CONFORMAL EQUIVALENCE OF VISUAL METRICS IN PSEUDOCONVEX DOMAINS

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Abstract. We refine estimates introduced by Balogh and Bonk, to show that the boundary extensions of isometries between smooth strongly pseudoconvex domains in $\mathbb{C}^n$ are conformal with respect to the sub-Riemannian metric induced by the Levi form. As a corollary we obtain an alternative proof of a result of Fefferman on smooth extensions of biholomorphic mappings between pseudoconvex domains. The proofs are inspired by Mostow's proof of his rigidity theorem and are based on the asymptotic hyperbolic character of the Kobayashi or Bergman metrics and on the Bonk-Schramm hyperbolic fillings.

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1. Introduction

Let $D \subset \mathbb{C}^n (n \geq 2)$ be a strongly pseudo-convex domain with C^∞-smooth boundary. Denote by d_K the distance function corresponding to a Finsler structure K satisfying suitable

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estimates, see (2.8). For example, one may consider the Bergman metric or the Kobayashi metric or the Carathéodory metric. In [BB00, BB99], Balogh and Bonk have proved that the metric space (D, d_K) is hyperbolic in the sense of Gromov and its visual boundary coincides with the topological boundary ∂D. They also show that the Carnot-Carathéodory metric d_{CC} corresponding to the Levi form on ∂D, determines the canonical class of snowflake equivalent visual metrics on ∂D. As a consequence, results from the theory of Gromov hyperbolic spaces can be immediately applied in this setting. Among these we recall that every quasi-isometry between such spaces extends to a quasi-conformal map between the visual boundaries, endowed with their families of visual metrics, see for instance [GdlH90, BH99] and references therein.

Our main contribution is to show that extensions of isometries are actually diffeomorphisms that are conformal with respect to the Carnot-Carathéodory metric. We only need to show that the extension is 1-quasi-conformal, as the smoothness then follows from the recent results in [CCLDO16].

As in [BB00], our strategy involves the Bonk-Schramm hyperbolic filling metric g defined in (1.2). This metric provides a stepping stone to connect the Carnot-Carathéodory distance, defined on the boundary by the Levi form (see Section 2.2), with the invariant metric defined in the domain.

Theorem 1.1. *Let $D_1, D_2 \subset \mathbb{C}^n$ be strongly pseudoconvex C^∞-smooth domains and denote by d_K the distance function corresponding to a Finsler structure K satisfying* (2.8)*, and by d_{CC} the Carnot-Carathéodory distance on the boundaries induced by the Levi form. If $f : (D_1, d_K) \to (D_2, d_K)$ is an isometry then the induced boundary map $F : (\partial D_1, d_{CC}) \to (\partial D_2, d_{CC})$ is a diffeomorphism, conformal with respect to the metric d_{CC}.*

We emphasize that the result holds when d_K is the Bergman, the Kobayashi, or the Carathéodory metrics. Indeed, these distances satisfy (2.8) in view of the work in [BB00, BB99, Ma91].

As we noted above, the proof of Theorem 1.1 is based on the study of the relation between the visual distances associated to d_K and the visual distance of an ad-hoc hyperbolic filing metric, built through the Carnot-Carathéodory distance: For $x \in D$ denote by $h(x) := \sqrt{d_E(x, \partial D)}$ and by $\pi(x) \in \partial D$ a closest point in ∂D with respect to the Euclidean distance $d_E(\cdot, \cdot)$, noting it is uniquely defined in a neighborhood of ∂D. Set

$$g(x, y) := 2 \log \left(\frac{d_{\mathrm{CC}}(\pi(x), \pi(y)) + \max(h(x), h(y))}{\sqrt{h(x)h(y)}} \right). \tag{1.2}$$

This is an hyperbolic filling metric built from the metric space $(\partial D, d_{\mathrm{CC}})$ (see Bonk and Schramm [BS00]). Balogh and Bonk [BB00, Corollary 1.3], showed that g is a metric in a neighborhood of ∂D and that g and the invariant distance function d_K are $(1, C)$-quasi-isometric. As a consequence, they give rise to quasi-conformally equivalent visual metrics.

The main technical point of our work is to refine this result in a quantitative fashion. We show that a particular visual quasi-distance ρ_o^K associated to the invariant metric d_K is in fact *pointwise and asymptotically* $(1+\epsilon)$-quasi-conformally equivalent to the Carnot-Carathéodory d_{CC} metric. By *pointwise and asymptotically* we mean that for every point $x \in \partial D$ in the boundary, and for every $\epsilon > 0$, one can choose a base point o for the definition of the visual distances so that the identity map has distortion less than $1+\epsilon$ at x. Following ideas in CAT(-1) spaces, given a pointed metric space (X,d,o) we consider the *Bourdon function*

$$\rho_o^d(x,y) = \exp(-\langle x,y\rangle_o), \tag{1.3}$$

where $\langle x,y\rangle_o$ denotes the Gromov product in (X,d), see Section 2. Usually, ρ_o^d is called Bourdon distance since for CAT(-1) spaces it satisfies the triangle inequality. In our setting, ρ_o^d may not be a distance.

Moreover, on a CAT(-1) space X Bourdon showed in [Bou95] that the visual boundaries $(\partial_\infty X, \rho_o^d)$ corresponding to diffent base points $o, o' \in X$ are conformally equivalent, thus implying immediately that any isometry of X extends to a conformal maps of its visual boundaries. Since pseudoconvex domains may not have negative curvature (see [Kra13]) and may not be simply connected, they are not CAT(-1) spaces and so one cannot apply Bourdon's result.

Theorem 1.1 is achieved in two steps: First one shows that the Carnot-Carathéodory distance is conformally equivalent[1] to the Bourdon function ρ_o^g associated to the hyperbolic filling metric g.

Proposition 1.4. *For any $o \in D$, the functions d_{CC} and ρ_o^g are conformally equivalent.*

In other words, the identity map $(\partial D, d_{CC}) \to (\partial D, \rho_o^g)$ has distortion that is identically equal to one. See (2.1) for the definition of distortion.

Next, we show that at every boundary point, and for any $\epsilon > 0$, one can find a base point $o \in D$ such that the corresponding visual functions ρ_o^K and ρ_o^g are $(1+\epsilon)$-biLipschitz equivalent in a neighborhood of that point. In the following we denote Euclidean balls in $\mathbb{C}^n$ with the notation $B(x,r)$.

Proposition 1.5. *For any $\bar{p} \in \partial D$ and $\bar{\epsilon} > 0$ there exists $r > 0$ such that for all $\omega \in \partial D \cap B(\bar{p}, r) \setminus \{\bar{p}\}$ there exists $r' > 0$ such that for all $o \in D \cap B(\omega, r')$ the two functions ρ_o^g and ρ_o^K are $(1+\bar{\epsilon})$-biLipschitz on $\partial D \cap B(\bar{p}, r')$.*

The proof of Proposition 1.5 and Theorem 1.1 are in Section 5. Theorem 1.1 follows rather directly from Propositions 1.4 and 1.5 and from the following diagram

$$\begin{tikzcd}
& (D_1, d_{\mathrm{K}}) \arrow[r, "f", "\mathrm{iso}"'] \arrow[d, squiggly] & (D_2, d_{\mathrm{K}}) \arrow[d, squiggly] & \\
(\partial D_1, \rho_o^{\mathrm{g}}) \arrow[r, "\mathrm{id}", "(1+\epsilon)\mathrm{BL}"'] & (\partial D_1, \rho_o^K) \arrow[r, "F", "\mathrm{iso}"'] & (\partial D_2, \rho_{f(o)}^K) \arrow[r, "\mathrm{id}", "(1+\epsilon)\mathrm{BL}"'] & (\partial D_2, \rho_{f(o)}^g) \arrow[d, "\mathrm{id}", "\mathrm{conf}"'] \\
(\partial D_1, d_{\mathrm{CC}}) \arrow[u, "\mathrm{id}", "\mathrm{conf}"'] \arrow[rrr, dashed, "F"] & & & (\partial D_2, d_{CC})
\end{tikzcd} \tag{D}$$

[1] The result holds for any hyperbolic filling as in the work of Bonk and Schramm. See Section 3.2

At the center of this chain of compositions there is an isometry, the rest of the links are either $(1+\epsilon)$ biLipschitz maps or conformal maps, so that the total distortion is at most ϵ away from being equal to 1 everywhere.

