# Stochastic Controls in Competitions and Mean Field Games

by

Jiaxuan Ye

A Dissertation

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfillment of the requirements for the

Degree of Doctor of Philosophy

in

Mathematical Sciences

by

---

April 2022

APPROVED:

Dr. Gu Wang, Advisor
Department of Mathematical Sciences
Worcester Polytechnic Institute

Dr. Stephan Sturm, Co-advisor
Department of Mathematical Sciences
Worcester Polytechnic Institute

Dr. Roger Lui
Department of Mathematical Sciences
Worcester Polytechnic Institute

Dr. Mihai Sîrbu
Department of Mathematics
The University of Texas at Austin

Dr. Qingshuo Song
Department of Mathematical Sciences
Worcester Polytechnic Institute

## Acknowledgements

First and foremost, I would like to express my gratitude to my advisors, Professor Gu Wang and Professor Stephan Sturm, for their support in my Ph.D. program. My work would not have been possible without their guidance. Meanwhile, I would also like to thank Professor Qingshuo Song and Ph.D. candidates Jiamin Jian and Peiyao Lai for their work on my third project. Besides, I would also like to extend my sincere appreciation to my dissertation committee members, Dr. Roger Lui and Dr. Mihai Sîrbu, for their great support and invaluable comments.

I would also like to thank all of the faculty members, staff from Mathematical Sciences Department of Worcester Polytechnic Institute for all their support, especially Dr. Luca Capogna, Dr. Darko Volkov, Dr. Zhongqiang Zhang, Dr. Marcel Blais, and Rhonda Podell for their supports during my six-year Ph.D. program.

I would like to thank my friends for their kindness and encouragement. In particular, I express my sincere gratitude to all those who have shared their ideas with me, including Xiaohui Chen, Yuchen Dong, Riuji Sato, Patchara Santawisook, and Elisa Negrini. I am also grateful to all my friends at WPI.

Last but not least, I would like to appreciate my parents, my father Shengzu Ye and my mother Yinqing Zhu, for their love, support, and effort to pursue my dream in the academy.

# Abstract

Problems combining games and controls for multiple players become widely studied due to the complexity of the world and the interactions among populations. In this thesis, we propose two models for fund managers who compete with their relative performance and one Mean Field Game model with common noise for cost minimization.

For the first model, we consider a group of managers competing for the cash flows based on their relative performance by choosing between an idiosyncratic and a common risky investment opportunity. Since investors may choose to invest or withdraw continuously conditional on the real-time performance of funds, the model is of continuous competition. The unique constant equilibrium is derived in closed form, which implies that funds generally decrease the investments in their idiosyncratic risky assets under competition, in order to lower the risk of the relative performance. It pushes all funds to herd and hurts their after-fee performance. However, sufficiently disadvantaged funds with poor idiosyncratic investment opportunities or highly risk-averse managers may take the excessive risk for a better chance of attracting new investments, and their performance may improve compared to the case without competition, which benefits the investors.

For the second model, we propose a principle agent model where the principle is a policy maker who decides the optimal capital gain tax rate and agents are fund managers who choose optimal portfolios in their investment opportunities. The optimal tax rate and unique portfolios are derived for one policy maker and one representative fund. Moreover, with one policy maker but N funds competing with each other based on the terminal relative performance, there exist multiple Nash equilibria and a unique Pareto optimal equilibrium can be found. Our findings also suggest that managers may take more risks with the higher tax rate, which is different from the existing literature.

For the third model, we study Mean Field Games with a common noise given by a continuous time Markov chain or an independent Brownian motion under a quadratic cost structure. The theory implies that the optimal path under the equilibrium is a Gaussian process conditional on the common noise. Interestingly, it reveals the Markovian structure of the random equilibrium measure flow, which can be characterized via a deterministic finite dimensional system. Moreover, the counterpart N-player game can be embedded in the probability space generated by two Brownian motions, which concludes the convergence of the N-player game to Mean Field Games, both in the sense of the processes and the empirical measure.

# Contents

# List of Notations

Following notations are used in the dissertation.

Common: for all $i = 1, \cdots, N$,

| | |
|---|---|
| $W_i, B, W, \tilde{W}$ | Brownian motion |
| $\mathcal{F}$ | Filtration |
| $\mathbb{P}, \mathbb{P}^{(N)}$ | Probability space |
| $\Omega, \Omega^{(N)}$ | Probability space |
| $\mathcal{A}, \mathcal{A}_i$ | Admissible set for controls |

Chapter 2 and 3: for all $i = 1, \cdots, N$,

| | |
|---|---|
| $r$ | risk free rate |
| $\mu_i, \mu, a$ | Risk premium for the asset |
| $\sigma_i, \sigma, b$ | Volatility for the asset |
| $\lambda_i, \lambda, \lambda_m$ | Sharpe ratio of the asset |
| $S_i, S_m, S$ | Asset price process |
| $X_i$ | Fund's value process |
| $\pi_i, \theta_i, \pi$ | Proportional portfolio of each asset |
| $\pi_i^M, \theta_i^M$ | Proportional portfolio of each asset under Merton's model (without competition) |
| $\psi_i, \psi$ | Management fee rate |
| $\ell_i, \alpha_i$ | Sensitivity of fund flows to absolute and relative performance |
| $\gamma_i, \gamma$ | Manager's relative risk aversion |
| $\sigma_i^*, \sigma_i^M$ | Volatility of fund's investment with and without competition |
| $\eta_i^*, \eta_i^M$ | Fund's after-fee Sharpe ratio with and without competition |
| $Beta_i^*, Beta_i^M$ | Beta coefficient of fund's after-fee return and the market average with and without competition |
| $Beta_{mi}^*, Beta_{mi}^M$ | Beta coefficient of fund's after-fee return and the common investment opportunity with and without competition |
| $I, F$ | Investor and manager's account value process |
| $A_T, A_{iT}$ | Terminal after-tax wealth |
| $N_T$ | Cumulative tax |
| $Z, Z_i$ | Fund's after-fee return rate process |
| $\tau_c, \tau_o$ | Capital gain tax rate and income tax rate |
| $\xi$ | Stochastic discount factor |

Chapter 4: for all $i = 1, \cdots, N$,

| | |
|---|---|
| $Y$ | Continuous Markov chain |
| $Q$ | Generator of the continuous time Markov chain |
| $X, X^{(N)}$ | Controlled process |
| $\alpha, \alpha_i$ | Control |
| $m$ | Measure flow |
| $[m]_k$ | $k$th moment of the measure |

# List of Figures

# Chapter 1

# Introduction

## 1.1 Motivations and Literature Review

The growing complexity of the world makes interactions among populations become increasingly complicated. Thus, the study to see the pattern with interplay becomes more essential and meaningful. Especially in the control problems with players choosing their optimal strategies, it is worth a close observation of how interactions change people's optimal controls and whether there is any spillover effect on society.

Since each player's optimal strategy depends on all the other players' actions, we need to introduce the idea of Nash equilibrium to define the optimality for all players. The Nash equilibrium was first introduced by John Forbes Nash Jr. in his famous paper "Non-cooperative games" in 1951. Since then, the idea is broadly used in various fields that include interplay among players. However, the pure strategy may not be unique, for example, in the matching pennies game shown in Table 1.1. In this dissertation, we add some restrictions in order to have the unique strategies for all players.

| | Heads | Tails |
|---|---|---|
| Heads | +1,+1 | -1,-1 |
| Tails | -1,-1 | 0,0 |

Table 1.1: Payoff matrix for matching pennies game. Pure strategy is not unique.

One application for stochastic controls among multiple players is in the financial industry. As shown in Table 1.2, the number of participants in the financial industry is increasing dramatically, even if we only look at the data from mutual funds alone. With more managers involved in financial companies, investors have more options from funds, which facilitates competition in the industry. Each fund/manager seeks a better performance to attract new investments and thus gains more profits. For mutual funds, this kind of competition between fund flows based on relative performance is well documented in the empirical literature as Gruber (1996), Chevalier and Ellison (1997), Sirri and Tufano (1998), Patel et al. (1991), Ippolito (1992). However, most of the theoretical analyses focus on the competition between two funds, or in discrete-time models Browne (2000), Taylor (2003), Huang et al. (2007), Palomino (2005), Basak et al. (2007), or on incentives for multiple interacting agents without fund flows Anthropelos et al. (2020), Bielagk et al. (2017), Frei and Dos Reis (2011), Lacker and Soret (2020), Han et al. (2022), Dal Forno and Merlone (2010), Siemsen et al. (2007), Lioui and Poncet (2013). Therefore, we build a continuous time model similar to Espinosa and Touzi (2015), Basak and Makarov (2015), Lacker and Zariphopoulou (2019), but instead of the comparison at the terminal time, we consider a competition for fund flows which happens continuously. Thus the relative performance does not enter into the utility function, but the dynamics of the assets under management of each fund.

| | 1945 | | 2004 | |
|---|---|---|---|---|
| Type of Fund | Number of funds | Assets[a] (millions) | Number of funds | Assets (billions) |
| Stock/hybrid funds | 49 | \$794 | 5100 | \$4266.9 |
| Bond funds | 19 | 88.0 | 2100 | 1246.8 |
| Money market funds | 0 | - | 970 | 1962.2 |
| Total | 68 | \$882.0 | 8170 | \$7475.9 |

Table 1.2: The mutual fund industry: growth in funds and assets. Source: Table 1 in Bogle (2005).
[a] Total assets of stock funds in 1945 estimated as 90 percent of industry total.

Competition among funds not only affects the funds themselves, but also the benefits of investors. Our model supplements the literature on the principal-agent relationship between the investors and managers, which usually focuses on the case of only one agent Ou-Yang (2003), Aivaliotis and Palczewski (2014). An interesting result is that, though for most funds the after fee performance, measured in Sharpe ratios, is lower with competition, compared to the case without fund flows, the performance of disadvantaged funds may increase in face of competition, which benefits the investors, because after all, fund flows based on relative performance push the manager to pursue superior returns over other funds.

Another spillover effect of funds' competition is on the policy makers. To decide the optimal capital gain tax rate for policy makers, we purpose a model with one policy maker and either one representative or $N$ funds being the counter-parties. Similar to Basak and Makarov (2015), Lacker and Zariphopoulou (2019), managers face a utility maximization problem at the terminal date, which includes the utility brought by the future cash flows based on the relative performance, while policy makers choose the capital gain tax to maximize their tax incomes under the Markowitz mean-variance setting. Due to the introduction of taxes, managers' utility functions are non-concave, which can be solved by the concavification technique introduced in Bichuch and Sturm (2014), Seifried (2010). Our findings show that the competition, in general, makes managers less aggressive so as to avoid the possible loss from poor relative performance and pushes the policy maker to increase the optimal capital gain tax in order to compensate for the loss of tax incomes.

Economic literature Feldstein (1969), Stiglitz (1975), Yost (2018) shows that an increment in capital gain tax results in a decrease in managers' risk-taking. This is due to both income and substitution effects described in Feldstein and Yitzhaki (1978), Balcer and Judd (1987). The income effect means that managers tend to be more aggressive to cover the loss from the higher tax, while the substitution effect refers to the more conservative strategy caused by less marginal after-tax incomes. Our results show a different pattern that managers tend to be more aggressive when it is closer to the terminal date while more conservative at the beginning of the period.

After the discussion about multi-player interactions, it is a natural question that what will happen if we let the number of players approach infinite. This is exactly the starting point of the Mean Field Game (MFG) theory. MFGs have attracted resurgent attention from numerous researchers in probability after the pioneering works of Lasry and Lions (2007), Huang et al. (2006).

An important recent development in this direction is Mean Field Games with a common noise and we refer to comprehensive descriptions in the book Carmona et al. (2018) and the references therein. Meanwhile, Linear-Quadratic (LQ) control problems have been widely recognized in the stochastic control theory due to their broad applications. The optimal path is Gaussian under the LQ structure and the problem is also called LQG to emphasize this Gaussian property, see for instance Yong and Zhou (1999). More importantly, LQ structure leads to solvability in a closed form, namely the Ricatti system, and this usually sheds light on many fundamental properties of the control theory. Thus, LQG MFGs are widely studied in Huang (2009/10), Nguyen and Huang (2012), Huang et al. (2014), Firoozi et al. (2020), Huang et al. (2012), Feng et al. (2019), Huang and Huang (2013), Bardi and Priuli (2013), Huang et al. (2015), Gao et al. (2020).

We study a class of MFG problems and the counterpart $N$-player game in the context of LQ structure with a common noise either as a continuous time Markov chain or an independent Brownian motion. The path dependence nature of MFGs makes this an infinite dimension problem, but we show that the solution can be described through a finite dimensional system. Some results relevant to our first contribution can be found in the recent papers Ahuja (2015), Tchuendom (2018). However, both papers have their mean field term only via the mean process, but not the second moment as in our paper, which makes the underlying control problem not a typical LQG setting and gives extra difficulties. Moreover, Tchuendom (2018) provides the solution via FBSDE, which is an infinite dimensional system, and Ahuja (2015) provides a finite dimensional system for the solution from the Master equation.

Our second contribution, yet the more important one compared to the first one, is the proof of the convergence of the $N$-player game to MFGs, both in the sense of the processes and the empirical measure. We embed the generic player's path of the $N$-player game to the sample space of the MFG setting, which enables us to compare the difference of paths and measures between the $N$-player game and MFGs in almost sure sense. We show that there exists a hidden algebraic pattern to the coefficients of value functions invariant to the number of players. This pattern eventually enables us to reduce the representation of the generic path from a functional of $N$-dimensional Brownian motion to a functional of two-dimensional Brownian motion, no matter how large the number $N$ is. Indeed, the pattern leading to the success of the above embedding procedure is precisely accounted for the dimension-invariant feature of the mean field terms at the equilibrium, which provides a new insight distinguished from the $\epsilon$-Nash equilibrium discussed in the literature, which does not require the same sample space.

The first model in Chapter 2 is joint work with my advisor Professor Gu Wang and the article is available on SSRN. The second model in Chapter 3 is joint work with my advisor Professor Stephan Sturm and the results are still under revision. The last model in Chapter 4 is a joint project with Professor Qingshuo Song and two other Ph.D. candidates, Jiamin Jian and Peiyao Lai in WPI. The paper is available on arxiv.

## 1.2 Some Preliminaries

In this section, we will introduce definitions relevant to the Nash equilibrium under different situations. For the ease of notation, with positive integer $n$, $v \in \mathbb{R}^n$ and $D \in \mathbb{R}^{n\times n}$, $v_{-i} \in \mathbb{R}^{n-1}$ is the vector after removing $v$'s $i$-th element, and $D_{-i} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the matrix after removing $D$'s $i$-th row and $i$-th column.

### 1.2.1 $N$-player Game

Consider a complete filtered space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ endowed with $N+1$ Brownian motions $W_1, W_2, \cdots, W_N$ and $B$, which generate the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Denote $\mathcal{A}_i$ and $\Theta$ as the admissible set including all the process $\pi_i$ and $\theta$ which are integrable with respect to $W_i$ and $B$ respectively. Let $\pi = \begin{bmatrix}\pi_1 & \ldots & \pi_N\end{bmatrix}^\top$ and $\theta = \begin{bmatrix}\theta_1 & \ldots & \theta_N\end{bmatrix}^\top$. The objective functional for each player is given by $J_i\left(\pi_i, \theta_i; \pi_{-i}, \theta_{-i}\right)$.

**Definition 1.** *Let $\mathcal{A}$ and $\Theta^N$ be the Cartesian product of $\mathcal{A}_i$'s, and the $N$-th Cartesian product of $\Theta$, respectively. $(\pi^*, \theta^*) \in \mathcal{A} \times \Theta^N$ is a Nash equilibrium if for every $i \in \{1, \cdots, N\}$, and any $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$,*

$$J_i(\pi_i, \theta_i; \pi^*_{-i}, \theta^*_{-i}) \leq J_i(\pi^*_i, \theta^*_i; \pi^*_{-i}, \theta^*_{-i}).$$

*Furthermore, $(\pi^*, \theta^*)$ is called a constant equilibrium if $\pi^*_i$ and $\theta^*_i$ are constants for each $i \in \{1, \cdots, N\}$.*

**Definition 2.** *$(\pi^*, \theta^*)$ is called Pareto optimal if there does not exist $(\pi, \theta) \in \mathcal{A} \times \Theta^N$ such that for all $i = 1, \cdots, N$,*

$$J_i(\pi_i, \theta_i; \pi_{-i}, \theta_{-i}) \leq J_i(\pi^*_i, \theta^*_i; \pi^*_{-i}, \theta^*_{-i}),$$

*with strict inequality with at least one $i$.*

**Definition 3.** *If there exist multiple Nash equilibria, $(\pi^*, \theta^*)$ is called a Pareto optimal Nash equilibrium if for it is Pareto optimal among all Nash equilibria.*

Note that in Chapter 2 and 3, we seek for maximization of the objective functional, so the Nash equilibrium achieves maxima of the objective functional. However, in Chapter 4, we search for its minimum, so $\leq$ is changed to $\geq$ in Definition 1. Meanwhile, In the proof of Nash equilibrium in Chapter 2, the following definition of reaction function is also required.

**Definition 4.** *A function $x_i = f(x_{-i})$ is called the reaction function of player $i$ such that for any $\tilde{x} \in \mathbb{R}$,*

$$J_i(\tilde{x}, x_{-i}) \leq J_i(f(x_{-i}), x_{-i}).$$

### 1.2.2 Mean Field Games with Common Noise

The following definitions are for the illustrations of Chapter 4 alone. Consider a complete filtered space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ endowed with 2 Brownian motions $W$ and $B$, and $Y$ representing the common noise, which generate the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Denote $L^p := L^p(\Omega, \mathbb{P})$ as the space of random variables $X$ on $(\Omega, \mathcal{F}_T, \mathbb{P})$ with finite $p$-th moment with norm $\|X\|_p = \left(\mathbb{E}\left[|X|^p\right]\right)^{1/p}$, and $L^p_{\mathbb{F}} := L^p_{\mathbb{F}}([0,T] \times \Omega)$ as the space of all $\mathbb{F}$-progressively measurable random processes $\alpha = (\alpha_t)_{0\leq t\leq T}$ satisfying

$$\mathbb{E}\left[\int_0^T |\alpha_t|^p dt\right] < \infty.$$

For any polish space $(S, \mathcal{B}(S), d)$, denote $\delta_x$ as the Dirac measure on the point $x \in S$. The collection of all probabilities $m$ on $(S, \mathcal{B}(S), d)$ having finite $k$-th moment is denoted by $\mathcal{P}_k(S)$, i.e. for any $m \in \mathcal{P}_k(S)$,

$$[m]_k := \int x^k m(dx) < \infty.$$

The equilibrium of Mean Field Games (MFGs) with the common noise yields the conditional distribution. For real valued random variables $X$ and $Z$ in $(\Omega, \mathcal{F}_T, \mathbb{P})$, the distribution of $X$ conditional on $\sigma(Z)$ is denoted by $\mathcal{L}(X|Z)$, or equivalently, for any $A \in \mathcal{F}_T$,

$$\mathcal{L}(X|Z)(A) = \mathbb{E}[I_A(X)|Z].$$

Note that $\mathcal{L}(X|Z)$ is $\sigma(Z)$-measurable random probability distribution with $k$-th moment $[\mathcal{L}(X|Z)]_k = \mathbb{E}\left[X^k \middle| Z\right]$, if it exists. We refer to more details on the conditional distribution in Volume II of Carmona et al. (2018).

Given the objective functional $J$ and some given random measure flow $m : (0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, a generic player chooses the optimal control and controlled process $\hat{\alpha}$ and $\hat{X}$. Note that to introduce MFG Nash equilibrium, it is often convenient to highlight the dependence of the optimal path and optimal control of the generic player and its associated value on the underlying density flow $m$, which are denoted by

$$\hat{X}_t[m], \hat{\alpha}_t[m], \text{ and } V[m],$$

respectively.

**Definition 5.** *Given an initial distribution $\mathcal{L}(X_0) = m_0 \in \mathcal{P}_2(\mathbb{R})$, a random measure flow $\hat{m}$ is said to be a MFG equilibrium measure if it satisfies fixed point condition, for any $t \in (0,T]$,*

$$\hat{m}_t = \mathcal{L}\left(\hat{X}_t[\hat{m}] \middle| \tilde{\mathcal{F}}_t\right), \tag{1.2.1}$$

*almost surely in $\mathbb{P}$, where $\tilde{\mathcal{F}}$ is the filtration generated by the common noise. The path $\hat{X}$ and the control $\hat{\alpha}$ associated to $\hat{m}$ is called as the MFG equilibrium path and equilibrium control, respectively.*

Next proposition and definition provide embedding approach to prove a convergence in distribution, which will be used later in Chapter 4 to show the convergence of the generic player of $N$-player game to MFGs.

**Proposition 6.** *Suppose $\left(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)}\right)$ is a complete probability space. Let $X^{(N)}$ and $X$ be random variables of $\Omega^{(N)} \mapsto S$ and $\Omega \mapsto S$, respectively. Then, $X^{(N)}$ is convergent in distribution to $X$, denoted by $X^{(N)} \Rightarrow X$, if there exists $Z^N : \Omega \mapsto S$ satisfying $\mathcal{L}(Z^N) = \mathcal{L}(X^{(N)})$, such that $Z^N \to X$ holds almost surely, i.e.*

$$\lim_{N\to\infty} d\left(Z^N, X\right) = 0,$$

*almost surely in $\mathbb{P}$, where $d$ represents the metric in $S$.*

**Definition 7.**

1. *The value function of player $i$ for $i = 1, 2, \ldots, N$ of the Nash game is defined by $V^N = (V_i^N : i = 1, 2, \ldots, N)$ satisfying the equilibrium condition*

$$V_i^N(y, x^N) \quad = J_i^N(y, x^N, \hat{\alpha}_i^{(N)}, \hat{\alpha}_{-i}^{(N)}) \le J_i^N(y, x^N, \alpha_i^{(N)},\ \hat{\alpha}_{-i}^{(N)}), \tag{1.2.2}$$

*for all $\alpha_i^{(N)} \in \mathcal{A}$.*

2. *The generic player's path at equilibrium is $\hat{X}_{ut}^{(N)}$, where $u := u^{(N)}$ is a uniform random variable on the set $\{1, 2, \ldots, N\}$ in $\Omega^{(N)}$ independent of $(W^{(N)}, Y^{(N)})$.*

## 1.3 Outlines

The organization of this dissertation is as follows:

Chapter 2 discusses a model among $N$ mutual fund managers competing for investment flows based on relative performance. We derive the unique constant equilibrium in closed form and give the comparison of after-fee Sharpe ratios and Beta coefficients between the solution to the Merton's model and ours. Section 2.1 gives an introduction to the model settings. Section 2.2 shows the main results for $N$-player game and some numerical results. Section 2.3 includes all the detailed proofs for Chapter 2.

Chapter 3 builds a model between policy makers and fund managers, where policy makers decide the best capital gain tax while the latter choose the optimal portfolio to maximize their terminal after-tax wealth. The unique (Pareto optimal) Nash equilibrium is derived for the managers' problem. Section 3.1 gives an introduction to the model settings of the policy maker's problem and managers' problem with one or $N$ players. Section 3.2 gives the (Pareto optimal) Nash equilibrium and sensitivity analysis for managers' problems. Section 3.3 shows some numerical results for both one and $N$ managers. Section 3.4 includes all the proofs for Chapter 3.

Chapter 4 studies Mean Field Games with a common noise given by either a continuous time Markov chain or an independent Brownian motion under a quadratic cost structure. Section 4.1 and Section 4.2 give the solution to MFGs with Markov chain and Brownian motion as the common noise respectively and proves the convergence of the counterpart $N$-player game. Section 4.3 gives the detailed proofs in Chapter 4 and some explicit solutions to the problem without common noise and extends the model to multidimensional.

Chapter 5 gives the conclusions and future work for the three models described above.

# Chapter 2

# Model I: Mutual Funds' Competitions

## 2.1 Model

### 2.1.1 Mutual Fund Investments and Flows

Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, endowed with $N+1$ Brownian Motions $W_1, W_2, \ldots, W_N$ and $B$, which generate the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Assume $\langle W_i, B\rangle_t = \rho_{im}t$ and $\langle W_i, W_j\rangle_t = \rho_{ij}t$, where $\rho_{im} \in (-1,1)$ and $\rho_{ij} \in (-1,1)$ are constants, for every $1 \leq i, j \leq N$. Denote $\rho$ as the $N\times N$ matrix with $(\rho)_{ij} = \rho_{ij}$ and $\rho_m$ as the $N$-dimensional vector with $(\rho_m)_i = \rho_{im}$.

Suppose that mutual fund $i$ ($i = 1, \ldots, N$), in addition to a risk-free asset $S_0$, which earns a constant rate of return $r$, allocates its assets under management between two risky investment opportunities: (i) $S_m$, which is accessible to all investors in the market, e.g. a market index, following the dynamics

$$\frac{dS_{mt}}{S_{mt}} = (r+a)dt + bdB_t, \tag{2.1.1}$$

with the constants $a, b > 0$, and (ii) $S_i$, which only fund $i$ can invest in, reflecting the fund manager's skill, is described by a geometric Brownian Motion

$$\frac{dS_{it}}{S_{it}} = (r+\mu_i)dt + \sigma_i dW_{it}, \tag{2.1.2}$$

where constants $\mu_i, \sigma_i > 0$. Let $\lambda_i := \frac{\mu_i}{\sigma_i}$ for $1 \leq i \leq N$ and $\lambda_m := \frac{a}{b}$. Denote $\pi_{it}$ and $\theta_{it}$ as the proportions of fund $i$'s assets invested in $S_i$ and $S_m$ at time $t$, which are integrable with respect to $W_i$ and $B$, and denote the collection of all such strategies as $\mathcal{A}_i$ and $\Theta$, respectively. Given $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$, $R_{it}$, the excessive return over the risk-free rate from these investments, follows

$$dR_{it} = \pi_{it}\left(\frac{dS_{it}}{S_{it}} - rdt\right) + \theta_{it}\left(\frac{dS_{mt}}{S_{mt}} - rdt\right) = \pi_{it}(\mu_i dt + \sigma_i dW_{it}) + \theta_{it}(adt + bdB_t). \tag{2.1.3}$$

The investors of the fund compensate the manager by management fees $\psi_i X_{it}$, where $\psi_i > 0$ is a constant, and $X_{it}$ is fund $i$'s value at time $t$.

Furthermore, assume that the $N$ mutual funds belong to the same peer group, e.g. because they have the same investment "style" characterized in Brown and Goetzmann (1997), or they belong to the same family, managed by different managers in the same firm. Therefore, investors can compare each fund's return with the rest of the group and move their investment accordingly. If fund's return is higher than the average, it will attract more investment from clients, and the clients will withdraw if the return is lower. The size of the flow at time $t$ is proportional to $X_t^i$, and the after-fee relative return over the industry average $(dR_{it} - \psi_i dt) - \frac{1}{N}\sum_{j=1}^{N}(dR_{jt} - \psi_j dt)$. Meanwhile, fund $i$ also attracts cash flows proportional to the fund size and its absolute after-fee return. Thus, $X_i$ follows

$$\frac{dX_{it}}{X_{it}} = (r - \psi_i)dt + dR_{it} + \ell_i(dR_{it} - \psi_i dt) + \alpha_i \left( (dR_{it} - \psi_i dt) - \frac{1}{N}\sum_{j=1}^{N}(dR_{jt} - \psi_j dt) \right), \quad (2.1.4)$$

where $\ell_i, \alpha_i > 0$ describe the sensitivity of fund flows to the absolute and relative performance of fund $i$ compared to its peers. We do not assume the sum of cash flows as zeros. Better performance of the funds can attract extra investment from new clients, and vice versa.

Note that the manager of each fund is assumed to have the full information about other funds' investment opportunities and their portfolio choices, which is also assumed in the literature on competition between asset managers Basak and Makarov (2015), Lacker and Zariphopoulou (2019). It agrees with the fact that investment strategies of mutual funds are public information, and can also model the competition in a fund family managed by the same company Kempf and Ruenzi (2008).

### 2.1.2 Preferences

The manager of fund $i$ chooses the investment strategies $(\pi_i, \theta_i)$ and maximizes the discounted expected power utility from management fees over the time interval $[0, T]$. Since there are fund flows based on relative performance, in addition to the fund $i$'s investment strategy $(\pi_i, \theta_i)$, the welfare of the manager also depends on the strategies her competitors are taking. Let $\pi = (\pi_1, \ldots, \pi_N)^\top$ and $\theta = (\theta_1, \ldots, \theta_N)^\top$, and manager $i$'s goal is $\sup_{(\pi_i,\theta_i)\in\mathcal{A}_i\times\Theta} J_i(\pi_i, \theta_i; \pi_{-i}, \theta_{-i})$, where

$$J_i(\pi_i, \theta_i; \pi_{-i}, \theta_{-i}) = \mathbb{E}\left[\int_0^T e^{-\beta_i t}\frac{(\psi_i X_{it})^{1-\gamma_i}}{1-\gamma_i}dt\right], \quad (2.1.5)$$

where $\beta_i$ is the manager $i$'s subjective discount factor, and $\gamma_i > 0$ $(\neq 1)$ is the coefficient of relative risk aversion. Since the utility is homogeneous in the initial value of $X_i$, without loss of generality, assume that $X_{i0} = 1$ for each $1 \le i \le N$.

## 2.2 Main Results

In this section, we want to discuss the Nash equilibrium among the $N$ mutual funds defined in Definition 1. The following theorem shows that there exists a unique constant equilibrium. Notice that though the equilibrium $(\pi^*, \theta^*)$ are constants, for each $1 \le i \le N$, $J_i(\tilde{\pi}_i, \tilde{\theta}_i; \pi^*_{-i}, \theta^*_{-i}) \le J_i(\pi^*_i, \theta^*_i; \pi^*_{-i}, \theta^*_{-i})$ for every $(\tilde{\pi}_i, \tilde{\theta}_i) \in \mathcal{A}_i \times \Theta$, i.e. $(\pi^*_i, \theta^*_i)$ is optimal among all admissible, may be stochastic investment strategies, given the constant equilibrium choices of other competitors.

**Theorem 8.** *There exists a unique constant equilibrium*

$$\pi^* = A_f P_f^{-1}\gamma^{-1}\lambda_f, \quad (2.2.1)$$

$$\theta^* = A_m P_m^{-1}\left(\gamma^{-1}\eta_m + C A_f^{-1}\pi^*\right), \quad (2.2.2)$$

*where $\lambda_f$ and $\eta_m$ are two $N$-dimensional vectors with $(\lambda_f)_i = \lambda_i - \rho_{im}\lambda_m$ and $(\eta_m)_i = \lambda_m - \rho_{im}\lambda_i$ respectively, for $1 \le i \le N$. $A_f, A_m$ and $\gamma$ are diagonal matrices with the diagonal elements $(A_f)_{ii} = \frac{N}{(N(1+\ell_i)+(N-1)\alpha_i)\sigma_i}$, $(A_m)_{ii} = \frac{N}{(N(1+\ell_i)+(N-1)\alpha_i)b}$ and $(\gamma)_{ii} = \gamma_i$, respectively, for $1 \le i \le N$. $P_f, P_m$ and $C$ are $N \times N$ matrices with*

$$(P_f)_{ij} = \begin{cases} 1-\rho_{im}^2 & \text{if } i = j, \\ -c_{ij}(\rho_{ij} - \rho_{im}\rho_{jm}) & \text{if } i \neq j, \end{cases} \quad (P_m)_{ij} = \begin{cases} 1-\rho_{im}^2 & \text{if } i = j, \\ -c_{ij}(1-\rho_{im}^2) & \text{if } i \neq j. \end{cases} \tag{2.2.3}$$

$$(C)_{ij} = \begin{cases} 0 & \text{if } i = j, \\ c_{ij}(\rho_{jm} - \rho_{im}\rho_{ij}) & \text{if } i \neq j, \end{cases} \quad c_{ij} = \frac{\alpha_i}{(1+\ell_j)N + (N-1)\alpha_j}, 1 \le i, j \le N. \tag{2.2.4}$$

Similar to Lacker and Zariphopoulou (2019), Basak and Makarov (2014), we searched for the equilibria in which portfolios of all funds are constants, and find the unique one. Note that it may not be the unique equilibrium if the investment strategies are allowed to be stochastic, but it is a natural choice for fund managers given the homogeneity of the power utilities and the constant investment opportunities.

