with any integer is also a multiple of n, i.e., it is in $n\mathbb{Z}$.

We will occasionally write for a two-sided ideal I of a ring R that $I \lhd R$, read "I is an ideal of R". For left and right ideals L and S respectively of a ring R we can write $L \lhd_L R$ and $S \lhd_R R$ analogously to two-sided ideals.

Remark 1.7. An ideal I of a k-*algebra* A is the same as an ideal of a ring except for the additional requirement that the subset also be closed under scalar multiplication. However, since we consider rings, and hence algebras, to be unital we always have $k\dot{1} \leqslant A$ and closure under scalar multiplication follows from the normal multiplicative property of ideals, provided I *is two-sided or a left ideal*, using the convention that the scalar product is applied on the left. In the case of a right scalar product, all two-sided or right ring ideals will also be algebra ideals.

So we can safely focus on ring ideals without having to worry about lifting them to algebra ideals, provided we stick to two-sided and left ideals.

As with rings algebra ideals are exactly the subsets that form kernels of homomorphisms. With this in mind we can state the first isomorphism theorem for algebras.

Theorem 1.8 (First Isomorphism Theorem for Algebras)**.** *Let $\phi : A \to B$ be a homomorphism of algebras. Then*

$$Im\,(\phi) \cong A/Ker\,(\phi)\,.$$

The proof of this version of the first isomorphism theorem is exactly analogous to that for rings.

Matrix Algebras

As we eluded to earlier, matrices valued in a field form an algebra under the standard addition, multiplication, and scalar product. Formally,

Definition 1.9. A *matrix algebra* $M_n(k)$ over a field k is the set of all $n \times n$ matrices with entries in k with the standard definitions for addition, multiplication, and scaling by an element of k.

Our goal is to study the subalgebras of matrix algebras, specifically their dimensions. We will investigate nilpotent, simple, semisimple, and parabolic matrix algebras, describing first their properties and general structure and then their possible dimensions and their densities.

Zero-Pattern Matrix Algebras

We now introduce a notation for describing matrix subalgebras that we will find useful.

Definition 1.10. Let Z be an $n \times n$ matrix whose entries are all either 0 or $*$. We build of subspace W of a full matrix algebra $M_n(k)$ from Z by taking a subspace basis consisting of all $E_{i,j}$ where $Z_{i,j}$ is $*$. We then take the multiplicative closure $\langle W \rangle$ (note this is a slightly different than the additive generator $\langle + \rangle$ we have discussed before) to create a subalgebra, the *zero-pattern subalgebra* associated with Z. We will, unless specifically stated, only concern ourselves with matrices Z such that their associated W is already closed under the ring operations, that is, that $W = \langle W \rangle$.

With this in mind, we can describe a zero-pattern subalgebra in simpler terms as the subalgebra of all matrices that have a 0 where Z has a zero and any element of k where Z has a $*$. We will also generally elide the mention of $\langle W \rangle$ itself, choosing instead to implicitly describe the resulting subalgebra by providing Z.

One can, if they prefer, view the $*$ terms in matrices Z as each being distinct variables allowed to range over the whole of k. Our notation merely expresses this more succinctly.

Example 1.11. $Z = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ describes the subalgebra of $M_3(k)$ consisting of diagonal matrices.

It is important to note that not all subalgebras of a matrix algebra are zero-pattern subalgebras, for example

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} a, b \in k \right\}$$

Can be verified to be a subalgebra of $M_3(k)$ but is not generated by a $*$ matrix, as the product of any two elements of A is of the form

$$MN = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c & d \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} a, b, c, d \in k$$

So A in this example has dimension 2 while a $*$ matrix in the same shape would have dimension 3.

Laffey proved a structure theorem for zero-pattern matrix algebras that contain an element with n distinct eigenvalues which states that such matrix algebras are two-generated. The general problem of determining when a matrix algebra is generated by a zero-pattern matrix is difficult, we pose it as a potential future question.

1.2 Simple Algebras over an Algebraically Closed Field

Lemma 1.12. *Let $E_{i,j} \in M_n(F)$ denote a* "mask" matrix*, a matrix with a one in the i, j position and zeros elsewhere. Left multiplication by a mask matrix $E_{i,i}$ by a matrix M results in a matrix with row i of M for its ith row and zeros elsewhere, and likewise right multiplication yields a matrix with the ith column of M preserved. Taking the product $E_{i,j}ME_{i,j}$ will result in a matrix that is all zeros except for entry i, j, which will be equal to the i, j entry of M.*

Proof. Consider the result of left (resp. right) multiplication as the inner product of rows of a mask matrix and columns of an arbitrary matrix (resp. columns and rows). The result is evident. ∎

Remark 1.13. The $E_{i,j}$ form the standard basis for $M_n(k)$ as a vector space.

We now give a definition for our objects of interest in this section.

Definition 1.14. An algebra is *simple* if it has no non-trivial two-sided ideals.

Proposition 1.15. *Any matrix algebra $M_n(k)$ over a field k is simple.*

Proof. Let $\mathcal{I}$ be a non-zero two-sided ideal of $M_n(k)$ and let $A = a_{i,j} \in \mathcal{I}$ be a matrix with $a_{r,k} \neq 0$ for some r, k. Then $N = \frac{1}{a_{r,k}} E_{r,r}AE_{k,k}$ is a matrix with a 1 in position r, k and zeros elsewhere. Further, $N \in \mathcal{A}$. We can then, by applying left and right permutation matrices to N, obtain each of the matrices $E_{i,i}$ which are again elements of the ideal. Then we have $\sum_{i=0}^{n} E_{i,i} = I \in \mathcal{A}$. But then $\mathcal{A}$ contains a unit and is therefore equal to the whole $M_n(k)$, so the only non-zero two-sided ideal of $M_n(k)$ is itself thus $M_n(k)$ is simple. ■

We will now show the converse is also true, i.e., that all simple algebras are (isomorphic to) matrix algebras. We will need a couple intermediary results first, adapted from Alperin and Bell's treatment of Wedderburn Theory [2].

Definition 1.16. For a given ring R a left *R-module* over an abelian group M is a binary operation $R \times M \to M$ called scalar multiplication which satisfies:

$\forall r, s \in R \quad m, n \in M$

1. $r(sm) = (rs)m$
2. $r(m + n) = rm + rn$
3. $(r + s)m = rm + sm$
4. $1_R m = m$

We insist on the fourth condition as all the rings we will be considering have unity. A right R-module is defined similarly by reversing the order of the binary operation and adjusting the axioms in the same way. All modules we consider will be left modules unless otherwise stated.

Modules are a natural generalization of vector spaces, we simply allow the field k in a vector space to be an arbitrary ring R. Notably, all vector spaces and hence all algebras are modules. Every ring R is also a module over itself: let $M = (R, +)$ as an abelian group, then we can define a natural R-module action on M for any $r \in R$ and $m \in M$ by $r \cdot m = rm$.

Remark 1.17. While modules naturally generalize vector spaces this generalization comes at the cost of losing many of the useful properties of vector spaces. For instance, a module need not have a basis and if it has multiple bases these need not have the same size.

Proposition 1.18. *If A is a k-algebra then any A-module M is also a k-module.*

Proof. By 1.3, any k-algebra A contains an isomorphic copy of k as a subalgebra. Then we can define a k-module structure on an A-module M by restricting the action of A on M to elements of A that lie in k. ■

Note that the structure in 1.18 is *not* an algebra as we have no way to multiply elements of the abelian group M (we do not know how to enrich M with a ring structure).

Example 1.19.

- The matrix algebra $A = M_n(k)$ is a $k-$algebra and thus a k-module, but also an A-module where the scalar product is given by matrix multiplication.

- Let L be the subset of $M_n(k)$ consisting of matrices where only the first column is non-zero. Then L is closed under addition, has inverses, and an additive identity, so L is a group. L is abelian due to the commutativity of k addition. Then L is a k-module and an A-module under scalar multiplication and matrix multiplication respectively. However L is not an algebra because it does not contain a multiplicative identity (L is in fact a left ideal of $M_n(k)$).

Definition 1.20. A *module homomorphism* $\phi : M \to N$ between R-modules M and N is a group homomorphism with the additional property that $\phi(rm) = r\phi(m)$ for all $r \in R$ and $m \in M$.

Definition 1.21. A *submodule* N of a module M is a subgroup of M such that $rn \in N$ for all $r \in R$ and $n \in N$. In this case we write $N \leqslant M$. A module is *simple* or *irreducible* if its only submodules are itself and the zero module.

Proposition 1.22. *If a ring R is viewed as a module over itself then the submodules of R are exactly the left ideals of R.*

Proof. Any left ideal of R is a submodule as it is closed under addition and multiplication from the left. Now let S be a submodule of R. Then S is closed under addition of scalar multiplication from the left, that is, S is a left ideal of R. ■

Proposition 1.23. *The kernel of a module homomorphism is a submodule.*

Proof. Let ϕ be a module homomorphism and a and b be elements of $\ker\phi$. Then $\phi(a-b) = \phi(a) - \phi(b) = 0 - 0 = 0$, so $(a-b) \in \ker\phi$ and $\ker\phi$ is a subgroup. To show it is a submodule let $r \in R$, then we have $\phi(ra) = r\phi(a) = r0 = 0$ and $\ker\phi$ is closed under multiplication. ■

Because M is always an abelian group every submodule N of M is also a normal subgroup of M, that is, we can form the quotient group M/N. We can extend the scalar multiplication on M to act on M/N, so M/N is also a module, the *quotient module* of M by N. Notable we have a first isomorphism theorem for modules and a correspondence between submodules and kernels of homomorphisms.