From the conformal equivalence theorem above and the results in [CCLDO16], one can immediately infer a result about boundary extensions for biholomorphisms between strictly pseudoconvex domains in $\mathbb{C}^n$, originally established by Fefferman [Fef74].

Corollary 1.6. *Let $D_1, D_2 \in \mathbb{C}^n (n \geq 2)$ be strongly pseudo-convex domains with C^∞-smooth boundaries. If $f : D_1 \to D_2$ is a biholomorphism then it extends to a smooth map $F : \partial D_1 \to \partial D_2$ that is conformal with respect to the corresponding subRiemannian contact structure. In particular, at every boundary point, its differential is a similarity between the maximally complex tangent planes.*

Since the publication of [Fef74] there have been several significative extensions and simplifications of the result. A small sample of this extensive line of inquiry can be found in the references [BL80, BC82, NWY80, Bar83, Kra15].

Rather than a simplification of Fefferman's original proof, our approach is a recasting of the result from the perspective of analysis in metric spaces and the circle of ideas at the core of Mostow rigidity [Mos73]. The differentiable structure is not used to show that the extension map is 1-quasi-conformal, and then it only enters in play coupled with the rigidity of 1-quasi-conformal mappings in higher dimension. Likewise, curvature enters into the arguments only in its synthetic (metric) form. In particular, our work can be seen as an instance of a dictionary, introduced by Bonk, Heinonen, and Koskela in [BHK01], translating back and forth problems in domains in Euclidean spaces by means of ad hoc hyperbolic or quasi-hyperbolic metrics, that endow such domains with an hyperbolic structure in the sense of Gromov. For more results along this line, see also the recent, interesting work of Zimmer in [Zim16].

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2. Preliminaries

In this section we recall some basic definitions and results. We start by discussing distortion and conformal maps on subRiemannian manifolds. Then we discuss pseudoconvex domains and their metrics. Finally we review hyperbolicity in the sense of Gromov.

2.1. **Distorsion in subRiemannian geometry.** By a previous work of the authors together with Ottazzi, we know that several definitions of conformal maps are equivalent in the setting of contact subRiemannian manifolds. We now recall the two definitions that we shall need in this paper.

For a homeomorphism $F : X \to Y$ between general metric spaces, we consider the following quantities

$$\mathrm{L}_F(x) := \limsup_{x' \to x} \frac{d(F(x), F(x'))}{d(x, x')} \qquad \text{and} \qquad \ell_F(x) := \liminf_{x' \to x} \frac{d(F(x), F(x'))}{d(x, x')}.$$

The quantity $\mathrm{L}_F(x)$ is sometimes denoted by $\mathrm{Lip}_F(x)$ and is called the pointwise Lipschitz constant. Within this paper, we define the *distortion* of f at a point $x \in X$ as

$$H^*(x, F, d_X, d_Y) := \frac{\mathrm{L}_F(x)}{\ell_F(x)}. \tag{2.1}$$

The homeomorphism f is said to be *quasi-conformal* if there exists K such that for all $x \in X$ one has

$$\limsup_{r \to 0} \frac{\sup\{d_Y(F(p), F(q)) : d_X(p, q) \le r\}}{\inf\{d_Y(F(p), F(q)) : d_X(p, q) \ge r\}} \le K.$$

It is well-known that in the literature there are several other equivalent definitions of quasi-conformality in 'geometrically nice' spaces, see [Wil12]. However, the equivalence is not quantitative, in the sense that each definition has an associated constant (like the K above) and the value of of these constants can be different from definition to definition. Thus we need to clarify what is a conformal map. To do this we invoke Theorem 1.3 and Theorem 1.19 from [CCLDO16]. Namely, the additional subRiemannian structure allows to an unambiguous definition of 1-quasiconformality.

Lemma 2.2 (C-L-O)**.** *Let $F : X \to Y$ be a quasi-conformal homeomorphism between two equiregular subRiemannian manifolds.*

(i) The requirement $H^*(\cdot, F, d_X, d_Y) \equiv 1$ is equivalent to other notions of 1-quasi-conformality.

(ii) If X and Y are contact manifolds, then 1-quasi-conformality of F is equivalent to F being conformal (i.e., smooth and with horizontal differential that is a homothety).

One of the advantages to work with (2.1) is that it immediately yields a chain rule:

$$H^*(x, F_1 \circ F_2) \le H^*(x, F_2) H^*(F_2(x), F_1). \tag{2.3}$$

The last equation follows from the fact that $\limsup a_n b_n \le \limsup a_n \limsup b_n$ whenever $a_n, b_n \ge 0$. Moreover, we trivially have that if f is an L-biLipschitz homeomorphism, then

$$H^*(x, F) \le L. \tag{2.4}$$

2.2. **Pseudoconvex domains and hermitian metrics.** We recall some of the basic definitions about pseudoconvex domains and hermitian metrics, as well as some key results proved by Balogh and Bonk in [BB00].

Let $D \subset \mathbb{C}^n$, $n \ge 2$ be a smooth, bounded open set. Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ denote the signed distance function from ∂D, negative in D and positive in its complement. Set $N_\delta = \{x \in D \mid d_E(x, \partial D) < \delta\}$.

Lemma 2.5 (Tubular Neighborhood Theorem)**.** *Let $D \subset \mathbb{C}^n$, $n \ge 2$ be a bounded domain with smooth boundary. There exists $\delta_0 > 0$ such that the projection $\pi : N_{\delta_o} \to \partial D$ is a smooth, well defined map and the distance function $d_E(\cdot, \partial D)$ is smooth on N_{δ_0}.*

We will denote by $n(x)$ the outer unit normal at $x \in \partial D$, so that the fiber $\pi^{-1}(x) \cap N_{\delta_0} = \{x + sn(x) | s \in (0, \delta_0)\}$.

For $p \in \partial D$, one can define the tangent space $T_p\partial D = \{Z \in \mathbb{C}^n | Re\langle \bar{\partial}\varphi(p), Z\rangle = 0\}$ and its maximal complex subspace $H_p\partial D = \{Z \in \mathbb{C}^n | \langle \bar{\partial}\varphi(p), Z\rangle = 0\}$, where $\langle Z, Z'\rangle = \sum_{i=1}^n Z_i \bar{Z}_i'$ is the hermitian product. By definition, the domain D is *strictly pseudoconvex* if for every $p \in \partial D$, the Levi form

$$L_\varphi(p, Z) := \sum_{\alpha,\beta=1}^n \partial^2_{z_\alpha \bar{z}_\beta}\varphi(p) Z_\alpha \bar{Z}_\beta \tag{2.6}$$

is positive definite on $H_p\partial D$.

For each $p \in \partial D$ one has a splitting $\mathbb{C}^n = H_p\partial D \oplus N_p\partial D$, where $N_p\partial D$ is the complex one-dimensional subspace orthogonal to $H_p\partial D$. This splitting at p induces a decomposition $Z = Z_H + Z_N$ for all $Z \in \mathbb{C}^n$.

Metrics that are invariant under the action of biholomorphisms play a key role in several complex variables. Important examples are the Bergman metric, the Kobayashi metric, and the Carathéodory metric (see [Kra13]). In all cases, for $x \in D$ the length of a complex vector $Z \in T_xD = \mathbb{C}^n$ is given by a Finsler structure $K(x, Z)$. We will rely on the following result, which can be found in [BB99] and also [BB00, Proposition 1.2].