Since every fund invests in $S_m$, fund $i$ has exposure to the risk in $S_m$ through the fund flows and the investments of other funds. Instead of $\theta_i$, the manager actually has to choose optimal effective investment $\zeta_i := \frac{(1+\ell_i)N+(N-1)\alpha_i}{N}\theta_i - \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\theta_j$ in $S_m$. Based on the equilibrium deduced in Theorem 8, for each fund $i$, the optimal effective investment gives $\zeta_i = \dfrac{\lambda_m - \rho_{im}\lambda_i}{\gamma_i b(1-\rho_{im}^2)} + \sum_{j\neq i} \dfrac{\rho_{jm} - \rho_{im}\rho_{ij}}{(1-\rho_{im}^2)} \dfrac{(1+\ell_i)N + (N+1)\alpha_i}{(1+\ell_j)N + (N-1)\alpha_j} \dfrac{\alpha_i\sigma_i}{N}\pi_j^*$, which consists of the strategy to Merton's problem defined in (2.2.5) and the risk premium of competition on the common investment opportunity described by the second term.

Without fund flows ($\alpha_i = \ell_i = 0$), the expected utility $J_i$ is independent of $\pi_{-i}$ and $\theta_{-i}$, and the manager essentially faces the Merton problem with two correlated risky assets, and the optimal investment strategies for the fund $i$ are also constants (the verification is omitted)

$$\pi_i^M = \frac{\lambda_i - \rho_{im}\lambda_m}{\gamma_i\sigma_i(1-\rho_{im}^2)}, \quad \theta_i^M = \frac{\lambda_m - \rho_{im}\lambda_i}{\gamma_i b(1-\rho_{im}^2)}, \tag{2.2.5}$$

which only depend on the investment opportunities $S_i$ and $S_m$ accessible to fund $i$. With the possibility of in/out flows, since managers maximize welfare from the management fees proportional to the assets under management, they care about the total return of the fund, including the flows. The equilibrium strategies $\pi_i^*$ and $\theta_i^*$ include hedging component that is against the risk exposure to other risky investment opportunities, and depend on their correlations and the rates of management fees of all funds. For example, if $\lambda_i = \rho_{im} = 0$, $\pi_i^M = 0$, because $S_i$ brings zero expected return, and cannot be used to hedge the risk in $S_m$. However, with competition based on relative performance, even if $\lambda_i = \rho_{im} = 0$, as long as $S_i$ is not independent to other $S_j$'s, $\pi_i^*$ is not necessarily zero – $S_i$ is worth the investment, not because of the return it provides, but the hedge it brings against the risks in other funds' investments.

In the following we discuss how the competition over flows based relative performance affects the fund managers' equilibrium investment strategies and the investment returns for fund investors, and how they compare the counterpart without competition, or in the other words, the Merton's problem. Note that from fund $i$'s investors' point of view, the return on their own investments is $dR^*_{it} - \psi_i dt$ corresponding to $\pi^*_i$ and $\theta^*_i$, instead of $\frac{dX^*_{it}}{X^*_{it}}$, which includes the flows based absolute performance and relative one of $dR^*_{it} - \psi_i dt$ over the industry average $\frac{1}{N}\sum_{j=1}^{N}(dR^*_{jt} - \psi_j dt)$. Thus when we discuss the fund performance and calculate the after-fee Sharpe ratios, the calculations do not take into account of fund flows. In particular, we compare the volatility of fund investment with and without competition, denoted as $\sigma^*_i$ and $\sigma^M_i$ respectively

$$\sigma^*_i = \sqrt{(\pi^*_i\sigma_i)^2 + 2\rho_{im}\pi^*_i\theta^*_i\sigma_i b + (\theta^*_i b)^2}, \quad \sigma^M_i = \sqrt{\left(\pi^M_i\sigma_i\right)^2 + 2\rho_{im}\pi^M_i\theta^M_i\sigma_i b + \left(\theta^M_i b\right)^2}, \tag{2.2.6}$$

and the corresponding after-fee Sharpe ratios of the fund investment

$$\eta^*_i = \frac{-\psi_i + \pi^*_i\mu_i + \theta^*_i a}{\sqrt{(\pi^*_i\sigma_i)^2 + 2\rho_{im}\pi^*_i\theta^*_i\sigma_i b + (\theta^*_i b)^2}}, \quad \eta^M_i = \frac{-\psi_i + \pi^M_i\mu_i + \theta^M_i a}{\sqrt{\left(\pi^M_i\sigma_i\right)^2 + 2\rho_{im}\pi^M_i\theta^M_i\sigma_i b + \left(\theta^M_i b\right)^2}}. \tag{2.2.7}$$

We are also interested in how each fund's return compares to the industry average. Fund's performance is measured in terms of difference between the individual fund's after-fee return $dR_{it} - \phi_i dt$ and the industry average $\frac{1}{N}\sum_{j=1}^{N}(dR_{jt} - \phi_j dt)$. The risk-return trade off of the competition tends to move individual fund's investment strategy in different directions: on one hand, the manager wants to deviate from the industry average, in order to outperform and attract new investments, which increases future management fees. On the other hand, the risk averse manager may also tend to mimic the competitors, which decreases the risk of outflows due to poor relative performance, and as a result, returns from different funds tend to be similar. The second effect of funds' competition is referred to as herding Graham (1999), and is discussed in Scharfstein and Stein (1990), Grinblatt et al. (1995) for institutional investors who have reputation concerns and make investment decisions based on past performance.

Let $\bar{\theta}^* := \frac{1}{N}\sum_{i=1}^{N}\theta^*_i$ and the average logarithmic return of the $N$ funds in equilibrium is $\bar{R}^*_t := \frac{1}{N}\sum_{i=1}^{N}R^*_{it}$ and $d\bar{R}^*_t = \left(r - \frac{1}{N}\sum_{i=1}^{N}\psi_i\right)dt + \frac{1}{N}\sum_{i=1}^{N}\pi^*_i(\mu_i dt + \sigma_i dW_{it}) + \bar{\theta}^*(a dt + b dB_t)$. We use the Beta coefficient of $R^*_i$ with respect to $\bar{R}^*$ to measure the "distance" between fund $i$ and the industry average, denoted as $\mathrm{Beta}^*_i$, and

$$\mathrm{Beta}^*_i = \frac{N\left(q_i'\Sigma\rho\Sigma\pi^* + Nq_i'\Sigma\rho_m\bar{\theta}^* b + (\pi^*)'\Sigma\rho_m\theta^*_i b + N\theta^*_i\bar{\theta}^* b^2\right)}{(\pi^*)'\Sigma\rho\Sigma\pi^* + 2N(\pi^*)'\Sigma\rho_m\bar{\theta}^* b + N^2\left(\bar{\theta}^*\right)^2 b^2}, \tag{2.2.8}$$

where $q_i$ is an $N$-dimensional vector with zero entries except that $(q_i)_i = \pi^*_i$, and $\Sigma$ is an $N \times N$ diagonal matrix with $(\Sigma)_{ii} = \sigma_i$. If there is no competition based on relative performance, the Beta coefficient between the corresponding return $R^M_i$ and their average $\bar{R}^M$, denoted as $\mathrm{Beta}^M_i$, can be similarly calculated. Let $\bar{\theta}^M = \frac{1}{N}\sum_{i=1}^{N}\theta^M_i$, $\pi^M$ be the $N$-dimensional vector with $\left(\pi^M\right)_i = \pi^M_i$, and $q^M_i$ be the $N$-dimensional vector with zero entries except that $\left(q^M_i\right)_i = \pi^M_i$,

$$\mathrm{Beta}^M_i = \frac{N\left(\left(q^M_i\right)'\Sigma\rho\Sigma\pi^M + N\left(q^M_i\right)'\Sigma\rho_m\bar{\theta}^M b + \left(\pi^M\right)'\Sigma\rho_m\theta^M_i b + N\theta^M_i\bar{\theta}^M b^2\right)}{\left(\pi^M\right)'\Sigma\rho\Sigma\pi^M + 2N\left(\pi^M\right)'\Sigma\rho_m\bar{\theta}^M b + N^2\left(\bar{\theta}^M\right)^2 b^2}. \tag{2.2.9}$$

The closer $\mathrm{Beta}^*_i$ (or $\mathrm{Beta}^M_i$) is to 1, the more closely fund $i$ mimics the industry average. If this is the case for most funds, then the herding effect is present.

The Beta coefficients of $R_i$ with respect to the common investment opportunity $\frac{dS_{mt}}{S_{mt}} = adt + bdB_t$ with and without competition, denoted as $\text{Beta}^*_{mi}$ and $\text{Beta}^M_{mi}$, can be computed similarly

$$\text{Beta}^*_{mi} = \frac{\pi^*_i \rho_{im}\sigma_i + \theta^*_i b}{b}, \quad \text{Beta}^M_{mi} = \frac{\pi^M_i \rho_{im}\sigma_i + \theta^M_i b}{b}. \tag{2.2.10}$$

They measure the "distance" between each fund's investment and $S_m$. The further away $\text{Beta}^*_{mi}$ (or $\text{Beta}^M_{mi}$) is from 1, the more fund $i$ specializes in its idiosyncratic investment opportunity $S_i$.

### 2.2.1 The Case of Two Funds

To illustrate the effect on the equilibrium portfolios of the changes in fund investment opportunities, we start from the case of two funds. In this case (with $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$)

$$\pi^*_i = \frac{2}{(2(1+\ell_i)+\alpha_i)\sigma_i\kappa_1}\left(\frac{1}{\gamma_i}(1-\rho^2_{jm})(\lambda_i - \rho_{im}\lambda_m) + \frac{1}{\gamma_j}\frac{\alpha_i(\rho_{12}-\rho_{1m}\rho_{2m})(\lambda_j - \rho_{jm}\lambda_m)}{2(1+\ell_j)+\alpha_j}\right), \tag{2.2.11}$$

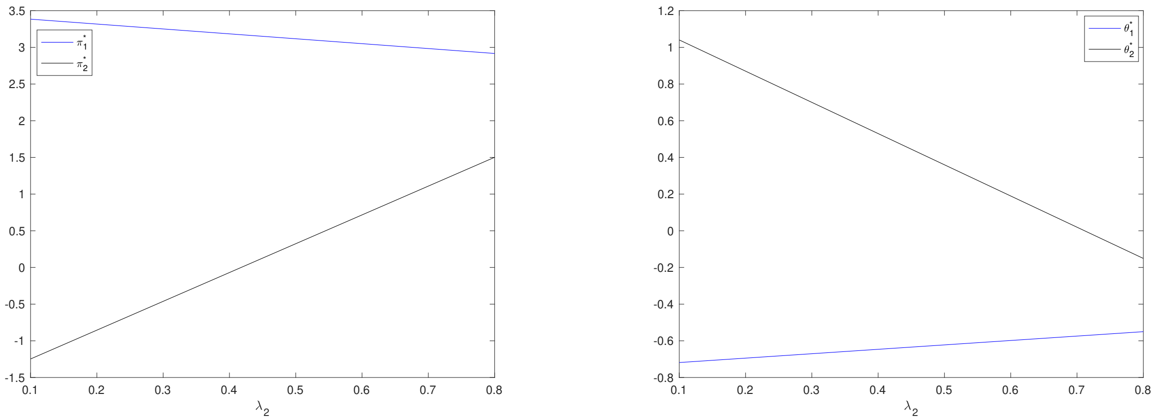
$$\begin{aligned}\theta^*_i = &\frac{2}{(2(1+\ell_i)+\alpha_i)b\kappa_2}\left(\left(\frac{1}{\gamma_i}(1-\rho^2_{jm})(\lambda_m - \rho_{im}\lambda_i) + \frac{1}{\gamma_j}\frac{\alpha_i}{2(1+\ell_j)+\alpha_j}(1-\rho^2_{im})(\lambda_m - \rho_{jm}\lambda_j)\right)\right.\\ &\left. + \frac{\alpha_1\alpha_2}{2(2(1+\ell_j)+\alpha_j)}(1-\rho^2_{im})(\rho_{im}-\rho_{12}\rho_{jm})\sigma_i\pi^*_i + \frac{\alpha_i}{2}(1-\rho^2_{jm})(\rho_{jm}-\rho_{12}\rho_{im})\sigma_j\pi^*_j\right),\end{aligned} \tag{2.2.12}$$

where $\kappa_1 = (1-\rho^2_{1m})(1-\rho^2_{2m}) - \frac{\alpha_1\alpha_2}{(2(1+\ell_1)+\alpha_1)(2(1+\ell_2)+\alpha_2)}(\rho_{12}-\rho_{1m}\rho_{2m})^2$ and $\kappa_2 = (1-\rho^2_{1m})(1-\rho^2_{2m})\frac{4+2\alpha_1(1+\ell_2)+2\alpha_2(1+\ell_1)}{(2(1+\ell_1)+\alpha_1)(2(1+\ell_2)+\alpha_2)}$. In addition to $\lambda_i$, $\rho_{im}$, $\lambda_m$, $\pi^*_i$ and $\theta^*_i$ also depend on the investment of the other fund, the fund's sensitivity to flows and the correlations between investment opportunities, while $\pi^*_i$ and $\theta^*_i$ reduce to $\pi^M_i$ and $\theta^M_i$, if $\alpha_i = \ell_i = 0$ for $i = 1, 2$.

Figure 2.1 shows how equilibrium portfolios change against the idiosyncratic opportunity of fund 2, summarized by $\lambda_2$, with $\rho_{1m} = 0.3, \rho_{2m} = 0.5, \rho_{12} = -0.6$, $\alpha_1 = \alpha_2 = 0.8$, $\ell_1 = \ell_2 = 0$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\gamma_1 = \gamma_2 = 2$. As $\lambda_2$ increases comparing to $\lambda_m$, $\pi^*_2$ becomes larger, from negative to positive, while $\theta^*_2$ decreases, because $\rho_{2m}$ is positive and fund 2 decreases the risk exposure to $S_m$, in order to hedge the increased risk taking in $S_2$, which coincide with results from the classical Merton problem. It shows that though fund 2's manager also needs to hedge against the risk in fund 1's investment, the risk-return trade off from its own investment opportunity dominates in the choice of the optimal portfolios.

On the other hand, fund 1's portfolios also change because the manager's compensation depends on the relative performance and thus the portfolios of fund 2. Since in the dynamics of $X_{1t}$ in (2.1.4), $dR_{2t}$ has a negative coefficient, the increasing $\lambda_2$ ($\pi^*_2$) leads to a larger negative exposure to $W_2$, which is negatively correlated with $W_1$, and fund 1 decreases $\pi^*_1$ to decrease the total risk exposure. For the investment in $S_m$, fund 1 optimizes over the effective exposure $\zeta_1 = \frac{2+\alpha_1}{2}\theta_1 - \frac{\alpha_1}{2}\theta_2$. Thus with larger $\theta^*_2$, $\theta^*_1$ tends to be larger. On the other hand, fund 1 also tends to increase $\zeta_1$ in order to hedge the increased risk in $dR_{2t}$, because of the larger $\pi^*_2$. The combined effect is that $\theta^*_1$ becomes larger as $\lambda_2$ increases.

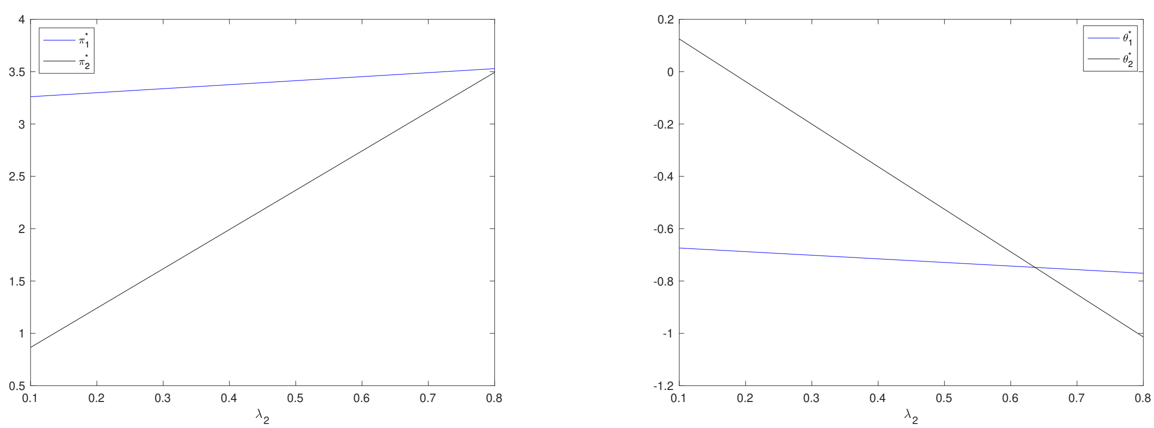
In Figure 2.2, $\rho_{12}$ is changed to 0.6 with other parameters the same as for Figure 2.1. $\pi^*_2$ and $\theta^*_2$ show the same pattern as in the previous case. On the other hand, since $\rho_{12}$ becomes positive, $\pi^*_1$ and $\theta^*_1$ show the opposite trend to the previous case, though the changes with respect to $\lambda_2$ are again relatively small, compared to $\pi^*_2$ and $\theta^*_2$.




(a) $\pi_1^*$ and $\pi_2^*$

(b) $\theta_1^*$ and $\theta_2^*$

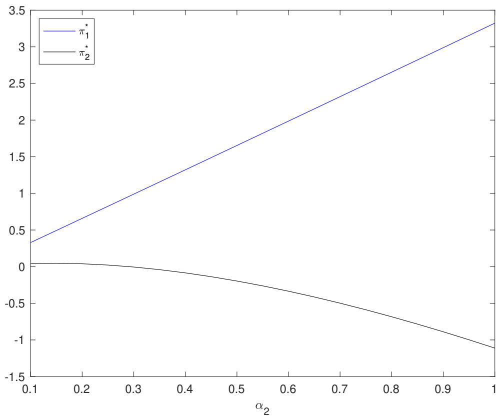
Figure 2.1: Equilibrium portfolios with $\rho_{1m} = 0.3, \rho_{2m} = 0.5, \rho_{12} = -0.6$, $\alpha_1 = \alpha_2 = 0.8$, $\ell_1 = \ell_2 = 0$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\gamma_1 = \gamma_2 = 2$, against $\lambda_2$.



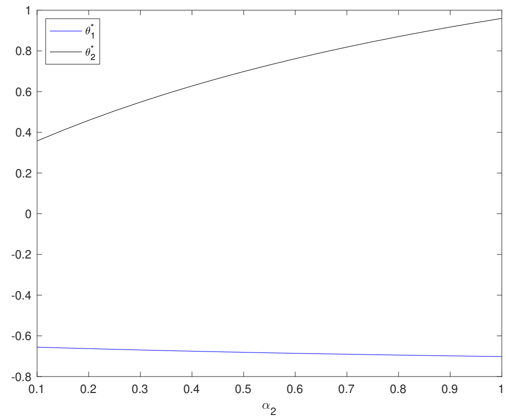

(a) $\pi_1^*$ and $\pi_2^*$

(b) $\theta_1^*$ and $\theta_2^*$

Figure 2.2: Equilibrium portfolios with $\rho_{1m} = 0.3, \rho_{2m} = 0.5, \rho_{12} = 0.6$, $\alpha_1 = \alpha_2 = 0.8$, $\ell_1 = \ell_2 = 0$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\gamma_1 = \gamma_2 = 2$, against $\lambda_2$.




(a) $\pi_1^*$ and $\pi_2^*$




(b) $\theta_1^*$ and $\theta_2^*$

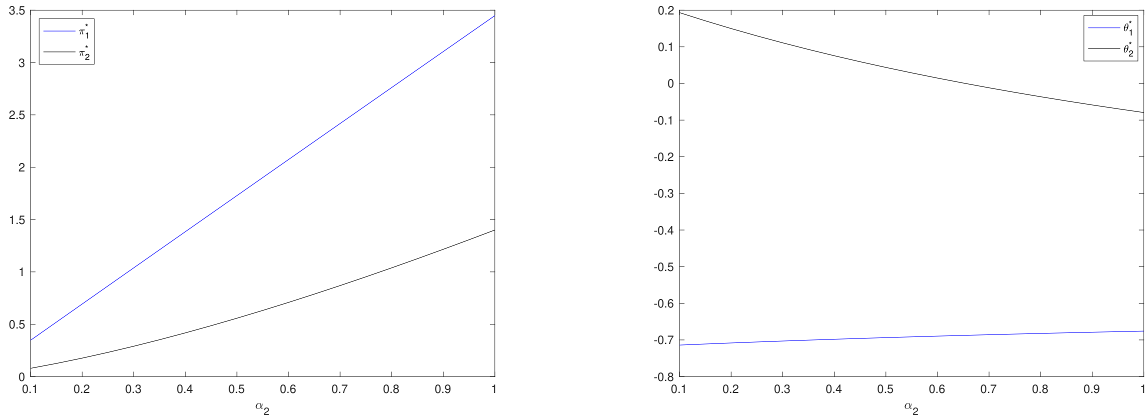
Figure 2.3: Equilibrium portfolios with $\rho_{1m} = 0.3, \rho_{2m} = 0.5, \rho_{12} = -0.6$, $\alpha_1 = 0.8$, $\ell_1 = \ell_2 = 0$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, against $\alpha_2$.

Regarding the effect of the sensitivity of fund flows, Figure 2.3 shows how equilibrium portfolios change against $\alpha_2$, with $\rho_{1m} = 0.3, \rho_{2m} = 0.5, \rho_{12} = -0.6$, $\alpha_1 = 0.8$, $\ell_1 = \ell_2 = 0$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$. For fund 2, the fund flows magnify the return and risk of its own investments as well as those of fund 1 as the benchmark. As $\alpha_2$ increases, this magnifying effect becomes larger. As a result, $\pi_2^*$ becomes more negative, so as to hedge the risk in $S_1$ (notice that $\lambda_2 < \lambda_1$ and $dW_{2t}$ is positively correlated with $-dR_{1t}$ for $\pi_1^* > 0$). Hence, $\pi_1^*$ becomes more positive to hedge more risks in the investment of fund 2, even though $\alpha_2$ does not enter the dynamics of fund 1 directly.

On the other hand, $\theta_2^*$ increases with $\alpha_2$, to hedge larger risks in the investment in $S_2$ and the fund 1's exposure to $S_1$. Similarly, $\theta_1^*$ moves in the opposite direction to $\pi_1^*$, while the effect is negligible, because $\alpha_2$ does not directly enter the dynamics of fund 1, and the increase in $\theta_2^*$ fulfills part of the hedging demands from the increase in $\pi_1^*$, which lowers fund 1's effective exposure to $S_m$, $\zeta_1 = \frac{2+\alpha_1}{2}\theta_1 - \frac{\alpha_1}{2}\theta_2$. In Figure 2.4, $\rho_{12}$ is changed to 0.6. $\pi_2^*$ and $\theta_2^*$ show the opposite pattern to Figure 2.3, while $\pi_1^*$ and $\theta_1^*$ behave similarly, following the same intuition as above.

Next we examine the comparison of the portfolios with and without competition in the case of $N = 2$, $\lambda_m = 0$, $\rho_{im} = 0$, $\alpha_i = \alpha$, $\ell_i = \ell$, $\psi_i = \psi$ for $i = 1, 2$. In this case, $\theta_i^* = \theta_i^M = 0$, and each fund invests in their own investment opportunities, which are correlated with each other – similar to the models in Basak and Makarov (2015), Lacker and Zariphopoulou (2019), which also leads to clearer conclusions, and sheds light on the results of $N > 2$. Let the risk aversion-adjusted Sharpe ratio for each fund's idiosyncratic investment opportunity as $\lambda_{i,\gamma_i} = \frac{\lambda_i}{\gamma_i}$ $(i = 1, 2)$, $\bar{\lambda} = \frac{\lambda_{2,\gamma_2}}{\lambda_{1,\gamma_1}}$ and without loss of generality assume $\bar{\lambda} \leq 1$. Then the equilibrium portfolios are

$$\pi_1^* = \frac{2\left(\lambda_{1,\gamma_1} + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\lambda_{2,\gamma_2}\right)}{(2(1+\ell)+\alpha)\sigma_1\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2\rho_{12}^2\right)}, \quad \pi_2^* = \frac{2\left(\lambda_{2,\gamma_2} + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\lambda_{1,\gamma_1}\right)}{(2(1+\ell)+\alpha)\sigma_2\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2\rho_{12}^2\right)}. \tag{2.2.13}$$



(a) $\pi_1^*$ and $\pi_2^*$ with $\rho_{2m} = 0.6$ (b) $\theta_1^*$ and $\theta_2^*$ with $\rho_{2m} = 0.6$

Figure 2.4: Equilibrium portfolios with $\rho_{1m} = 0.3, \rho_{2m} = 0.5, \rho_{12} = 0.6$, $\alpha_1 = 0.8$, $\ell_1 = \ell_2 = 0$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, against $\alpha_2$.

**Proposition 9.** *$\pi_1^* < \pi_1^M$, $\eta_1^* < \eta_1^M$, and*

*(i) If $\rho_{12} > 0$ and $\bar\lambda < \frac{2\alpha\rho_{12}}{(2+4\ell)\alpha+\alpha^2(1-\rho_{12}^2)+4\ell(1+\ell)}$, then $\pi_2^* > \pi_2^M$ and $\eta_2^* > \eta_2^M$. If $\rho_{12} \geq 0$ and $\bar\lambda \geq \frac{2\alpha\rho_{12}}{(2+4\ell)\alpha+\alpha^2(1-\rho_{12}^2)+4\ell(1+\ell)}$, then $\pi_2^* \leq \pi_2^M$ and $\eta_2^* \leq \eta_2^M$.*

*(ii) If $\rho_{12} < 0$, $\pi_2^* < \pi_2^M$, and if $\bar\lambda < -\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}$, $\eta_2^* > \eta_2^M$. Otherwise $\eta_2^* \leq \eta_2^M$.*

In addition to the total risks in the fund investment, fund managers also care about the risk in the relative performance, because it affects the fund flows and thus management fees in the future. On one hand, they want to keep investment strategy $\pi_i^M$ which brings the best risk-return trade off according to their own risk attitude. On the other hand, they may want to invest less in $S_i$, in order to decrease the risk of poor performance against their competitors. Proposition 9 shows that for the fund 1 with the larger risk aversion-adjusted Sharpe ratio $\frac{\lambda_1}{\gamma_1}$, though $S_1$ is a better investment opportunity, since it is relatively easier to outperform, the concerns for the risks in the relative performance dominates and the manager takes less risk ($\pi_1^* \leq \pi_1^M$). The fund investment is not taking full advantage of the good investment opportunity, which hurts the fund's performance ($\eta_1^* < \eta_1^M$). The same could happen if $\gamma_1$ is small, and therefore the manager tends to take large risk without competition, and has a better chance of outperform the competitor. It is consistent with the results in Basak and Makarov (2015) that more risk tolerant managers may decrease the volatility of the fund.

On the other hand, the relatively disadvantaged manager (with smaller $\lambda_2$ or large risk aversion $\gamma_2$) behaves differently under different conditions. If $\rho_{12} \geq 0$, the portfolio choice of the competitor hedges part of the risk in the fund's own investment. Thus if $\bar\lambda$ is small, i.e. the disadvantage is big, the eagerness for new investments dominates, and the manager increases the fund's risk for a better chance of winning the competition. This may not be a bad news for the clients, because the after-fee Sharpe ratio of the fund actually increases. If $\bar\lambda$ is sufficiently large, then the peer pressure is lighter and fund 2 invests similarly to fund 1, by decreasing the risky investment, and thus lowers the performance. If $\rho_{ij} < 0$, the introduction of fund flows increases the total risks in the fund, and the concern for the fund's absolute performance leads fund 2 to decrease the investment in $S_2$ to hedge against the risk in $S_1$. $\frac{\pi_2^*}{\pi_2^M}$ is increasing in $\bar{\lambda}$, and equals to $\frac{2\left(1+\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\right)}{(2(1+\ell)+\alpha)\left(1-\left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2\rho_{12}^2\right)} > 0$ at $\bar{\lambda} = 1$ and $-\infty$ at $\bar{\lambda} = 0$. Thus similar to the case of $\rho_{12} > 0$, if $\bar{\lambda}$ is small (with a threshold different to the previous case), the big disadvantage leads fund manager 2 to take a large risk aiming to win the competition. If $\bar{\lambda}$ is sufficiently large, then fund 2 already has a good chance of outperform fund 1. Therefore the decrease in the investment in $S_2$ cannot be too large, especially if it is negative, because it increases the risk in the relative performance.

**Proposition 10.** *Let* $\Delta = \left(\frac{\alpha}{1+\ell+\alpha}\rho_{12} + \frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}}\right)^2 - 4 \geq 0$. *Then* $|\mathrm{Beta}_i^* - 1| - \left|\mathrm{Beta}_i^M - 1\right| \leq 0$ *for both* $i = 1$ *and* $2$ *if and only if one of the following holds: (i)* $\rho_{12} \geq 0$*, and (ii)* $\rho_{12} < 0$*,* $\bar{\lambda} \leq \frac{-\left(\frac{\alpha}{1+\ell+\alpha}\rho_{12}+\frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}}\right)-\sqrt{\Delta}}{2}$.

The changes in the Beta coefficients can also be characterized in terms of the correlation between the funds' investment opportunities and their risk aversion-adjusted Sharpe ratios. Though fund 2 may behave differently according to Proposition 9, in most cases the competition pushes both funds' investments closer to their average. If $\rho_{12} > 0$ and $\bar{\lambda}$ is small, then $\pi_2^M$ is small comparing to $\pi_1^M$, and with competition $\pi_1^*$ and $\pi_2^*$ move toward each other. If $\bar{\lambda}$ is large, then $\pi_1^*$ and $\pi_2^*$ both become smaller positive numbers, and are closer to the average. If $\rho_{12} < 0$ and $\bar{\lambda}$ is small, $\pi_2^*$ tends to be negative with large absolute values. Then with negative correlation between $S_1$ and $S_2$, two funds actually becomes closer. Only in the case of $\rho_{12} < 0$ and sufficiently large $\bar{\lambda}$, i.e. fund 2 has less peer pressure, decrease of the position in $S_2$ is limited, and fund 2 stays sufficiently negatively correlated with fund 1. As a result, both funds are further away from their average.

### 2.2.2 The Equilibrium among $N$ Funds

For more than two funds, the equilibrium depends on the model parameters, especially the correlation structure, in a complex way, and explicit characterization is no longer available in terms of the risk aversion-adjusted Sharpe ratio as in the case of $N = 2$. For example, it is not likely that the Beta coefficients of all funds move in the same direction as in Proposition 10. With the explicit solution to the equilibrium, we check the effect of competition by numerical experiments. The results, which are largely consistent with the conclusions for the case of $N = 2$, show that the competition tends to push funds to decrease risk takings in their idiosyncratic investment opportunities, in order to decrease the risk in the relative performance. This usually leads to worse performance in terms of Sharpe ratios. However, managers with big disadvantages tend to take larger idiosyncratic risk in order to beat the average. In terms of herding effect, while the fund flows generally pushes funds to become closer to the industry average, if no funds are severely disadvantaged and some funds investment opportunities are negatively correlated, then the competition may push all competitors to move away from the industry average.