Schur's Lemma is a basic but very useful result on simple modules over an algebraically closed field.

Lemma 1.24 (Schur [2, p. 111])**.**

1. *The only homomorphisms of simple modules are isomorphisms or the zero map. That is, a non-zero homomorphism $M \to N$ of simple modules M and N is necessarily an isomorphism.*

2. *For a simple A-module S we have*

$$\mathrm{End}_A(S) = \lambda I$$

where I is the identity map and $\lambda \in k$.

Proof.

1. Let $\phi : M \to N$ be a module homomorphism of simple modules M and N. Then by simplicity of M either $\ker \phi = \{ 0 \}$, in which case ϕ is injective and $\mathrm{Im}\, \phi = N$ by simplicity of N, or $\ker \phi = M$ in which case ϕ is the zero map.

2. Let $\phi \in \mathrm{End}_A(S)$ and let λ be an eigenvalue with corresponding eigenvector v of ϕ ($\mathrm{End}_A(S) \subseteq \mathrm{End}_k(S) \cong M_n(k)$, so eigenvectors are sensible). Such an eigenvalue and eigenvector exist as k is algebraically closed. Then $\phi - \lambda I$ is a map $S \to S$ with non-zero kernel, but since S simple this implies that $\phi - \lambda I = 0$ and $\phi = \lambda I$.

■

Definition 1.25. For k-algebras $L \leqslant A$ denote the set of L-linear endomorphisms of an algebra A by $\mathrm{End}_L(A)$. That is, $\mathrm{End}_L(A) = \{ \phi : A \to A \mid \phi(la) = l\phi(a) \}$.

$\mathrm{End}_L(A)$ is a k-algebra, the *endomorpism algebra*, with addition given by function addition, multiplication by function composition, and scalar multiplication given by linearity.

Definition 1.26. The *opposite algebra* A^{op} of an algebra A is obtained by reversing the order of multiplication. That is, ab in A^{op} is equal to ba in A.

If A is commutative then A^{op} is identical to A, and there is a natural isomorphism between A and $A^{op^{op}}$.

Lemma 1.27. *If A is an algebra then $A^{op} \cong \mathrm{End}_A(A)$.*

Proof. Let $\phi \in \mathrm{End}_A(A)$. Then ϕ is completely determined by $\phi(1)$, for $\phi(a) = a\phi(1)$ for all $a \in A$, that is, A and $\mathrm{End}_A(A)$ are in bijection. Define a map $\psi : A^{op} \to \mathrm{End}_A(a)$ by $\psi(a) = - \times a$ (right multiplication). Then ψ is a homomorphism of algebras. (We require the domain of ψ to be A^{op} as we have for ϕ_a, $\phi_b \in \mathrm{End}_A(A)$ that $\phi_a\phi_b = - \times ba = \phi_{ba}$). Likewise, for any $m \in A$ we can construct a valid endomorphism $\phi_a = - \times m$, so ψ is surjective and $A^{op} \cong \mathrm{End}_A(A)$. ■

Lemma 1.28. *Let A be a simple k algebra. Then as an A module we have*

$$A \cong \bigoplus^n S$$

where S is a simple A-module.

Proof. We begin by considering A as an A-module over itself. Let A' be the (not-necessarily direct) sum of all the simple A-sub-modules (that is, left ideals of A as an algebra) and let S be a simple A-module. Then for $a \in A$, we have that Sa is either 0 or a simple A-module. Then $Sa \subseteq A'$ and A' is a right ideal. But A' is also a left ideal as it is a sum of left ideals, therefore A' is a two sided ideal and by the simplicity of A either 0 or A. A' is not 0 so we conclude that $A = A'$ and that A is a (not-necessarily direct) sum of simple modules.

To see that A is in fact a direct sum of simple modules, let B be a simple submodule of A and C be a maximum submodule such that $B \cap C = \{ 0 \}$. We want to establish that $A = B + C$. Suppose this is not the case. Then there is a simple submodule S not contained in $B + C$. By simplicity of S, $S \cap C = \{ 0 \}$ and $C \subset S + C$.

By maximality of C the set $B \cap (S + C)$ is non-empty. Let $b \in B \cap (S + C)$ such that $b = s + c$ for some $s \in S$ and $c \in C$. Then $s = b - c \in S \cap (B + C)$ so $s = 0$ as S is disjoint from $B + C$, and $b = c$ which implies $b = 0$ as B and S are also disjoint. But then $B \cap (C + S) = \{ 0 \}$ and C is not maximal, a contradiction! So we must have $A = B + C$, which as B and C are disjoint means $A = B \oplus C$. Applying this process to each simple submodule of A results in the decomposition $A = \oplus_{i=1}^{n} S_i$ for simple submodules S_i.

Finally, to show that each of the S_i are isomorphic simply note that $S_i S_j \neq 0$ then the map $\phi : S_i \to S_j$ given by $\phi(s_i) = s_i s$ for some fixed $s \neq 0 \in S_j$ is a homomorphism, and hence by Schur's Lemma an isomorphism.

To see that $S_i S_j$ must be nonzero observe that if $S_i S_j$ were to equal 0 we would have that $(S_i A)(S_j A) = S_i(AS_j)A \subseteq S_i S_j A = 0A = 0$, but $S_i A$ and $S_j A$ are two-sided ideals and hence by simplicity of A either 0 or A. Then one of S_i and S_j is zero as otherwise we would have for $x \in S_i$ $x1 = x \neq 0 \in S_i A$ and likewise for S_j. So $S_i S_j \neq 0$. ∎

Remark 1.29. In the previous lemma the submodule S corresponds to a left ideal of A. Specifically as S is simple it is a *minimal* left ideal, meaning it does not contain another non-zero left ideal.

Lemma 1.30. *Let A be a simple algebra and S be a simple A-module . Then* $\mathrm{End}_A \left(\bigoplus^n S \right) \cong M_n \left(\mathrm{End}_A (S) \right)$.

Proof. Consider $s = s_i \in \oplus_{j=1}^{n} S$ as a n vector and let $\Phi = (\phi_{i,j}) \in M_n \left(\mathrm{End}_A (S) \right)$. We define a map $\rho : M_n \left(\mathrm{End}_A (S) \right) \to \mathrm{End}_A \left(\bigoplus^n S \right)$ by $\rho (\Phi) = \Phi s$ be the action by the matrix product where multiplication of $\phi_{i,j} s_i = \phi_{i,j}(s)$ (function application). ρ is a homomorphism by the distributivity and associativity of the matrix product. Additionally, $\ker \rho = 0$ as matrix algebras are simple and ρ is not the zero map. So ρ is an injective homomorphism.

To show ρ is surjective we define a preimage for each $\Psi \in \mathrm{End}_A \left(\bigoplus^n S \right)$ by

$$\Psi^{-1} = \begin{bmatrix} \psi_{1,1} & \dots & \psi_{1,n} \\ \vdots & \ddots & \vdots \\ \psi_{n,1} & \dots & \psi_{n,n} \end{bmatrix}$$

where $\psi_{i,j}$ for some fixed j is defined implicitly as

$$\Psi s_i = \begin{bmatrix} \psi_{1,j}(s_i) \\ \vdots \\ \psi_{n,j}(s_i) \end{bmatrix}$$

So ρ is an isomorphism of algebras. ■

We can now describe the simple finite-dimensional algebras over algebraically closed fields.

Theorem 1.31. *Every finite-dimensional $\mathbb{C}$-algebra is isomorphic to a matrix algebra $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

Proof. We have

$$\begin{aligned}
A &\cong \mathrm{End}_{A^{op}}\left(A^{op}\right) && \text{By Lemma 1.27} \\
&\cong \mathrm{End}_{A^{op}}\left(\oplus^n S\right) && \text{By Lemma 1.28} \\
&\cong M_n\left(\mathrm{End}_{A^{op}}\left(S\right)\right) && \text{By Lemma 1.30} \\
&\cong M_n(k) && \text{By Schur's Lemma}
\end{aligned}$$

■

This result has a simple corollary that answers our question of what the possible dimensions of matrix algebras are for the class of simple algebras.

Corollary 1.32. *$M_n(k)$ has simple subalgebras of dimension m^2 for all $0 \leqslant m \leqslant n$.*

Remark 1.33. Wedderburn's Theorem can be easily generalized to fields that are not algebraically closed, we just note that $\mathrm{End}_{A^{op}}\left(S\right)$ is itself a division algebra. The theorem then becomes that every simple k-algebra is isomorphic to a matrix algebra over a division algebra of k.

Chapter 2

Nilpotent Matrix Subalgebras

We now wish to turn our attention to another class of matrix subalgebras–those that are nilpotent. It is fairly straightforward to provide a description of the dimensions of nilpotent matrix subalgebras, and we will and we will in fact find that for a matrix algebra $M_n(k)$ each "reasonable" dimension is realizable in a nilpotent subalgebra. This will stand in contrast with later chapters.