Proposition 2.7 (Balogh-Bonk). *Let $D \subset \mathbb{C}^n$, $n \geq 2$ be a bounded, strictly pseudoconvex domain with smooth boundary and let $K(x, Z)$ be the Finsler structure associated to the Bergman metric or the Kobayashi metric or the Carathéodory metric. For every $\bar{\epsilon} > 0$ there exists $\delta_0, C > 0$ such that for all $x \in D$ with $d_E(x, \partial D) \leq \delta_0$ and $Z \in \mathbb{C}^n$ one has*

$$\begin{aligned}(1 - C\sqrt{d_E(x, \partial D)})\left(\frac{|Z_N|^2}{4d_E^2(x, \partial D)} + (1 - \bar{\epsilon})\frac{L_\varphi(\pi(x), Z_H)}{d_E(x, \partial D)}\right)^{\frac{1}{2}} &\leq K(x, Z) \\ &\leq (1 + C\sqrt{d_E(x, \partial D)})\left(\frac{|Z_N|^2}{4d_E^2(x, \partial D)} + (1 + \bar{\epsilon})\frac{L_\varphi(\pi(x), Z_H)}{d_E(x, \partial D)}\right)^{\frac{1}{2}},\end{aligned} \tag{2.8}$$

where $Z = Z_H + Z_N$ is the splitting at $\pi(x)$.

The subbundle $H\partial D$ is a contact distribution on ∂D and the triplet $(\partial D, H\partial D, L_\varphi)$ yields a contact subRiemannian manifold. In this structure, the *horizontal* curves are those arcs in ∂D that are tangent to the contact distribution, and the *Carnot-Carathéodory* distance $d_{CC}(p, q)$ between $p, q \in \partial D$ is defined as the minimum time it takes to reach one point from the other traveling along horizontal curves at unit speed with respect to the Levi form, see [Gro96].

As in [BB00], we will need to use a family of Riemannian metrics on ∂D that approximate the sub-Riemannian metric associated to the Levi form, and that in fact have corresponding distance functions that converge in the sense of Gromov-Hausdorff to the Carnot-Carathéodory distance. For every $k > 0$ we define a Riemannian metric g_k on $T\partial D$ as

$$g_k^2(p, Z) := L_\varphi(p, Z_H) + k^2|Z_N|^2, \tag{2.9}$$

for every $p \in \partial D$ and every $Z = Z_H + Z_N \in T_p\partial D$. Here we just recall a basic comparison result (see for instance [BB00, Lemma 3.2]) relating the distance function d_k associated to g_k to the Carnot-Carathéodory distance d_{CC}.

Lemma 2.10. *There exists a constant $C > 0$ such that for all $k > 0$, and for all points $p, q \in \partial D$, with $d_{CC}(p, q) \geq k^{-1}$ one has*

$$C^{-1} d_k(p, q) \leq d_{CC}(p, q) \leq C d_k(p, q). \tag{2.11}$$

2.3. **Gromov Hyperbolicity.** Let x, y, o be three points in a metric space (X, d). Then the Gromov product of x and y at o, denoted $\langle x, y\rangle_o$, is defined by

$$\langle x, y\rangle_o = \frac{1}{2}\Big(d(x, o) + d(y, o) - d(x, y)\Big).$$

Then X is called *Gromov hyperbolic* if there exists $\delta \geq 0$ such that

$$\langle x, y\rangle_o \geq \min\{\langle x, z\rangle_o, \langle z, y\rangle_o\} - \delta, \qquad \text{for all } x, y, z, o \in X.$$

For a Gromov hyperbolic space X one can define a boundary set $\partial_\infty X$ as follows, see [BH99, p.431-2]. Fix a basepoint $o \in X$. A sequence (x_i) in X is said to *converge at infinity* if $\lim_{i,j\to\infty}\langle x_i, x_j\rangle_o = \infty$. Two sequences (x_i) and (y_i) converging at infinity are called equivalent if $\lim\langle x_i, y_i\rangle_o = \infty$. These notions do not depend on the choice of the basepoint o. The set $\partial_\infty X$ is now defined as the set of equivalence classes of sequences converging at infinity.

For $p, q \in \partial_\infty X$ and $o \in X$ we define

$$\langle p, q\rangle_o = \sup \liminf_{i\to\infty}\langle x_i, y_i\rangle_o,$$

where the supremum is taken over all sequences (x_i) and (y_i) representing the boundary points p and q, respectively. Actually, there exists such sequences (x_i) and (y_i) for which $\langle p, q\rangle_o = \lim_{i\to\infty}\langle x_i, y_i\rangle_o$, see [BH99, Remark 3.17].

Balogh and Bonk have proved that if $D \subset \mathbb{C}^n$, $n \geq 2$ is a bounded, strictly pseudoconvex domain with smooth boundary, and $K(x, Z)$ is a norm satisfying (2.8), then the corresponding metric space (D, d_K) is Gromov hyperbolic and its visual boundary coincides with the topological boundary. See [BB00, Theorem 1.4].

3. Conformal equivalence of boundary metrics

3.1. **Proof of Proposition 1.4.** In this section we prove Proposition 1.4, and then show that the conformal equivalence result holds more in general for every hyperbolic filling.

Let g be as defined in (1.2) and let ρ_o^g be its Bourdon distance, as defined by (1.3). We begin by giving a computation of the distance ρ_o^g on two points $p, q \in \partial D$. We represent p and q by two sequences x_i and $y_i \in D$, respectively. Notice that since $x_i \to p$ in $\mathbb{C}^n$ then $\pi(x_i) \to p$ and $h(x_i) \to 0$. In particular, we also have that $\max(h(x_i), h(o)) = h(o)$. Similar considerations apply to y_i and q. We compute

$$\begin{aligned}
\rho_o^g(p,q) &= \exp(-\langle p,q\rangle_o) \\
&= \lim_{i\to\infty} \exp(-\langle x_i,y_i\rangle_o) \\
&= \lim_{i\to\infty} \exp(-\frac{1}{2}(g(x_i,o)+g(y_i,o)-g(x_i,y_i))) \\
&= \lim_{i\to\infty} \exp\Bigg(-\log(\frac{d_{\mathrm{CC}}(\pi(x_i),\pi(o))+\max(h(x_i),h(o))}{\sqrt{h(x_i)h(o)}}) \\
&\qquad\qquad -\log(\frac{d_{\mathrm{CC}}(\pi(y_i),\pi(o))+\max(h(y_i),h(o))}{\sqrt{h(y_i)h(o)}}) \\
&\qquad\qquad\qquad +\log(\frac{d_{\mathrm{CC}}(\pi(x_i),\pi(y_i))+\max(h(x_i),h(y_i))}{\sqrt{h(x_i)h(y_i)}})\Bigg) \\
&= \lim_{i\to\infty} \left(\frac{d_{\mathrm{CC}}(p,\pi(o))+h(o)}{\sqrt{h(x_i)h(o)}}\right)^{-1} \left(\frac{d_{\mathrm{CC}}(q,\pi(o))+h(o)}{\sqrt{h(y_i)h(o)}}\right)^{-1} \frac{d_{\mathrm{CC}}(p,q)}{\sqrt{h(x_i)h(y_i)}} \\
&= \frac{d_{\mathrm{CC}}(p,q)\ h(o)}{(d_{\mathrm{CC}}(p,\pi(o))+h(o))(d_{\mathrm{CC}}(q,\pi(o))+h(o))}.
\end{aligned}$$

For every $p\in\partial D$ one has

$$\lim_{q\to p}\frac{\rho_o^g(p,q)}{d_{CC}(p,q)} = \lim_{q\to p}\frac{h(o)}{(d_{\mathrm{CC}}(p,\pi(o))+h(o))(d_{\mathrm{CC}}(q,\pi(o))+h(o))} = \frac{h(o)}{(d_{\mathrm{CC}}(p,\pi(o))+h(o))^2}$$

so the limit exists, and the identity map $(\partial D, d_{CC}) \to (\partial D, \rho_o^g)$ is 1-quasi-conformal. □

3.2. **Boundary distances of hyperbolic fillings.** An important contribution of Bonk and Schramm [BS00], is that the functor $X\to\partial_\infty X$ has an inverse functor, in the form of *hyperbolic filling* spaces $Con(Z)$. To be more precise, one defines $Con(Z) = Z\times(0,D)$, endowed with the metric given by

$$d_2((x,u),(y,v)) = 2\log\left(\frac{d_1(x,y)+\max(u,v)}{\sqrt{uv}}\right). \tag{3.1}$$

The space $(Con(Z), d_2)$ is Gromov hyperbolic, and its visual boundary is Z, with the canonical class of snowflake equivalent metrics given by d_1. Here we note that a particular visual metric is actually conformal to d_1. We will consider the particular visual metric generated by g given by the Bourdon distance. Choose a generic base point choose a base point $o=(z,s)$, with $z\in Z$ and $s\in(0,D)$. For any two points $x,y\in Z$ so that $d_1(x,y)<s$. consider $u,v\in(0,d_1(x,y))$. Following (1.3), the Bourdon distance $d_2(x,y)$ is defined as follows

$$d_2(x,y) = \lim_{u,v\to 0} e^{-\langle (x,u),(y,v)\rangle_o}.$$

Notice that in general, Bourdon distances associated to the hyperbolic fillings are a quasi-distance. By quasi-distance we intend that the triangle inequality is satisfied modulo a multiplicative constant.