In the case of $N = 5$, Figure 2.5 plots the funds' portfolios with competition ($\pi_i^*$'s and $\theta_i^*$'s) and without competition ($\pi_i^M$'s and $\theta_i^M$'s), the corresponding Sharpe ratios $\eta_i^*$'s and $\eta_i^M$'s, and volatilities $\sigma_i^*$ and $\sigma_i^M$, with $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\gamma_i = 2$, $\rho_{im} = 0.1$ for every $1 \leq i \leq 5$, $\lambda_i$'s forming an arithmetic sequence from 0.1 to 0.5, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$. Compared to the case where the managers do not have to care about relative performance, all the funds have lower after fee Sharpe ratios. The main reason is similar to the case of $N = 2$ that the managers are concerned about the risk of under performance. Thus they take

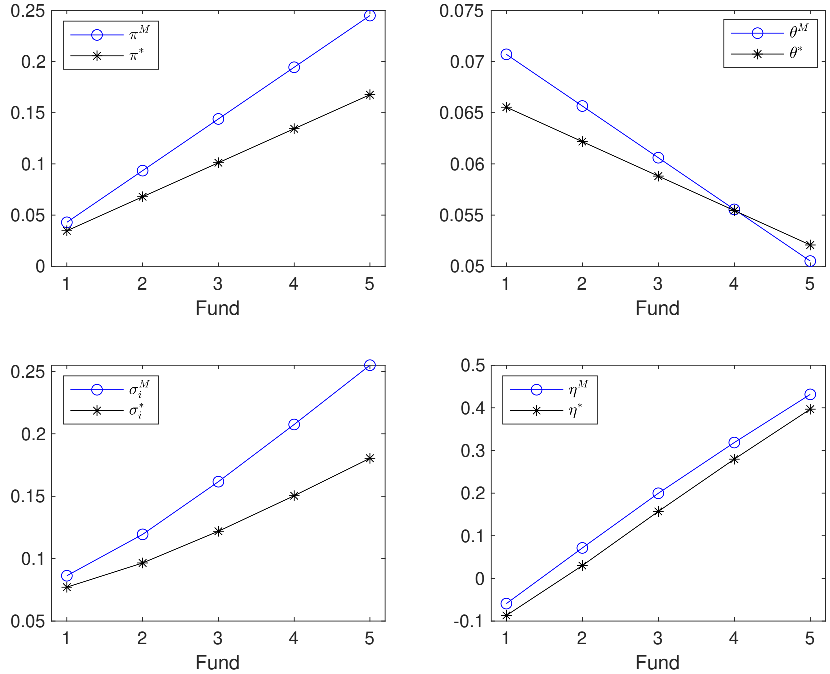



Figure 2.5: Funds' portfolios, volatility and Sharpe ratios, with $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\gamma_i = 2$, $\rho_{im} = 0.1$ for every $1 \leq i \leq 5$, $\lambda_i$'s form an arithmetic sequence from 0.1 to 0.5, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$.

less risk in the idiosyncratic opportunity $S_i$. This change is larger for funds with better investment opportunities. It lowers the expected return of the fund. On the other hand, even for the case when $\theta^* > \theta^M$, since the decrease in $\pi_i^*$ from $\pi_i^M$ is much larger than the increase in $\theta_i^*$ from $\theta_i^M$, the total risk that the fund is taking is smaller, and thus the Sharpe ratio decreases, but not as much.

Figure 2.6 illustrates the case where $\lambda_i = 0.3$ and $\rho_{im}$'s form an arithmetic sequence from $-0.2$ to 0.6. Similar to the previous case, all the funds take less risk in their idiosyncratic investment opportunities in face of competition. If $S_i$ is more positively correlated with $S_m$, fund $i$ invests more in $S_m$, even from negative to positive amount in some cases. Only for fund 1 with $\rho_{1m} < 0$, $\theta_1^* < \theta_1^M$, because the need for hedging the risk in $S_1$ is reduced. Due to the less risk taking in the fund investment, similar to the previous case, the total risk of the funds and the Sharpe ratios decrease. Notice that in this case $\lambda_i/\gamma_i$ is a constant across 5 funds, and the result agrees with Proposition 9 in the change of Sharpe ratios.

Figure 2.7 illustrates the case where $\lambda_i$'s form an arithmetic sequence from 0.1 to 1.3, $\gamma_i$'s form an arithmetic sequence from 0.5 to 4, $\rho_{im} = 0$ for every $1 \leq i \leq 5$, $\rho_{i5} = -0.2$ $(i \neq 5)$, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, and other parameters are the same as in the previous cases. It shows some features that we do not see in the case of $N = 2$, due to the complex dependence on the correlation structure. $\lambda_i$'s have larger differences than in the previous cases, but $\lambda_i/\gamma_i$ actually become closer than in Figure 2.5. With negative correlations between the funds' idiosyncratic investment opportunities, while other funds behave similarly as in 2.5, the most disadvantaged manager (of fund 1) takes

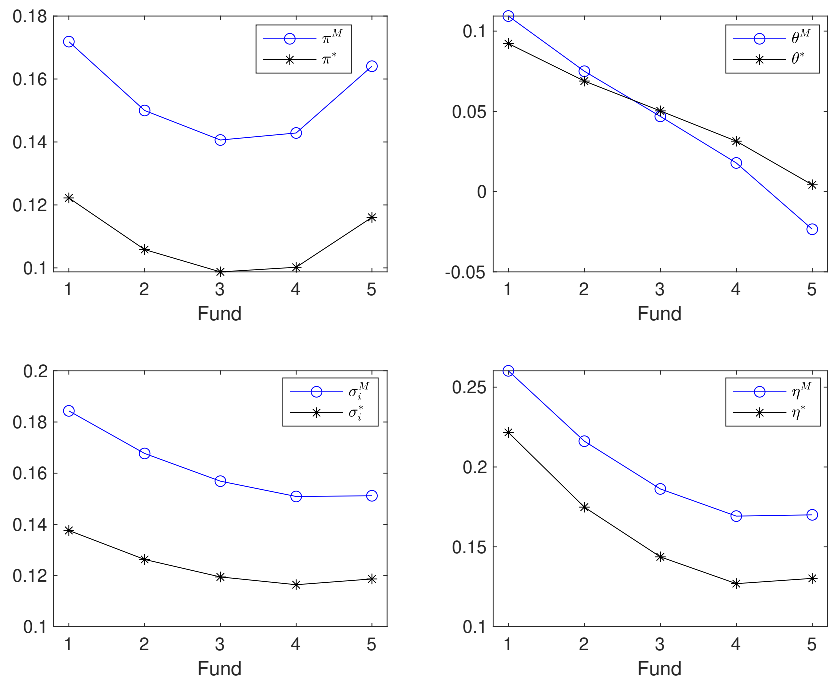



Figure 2.6: Funds' portfolios, volatility and Sharpe ratios, with $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\gamma_i = 2$, $\lambda_i = 0.3$ for every $1 \leq i \leq 5$, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, $\rho_{im}$'s form an arithmetic sequence from $-0.2$ to $0.6$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$.

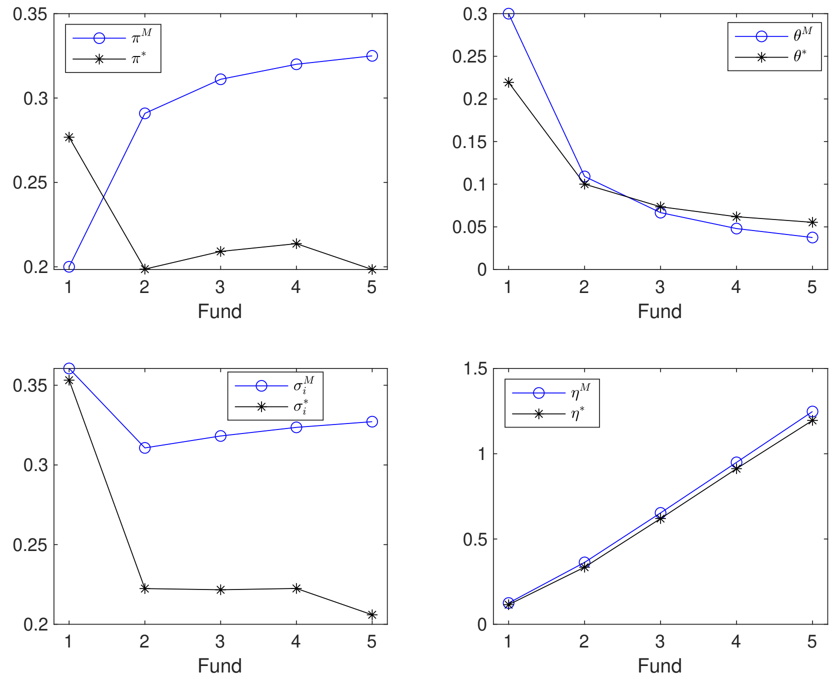



Figure 2.7: Funds' portfolios, volatility and Sharpe ratios, with $N = 5$, $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\rho_{im} = 0$ for every $1 \leq i \leq 5$, $\lambda_i$'s form an arithmetic sequence from 0.1 to 1.3, $\gamma_i$'s form an arithmetic sequence from 0.5 to 4, $\rho_{i5} = -0.2$ $(i \neq 5)$, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$.

larger risk in the fund's idiosyncratic investment opportunity, and less in $S_m$, in order to have a better chance of beating the competitors and attract new investments. As a result, the total risk of fund 1's investments decrease least with competition comparing to the other funds.

Next, we use the Beta coefficient with respect to the industry average and $S_m$ to measure the herding and specialization effect respectively, of the competition. In particular, if $|\text{Beta}_i^* - 1| < |\text{Beta}_i^M - 1|$, fund $i$ is closer to the industry average, and if $|\text{Beta}_{mi}^* - 1| > |\text{Beta}_{mi}^M - 1|$, then fund $i$ tends to focus more on its idiosyncratic investment opportunity rather than $S_m$, in order to have a better chance of beating the industry average.

Figure 2.8 illustrates the case with the same parameters as for Figure 2.5. From Figure 2.5, to hedge the risk from relative performance, the decrease of $\pi_i^*$ from $\pi_i^M$ is more than the increase of $\theta_i^*$ from $\theta_i^M$. Since $\rho_{im} > 0$ for every $1 \leq i \leq 5$, combined effect is that each fund's investment is further away from $S_m$ ($\text{Beta}_{mi}^*$ is further away from 1 than $\text{Beta}_{mi}^M$), and at the same time closer to the industry average. In Figure 2.9, the model parameters are the same as for Figure 2.6. As shown in Figure 2.6, each fund deceases its investment in the idiosyncratic investment opportunity, and most of them increase the investment in $S_m$, except fund 1, because $S_1$ is negative correlated with $S_m$. Thus though the Beta coefficients with respect to $R_m$ do not change much with and without competition, $\text{Beta}_i^*$'s are always closer to 1 than $\text{Beta}_i^M$'s. Both the above results show that in general the competition pushes mutual funds to herd. Also, though $\theta_i^*$ in most cases are greater than $\theta_i^M$, because of the positive correlations between $S_i$'s and $S_m$, each fund is further away from

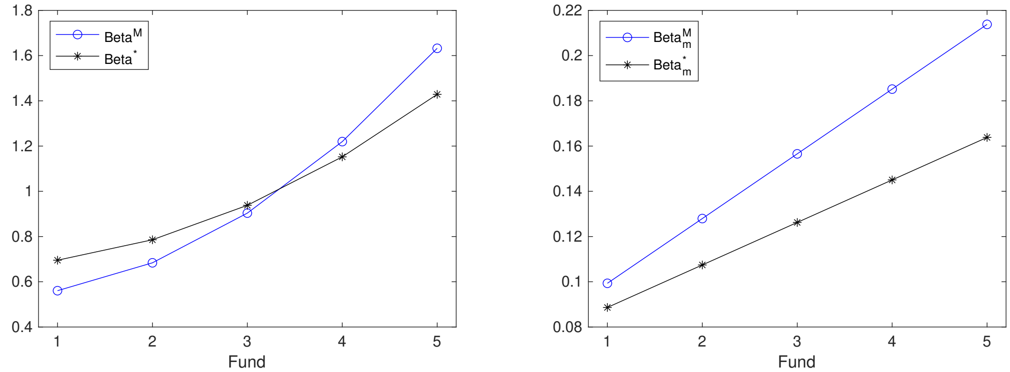



Figure 2.8: Funds' Beta coefficients with and without competition, with $N = 5$, $\gamma_i = 2$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\sigma_i = 0.2$ and $\rho_{im} = 0.2$, $1 \le i \le 5$, $\lambda_i$'s form an arithmetic sequence from 0.1 to 0.5, and $\rho_{ij} = 0.1$ for $1 \le i \ne j \le 5$.

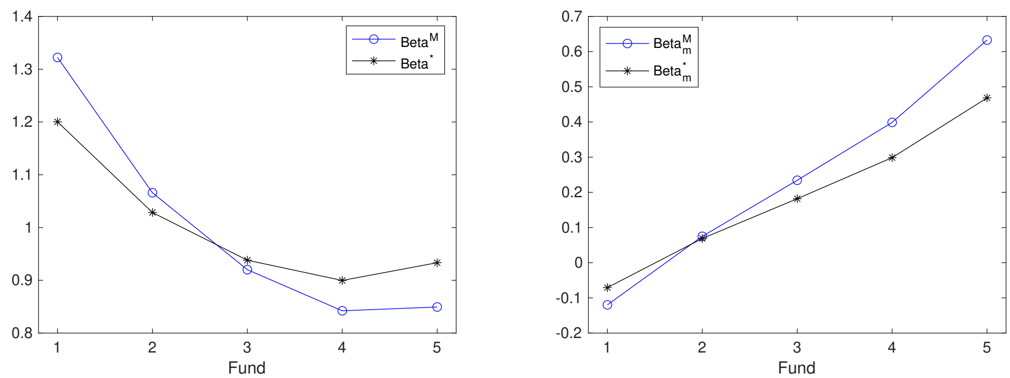



Figure 2.9: Funds' Beta coefficients with and without competition, with $N = 5$, $\lambda_i = 0.3$, $\gamma_i = 2$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\sigma_i = 0.2$, $1 \le i \le 5$, $\rho_{im}$'s form an arithmetic sequence from $-0.2$ to 0.6, and $\rho_{ij} = 0.2$ for $1 \le i \ne j \le 5$. $\lambda_m = 0.15$, $b = 0.15$.

the common investment opportunity.

In Figure 2.10, the model parameters are the same as for Figure 2.7, and the funds' behaviors change drastically. While funds 2-5 decreases their investment in their idiosyncratic risk, similarly to the previous two cases, the disadvantaged fund 1 takes excessive risk by large exposure in $S_1$. It shifts the industry average so large that $|\text{Beta}_i^* - 1| > |\text{Beta}_i^M - 1|$ for every $1 \le i \le 5$, which means that competition actually increases the risk in relative performance for every fund, even though most of them choose the optimal portfolios to avoid this. In this case, funds 2-5 take more exposure in $S_m$ and only fund 1 specializes more in its idiosyncratic risk.

Finally, let us consider a special case in which $S_m$ is the only risky investment opportunity for each fund and investors only care about the relative performance ($\ell_i = 0$). With the investment

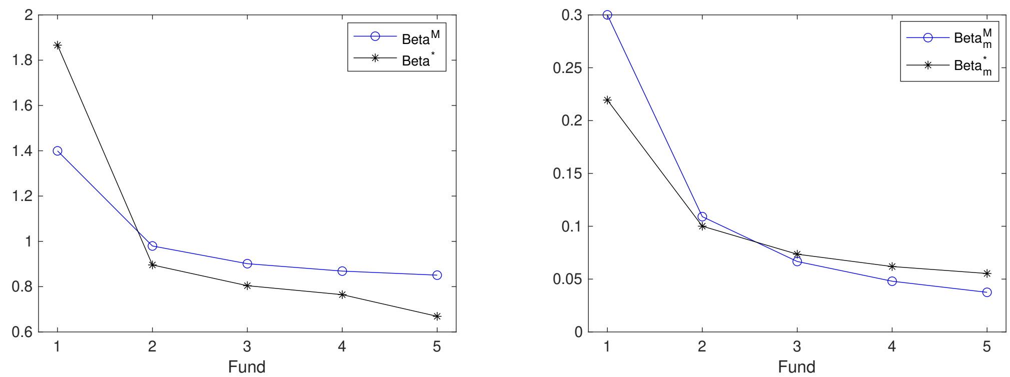



Figure 2.10: Funds' Beta coefficients with and without competition, with $N = 5$, $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha_i = 0.5$, $\ell_i = 0.1$, $\rho_{im} = 0$ for every $1 \le i \le 5$, $\lambda_i$'s form an arithmetic sequence from 0.1 to 1.3, $\gamma_i$ form an arithmetic sequence from 0.5 to 4, $\rho_{i5} = -0.2$ $(i \neq 5)$, $\rho_{ij} = 0.2$ $(1 \le i \neq j \le 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$.

strategy $\theta_i$ in $S_m$, the dynamics of $X_i$ is

$$\frac{dX_{it}}{X_{it}} = \left( r - \frac{N + (N-1)\alpha_i}{N}\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\psi_j \right) dt + \left( \frac{N + (N-1)\alpha_i}{N}\theta_{it} - \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\theta_{jt} \right)(adt + bdB_t), \tag{2.2.14}$$

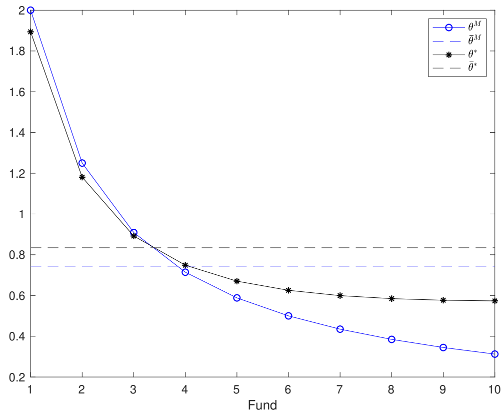
which allows more concrete discussions about how close are each fund to the industry average.

**Proposition 11.** *If $S_m$ is the only risky investment opportunity for every fund (as described in (2.2.14)), then there exists a unique equilibrium $\theta^* \in \Theta^N$ such that for each $1 \le i \le N$,*

$$\theta_i^* = \frac{\lambda_m}{b}\left( \frac{1}{1+\alpha_i}\frac{1}{\gamma_i} + \frac{\alpha_i}{1+\alpha_i}\frac{1}{\bar{\gamma}} \right), \tag{2.2.15}$$

*where* $\bar{\gamma} = \frac{N}{\sum_{i=1}^{N} \frac{1+\bar{\alpha}}{(1+\alpha_i)\gamma_i}}$ *and* $\bar{\alpha} = \frac{N}{\sum_{i=1}^{N} \frac{1}{1+\alpha_i}} - 1.$

Notice that in this case, though $\theta_i^*$'s in (2.2.15) are constants, they are unique among all admissible strategies. Also, if there is no fund flows ($\alpha_i = 0$), the manager is facing essentially a Merton problem with $S_m$ as the only risky investment opportunity. The optimal strategy is $\theta_i^M = \frac{\lambda_m}{b\gamma_i}$ for each $1 \le i \le N$. Comparing to $\theta_i^M$, $\theta_i^*$ is also of the Merton type. However, the manager's risk tolerance shifts to a linear combination between the manager's own risk tolerance $\frac{1}{\gamma_i}$ and $\frac{1}{\bar{\gamma}}$, the average of the risk tolerance of all competing fund managers, weighted by the sensitivity of fund flows to the relative performance. In the following we compare the fund's investment and check the herding effect of competition. Since the return of every fund is driven by a common Brownian Motion $B$, to analyze the similarities between each fund and its competitors, it suffices to compare $\theta_i^*$'s and $\theta_i^M$'s, and their industry average $\bar{\theta}^*$ and $\bar{\theta}^M$, respectively. The next proposition shows that, relatively more risk averse managers may take larger risk in face of competition. Also, if the fund with more risk averse manager has larger flow sensitivity to the relative performance, then




(a) $\alpha_i$'s form an arithmetic sequence from 0.1 to 1

(b) $\alpha_i$'s form an arithmetic sequence from 1 to 0.1

Figure 2.11: $\hat{\theta}_i$'s, $\theta_i^*$'s and their average if managers only invest in $S_m$. $N = 10$, $\lambda_m = 0.15$, $b = 0.15$, $\ell_i = 0$ and $\gamma_i$'s form an arithmetic sequence from 0.5 to 3.2.

in average the investment of the whole group becomes more risky. In the special case of constant fund flow sensitivities, $\theta_i^*$ is always closer to the industry average, than the counterpart without competition.

**Proposition 12.** *(i) If $\gamma_i > \bar{\gamma}$, $\theta_i^* > \theta_i^M$, and vice versa.*

*(ii) If $(\gamma_i - \gamma_j)(\alpha_i - \alpha_j) \geq 0$ for every pair of $1 \leq i \leq j \leq N$, $\bar{\theta}^* \geq \bar{\theta}^M$, and vice versa.*

*(iii) If $\alpha_i$ equals a constant $\alpha > 0$ for every $1 \leq i \leq N$, then $\bar{\theta}^* = \bar{\theta}^M$, and $\theta_i^* - \bar{\theta}^* = \frac{1}{1+\alpha}\left(\hat{\theta}_i - \bar{\theta}^M\right)$.*

The intuition for these results is that facing the same investment opportunity, the funds with more risk averse managers tend to take less risk, and are thus less likely to win in the competition. Thus the concern for relative performance pushes them to be more aggressive to keep up. On the other hand, funds with less risk averse managers are at a better position in the competition, and are thus more concerned about the risk of poor performance. They invest less in $S_m$ to avoid possible outflow due to the loss in the risky asset. Furthermore, if the high risk aversion is accompanied by high sensitivity $\alpha_i$ of fund flows, then the effect of more risk taking for more risk averse managers is magnified compared to the effect of less risk taking for less risk averse managers, and the average risk taking of all funds with competition is higher than the counterpart without.

Figure 2.11 shows $\hat{\theta}_i$'s, $\theta_i^*$'s and their average, with $N = 10$, $\lambda_m = 0.15$, $b = 0.15$ and $\gamma_i$'s being equally spaced between 0.5 and 3.2. The left panel shows the case of increasing $\alpha_i$'s and $\bar{\theta}^* > \bar{\theta}^M$. In the right panel, $\alpha_i$'s are decreasing, and the inequality is reversed. In both graphs, similar to previous examples, $\theta_i^*$'s are closer to $\bar{\theta}^*$, compared to the distance between $\theta_i^M$ and $\bar{\theta}^M$, with an exception of 2 out of the 10 funds. In the special case of $\alpha_i$ being a constant, Part (iii) of Proposition 12 confirms that this comparison holds for every fund, and the competition based on relative performance does have herding effect on the fund investment.

In conclusion, the managers have two considerations in his/her portfolio choice, one is the total risk taking of the fund, which decides the return, and the other is the risk in the relative performance, which decides the fund flow. Our results show that in most cases the concern for the poor relative performance dominates, and managers take less risk in their idiosyncratic investment opportunity, so that the fund behaves more closely to the industry average but hurts the after-fee performance compared to the case without competition. It indicates that competition pushes funds to herd, which agrees with the results in Maug and Naik (2011). However, if the fund is disadvantaged with poor idiosyncratic investment opportunity or the manager is of relatively high risk aversion among the group (so that he/she takes low risk without competition), then to beat the competitors and attract new investment, the fund increases the risk taking in its idiosyncratic investment opportunity and increases the after-fee performance, which benefits the investors. It supports the conclusion in Basak and Makarov (2015) that competition can lead to specialization, which is also discussed in Brennan (1975), Uppal and Wang (2003), Van Nieuwerburgh and Veldkamp (2009), Boyle et al. (2012), Liu (2014). It also partially agrees with the results in Basak and Makarov (2015), Lacker and Zariphopoulou (2019) that more risk averse fund managers tend to take more risk under competition, than those with lower risk aversion. However, with appropriate correlation structures, if the funds' investment opportunities and the managers' risk aversions are close to each other, it could happen that every fund is further away from the average, comparing to the case without competition.

## 2.3 Appendix

*The proof of Theorem 8.* The first step is to find the optimal portfolio choice of the fund $i$, given the investment strategies $\pi_{-i}$ and $\theta_{-i}$ of other funds. Since we focus on constant equilibria, assume that $\pi_{-i}$ and $\theta_{-i}$ are constants. Then with $\tilde{X}_{it} := \exp\left(-\left(r - \left(1 + \ell_i + \frac{N-1}{N}\alpha_i\right)\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\psi_j\right)t\right) X_{it}$, Fubini's theorem implies that

$$\mathbb{E}\left[\int_0^T e^{-\beta_i t}\frac{(\psi_i X_{it})^{1-\gamma_i}}{1-\gamma_i}dt\right] = \int_0^T e^{\left(-\beta_i + r - \left(1+\ell_i+\frac{N-1}{N}\alpha_i\right)\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\psi_j\right)t}\psi_i^{1-\gamma_i}\frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i}dt \quad (2.3.1)$$

For any $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$ $(1 \le i \le N)$, let $\tilde{\pi}_{it} := \left(1 + \ell_i + \frac{N-1}{N}\alpha_i\right)\sigma_i\pi_{it}$, $\tilde{\theta}_{it} := \left(1 + \ell_i + \frac{N-1}{N}\alpha_i\right)b\theta_{it}$, so that $\pi := (\pi_1, \cdots, \pi_N) = A_f\tilde{\pi}$ and $\theta := A_m\tilde{\theta}$, where $\tilde{\pi}$ and $\tilde{\theta}$ are $N$-dimensional vectors with $(\tilde{\pi})_i = \tilde{\pi}_i$ and $(\tilde{\theta})_i = \tilde{\theta}_i$. Thus

$$\frac{d\tilde{X}_{it}}{\tilde{X}_{it}} = \tilde{\pi}_{it}(\lambda_i dt + dW_{it}) + \left(\tilde{\theta}_{it} - \sum_{j\neq i}^{N} c_{ij}\tilde{\theta}_j\right)(\lambda_m dt + dB_t) - \sum_{j\neq i}^{N} c_{ij}\left(\tilde{\pi}_j(\lambda_j dt + dW_{jt})\right). \quad (2.3.2)$$

With $\phi_{it} = \begin{bmatrix} \tilde{\pi}_{it} \\ \tilde{\theta}_{it} - \sum_{j\neq i}^{N} c_{ij}\tilde{\theta}_j \end{bmatrix}$, $h_i = \begin{bmatrix} \lambda_i \\ \lambda_m \end{bmatrix}$, $F_{it} = \begin{bmatrix} W_{it} \\ B_t \end{bmatrix}$, $\lambda_{-i} = \begin{bmatrix} \cdots & \lambda_{i-1} & \lambda_{i+1} & \cdots \end{bmatrix}'$, and $W_{-it} = \begin{bmatrix} \cdots & W_{(i-1)t} & W_{(i+1)t} & \cdots \end{bmatrix}'$, the dynamics of $\tilde{X}$ is

$$\frac{d\tilde{X}_{it}}{\tilde{X}_{it}} = \phi_{it}'(h_i dt + dF_{it}) - \tilde{\pi}_{-i}'C_i(\lambda_{-i}dt + dW_{-it}),$$

where $C_i$ is an $(N-1)$-dimensional matrix with diagonal entries $c_{ij}$ for $1 \le j \le N$ and $j \neq i$.

Lemma 13 shows that $\hat{\phi}_i = \frac{1}{\gamma_i} w_i^{-1} h_i + w_i^{-1} w_{-i} C_i \pi_{-i}$ maximizes $\frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i}$. Since it is a constant strategy independent of $t$, it also maximizes the discounted expected utility from management fees for each manager $i$

$$\int_0^T e^{\left(-\beta_i + r - \left(1+\ell_i+\frac{N-1}{N}\alpha_i\right)\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^N \psi_j\right)t} \psi_i^{1-\gamma_i} \frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i} dt.$$

$\hat{\phi}_i = \frac{1}{\gamma_i} w_i^{-1} h_i + w_i^{-1} w_{-i} C_i \tilde{\pi}_{-i}$ for $1 \leq i \leq N$ are $2N$ equations of constants $\tilde{\pi} = (\tilde{\pi}_1, \cdots, \tilde{\pi}_N)$ and $\tilde{\theta} = (\tilde{\theta}_1, \cdots, \tilde{\theta}_N)$:

$$P_f \tilde{\pi} = \gamma^{-1}\lambda_f, \quad P_m \tilde{\theta} = \gamma^{-1}\eta_m + C\tilde{\pi}, \tag{2.3.3}$$

of which the solution corresponds to the equilibrium strategies of the $N$ funds. Since Lemma 14 shows that $P_f$ and $P_m$ are invertible, there exists a unique solution $\tilde{\pi} = P_f^{-1}\gamma^{-1}\lambda_f$, $\tilde{\theta} = P_m^{-1}\left(\gamma^{-1}\eta_m + C\tilde{\pi}\right)$. Therefore $\pi^* = A_f P_f^{-1}\gamma^{-1}\lambda_f$, and $\theta^* = A_m P_m^{-1}\left(\gamma^{-1}\eta_m + C A_f^{-1}\pi^*\right)$. $\square$

**Lemma 13.** *Given constant* $\pi_{-i}$ *and* $\theta_{-i}$, $\underset{\phi_i:(\pi_i,\theta_i)\in\mathcal{A}_i\times\Theta}{\arg\max} \frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i} = \hat{\phi}_i = \frac{1}{\gamma_i} w_i^{-1} h_i + w_i^{-1} w_{-i} C_i \pi_{-i}$, *for every* $0 \leq t \leq T$, *where* $w_i = \begin{bmatrix} 1 & \rho_{im} \\ \rho_{im} & 1 \end{bmatrix}$, $w_{-i} = \begin{bmatrix} (\rho_i)'_{-i} \\ (\rho_m)'_{-i} \end{bmatrix}$, *and* $\rho_i$ *is the* $N$*-dimensional vector with* $(\rho_i)_j = \rho_{ij}$.

*Proof.* We prove the case of $0 < \gamma_i < 1$ and focus on $\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$ because $1-\gamma_i > 0$. The case of $\gamma > 1$ follows similarly. Define a stochastic process $\xi$ such that $\xi_0 = 1$ and

$$-\frac{d\xi_t}{\xi_t} = \left(M_i' w_{-i} + M_{-i}'\rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i} dt + M_i' dF_{it} + M_{-i}' dW_{-it}, \tag{2.3.4}$$

where $M_i$ and $M_{-i}$ are two constant vectors to be determined later, which satisfy $w_i M_i + w_{-i} M_{-i} = h_i$. Then

$$\frac{d\xi_t \tilde{X}_{it}}{\xi_t \tilde{X}_{it}} = -\left(M_i' w_i' + M_{-i}' w_{-i}' - h_i'\right)\phi_{it} dt + (\phi_{it}' - M_i') dF_{it} - (\pi_{-i}' C_i + M_{-i}') dW_{-it}. \tag{2.3.5}$$

Thus $\xi_t \hat{X}_{it}$ is a non-negative local martingale, and hence a supermartingale. Therefore (ignoring the positive $1-\gamma_i$), by Hölder's inequality and noticing that $\tilde{X}_{i0} = X_{i0} = 1$,

$$\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right] \leq \mathbb{E}\left[\xi_t \tilde{X}_{it}\right]^{1-\gamma_i} \mathbb{E}\left[\xi_t^{\frac{\gamma_i - 1}{\gamma_i}}\right]^{\gamma_i} \leq \mathbb{E}\left[\xi_t^{\frac{\gamma_i - 1}{\gamma_i}}\right]^{\gamma_i} \tag{2.3.6}$$

$$= \exp\left((1-\gamma_i)\left(\left(M_i' w_{-i} + M_{-i}'\rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i} + \frac{1}{2\gamma_i} M_i' w_i M_i + \frac{1}{2\gamma_i} M_{-i}' \rho_{-i} M_{-i} + \frac{1}{\gamma_i} M_i' w_{-i} M_{-i}\right) t\right), \tag{2.3.7}$$

which is an upper bound for $\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$ corresponding to any $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$.