2.1 Nilpotent Algebras

Definition 2.1. An element x of an algebra A is said to be *nilpotent of class* t if $x^t = 0$. The algebra itself is said to be nilpotent of class t if every product of t or more elements is zero.

The smallest nilpotency class of an algebra is called its *rank*.

Note that an algebra being nilpotent of class t implies that it is nilpotent of class $t+1$, and that every element of the algebra has nilpotency class at most t.

Example 2.2. The algebras of order n strictly upper and lower triangular matrices are nilpotent of rank $n-1$.

This can be proved with the assistance of the following lemma and definition.

Definition 2.3. An r *super upper triangular* matrix is a matrix where the only non-zero entries are those above the r super diagonal (the set of entries for which the row index plus r equals the column index), that is, the (i, j) entry of the matrix being nonzero implies that $i + r \leqslant j$.

Note the 1 super upper triangular matrices are equivalent to the strictly upper triangular matrices.

We will abbreviate r super upper triangular to r-SUT for readability purposes.

Lemma 2.4. *Let M be an r-SUT matrix and N be a q-SUT matrix. Then MN is an $(r+q)$-SUT matrix.*

Proof. Let $M = (m_{i,j})$ and $N = (n_{i,j})$. From the definition of super upper triangularity, $m_{i,j} \neq 0$ only if $i + r \leqslant j$, and likewise $n_{i,j} \neq 0$ only if $i + q \leqslant j$. Let $P = MN = (p_{i,j})$. Then $p_{i,j} = \sum_{k=1}^{n} m_{i,k} n_{k,j}$. If $p_{i,j} \neq 0$ then there must exist a k such that $m_{i,k} \neq 0$ and $n_{k,j} \neq 0$, that is, a k such that $i + r \leqslant k$ and $k + q \leqslant j$. Assume such a k exists. Then we have

$$\begin{aligned} i + r &\leqslant k \\ i + r + q &\leqslant k + q \\ i + (r + q) &\leqslant j \end{aligned}$$

Which is exactly the same as $P = MN$ being $r+q$ super upper triangular. ∎

We can now show that the strictly upper triangular matrices are nilpotent of rank $n - 1$.

Proposition 2.5. *Let $N < M_n(k)$ be the subalgebra of strictly upper triangular matrices (that is, 1-SUT), then N has nilpotency rank $n - 1$.*

Proof. Let $N_1 \in N$ be 1-SUT and N_r be r-SUT. By the previous lemma $N_1 N_r$ and $N_r N_1$ are both $(r + 1)$-SUT. It follows inductively that an r-fold product of strictly upper triangular, that is, 1-SUT, matrices is r SUT.

Note that an $(n - 1)$-SUT matrix is the zero matrix, for there are no (i, j) such that $i + n - 1 \leqslant j$. Thus $N_1^{n-1} = 0$ and N_1 has nilpotency class $n - 1$.

To show that N is in fact nilpotent of rank $n - 1$ we consider the product D^{n-2} where $D = (d_{i,j})$ is the 1 super diagonal matrix of all ones ones. $D^2 = \sum_{k=1}^{n} d_{i,k} d_{k,j}$ which has a nonzero term if and only if $i + 1 = k$ and $k + 1 = j$, that is, if $i + 2 = j$. A simple inductive argument then shows that $D^r \neq 0$ for $r < n - 1$ and thus the strictly upper triangular matrices have nilpotency rank $n - 1$. ∎

In fact, all nilpotent matrix algebras are conjugate to the algebra of strictly upper triangular matrices.

Theorem 2.6 (Lie-Kolchin for Associative Algebras)**.** *A nilpotent matrix algebra N over an algebraically closed field is conjugate to an algebra of strictly upper-triangular matrices.*

Proof. Assume there exists a counterexample $N < M_n(k)$ and that this counterexample is minimal with respect to n and nilpotency rank. We say that an algebra is *reducible* if its action on itself preserves a non-trivial subspace.

Assume N is reducible. Since N preserves a subspace, under suitable basis any element $N' \in N$ must take the form

$$\left[\begin{array}{c|c} N_1 & B \\ \hline 0 & N_2 \end{array}\right].$$

Now since N is nilpotent both N_1 and N_2 must be nilpotent as well. But since N is a minimal counterexample to the theorem, N_1 and N_2 must be conjugate to strictly upper triangular matrix algebras, so N is also conjugate to a strictly upper triangular matrix algebra.

Assume on the other hand that N is irreducible. Fix $X = A_1 A_2 ... A_{r-1} \in N$ such that $X \neq 0$ and r is the nilpotency rank of N. Schur's Lemma (Lemma 1.24) tells us that because N is irreducible the only elements that commute with all of N are scalar matrices, but for any $M \in N$ $XM = MX = 0$, so X must be scalar. The only nilpotent scalar matrix is the zero matrix, but $X \neq 0$ so we have the required contradiction. ∎

Remark 2.7. The Lie-Kolchin theorem is typically presented as a result on Lie Algebras, but analogs hold for other categories of algebraic structures. Notable it will also hold for algebras over a non-algebraically closed field, (our proof only used algebraically closed in the invocation of Schur's Lemma), however the proof of this fact is more complicated than that of the algebraically closed case.

Corollary 2.8. *The maximum dimension of a nilpotent matrix subalgebra is* $\frac{n^2-n}{2} = \binom{n}{2}$.

2.2 Dimensions of Nilpotent Matrix Subalgebras

In this section we wish to demonstrate that for a matrix algebra $M_n(k)$ there exists a nilpotent subalgebra for each dimension $0 \leqslant m \leqslant \frac{n^2-n}{2}$. We begin by defining a total order $\mathcal{D}$ on the set $E = \{ (i,j) \mid i < j, 0 < i, j \leqslant n \}$. Define a sequence $D_n = (r, k)$ on E where we denote $D_n^i = r$ and $D_n^j = k$ by

$$D_1 = (1, n)$$
$$D_n = \begin{cases} (1, n - D_{n-1}^i) & D_{n-1}^j = n \\ (D_{n-1}^i + 1, D_{n-1}^j + 1) & \text{otherwise} \end{cases}$$

Then take the order $\mathcal{D}$ to be the order on E defined by each $e \in E$'s position in the sequence D_n. Visually this order traces out each superdiagonal of $M_n(k)$ starting at the $n-1$ super diagonal and working inwards, for example in $M_4(k)$:

$$\begin{bmatrix} & 4 & 2 & 1 \\ & & 5 & 3 \\ & & & 6 \\ & & & \end{bmatrix}$$

Notable, if $s < t$ and e_1 is on the s super diagonal and e_2 is on the t super diagonal, $e_2 < e_1$ in the order $\mathcal{D}$. Additionally, if the span of all D_j is taken for $0 \leqslant j \leqslant i$ and D_i is t super diagonal, then the span contains at least the $t+1$-SUT matrices.

Theorem 2.9. *There exists a nilpotent subalgebra of $M_n(k)$ of dimension m for each $0 \leqslant m \leqslant \frac{n^2-n}{2} = \binom{n}{2}$.*

Proof. Let ψ be the function that sends (i,j) to $E_{i,j}$ for all $0 \leqslant i,j \leqslant n$. Define an ascending chain of subspaces A_i by

$$A_i = \{\ \psi(D_j) \mid 1 \leqslant j \leqslant i\ \}$$

with respect to the order $\mathcal{D}$. We define A_0 as the zero set.

We will show inductively each A_i is a subalgebra. Since $A_0 = \{\ 0\ \}$ is clearly a subalgebra, so assume the hypothesis holds for all A_k when $k < i$.

Let M and N be elements of A_i, we need to show that the product MN lies in A_i. Let t be lowest superdiagonal containing an element of A_i. Then M and N are both t-SUT. By Lemma 2.4, MN is $2t$-sut and $MN \in A_i$. So A_i is closed and multiplication and is a subalgebra.

Then each A_i is a subalgebra of dimension i and we have constructed nilpotent subalgebras of dimension m for each $0 \leqslant m \leqslant \frac{n^2-n}{2}$. ■

Remark 2.10. There are many ways to build the ascending chain of subalgebras, ours was chosen simply for ease of proof. In theory any chain with the property that under the standard basis of $E'_{r,k}s$ that the presence of $E_{r,k}$ in the basis implied the presence of $E_{i,k}$ and $E_{r,j}$ for $1 \leqslant i < r$ and $k < j \leqslant n$ would work.

2.3 Schur Subalgebras

A straightforward result attributed to Schur shows the existence of nilpotentency class 2 subalgebras of $M_n(k)$ of dimension m for all $0 \leqslant m \leqslant \lfloor \frac{n}{4} \rfloor$.

Definition 2.11. A *Schur subalgebra* of $M_n(k)$ is a subalgebra with the property that the product of any two elements is zero, that is, a subalgebra that is nilpotent of rank 2.

It is easy to construct a Schur subalgebra for each $0 \leqslant m \leqslant \lfloor \frac{n}{4} \rfloor$. Start by writing an $n \times n$ matrix in blocks.