Proposition 3.2. *Let d_1 a distance on a bounded space Z. If d_2 denotes the Bourdon distance associated to the hyperbolic filling for d_1, then d_1 and d_2 are conformally equivalent.*

Proof. In order to show that d_1, d_2 are conformally equivalent it suffices to prove that the limit $\lim_{y\to x} d_1(x,y)/d_2(x,y)$ exists for every $x \in Z$. Fix any $z \in Z$ and $s \in (0, D)$. Let $o = (z, s)$. Take two points $x, y \in Z$ so that $d_1(x,y) < s$. Take $u, v \in (0, d_1(x,y))$.

The rest of the proof follows from

$$\langle (x,u),(y,v)\rangle_o = \frac{1}{2}(d_2((x,u),o) + d_2((y,v),o) - d_2((x,u),(y,v)))$$

$$= \log\left(\frac{d_1(x,z)+\max(u,s)}{\sqrt{us}}\right) + \log\left(\frac{d_1(y,z)+\max(v,s)}{\sqrt{vs}}\right) - \log\left(\frac{d_1(x,y)+\max(u,v)}{\sqrt{uv}}\right)$$

$$= \log\frac{(d_1(x,z)+s)(d_1(y,z)+s)}{s(d_1(x,y)+\max(u,v))}$$

$$= \log\left(\frac{(d_1(x,z)+s)(d_1(y,z)+s)}{s(d_1(x,y)+\max(u,v))}\frac{d_1(x,y)}{d_1(x,y)}\right)$$

$$= -\log\Big(d_1(x,y)\Big) + \log\left(\frac{d_1(x,y)(d_1(x,z)+s)(d_1(y,z)+s)}{s(d_1(x,y)+\max(u,v))}\right)$$

We calculate $\lim_{y\to x} d_1(x,y)/d_2(x,y)$. Consider the quotient

$$\begin{aligned}
d_2(x,y)/d_1(x,y) &= \lim_{u,v\to 0}\frac{e^{-\langle (x,u),(y,v)\rangle_o}}{d_1(x,y)} \\
&= \lim_{u,v\to 0}\frac{e^{\log\left(d_1(x,y)\right)}e^{-\log\left(\frac{d_1(x,y)(d_1(x,z)+s)(d_1(y,z)+s)}{s(d_1(x,y)+\max(u,v))}\right)}}{d_1(x,y)} \\
&= \lim_{u,v\to 0}\left(\frac{d_1(x,y)(d_1(x,z)+s)(d_1(y,z)+s)}{s(d_1(x,y)+\max(u,v))}\right)^{-1} \\
&= \frac{sd_1(x,y)}{d_1(x,y)(d_1(x,z)+s)(d_1(y,z)+s)} \\
&= \frac{s}{(d_1(x,z)+s)(d_1(y,z)+s)}.
\end{aligned}$$

The latter implies that

$$\lim_{y\to x}\frac{d_2(x,y)}{d_1(x,y)} = \frac{s}{(d_1(x,z)+s)^2},$$

which gives the conclusion. ∎

4. Comparing d and g, after Balogh and Bonk

The quantitative bounds on the distortion of the identity map in Proposition 1.5 follow from the following result, which is a refinement of an analogue statement of Balogh and Bonk [BB00, Corollary 1.3]. We follow largely their arguments, but where in [BB00] the noise due to the rough geometry would yield an additive constant, here instead we need to exploit the fact that the geometry is asymptotically hyperbolic to show that such constants can be chosen arbitrarily small the closer one gets to the boundary.

Theorem 4.1. *For every $\bar{p} \in \partial D$ and $\epsilon > 0$ there exists $r > 0$ such that for all distinct $p, q \in \partial D \cap B(\bar{p}, r)$ there exists $r' > 0$ such that for all $x \in D \cap B(p, r')$ and all $y \in D \cap B(q, r')$*

$$g(x, y) - \epsilon \leq d_K(x, y) \leq g(x, y) + \epsilon. \tag{4.2}$$

In the rest of the paper we will refer to this result in connection with the quintuplet $(\bar{p}, p, q, x, y)$.

4.1. Lemmata.

The proof of Theorem 4.1 is based on preliminary estimates established in Lemma 4.3, Lemma 4.8, and Lemma 4.12 below.

Lemma 4.3. *Let $\delta_0 > 0$ to be the constant in Lemma 2.5. For $x_1, x_2 \in D$ with $d_E(x_i, \partial D) < \delta_0$, and $h(x_1) \geq h(x_2)$, consider a piecewise C^1 curve $\gamma : [0, 1] \to N_{\delta_0}$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The length $l_K(\gamma)$ of γ with respect to the metric d_K satisfies*

$$l_K(\gamma) \geq \ln \frac{h(x_1)}{h(x_2)} - C\Big(h(x_1) - h(x_2)\Big), \tag{4.4}$$

where C is the same constant as in (2.8). Moreover, if the curve is a segment $\gamma(t) = x_1 + t(x_2 - p_1) \subset \pi^{-1}(p)$ for some $p \in \partial D$ then one has

$$l_K(\gamma) \leq \ln \frac{h(x_1)}{h(x_2)} + C\Big(h(x_1) - h(x_2)\Big), \tag{4.5}$$

Proof. Recall from [BB00, page 517] that

$$\left|\frac{d}{dt} h(\gamma(t))\right| = \frac{|Re\langle n(\pi(\gamma(t))), \gamma'(t)\rangle|}{2h(\gamma(t))} \leq \frac{|\gamma'_N(t)|}{2h(\gamma(t))}.$$

The latter and (2.8) yield

$$\begin{aligned}
l_K(\gamma) &= \int_0^1 K(\gamma(t), \gamma'(t)) dt \\
&\geq \int_0^1 (1 - C d_E^{\frac{1}{2}}(\gamma(t), \partial D)) \left(\frac{|\gamma'_N(t)|^2}{4 d_E^2(\gamma(t), \partial D)} + (1 - \bar{\epsilon}) \frac{L_\varphi(\pi(\gamma(t), \gamma'_H(t))}{d_E(\gamma(t), \partial D)} \right)^{\frac{1}{2}} dt \\
&\geq \int_0^1 (1 - C d_E^{\frac{1}{2}}(\gamma(t), \partial D)) \frac{|\gamma'_N(t)|}{2 d_E(\gamma(t), \partial D)} dt \geq \int_0^1 \frac{(1 - C h(\gamma(t))}{h(\gamma(t))} \frac{d}{dt} h(\gamma(t)) dt \\
&= \ln \frac{h(x_1)}{h(x_2)} - C(h(x_1) - h(x_2)),
\end{aligned}$$

which gives (4.4).