Next we search for the minimum among all such upper bounds corresponding to different choices of $M_i$ and $M_{-i}$, by considering the following constrained minimization problem:

$$\min_{\{M_i, M_{-i}\}} \frac{1}{2\gamma_i} M_i' w_i M_i + \frac{1}{2\gamma_i} M_{-i}' \rho_{-i} M_{-i} + \frac{1}{\gamma_i} M_i' w_{-i} M_{-i} + \left(M_i' w_{-i} + M_{-i}' \rho_{-i}\right) C_i \pi_{-i}, \tag{2.3.8}$$

$$\text{subject to: } w_i M_i + w_{-i} M_{-i} = h_i. \tag{2.3.9}$$

The corresponding Lagrangian function, with Lagrange multiplier $l$, is

$$\mathcal{L} = \frac{1}{2\gamma_i}\left(M_i' w_i M_i + M_{-i}'\rho_{-i}M_{-i} + 2M_i' w_{-i}M_{-i}\right) + \left(M_i' w_{-i} + M_{-i}'\rho_{-i}\right) C_i\pi_{-i} + l'\left(h_i - w_i M_i - w_{-i}M_{-i}\right). \tag{2.3.10}$$

The first order conditions for $M_i$, $M_{-i}$ and $l$ are

$$M_i = \gamma_i l - w_i^{-1} w_{-i} M_{-i} - \gamma_i w_i^{-1} w_{-i} C_i \pi_{-i}, \tag{2.3.11}$$

$$0 = \frac{1}{\gamma_i}\rho_{-i}M_{-i} + \frac{1}{\gamma_i} w_{-i}' M_i + \rho_{-i} C_i \pi_{-i} - w_{-i}' l, \tag{2.3.12}$$

$$0 = h_i - w_i M_i - w_{-i} M_{-i}. \tag{2.3.13}$$

Plugging (2.3.11) into (2.3.12) implies that

$$0 = \left(\rho_{-i} - w_{-i}' w_i^{-1} w_{-i}\right)\left(M_{-i} + \gamma_i C_i \pi_{-i}\right). \tag{2.3.14}$$

Instead of discussing the uniqueness of solutions to the above equation, we pick out one of them $M_{-i} = -\gamma_i C_i \pi_{-i}$, $M_i = \gamma_i \hat{\phi}_i$, $l = \hat{\phi}_i$, and verify that the candidate strategy $\hat{\phi}_i$ can achieve the upper-bound corresponding to $M_{-i}$ and $M_i$, which verifies that $\hat{\phi}_i$ is indeed the maximizer of $\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$.

The upper bound corresponding to $M_{-i} = -\gamma_i C_i \pi_{-i}$ and $M_i = \gamma_i \hat{\phi}_i$ is

$$\exp\left((1-\gamma_i)\left(\left(M_i' w_{-i} + M_{-i}'\rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i} + \frac{1}{2\gamma_i} M_i' w_i M_i + \frac{1}{2\gamma_i} M_{-i}' \rho_{-i} M_{-i} + \frac{1}{\gamma_i} M_i' w_{-i} M_{-i}\right) t\right) \tag{2.3.15}$$

$$= \exp\left((1-\gamma_i)\left(\left(\gamma_i \hat{\phi}_i' w_{-i} - \gamma_i \pi_{-i}' C_i \rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i} + \frac{\gamma_i}{2}\left(\hat{\phi}_i' w_i \hat{\phi}_i + \pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} - 2\hat{\phi}_i' w_{-i} C_i \pi_{-i}\right)\right) t\right) \tag{2.3.16}$$

$$= \exp\left(\left(-(1-\gamma_i)\lambda_{-i}' C_i \pi_{-i} + \frac{(1-\gamma_i)\gamma_i}{2}\left(\hat{\phi}_i' w_i \hat{\phi}_i - \pi_{-i}' C_i \rho_{-i} C_i \pi_{-i}\right)\right) t\right). \tag{2.3.17}$$

On the other hand, for $\tilde{X}_i$ corresponding to $\hat{\phi}_i$,

$$\tilde{X}_{it} = \exp\left(\hat{\phi}_i'(h_i t + F_{it}) - \pi_{-i}' C_i(\lambda_{-i} t + W_{-it}) + \left(-\frac{1}{2}\hat{\phi}_i' w_i \hat{\phi}_i - \frac{1}{2}\pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} + \hat{\phi}_i' w_{-i} C_i \pi_{-i}\right) t\right).$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right] \\
&= \exp\Bigg((1-\gamma_i)\Bigg(\hat{\phi}_i' h_i - \pi_{-i}' C_i \lambda_{-i} - \frac{1}{2}\hat{\phi}_i' w_i \hat{\phi}_i - \frac{1}{2}\pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} + \hat{\phi}_i' w_{-i} C_i \pi_{-i} \\
&\quad + \frac{(1-\gamma_i)}{2}\hat{\phi}_i' w_i \hat{\phi}_i' + \frac{(1-\gamma_i)}{2}\pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} - (1-\gamma_i)\hat{\phi}_i' w_{-i} C_i \pi_{-i}\Bigg) t\Bigg) && (2.3.18) \\
&= \exp\left(\left(-(1-\gamma_i)\pi_{-i}' C_i \lambda_{-i} + \frac{\gamma_i(1-\gamma_i)}{2}\left(\hat{\phi}_i' w_i \hat{\phi}_i' - \pi_{-i}' C_i \rho_{-i} C_i \pi_{-i}\right)\right) t\right), && (2.3.19)
\end{aligned}$$

which coincides with the upper bound corresponding to $M_{-i} = -\gamma_i C_i \pi_{-i}$ and $M_i = \gamma_i \hat{\phi}_i$. $\square$

**Lemma 14.** *$P_f$ and $P_m$ are invertible.*

*Proof.* $P_f$ and $P_m$ can be rewritten as $P_f = A_1 P_{diag} P_1 P_{diag} A_2$, and $P_m = P_{diag}^2 A_1 P_2 A_2$, where $A_1$, $A_2$ and $\mathbf{P}_{diag}$ are $N \times N$ diagonal matrices with $(A_1)_{ii} = \frac{1}{N(1+\ell_i)+(N-1)\alpha_i}$, $(A_2)_{ii} = \alpha_i$ and $(P_{diag})_{ii} = \sqrt{1-\rho_{im}^2}$, and $P_1$ and $P_2$ are $N \times N$ matrices with

$$(P_1)_{ij} = \begin{cases} \frac{1}{c_{ii}} & \text{if } i = j, \\ -\frac{\rho_{ij}-\rho_{im}\rho_{jm}}{\sqrt{1-\rho_{im}^2}\sqrt{1-\rho_{jm}^2}} & \text{if } i \neq j, \end{cases} \quad (P_2)_{ij} = \begin{cases} \frac{1}{c_{ii}} & \text{if } i = j, \\ -1 & \text{if } i \neq j, \end{cases}. \tag{2.3.20}$$

On the other hand, for $i \neq j$, Brownian Motions $W_i$ and $W_j$ can be written as

$$W_{it} = \rho_{im} B_t + \sqrt{1-\rho_{im}^2} Z_{it}, \quad W_{jt} = \rho_{jm} B_t + \sqrt{1-\rho_{jm}^2} Z_{jt}, \tag{2.3.21}$$

where $Z_i, Z_j$ are Brownian motions independent of $B$. Suppose that $\langle Z_i, Z_j \rangle_t = \rho_{ij}^z t$, and then $\rho_{ij} = \rho_{im}\rho_{jm} + \sqrt{(1-\rho_{im}^2)(1-\rho_{jm}^2)}\rho_{ij}^z$, which implies that $\frac{(\rho_{ij}-\rho_{im}\rho_{jm})^2}{(1-\rho_{im}^2)(1-\rho_{jm}^2)} = (\rho_{ij}^z)^2 \leq 1$. Since $c_{ii} = \frac{\alpha_i}{(1+\ell_i)N+(N-1)\alpha_i} < \frac{1}{N-1}$, both $P_1$ and $P_2$ are strictly diagonally dominated matrices, and hence invertible. Therefore $P_f$ and $P_m$ are invertible, because the diagonal matrices $A_1$, $A_2$ and $P_{diag}$ are also invertible. □

*The proof of Proposition 9.* Following (2.2.5) and (2.2.13), with $\rho_{12} \in (-1, 1)$ and $\bar{\lambda} \leq 1$,

$$\frac{\pi_1^*}{\pi_1^M} = \frac{2\left(1 + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)}{(2(1+\ell)+\alpha)\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2 \rho_{12}^2\right)} = \frac{2(2(1+\ell)+\alpha)\left(1 + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)}{4\alpha(1+\ell)+\alpha^2(1-\rho_{12}^2)+4(1+\ell)^2} \tag{2.3.22}$$

$$\leq \frac{4(1+\ell)+4\alpha}{\alpha^2(1-\rho_{12}^2)+4\alpha(1+\ell)+4(1+\ell)^2} < 1. \tag{2.3.23}$$

$$\eta_1^* - \eta_1^M = \lambda_1 - \frac{\psi(2(1+\ell)+\alpha)\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2 \rho_{12}^2\right)}{2\lambda_{1,\gamma_1}\left(1 + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)} - \left(\lambda_1 - \frac{\psi}{\lambda_{1,\gamma_1}}\right) \tag{2.3.24}$$

$$\leq - \frac{\alpha\psi\left((2+4\ell+\alpha(1-\rho_{12}^2)) - 2\rho_{12}\bar{\lambda}\right)}{2(2(1+\ell)+\alpha)\lambda_{1,\gamma_1}\left(1 + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)} \tag{2.3.25}$$

$$< - \frac{\alpha\psi\left(2 - 2\bar{\lambda}\right)}{2(2(1+\ell)+\alpha)\lambda_{1,\gamma_1}\left(1 + \frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)} \leq 0. \tag{2.3.26}$$

On the other hand,

$$\frac{\pi_2^*}{\pi_2^M} - 1 = \frac{2\left(\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12} + \bar{\lambda}\right)}{\bar{\lambda}(2(1+\ell)+\alpha)\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2 \rho_{12}^2\right)} - 1 = \frac{\frac{2\alpha\rho_{12}}{2(1+\ell)+\alpha} + \left(\alpha\left(\frac{\alpha\rho_{12}^2}{2(1+\ell)+\alpha} - 1\right) - 2\ell\right)\bar{\lambda}}{\bar{\lambda}(2(1+\ell)+\alpha)\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2 \rho_{12}^2\right)}. \tag{2.3.27}$$

Thus $\pi_2^* \leq \pi_2^M$ is equivalent to $\frac{2\alpha\rho_{12}}{2(1+\ell)+\alpha} + \left(\alpha\left(\frac{\alpha\rho_{12}^2}{2(1+\ell)+\alpha} - 1\right) - 2\ell\right)\bar{\lambda} \leq 0$, which, since $\frac{\alpha\rho_{12}^2}{2(1+\ell)+\alpha} < 1$, always holds for $\rho_{12} < 0$, or implies that $\bar{\lambda} \geq \frac{2\alpha\rho_{12}}{(2+4\ell)\alpha+\alpha^2(1-\rho_{12}^2)+4\ell(1+\ell)}$ for $\rho_{12} \geq 0$. Finally,

$$\eta_2^* - \eta_2^M = -\frac{\psi\left(\left(4\ell(1+\ell)+\alpha(4\ell+2)+\alpha^2(1-\rho_{12}^2)\right)\bar{\lambda} - 2\alpha\rho_{12}\right)}{2(2(1+\ell)+\alpha)\lambda_{2,\gamma_2}\left(\bar{\lambda} + \frac{\alpha\rho_{12}}{2(1+\ell)+\alpha}\right)}. \tag{2.3.28}$$

Thus, if $\rho_{12} \geq 0$, $\bar{\lambda} + \frac{\alpha\rho_{12}}{2(1+\ell)+\alpha} > 0$, and $\eta_2^* \leq \eta_2^M$ if and only if $\bar{\lambda} \geq \frac{2\alpha\rho_{12}}{(2+4\ell)\alpha+\alpha^2(1-\rho_{12}^2)+4\ell(1+\ell)}$. If $\rho_{12} < 0$, $\left(\left(4\ell(1+\ell)+\alpha(4\ell+2)+\alpha^2(1-\rho_{12}^2)\right)\bar{\lambda} - 2\alpha\rho_{12}\right) \geq 0$, and $\eta_2^* \leq \eta_2^M$ if and only if $\bar{\lambda} \geq -\frac{\alpha\rho_{12}}{2(1+\ell)+\alpha}$. $\square$

*The proof of Proposition 10.* Following (2.2.8) and (2.2.9),

$$\mathrm{Beta}_1^* - 1 = \frac{\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2\rho_{12}^2\right)(1-\bar{\lambda}^2)}{\left(1+\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)^2 + 2\rho_{12}\left(1+\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)\left(\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}+\bar{\lambda}\right) + \left(\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}+\bar{\lambda}\right)^2} \geq 0, \tag{2.3.29}$$

$$\mathrm{Beta}_1^M - 1 = \frac{1-\bar{\lambda}^2}{1+2\rho_{12}\bar{\lambda}^2+\bar{\lambda}^2} \geq 0. \tag{2.3.30}$$

Therefore if $\bar{\lambda} = 1$, $\mathrm{Beta}_1^* = \mathrm{Beta}_1^M = 1$. Otherwise, since $\bar{\lambda} < 1$, the sign of $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| = \mathrm{Beta}_1^* - \mathrm{Beta}_1^M$ is the same as that of

$$\begin{aligned}
&\left(1 - \left(\frac{\alpha}{2(1+\ell)+\alpha}\right)^2\rho_{12}^2\right)(1+2\rho_{12}\bar{\lambda}+\bar{\lambda}^2) - \left(1+\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)^2 \\
&\quad - 2\rho_{12}\left(1+\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}\bar{\lambda}\right)\left(\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}+\bar{\lambda}\right) - \left(\frac{\alpha}{2(1+\ell)+\alpha}\rho_{12}+\bar{\lambda}\right)^2 \\
&= -\frac{4\alpha(1+\ell+\alpha)}{(2(1+\ell)+\alpha)^2}\rho_{12}^2\left(1+\left(\frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}} + \frac{\alpha}{1+\ell+\alpha}\rho_{12}\right)\bar{\lambda}+\bar{\lambda}^2\right).
\end{aligned} \tag{2.3.31}$$

If $\rho_{12} \geq 0$, $1+\left(\frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}} + \frac{\alpha}{1+\ell+\alpha}\rho_{12}\right)\bar{\lambda}+\bar{\lambda}^2 \geq 0$, and hence $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| \leq 0$. If $\rho_{12} < 0$, $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| \leq 0$ is equivalent to

$$1+\left(\frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}} + \frac{\alpha}{1+\ell+\alpha}\rho_{12}\right)\bar{\lambda}+\bar{\lambda}^2 \geq 0. \tag{2.3.32}$$

Since $\frac{\alpha}{1+\ell+\alpha}\rho_{12} + \frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}}$ is negative and is decreasing in $\rho_{12} \in [-1, 0)$, the maximum value at $\rho_{12} = -1$ is $-2$, and $\Delta \geq 0$. Since $\lambda \leq 1$, the two roots of the left hand side of (2.3.32) are

$$\frac{-\left(\frac{\alpha}{1+\ell+\alpha}\rho_{12} + \frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}}\right)+\sqrt{\Delta}}{2} \geq 1 \text{ and } 0 \leq \frac{-\left(\frac{\alpha}{1+\ell+\alpha}\rho_{12} + \frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}}\right)-\sqrt{\Delta}}{2},$$

and $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| \leq 0$ if $\bar{\lambda} \leq \frac{-\left(\frac{\alpha}{1+\ell+\alpha}\rho_{12} + \frac{2(1+\ell)+\alpha}{1+\ell+\alpha}\frac{1}{\rho_{12}}\right)-\sqrt{\Delta}}{2}$, and otherwise the inequality is reversed.

On the other hand, algebraic calculations show that $\text{Beta}_2^* - 1 = -\left(\text{Beta}_1^* - 1\right) \leq 0$ and $\text{Beta}_2^M - 1 = -\left(\text{Beta}_1^M - 1\right) \leq 0$. Thus the difference between $\text{Beta}_2^*$, $\text{Beta}_2^M$ and $\text{Beta}_1^*$, $\text{Beta}_1^M$ is at the numerator, which is now $\bar{\lambda}^2 - 1$. Therefore, the sign of $|\text{Beta}_2^* - 1| - |\text{Beta}_2^M - 1| = \text{Beta}_1^* - \text{Beta}_1^M$ is the same as (2.3.31), and equivalent conditions for Fund 1 still hold. $\square$

*The proof of Proposition 11.* Let $\tilde{X}_{it} = \exp\left(-\left(r - \left(1 + \frac{N-1}{N}\alpha_i\right)\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^N \psi_j\right)t\right)X_{it}$. Then with $\zeta_{it} = \frac{N+(N-1)\alpha_i}{N}\theta_{it} - \frac{\alpha_i}{N}\sum_{j\neq i}^N \theta_{jt}$, $\tilde{X}_i$ follows

$$\frac{d\tilde{X}_{it}}{\tilde{X}_{it}} = \zeta_{it}(adt + bdB_t), \quad \tilde{X}_{i0} = X_{i0} = 1. \tag{2.3.33}$$

We first calculate the optimal $\zeta_i$ (or equivalently the optimal $\theta_i$) given $\theta_j$'s ($j \neq i$) of other funds. With $d\xi_t/\xi_t = -\lambda_m dB_t$ and $\xi_0 = 1$, $d(\xi_t\tilde{X}_{it}) = \xi_t\tilde{X}_{it}\left(\zeta_{it}b - \lambda_m\right)dB_t$, which is a non-negative local martingale, and thus a supermartingale. Then for $0 < \gamma_i < 1$ (the case of $\gamma_i > 1$ follows similarly), by Hölder's inequality, for any $\theta = (\theta_1, \ldots, \theta_N) \in \Theta^N$,

$$\mathbb{E}\left[\frac{\tilde{X}_{it}^{1-\gamma_i}}{1-\gamma_i}\right] = \frac{\mathbb{E}\left[\left(\xi_t\tilde{X}_{it}\right)^{1-\gamma_i}\xi_t^{\gamma_i-1}\right]}{1-\gamma_i} \leq \frac{\mathbb{E}\left[\xi_t\hat{X}_{it}\right]^{1-\gamma_i}\mathbb{E}\left[\xi_t^{-\frac{1-\gamma_i}{\gamma_i}}\right]^{\gamma_i}}{1-\gamma_i} \leq \frac{\exp\left(\frac{1-\gamma_i}{2\gamma_i}\lambda_m^2 t\right)}{1-\gamma_i}, \tag{2.3.34}$$

which gives an upper bound of $\mathbb{E}\left[\frac{\tilde{X}_{it}^{1-\gamma_i}}{1-\gamma_i}\right]$. On the other hand, with $\theta_{it} = \frac{N\left(\frac{\lambda_m}{\gamma_i b} + \frac{\alpha_i}{N}\sum\limits_{j\neq 1}^N \theta_{jt}\right)}{N+(N-1)\alpha_i}$, and thus $\zeta_{it} = \frac{\lambda_m}{\gamma_i b}$, $\mathbb{E}\left[\frac{1}{1-\gamma_i}\tilde{X}_{it}^{1-\gamma_i}\right] = \frac{\exp\left(\frac{1-\gamma_i}{2\gamma_i}\lambda_m^2 t\right)}{1-\gamma_i}$, which indicates that $\zeta_i$ is the maximizer of $\mathbb{E}\left[\frac{\tilde{X}_{it}^{1-\gamma_i}}{1-\gamma_i}\right]$. Since $\zeta_{it} = \frac{\lambda_m}{\gamma_i b}$ is a constant strategy, independent of $t$, it also maximizes manager $i$'s expected utility

$$\int_0^T e^{\left(-\beta_i + r - \left(1+\frac{N-1}{N}\alpha_i\right)\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^N \psi_j\right)t}\psi_i^{1-\gamma_i}\frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i}dt,$$

and the optimal strategy given $\theta_j$'s ($j \neq i$) is $\theta_{it} = \frac{N\left(\frac{\lambda_m}{\gamma_i b} + \frac{\alpha_i}{N}\sum\limits_{j\neq i}^N \theta_{jt}\right)}{N+(N-1)\alpha_i}$.

To find the equilibrium, it suffices to solve the system of $N$ equations, each representing the optimal strategy given the portfolio of other funds: $P_I\theta_t = \frac{\lambda_m}{b}\gamma^{-1}e$, where $\theta_t = (\theta_{1t}, \cdots, \theta_{Nt})'$, $e$ is the $N$-dimensional vector with all entries equal to 1, $P_I$ the $N \times N$ matrix with $(P_I)_{i,j} = \begin{cases} \frac{N+(N-1)\alpha_i}{N} & \text{if } i = j \\ -\frac{\alpha_i}{N} & \text{if } i \neq j, \end{cases}$ and $\gamma$ is defined in Theorem 8. Since $P_I$ is strictly diagonally dominated and thus invertible, there exists a unique solution $\theta^* = \frac{\lambda_m}{b}P_I^{-1}\gamma^{-1}e$ for every $0 \leq t \leq T$. Furthermore, since $P_I = -\frac{1}{N}A_2(D + ee')$, where $A_2$ is the diagonal matrix defined in the proof of Lemma 14, $D$ is an $N \times N$ diagonal matrix with $(D)_{ii} = -\frac{1}{c_{ii}} - 1$, $1 \leq i \leq N$, by Sherman-Morrison-Woodbury formula (See equation 2.1.4 in Golub and Van Loan (1996)),

$$(P_I^{-1})_{i,j} = \begin{cases} \frac{1}{1+\alpha_i} + \frac{1+\bar{\alpha}}{N}\frac{\alpha_i}{(1+\alpha_i)^2} & , i = j \\ \frac{1+\bar{\alpha}}{N}\frac{\alpha_i}{(1+\alpha_i)(1+\alpha_j)} & , i \neq j \end{cases},$$

and this solution reduces to $\theta_i^* = \frac{\lambda_m}{b}\left(\frac{1}{1+\alpha_i}\frac{1}{\gamma_i} + \frac{\alpha_i}{1+\alpha_i}\frac{1}{\bar{\gamma}}\right)$ for $1 \leq i \leq N$. $\square$

*The proof of Proposition 12.* (i) Both $\theta_i^*$ and $\theta_i^M$ are proportion to $\frac{\lambda_m}{b}$, while the coefficient for $\theta_i^M$ is $\frac{1}{\gamma_i}$ and that for $\theta_i^*$ is a convex combination between $\frac{1}{\gamma_i}$ and $\frac{1}{\bar{\gamma}}$. The claim follows by the comparing the two coefficients. (ii) Since $\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{1+\alpha_i}\frac{1}{\gamma_i}+\frac{\alpha_i}{1+\alpha_i}\frac{1}{\bar{\gamma}}\right)=\frac{1}{\bar{\gamma}}$, $\bar{\theta}^*=\frac{\lambda_m}{b\bar{\gamma}}$, and

$$\begin{aligned}\bar{\theta}^*-\bar{\theta}^M &= \frac{\lambda_m}{b}\left(\frac{1}{\bar{\gamma}}-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma_i}\right)=\frac{\lambda_m}{b}\left(\frac{1+\bar{\alpha}}{N}\sum_{i=1}^{N}\frac{1}{(1+\alpha_i)\gamma_i}-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma_i}\right)\\ &= \frac{\lambda_m}{b}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1+\bar{\alpha}}{1+\alpha_i}-1\right)\frac{1}{\gamma_i}. \end{aligned}\tag{2.3.35}$$

Since $(\gamma_i-\gamma_j)(\alpha_i-\alpha_j)\geq 0$ for every pair of $i$ and $j$, $\left(\frac{1+\bar{\alpha}}{1+\alpha_i}-1\right)$'s and $\frac{1}{\gamma_i}$'s are similarly ordered, and from Tchebychef's inequality (Hardy et al. 1952, 2.17.1), the above is greater than or equal to

$$\frac{\lambda_m}{b}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1+\bar{\alpha}}{1+\alpha_i}-1\right)\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma_i}=0,\tag{2.3.36}$$

and the inequality is reversed if $(\gamma_i-\gamma_j)(\alpha_i-\alpha_j)\leq 0$ for every pair of $i$ and $j$.

(iii) $\bar{\theta}^*=\bar{\theta}^M$ follows from (ii). Furthermore, since $\frac{1}{\bar{\gamma}}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma_i}$,

$$\begin{aligned}\theta_i^*-\bar{\theta}^* =&\frac{\lambda_m}{b}\left(\frac{1}{1+\alpha}\frac{1}{\gamma_i}+\sum_{j=1}^{N}\left(\frac{\alpha}{N(1+\alpha)}\frac{1}{\gamma_j}\right)-\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\gamma_j}\right)=\frac{\lambda_m}{b}\left(\frac{1}{1+\alpha}\frac{1}{\gamma_i}-\frac{1}{1+\alpha}\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\gamma_j}\right)\\ =&\frac{1}{1+\alpha}\frac{\lambda_m}{b}\left(\frac{1}{\gamma_i}-\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\gamma_j}\right)=\frac{1}{1+\alpha}\left(\theta_i^M-\bar{\theta}^M\right). \quad \square\end{aligned}\tag{2.3.37}$$

# Chapter 3

# Model II: Optimal Capital Gain Tax

## 3.1 Model

### 3.1.1 Optimization Problems

First, we describe the optimization problem for the managers and the policy makers before giving the details of the model. We assume that the policy makers can decide the capital gain tax rate $\tau_c \in [0, 1]$ at time 0. The representative fund manager's problem is, given the $\tau_c$, choosing the optimal portfolio $\pi \in \mathcal{A}$ to maximize the terminal utility,

$$J_m(\pi; \tau_c) = \mathbb{E}\left[u_0(A_T)\right],$$

where $\mathcal{A}$ is the admissible set for $\pi$, $u(\cdot)$ is manager's utility function, and $A_T$ is the terminal after-tax wealth, which will be discussed in more details below. We assume that the utility function is of constant relative risk aversion type with risk aversion $\gamma > 0 (\neq 1)$, $u_0(x) = \frac{1}{1-\gamma} x^{1-\gamma}$. $A_T$ consists of the performance fees given by the investors and the capital gain taxes deducted at the terminal.

Meanwhile, since investors may not be in the country, policy makers want to maximize the tax incomes from the funds – both the income tax and the capital gain tax with rate $\tau_c$. Since income tax rate $\tau_o$ is related to many other aspects of the society, like employment rate, and is hard to alter in short time, $\tau_o$ is regarded as constant in our model. Therefore, policy makers face a mean variance problem, which maximizes the tax incomes collected from time 0 to $T$ by choosing the optimal $\tau_c \in [0, 1]$,

$$J_p(\tau_c) = \mathbb{E}\left[N_T\right] - \frac{1}{2} w \mathrm{Var}\left[N_T\right], \tag{3.1.1}$$

where $N_T$ is the discounted cumulative collected taxes including both income taxes and capital gain taxes, and $w$ is the risk aversion of the policy makers.

### 3.1.2 Model Details

Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ which is generated by Brownian Motion $W$. Suppose that the fund allocates its asset between the risk-free asset $S^0$, which earns a constant rate of return $r$, and its own risky investment opportunity $S$, which is public information among market, and follows the dynamics

$$\frac{dS_t}{S_t} = (r + \mu)dt + \sigma dW_t, \tag{3.1.2}$$

with $\mu$ and $\sigma$ as the excess return and the volatility of the investment opportunity. Denote $\theta := \frac{\mu}{\sigma}$ and $\pi_t$ as the proportions of fund's assets invested in $S$ at time $t$, which is integrable with respect to $W$. Denote the collection of all such strategies as $\mathcal{A}$. The investors of the fund compensate the manager by management fees $\psi I_t$ and the performance fees $\alpha(I_T - I_0)_+$ at the end of the period if values in their accounts increase, where $\psi, \alpha > 0$ are constants representing the management and performance fee rates, and $I_t$ is the value at time $t$ in investors' accounts. Given $\pi \in \mathcal{A}$, the values in manager's own account $F_t$ and investors' account $I_t$ follow

$$dI_t = (r - \psi) I_t dt + \pi_t I_t(\mu dt + \sigma dW_t), \tag{3.1.3}$$

$$dF_t = \psi I_t (1 - \tau_o) dt + r F_t dt + \pi_t F_t (\mu dt + \sigma dW_t). \tag{3.1.4}$$

The terminal after-tax wealth $A_T$ and the cumulative tax collected $N_T$ can be represented as

$$\begin{aligned} A_T &= F_T + \alpha(I_T - I_0)_+ - \tau_c \left(F_T + \alpha(I_T - I_0)_+ - F_0\right)_+, \\ N_T &= \int_0^T e^{-rt}\tau_o \psi I_t dt + e^{-rT}\tau_c \left(F_T + \alpha(I_T - I_0)_+ - F_0\right)_+. \end{aligned}$$

Define

$$dZ_t = Z_t \left[(r - \psi)dt + \pi_t(\mu dt + \sigma dW_t)\right] \ , \ Z_0 = 1.$$

$Z_t$ represents the after-fee return rate of manager's portfolio since $I_t = I_0 Z_t$. Accordingly, $F_T = A Z_T$ and

$$A_T = \begin{cases} A Z_T & , \ Z_T \le \frac{D}{A} \\ (1 - \tau_c) A Z_T + \tau_c D & , \ \frac{D}{A} < Z_T \le 1 \ , \\ (1 - \tau_c)(A + B) Z_T + \tau_c D - (1 - \tau_c) B & , \ Z_T > 1 \end{cases} \tag{3.1.5}$$

where

$$A = F_0 e^{\psi T} + I_0 (1 - \tau_o) \left(e^{\psi T} - 1\right) \ , \ B = \alpha I_0 \ , \ D = F_0.$$

Note that the terminal after-tax wealth is a function of $Z_T$ alone, and therefore, manager's value function is Markovian under the return rate $Z$. Denote $u(Z_T) := u_0(A_T)$. For the following discussions, we will base our analyses on the process $Z$ and the according value function $V(t, Z)$,

$$V(t, Z) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[u(Z_T) \middle| Z_t = Z\right].$$

### 3.1.3 $N$-player Game

It is natural to extend the model to $N$ funds with one representative policy maker, and each fund has different risk aversion $\gamma_i$, performance fee rates $\alpha_i$, management fee rate $\psi_i$, and initial values for both investors' and managers' account $I_{i0}$ and $F_{i0}$. Due to the solvability of the model, we assume $\gamma_i > 1$ for $i = 1, \cdots, N$. This is actually a reasonable assumption due to the empirical results in Chetty (2006), Friend and Blume (1975), Szpiro (1986), Vissing-Jørgensen and Attanasio (2003). Dynamics for both processes are the same as (3.1.3) and (3.1.4) except that those parameters are related to player $i$. For the tractability of the model, we assume that all managers can invest in the risk-free asset $S^0$ and the common investment opportunity $S$, which shares the same dynamic as (3.1.2). The admissible set $\mathcal{A}$ thus remains the same as well. This reflects the case when competitions are among passive funds who track one common investment opportunity such as S&P 500.