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Such that B is a $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$ matrix, A is $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$, C is $\lceil \frac{n}{2} \rceil \times \lfloor \frac{n}{2} \rfloor$, and D is $\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil$.

Now let $\mathcal{Z} \subset M_n(k)$ be the set of matrices that when written in this form have A, C, and D be zero matrices. $\mathcal{Z}$ is clearly closed under addition and scalar multiplication. Let $M, N \in \mathcal{Z}$. Then we have

$$MN = \left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & X \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

So $\mathcal{Z}$ is closed under multiplication and hence is a subalgebra.

Note that the structure of B and X had no effect on the product of the two matrices, meaning any subset of $\mathcal{Z}$ that is both additively and scalar closed is itself a subalgebra. This means that *any* zero-pattern matrix whose block form matches that of $\mathcal{Z}$ induces a subalgebra. This demonstrates the following result.

Proposition 2.12 (Schur). *Let $M_n(k)$ be a full matrix algebra. Then for $0 \leqslant m \leqslant \lfloor \frac{n^2}{4} \rfloor$ there exists an m-dimensional nilpotent subalgebra of $M_n(k)$.*

Proof. Given m simply select an $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$ ZPM Z with exactly m $*$ entries and the rest 0. Then

$$\left[\begin{array}{c|c} 0 & Z \\ \hline 0 & 0 \end{array}\right]$$

induces an m dimensional Schur subalgebra. We thus have Schur subalgebras of each dimension $0 \leqslant m \leqslant \lfloor \frac{n^2}{4} \rfloor$, with the upper bound being achieved when Z generates the full $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$ matrix. ∎

Example 2.13. The zero-pattern matrix

$$Z = \begin{bmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

generates a three dimensional Schur subalgebra of $M_4(k)$.

Mirzakhani showed that $\lfloor \frac{n^2}{4} \rfloor + 1$ is the maximum dimension of an abelian matrix subalgebra [4]. (The factor of 1 is from the identity matrix, as for any Schur matrices A and B we certainly have $(I + A)(I + B) = I + A + B + AB = I + A + B + BA = (I + B)(I + A)$ as $AB = BA = 0$). Since nilpotent rank-two subalgebras are trivially abelian and as we established these subalgebras exist for each dimension $0 \leqslant m \leqslant \lfloor \frac{n^2}{4} \rfloor$ we may conclude that these are in fact all the dimensions of rank-2 nilpotent subalgebras.

Remark 2.14. Mutually-commuting is *exactly* the meaning of rank-two nilpotent in the context of Lie Algebras.

We can extend this to nilpotent subalgebras of higher rank by taking our block matrix and partitioning each block again into four pieces, resulting in an overall 4×4 block matrix.

$$\left[\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right] \mapsto \left[\begin{array}{c|c|c|c} 0 & 0 & A_1 & A_2 \\ \hline 0 & 0 & A_3 & A_4 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

Note: at this point we will cease drawing dividing lines for the blocks in the interest of visual clarity, the reader should keep in mind that the entries of the matrices we consider for the remainder of this section are block matrices.

We then repeat the Schur construction on the upper and lower right $2x2$ block sections, that is, we build a matrix of the form

$$N_4 = \begin{bmatrix} 0 & B & A_1 & A_2 \\ 0 & 0 & A_3 & A_4 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Where all of the A_i, B, and C are arbitrary zero-pattern matrices and B and C are $\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \rfloor \times \lfloor \frac{\lceil \frac{n}{2} \rceil}{4} \rfloor$ and $\lfloor \frac{\lceil \frac{n}{2} \rceil}{4} \rfloor \times \lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \rfloor$ respectively.

Proposition 2.15. *A matrix of the form N_4 has nilpotency rank four.*

Proof. Note that

$$N_4^2 = \begin{bmatrix} 0 & 0 & BA_3 & A_1C + BA_4 \\ 0 & 0 & 0 & A_3C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has nilpotency rank two, implying N_4 has nilpotency rank of at most four. To see N_4 does not have nilpotency rank three simply observe

$$N_4^3 = \begin{bmatrix} 0 & 0 & 0 & BA_3C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0$$

. ■

This fact implies that we can always construct nilpotent class four (that is, nilpotent rank at most four) matrix subalgebras for all dimensions

$$0 \leqslant m \leqslant \lfloor \frac{n^2}{4} \rfloor + 2\lfloor \frac{n^2}{16} \rfloor \to \frac{3}{8}n^2 \text{ asymptotically as } n \text{ increases}$$

.

Note that if BA_3C were 0 this would be a subalgebra of nilpotency rank 3, in particular this would occur if either B or C were zero. This gives us a lower bound on the number of dimensions of nilpotent class three subalgebras:

$$0 \leqslant m \leqslant \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n^2}{16} \rfloor \to \frac{5}{16}n^2 \text{ asymptotically as } n \text{ increases}$$

.

We can imagine continuing this construction in a fractal-like manner, dividing the matrix next into 64 blocks and extending it by four new zero-pattern blocks, then dividing into 256 blocks and extending by eight zero-pattern blocks, etc.

Question:

Let $\mathcal{Z}(r,n)$ denote the maximum dimension of a nilpotency rank r subalgebra of $M_n(k)$. The results in this chapter prove that

$$\begin{aligned}
\mathcal{Z}(r,n) &\leqslant \frac{n^2-n}{2} \\
\mathcal{Z}(2,n) &= \lfloor \frac{n^2}{4} \rfloor \\
\mathcal{Z}(3,n) &\geqslant \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n^2}{16} \rfloor \\
\mathcal{Z}(4,n) &\geqslant \lfloor \frac{n^2}{4} \rfloor + 2\lfloor \frac{n^2}{16} \rfloor
\end{aligned}$$

What is the behavior of

$$\mathcal{Z}(r,n)$$

in general? Are the previously established bounds sharp?

2.4 Integer Sequences Associated with Nilpotent Algebras

A *nilpotent matrix subalgebra* of nilpotency class r is a subalgebra of the full matrix algebra $M_n(k)$ such that any product of r elements equals 0.

Dimensions of Nilpotent Subalgebras

There are nilpotent subalgebras of $M_n(k)$ of dimension m for all $0 \leqslant m \leqslant \frac{n^2-n}{2} = \binom{n}{2}$. The number of dimensions of nilpotent subalgebras of $M_n(k)$ is thus $\frac{n^2-n}{2} + 1$.

These are exactly the triangular numbers plus one.

Sequence of Number of Dimensions

OEIS sequence A000124. [6]

$$A_n = \frac{n^2 - n}{2} + 1$$

First 100 Terms

1	2	4	7	11	16	22	29	37	46
56	67	79	92	106	121	137	154	172	191
211	232	254	277	301	326	352	379	407	436
466	497	529	562	596	631	667	704	742	781
821	862	904	947	991	1036	1082	1129	1177	1226
1276	1327	1379	1432	1486	1541	1597	1654	1712	1771
1831	1892	1954	2017	2081	2146	2212	2279	2347	2416
2486	2557	2629	2702	2776	2851	2927	3004	3082	3161
3241	3322	3404	3487	3571	3656	3742	3829	3917	4006
4096	4187	4279	4372	4466	4561	4657	4754	4852	4951

Density of Dimensions of Nilpotent Subalgebras

There are n^2 possible dimensions of subalgebras of $M_n(k)$. We see that

$$\lim_{n\to\infty} \frac{A_n}{n^2} = \frac{n^2 - n + 2}{2n^2} = \frac{1}{2}$$

Dimensions of Nilpotent Rank 2 Subalgebras

The *nilpotency rank* of a nilpotent subalgebra is the smallest integer r such that all length r products of elements are 0. There are rank 2 nilpotent subalgebras of $M_n(k)$ of dimensions m for all $0 \leqslant m \leqslant \lfloor \frac{n^2}{4} \rfloor$. This is 1 less than the maximum number of linearly-independent mutually-commuting square $n \times n$ matrices.

Sequence of Number of Dimensions

OEIS sequence A002620. [7]

$$A_n = \lfloor \frac{n^2}{4} \rfloor$$

First 100 Terms

0	1	2	4	6	9	12	16	20	25
30	36	42	49	56	64	72	81	90	100
110	121	132	144	156	169	182	196	210	225
240	256	272	289	306	324	342	361	380	400
420	441	462	484	506	529	552	576	600	625
650	676	702	729	756	784	812	841	870	900
930	961	992	1024	1056	1089	1122	1156	1190	1225
1260	1296	1332	1369	1406	1444	1482	1521	1560	1600
1640	1681	1722	1764	1806	1849	1892	1936	1980	2025
2070	2116	2162	2209	2256	2304	2352	2401	2450	2500

Chapter 3

Semisimple Matrix Subalgebras

In this chapter we investigate the class of semisimple matrix algebras. We find that their dimensions are elegantly modeled by a recurrence relation, but that exact computations and bounds on the dimension set are difficult in general. We provide a lower bound on the size of the dimension set using several simplifying estimations.

3.1 Semisimple Algebras

Over any field, $M_n(k)$ is a simple algebra by Proposition 1.15, and if k is algebraically closed then any (finite dimensional) simple k-algebra is isomorphic to some full matrix algebra over k by Theorem 1.31. It is often interesting and useful to consider *semisimple* algebras, algebras which are direct sums of simple algebras.