On the other hand, if $\gamma(t) = x_1 + t(x_2 - x_1)$, then we observe that γ' is parallel to the unit normal at $\pi(x_i)$ and so has no tangent component, hence no horizontal component with respect to the splitting at $\pi(x_i)$. Using the fact that

$$d_E(\gamma(t), \partial D) = |\gamma(t) - p| = |t(x_2 - x_1)| + |x_1 - p|,$$

and (2.8) one has

$$\begin{aligned} l_K(\gamma) &= \int_0^1 K(\gamma(t), \gamma'(t))dt \\ &\leq \int_0^1 (1 + Cd_E^{\frac{1}{2}}(\gamma(t), \partial D))\left(\frac{|\gamma_N'(t)|^2}{4d_E^2(\gamma(t), \partial D)} + (1+\bar{\epsilon})\frac{L_\varphi(\pi(\gamma(t), \gamma_H'(t))}{d_E(\gamma(t), \partial D)}\right)^{\frac{1}{2}} dt \\ &= \frac{1}{2}\int_0^1 \frac{|x_2 - x_1|}{t|x_2 - x_1| + |x_1 - p|}dt + \frac{C}{2}\int_0^1 \frac{1}{\sqrt{t|x_2 - x_1| + |x_1 - p|}}dt \\ &= \ln \frac{h(x_1)}{h(x_2)} + C(h(x_1) - h(x_2)), \end{aligned}$$

which gives (4.5). □

An immediate consequence of Lemma 4.3 is the following.

Corollary 4.6. *Let $\delta_0 > 0$ to be the constant in Proposition 2.5. If $x_1, x_2 \in D$, with $\delta_0 > h(x_1) \geq h(x_2)$, then*

$$\ln \frac{h(x_1)}{h(x_2)} - C\big(h(x_1) - h(x_2)\big) \leq d_K(x_1, x_2).$$

Moreover, if $\pi(x_1) = \pi(p_2)$, then we also have

$$d_K(x_1, x_2) \leq \ln \frac{h(x_1)}{h(x_2)} + C\big(h(x_1) - h(x_2)\big) \tag{4.7}$$

where C is the same constant as in (2.8).

The next lemma provides an upper bound for $d_K(x_1, x_2)$ in the case when both points x_1, x_2 are at the same distance from the boundary and equal to the Carnot-Carathéodory distance between the projections $\pi(x_1), \pi(x_2)$.

Lemma 4.8. *Let $p_1, p_2 \in \partial D$. If we set $x_i := p_i - d_{CC}(p_1, p_2)n(p_i)$, $i = 1, 2$, then*

$$d_K(x_1, x_2) \to 0, \qquad \text{as } d_{CC}(p_1, p_2) \to 0. \tag{4.9}$$

Proof. Let $\eta > 0$ and let $\alpha : [0, 1] \to \partial D$ be any horizontal curve with $\alpha(0) = p_1$ and $\alpha(1) = p_2$, such that its subRiemannian length l_{CC}, satisfies

$$l_{CC}(\alpha) = \int_0^1 L_\varphi^{\frac{1}{2}}(\alpha(t), \alpha'(t))dt \leq d_{CC}(p_1, p_2)(1 + \eta).$$

Define a new curve $\gamma : [0, 1] \to D$ as a *lift* at height $h \in (0, \delta_0)$ of α by the formula

$$\gamma(t) := \alpha(t) - h\ n(\alpha(t)). \tag{4.10}$$

Arguing as in the proof of [BB00, Lemma 2.2] yields the following relations between α' and γ',

$$
\begin{aligned}
L(\alpha(t),\gamma'_H(t)) &= L(\alpha(t),\alpha'(t)) + O(h|\alpha'(t)|^2) \\
|\gamma'_N(t)| &= O(h|\alpha'(t)|).
\end{aligned} \tag{4.11}
$$

In fact, from (4.10) one has $\gamma'(t)|_H = \alpha'(t) - [dn|_{\alpha(t)}\alpha'(t)]_H$ which, together with the bilinearity of the Levi form, yields (4.11). Consequently we have

$$
\begin{aligned}
l_K(\gamma) &= \int_0^1 K(\gamma(t),\gamma'(t))dt \\
&\le \int_0^1 (1+Cd_E^{\frac{1}{2}}(\gamma(t),\partial D))\left(\frac{|\gamma'_N(t)|^2}{4d_E^2(\gamma(t),\partial D)} + (1+\bar\epsilon)\frac{L_\varphi(\pi(\gamma(t),\gamma'_H(t))}{d_E(\gamma(t),\partial D)}\right)^{\frac{1}{2}} dt \\
&= \int_0^1 (1+Ch^{\frac{1}{2}})\left(\frac{|\gamma'_N(t)|^2}{4h^2} + (1+\bar\epsilon)\frac{L_\varphi(\pi(\gamma(t),\gamma'_H(t))}{h}\right)^{\frac{1}{2}} dt \\
&\le \int_0^1 (1+Ch^{\frac{1}{2}})\left(C|\alpha'(t)|^2 + (1+\bar\epsilon)\frac{L_\varphi(\alpha(t),\alpha'(t))}{h}\right)^{\frac{1}{2}} dt \\
&\le (1+C\sqrt{h})(1+\eta)\left[Cd_{CC}(p_1,p_2) + (1+\bar\epsilon)^{\frac{1}{2}}h^{-\frac{1}{2}}d_{CC}(p_1,p_2)\right].
\end{aligned}
$$

Setting $h = d_{CC}(x_1,x_2)$ in the latter yields the conclusion. □

The next lemma will be instrumental in establishing a lower bound for $d_K(x_1,x_2)$ in the case when a length minimizing arc γ joining two points $x_1,x_2 \in D$ will travel at a distance further than the Carnot-Carathéodory distance between their projections.

Lemma 4.12. *Let $\delta_0 > 0$ be smaller than the similarly named constants in Propositions 2.5 and 2.7. Consider two points $x_1,x_2 \in D$ with $d_E(x_i,\partial D) < \delta_0$. Set $p_i = \pi(x_i) \in \partial D$, and let $\gamma : [0,1] \to D$ denote an arc joining x_1 and x_2. If $\max_{z\in\gamma} h(z) \ge d_{CC}(p_1,p_2)$ then*

$$
l_K(\gamma) \ge 2\ln\left(\frac{d_{CC}(p_1,p_2)}{\sqrt{h(x_1)h(x_2)}}\right) - C(2d_{CC}(p_1,p_2) - h(x_1) - h(x_2)), \tag{4.13}
$$

where C is the same constant as in (2.8).

Proof. Choose $t_0 \in [0,1]$ such that $h(\gamma(t_0)) = \max_{z\in\gamma} h(z)$. Set γ_1,γ_2 be the two branches of the curve γ corresponding to the subintervals $[0,t_0]$ and $[t_0,1]$. Set also $\bar\gamma_1$ and $\bar\gamma_2$ to be the connected components on γ_1 and γ_2 joining x_i to the closest points $y_i \in \gamma$ such that $h(y_i) = d_{CC}(p_1,p_2)$, for $i = 1,2$. More formally, $y_1 = \gamma(t_1)$, with $t_1 = \inf\{t \in [0,t_1]$ such that $h(\gamma(t)) \ge d_{CC}(p_1,p_2)\}$. The point y_2 is defined analogously.

Next we invoke Lemma 4.3 to deduce

$$\begin{aligned} l_K(\gamma) &\geq l_K(\bar\gamma_1) + l_K(\bar\gamma_2) \\ &\geq \ln\frac{d_{CC}(p_1,p_2)}{h(x_1)} + \ln\frac{d_{CC}(p_1,p_2)}{h(x_2)} - C(2d_{CC}(p_1,p_2) - h(x_1) - h(x_2)) \\ &= 2\ln\left(\frac{d_{CC}(p_1,p_2)}{\sqrt{h(x_1)h(x_2)}}\right) - C(2d_{CC}(x_1,x_2) - h(x_1) - h(x_2)), \end{aligned}$$

which is the desired bound (4.13). □

4.2. **Proof of Theorem 4.1.** Thanks to the previous lemmata we can now prove the main result of the section.

Proof of Theorem 4.1. We shall show that for all $\bar p \in \partial D$ and $\epsilon > 0$ one can choose $r > 0$ small enough so that for all distinct $p, q \in \partial D \cap B(\bar p, r)$ one can find $r' \in (0, r)$ such that (4.2) holds for all $x \in D \cap B(p, r')$ and all $y \in D \cap B(q, r')$. In our proof we begin with arbitrary values of r and r' and then put several constrains on them.