Competitions are introduced among all managers by combining the relative performances in the terminal utility functions. For manager $i$, given $\pi_i \in \mathcal{A}$, define

$$dZ_{it} = Z_{it}\left[(r-\psi_i)dt + \pi_{it}(\mu dt + \sigma dW_t)\right] \ , \ Z_{i0} = 1.$$

Similarly, $Z_{iT}$ represents fund $i$'s after-fee return rate at terminal, so the relative performance for manager $i$ is

$$R_{iT} = \frac{Z_{-iT}}{Z_{iT}} \ , \ Z_{-iT} = \frac{1}{N-1}\sum_{j\neq i}^{N} Z_{jT}.$$

Now manager $i$'s utility function $u_{i0}$ is based on the terminal after-tax wealth $A_{iT}$ and the relative performance $R_{iT}$ with

$$u_{i0}(x,r) = \lambda_i \frac{1}{1-\gamma_i} x^{1-\gamma_i} - (1-\lambda_i)\zeta_i r, \tag{3.1.6}$$

with $\lambda_i \in [0,1]$ as the weight between the utility brought by the terminal wealth and the relative performance and $\zeta_i \geq 0$ as the sensitivity to the relative performance. Positiveness of $\zeta_i$ reflects the observation that better relative performance brings cash inflow to the fund as well as builds fund's reputations and gains further income, and vice versa. More precisely, $-\zeta_i r$ can be changed to $-\zeta_i(r-1)$, representing the gain and loss caused by the relative performance, but since $\zeta_i$ is a constant, it is equivalent to solve (3.1.6). Similarly, each manager chooses $\pi_i \in \mathcal{A}$ to maximize their terminal utility,

$$J_{im}(\pi_i, \pi_{-i}; \tau_c) = \mathbb{E}\left[u_{i0}(A_{iT}, R_{iT})\right],$$

where $\pi = \begin{bmatrix}\pi_1 & \cdots & \pi_N\end{bmatrix}$, $\pi_{-i}$ is the vector in $\mathbb{R}^{N-1}$ by removing the $i$th element from $\pi$, and $A_{iT}$ is defined as

$$A_{iT} = \begin{cases} A_i Z_{iT} & , \ Z_{iT} \leq \frac{D_i}{A_i} \\ (1-\tau_c)A_i Z_{iT} + \tau_c D_i & , \ \frac{D_i}{A_i} < Z_{iT} \leq 1 \\ (1-\tau_c)(A_i+B_i)Z_{iT} + \tau_c D_i - (1-\tau_c)B_i & , \ Z_{iT} > 1 \end{cases},$$

where

$$A_i = F_{i0}e^{\psi_i T} + I_{i0}(1-\tau_o)\left(e^{\psi_i T} - 1\right) \ , \ B_i = \alpha_i I_{i0} \ , \ D_i = F_{i0}.$$

Note that $A_{iT}$ and $R_{iT}$ can be represented as functions of $Z_{iT}$ and $Z_{-iT}$, and thus we define

$$u_i(Z_{iT}, Z_{-iT}) := u_{i0}(A_{iT}, R_{iT}),$$

which is used in the following discussions instead of $u_{i0}$.

We will discuss the Pareto optimal Nash equilibrium among $N$ funds as defined in Definition 3. Note that in this Chapter, there is only one risky investment opportunity for managers, so

$\theta_i$ should be omitted in the definition of Pareto optimal Nash equilibrium. For the existence of Nash equilibrium, we need to introduce the weighted $L^2$ space $L^{2,\phi}$ with the norms $\|f\|_{\phi,2} = \left(\int_{\mathbb{R}} \|f(x)\|_2^2 \phi(x)dx\right)^{\frac{1}{2}}$, where $\phi$ is the one-dimensional normal density function with mean 0 and variance $T$ and $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^2$. Note that the space is complete. Denote $L^{2,\phi,+}$ as the subset of $L^{2,\phi}$ where functions are non-negative.

In the extension of $N$-player game, policy maker faces the same problem but $N_t$ becomes the sum of all managers' taxes collected,

$$N_T = \sum_{i=1}^{N} \int_0^T e^{-rt}\tau_o \psi_i I_{it} dt + e^{-rT}\tau_c \left(F_{iT} + \alpha_i (I_{iT} - I_{i0})_+ - F_{i0}\right)_+ .$$

## 3.2 Main Results

In this section we will present the main results of this paper and discuss their implications. The following theorems show that there exists an optimal control $\hat{\pi}$ for one representative manager or a unique Pareto optimal Nash Equilibrium for $N$-player game and an optimal $\tau_c$ for the policy makers. There exist multiple equilibria if competitions are introduced. The equilibria are totally ordered, both element-wise and in the sense of the value of utility functions, and thus the Pareto optimal Nash equilibrium can be found.

### 3.2.1 Funds' and Policy Makers' Problems

**Theorem 15.** *Given the capital gain tax rate $\tau_c \in [0, 1]$,*

*(i) One-player model: the optimal strategy $\hat{Z}_T$ for the manager is*

$$\hat{Z}_T = \hat{x}(\hat{\eta} e^{\psi T} \xi_T),$$

*where $\xi_t$ is the unique stochastic discount factor $\xi_t = \exp\left(-\left(r + \frac{1}{2}\theta^2\right)t - \theta W_t\right)$ and the unique $\hat{\eta}$ solves $\mathbb{E}\left[e^{\psi T} \xi_T \hat{Z}_T\right] = 1$. Moreover, the according optimal portfolio strategy can be described as*

$$\hat{\pi}_t = \frac{\hat{l}_x(t, W_t + \theta t)}{\sigma \hat{l}(t, W_t + \theta t)},$$

*where $\hat{x}(\cdot)$ is defined in (3.2.1)-(3.2.3) in section 3.2.3.1 and*

$$\hat{l}(t, x) = \frac{e^{-(r-\psi)(T-t)}}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} e^{-\frac{(y-x)^2}{2(T-t)}} \hat{x}\left(\hat{\eta} e^{\psi T} e^{-\left(r+\frac{1}{2}\theta^2\right)T - \theta(y - \theta T)}\right) dy.$$

*(ii) $N$-player game with $\min_i \gamma_i > 1$: the unique Pareto optimal Nash equilibrium for the optimal strategy $\hat{\pi}$ can be described as*

$$\hat{\pi}_{it} = \frac{(\hat{l}_i)_x(t, W_t + \theta t)}{\sigma \hat{l}_i(t, W_t + \theta t)},$$

*where $l_i$ is the feedback form of $Z_{iT}$ such that $Z_{iT} = l_i(\hat{\eta}, W_T)$, where $\hat{\eta}$ uniquely solves $\mathbb{E}\left[e^{\psi_i T}\xi_T Z_{iT}\right] = 1$ for $i = 1, \cdots, N$ and*

$$\hat{l}_i(t,x) = \frac{e^{-(r-\psi_i)(T-t)}}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} e^{-\frac{(y-x)^2}{2(T-t)}} l_i(\hat{\eta}, y - \theta T) dy.$$

The optimal $\hat{\pi}$ for one player model is the result of the concavification, see the general theory in Bichuch and Sturm (2014) and Seifried (2010). Since $u(A_T)$ with (3.1.5) is a piece-wise concave function of $Z_T$, but has kinks at $\frac{D}{A}$ and 1, the dual of $u(A_T)$ has different expressions depending on the relationship between $A$ and $(1-\tau_c)(A+B)$. Note that $\tau_c$ has two effects on managers' terminal wealth and according optimal portfolio. On one side, larger $\tau_c$ means less managers' terminal wealth, and managers' strategies are more cautious. For example, in the extreme case when $\tau_c = 1$, there is a hard stop for managers' wealth at $D = F_0$, and a hard stop of optimal $Z_T$ at 1. In this case, managers do not try to earn money for both themselves and the investors. On the contrary, they are aiming for losing certain amount of money and maximize their incomes from the management fees. On the other hand, larger $\tau_c$ may stimulate managers to be more aggressive to maintain similar earnings compared to the lower $\tau_c$ case. For example, when $\tau_c = 0$, managers' wealth increases more quickly when they try to make profits than when they lose money according to (3.1.5), and thus their strategies become aggressive to have a better position in the terminal wealth. Those two effects are summarized as the substitution and income effect in Feldstein and Yitzhaki (1978), Balcer and Judd (1987).

For $N$-player game, we solve the fixed point problem for $l = Pl$ where $\hat{Z}_{iT} = l_i(\hat{\eta}, W_T)$. The details will be discussed in the section 3.2.3.2. Note that for fixed $\omega \in \Omega$ and $\eta \in \mathbb{R}^N$, $Z_{iT}(\omega)$ (or $l_i(\eta, x)$) forms a system of non-linear equations in $\mathbb{R}^N$ through their reaction functions, and the solution of Nash equilibrium can be constructed through each point. Hence, the existence of multiple intersections of the reaction functions for each point gives the possibility of multiple equilibria for $\hat{Z}_T$ as well.

Consider one example of two players with $\lambda_1 = 0$ and $\tau_c = 0$ for simplicity. The utility functions are

$$u_1(x_1, x_2) = -\frac{x_2}{x_1}\ ,\ u_2(x_1, x_2) = -\frac{1}{10}\left(\frac{9}{A_2} + \frac{x_1}{x_2}\right),$$

where $A_2 = \begin{cases} x_2 & , x_2 \le 1, \\ 2x_2 - 1 & , x_2 > 1. \end{cases}$ Define the dual of two functions as

$$\hat{u}_1(x_2, y) = \sup_{x_1 \ge 0} u_1(x_1, x_2) - x_1 y\ ,\ \hat{u}_2(x_1, y) = \sup_{x_2 \ge 0} u_2(x_1, x_2) - x_2 y,$$

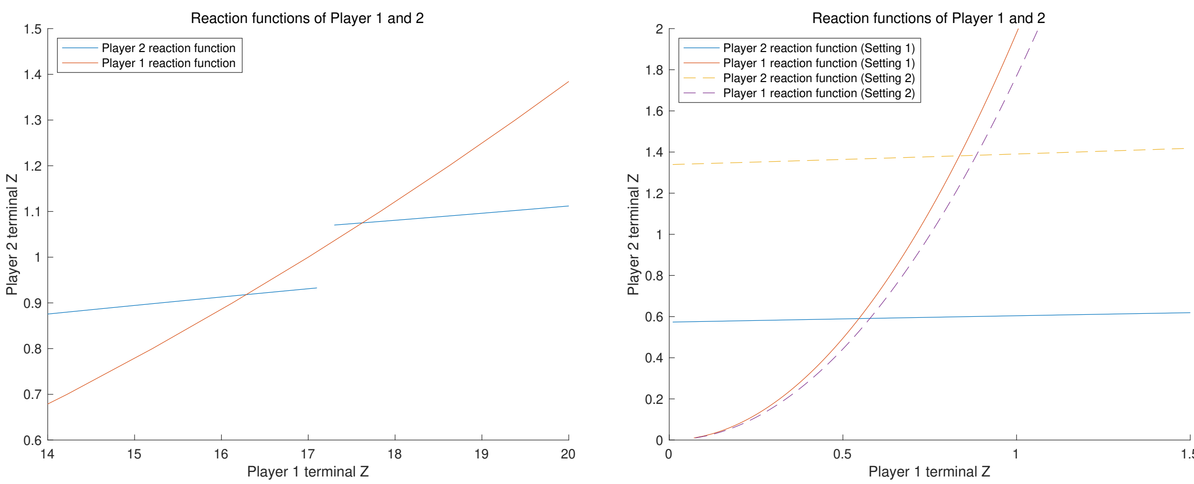
The optimal reaction function for player 1 is $x_1^* = \sqrt{\frac{x_2}{y}}$, and the optimal reaction function for player 2 is $x_2^*$, which is determined by the maximum of the following two functions and the according optimal $x_2$:

$$\sup_{x_2 \le 1} u_2(x_1, x_2) - x_2 y = \begin{cases} -2\sqrt{y}\sqrt{\frac{9+x_1}{10}} & , y > \frac{9+x_1}{10}, \\ -\frac{9+x_1}{10} - y & , y \le \frac{9+x_1}{10}, \end{cases}\ ,\ x_2 = \begin{cases} \sqrt{\frac{9+x_1}{10y}} & , y > \frac{9+x_1}{10}, \\ 1 & , y \le \frac{9+x_1}{10}, \end{cases},$$

$$\sup_{x_2 > 1} u_2(x_1, x_2) - x_2 y = \begin{cases} -\frac{1}{10}\left(\frac{9}{2x_{20}-1} + \frac{x_1}{x_{20}}\right) - x_{20} y & , y \le \frac{18+x_1}{10}, \\ -\frac{9+x_1}{10} - y & , y > \frac{18+x_1}{10}, \end{cases}\ ,\ x_2 = \begin{cases} x_{20} & , y \le \frac{18+x_1}{10}, \\ 1 & , y > \frac{18+x_1}{10}, \end{cases},$$

where $x_{20}$ solves $\frac{1}{10}\left(\frac{18}{(2x_{20}-1)^2}+\frac{x_1}{x_{20}^2}\right)=y$. Since $y$ for each equation equals to $\eta_i e^{\psi T}\xi_T$ for the final reaction function, $y$'s are different for each player as long as $\eta_i$'s are different. Here, suppose for fixed $W_T$ value, $\eta_1 e^{\psi T}\xi_T = 3$ and $\eta_2 e^{\psi T}\xi_T = 0.1$, and the reaction functions are represented in Figure 3.1 (a). This shows the possible multi fixed points for the pointwise problem with fixed $\eta$.

We show in section 3.2.3.2 that the multiple Nash equilibria are totally ordered, in the sense of both the value function and element-wise comparisons. When $\gamma_i > 1$, after we substitute $Z_{iT}$ as the reaction function of $Z_{-iT}$, the utility function of player $i$ is a non-increasing function of $Z_{-iT}$, and therefore smaller $Z_T$ makes everyone's utility larger. Hence, the Pareto optimal strategy is the smallest fixed point of $N$ reaction functions. Figure 3.1 (b) shows the possible multi fixed points of the reaction functions for different $\eta$'s, which shows the element-wise order of two equilibria. The smaller one is the Pareto optimal Nash equilibrium. Note that the $\eta$'s are not given, but calculated through the budget constraints $\mathbb{E}\left[e^{\psi T}\xi_T Z_{iT}\right]=1$ for all $i=1,\cdots,N$.




(a) Reaction functions with $y_1 = 1/17^2$ and $y_2 = 3$. (b) Reaction functions with $\eta$'s satisfying the budget constraints.

Figure 3.1: Reaction functions for both players. (a) shows the possible multiple fixed points for some fixed $\eta$'s (b) shows two different equilibrium points, where both sets of $\eta$'s satisfy the budget constraints. Setting 1 gives $\eta_1 = 0.9897, \eta_2 = 1.3699$; and Setting 2 gives $\eta_1 = 0.8852; \eta_2 = 0.3197$. The left intersection point is the Pareto optimal one. $r = 0.05$, $\mu = 0.15$, $\sigma = 0.2$, $\psi_1 = \psi_2 = 0.02$, $T = 1$.

Given the optimal portfolio(s) for both 1-player model and $N$-player games, $J_p(\tau_c)$ is a continuous function of $\tau_c$. Since $\tau_c \in [0,1]$, there must exist a maximum point in the interval. Therefore, the following theorem can be concluded.

**Theorem 16.** *Given the optimal $\hat{\pi}$ defined in Theorem 15, there exists an optimal $\tau_c \in [0,1]$ such that $J_p$ achieves its maximum.*

As discussed above, $\tau_c$ has a negative effect on managers' strategies, which brings down the tax basis since managers are reluctant to gain more profits. However, higher tax rate $\tau_c$ also means potential higher total tax income. The two effects have the opposite effect on policy makers' decisions of optimal tax rate. It is hard to get the exact value of the optimal $\tau_c$ due to the complexity of $J_p$ as a function of $\tau_c$. We will discuss in section 3.3 how the optimal $\tau_c$ changes numerically according to different risk aversions of managers and policy makers.

### 3.2.2 Sensitivity Analysis

In reality, risk aversions and sensitivity coefficients are hard to estimate, especially for the $N$-player game. Estimating other managers' risk aversions and sensitivity coefficients could have potential errors, which bring biases in the Pareto optimal Nash equilibrium. In this section, we will show that the deviations from parameters do not cause huge changes in the optimal strategy, or in other words, the equilibrium is stable.

**Theorem 17.** *For $N$-player games with $\min_i \gamma_i > 1$, consider the parameter triple $(\gamma, \zeta, \lambda)$ and its perturbation $(\gamma + \Delta\gamma, \zeta + \Delta\zeta, \lambda + \Delta\lambda)$ where each element is an $N$-dimensional vector. Denote the according unique Pareto optimal Nash equilibrium as $\hat{\pi}$ and $\hat{\pi}_\Delta$ and $\Delta = \max\{\|\Delta\gamma\|_2, \|\Delta\zeta\|_2, \|\Delta\lambda\|_2\}$. Then, there exists a function $K$ depending on $t$, $W_t$, $\gamma$, $\zeta$, $\lambda$ such that*

$$\|\hat{\pi}_t - \hat{\pi}_{\Delta t}\|_2 \leq K(t, W_t, \gamma, \zeta, \lambda)\Delta|\ln \Delta|.$$

The detailed proof is shown in Appendix 3.4.3. Based on the result of Theorem 17, the optimal portfolios for $N$-player games are useful even if managers can only approximate the other managers' parameters. As long as the approximation is not too far away, the optimal strategy is stable and applicable. Meanwhile, the difference between two equilibria is measured pointwisely, and thus, $K$ is function of $W_t$. As shown in the proof of Theorem 17, $K$ will explode when $W_t$ achieves both positive and negative extreme value, but it is of order $\mathcal{O}(W_t^2)$. Combining with the density of Brownian motion at time $t$, we can conclude that the difference between the optimal $\hat{\pi}$'s in $L^2(\mathbb{P})$ is also of order $\mathcal{O}(\Delta|\ln \Delta|)$.

Meanwhile, the $|\ln \Delta|$ term actually comes from the perturbation of $\gamma$. Hence, the optimal strategies are more sensitive to the $\gamma$. In practice, this gives managers a hint that approximation of $\gamma$ is more important than the other two parameters, although it is hard to estimate people's risk aversion in general. This also indicates that for the policy makers, they need to take the upper bound of the perturbations into consideration when deciding the optimal tax rate.

### 3.2.3 Proof of Theorem 15

We will mainly use the concavification technique to deal with the non-concave property of the utility function on $Z_T$. Since our models only depend on the terminal wealth, the first step is to find the optimal terminal wealth $Z_T$ or the Pareto optimal Nash equilibrium $Z_{iT}$ in $N$-player game. Then based on the martingale property of $Z_T$ ($Z_{iT}$), the optimal portfolio can be constructed. Note that the Pareto optimality of $Z_{iT}$ is in the path-wise sense: for each $\omega \in \Omega$, the Pareto optimal $\hat{Z}_{iT}(\omega) \geq \tilde{Z}_{iT}(\omega)$ for any Nash equilibrium $\tilde{Z}_{iT}$ and $i = 1, \cdots, N$.

#### 3.2.3.1 One-player model

Define $\hat{x}(y)$ as

(i) If $y_0 \le (1-\tau_c)AD^{-\gamma}$,

$$\hat{x}(y) = \begin{cases} \left(\frac{1}{(1-\tau_c)(A+B)}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} + \frac{B}{A+B} - \frac{\tau_c}{1-\tau_c}\frac{D}{A+B} & , y < y_0, \\ \left(\frac{1}{(1-\tau_c)A}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} - \frac{\tau_c}{1-\tau_c}\frac{D}{A} & , y_0 \le y < (1-\tau_c)AD^{-\gamma}, \\ \frac{D}{A} & , (1-\tau_c)AD^{-\gamma} \le y < AD^{-\gamma}, \\ \left(\frac{1}{A}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} & , y \ge AD^{-\gamma}, \end{cases} \tag{3.2.1}$$

(ii) If the condition in i) is not satisfied and $y_1 \le AD^{-\gamma}$ or $(1-\tau_c)AD^{-\gamma} < (1-\tau_c)(A+B)\left[(1-\tau_c)A+\tau_c D\right]^{-\gamma} \le AD^{-\gamma}$,

$$\hat{x}(y) = \begin{cases} \left(\frac{1}{(1-\tau_c)(A+B)}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} + \frac{B}{A+B} - \frac{\tau_c}{1-\tau_c}\frac{D}{A+B} & , y < y_1, \\ \frac{D}{A} & , y_1 \le y < AD^{-\gamma}, \\ \left(\frac{1}{A}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} & , y \ge AD^{-\gamma}, \end{cases} \tag{3.2.2}$$

(iii) If conditions in i) and ii) are not satisfied,

$$\hat{x}(y) = \begin{cases} \left(\frac{1}{(1-\tau_c)(A+B)}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} + \frac{B}{A+B} - \frac{\tau_c}{1-\tau_c}\frac{D}{A+B} & , y < y_2, \\ \left(\frac{1}{A}\right)^{-\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} & , y \ge y_2, \end{cases} \tag{3.2.3}$$

where

$$y_0 = (1-\tau_c)[(1-\tau_c)A+\tau_c D]^{-\gamma} \left[\frac{-\frac{\gamma}{1-\gamma}\left(\left(\frac{1}{A}\right)^{-\frac{1-\gamma}{\gamma}} - \left(\frac{1}{A+B}\right)^{-\frac{1-\gamma}{\gamma}}\right)}{\frac{1}{A} - \frac{1}{A+B}}\right]^{\gamma}, \tag{3.2.4}$$

$$y_2 = A^{-\gamma}\left(\frac{(1-\tau_c)B-\tau_c A}{(1-\tau_c)B-\tau_c D}\right)^{\gamma} \left[\frac{-\frac{\gamma}{1-\gamma}\left(\left(\frac{1}{A}\right)^{-\frac{1-\gamma}{\gamma}} - \left(\frac{1}{(1-\tau_c)(A+B)}\right)^{-\frac{1-\gamma}{\gamma}}\right)}{\frac{1}{A} - \frac{1}{(1-\tau_c)(A+B)}}\right]^{\gamma}, \tag{3.2.5}$$

and $y_1$ uniquely solves

$$\frac{1}{1-\gamma}D^{1-\gamma} - y\frac{D}{A} = \frac{\gamma}{1-\gamma}\left(\frac{y}{(A+B)(1-\tau_c)}\right)^{-\frac{1-\gamma}{\gamma}} + \left(\frac{\tau_c}{1-\tau_c}\frac{D}{A+B} - \frac{B}{A+B}\right)y. \tag{3.2.6}$$

*Proof of Theorem 15 (i):* First, let us show the existence and uniqueness of the $\hat{\eta}$. The uniqueness is obvious since $\hat{Z}_T$ is non-increasing function of $\hat{\eta}$, and so is $\mathbb{E}\left[e^{\psi T}\xi_T\hat{Z}_T\right]$. For the existence, in either case, when $\hat{\eta}$ goes to 0, $\mathbb{E}\left[e^{\psi T}\xi_T\hat{Z}_T\right]$ goes to infinite, and vice versa. Combining the above argument, the equation has a unique solution $\hat{\eta}$.

Under each condition, as shown in Lemma 21, the $\hat{x}$ achieving $\hat{u}(y) = \sup_x u(x) - xy$ is (3.2.1) - (3.2.3). Meanwhile, since $d(e^{\psi T}\xi_t Z_t) = e^{\psi T}\xi_t Z_t(\pi_t\sigma - \theta)dW_t$, and $e^{\psi T}\xi_t Z_t \geq 0$, it is a supermartingale. Thus,

$$\mathbb{E}\left[e^{\psi T}\xi_T Z_T\right] \leq 1,$$

and $\mathbb{E}\left[\int_0^T e^{\psi T}\xi_t Z_t(\pi_t\sigma - \theta)dW_t\right] \leq 0$. Define

$$\Lambda(\eta\xi_T) = \sup_{Z_T \geq 0} \mathbb{E}\left[u(Z_T) - \eta e^{\psi T}\xi_T Z_T + \eta + \int_0^T \eta e^{\psi T}\xi_t Z_t(\pi_t\sigma - \theta)dW_t\right] \leq \eta + \mathbb{E}\left[\hat{u}\left(\eta e^{\psi T}\xi_T\right)\right], \tag{3.2.7}$$

due to the conclusions in Lemma 21. Thus, $J_m \leq \inf_{\eta\geq 0}\Lambda(\eta\xi_T) \leq \Lambda(\hat{\eta}\xi_T) \leq \hat{\eta} + \mathbb{E}\left[\hat{u}(\hat{\eta}\xi_T)\right]$, and the equality in (3.2.7) is achieved with $\eta$ changed to $\hat{\eta}$ when $Z_T$ arrives at $\hat{Z}_T = \hat{x}\left(\hat{\eta}e^{\psi T}\xi_T\right)$.

After changing to the risk-neutral measure, where $dW_t^{\mathbb{Q}} = dW_t + \theta dt$, we want to show that $\hat{Z}_t = \hat{l}(t, W_t^{\mathbb{Q}})$. Since $d\hat{Z}_t/\hat{Z}_t = (r-\psi)dt + \hat{\pi}_t(\mu dt + \sigma dW_t)$, with $\hat{\pi}_t = \frac{\hat{l}_x(t,W_t^{\mathbb{Q}})}{\sigma\hat{l}(t,W_t^{\mathbb{Q}})}$,

$$d\ln\left(e^{-(r-\psi)t}\hat{Z}_t\right) = -\frac{1}{2}\left(\frac{\hat{l}_x}{\hat{l}}\right)^2 dt + \frac{\hat{l}_x}{\hat{l}}dW_t^{\mathbb{Q}}.$$

At the same time, denote $l(t,x) := e^{(r-\psi)(T-t)}\hat{l}(t,x)$. Due to the fact that $\partial_t l + \frac{1}{2}\partial_{xx} l = 0$,

$$d\ln\left(l(t,W_t^{\mathbb{Q}})\right) = \left(\frac{1}{l}\frac{\partial l}{\partial t} + \frac{1}{l}\frac{1}{2}\frac{\partial^2 l}{\partial x^2} - \frac{1}{2}\frac{l_x^2}{l^2}\right)dt + \frac{l_x}{l}dW_t^{\mathbb{Q}} = -\frac{1}{2}\left(\frac{\hat{l}_x}{\hat{l}}\right)^2 dt + \frac{\hat{l}_x}{\hat{l}}dW_t^{\mathbb{Q}},$$

since $l_x/l = \hat{l}_x/\hat{l}$. Combining with the fact that $\hat{Z}_T = \hat{l}\left(T, W_T^{\mathbb{Q}}\right)$, we can conclude that $\hat{Z}_t$ and $\hat{l}(t,W_t^{\mathbb{Q}})$ are modifications of each other, which verifies the optimality of $\hat{\pi}_t$. ☐

#### 3.2.3.2 *N*-player Game

Define $\hat{u}_i(y,z) := \sup_x u_i(x,z) - xy$ for fixed $z$. Denote

$$\hat{u}_{i1}(y,z) = \sup_{x\leq\frac{D_i}{A_i}} u_i(x,z) - xy \ , \ \hat{u}_{i2}(y,z) = \sup_{\frac{D_i}{A_i}<x\leq 1} u_i(x,z) - xy \ , \ \hat{u}_{i3}(y,z) = \sup_{x>1} u_i(x,z) - xy.$$

They can be explicitly calculated in the following expressions.

$$\hat{u}_{i1} = \begin{cases} \frac{\lambda_i}{1-\gamma_i}(A_i x_1^*)^{1-\gamma_i} - (1-\lambda_i)\zeta_i \frac{z}{x_1^*} - x_1^* y & , y \geq \lambda_i A_i D_i^{-\gamma_i} + (1-\lambda_i)\zeta_i \left(\frac{D_i}{A_i}\right)^2 z, \\ \frac{\lambda_i}{1-\gamma_i} D_i^{1-\gamma_i} - (1-\lambda_i)\zeta_i \frac{A_i}{D_i} z - \frac{D_i}{A_i} y & , \text{otherwise,} \end{cases}$$

$$\hat{u}_{i2} = \begin{cases} \frac{\lambda_i}{1-\gamma_i} D_i^{1-\gamma_i} - (1-\lambda_i)\zeta_i \frac{A_i}{D_i} z - \frac{D_i}{A_i} y & , y \geq \lambda_i(1-\tau_c) A_i D_i^{-\gamma_i} + (1-\lambda_i)\zeta_i \left(\frac{A_i}{D_i}\right)^2 z, \\ \frac{\lambda_i}{1-\gamma_i}\left((1-\tau_c)A_i x_2^* + \tau_c D_i\right)^{1-\gamma_i} - & , \lambda_i(1-\tau_c)A_i\left((1-\tau_c)A_i + \tau_c D_i\right)^{-\gamma_i} + (1-\lambda_i)\zeta_i z \\ (1-\lambda_i)\zeta_i \frac{z}{x_2^*} - x_2^* y & \leq y < \lambda_i(1-\tau_c)A_i D_i^{-\gamma_i} + (1-\lambda_i)\zeta_i \left(\frac{A_i}{D_i}\right)^2 z, \\ \frac{\lambda_i}{1-\gamma_i}\left((1-\tau_c)A_i + \tau_c D_i\right)^{1-\gamma_i} & , \text{otherwise,} \\ -(1-\lambda_i)\zeta_i z - y & \end{cases}$$

$$\hat{u}_{i3} = \begin{cases} \frac{\lambda_i}{1-\gamma_i}\left((1-\tau_c)A_i + \tau_c D_i\right)^{1-\gamma_i} - (1-\lambda_i)\zeta_i z - y & , y \geq \lambda_i(1-\tau_c)(A_i+B_i) \cdot \\ & [(1-\tau_c)A_i + \tau_c D_i]^{-\gamma_i} + (1-\lambda_i)\zeta_i z, \\ \frac{\lambda_i}{1-\gamma_i}\left((1-\tau_c)(A_i+B_i)x_3^* + \tau_c D_i - (1-\tau_c)B_i\right)^{1-\gamma_i} & , \text{otherwise,} \\ -(1-\lambda_i)\zeta_i \frac{z}{x_3^*} - x_3^* y & \end{cases}$$

where $x_i^*$ $(i = 1, 2, 3)$ are functions of $y$ and $z$ and uniquely solve

$$y = \lambda_i A_i^{1-\gamma_i}(x_1^*)^{-\gamma_i} + (1-\lambda_i)\zeta_i z (x_1^*)^{-2}, \tag{3.2.8}$$
$$y = (1-\tau_c)A_i\lambda_i\left((1-\tau_c)A_i x_2^* + \tau_c D_i\right)^{-\gamma_i} + (1-\lambda_i)\zeta_i z (x_2^*)^{-2}, \tag{3.2.9}$$
$$y = (1-\tau_c)(A_i+B_i)\lambda_i\left((1-\tau_c)(A_i+B_i)x_3^* + \tau_c D_i - (1-\tau_c)B_i\right)^{-\gamma_i} + (1-\lambda_i)\zeta_i z (x_3^*)^{-2}. \tag{3.2.10}$$

Denote $\hat{x}_{ik}$ (as a function of $y$ and $z$) as the optimizer of $\hat{u}_{ik}$:

$$\hat{x}_{i1} = \begin{cases} x_1^* & , y \geq \lambda_i A_i D_i^{-\gamma_i} + \\ & (1-\lambda_i)\zeta_i \left(\frac{D_i}{A_i}\right)^2 z, \\ \frac{D_i}{A_i} & , \text{otherwise,} \end{cases}, \quad \hat{x}_{i3} = \begin{cases} 1 & , y \geq \lambda_i(1-\tau_c)(A_i+B_i)((1-\tau_c)A_i + \tau_c D_i)^{-\gamma_i} + \\ & (1-\lambda_i)\zeta_i z, \\ x_3^* & , \text{otherwise,} \end{cases}$$

$$\hat{x}_{i2} = \begin{cases} \frac{D_i}{A_i} & , y \geq \lambda_i(1-\tau_c)A_i D_i^{-\gamma_i} + (1-\lambda_i)\zeta_i \left(\frac{A_i}{D_i}\right)^2 z, \\ x_2^* & , \lambda_i(1-\tau_c)A_i((1-\tau_c)A_i + \tau_c D_i)^{-\gamma_i} + (1-\lambda_i)\zeta_i z \leq y < \lambda_i(1-\tau_c)A_i D_i^{-\gamma_i} + (1-\lambda_i)\zeta_i \left(\frac{A_i}{D_i}\right)^2 z, \\ 1 & , \text{otherwise.} \end{cases}$$

Hence, $\hat{u}_i(y,z) = \max_{k=1,2,3}\{\hat{u}_{ik}\}$, and the according value of $x$ to achieve the supremum can be described as $\hat{x}_i(y,z)$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$.

**Lemma 18.** *Given* $\min_i \gamma_i > 1$, *for fund* $i$, *the reaction function* $\hat{Z}_{iT}$ *of the other parties* $Z_{-i}$ *has the following expression,*

$$\hat{Z}_{iT} = \hat{x}_i\left(e^{\psi_i T}\hat{\eta}_i \xi_T, Z_{-i}\right), \tag{3.2.11}$$

*for some* $\hat{\eta}$ *satisfying the budget constraint* $\mathbb{E}\left[e^{\psi_i T}\xi_T \hat{Z}_{iT}\right] = 1$ *for* $i = 1, \cdots, N$.