Definition 3.1. An algebra A is a *direct sum* of two subalgebras M_1 and M_2 if $M_1 \cap M_2 = \{ 0 \}$ and we have $(m_1 + m_2)(m_1^{'} + m_2^{'}) = m_1 m_1^{'} + m_2 m_2^{'}$ for all $m_1, m_1^{'} \in M_1$ and $m_2, m_2^{'} \in M_2$.

Definition 3.2. An algebra A over a field k is said to be *semisimple* if it is isomorphic to a direct sum of simple algebras. [2, p. 121]

For an algebraically closed field the semisimple algebras are exactly direct sums of full matrix algebras by the previously mentioned results. Recall that the direct sum of two matrices A and B over the same ring is defined as

$$A \oplus B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right].$$

(That this definition of direct sum satisfies the definition given earlier follows from the properties of block matrices.) From these facts it is easy to see that any semisimple matrix algebra $\mathcal{A}$ can, up to conjugation, be expressed in the form

$$\mathcal{A} = \{ \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & Z \end{bmatrix} |A \in M_{n_1}(k),\ B \in M_{n_2}(k),\ \dots,\ Z \text{ all zeros}\}$$

Note that $\mathcal{A}$ need not be of full rank due to the inclusion of Z in the direct sum, we can think of this as an embedding of semisimple subalgebra of a smaller matrix algebra into a larger $M_n(k)$. In fact this is a complete description of semisimple matrix algebras, every set of matrices in this form is a semisimple subalgebra.

Proposition 3.3. *If*

$$\mathcal{A} = \left\{ \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & Z \end{bmatrix} \middle| A \in M_{n_1}(k),\ B \in M_{n_2}(k),\ \dots,\ Z \textit{ all zeros} \right\}$$

then $\mathcal{A}$ is semisimple.

Proof. The properties of block matrices show us that $\mathcal{A}$ is closed under addition, multiplication, and scalar products, so it is an algebra. Since $\mathcal{A}$ splits into a direct sum of $M_{n_i}(k)$, each of which is simple, $\mathcal{A}$ is an algebra composed of a direct sum of simple algebras and is therefore semisimple. ■

Proposition 3.4. *A semisimple matrix algebra $\mathcal{A}$ is isomorphic to a subalgebra of $M_n(k)$ if and only if $W(\mathcal{A}) \leqslant n$.*

Proof. Let $\mathcal{A}$ be a semisimple matrix algebra. If $\mathcal{A}$ is isomorphic to a subalgebra of $M_n(k)$, then $W(\mathcal{A})$ is evidently no greater than n. Conversely if $n \geqslant W(\mathcal{A})$ then we can embed $\mathcal{A}$ into $M_n(K)$ as a block-diagonal subalgebra. ■

Theorem 3.5. *$M_n(k)$ has a semisimple subalgebra of dimension m if and only if we can write $m = \sum_{i=0}^{j} k_i^2$ such that $\sum_{i=0}^{j} n_i \leqslant n$.*

Proof. Assume $M_n(k)$ has a semisimple subalgebra A of dimension m. Then $A = \oplus_{i=0}^{j} M_{n_i}(k)$. The sum $\sum_{i=0}^{j} n_i$ must be less than n or else the subalgebra would not fit in $M_n(k)$. Then as the dimension of a full matrix algebra is n^2 we have $m = \dim A = \sum_{i=0}^{j} n_i^2$.

To show the converse, let m be expressed as $m = \sum_{i=0}^{j} n_i^2$ such that $\sum_{i=0}^{j} n_i \leqslant n$. Then we can construct a semisimple algebra of $M_n(k)$ by taking $\oplus_{i=0}^{j} M_{n_i}(k)$, which will evidently have dimension m. ■

Definition 3.6. Let $\mathcal{A} = \bigoplus_{i=1}^{j} M_{n_i}(k)$ be a semisimple matrix algebra. Then the *width* of $\mathcal{A}$, denoted by $W(\mathcal{A})$ is the sum of the orders of its direct summands, i.e., $W(\mathcal{A}) = \sum_{i=1}^{j} n_i$.

There are a few observations we can make about the width of a semisimple matrix algebra when we consider them as subalgebras of a complete matrix algebra $M_n(K)$.

Definition 3.7. The *commutator* of two matrices $A, B \in M_n(k)$, written $[A, B]$, is the difference $AB - BA$.

Two matrices have a commutator of zero if and only if they commute. In general, the commutator can be viewed as a measure of "how commutative" a pair of matrices are.

Remark 3.8. The commutator is an example of a more general class of bilinear maps known as *Lie brackets.* The study of Lie algebras is the study of vector spaces equipped with a Lie bracket.

It will be useful to consider commutators of sets as well, in this case the commutator should be considered as the set of all commutators of pairs of elements of each set. For example, $[M_n(k), M_n(k)] = \{[M_1, M_2]\,|M_1, M_2 \in M_n(k)\}$ is the commutator of the whole algebra $M_n(k)$. This is a useful invariant, if two algebras have different commutators we know they are not isomorphic.

For a given dimension and width there are in general numerous non-equivalent semisimple algebras. We have already seen that we can pad out a semisimple algebra with zero algebras to increase the order of the matrix algebra it is contained in without changing any of its other algebraic properties (this can be though of as, considering the semisimple algebra as a subalgebra of a full matrix algebra, embedding the semisimple space into a larger full algebra). We can similarly replace any zero algebras in a semisimple algebra with $M_1(k)$ to increase the dimension without changing the order of the matrix algebra containing it . More generally we will find there exist non-isomorphic algebras with coinciding dimension and width contained in the same full matrix algebra.

Example 3.9. Semisimple Matrix Algebras

- $\mathcal{B} = M_3(\mathbb{R}) = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} * \in \mathbb{R}$ is a simple algebra, and therefore is a trivially semisimple algebra of one summand. $dim(\mathcal{B}) = 9$, $width((B)) = 3$. (For a simple algebra width will simply equal order).

- $\mathcal{C} = M_3(\mathbb{F}_2) \oplus M_2(\mathbb{F}_2) = \begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} * \in \mathbb{F}_2$ is a semisimple algebra of dimension 13 and width 5.

- $\mathcal{D} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix} * \in \mathbb{C}$ is a semisimple algebra of dimension 12 and width 6.

- $\mathcal{E} = M_3(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} * \in \mathbb{C}$ is also a semisimple algebra of dimension 12 and width 6. Note that the structure is distinct from $\mathcal{D}$ in a non-trivial way: the commutator subalgebra

$$[\mathcal{E}, \mathcal{E}] = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  has dimension 9 while the commutator subalgebra

$$[\mathcal{D}, \mathcal{D}] = \mathcal{D}$$

  has dimension 12 because the commutator subalgebra of a non-abelian simple group S is S and thus the direct product of non-abelian simple groups is also its own commutator subalgebra.

- $\mathcal{F} = M_1(\mathbb{Q}) \oplus \mathbf{0} = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} * \in \mathbb{Q}$ is a semisimple algebra of dimension 1 and width 2. The natural width of this algebra is 1, and it could be equally realized as $M_1(\mathbb{Q}) \cong \mathbb{Q}$.

3.2 Dimensions of Semisimple Subalgebras

As we saw in 3.5 the dimensions of semisimple algebras correspond exactly to sums of squares $\sum_{i=1}^{l} k_i^2$. Viewed as subalgebras of a full matrix algebra $M_n(k)$ we see that they are exactly those sums of squares such that the width $\sum_{i=1}^{l} k_i$ does not exceed n. Let A_n denote the set of dimensions of semisimple subalgebras that are contained in an order n full matrix algebra. The following theorem gives a recursive description of A_n.

Theorem 3.10 (Dimensions of Semisimple Subalgebras)**.** *For a fixed n there exists a semisimple subalgebra of $M_n(k)$ of dimension $m \leqslant n^2$ if and only if either $m = n^2$ or there exist α and β such that $m = \alpha^2 + \beta$ and there is a semisimple subalgebra of dimension β of $M_{n-\alpha}(k)$.*

Proof. Let A be a semisimple subalgebra of $M_n(k)$. If A is simple them $\dim A = n^2$. Assume A is not simple, then A contains a direct summand $S \neq 0$ such that S is simple and $\dim S = \alpha^2$. Then A/S is a semisimple subalgebra of $M_{n-\alpha}(k)$ of dimension $\beta = m - \alpha^2$.

The converse is trivial.