If p and q are distinct, then the value $d_1 := d_{CC}(p, q)$ is strictly positive. We shall choose r smaller that the constants δ_0 in Propositions 2.5 and 2.7 and so that d_1 is small enough to be determined later. Denote by $\bar x$, and $\bar y$ the projections on the boundary of x and y, respectively. Note that since the projections are the closest points in ∂D, then $\bar x \in B(p, 2r')$ and $\bar y \in B(q, 2r')$. Set $d_2 := d_{CC}(\bar x, \bar y)$. Notice that as $r' \to 0$ we have $d_2 \to d_1$. We shall choose r' sufficiently small so that $r' < d_2$ and $d_2 \in (d_1/2, 2d_1)$. In particular, if r was chosen small enough, then d_2 is positive and smaller than the constants δ_0 in Propositions 2.5 and 2.7.

Proof of the upper bound in (4.2). Set $x' := \bar x - d_2 n(\bar x)$ and $y' := \bar y - d_2 n(\bar y)$, so x', y' are points in D at distance d_2 from ∂D and with the same projection on ∂D as x, y, respectively.

By Lemma 4.8 we can choose d_1 sufficiently small so that $d_K(x', y') < \epsilon/3$. Invoking (4.7), since $h(x') = h(y') = d_2 > \max\{h(x), h(y)\}$, yields

$$d_K(x, x') \leq \log(d_2/h(x)) + C(d_2 - h(x)) \qquad \text{and} \qquad d_K(y, y') \leq \log(d_2/h(y)) + C(d_2 - h(y)).$$

Choose d_1 chosen sufficiently small so that $Cd_2 \leq \epsilon/3$.

Combining the previous bounds with the definition of g, we obtain the following estimates

$$\begin{aligned} d_K(x,y) - g(x,y) &\leq d_K(x,x') + d_K(x',y') + d_K(y',y) - g(x,y) \\ &\leq \log(d_2/h(x)) + C(d_2 - h(x)) + \epsilon/3 + \log(d_2/h(y)) + C(d_2 - h(y)) \\ &\qquad -2\ln\left(\frac{d_2 + h(x)\wedge h(y))}{\sqrt{h(x)h(y)}}\right) \\ &\leq \epsilon - Ch(x) - Ch(y) - 2\ln\left(1 + \frac{h(x)\wedge h(y)}{d_2}\right) < \epsilon, \end{aligned}$$

where we used that the terms $h(x), h(y), \ln(1 + \frac{h(x)\wedge h(y)}{d_2})$ are positive. This conclude the proof of the upper bound in (4.2).

Proof of the lower bound in (4.2). Choose $\delta > 0$ such that $\ln(1/(1+\delta)) < \epsilon$ and $r' > 0$ small enough so that $\frac{\max(h(x),h(y))}{d_{CC}(p,q)} \leq \delta$, for all $x \in D \cap B(p,r')$ and all $y \in D \cap B(q,r')$. Consider any arc $\gamma : [0,1] \to D$ joining x and y, and set $H := \max_{z\in\gamma} h(z)$.

- *If* $H \geq d_{CC}(\bar{x},\bar{y})$ then in view of Lemma 4.12 we have

$$\begin{aligned} d_K(x,y) - g(x,y) &\geq 2\ln\Big(\frac{d_2}{\sqrt{h(x)h(y)}}\Big) - C(2d_2 - h(x) - h(y))) - 2\ln\Big(\frac{d_2 + h(x)\wedge h(y))}{\sqrt{h(x)h(y)}}\Big) \\ &\geq 2\ln\left(\frac{1}{1+\frac{\max(h(x),h(y))}{d_2}}\right) - C(2d_2 - h(x) - h(y)) \qquad (4.14) \\ &\geq -(C+2)\epsilon. \end{aligned}$$

In this case the proof is concluded.

- *If* $H \leq d_{CC}(\bar{x},\bar{y})$ then it follows that H is smaller than the constants δ_0 in Propositions 2.5 and 2.7. In particular we can assume without loss of generality that $CH < 1/2$, where C is as in (2.8). Let $t_0 \in [0,1]$ be such that $h(\gamma(t_0)) = H$ and consider the two branches γ_1, γ_2 of γ given by restrictions to $[0,t_0]$ and $[t_0,1]$. Given $\epsilon > 0$ as in the statement, let $\theta \in (1,2]$ so that $\ln\theta < \epsilon$ and define $k \in \mathbb{N}$ such that

$$h(\gamma(0)) \in \left[\frac{H}{\theta^k}, \frac{H}{\theta^{k-1}}\right].$$

Following [BB00], we define $s_0, s_1, ..., s_k \in [0,t_0]$ such that $s_0 = 0$ and

$$s_l = \min\left\{ s \in [0,t_0] \text{ such that } h(\gamma(s)) = \frac{H}{\theta^{k-l}} \right\}.$$

Set $t_1 = s_k \leq t_0$ and for each $l = 1, ..., k$,

$$\nu_l^{-1} = \frac{d_{CC}(\bar{x},\bar{y})\cdot(\theta-1)}{8\theta^{k-l}}.$$

For each of the two branches γ_1, γ_2, we distinguish two alternatives:

- **Alternative #1 (All sub-arcs have large slope)** In this alternative we assume that for every $l = 1, ..., k$ one has

$$d_{CC}(\pi(\gamma(s_{l-1})), \pi(\gamma(s_l))) \leq \nu_l^{-1} \qquad (4.15)$$

From the latter we draw two conclusions. The first is a simple application of the triangle inequality,

$$\begin{aligned} \text{(A1 (i))} \quad & d_{CC}(\bar{z}, \pi(\gamma(t_1)) \\ & \leq \sum_{l=1}^{k} d_{CC}(\pi(\gamma(s_{l-1})), \pi(\gamma(s_l))) \leq (\theta-1)\frac{d_{CC}(\bar{x},\bar{y})}{8\theta^k}\sum_{l=1}^{k}\theta^l \leq \frac{d_{CC}(\bar{x},\bar{y})}{4}. \end{aligned}$$

On the other hand, in view of Lemma 4.3 one has

$$\text{(A1 (ii))} \qquad l_K(\gamma|_{[0,t_1]}) \geq \ln\frac{h(\gamma(t_1))}{h(x)} - C(h(\gamma(t_1)) - h(x)) = \ln\frac{H}{h(x)} - C(H - h(x)).$$

- **Alternative #2 (One sub-arc has small slope)** In this alternative, we assume that there exists $l \in \{1, ..., k\}$ such that

$$d_{CC}(\pi(\gamma(s_{l-1})), \pi(\gamma(s_l))) > \nu_l^{-1} \tag{4.16}$$

Note that if $s \in [s_{l-1}, s_l]$ then from the definition of the points s_l, one has

$$h(\gamma(s)) \leq \theta^{l-k} H \leq \frac{8}{\theta - 1} \nu_l^{-1}.$$

We then claim that there exists a constant $\mathcal{C} > 0$ depending only on the defining function φ such that

$$l_K(\gamma|_{[s_{l-1}, s_l]}) \mathcal{C}(\theta - 1)^2 \frac{d_{CC}(\bar{x}, \bar{y})}{H}. \tag{4.17}$$

Indeed, arguing as in [BB00, page 521] we invoke (2.8) and Lemma 2.10 and we bound as follows:

$$\begin{aligned}
&l_K(\gamma|_{[s_{l-1}, s_l]}) \\
&\geq \mathcal{C} \frac{(1 - CH)\theta^{k-l}}{H} \int_{s_{l-1}}^{s_l} \left[L_\varphi(\pi(\gamma(s)), [\pi(\gamma(s))]'_H) + (\theta - 1)^2 \nu_l^2 |[\pi(\gamma(s))]'_N|^2 \right]^{\frac{1}{2}} ds \\
&\geq \frac{\mathcal{C}}{2}(\theta - 1) \frac{\theta^{k-l}}{H} \int_{s_{l-1}}^{s_l} \left[L_\varphi(\pi(\gamma(s)), [\pi(\gamma(s))]'_H) + \nu_l^2 |[\pi(\gamma(s))]'_N|^2 \right]^{\frac{1}{2}} ds \\
&\geq \mathcal{C}(\theta - 1) \frac{\theta^{k-l}}{H} d_{\nu_l}(\pi(\gamma(s_{l-1})), \pi(\gamma(s_l))) \\
&\geq \mathcal{C}(\theta - 1) \frac{\theta^{k-l}}{H} d_{CC}(\pi(\gamma(s_{l-1})), \pi(\gamma(s_l))) \\
&\geq \mathcal{C}(\theta - 1)^2 \frac{d_{CC}(\bar{x}, \bar{y})}{H},
\end{aligned}$$

where d_{ν_l} denotes the approximation of the Carnot-Caratheodory metric defined in (2.9).