Since we suppose that $Z_{iT}$ has the feedback form of $Z_{iT} = l_i(\eta, W_T)$, $l_i$'s form the following system $l = Pl$, where each $l_i$, it follows

$$l_i(\eta, x) = \hat{x}_i \left( \eta_i e^{\psi_i T} e^{-\left(r+\frac{1}{2}\theta^2\right)T} e^{-\theta x}, \frac{1}{N-1} \sum_{j \neq i}^{N} l_j(\eta, x) \right) \text{ , for all } i = 1, \cdots, N. \tag{3.2.12}$$

Define the weighted $L^2$ space $L^{2,\phi}$ with the norms $\|f\|_{\phi,2} := \left(\int_{\mathbb{R}} \|f(x)\|_2^2 \phi(x) dx\right)^{\frac{1}{2}}$, where $\phi$ is the one-dimensional normal density function with mean 0 and variance $T$ and $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^2$. Note that the space $L^{2,\phi}$ is complete. Denote $L^{2,\phi,+}$ as the subset of $L^{2,\phi}$ where functions are non-negative.

**Lemma 19.** *Given* $\min_i \gamma_i > 1$*, for fixed* $\eta$*, there exists a solution to* (3.2.12) *in* $L^{2,\phi,+}$*. Furthermore, if there exists multiple fixed points, there exists the unique best fixed point in the sense of the value of utility functions, i.e.* $\hat{l}$ *is the solution to* (3.2.12) *and*

$$\hat{l}_i(\eta, x) = \underset{l=Pl}{\arg\max}\, u_i \left( l_i(\eta, x), \frac{1}{N-1} \sum_{j \neq i}^{N} l_j(\eta, x) \right).$$

The detailed proof is in Appendix 3.4.2. For fixed $\eta$, we choose the unique best fixed point as the fixed point generated from $\eta$. The next step is to show there exists $\eta$ such that the budget constraint is satisfied. By such construction, we can have the solution satisfying Lemma 18 and it is the best one in the sense of the value of utility functions. In other word, the solution is Pareto optimal.

**Lemma 20.** *Given* $\min_i \gamma_i > 1$ *and the unique best fixed point generated by* $\eta$*, there exists a unique* $\hat{\eta}$ *such that the budget constraints*

$$\mathbb{E}\left[e^{\psi_i T} \xi_T l_i(\hat{\eta}, W_T)\right] = 1,$$

*for all* $i = 1, \cdots, N$*.*

Therefore, $\hat{\eta}$ and the according $\hat{l}$ constructed in Lemma 20 give the required Pareto optimal strategy, and the according portfolio can be derived through the heat kernel. The proof is similar to the proof for one-player model, which is omitted here.

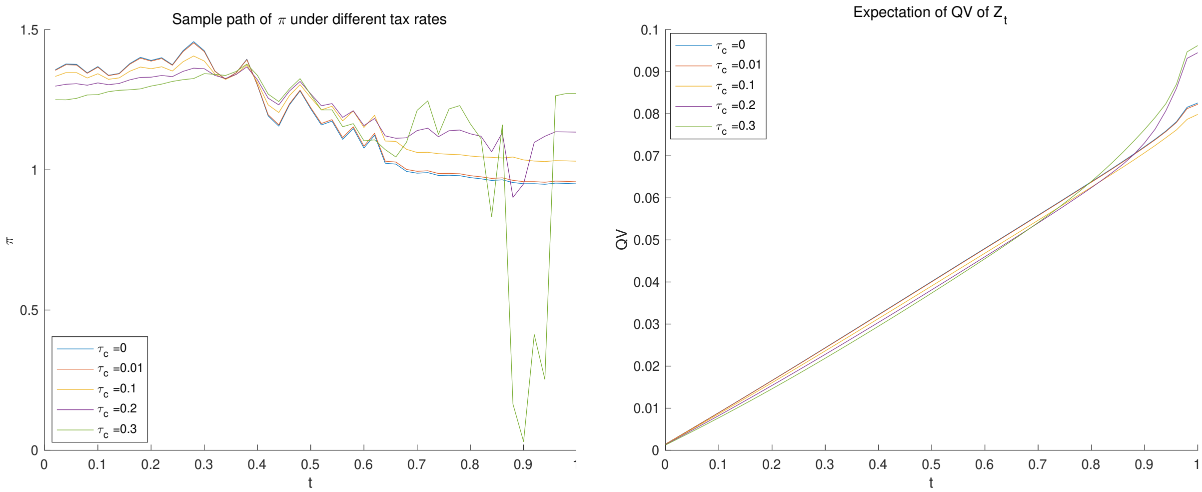
## 3.3 Numerical Examples

To illustrate how tax rates change managers' behaviors and what is the optimal tax rate for policy makers, it is simpler to present the result of one-player model. The followings are the parameters that we use for the approximation of one-player model:

$r = 0.01$ , $\mu = 0.1$ , $\sigma = 0.2$ , $T = 1$ , $\gamma = 2$ , $F_0 = 1$ , $I_0 = 2$ , $\alpha = 0.2$ , $w = 10$ , $\psi = 0.01$ , $\tau_o = 0.3$.

We transform the expression of $\hat{l}$ to the expectation according to some normal random variables with normal distribution and use the Monte Carlo simulation to approximate $J_p$ defined in (3.1.1). For the calculation of $\hat{\pi}_t$, we use $\delta = 10^{-4}$ and

$$\pi_t \approx \frac{\ln\left(\hat{l}(t, W_t + \theta t + \delta)\right) - \ln\left(\hat{l}(t, W_t + \theta t)\right)}{\sigma \delta}.$$




(a) One sample path of $\pi$.

(b) Quadratic variation of $Z_t$.

Figure 3.2: (a) Sample path of $\pi$ and (b) quadratic variation of $Z_t$ with respect to different tax rates.

Figure 3.2 shows the perspective of managers with respect to different tax rates. The left one shows one sample path of $\pi$ according to tax rates. The optimal portfolio is less aggressive with larger $\tau_c$ at the beginning and increases when the time is closer to the terminal time. Figure 3.2 (b) has a better illustration of this pattern with quadratic variation of $Z_t$. The curves with respect to different $\tau_c$ come across with each other at around time 0.68. This is the result of substitution and income effect mentioned in Section 3.2.1: managers incline to be more conservative due to the less marginal after-tax income with high tax rate, while they also tend to be more aggressive to overcome the loss from taxes. From figure 3.2, the closer to the terminal time, the more aggressive managers tend to be, since their utilities are based on the terminal wealth, and their urges to overcome the tax loss are more intensive.

Figure 3.3 shows how the policy makers' objective function changes according to different $\tau_c$. The left figure shows that $J_p$ achieves the maximum at around $\tau_c = 20\%$. When $\tau_c$ is small, the extra income brought by the higher tax rates dominates, and the objective function increases with larger $\tau_c$. However, since higher $\tau_c$ also prevents managers to invest, which causes a smaller base to tax on, when $\tau_c$ is closer to 1, this effect dominates, and the objective function drops dramatically. Same argument works for any combination of managers' and policy makers' risk aversion $\gamma$ and $w$, which gives the optimal $\tau_c$ shown in the right figure. The optimal $\tau_c$ is a decreasing function of $w$, which means that policy makers with higher risk aversions set the tax rate lower. At the same time, for fixed policy makers' risk aversion $w$, optimal $\tau_c$ is an increasing function of $\gamma$. The phenomenon can be explained by figure 3.2 and figure 3.4. When facing higher tax rate, managers tend to be more aggressive at the end of the period, which increase the volatility of the market as shown in Figure 3.2. Since the major part of tax income for policy makers are at the terminal time, the volatility of $N_T$ becomes larger as shown in figure 3.4 (b), and thus push the policy makers to choose lower tax rate to maximize their utilities. Meanwhile, when comparing figure 3.4 (a) and (b), we notice that the change of variance is relatively smaller compared to that of the return (mean) of $N_T$, which gives another reason why policy makers will pursue a smaller $\tau_c$.

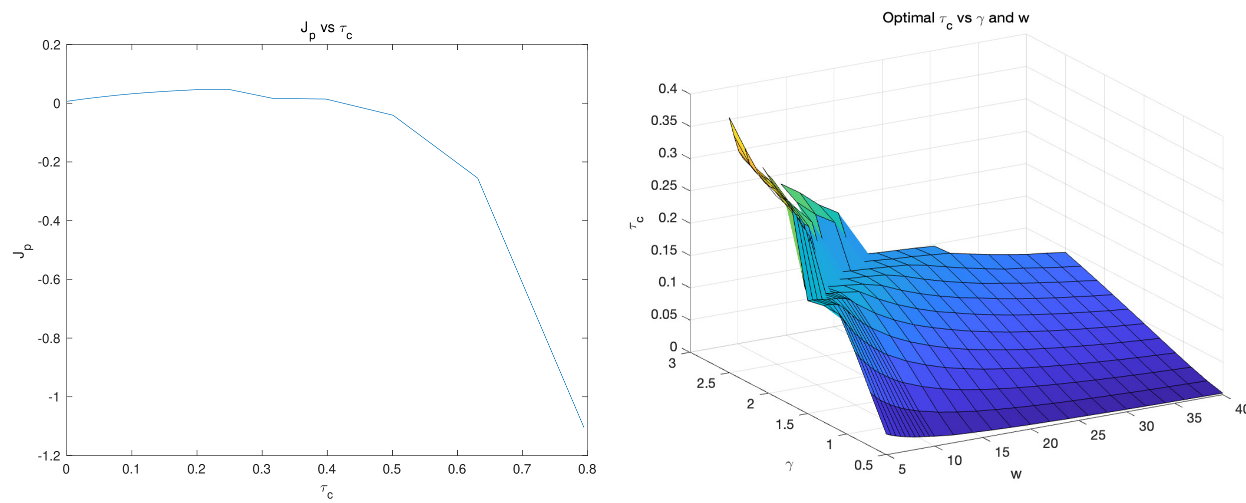


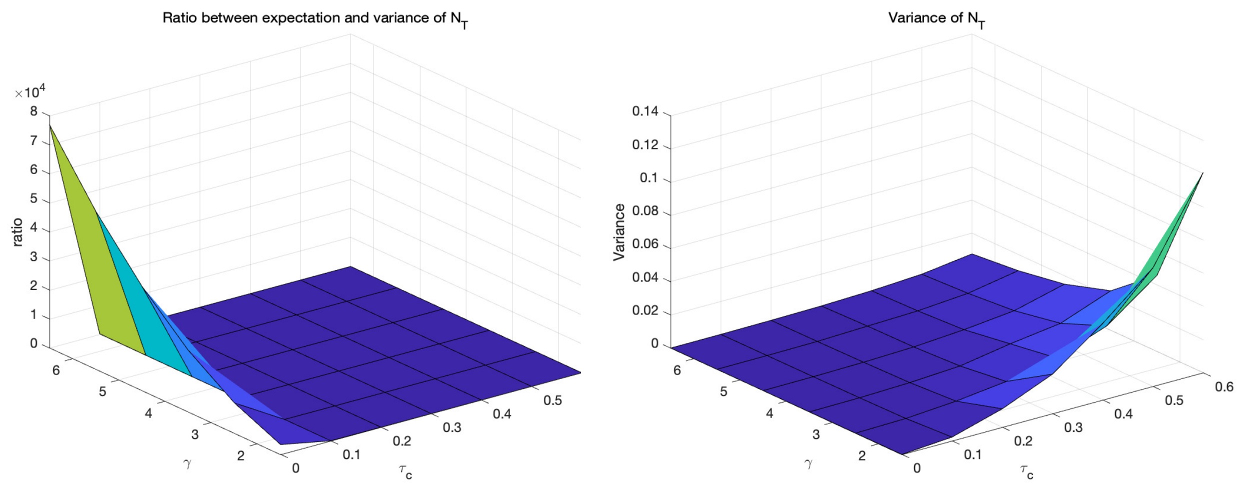

Figure 3.3: Policy makers' objective function $J_p$ v.s. different $\tau_c$ with $\gamma = 2$, $w = 10$. (Left) Optimal $\tau_c$ v.s. managers' and policy makers' risk aversion $\gamma$ and $w$. (Right)

We want to mention that in most literature like Stiglitz (1975), Yost (2018), Feldstein (1969), Falsetta et al. (2013) show that increment in capital gain tax would result in the decrease of managers' or householders' risk-taking. However, this is not the case in our model. Managers increase their risk-taking when $\tau_c$ is larger, which makes policy makers to lower the capital gain tax to decrease the volatility of their tax income.

For $N$-player game, we will use the following example with 3 players:

$$\begin{aligned} &u_1(Z_1, Z_2, Z_3) = -\frac{1}{10}\left(\frac{9}{A_1} + \frac{Z_2 + Z_3}{2Z_1}\right), \; u_2(Z_1, Z_2, Z_3) = -\frac{1}{2}\left(\frac{1}{2A_2^2} + \frac{Z_1 + Z_3}{Z_2}\right), \\ &u_3(Z_1, Z_2, Z_3) = -\frac{1}{9A_3^3} - \frac{Z_1 + Z_2}{3Z_3}, \; A_i = \begin{cases} Z_i & , Z_i \leq 1, \\ 2Z_i - 1 & , Z_i > 1. \end{cases} \quad (i = 1, 2, 3) \end{aligned} \tag{3.3.1}$$

Since in both the cases with and without competition ($\lambda_i = 1$), the optimal portfolio is calculated through the heat kernel based on $Z_T$, it is easier to see how managers change their behaviors with or without competition based on the distribution of $Z_T$. Figure 3.5 shows that under competition, managers tend to be less aggressive to avoid the possible loss brought by competition. Thus, the distributions of optimal $Z_{iT}$ move left compared with the case without competition. The result coincides with the results shown in Chapter 2. Meanwhile, less aggressive means higher risk aversion if we use a representative player to stand for the $N$ players. Figure 3.6 shows that managers are away from the market average with competition. This is due to incentives brought by competition such that managers are more diversified to have a better chance to beat the other parties. Combining with figure 3.3, we conclude that competition pushes policy maker to increase optimal tax rates so as to compensate for the loss of tax incomes.




(a) Sharpe ratio of $N_T$.

(b) Variance of $N_T$.

Figure 3.4: (a) Optimal Sharpe ratio and (b) variance of policy makers' income $N_T$.

## 3.4 Appendix

### 3.4.1 One-player Model

**Lemma 21.** *$\hat{u}(y) = \sup_x u(x) - xy$ can be expressed as*

*(i) If $y_0 \leq (1-\tau_c)AD^{-\gamma}$,*

$$\hat{u}(y) = \begin{cases} v_1(y) & , y < y_0, \\ v_2(y) & , y_0 \leq y < (1-\tau_c)AD^{-\gamma}, \\ v_3(y) & , (1-\tau_c)AD^{-\gamma} \leq y < AD^{-\gamma}), \\ v_4(y) & , y \geq AD^{-\gamma}, \end{cases} \tag{3.4.1}$$

*(ii) If the condition of i) is not satisfied and $y_1 \leq AD^{-\gamma}$ or $(1-\tau_c)AD^{-\gamma} < (1-\tau_c)(A + B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq AD^{-\gamma}$,*

$$\hat{u}(y) = \begin{cases} v_1(y) & , y < y_1, \\ v_3(y) & , y_1 \leq y < AD^{-\gamma}, \\ v_4(y) & , y \geq AD^{-\gamma}, \end{cases} \tag{3.4.2}$$

*(iii) If the conditions of i) and ii) are not satisfied,*

$$\hat{u}(y) = \begin{cases} v_1(y) & , y < y_2, \\ v_4(y) & , y \geq y_2, \end{cases} \tag{3.4.3}$$

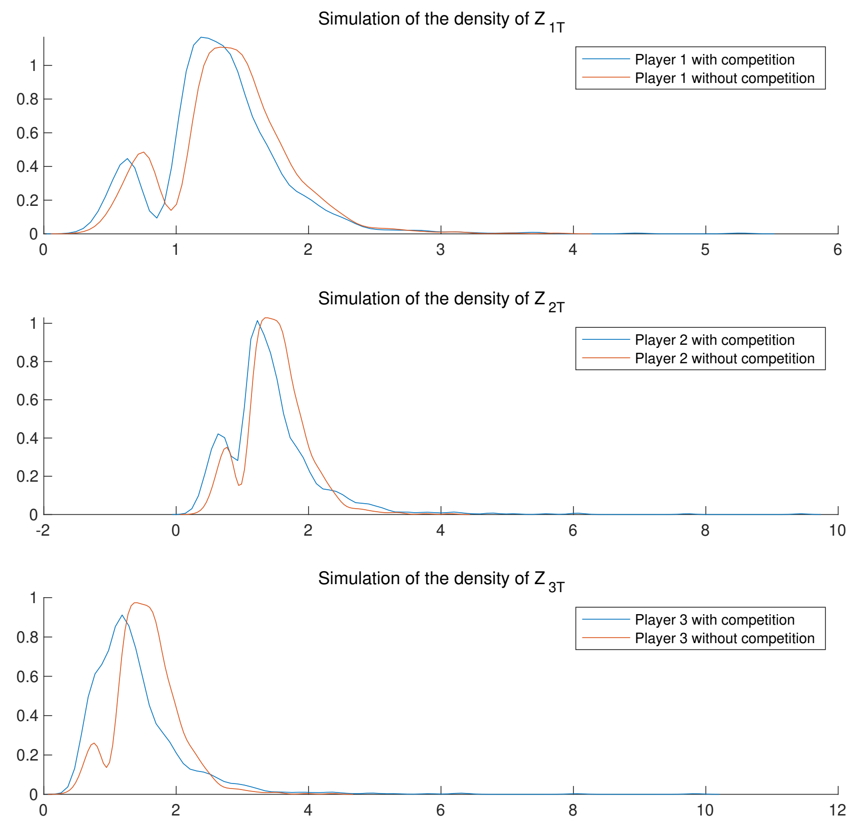



Figure 3.5: Histogram of $Z_{iT}$ $(i = 1, 2, 3)$ with and without competition. Settings are listed in (3.3.1) with $r = 0.05$, $\mu = 0.15$, $\sigma = 0.2$, $\psi_1 = \psi_2 = \psi_3 = 0.02$, $T = 1$.

*where*

$$
\begin{aligned}
v_1(y) &= \frac{\gamma}{1-\gamma}\left(\frac{y}{(A+B)(1-\tau_c)}\right)^{-\frac{1-\gamma}{\gamma}} + \left(\frac{\tau_c}{1-\tau_c}\frac{D}{A+B} - \frac{B}{A+B}\right)y, \\
v_2(y) &= \frac{\gamma}{1-\gamma}\left(\frac{y}{A(1-\tau_c)}\right)^{-\frac{1-\gamma}{\gamma}} + \frac{\tau_c}{1-\tau_c}\frac{D}{A}y, \\
v_3(y) &= \frac{1}{1-\gamma}D^{1-\gamma} - y\frac{D}{A}\ ,\ v_4(y) = \frac{\gamma}{1-\gamma}\left(\frac{y}{A}\right)^{-\frac{1-\gamma}{\gamma}}.
\end{aligned}
$$

*Meanwhile, this is achieved at $x = \hat{x}(y)$ defined in (3.2.1)-(3.2.3) and the unique $y_0, y_1, y_2$ defined in (3.2.4)-(3.2.5).*

*Proof.* Define $\hat{u}_1 = \sup_{x \le \frac{D}{A}} u(x) - xy$, $\hat{u}_2 = \sup_{\frac{D}{A} < x \le 1} u(x) - xy$, and $\hat{u}_3 = \sup_{x>1} u(x) - xy$, and

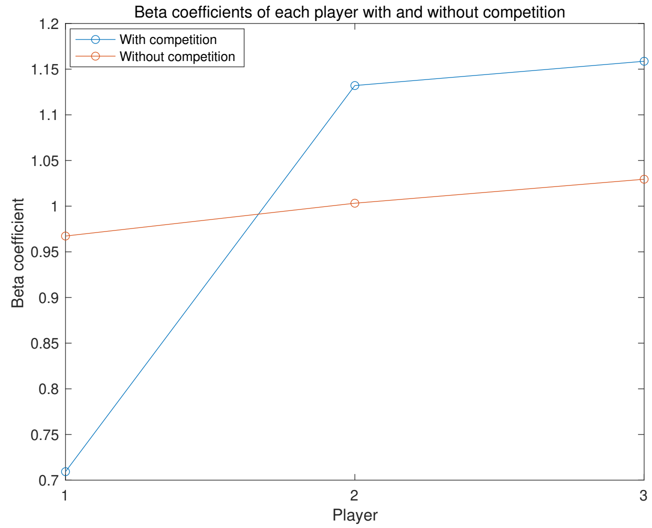



Figure 3.6: Beta coefficients of funds with respect to the market average with and without competition. Settings are listed in (3.3.1) with $r = 0.05$, $\mu = 0.15$, $\sigma = 0.2$, $\psi_1 = \psi_2 = \psi_3 = 0.02$, $T = 1$.

then $\hat{u} = \max\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$. Denote $v_5(y) = \frac{1}{1-\gamma}\left[(1-\tau_c)A + \tau_c D\right]^{1-\gamma} - y$.

$$\hat{u}_1 = \begin{cases} v_4(y) & , y \geq AD^{-\gamma}, \\ v_3(y) & , y < AD^{-\gamma}, \end{cases}, \; \hat{u}_3 = \begin{cases} v_1(y) & , y \leq (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}, \\ v_5(y) & , y > (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}. \end{cases}$$

$$\hat{u}_2 = \begin{cases} v_5(y) & , y < (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}, \\ v_2(y) & , (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)AD^{-\gamma}, \\ v_3(y) & , y \geq (1-\tau_c)AD^{-\gamma}, \end{cases}$$

Meanwhile, since $v_2((1-\tau_c)AD^{-\gamma}) = v_3((1-\tau_c)AD^{-\gamma})$ and the first order condition for $v_2$ on $(1-\tau_c)A[(1-\tau_c)A + \tau_c D]^{-\gamma} \leq y \leq (1-\tau_c)AD^{-\gamma}$ is

$$v_2'(y) = -(A(1-\tau_c))^{\frac{1-\gamma}{\gamma}} y^{-\frac{1}{\gamma}} + \frac{\tau_c}{1-\tau_c}\frac{D}{A} \leq -\frac{1}{A(1-\tau_c)}((1-\tau_c)A + \tau_c D) + \frac{\tau_c}{1-\tau_c}\frac{D}{A} \leq -1 \leq -\frac{D}{A},$$

due to $A > D$, $v_2 > v_3$ on this interval. Due to the fact that $v_3'(y) > v_5'(y)$ and

$$\begin{aligned} &(v_3 - v_5)((1-\tau_c)A[(1-\tau_c)A + \tau_c D]^{-\gamma}) \\ =&\frac{1}{1-\gamma}D^{1-\gamma} - \frac{1}{1-\gamma}[(1-\tau_c)A + \tau_c D]^{1-\gamma} - (1-\tau_c)(D-A)[(1-\tau_c)A + \tau_c D]^{-\gamma} < 0, \end{aligned}$$

by the concavity of $f(x) = \frac{1}{1-\gamma}x^{1-\gamma}$, $v_3 < v_5$ on $y \leq (1-\tau_c)A[(1-\tau_c)A + \tau_c D]^{-\gamma}$. Combining all the calculations together, we have

$$\max\{\hat{u}_1, \hat{u}_2\} = \begin{cases} v_5(y) & , y < (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}, \\ v_2(y) & , (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)AD^{-\gamma}, \\ v_3(y) & , (1-\tau_c)AD^{-\gamma} \leq y < AD^{-\gamma}, \\ v_4(y) & , y \geq AD^{-\gamma}. \end{cases}$$

For the simplification of the following calculations, by the construction of $\hat{u}_3$ and $\max\{\hat{u}_1, \hat{u}_2\}$, we list the following useful comparisons among $v_i (i = 1, \cdots, 5)$:

(a) For $y \leq (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, $v_5(y) \leq v_1(y)$.

(b) $v_3 \leq v_4$, $v_5 \leq v_2$, $v_3 \leq v_2$, and $v_5 \leq v_1$ since $v_4$, $v_2$ and $v_1$ are constructed with zero first order conditions, but $v_3$, $v_5$ are not.

The following discussions are split into 3 cases: i)$(1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq (1-\tau_c)AD^{-\gamma}$; ii)$(1-\tau_c)AD^{-\gamma} < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq AD^{-\gamma}$; iii)$(1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} > AD^{-\gamma}$.

(i) When $y < (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, by the relationship shown in (a), $\hat{u}(y) = v_1(y)$.

For $(1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, (3.2.4) is equivalent to $v_1(y_0) = v_2(y_0)$. By the convexity of $f(x) = \frac{\gamma}{1-\gamma}x^{-\frac{1-\gamma}{\gamma}}$, $(1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} < y_0 < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$. Therefore,

$$\hat{u}(y) = \begin{cases} v_1(y) & , (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < y_0, \\ v_2(y) & , y_0 \leq y < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}. \end{cases}$$

For $(1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)AD^{-\gamma}$, by (b), $v_5 \leq v_2$, and $\hat{u}(y) = v_2(y)$.

For $(1-\tau_c)AD^{-\gamma} \leq y < AD^{\gamma}$, due to the concavity, $A > D$ and $y \geq (1-\tau_c)AD^{-\gamma}$,

$$\frac{1}{1-\gamma}\left[((1-\tau_c)A + \tau_c D)^{1-\gamma} - D^{1-\gamma}\right] < D^{-\gamma}(1-\tau_c)(A-D) \leq y\left(1 - \frac{D}{A}\right),$$

and thus, $\hat{u}(y) = v_3(y)$. Note the above inequality does not require the upper bound of $y$.

For $y > AD^{-\gamma}$, since by (b), $v_4(y) \geq v_3(y) \geq v_5(y)$, and thus $\hat{u}(y) = v_4(y)$. In conclusion, $\hat{u}$ can be expressed as (3.4.1).

(ii) For $y < (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, as shown in the calculations of case i), $\hat{u}(y) = v_1(y)$.

For $(1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)AD^{-\gamma}$, if $y_0 \leq (1-\tau_c)AD^{-\gamma}$,

$$\hat{u}(y) = \begin{cases} v_1(y) & , (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < y_0, \\ v_2(y) & , y_0 \leq y < (1-\tau_c)AD^{-\gamma}, \end{cases}$$

and the following calculations for the other intervals are the same as case i).

Now we consider the case when $y_0 > (1-\tau_c)AD^{-\gamma}$. For $(1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)AD^{-\gamma}$, $\hat{u}(y) = v_1(y)$.

For $(1-\tau_c)AD^{-\gamma} \leq y < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, note that $y_0 > (1-\tau_c)AD^{-\gamma}$ indicates

$$\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} > \frac{A}{A+B}D^{-\gamma}, \tag{3.4.4}$$

since $(1-\tau_c)\left[(1-\tau_c)A + \tau_c D\right](A+B) \geq y_0 > (1-\tau_c)AD^{-\gamma}$, where the first inequality is due to the concavity of $f(x) = -\frac{\gamma}{1-\gamma}x^{-\frac{1-\gamma}{\gamma}}$. Meanwhile, $y_1$ defined in (3.2.6) uniquely solves $v_1(y) = v_3(y)$, and $y_1$ is in this interval because

(a) The monotonicity of $(v_1 - v_3)(y)$:

$$
\begin{aligned}
(v_1 - v_3)' &= -\frac{1}{(A+B)(1-\tau_c)}\left(\frac{y}{(1-\tau_c)(A+B)}\right)^{-\frac{1}{\gamma}} + \left(\frac{\tau_c}{1-\tau_c}\frac{D}{A+B} - \frac{B}{A+B} + \frac{D}{A}\right) \\
&\leq -\frac{(1-\tau_c)A + \tau_c D}{(1-\tau_c)(A+B)} + \frac{(1-\tau_c)A + \tau_c D}{(1-\tau_c)(A+B)} - \frac{A-D}{A} < 0.
\end{aligned}
$$

(b) $v_1 - v_3 < 0$ at $y = (1-\tau_c)(A+B)\left((1-\tau_c)AC + \tau_c D\right)^{-\gamma}$. If not, it is equivalent to say

$$
\frac{\frac{1}{1-\gamma}\left(((1-\tau_c)A + \tau_c D)^{1-\gamma} - D^{1-\gamma}\right)}{(1-\tau_c)(A-D)} > [(1-\tau_c)AC + \tau_c D]^{-\gamma}\frac{A+B}{A} \geq D^{-\gamma},
$$

where the last inequality is due to (3.4.4), and it contradicts the concavity of the function $y = \frac{1}{1-\gamma}x^{1-\gamma}$.

(c) $v_1 - v_3 \geq 0$ at $y = (1-\tau_c)AD^{-\gamma}$ since $v_1 \geq v_2 = v_3$ at this point.

Hence, in this interval,

$$
\hat{u}(y) = \begin{cases} v_1(y) & , (1-\tau_c)AD^{-\gamma} \leq y < y_1, \\ v_3(y) & , y_1 \leq y < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}. \end{cases}
$$

For $y \geq (1-\tau_c)(A+B)\left((1-\tau_c)A + \tau_c D\right)^{-\gamma}$, it is the same as (i),

$$
\hat{u}(y) = \begin{cases} v_3(y) & , (1-\tau_c)(A+B)\left((1-\tau_c)A + \tau_c D\right)^{-\gamma} \leq y < AD^{-\gamma} \\ v_4(y) & , y \geq AD^{-\gamma} \end{cases}.
$$

In conclusion, if $y_0 \leq (1-\tau_c)AD^{-\gamma}$, $\hat{u}$ can be expressed as (3.4.1). Otherwise, $\hat{u}$ is (3.4.2).

(iii) For $y < (1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, it is the same as (i) and (ii).

For $(1-\tau_c)A\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma} \leq y < (1-\tau_c)AD^{-\gamma}$, the expression of $\hat{u}$ has the same result as in case (ii) depending on whether $y_0 < (1-\tau_c)AD^{-\gamma}$.

For $(1-\tau_c)AD^{-\gamma} \leq y < AD^{-\gamma}$, according to the calculation in case (ii), if $y_1 > AD^{-\gamma}$, $\hat{u}(y) = v_1(y)$. Otherwise,

$$
\hat{u}(y) = \begin{cases} v_1(y) & , (1-\tau_c)AD^{-\gamma} \leq y < y_1, \\ v_3(y) & , y_1 \leq y < AD^{-\gamma}. \end{cases}
$$

For $AD^{-\gamma} \leq y < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, note that case (iii) indicates that

$$
(1-\tau_c)(A+B) \geq A, \tag{3.4.5}
$$

since $\frac{A}{(1-\tau_c)(A+B)} \leq \left(\frac{D}{(1-\tau_c)A + \tau_c D}\right)^{\gamma} < 1$. Meanwhile, $v_1(y_2) = v_4(y_2)$ for $y_2$ defined in (3.2.5) and

$$
y_2 \leq A^{-\gamma}(1-\tau_c)(A+B) \leq (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma},
$$

due to the concavity of $f(x) = -\frac{\gamma}{1-\gamma}x^{-\frac{1-\gamma}{\gamma}}$, $A > D$ and (3.4.5). Meanwhile, at $y = y_2$,

$$\begin{aligned}(v_1 - v_4)' &= \left(\left(\frac{1}{A}\right)^{-\frac{1-\gamma}{\gamma}} - \left(\frac{1}{(1-\tau_c)(A+B)}\right)^{-\frac{1-\gamma}{\gamma}}\right) y_2^{-\frac{1}{\gamma}} - \frac{(1-\tau_c)B - \tau_c D}{(1-\tau_c)(A+B)}\\ &= -\frac{1}{\gamma}\frac{(1-\tau_c)B - \tau_c D}{(1-\tau_c)(A+B)} < 0.\end{aligned}$$

Therefore, if $y_1 < AD^{-\gamma}$, $v_1 < v_3 = v_4$ at $y = AD^{-\gamma}$, which gives $\hat{u}(y) = v_4(y)$. Otherwise, $v_1 \geq v_3 = v_4$ at $y = AD^{-\gamma}$. which gives $y_2 > AD^{-\gamma}$, and thus

$$\hat{u}(y) = \begin{cases} v_1(y) & , AD^{-\gamma} \leq y < y_2, \\ v_4(y) & , y_2 \leq y < (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}. \end{cases}$$

For $y \geq (1-\tau_c)(A+B)\left[(1-\tau_c)A + \tau_c D\right]^{-\gamma}$, note that $v_4 \geq v_1$ in this interval due to the uniqueness of $y_2$ and its negative first order derivative at this point. Combining with (b), we have $v_4 \geq v_1 \geq v_5$ on this interval. Hence, $\hat{u}(y) = v_4(y)$.