■

The previous theorem implies that we can express the set sequence of dimensions of semisimple subalgebras through the recurrence relation

$$A_1 = \{ 0, 1 \}$$
$$A_n = \bigcup_{k=1}^{n} \{ k^2 + \alpha \mid \alpha \in A_{n-k} \} \cup \{ 0 \}$$

A similar result holds if we only wish to consider those semisimple subalgebras that have their natural width, that is, do not contain zero algebras in their direct sum nor zeros on the diagonals of their matrices (equivalently, algebras which contain the identity I). We simply must modify the initial values of the recurrence relation to not include zero:

$$B_1 = \{ 1 \}$$
$$B_n = \bigcup_{k=1}^{n} \{ k^2 + \alpha \mid \alpha \in B_{n-k} \} \cup \{ 0 \}$$

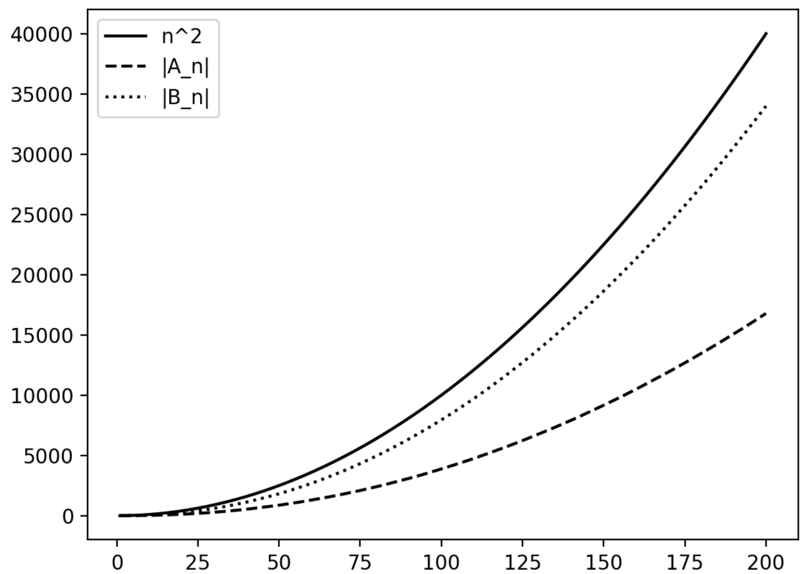
The table below gives the values of A_n and B_n for small n.

n	1	2	3	4
A_n	$\{ 0, 1 \}$	$\{ 0, 1, 2, 4 \}$	$\{ 0, 1, 2, 3, 4, 5, 9 \}$	$\{ 0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 16 \}$
B_n	$\{ 1 \}$	$\{ 2, 4 \}$	$\{ 3, 5, 9 \}$	$\{ 4, 6, 10, 16 \}$

If we inspect the table we can make a few observations. First, that for any non-trivial n there will be integers $m \leqslant n^2$ that are not the dimensions of semisimple subalgebras of $M_n(k)$. For example $M_2(k)$ has no semisimple subalgebra of dimension 3. We also note that there seems to be a region of small values for each n where each dimension $0 \leqslant m \leqslant N$ occurs for some N and N is maximal. We will call this the *continuous region.* It is easy to see that the continuous region contains at least the dimensions 0 through n, though the actual area is much larger, we will later show that it grows quadratically with n.

For an integer n we say that $G(n)$ is the first dimension of a semisimple algebra that cannot be realized as a subalgebra of $M_n(k)$. So $G(n) = N + 1$ where the N is the upper bound for the continuous region.

It is not clear how fast the sets A_n and B_n grow. Below is a figure of $|A_n|$ and $|B_n|$ for small values of n along with n^2 for comparison (as $|A_n|$ and $|B_n|$ are clearly less than n^2).



3.3 Density Analysis

We are interested in counting the number of dimensions of semisimple subalgebras for each $M_n(k)$. This is difficult in general, while we have a recurrence relation for the set of subalgebra dimensions for each n it does not easily yield a closed form expression or even an asymptotic bound. Our aim here is to simplify the problem of counting the number of dimensions to a point where we can give a lower bound for the function. To do this we will use an argument adapted from *A Note on the Symmetric Powers of the Standard Representations of S_n* by Savitt and Stanley. [5] First we need some additional notation.

Definition 3.11. Let $\Lambda_m = \{ A \in M_n(k) \mid A \text{ semisimple}, dim(A) = m \}$, that is, the set of all semisimple algebras of dimension m. Then $\mathcal{W}(m) = \inf_{\Lambda_m} w(A)$, the smallest width of an m-dimensional algebra.

It is clear that $\mathcal{W}(m)$ is the smallest $M_n(k)$ containing a m-dimensional subalgebra by Proposition 3. $\mathcal{W}$ is difficult to compute in general, so we define a function that, while not the smallest width, is a good estimate and is easy to calculate.

Definition 3.12. For a given m, write m as a sum of squares *greedily*, that is, as $m = a_0^2 + ... + a_r^2$ where each successive a_j is chosen as large as possible. Note that this is the dimension of a semisimple algebra $A = \bigoplus_{j=0}^r M_{a_j}(k)$. The *greedy width* $\mathcal{GW}(m)$ is the sum of the a_j's, $\mathcal{GW}(m) = \sum_{j=0}^r a_j$.

The greedy width is not always the minimal width, but it is close. For $m \leqslant 100$ the only values where the greedy width differs from the minimal width are $m = 32$ where $\mathcal{W}(32) = 8$ and $\mathcal{GW}(32) = 10$ (corresponding to $32 = 4^2 + 4^2$ and $32 = 5^2 + 2^2 + 1^2 + 1^2 + 1^2$ respectively) and $m = 61$ where $\mathcal{W}(61) = 11$ while $\mathcal{GW}(61) = 13$ (corresponding to $61 = 6^2 + 5^2$ and $61 = 7^2 + 3^2 + 1^2 + 1^2 + 1^2$ respectively).

We can now construct a lower bound for the number of dimensions of semisimple subalgebras of $M_n(k)$. Specifically we will bound $G(n)$ below, which while smaller than the true value will be significantly easier to compute. In terms of the width function $\mathcal{W}(m)$, this amounts to finding the smallest m such that $\mathcal{W}(m) > n$. Since $\mathcal{W}(m)$ is also difficult to compute, we instead find an upper bound for this m using the greedy width function $\mathcal{GW}(m)$. Since the smallest m for which $\mathcal{GW} > n$ is less than or equal to the smallest mm for which $\mathcal{W}(m) > n$, this will constitute a lower bound for the number of dimensions of subalgebras.

We begin with the following propositions.

Proposition 3.13. *$\sqrt{m} + 2m^{\frac{1}{4}} \leqslant \sqrt{2m}$ for $m > 544$.*

Proof. Suppose that $\sqrt{m} + 2m^{\frac{1}{4}} \leqslant \sqrt{2m}$. Then $2m^{\frac{1}{4}} \leqslant \left(\sqrt{2} - 1\right)\sqrt{m}$. Raising both sides to the fourth power we find $16m \leqslant \left(\sqrt{2} - 1\right)^4 m^2$. The quadratic $\left(\sqrt{2} - 1\right)^4 m^2 - 16m = 0$ has roots at 0 and $\frac{16}{\left(\sqrt{2}-1\right)^4} \approx 544$, hence the result follows for $m > 544$ as required. ■

Theorem 3.14. *$\mathcal{W}(m) \leqslant \sqrt{2m}$ for $m \geqslant 673$.*

Proof. First, we establish two facts we will need later on.

1. For all $1 \leqslant m \leqslant 673$ we have $\mathcal{W}(m) \leqslant \mathcal{GW}(m) \leqslant 37$.
2. For all $674 \leqslant m \leqslant 8000$ we have $\mathcal{W}(m) \leqslant \mathcal{GW}(m) \leqslant \sqrt{2m}$.

Both of these facts can be verified computationally.

Now suppose $m > 8000$ and that for all $8000 < t \leqslant m$ we have $\mathcal{W}(t) \leqslant \sqrt{2t}$ (we established this inequality for $674 \leqslant t \leqslant 8000$ in (2)). Take s such that

$$s^2 \leqslant m \leqslant (s+1)^2.$$

Then we have

$$m - s^2 \leqslant 2s.$$

By the induction hypothesis,

$$\begin{aligned}\mathcal{W}(m) &\leqslant \mathcal{W}(s^2) + \mathcal{W}(m - s^2) \\ &\leqslant s + \max\left\{\sqrt{2(m - s^2)}, 37\right\}\end{aligned}$$

For $m > 8000$ we have $s + 37 \leqslant \sqrt{2m}$, so we only need to show that $s + \sqrt{2(m - s^2)} \leqslant \sqrt{2m}$, which as $m - s^2 \leqslant 2s$ and s is less than or equal to $\sqrt{m}$ is equivalent to showing that

$$\sqrt{m} + 2\sqrt{\sqrt{m}} \leqslant \sqrt{2m},$$

which we proved was true for $m > 544$ in Lemma 3.13. ∎

We now establish a lower bound for the continuous region.

Proposition 3.15. *There exists an m dimensional semisimple subalgebra of $M_n(k)$ for all $0 \leqslant m \leqslant \frac{n^2}{2}$ for $n > 37$.*

Proof. Let m be an integer such that $m \leqslant \frac{n^2}{2}$. By Theorem 3.14, $\mathcal{W}(m) \leqslant \sqrt{n^2} = n$, hence $M_n(k)$ has a semisimple subalgebra of dimension m. ∎

We can apply this result recursively to obtain a better estimate of the size of the continuous region and therefore the number of dimensions of semisimple subalgebras of $M_n(k)$. Define a family of intervals

$$S_i = \left\{ i^2, ..., i^2 + \frac{(n-i)^2}{2} \right\}$$

Each S_i is precisely the set of dimensions obtained by fixing a $k \times k$ simple block then invoking Theorem 3.15 to bound the continuous region of the remainder; there is a semisimple subalgebra of $M_n(k)$ of dimension m for each $m \in S_i$.