Next we claim that

$$l_L(\gamma|_{[0,t_1]}) \geq \ln\left(\frac{H}{h(y)}\right) + \frac{\mathcal{C}(\theta - 1)^2}{H} d_{CC}(\bar{x}, \bar{y}) - C\big(H - h(y)\big) - \epsilon. \tag{A2}$$

Indeed, Lemma 4.3 and (4.17) yields

$$\begin{aligned}
l_L(\gamma|_{[0,t_1]}) &= l_K(\gamma|_{[0,s_{l-1}]}) + l_K(\gamma|_{[s_{l-1},s_l]}) + l_K(\gamma|_{[s_l,t_1]}) \\
&\geq \ln\left(\frac{H}{h(\gamma(s_l))} \frac{h(\gamma(s_{l-1}))}{h(x)}\right) + \frac{\mathcal{C}(\theta - 1)^2}{H} d_{CC}(\bar{x}, \bar{y}) \\
&\qquad - C\big(H - h(\gamma(s_l) + h(\gamma(s_{l-1})) - h(x)\big) \\
&\geq \ln\left(\frac{H}{h(x)} \theta^{-1}\right) + \frac{\mathcal{C}(\theta - 1)^2}{H} d_{CC}(\bar{x}, \bar{y}) - C\big(H - h(x)\big) \\
&\geq \ln\left(\frac{H}{h(x)}\right) + \frac{\mathcal{C}(\theta - 1)^2}{H} d_{CC}(\bar{x}, \bar{y}) - C\big(H - h(x)\big) - \epsilon.
\end{aligned}$$

Applying similar consideration to the branch γ_2 one obtains a $t_2 \in [t_0, 1]$ such that one of the following two alternatives hold: Either

$$\text{(B1)} \qquad d_{CC}(\bar{y}, \pi(\gamma(t_2)) \leq \frac{d_{CC}(\bar{x}, \bar{y})}{4} \quad \text{and} \quad l_K(\gamma|_{[t_2,1]}) \geq \ln \frac{H}{h(y)} - C(H - h(y)).$$

or

$$\text{(B2)} \qquad l_L(\gamma|_{[t_2,1]}) \geq \ln \left(\frac{H}{h(y)} \right) + \frac{\mathcal{C}(\theta - 1)^2}{H} d_{CC}(\bar{x}, \bar{y}) - C\big(H - h(y)\big) - \epsilon.$$

To conclude the proof we need to examine all possible combinations of these alternatives. We will show that in each case one obtains

$$l_K(\gamma) \geq 2 \ln \left(\frac{d_{CC}(\bar{x}, \bar{y})}{\sqrt{h(x)h(y)}} \right) - C(2d_{CC}(\bar{x}, \bar{y}) - h(x) - h(y)) - \epsilon. \tag{4.18}$$

- **Suppose both (A1) and (B1) hold.** Observe that
$$\begin{aligned} d_{CC}(\pi(\gamma(t_1)), \pi(\gamma(t_2))) \quad &\geq \quad d_{CC}(\bar{x}, \bar{y}) - d_{CC}(\bar{x}, \pi(\gamma(t_1))) - d_{CC}(\bar{y}, \pi(\gamma(t_2))) \\ &\geq \quad \frac{d_{CC}(\bar{x}, \bar{y})}{2}. \end{aligned}$$
Repeating the argument in (4.17) for $l = k$ and invoking the Riemannian approximation lemma [BB00, Lemma 2.2] one has
$$l_L(\gamma|_{[t_1,t_2]}) \geq \mathcal{C}(\theta - 1) \frac{d_{\nu_k}(\pi(\gamma(t_1)), \pi(\gamma(t_2)))}{H} \geq \mathcal{C}(\theta - 1) \frac{d_{CC}(\bar{x}, \bar{y})}{H}.$$
The latter, together with (A1 (ii)), and the second inequality in (B1) yields
$$l_K(\gamma) \geq 2 \ln \left(\frac{H}{\sqrt{h(x)h(y)}} \right) + \mathcal{C}(\theta - 1) \frac{d_{CC}(\bar{x}, \bar{y})}{H} - C(2H - h(x) - h(y)).$$
Since the right hand side is monotone decreasing in $H \leq d_{CC}(\bar{x}, \bar{y})$ then one has
$$\begin{aligned} l_K(\gamma) \quad &\geq \quad 2 \ln \left(\frac{d_{CC}(\bar{x}, \bar{y})}{\sqrt{h(x)h(y)}} \right) + \mathcal{C}(\theta - 1) - C(2d_{CC}(\bar{x}, \bar{y}) - h(x) - h(y)) \\ &\geq \quad 2 \ln \left(\frac{d_{CC}(\bar{x}, \bar{y})}{\sqrt{h(x)h(y)}} \right) - C(2d_{CC}(\bar{x}, \bar{y}) - h(x) - h(y)) \end{aligned}$$
completing the proof of (4.18).
- **Suppose both (A1) and (B2) hold.** One immediately has
$$\begin{aligned} l_K(\gamma) &\geq l_K(\gamma_{[0,t_1]}) + l_K(\gamma|_{[t_2,1]}) \\ &\geq \ln \left(\frac{H}{h(x)} \right) + \frac{\mathcal{C}(\theta - 1)}{H} d_{CC}(\bar{x}, \bar{y}) - C[H - h(x)] - \epsilon + \ln \frac{H}{h(y)} - C(H - h(y)). \end{aligned}$$
Applying the same consideration as above we immediately deduce (4.18).
- **Suppose both (A2) and (B1) hold.** This combination is dealt with analogously to the previous case.

- **Suppose both (A2) and (B2) hold.** Estimate (4.18) follows immediately from (A2) and (B2).

To conclude the proof we need to consider the infimum of $l_K(\gamma)$ among all arcs γ joining x and y and apply (4.18) to each. One has

$$\begin{aligned} d_K(x,y) - g(x,y) &\geq 2\ln\left(\frac{d_{CC}(\bar{x},\bar{y})}{\sqrt{h(x)h(y)}}\frac{1}{\frac{d_{CC}(\bar{x},\bar{y})}{\sqrt{h(x)h(y)}}+\frac{\max\{h(x),h(y)\}}{\sqrt{h(x)h(y)}}}\right) \\ &\qquad -C(2d_{CC}(\bar{x},\bar{y})-h(x)-h(y))-\epsilon \\ &= -2\ln\left(1+\frac{\max\{h(x),h(y)\}}{d_{CC}(\bar{x},\bar{y})}\right) \\ &\qquad -C(2d_{CC}(\bar{x},\bar{y})-h(x)-h(y))-\epsilon. \end{aligned}$$

The proof is then concluded by applying the same argument as in (4.14). □

5. Local biLipschitz equivalence of Bourdon functions and proof of main result

In this section we prove Proposition 1.5 and the main result, Theorem 1.1.