In conclusion, if $y_1 < AD^{-\gamma}$, $\hat{u}$ is (3.4.2). Otherwise, it is (3.4.3).

Combine all three cases and the results are same as the third case (3.4.1) - (3.4.3). □

### 3.4.2 $N$-player Game

*Proof of Lemma 18:* Note that $\hat{x}_i(y,z)$ achieves supremum of $u_i(x,z) - xy$. Due to the non-negativeness of $e^{\psi_i T}\xi_t Z_{it}$ and

$$d(e^{\psi_i T}\xi_t Z_{it}) = e^{\psi_i T}\xi_t Z_{it}(\pi_{it}\sigma - \theta)dW_t,$$

it is a supermartingale. Therefore,

$$\mathbb{E}\left[e^{\psi_i T}\xi_T Z_{iT}\right] \leq 1 \ , \ \mathbb{E}\left[\int_0^T e^{\psi_i t}\xi_t Z_{it}(\pi_{it}\sigma - \theta)dW_t\right] \leq 0.$$

Define

$$\Lambda_i\left(e^{\psi_i T}\eta_i\xi_T\right) = \sup_{\{Z_{iT}\}} \mathbb{E}\left[u_i(Z_T) - \eta_i e^{\psi_i T}\xi_T Z_{iT} + \eta_i + \int_0^T e^{\psi_i t}\eta_i\xi_t Z_{it}\pi_{it}\sigma dW_t\right] \leq \eta_i + \mathbb{E}\left[\hat{u}_{i0}\left(e^{\psi_i T}\eta_i\xi_T, Z_{-i}\right)\right],$$

due to the conclusions in Lemma 21. Meanwhile, since

$$J_{im} \leq \inf_{\eta_i \geq 0} \Lambda_i\left(e^{\psi_i T}\eta_i\xi_T\right) \leq \Lambda\left(e^{\psi_i T}\hat{\eta}_i\xi_T\right) \leq \hat{\eta}_i + \mathbb{E}\left[\hat{u}_{i0}\left(e^{\psi_i T}\hat{\eta}_i\xi_T, Z_{-i}\right)\right],$$

the equality with $\eta_i$ changed to $\hat{\eta}_i$ is achieved when $Z_{iT}$ arrives $\hat{Z}_{iT}$ defined in (3.2.11). □

Define the $\epsilon$-truncated operator $P_\epsilon$: $L^{2,\phi,+} \to L^{2,\phi,+}$ satisfying

$$l_\epsilon = P_\epsilon l_\epsilon, \tag{3.4.6}$$

where $l_\epsilon$ is $N$-dimensional function with $i$th element as $l_{i\epsilon} = l_i \vee \frac{\epsilon}{N-1} \wedge \frac{\epsilon^{-1}}{N-1}$. Denote $\hat{x}_{i\epsilon} = \hat{x}_i \vee \frac{\epsilon}{N-1} \wedge \frac{\epsilon^{-1}}{N-1}$. Note that $P_\epsilon$ is actually an operator mapping from $U_\epsilon$ to $U_\epsilon$, where

$$U_\epsilon = \left\{f \in L^{2,\phi,+} : \epsilon \leq |f| \leq \epsilon^{-1}\right\} \subset L^{2,\phi,+}. \tag{3.4.7}$$

**Lemma 22.** *Given* $\min_i \gamma_i > 1$*, for fixed* $\eta_i$ $(i = 1, \cdots, N)$*,* $\hat{x}_{i\epsilon}(y, z)$ *satisfies the following properties:*

*(i)* $\hat{x}_{i\epsilon}$ *is a continuous function except some negligible set, and for fixed* $z$*,* $\hat{x}_{i\epsilon}$ *is a piecewise convex function which has the expression among* $1$*,* $\frac{D_i}{A_i}$*,* $\epsilon$*,* $\epsilon^{-1}$ *and* $x_i^*$ *defined in* (3.2.8)*,* (3.2.9) *and* (3.2.10)*.*

*(ii) For fixed* $y$*,* $\hat{x}_i$ *is a non-decreasing function of* $z$*.*

*(iii) For arbitrary* $\delta > 0$*, define*

$$U_{1\delta} = \left\{ (x, y) : (x, y) \notin O\left((x_0, y_0), \delta\right), \liminf_{(x,y)\to(x_0,y_0)} \hat{x}_{i\epsilon}(x, y) \neq \limsup_{(x,y)\to(x_0,y_0)} \hat{x}_{i\epsilon}(x, y) \right\},$$

$$U_{2\delta} = \left\{ (x, y) : \hat{x}_{i\epsilon}(x, y) \notin O(1, \delta) \cup O\left(\frac{D_i}{A_i}, \delta\right) \cup O\left(\epsilon, \delta\right) \cup O\left(\epsilon^{-1}, \delta\right) \right\},$$

*There exist* $F_1(\epsilon, \delta) < \infty$ *and* $F_2(\epsilon, \delta) > 0$ *such that for* $\epsilon \le y \le \epsilon^{-1}$ *and* $(x, y) \in U_{1\delta}$*,*

$$\left| \hat{x}_{i\epsilon}\left(G_i e^{-\theta x}, z^{(1)}\right) - \hat{x}_{i\epsilon}\left(G_i e^{-\theta x}, z^{(2)}\right) \right| \le F_1(\epsilon, \delta) \left| z^{(1)} - z^{(2)} \right|, \tag{3.4.8}$$

*and furthermore, if* $(x, y) \in U_{2\delta}$*,*

$$\left| \hat{x}_{i\epsilon}\left(G_i e^{-\theta x}, z^{(1)}\right) - \hat{x}_{i\epsilon}\left(G_i e^{-\theta x}, z^{(2)}\right) \right| \ge F_2(\epsilon, \delta) \left| z^{(1)} - z^{(2)} \right|. \tag{3.4.9}$$

**Remark 23.** *Note that* $\lim_{\delta\to 0} F_1(\epsilon, \delta) = \infty$ *and* $\lim_{\delta\to 0} F_2(\epsilon, \delta) = 0$*.*

*Proof.* (i) Note that $\hat{x}_{ik}$ $(k = 1, 2, 3)$ is a continuous function of $y$ and $z$, and hence, the only points where $\hat{x}_{i\epsilon}$ is not continuous are those where the values of $x_i^*$ and $x_j^*$ or $x_i^*$ and 1 or $\frac{D_i}{A_i}$ meet, which has measure 0.

Meanwhile, note for $x_1^*$ in (3.2.8),

$$1 = -\left(\gamma_i \lambda_i A_i^{1-\gamma_i} (x_1^*)^{-\gamma_i - 1} + 2(1 - \lambda_i)\zeta_i z (x_1^*)^{-3}\right) \frac{\partial x_1^*}{\partial y},$$

$$\frac{\partial^2 x_1^*}{\partial y^2} = \frac{\gamma_i(\gamma + 1)\lambda_i A_i^{1-\gamma_i} (x_1^*)^{-\gamma_i - 2} + 6(1 - \lambda_i)\zeta_i z (x_1^*)^{-4}}{\gamma_i \lambda_i A_i^{1-\gamma_i} (x_1^*)^{-\gamma_i - 1} + 2(1 - \lambda_i)\zeta_i z (x_1^*)^{-3}} \frac{\partial x_1^*}{\partial y}.$$

It is easy to see that $\frac{d^2 x_1^*}{dy^2} > 0$. Similar results can be found for $x_2^*$ and $x_3^*$. Thus, the second part of the statement is obvious since $x_i^*$ are convex and there are only finite possibilities of the expression of $\hat{x}_{i\epsilon}$ based on $\hat{u}_{ik}$ $(k = 1, 2, 3)$.

(ii) For $\hat{x}_{ik}$ $(k = 1, 2, 3)$, take $\hat{x}_{i1}$ as an example, if the graph is on the segment of $x_1^*$

$$\frac{\partial x_1^*}{\partial z} = \frac{(1 - \lambda_i)\zeta_i (x_1^*)^{-2}}{\gamma_i \lambda_i A_i (x_1^*)^{-\gamma_i - 1} + 2(1 - \lambda_i)\zeta_i z (x_1^*)^{-3}} = \frac{(1 - \lambda_i)\zeta_i}{\gamma_i \lambda_i A_i (x_1^*)^{1-\gamma_i} + 2(1 - \lambda_i)\zeta_i \frac{z}{x_1^*}} \ge 0. \tag{3.4.10}$$

Hence, it is a non-decreasing function of $z$ on each segment. Meanwhile, if $z$ increases, the region for $\hat{x}_{i1} = x_1^*$ becomes smaller, making the function value larger for those points which switch region for bigger $z$. Hence, in general, $\hat{x}_{i1}$ is a non-decreasing function of $z$. Similar results can be found for $\hat{x}_{i2}$ and $\hat{x}_{i3}$. Combining these three functions, we can conclude that $\hat{x}_i$ is also a non-decreasing function of $z$.

(iii) Note that for fixed $x \in U_{1\delta} \cup U_{2\delta}$, if $(y, z)$ is on the same segment of the graph, for example, on the graph of $x_1^*$: (3.2.8), the partial derivative is shown in (3.4.10). Since both $x_1^*$ and $z$ are in $[\epsilon, \epsilon^{-1}]$ (otherwise, $\hat{x}_i$ cannot be $x_1^*$), since $\gamma_i > 1$,

$$\frac{(1-\lambda_i)\zeta_i}{\gamma_i\lambda_i A_i \epsilon^{1-\gamma_i} + 2(1-\lambda_i)\zeta_i\epsilon^{-2}} \le \frac{\partial x_1^*}{\partial z} \le \frac{(1-\lambda_i)\zeta_i}{\gamma_i\lambda_i A_i \epsilon^{\gamma_i-1} + 2(1-\lambda_i)\zeta_i\epsilon^{2}}.$$

Similar results can be found for $x_2^*$ and $x_3^*$. If $(x, y)$ is around a discontinuous point and in $U_{1\delta}$, the maximal slope of the tangent line is the (finite) jump size over $2\delta$. Similarly, if $(x, y)$ are close to the boundary of $U_{2\delta}$, the minimal slope of the tangent line is $2\delta$ over the Lebesgue measure of the flat area. Therefore, we can always find $F_1(\epsilon, \delta)$ and $F_2(\epsilon, \delta)$ such that the inequalities hold.

□

Before the proof of the existence of the solution to (3.4.6), we need the following theorem.

**Lemma 24.** *(Kolmogorov-Riesz Theorem, Theorem 5 in Hanche-Olsen and Holden (2010)) Let $1 \le p < \infty$. A subset of $G$ of $L^p(\mathbb{R}^n)$ is totally bounded if and only if*

*(i) $G$ is bounded,*

*(ii) for every $\varepsilon > 0$, there is some $R$ so that, for every $g \in G$, $\int_{|x|>R} |g(x)|^p dx < \varepsilon$.*

*(iii) for every $\varepsilon > 0$, there is some $\rho > 0$ so that, for every $g \in G$ and $y \in \mathbb{R}^n$ with $|y| < \rho$, $\int_{\mathbb{R}^n} |g(x+y) - g(x)|^p dx < \varepsilon$.*

**Lemma 25.** *For any function $f \in U_\epsilon$ defined in (3.4.7), denote $\mu(x)$ as the cumulative distribution function of $\phi(x)$, and then,*

$$\lim_{y\to 0}\int_{\mathbb{R}} \|f(x+y) - f(x)\|_2^2 d\mu(x) = 0.$$

*Proof.* By Lusin's Theorem, for arbitrary $\rho > 0$, there exists some subset $E$ such that $\mu(E) < \rho$ and $f$ is continuous on $E^c$. Since $f(x)$ is bounded,

$$\begin{aligned}
&\lim_{y\to 0}\int_{\mathbb{R}} \|f(x+y) - f(x)\|_2^2 d\mu(x) \\
=&\lim_{y\to 0}\int_{E^c} \|f(x+y) - f(x)\|_2^2 d\mu(x) + \int_{E} \|f(x+y) - f(x)\| d\mu(x) \\
\le& \int_{E^c} \lim_{y\to 0} \|f(x+y) - f(x)\|_2^2 d\mu(x) + \lim_{y\to 0} 2N\epsilon^{-1}\mu(E) = 2N\epsilon^{-1}\rho.
\end{aligned}$$

By the arbitrariness of $\rho$, $\lim_{y\to 0}\int_{\mathbb{R}^N} \|f(x+y) - f(x)\|_2^2 d\mu(x) = 0$. □

**Lemma 26.** *Given $\min_i \gamma_i > 1$, for fixed $\eta$, there exists a solution to (3.4.6) in $L^{2,\phi,+}$.*

*Proof.* First, let us show that $P_\epsilon$ is a compact operator from $U_\epsilon$ to $U_\epsilon$.

(i) Totally Boundedness: First, $\{P_\epsilon l : l \in L^{2,\phi,+}\}$ is bounded since for any $n \in \mathbb{N}$,

$$\|P_\epsilon l\|_{\phi,2} \leq \epsilon^{-1}.$$

Meanwhile, for any $l$ in $L^{2,\phi,+}$,

$$\int_{|x|>R} \|P_\epsilon l\|_2^2 \cdot \phi(x)dx \leq \epsilon^{-2} \int_{|x|>R} \phi(x)dx \leq \epsilon^{-2}\sqrt{\frac{2}{\pi}}\frac{1}{R}e^{-R^2/2}.$$

Therefore, for any $h > 0$ there exists some $R$ such that $\int_{|x|>R} \|P_\epsilon l\|_2^2 \cdot \phi(x)dx \leq h$. By Lemma 25,

$$\lim_{y\to 0} \int_{\mathbb{R}} \|P_\epsilon l(x+y) - P_\epsilon l(x)\|_2^2 \, \phi(x)dx = 0.$$

Thus, it is natural to conclude that for any $h > 0$, there exist some $\rho > 0$ such that for any $|y| < \rho$ and $\int_{\mathbb{R}} \|P_\epsilon l(x+y) - P_\epsilon l(x)\|_2^2 \, \phi(x)dx < h$. Combining the above three calculations with Lemma 24, we show that $\{P_\epsilon l : l \in L^{2,\phi,+}\}$ is totally bounded.

(ii) Completeness: Consider a Cauchy sequence $\{(P_\epsilon l_n)\}_{n\in\mathbb{N}}$ and the corresponding $l_n$ consisting of $l_{in}$ as each element. Since $L^{2,\phi,+}$ is complete, the sequence has a limit, denoted as $v$. There also exists a subsequence $P_\epsilon l_{n_q}$ converge pointwise to $v$. We want to show that there exists some $l$ in $U_\epsilon$ such that $P_\epsilon l = v$. The proof is split into two steps. Denote

$$U^\delta_{m,n,i} = \left\{ x \in \mathbb{R} : (P_\epsilon l_m)_i(x) = (P_\epsilon l_n)_i(x) \in O(1,\delta) \cup O\left(\frac{D_i}{A_i}, \delta\right) \cup O\left(\epsilon, \delta\right) \cup O\left(\epsilon^{-1}, \delta\right) \right\},$$
$$U^\delta_{m,n} = \bigcap_{i=1}^{N} U_{m,n,i} \ , \ U^\delta = \bigcup_{p=1}^{\infty} \bigcap_{m,n\geq p} U_{m,n} \ , \ U = \lim_{\delta\to 0} U^\delta.$$

The first step is to construct $l$ on $\mathbb{R}/U$ based on the completeness of $L^{2,\phi,+}$ for $l_m$. The second step is to construct $l$ on $U$ given that $P_\epsilon l_m = P_\epsilon l_n$ for $m, n > p$ for some $p > 0$.

For the first step, note that for any $m, n \in \mathbb{N}$ and $i$th component, $(P_\epsilon l_m)_i \neq (P_\epsilon l_n)_i$, and

(3.4.9) on $\mathbb{R}/U^{\delta}_{m,n,i}$. Therefore,

$$
\begin{aligned}
h \geq \int_{\mathbb{R}} |(P_\epsilon l_n)_i - (P_\epsilon l_m)_i|^2 \phi(x)dx \geq & \int_{\mathbb{R}/U^{\delta}_{m,n,i}} F_2^2(\epsilon,\delta) \left|\sum_{k\neq i}^{N} l_{km}(x) - \sum_{k\neq i}^{N} l_{kn}(x)\right|^2 \phi(x)dx \\
\geq & F_2^2(\epsilon,\delta)\left(\int_{\mathbb{R}/U^{\delta}_{m,n,i}} \left|\sum_{k\neq i}^{N} l_{km}(x) - \sum_{k\neq i}^{N} l_{kn}(x)\right| \phi(x)dx\right)^2 \qquad (3.4.11) \\
\geq & F_2^2(\epsilon,\delta)\left(\int_{\mathbb{R}} \phi(x)dx\right)^{-1}\left(\int_{\mathbb{R}/U^{\delta}_{m,n,i}} \left(\sum_{k\neq i}^{N} (l_{kn}-l_{km})(x)\right)^{\frac{2}{3}} \phi(x)dx\right)^3 \qquad (3.4.12) \\
\geq & F_2^2(\epsilon,\delta)\left(\int_{\mathbb{R}/U^{\delta}_{m,n,i}} \left(\sum_{\substack{k\neq i \\ l_{km}(\cdot)\neq l_{kn}(\cdot)}}^{N} (l_{kn}-l_{km})^3 (l_{kn}-l_{km})^{-2}\right)^{\frac{2}{3}} \phi(x)dx\right)^3 \\
\geq & F_2^2(\epsilon,\delta)\left(\int_{\mathbb{R}/U^{\delta}_{m,n,i}} \sum_{\substack{k\neq i \\ l_{km}(\cdot)\neq l_{kn}(\cdot)}}^{N} (l_{kn}-l_{km})^2 \left(\sum_{\substack{k\neq i \\ l_{km}(\cdot)\neq l_{kn}(\cdot)}}^{N} (l_{kn}-l_{km})^4\right)^{-\frac{1}{3}} \phi(x)dx\right)^3 \qquad (3.4.13) \\
\geq & F_2^2(\epsilon,\delta)\cdot\left(N(\epsilon^{-1}-\epsilon)^4\right)^{-\frac{1}{3}}\left(\sum_{k\neq i}^{N}\int_{\mathbb{R}/U^{\delta}_{m,n,i}} (l_{kn}-l_{km})^2(x)\phi(x)dx\right)^3.
\end{aligned}
$$

(3.4.11) is due to Jensen's inequality and (3.4.12) and (3.4.13) are due to the reversed Hölder inequality. Based on the calculations, $l_{mj} - l_{nj}$ $(j \neq i)$ is of the order $h^{1/3}$ in $L^{2,\phi,+}$ on $\mathbb{R}/U^{\delta}_{m,n,i}$. Hence, combining all the components, we can conclude that $l_m - l_n$ is of of the order $h^{1/3}$ on $\mathbb{R}/U^{\delta}_{m,n}$. Meanwhile, since for any points in $\mathbb{R}/U^{\delta}$, there are at most finite many combinations of $m$ and $n$ such that $(P_\epsilon l_m)_i = (P_\epsilon l_n)_i$ which are around the flat part of the graph, for any $i \in \{1, \cdots, N\}$, we can safely say that $l_n$ forms a Cauchy sequence on $\mathbb{R}/U^{\delta}$, which has a limit $l$ on $\mathbb{R}/U^{\delta}$. Meanwhile, by the arbitrariness of $\delta$, we can conclude that the limit exists on $\mathbb{R}/U$. Note that $U$ is the space where there exists some $p > 0$ such that $(P_\epsilon l_m)_i = (P_\epsilon l_n)_i = 1, \frac{D_i}{A_i}, \epsilon, \epsilon^{-1}$ for all $m, n \geq p$.

For the second step, consider the construction of $l$ on $U$. For any point $x_0$ in $U$, there exists some $p > 0$ such that $x_0 \in \bigcap_{m,n\geq p} U_{m,n}$. Define $l(x_0) = l_p(x_0)$, and $P_\epsilon l = P_\epsilon l_m$ for any

$m \geq p$. Combining two steps, we can conclude that $l \in U_\epsilon$, and

$$\begin{aligned}
&\int_{\mathbb{R}} |v_i - (P_\epsilon l)_i|^2 \phi(x)dx = \lim_{q\to\infty} \int_{\mathbb{R}} \left|(P_\epsilon l_{n_q})_i - (P_\epsilon l)_i\right|^2 \phi(x)dx \\
\leq& \lim_{q\to\infty} F_1^2(\epsilon,\delta) \int_{U_{1\delta}} \left|\sum_{k\neq i}^N l_{kn_q}(x) - \sum_{k\neq i}^N l_k(x)\right|^2 \phi(x)dx + \int_{\mathbb{R}/U_{1\delta}} \left|(P_\epsilon l_{n_q})_i - (P_\epsilon l)_i\right|^2 \phi(x)dx \\
\leq& \lim_{q\to\infty} F_1^2(\epsilon,\delta) \sum_{k\neq i}^N \int_{\mathbb{R}} \left(l_{kn_q}(x) - l_k(x)\right)^2 \phi(x)dx + 4\epsilon^{-2}\mu(\mathbb{R}/U_{1\delta}) \leq K_1\delta,
\end{aligned}$$

where $U_{1\delta}$ is defined in Lemma 22. By the arbitrariness of $\delta$, $\int_{\mathbb{R}} |v_i - (P_\epsilon l)_i|^2 \phi(x)dx = 0$, and thus, $v = P_\epsilon l$, which gives the completeness of the space.

We conclude that $P_\epsilon$ is a compact operator, and by Schauder fixed point theorem, there exists at least one solution to (3.4.6) in $L^{2,\phi,+}$. ☐

Now we need to construct a solution to (3.2.12), and prove that $l \in L^{2,\phi,+}$.

**Lemma 27.** *Given* $\min_i \gamma_i > 1$*, for fixed* $\eta$*, there exists a solution* $l$ *to* (3.2.12) *and* $l \in L^{2,\phi,+}$*.*

*Proof.* Denote $l_n$ as the solution to $l_n = P_{\epsilon_n} l_n$ with $\epsilon_n \xrightarrow{n\to\infty} 0$, and

$$l = \liminf_{n\to\infty} l_n.$$

Note that since $\hat{x}_i$'s are non-decreasing function of the second variable,

$$\begin{aligned}
l_i =& \liminf_{n\to\infty} \hat{x}_i \left(\eta_i e^{\psi_i T} e^{-\left(r+\frac{1}{2}\theta^2\right)T} e^{-\theta x}, \frac{1}{N-1}\sum_{j\neq i}^N l_{jn}(\eta,x)(x)\right) \vee \frac{1}{N-1}\epsilon_n \wedge \frac{1}{N-1}\epsilon_n^{-1} \\
=& \hat{x}_i \left(\eta_i e^{\psi_i T} e^{-\left(r+\frac{1}{2}\theta^2\right)T} e^{-\theta x}, \frac{1}{N-1}\sum_{j\neq i}^N l_j(\eta,x)\right). \qquad (3.4.14)
\end{aligned}$$

Hence, $l$ is the required solution to (3.2.12).

The next step is to prove $l \in L^{2,\phi,+}$. It is easy to see $l \geq 0$, and therefore, the goal is to prove that $\int_{\mathbb{R}} \|l\|_2^2(x)\phi(x)dx$ has an upper bound. For $M$ large enough, consider the set

$$U_i = \left\{x : l_{\rho(j)}(x) \geq M, \text{ for } j \leq i, l_{\rho(k)}(x) < M, \text{ for } k > i, \rho(\cdot) \text{ is a permutation of } 1,\cdots,N.\right\}.$$

Without loss of generality, suppose that the first $i$ components of $l(x)$ are no less than $M$. We only need to consider $x_3^*$ since this is the only possibility that $\hat{x}_i$ achieves numbers greater than $M$. From (3.2.10), for $l_j$ $(j \leq i)$, denote $G_j = \eta_i e^{\psi_i T} e^{-\left(r+\frac{1}{2}\theta^2\right)T}$, and

$$\begin{aligned}
G_j e^{-\theta x} =& (1-\tau_c)(A_j+B_j)\lambda_j\left((1-\tau_c)(A_j+B_j)l_j(x) + \tau_c D_j - (1-\tau_c)B_j\right)^{-\gamma_j} + (1-\lambda_j)\zeta_j Z_{-j} l_j^{-2}(x) \\
\leq& K_j M^{-\gamma_j} + (1-\lambda_j)\zeta_j \frac{1}{N-1}\sum_{k\neq j} l_k(x) l_j^{-2}(x),
\end{aligned}$$

where $K_j$ are some constant. Note that if $Z_{-j} = \frac{1}{N-1}\sum_{k\neq j} l_k(x) \leq K_j l_j^{2-h}(x)$ for some small $h > 0$,

$$G_j e^{-\theta x} \leq \tilde{K}_j \min\{M^{-\gamma_j}, M^{-h}\}, \tag{3.4.15}$$

where $\tilde{K}_j$ and $K_j$ are some constant less than $M$ if $M$ is large enough. Meanwhile, as long as one of $l_j$ ( $j = 1, \cdots, i$) has the property described above, $U_i \subset \{G_j e^{-\theta x} \leq \tilde{K}_j \min\{M^{-\gamma_j}, M^{-h}\}\}$.

Next, we show that there exists some large $M_0$ such that for any $M > M_0$ at least one of $l_j$ satisfies (3.4.15). Suppose not, and for all $M > 0$,

$$\frac{1}{N-1}\sum_{k\neq j} l_k(x) \geq K_j l_j^2(x) \text{ , for } j = 1, \cdots, i.$$

Hence, summing both sides from 1 to $i$, since

$$\frac{1}{N-1}\sum_{j=1}^{i}\sum_{k\neq j}^{N} l_k(x) = \frac{i-1}{N-1}\sum_{j=1}^{i} l_j(x) + \frac{i}{N-1}\sum_{m=i+1}^{N} l_m(x) \leq \frac{i-1}{N-1}\sum_{j=1}^{i} l_j(x) + \frac{i(N-i)}{N-1}M,$$

we can get

$$\frac{i-1}{N-1}\sum_{j=1}^{i} l_j(x) + \frac{i(N-i)}{N-1}M \geq \sum_{j=1}^{i} K_j l_j^2(x) \geq \min_j K_j \cdot \sum_{j=1}^{i} l_j^2(x).$$

Note that $l_j(x)$'s are greater than $M$, and $\min_j K_j$ should not grow at the same speed as $M$. Therefore,

$$\inf_{l_j(x)\geq M}\left[\frac{i-1}{N-1}\sum_{j=1}^{i} l_j(x) + \frac{i(N-i)}{N-1}M - \min_j K_j \cdot \sum_{j=1}^{i} l_j^2(x)\right] \geq 0,$$

or equivalently,

$$\frac{i(i-1)}{N-1}M + \frac{i(N-i)}{N-1}M - \min_j K_j \cdot iM^2 \geq 0.$$

However, the above inequality cannot hold for all $M > 0$. Contradiction, and $M_0$ exists.

With the $M_0$ shown above, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \|l(x)\|_2^2 \phi(x)dx = \int_{l(x)\leq M_0} \|l(x)\|_2^2 \phi(x)dx + \int_{l(x)>M_0} \|l(x)\|_2^2 \phi(x)dx \\
\leq & M_0 + \int_{M_0}^{\infty} K\mu(\|l\|_2 > K)dK \\
\leq & M_0 + \int_{\mathbb{R}^+}\int_{M_0}^{\infty} K\mu\left(G_j e^{-\theta x} \leq \tilde{K}_j \min\{K^{-\gamma_j}, M^{-h}\}\right) dK < \infty.
\end{aligned}$$

The last line is due to the upper bound for the tail of normal distribution. $\square$

*Proof of Lemma 19:* Existence is proved through Lemma 27. Consider (3.2.12) at arbitrary point $x \in \mathbb{R}$, and it becomes a non-linear system on $\mathbb{R}^N$. Denote $l_i(z) = l_i(x, z) \in \mathbb{R}$, and note that for any fixed point, $l_i$'s satisfy the reaction function, or in other words,

$$\hat{u}_i(y, z) = u_i(l_i(z), z) - l_i(z)y.$$

Take $l_i(z) = x_1^*(z)$ as example,

$$\begin{aligned}
&\frac{\partial\left(\hat{u}_i(y,z) + l_i(z)y\right)}{\partial z} = y\frac{\partial x_1^*}{\partial z} - (1-\lambda_i)\zeta_i\frac{1}{x_1^*} \\
=&\frac{(1-\lambda_i)\zeta_i(x_1^*)^{-1}y}{\gamma_i\lambda_i A_i(x_1^*)^{-\gamma_i} + 2(1-\lambda_i)\zeta_i z(x_1^*)^{-2}} - (1-\lambda_i)\zeta_i\frac{1}{x_1^*} \\
=&\frac{(1-\lambda_i)\zeta_i\left(y - \gamma_i\lambda_i A_i(x_1^*)^{-\gamma_i} - 2(1-\lambda_i)\zeta_i z(x_1^*)^{-2}\right)}{x_1^*\left(\gamma_i\lambda_i A_i(x_1^*)^{-\gamma_i} + 2(1-\lambda_i)\zeta_i z(x_1^*)^{-2}\right)} \\
=&\frac{(1-\lambda_i)\zeta_i\left((1-\gamma_i)\lambda_i A_i(x_1^*)^{-\gamma_i} - (1-\lambda_i)\zeta_i z(x_1^*)^{-2}\right)}{x_1^*\left(\gamma_i\lambda_i A_i(x_1^*)^{-\gamma_i} + 2(1-\lambda_i)\zeta_i z(x_1^*)^{-2}\right)} \le 0,
\end{aligned}$$

where the first and last line are due to (3.2.8) and $\gamma_i > 1$. Similar result can be found for $x_2^*$ and $x_3^*$. Hence, $u_i(l_i(z), z)$ is a non-increasing function of $z$. Meanwhile, due to the positive relationship between $Z_i$ and $Z_{-i}$ from (3.4.10), it is impossible that $l_i^{(1)} > l_i^{(2)}$ while $l_j^{(1)} < l_j^{(2)}$ for some $j \neq i$. Since otherwise, from player $i$, $\frac{1}{N-1}\sum_{k\neq i} l_k^{(1)} > \frac{1}{N-1}\sum_{k\neq i} l_k^{(2)}$ and $\frac{1}{N}\sum_{k=1}^N l_k^{(1)} > \frac{1}{N}\sum_{k=1}^N l_k^{(2)}$. However, similarly, from player $j$, $\frac{1}{N}\sum_{k=1}^N l_k^{(1)} < \frac{1}{N}\sum_{k=1}^N l_k^{(2)}$, which contradicts the result from player $i$. Hence, if there exist two different fixed points $l^{(1)}$ and $l^{(2)}$, $l^{(1)} - l^{(2)}$ has the same sign for each entry. Now we can conclude that (element-wise) lowest $l$ has the largest value for utility function for all player. Consider $l_n$ are sequence of the fixed point, and $l^*(\eta, x) = \liminf_{n\to\infty} l_n(\eta, x)$ is also a fixed point due to (3.4.14) in the proof of Lemma 27. Since $x$ is chosen arbitrarily, we can construct $l^*$ pointwisely by choosing the lowest fixed point for each point in $\mathbb{R}$. $\square$

Before the proof of Lemma 20, we should first find the partial derivative of $l_i$ with respect to $\eta_k$ for $i, k = 1, \cdots, N$. Note that $\hat{x}_i$ has only finite many jump points. Hence, except those points, the derivatives are valid for (3.2.12):

$$\begin{bmatrix} \frac{\partial l_1}{\partial \eta_k} \\ \frac{\partial l_2}{\partial \eta_k} \\ \vdots \\ \frac{\partial l_2}{\partial \eta_k} \\ \vdots \\ \frac{\partial l_N}{\partial \eta_k} \end{bmatrix} = \frac{1}{N-1}\begin{bmatrix} 0 & a_1 & \cdots & a_1 & \cdots & a_1 \\ a_2 & 0 & \cdots & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_k & a_k & \cdots & 0 & \cdots & a_k \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_N & a_N & \cdots & a_N & \cdots & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial l_1}{\partial \eta_k} \\ \frac{\partial l_2}{\partial \eta_k} \\ \vdots \\ \frac{\partial l_2}{\partial \eta_k} \\ \vdots \\ \frac{\partial l_N}{\partial \eta_k} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_k \\ \vdots \\ 0 \end{bmatrix}, \tag{3.4.16}$$

where $a_i$ can be one of the following values: 0 or

$$\frac{\partial x_{i1}^*}{\partial z} = \frac{(1-\lambda_i)\zeta_i}{\gamma_i\lambda_i(A_i x_{i1}^*)^{1-\gamma_i} + 2(1-\lambda_i)\zeta_i\frac{z}{x_{i1}^*}},$$
$$\frac{\partial x_{i2}^*}{\partial z} = \frac{(1-\lambda_i)\zeta_i}{\gamma_i\lambda_i((1-\tau_c)A_i)^2((1-\tau_c)A_i x_{i2}^* + \tau_c D_i)^{-\gamma_i-1}(x_{i2}^*)^2 + 2(1-\lambda_i)\zeta_i\frac{z}{x_{i2}^*}},$$
$$\frac{\partial x_{i3}^*}{\partial z} = \frac{(1-\lambda_i)\zeta_i}{\gamma_i\lambda_i((1-\tau_c)(A_i+B_i))^2((1-\tau_c)(A_i+B_i)x_{i3}^* + \tau_c D_i - (1-\tau_c)B_i)^{-\gamma_i-1}(x_{i3}^*)^2 + 2(1-\lambda_i)\zeta_i\frac{z}{x_{i3}^*}}.$$

Note that $a_i \in \left(0, \frac{l_i}{2z}\right)$. $b_i$ can be 0 or the product of $e^{\psi_i T}e^{-\theta x}$ and $\frac{\partial x_{ik}^*}{\partial y} < 0$ ($k = 1,2,3$) and $\frac{b_i}{a_i} = -\frac{l_i^2}{(1-\lambda_i)\zeta_i}e^{\psi_i T}e^{-\theta x}$. $b_i = 0$ if and only if $a_i = 0$, and $b_i = \frac{\partial x_{ik}^*}{\partial \gamma_i}$ if and only if $a_i = \frac{\partial x_{ik}^*}{\partial z}$. Without loss of generality, suppose that all $a_i$ are not 0, since otherwise, $\frac{\partial l_i}{\partial \gamma_k} = 0$ for $i \neq k$, or all partial derivatives are 0 for $i = k$, and we can delete those rows. Note that (3.4.16) can be rewritten as $Ul_{\eta_k} = -v_k$, where $U$ is an $N \times N$ all one matrix except that the diagonal is the vector consisting of $-\frac{N-1}{a_i}$ as $i$th entry, $l_{\eta_k}$ is an $N$-dimensional vector with $i$th entry as $\frac{\partial l_i}{\partial \eta_k}$, and $v_k$ is an $N$-dimensional all zero vector except $k$th element as $\frac{(N-1)b_k}{a_k}$.