Lemma 3.16. *For $i \leqslant n - \sqrt{2}\sqrt{2n+3} + 1$ we have that $S_i \cap S_{i+1} \neq \varnothing$.*

Proof. It will be sufficient to show that the upper bound of S_i is less than the lower bound of S_{i+1}, that is

$$\begin{aligned}
(i+1)^2 &\leqslant i^2 + \frac{(n-i)^2}{2} \\
2i^2 + 4i + 2 &\leqslant 2i^2 + (n-i)^2 \\
2i^2 + 4i + 2 &\leqslant 2i^2 + n^2 - 2ni + i^2 \\
-i^2 + 4i + 2ni - n^2 + 2 \leqslant 0 &
\end{aligned}$$

Which we can solve to obtain

$$i \leqslant n - \sqrt{2}\sqrt{2n+3} + 2$$ ∎

We can now prove the following result.

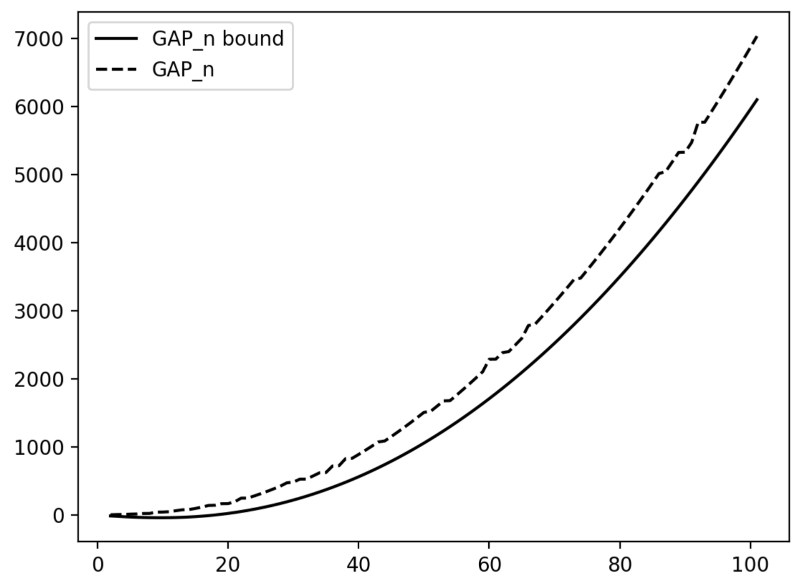
Theorem 3.17. *For $n > 40$ there exists an m dimensional semisimple subalgebra of $M_n(k)$ for all $0 \leqslant m \leqslant n^2 - 4n\sqrt{n+2}$.*

Proof. By the previous lemma we know that S_i and S_{i+1} overlap for $i \leqslant n - \sqrt{2}\sqrt{2n+3}+2$. Let $k = n - \sqrt{2}\sqrt{2n+3}+2$. Provided $\lfloor \frac{n}{\sqrt{2}} \rfloor \leqslant k$ we can extend the continuous region established previously by taking the union $0, ..., \frac{n^2}{2}$ with the S_i for $i \leqslant k$. This is possible for any $n > 40$, for which we obtain an extended continuous region of

$$0 \leqslant m \leqslant n^2 - 2\sqrt{2}n\sqrt{2n+3} + \mathcal{O}(n)$$

The result follows. ■

We can compare this bound to the true value for small n by plotting it against the first gap in the set of dimensions of semisimple subalgbras of $M_n(k)$.



We see that the difference between the bound and the true value seems to be increasing, suggesting that further improvements on the bound are possible.

Corollary 3.18. *Recall A_n is the set of dimensions of semisimple subalgebras of $M_n(k)$. Then by the previous theorem we have*

$$\lim_{n\to\infty} \frac{|A_n|}{n^2} = 1$$

3.4 Summary

Semisimple Matrix Subalgebras

A *semisimple matrix subalgebra* of $M_n(k)$ for an algebraically closed field k is a direct sum of $M_{n_i}(k)$ such that $\sum n_i \leqslant n$.

Dimensions of Subalgebras

For all $0 \leqslant k \leqslant n$ let λ_i be a partition of k indexed by i. Then the set of dimensions of semisimple subalgebras of $M_n(k)$ is the set

$$\left\{ \sum_i \lambda_i^2 \;\middle|\; \lambda \vdash k \leqslant n \right\}.$$

Sequence of Number of Dimensions

$$\begin{aligned} A_0 &= \{\, 0, 1 \,\} \\ A_n &= \{\, \alpha + \beta \mid \alpha \in A_a,\ \beta \in A_b,\ a + b = n \,\} \cup \{\, n^2 \,\} \end{aligned}$$

First 100 Terms

2	4	7	11	16	22	29	39	50	60
73	88	103	120	139	160	181	203	229	256
284	313	343	377	412	448	487	528	569	610
653	699	748	797	849	904	959	1014	1070	1129
1191	1255	1321	1388	1456	1526	1598	1672	1746	1821
1899	1981	2064	2148	2235	2322	2411	2503	2597	2690
2783	2881	2982	3086	3193	3298	3403	3512	3623	3734
3847	3964	4081	4199	4321	4446	4573	4700	4830	4961
5092	5225	5359	5498	5638	5779	5922	6067	6215	6365
6518	6670	6823	6978	7134	7293	7453	7615	7780	7947

Density of Dimensions

As we proved in Theorem 3.17, $|A_n| \leqslant n^2 - \sqrt{2}\sqrt{2n+3}n$. We then have

$$\lim_{n\to\infty} |A_n| \leqslant n^2 - \sqrt{2}\sqrt{2n+3}n = n^2$$

So A_n has density 1.

Semisimple Matrix Subalgebras of Natural Width

A *semisimple matrix subalgebra of natural width* is a semisimple sublagebra of $M_n(k)$ for an algebraically closed field k such that the subalgebra contains the identity or, equivalently, is not contained in a smaller $M_n(k)$.

Dimensions of Subalgebras

Let λ_i be a partition of n indexed by i. Then the set of dimensions of semisimple subalgebras of $M_n(k)$ is the set

$$\left\{ \sum_i \lambda_i^2 \;\middle|\; \lambda \vdash n \right\}.$$

Sequence of Number of Dimensions

OEIS sequence A069999. [3]

$$\begin{aligned} B_0 &= \{ 1 \} \\ B_n &= \{ \alpha + \beta \mid \alpha \in B_\alpha,\ \beta \in B_\beta,\ a + b = n \} \cup \{ n^2 \} \end{aligned}$$

First 100 Terms

1	2	3	5	7	9	13	18	21	27
34	39	46	54	61	72	83	92	106	118
130	145	162	176	193	209	226	246	265	284
308	330	352	375	402	426	453	480	508	538
570	598	631	661	694	730	765	800	835	872
911	951	992	1030	1071	1115	1158	1203	1251	1295
1343	1392	1440	1491	1541	1590	1642	1695	1750	1806
1861	1917	1977	2033	2092	2154	2216	2276	2340	2404
2467	2535	2605	2672	2741	2812	2882	2951	3024	3096
3170	3245	3319	3394	3474	3553	3634	3716	3798	3881

First Gap in Dimensions of Semisimple Subalgebras of $M_n(k)$

Define the sequence GAP_n to be the smallest integer that is *not* the dimension of a semisimple subalgebra of $M_n(k)$. This is one more than the upper endpoint of the continuous region of $M_n(k)$. Because when $n = 1$ there are no gaps this sequence begins at $n = 2$.

We proved in Theorem 3.17 that $GAP_n > n^2 - 4\sqrt{n+2}$.

First 100 Terms

3	6	7	12	15	22	23	42	43	48
63	76	79	96	115	140	143	166	167	192
247	248	279	312	347	384	423	472	483	526
527	572	619	624	719	724	827	832	889	948
1009	1072	1087	1152	1219	1288	1359	1432	1507	1520
1597	1676	1679	1760	1843	1928	2015	2104	2287	2288
2383	2400	2497	2596	2783	2800	2905	3012	3121	3232
3345	3460	3479	3596	3715	3836	3959	4084	4211	4340
4471	4604	4739	4876	5015	5040	5181	5324	5327	5472
5767	5768	5919	6072	6227	6384	6543	6704	6867	7032

Smallest Width of an m Dimensional Semisimple Subalgebra

Let $\Lambda_m = \{ A \in M_n(k) \mid A \text{ semisimple}, dim(A) = m \}$, that is, the set of all semisimple algebras of dimension m. Then $\mathcal{W}(m) = \inf_{\Lambda_m} w(A)$, the smallest width of an m-dimensional algebra. This is the sequence of values $\mathcal{W}(m)$ for 1..