Proof of Proposition 1.5. Let $\bar{p}$ as in the statement and choose $\epsilon > 0$ such that $\exp(\frac{3}{2}\epsilon) \leq 1+\bar{\epsilon}$. Invoke Theorem 4.1 in correspondence to the choice of $\bar{p}$ and ϵ, to obtain the value $r > 0$ and select any $\omega \in \partial D \cap B(\bar{p}, r) \setminus \{\bar{p}\}$. In correspondence to this choice of ω, Theorem 4.1 yields a smaller radius $0 < r' < r$, so that if we choose $y \in D \cap B(\bar{p}, r')$ and $o \in D \cap B(\omega, r')$ and then apply Theorem 4.1 to the quintuplet $(\bar{p}, \bar{p}, \omega, y, o)$ we obtain

$$|g(y,o) - d_K(y,o)| < \epsilon, \qquad \text{for all } y \in D\cap B(\bar{p}, r'), \text{ and } o \in D \cap B(\omega, r')$$

Next, given $p, q \in \partial D \cap B(\bar{p}, r')$ we similarly use Theorem 4.1 to infer the existence of a $r'' > 0$ for which, applying Theorem 4.1 to the quintuplet $(\bar{p}, p, qx, y)$

$$|d_K(x,y) - g(x,y)| \leq \epsilon, \qquad \text{for all } x \in D \cap B(p, r''), \text{ and for all } y \in D\cap B(q, r'').$$

If x_i (resp., y_i) is a sequence in D converging to p (resp., q), then for i large enough $x_i \in D \cap B(p, r'')$ and $y_i \in D \cap B(q, r'')$ and $x_i, y_i \in B(\bar{p}, r')$. From the above bounds one obtains

$$\begin{aligned} \left|\langle y_i, x_i\rangle_o^g - \langle y_i, x_i\rangle_o^K\right| &= \frac{1}{2}\left|g(y_i,o) - d_K(y_i,o) + g(x_i,o) - d_K(x_i,o) + d_K(x_i,y_i) - g(x_i,y_i)\right| \\ &\leq \frac{3}{2}\epsilon. \end{aligned}$$

Consequently, if the sequences x_i, y_i are taken so that $\langle p,q\rangle_o^g = \lim_{i\to\infty}\langle y_i, x_i\rangle_o^g$, we have

$$\begin{aligned} \frac{\rho_o^K(p,q)}{\rho_o^g(p,q)} &\leq \frac{\lim_{i\to\infty}\exp(-\langle y_i,x_i\rangle_o^K)}{\lim_{i\to\infty}\exp(-\langle y_i,x_i\rangle_o^g)} \\ &= \lim_{i\to\infty}\exp\left(\langle y_i,x_i\rangle_o^g - \langle y_i,x_i\rangle_o^K\right) \\ &\leq \exp(\frac{3}{2}\epsilon) \leq 1+\bar{\epsilon}. \end{aligned}$$

And similarly, $\rho^g_o(p,q)/\rho^K_o(p,q)$ is bounded by $1+\bar\epsilon$. $\square$

Proof of Theorem 1.1. For any $\bar p \in \partial D_1$ and $\bar\epsilon > 0$ we show that the boundary extension is $(1+\bar\epsilon)-$quasi-conformal at $\bar p$, i.e. $H^*(\bar p, F, d_{CC}, d_{CC}) \le 1+\bar\epsilon$, where H^* is defined as in (2.1). Following the diagram (D) in the introduction, from (2.3) for every $o \in D_1$ we have

$$\begin{aligned}(5.1)\quad & H^*(\bar p, F, d_{CC}, d_{CC}) \\ & \le H^*(\bar p, \mathrm{Id}_{\partial D_1}, d_{\mathrm{CC}}, \rho^g_o) H^*(\bar p, \mathrm{Id}_{\partial D_1}, \rho^g_o, \rho^K_o) H^*(\bar p, F, \rho^K_o, \rho^K_{f(o)}) \\ & \quad \cdot H^*(F(\bar p), \mathrm{Id}_{\partial D_2}, \rho^K_{f(o)}, \rho^g_{f(o)}) H^*(F(\bar p), \mathrm{Id}_{\partial D_2}, \rho^g_{f(o)}, d_{\mathrm{CC}}).\end{aligned}$$

Start by observing that for any $o \in D_1$ the pointed metric spaces (D_1, d_K, o) and $(D_2, d_K, f(o))$ are isometric. Thus they give rise to visual boundaries that are isometric with respect to the induced distances ρ^K_o an $\rho^K_{f(o)}$, as defined in (1.3). Consequently the induced extension map $F : (\partial D_1, \rho^K_o) \to (\partial D_2, \rho^K_{f(o)})$ is an isometry, and hence from (2.4)

$$(5.2)\qquad H^*(\bar p, F, \rho^K_o, \rho^K_{f(o)}) = 1.$$

Regarding the first and last term in the right-hand side of (5.1), in view of Proposition 1.4 we have that

$$(5.3)\qquad H^*(\bar p, \mathrm{Id}_{\partial D_1}, d_{\mathrm{CC}}, \rho^g_o) = H^*(F(\bar p), \mathrm{Id}_{\partial D_2}, \rho^g_{f(o)}, d_{\mathrm{CC}}) = 1.$$

We shall then prove that

$$(5.4)\qquad H^*(\bar p, \mathrm{Id}_{\partial D_1}, \rho^g_o, \rho^K_o) \le 1+\bar\epsilon \text{ and } H^*(F(\bar p), \mathrm{Id}_{\partial D_2}, \rho^K_{f(o)}, \rho^g_{f(o)}) \le 1+\bar\epsilon,$$

for some suitable choice of o. To prove this we will need to invoke Proposition 1.5 twice, in D_1 and in D_2, together with the observation (2.4). Namely, we shall prove that for a suitable choice of o The maps considered in (5.4) are $(1+\bar\epsilon)$-biLipschitz in a neighborhood of the considered points.

First we apply Proposition 1.5 in a neighborhood of $F(\bar p) \in \partial D_2$, thus yielding $r_2 > 0$ such that for all $\omega_2 \in \partial D_2 \cap B(F(\bar p), r_2) \setminus \{F(\bar p)\}$ there exists $r_2' > 0$ such that for all $o_2 \in D_2 \cap B(\omega_2, r_2')$ one has that $\rho^g_{o_2}$ and $\rho^K_{o_2}$ are $(1+\bar\epsilon)$-biLipschitz in $\partial D_2 \cap B(F(\bar p), r_2')$. For the moment we do not choose any specific ω_2 and o_2, so r_2' is still to be determined.

Next, we apply Proposition 1.5 to D_1 in a neighborhood of $\bar p$ and use it to choose $r_1 > 0$ such that for all $\omega_1 \in \partial D_1 \cap B(\bar p, r_1) \setminus \{\bar p\}$ there exists $r_1' > 0$ such that $o_1 \in D_1 \cap B(\omega_1, r_1')$ one has that $\rho^g_{o_1}$ and $\rho^K_{o_1}$ are $(1+\bar\epsilon)$-biLipschitz in $\partial D_1 \cap B(\bar p, r_1')$. By continuity of the map F we may have chosen r_1 small enough that $F(B(\bar p, r_1) \cap D_1) \subset B(F(\bar p), r_2) \cap D_2$.

We set $\omega_2 := F(\omega_1)$, which is then in $B(F(\bar p), r_2) \cap D_2$ and is different than $F(\bar p)$ since F is a homeomorphism. Now we fix r_2' accordingly, as we explained above. If needed we will select a smaller value for r_1' so that we can assume $F(B(\omega_1, r_1') \cap D_1) \subset B(F(\omega_1), r_2') \cap D_2$.

To conclude, we can now select any base point $o \in B(\omega_1, r_1') \cap D_1$, so that $f(o) \in B(\omega_2, r_2') \cap D_2$ and and hence $\rho^g_{o_1}$ and $\rho^K_{o_1}$ are $(1+\bar\epsilon)$-biLipschitz in $\partial D_1 \cap B(\bar p, r_1')$ and $\rho^g_{o_2}$ and $\rho^K_{o_2}$ are $(1+\bar\epsilon)$-biLipschitz in $\partial D_2 \cap B(F(\bar p), r_2')$. Thus, (2.4) gives (5.4).

Using the estimates (5.2), (5.3), and (5.4) in (5.1) we get $H^*(\bar{p}, F, d_{CC}, d_{CC}) \leq 1 + \bar{\epsilon}$. By the arbitrariness of $\bar{\epsilon}$ we deduce $H^*(\bar{p}, F, d_{CC}, d_{CC}) = 1$. Finally, from Lemma 2.2 we conclude. $\square$

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