Sequence of Smallest Widths

OEIS sequence A138554 [1].

$$\mathcal{W}(m) = \inf_{\Lambda_m} w(A)$$
$$\text{where } \Lambda_m = \{ A \in M_n(k) \mid A \text{ semisimple}, \dim(A) = m \}$$

First 100 Terms

1	2	3	2	3	4	5	4	3	4
5	6	5	6	7	4	5	6	7	6
7	8	9	8	5	6	7	8	7	8
9	8	9	8	9	6	7	8	9	8
9	10	11	10	9	10	11	12	7	8
9	10	9	10	11	12	11	10	11	12
11	12	13	8	9	10	11	10	11	12
13	12	11	12	13	14	13	14	15	12
9	10	11	12	11	12	13	14	13	12
13	14	15	14	15	16	13	14	15	10

Chapter 4

Conclusion

We investigated the dimensions of simple, semisimple, and nilpotent matrix subalgebras of $M_n(K)$. We obtained closed form expressions for the dimension sets of simple and nilpotent matrix algebras and a recurrence relation for the dimensions of semisimple matrix algebras. During our work we also found several interesting integer sequences associated with these classes of matrix algebra. The sequences

1. A000124, number of dimensions of nilpotent subalgebras of $M_n(k)$
2. A002620, number of dimensions of rank 2 nilpotent matrix subalgebras
3. A069999, number of dimensions of semisimple matrix subalgebras that contain the identity
4. A138544, smallest width of an m dimensional semisimple matrix subalgebra

were already known and included in the OEIS, while the sequences

1. $|A_n|$, number of dimensions of semisimple matrix subalgebras
2. GAP_n, first integer m that is not the dimension of a semisimple subalgebra of $M_n(k)$

are not located in the OEIS. We plan to submit the later list of sequences, as well as update the entries of the former list of sequences to include their relation to matrix subalgebras.

4.1 Further Questions

General Questions

- Which matrix subalgebras are conjugate to a zero-pattern matrix algebra? Which ones are not conjugate by are isomorphic?

In Relation to Nilpotent Matrix Subalgebras

Let $\mathcal{Z}(r,n)$ denote the maximum dimension of a nilpotency rank r subalgebra of $M_n(k)$. The results in this chapter prove that

$$\mathcal{Z}(r,n) \leqslant \frac{n^2-n}{2},$$

$$\mathcal{Z}(2,n) = \lfloor \frac{n^2}{4} \rfloor,$$

and

$$\mathcal{Z}(3,n) \geqslant \lfloor \frac{n^2}{4} \rfloor + 2\lfloor \frac{n^2}{16} \rfloor.$$

What is the behavior of

$$\mathcal{Z}(r,n)$$

in general? Are the previously established bounds sharp?

In Relation to Semisimple Subalgebras

What is the density of the sequence B_n of dimensions of semisimple subalgebras of $M_n(k)$ containing the identity?

In Relation to Parabolic Subalgebras

A *parabolic matrix subalgebra* is the stabilizer of a flag of subspaces (See Appendix B). Parabolic matrix subalgebras are related to semisimple subalgebras. What are the dimensions of the parabolic subalgebras and can these dimensions be related to the dimensions of semisimple subalgebras?

Appendix A

Zero-Pattern Subalgebras as Relations

Recall that we used a matrix Z valued in $S = \{ 0, * \}$ to describe certain matrix subalgebras. The operations we gave for S were that $* + * = * \times * = *$ with zero behaving as we expect an additive identity to. Formally as an algebraic structure, S equipped with the operations of $+$ and $\times$ as described is an *semiring*. Semirings arise from a relaxation of the ring axioms, we simply drop the requirement that elements have additive inverses.

Definition A.1. A *semiring* is a set S equipped with two associative binary operations $+$ and $\times$ such that:

- $+$ is commutative.
- $\times$ is left and right-distributive over $+$
- There is an additive identity 0 such that $s + 0 = 0 + s = s \; \forall s \in S$.
- $s \times 0 = 0 \times s = 0 \; \forall s \in S$.

Note that the fourth condition is true for proper rings but can be proved from the other ring axioms, this is not the case with semirings, as proofs that 0 annihilates the whole ring depend on the ability to take additive inverses.

Just as a ring can be considered an abelian group augmented with a multiplication operation, a semiring is an abelian monoid agumented with a multiplication operation.

Our matrices Z then can be viewed as matrices over the semiring

$$(\{ 0, * \}, +, \times)$$

such that $* + * = * \times * = *$. This semiring is called the *binary semiring*. Matrices over the binary semiring are of interest because they exactly encode binary relations on finite sets, and $n \times n$ matrix $Z = z_{i,j}$ defines a subset of the Cartesian product of an n element set with itself. Specifically we have $\mathcal{R} = \{\ (i, j) \mid z_{i,j} = *\ \}$. That is, $i\ j$ if and only if $z_{i,j} = *$. Matrix multiplication over the binary semiring corresponds to taking a composition of relations, and matrix addition corresponds to taking the disjunction of relations.

This viewpoint allows us to reinterpret our zero-pattern subalgebras in terms of relations.

- A matrix Z corresponds to a subalgebra if its corresponding relation is transitive. Taking the multiplicative closure of a matrix Z to complete a subaglebra is equivalent to taking the transitive closure of its corresponding relation.
- We showed that simple algebras are isomorphic to full matrix algebras over a division ring, these in turn correspond with universal relations. Semisimple algebras correspond with direct sums of universal relations (up to conjugation).
- A nilpotent algebra corresponds to a relation with the property that every relation chain a ... c terminates. The nilpotent rank is an upper bound on the length of relation chains.

Appendix B

Parabolic Matrix Subalgebras

The final class of matrix subalgebras we wish to consider are the parabolics. Parabolic subalgebras are closely related to semisimple subalgebras, as we will see. There are also a number of relations between their dimension sets and the methods we can use for computing and estimating them.

B.1 Flags and Stabilizers

Definition B.1. A *flag* of a vector space V is an ascending series of subspaces $0 \subset W_0 \subset \cdots \subset V$. We sometimes say a flag of a vector space is a flag *on* V.

Example B.2. The family of subspaces $\{ 0 \} \cup \{ W_i \mid W_i = \mathit{span} \{ e_0, \cdots, e_i \} \}$ is a flag on V called the *standard flag* where each e_i is a standard basis vector.

A proper flag will terminate in the whole vector space V. For our purposes it will be useful to relax this requirement and allow ourselves to consider "near flags" that do not necessarily end with V. Near flags can be thought of as flags of subspaces of V.

to-do: is this still the right definition of a near-flag?

Definition B.3. The *stabilizer* of a subspace W is the set

$$S_W = \{ \ \mid wM \in W \ \forall\, w \in W,\ M \in M_n(k) \} .$$

That is, the set of matrices whose action preserves W.

Remark B.4. When considering matrices acting on a vector space we will always assume the action to be on *row vectors*, that is, matrices act on vectors on the right under the normal matrix multiplication.

The stabilizer of a subspace is a subalgebra of $M_n(k)$, it is closed under scalar products (as the subspace is closed under scalar products), if $M_1 w \in W$ and $M_2 w \in W$ then $(M_1 + M_2)w = M_1 w + M_2 w \in W$ as subspaces are closed under addition and $M_1 M_2 w = M_1(M_2 w) \in W$ by associativity of the matrix product.

The stabilizer of a flag is simply the intersection of the stabilizers of its subspaces, that is, the set of matrices whose action preserves each subspace of the flag.

Definition B.5. The stabilizer of a flag is called a *parabolic subalgebra*. The stabilizer of a near-flag is a *near-parabolic subalgebra*.

Example B.6.

- The subalgebra of upper-triangular matrices is the stabilizer of the standard flag, and thus a parabolic subalgebra.
- The subalgebra $A = \begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$

  is *not* a parabolic subalgebra, while it *stabilizes* a flag it is not the *stabilizer* of one, only a subset of it.
- The subalgebra $A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

  stabilizes the flag $\{ 0 \} \subset \{ e_2, e_3 \} \subset \{ e_1, e_2, e_3 \} = V$.

Parabolic subalgebras are related to semisimple subalgebras, we can think of constructing a parabolic subalgebra from a semisimple subalgebra by filling in all the zeros above the diagonal of the semisimple subalgebra with $*$. (In fact parabolic subalgebras are the normalizers of semisimple subalgebras).

Appendix C

Code Used

Haskell code used in the proof of Theorem 5:

```
import Data.List

-- Find the greedy representation of a
-- natural number by a sum of squares
greedy :: Int -> [Int]
greedy 0 = []
greedy n = k : greedy (n - k^2)
  where k = (floor . sqrt) (fromIntegral n)

-- Greedy weight
gw :: Int -> Int
gw = (sum . greedy)

-- Theorem 5 condition
hypothesis :: Int -> Bool
hypothesis n = (fromIntegral (gw n)) <= sqrt (2 * (fromIntegral n))

-- Find the integer x < n such that the condition
-- h holds for any value greater than x (but less than n)
holdsAfter :: (Int -> Bool) -> Int -> Int
holdsAfter h n = head $ filter (not . h) [n, (n - 1)..1]

argmax :: Ord b => (a -> b) -> [a] -> a
argmax f xs = maximumBy (\a b -> f a `compare` f b) xs

main = do
  let b = holdsAfter hypothesis 7979
  let g = argmax gw [1..b]
```

```
putStrLn $ "Hypothesis holds for m >= " ++ (show b)
putStrLn $ "Maximum GW(m) for m < " ++ (show b) ++
           " is GW(" ++ (show g) ++ ") = " ++ (show $ gw g)
```

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