**Lemma 28.** *$U$ is almost surely invertible, and*

$$\left(U^{-1}\right)_{ij} = \begin{cases} -\left(1+\frac{N-1}{a_i}\right)^{-1}\left(1 + \frac{\left(1+\frac{N-1}{a_i}\right)^{-1}}{1-\sum_{j=1}^N\left(1+\frac{N-1}{a_j}\right)^{-1}}\right) & , \text{ if } i = j, \\ -\frac{\left(1+\frac{N-1}{a_i}\right)^{-1}\left(1+\frac{N-1}{a_j}\right)^{-1}}{1-\sum_{j=1}^N\left(1+\frac{N-1}{a_j}\right)^{-1}} & , \text{ if } i \neq j. \end{cases}$$

*Proof.* Note that $U$ can be decomposed as $\tilde{U} + \mathbb{1}\cdot\mathbb{1}^T$, where $\mathbb{1}$ is an $N$-dimensional all one vector, and $\tilde{U}$ is an $N \times N$ diagonal matrix with $\tilde{U}_{ii} = -\left(1+\frac{N-1}{a_i}\right) \leq -1$. Hence, $\tilde{U}$ is invertible. Meanwhile,

$$\begin{aligned} 1 + \mathbb{1}^T\tilde{U}^{-1}\mathbb{1} =& 1 - \sum_{i=1}^N\left(1+\frac{N-1}{a_i}\right)^{-1} \geq 1 - \sum_{i=1}^N\left(1+\frac{2(N-1)Z_{-i}}{Z_i}\right)^{-1} \\ =& 1 - \sum_{i=1}^N \frac{Z_i}{\sum_{j=1}^N Z_j + \sum_{j\neq i}^N Z_j} \geq 1 - \sum_{i=1}^N \frac{Z_i}{\sum_{j=1}^N Z_j} = 0, \end{aligned}$$

where $Z_i = l_i, z = Z_{-i}$. Note that the equality holds only when one of $Z_i = \infty$, which has probability 0. Therefore, by Sherman-Morrison formula, $U$ is almost surely invertible and

$$U^{-1} = \tilde{U}^{-1} - \frac{\tilde{U}^{-1}\mathbb{1}\mathbb{1}^T\tilde{U}^{-1}}{1 - \sum_{i=1}^N\left(1+\frac{N-1}{a_i}\right)^{-1}}.$$

□

Note that for any small $\epsilon \leq \frac{1}{8(N-1)}$,

$$\begin{aligned}
\epsilon \geq & 1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1} \geq 1-\sum_{j=1}^{N}\left(1+\frac{2(N-1)Z_{-j}}{Z_j}\right)^{-1} = 1-\sum_{j=1}^{N}\frac{Z_j}{\sum_{i=1}^{N}Z_i+\sum_{i\neq j}Z_i} \\
= & \sum_{j=1}^{N}Z_j\frac{\sum_{i\neq j}Z_i}{\sum_{i=1}^{N}Z_i\left(\sum_{i=1}^{N}Z_i+\sum_{i\neq j}Z_i\right)} \geq \sum_{j=1}^{N}Z_j\frac{\sum_{i\neq j}Z_i}{2\left(\sum_{i=1}^{N}Z_i\right)^2} = \frac{1}{2}\sum_{i\neq j}\frac{Z_i}{\sum_{i=1}^{N}Z_i}\frac{Z_j}{\sum_{j=1}^{N}Z_j} \\
\geq & \frac{1}{2}\max_{i\neq j}\frac{Z_i}{\sum_{i=1}^{N}Z_i}\frac{Z_j}{\sum_{j=1}^{N}Z_j} \geq \frac{1}{2(N-1)}\left(1-\frac{Z_j}{\sum_{j=1}^{N}Z_j}\right)\frac{Z_j}{\sum_{j=1}^{N}Z_j},
\end{aligned}$$

where $j = \arg\max_k Z_k$ and $\frac{Z_j}{\sum_{j=1}^{N}Z_j} \geq \frac{1}{N}$. Hence, $\frac{Z_j}{\sum_{j=1}^{N}Z_j} \geq \frac{1+\sqrt{1-8(N-1)\epsilon}}{2} \geq 1-K\epsilon$ for small $\epsilon$ and constant $K > 2(N-1)$. Equivalently, $Z_j \geq \frac{1-K\epsilon}{K\epsilon}(N-1)Z_{-j}$. When $Z_{-j} > \epsilon^{1/2}$, $Z_j \geq \frac{1-K\epsilon}{K}\epsilon^{-1/2}$. In this case, from (3.2.10),

$$\begin{aligned}
G_je^{-\theta x} = & (1-\tau_c)(A_j+B_j)\lambda_j\left((1-\tau_c)(A_j+B_j)l_j(x)+\tau_cD_j-(1-\tau_c)B_j\right)^{-\gamma_j}+(1-\lambda_j)\zeta_jZ_{-j}l_j^{-2}(x) \\
\leq & K_j\epsilon^{\gamma_j/2}+(1-\lambda_j)\zeta_j\frac{Z_{-j}}{l_j}l_j^{-1} \leq K_j\epsilon^{\gamma_j/2}+(1-\lambda_j)\zeta_j(N-1)\frac{K\epsilon}{1-K\epsilon}\frac{K}{1-K\epsilon}\epsilon^{1/2} \leq K_j\epsilon^{\max\{3,\gamma_j\}/2},
\end{aligned}$$

where $K_j$ are constant varying lines to lines. Meanwhile, if $Z_{-j} \leq \epsilon^{1/2}$, there exist some $i \neq j$ such that $\epsilon^{1/2} \geq Z_i$. Considering (3.2.8), we have

$$G_ie^{-\theta x} = \lambda_iA_i^{1-\gamma_i}l_i^{-\gamma_i}(x)+(1-\lambda_i)\zeta_iZ_{-i}l_i^{-2}(x) \geq \lambda_iA_i^{1-\gamma_i}\epsilon^{-1/2}.$$

Therefore,

$$\left\{\epsilon \geq 1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right\} \subset \left\{G_je^{-\theta x} \leq K_j\epsilon^{\max\{3,\gamma_j\}/2}\right\}\bigcup\left\{G_ie^{-\theta x} \geq \lambda_iA_i^{1-\gamma_i}\epsilon^{-1/2}\right\},$$

for $j = \arg\max_k Z_k$ and some $i \neq j$.

**Lemma 29.** $\displaystyle\int_{\mathbb{R}}\left|\frac{\partial l_i}{\partial \eta_k}\right|^2(x)\phi(x)dx < \infty$ *for all* $i, k = 1, \cdots, N$.

*Proof.* By Lemma 28,

$$\frac{\partial l_i}{\partial \eta_k} = \begin{cases} \frac{(N-1)b_k}{a_k+(N-1)}\left(1+\frac{\left(1+\frac{N-1}{a_k}\right)^{-1}}{1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}}\right) & \text{, if } i = k, \\ \frac{(N-1)b_k}{a_k+(N-1)}\frac{\left(1+\frac{N-1}{a_i}\right)^{-1}}{1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}} & \text{, if } i \neq k. \end{cases}$$

Consider arbitrary $\Delta > 0$ and $\varepsilon \in (0, 1/2)$. Note that $\frac{b_i}{a_i} = -\frac{l_i^2}{(1-\lambda_i)\zeta_i}e^{\psi T}e^{-\theta x}$.

(i) On $U_1 \equiv \left\{\Delta^{\varepsilon} \leq 1 - \sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right\} \cap \{l_i(x) \leq \Delta^{-\varepsilon}\}$,

$$\int_{U_1}\left|\frac{\partial l_i}{\partial \eta_k}\right|^2 (x)\phi(x)dx \leq \int_{\mathbb{R}} 2(N-1)^2\left(\frac{b_i}{a_i}\right)^2\left(1+\frac{\left(1+\frac{N-1}{a_i}\right)^{-2}}{\left(1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right)^2}\right)\phi(x)dx$$

$$\leq \int_{\mathbb{R}} K_1\Delta^{-2\varepsilon}\left(1+\Delta^{-2\varepsilon}\right)e^{-2\theta x}\phi(x)dx \leq K_1\Delta^{-4\varepsilon},$$

where $K_1$ is a constant which may vary from lines to lines.

(ii) On $U_2 \equiv \left\{\Delta^{\varepsilon} \geq 1 - \sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right\} \cap \{l_i(x) \geq \Delta^{-\varepsilon}\}$, note that for $j = \arg\max_k l_k(\eta, x)$,

$$U_2 \subset \left\{G_j e^{-\theta x} \leq K_j\Delta^{\max\{3,\gamma_j\}\varepsilon/2}\right\} \bigcup \left\{G_i e^{-\theta x} \geq \lambda_i A_i^{1-\gamma_i}\Delta^{-\varepsilon/2}\right\} \bigcap \left\{G_i e^{-\theta x} \leq K_i\Delta^{\min\{\gamma_i,h\}\varepsilon}\right\}$$
$$\subset \left\{G_j e^{-\theta x} \leq K_j\Delta^{\max\{3,\gamma_j\}\varepsilon/2}\right\} \equiv \tilde{U}_2.$$

Hence, denoting $u_2 = -\frac{1}{\theta}\ln\left(\frac{K_j}{G_j}\Delta^{\max\{3,\gamma_j\}\varepsilon/2}\right)$, we get

$$\int_{U_2}\left|\frac{\partial l_i}{\partial \eta_k}\right|^2 (x)\phi(x)dx = \int_{\tilde{U}_2}\left|\frac{\partial l_i}{\partial \eta_k}\right|^2 (x)\phi(x)dx \leq 2\int_{u_2}^{\infty} z\Phi\left\{\left|\frac{\partial l_i}{\partial \eta_k}\right| > z\right\}dz$$
$$\leq 2\int_{u_2}^{\infty} z\Phi\left(\left\{\frac{(N-1)b_i}{a_i+(N-1)} \geq \sqrt{z}\right\} \bigcup \left\{\frac{\left(1+\frac{N-1}{a_i}\right)^{-1}}{1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}} \geq \sqrt{z}\right\}\right)dz$$
$$\leq 2\int_{u_2}^{\infty} z\left(\Phi\left\{\frac{b_i}{a_i} \geq \frac{\sqrt{z}}{N-1}\right\} + \Phi\left\{1-\sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1} \leq z^{-1/2}\right\}\right)dz$$
$$\leq 2\int_{u_2}^{\infty} z\left(\Phi\left(G_i e^{-\theta x} \leq K_i z^{-\frac{\min(\gamma_i,h)}{2(3-\gamma_i)}}\right) + \Phi\left(G_j e^{-\theta x} \leq K_j z^{-\max\{3,\gamma_j\}/4}\right) + \Phi_1\left(G_i e^{-\theta x} \geq \lambda_i A_i^{1-\gamma_i} z^{1/4}\right)\right)dz$$
$$\leq 2\int_{u_2}^{\infty} z\left(\Phi\left(x \geq K_{i1}\ln z\right) + \Phi(x \geq K_j\ln z) + \Phi(x \geq K_{i2}\ln z)\right)dz$$
$$\leq 6\int_{u_2}^{\infty} z\Phi(x \geq K\ln z)dz \leq K_2\int_{u_2}^{\infty} z\frac{e^{-K_3\ln^2(z)}}{\ln(z)}dz$$
$$= K_2\int_{\ln u_2}^{\infty} e^{2u}\frac{e^{-K_3u^2}}{u}du \leq K_2\int_{\ln u_2-\frac{1}{K_3}}^{\infty}\frac{1}{v}e^{-K_3v^2}dv \tag{3.4.17}$$
$$\leq K_2\frac{1}{(\frac{1}{2}\ln u_2)^2}\exp\left(-\frac{K_3}{4}\ln^2 u_2\right) \leq K_2\exp\left(-K_3\ln(u_2)\right) = K_2u_2^{-K_3} \tag{3.4.18}$$
$$\leq K_2(\ln(\Delta^{-1}))^{-K_3} \leq K_2|\ln\Delta|^{-K_3},$$

where $\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$, $K = \min\{K_{i1}, K_{i2}, K_j\}$ and $K_i, K_j, K_{i1}, K_{i2}, K_2, K_3$ are some constant, and may vary from lines to lines. (3.4.17) is due to change of variable: $u = \ln z$ and $v = u - \frac{1}{K_3}$. (3.4.18) is because of the upper bound for the tail of normal distribution and the fact that $\Delta$ can be chosen small enough such that $\ln(u_2) - \frac{1}{K_3} \geq \frac{1}{2}\ln(u_2) > 1$.

(iii) On $U_3 \equiv \left\{\Delta^{\varepsilon} \leq 1 - \sum_{j=1}^{N}\left(1 + \frac{N-1}{a_j}\right)^{-1}\right\} \cap \{l_i(x) \geq \Delta^{-\varepsilon}\} \subset \left\{G_i e^{-\theta x} \leq K_i \Delta^{\min\{\gamma_i, h\}\varepsilon}\right\} \equiv \tilde{U}_3$, denote $u_3 = -\frac{1}{\theta}\ln\left(\frac{K_i}{G_i}\Delta^{\min\{\gamma_i, g\}\varepsilon}\right)$.

$$\int_{U_3}\left|\frac{\partial l_i}{\partial \eta_k}\right|^2(x)\phi(x)dx \leq \int_{\tilde{U}_3} 2(N-1)^2\left(\frac{b_i}{a_i}\right)^2\left(1 + \frac{\left(1+\frac{N-1}{a_i}\right)^{-2}}{\left(1 - \sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right)^2}\right)\phi(x)dx$$

$$\leq K_4\left(1 + \Delta^{-2\varepsilon}\right)\int_{\tilde{U}_3}\left(\frac{b_i}{a_i}\right)^2\phi(x)dx \leq K_4\Delta^{-2\epsilon}\int_{u_3}^{\infty} z\Phi\left(\frac{b_i}{a_i} \geq z\right)dz$$

$$\leq K_4\Delta^{-2\varepsilon}\int_{u_3}^{\infty} z\Phi\left(x \geq K\ln z\right)dz \leq K_4\Delta^{-2\varepsilon}|\ln\Delta|^{-K_3},$$

where $K_4$ is the constant which may vary from lines to lines, and $K_3$ is the same as in case (ii).

(iv) On

$$\begin{aligned} U_4 \equiv & \left\{\Delta^{\varepsilon} \geq 1 - \sum_{j=1}^{N}\left(1 + \frac{N-1}{a_j}\right)^{-1}\right\} \cap \left\{l_i(x) \leq \Delta^{-\varepsilon}\right\} \\ \subset & \left\{G_j e^{-\theta x} \leq K_j \Delta^{\max\{3,\gamma_j\}\varepsilon/2}\right\} \bigcup \left\{G_i e^{-\theta x} \geq \lambda_i A_i^{1-\gamma_i}\Delta^{-\varepsilon/2}\right\} \\ \subset & \left\{x \geq -\frac{1}{\theta}\max\left\{\ln\left(\frac{K_j}{G_j}\Delta^{\max\{3,\gamma_j\}\varepsilon/2}\right), \ln\left(\frac{G_i}{\lambda_i A_i^{1-\gamma_i}}\Delta^{\varepsilon/2}\right)\right\}\right\} \equiv \tilde{U}_4, \end{aligned}$$

denote $u_4 = -\frac{1}{\theta}\max\left\{\ln\left(\frac{K_j}{G_j}\Delta^{\max\{3,\gamma_j\}\varepsilon/2}\right), \ln\left(\frac{G_i}{\lambda_i A_i^{1-\gamma_i}}\Delta^{\varepsilon/2}\right)\right\}$. Note that the last line is due to the symmetry of the centered normal distribution.

$$\int_{U_4}\left|\frac{\partial l_i}{\partial \eta_k}\right|^2(x)\phi(x)dx \leq \int_{\tilde{U}_4} 2(N-1)^2\left(\frac{b_i}{a_i}\right)^2\left(1 + \frac{\left(1+\frac{N-1}{a_i}\right)^{-2}}{\left(1 - \sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right)^2}\right)\phi_1(x)dx$$

$$\leq K_5\Delta^{-2\varepsilon}\int_{\tilde{U}_4}\frac{1}{\left(1 - \sum_{j=1}^{N}\left(1+\frac{N-1}{a_j}\right)^{-1}\right)^2}\phi_1(x)dx$$

$$\leq K_5\Delta^{-2\varepsilon}\int_{u_4}^{\infty} z\Phi_1\left(\left\{1 - \sum_{j=1}^{N}\left(1 + \frac{N-1}{a_j}\right)^{-1} \leq z^{-1/2}\right\}\right)dz \leq K_5\Delta^{-2\varepsilon}|\ln\Delta|^{-K_3},$$

where $K_5$ is the constant which may vary from lines to lines, and $K_3$ is the same as in case (ii).

In conclusion, $\int_{\mathbb{R}} \left|\frac{\partial l_i}{\partial \eta_k}\right|^2 (x)\phi(x)dx < \infty$. $\square$

*Proof of Lemma 20:* Define a new measure $\tilde{\mathbb{P}}$ such that $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(aW_t - \frac{1}{2}a^2T\right)$, where $a$ will be determined later. Denote the Brownian motion and the expectation under $\tilde{\mathbb{P}}$ as $\tilde{W}_t$ and $\tilde{\mathbb{E}}[\cdot]$ respectively. Hence the budget constraints for manager $i$ becomes

$$e^{-\left(\frac{1}{2}a^2+\theta a\right)T}\tilde{\mathbb{E}}\left[e^{\psi_i T}e^{-\left(r+\frac{1}{2}\theta^2\right)T}e^{-(\theta+a)\tilde{W}_T}l_i\left(\eta, \tilde{W}_T + aT\right)\right] = 1.$$

Define

$$P_{im}(\eta) = e^{-\left(\frac{1}{2}a^2+\theta a\right)T}\eta_i\tilde{\mathbb{E}}\left[e^{\psi_i T}e^{-\left(r+\frac{1}{2}\theta^2\right)T}e^{-(\theta+a)\tilde{W}_T}l_i\left(\eta, \tilde{W}_T + aT\right)\right] \wedge m \vee \frac{1}{m}.$$

The proof is split into two parts: the first part is to prove the existence and uniqueness of the solution to

$$\eta = P_m\eta, \tag{3.4.19}$$

where $P_m$ maps $\mathbb{R}^N$ to $\mathbb{R}^N$ and its $i$th entry is $P_{im}$. The second step is to show that there exists an $m_0$ such that for any $m \geq m_0$, the solution $\eta \in [1/m_0, m_0]$. Note that (3.4.19) is equivalent to the budget constraints for manager $i$.

(i) Define the space $Q_m = \left\{\eta \in \mathbb{R}^N, \eta_i \in [1/m, m]\right\}$. Then $P_m$ is an operator mapping from $Q_m$ to $Q_m$. Consider a sequence of $\eta^{(n)}$ and $P_m\eta^{(n)}$. Since for each $i = 1, \cdots, N$, except some negligible set,

$$\begin{aligned}\left|\frac{\partial P_{im}\eta^{(n)}}{\partial \eta_k^{(n)}}\right| \leq & e^{-\left(\frac{1}{2}a^2+\theta a\right)T}\left(\tilde{\mathbb{E}}\left[e^{\psi_i T}e^{-\left(r+\frac{1}{2}\theta^2\right)T}e^{-(\theta+a)\tilde{W}_T}l_i\left(\eta^{(n)}, \tilde{W}_T + aT\right)\right]\delta_{ik} + \right. \\ & \left. \eta_i^{(n)}\tilde{\mathbb{E}}\left[e^{\psi_i T}e^{-\left(r+\frac{1}{2}\theta^2\right)T}e^{-(\theta+a)\tilde{W}_T}\left|\frac{\partial l_i}{\partial \eta_k^{(n)}}\right|\right]\right),\end{aligned}$$

where $\delta_{ik} = \begin{cases} 1 & , i = k \\ 0 & , \text{otherwise} \end{cases}$. Note that since $\eta_i$'s are bounded and $l_i \in L^{2,\phi,+}$, the first term in the above inequality is bounded. Meanwhile, by Cauchy Schwarz inequality and Lemma 29, the second term in the inequality is also bounded. Therefore, $P_m(\eta)$ is a Lipschitz continuous function with Lipschitz constant independent of $n$. By choosing large enough $a$ such that the Lipschitz constant less than 1, we get a contraction mapping from $Q^m$ to $Q^m$, and hence there exists a unique fixed point $\eta$ to (3.4.19).

(ii) We will show by contradiction. Suppose not, and then for any $m > 0$, $\eta_i\mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] > m$ or $\eta_i\mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] < \frac{1}{m}$. Note that in these cases, $\eta_i = m$ or $\frac{1}{m}$.
If $\eta_i\mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] > m$, $\mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] > 1$. However, when $\eta_i = \infty$, according to (3.2.8), the optimal $Z_{iT} = 0$, which gives $\mathbb{E}\left[e^{\psi_i T}\xi_T l_i[\eta](W_T)\right] = 0 < 1$. By the continuity of the function, when $\eta_i$ is small, the expression should be close to 0, and less than 1. Contradiction.

If $\eta_i \mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] < \frac{1}{m}$, $\mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] < 1$. However, when $\eta = 0$, according to (3.2.10), the optimal $Z_{iT} = \infty$, which gives $\mathbb{E}\left[e^{\psi_i T}\xi_T l_i[\eta](W_T)\right] = \infty > 1$. By the continuity of the function, when $\eta_i$ is large, the expression should be larger than 1. Contradiction.

Combining the two arguments, we can conclude that there exists an $\eta$ such that for all $i = 1, \cdots, N$, $\mathbb{E}\left[e^{\psi_i T}\xi_T l_i(\eta, W_T)\right] = 1$, which satisfies the budget constraint. $\square$

### 3.4.3 Proof of Theorem 17

First, note that the partial derivative of $l_i$ with respect to $\gamma_k$, $\zeta_k$ and $\lambda_k$ has the same expression as (3.4.16), except that the partial derivatives and $b_i$ are changed accordingly. Based on Lemma 28, for $\gamma$,

$$\frac{\partial l_i}{\partial \gamma_k} = \begin{cases} \frac{(N-1)b_k}{a_k+(N-1)}\left(1 + \frac{\left(1+\frac{N-1}{a_k}\right)^{-1}}{1-\sum_{j=1}^N \left(1+\frac{N-1}{a_j}\right)^{-1}}\right) & \text{, if } i = k, \\ \frac{(N-1)b_k}{a_k+(N-1)} \frac{\left(1+\frac{N-1}{a_i}\right)^{-1}}{1-\sum_{j=1}^N \left(1+\frac{N-1}{a_j}\right)^{-1}} & \text{, if } i \neq k. \end{cases}$$

**Lemma 30.** *For* $\min_i \gamma_i > 1$*, denote the perturbation in* $\gamma$ *as* $\Delta\gamma$*, and the according Pareto optimal Nash equilibrium as* $l_{\Delta\gamma}$*. Let* $\Delta = \|\Delta\gamma\|_2$*. Then, there exists some function* $K_1$ *depending on* $\gamma$ *such that*

$$\|l - l_{\Delta\gamma}\|_{2,\phi} \leq K_1(\gamma)\Delta|\ln \Delta|.$$

*Proof.* The proof is similar to the proof of Lemma 29. We will split it four cases. The difference is at $b_i/a_i$ and we will approximate the difference between $l$ and $l_{\Delta\gamma}$ of manager $i$ in $L^{2,\phi,+}$ by

$$\int_{\mathbb{R}} \left|\frac{\partial l_i}{\partial \gamma_k}\right|^2 (x)\phi_1(x)dx.$$

If $\gamma_i < 2$,

$$\frac{b_i}{a_i + (N-1)} \leq \frac{b_i}{a_i} \leq \frac{\lambda_i(1-\tau_c)(A_i+B_i)\ln\left((1-\tau_c)(A_i+B_i)x_{i3}^* + \tau_c D_i - (1-\tau_c)B_i\right)}{(1-\lambda_i)\zeta_i(x_{i3}^*)^{-2}\left((1-\tau_c)(A_i+B_i)x_{i3}^* + \tau_c D_i - (1-\tau_c)B_i\right)^{\gamma_i}} \leq K_{i1}(x_{i3}^*)^{2-\gamma_i}\ln(x_{i3}^*),$$

for some constant $K_{i1}$ and large $x_{i3}^*$. If $l_i(x) = x_{i3}^*$ is very large, the upper bound for $\frac{b_i}{a_i}$ will explode. If $\gamma_i \geq 2$,

$$\left|\frac{b_i}{a_i + (N-1)}\right| \leq \left|\frac{b_i}{a_i}\right| \leq -\frac{\lambda_i A_i^{1-\gamma_i}\ln(A_i x_{i1}^*)}{(1-\lambda_i)\zeta_i(x_{i1}^*)^{\gamma_i-2}} \leq K_{i2}(x_{i1}^*)^{2-\gamma_i}|\ln(x_{i1}^*)|.$$

for some constant $K_{i2}$ and small $x_{i1}^*$. If $l_i(x) = x_{i1}^*$ is close to 0, the upper bound for $\frac{b_i}{a_i}$ will explode as well. Hence, we want to show that the probability of the large $\left|\frac{b_i}{a_i}\right|$ is small. For $\gamma_i < 2$, following the proof in Lemma 27, and if $l_i(x) \geq \epsilon^{-1}$ for small enough $\epsilon$, we have

$$G_i e^{-\theta x} \leq K_i \min\left\{\epsilon^{\gamma_i}, \epsilon^h\right\},$$

where $K_i$ is some constant and $h > 0$ is some small number.

For $\gamma_i < 2$ and $\varepsilon \in (0, 1/2)$, on (i) $U_1 = \left\{\Delta^\varepsilon \leq 1 - \sum_{j=1}^N \left(1 + \frac{N-1}{a_j}\right)^{-1}\right\} \cap \{l_i(x) \leq \Delta^{-\varepsilon}\}$,

$$\int_{U_1} \left|\frac{\partial l_i}{\partial \gamma_k}\right|^2 (x)\phi(x)dx \leq \int_{\mathbb{R}} 2(N-1)^2 \left(\frac{b_i}{a_i}\right)^2 \left(1 + \frac{\left(1 + \frac{N-1}{a_i}\right)^{-2}}{\left(1 - \sum_{j=1}^N \left(1 + \frac{N-1}{a_j}\right)^{-1}\right)^2}\right) \phi(x)dx$$

$$\leq \int_{\mathbb{R}} K_1 \varepsilon^2 \ln^2(\Delta) \Delta^{(2-\gamma_i)\varepsilon} \left(1 + \Delta^{-2\varepsilon}\right) \phi(x)dx \leq K_1 \ln^2(\Delta) \Delta^{-\gamma_i \varepsilon},$$

where $K_1$ is constant which may vary from lines to lines. Case (ii) - (iii) are the same as in the proof of Lemma 27. On (iv) $U_4$, the difference is that

$$\int_{U_4} \left|\frac{\partial l_i}{\partial \gamma_k}\right|^2 (x)\phi_1(x)dx \leq \int_{\tilde{U}_4} 2(N-1)^2 \left(\frac{b_i}{a_i}\right)^2 \left(1 + \frac{\left(1 + \frac{N-1}{a_i}\right)^{-2}}{\left(1 - \sum_{j=1}^N \left(1 + \frac{N-1}{a_j}\right)^{-1}\right)^2}\right) \phi_1(x)dx$$

$$\leq K_5 \ln^2(\Delta) \int_{\tilde{U}_4} \frac{1}{\left(1 - \sum_{j=1}^N \left(1 + \frac{N-1}{a_j}\right)^{-1}\right)^2} \phi_1(x)dx$$

$$\leq K_5 \ln^2(\Delta) \int_{u_4}^{\infty} z\Phi_1 \left(\left\{1 - \sum_{j=1}^N \left(1 + \frac{N-1}{a_j}\right)^{-1} \leq z^{-1/2}\right\}\right) dz \leq K_5 \ln^2(\Delta) |\ln \Delta|^{-K_3},$$

where $K_5$ is some constant which may vary from lines to lines and $\tilde{U}_4$ and $u_4$ are defined in case (iv) of the proof of Lemma 27.

Combining the above four cases, we can have

$$\int_{\mathbb{R}} \left|\frac{\partial l_i}{\partial \gamma_k}\right|^2 (x)\phi_1(x)dx \leq K_1(\gamma) \Delta^{-2\varepsilon} \ln^2 \Delta,$$

where $K$ is some constant depending on $\gamma$.

For $\gamma \geq 2$, similar results can be found by replacing the term $\{l_i(x) \geq \Delta^{-\varepsilon}\}$ to $\{l_i(x) \leq \Delta^{\varepsilon}\}$ in case (i), and accordingly for the other cases. Since both terms turn out to have the form of $\{x \geq -K \ln \Delta\}$ since $l_i(x) < \epsilon$ is equivalent to

$$G_i e^{-\theta x} \geq \lambda_i A_i^{1-\gamma_i} \epsilon^{-\gamma_i} + (1 - \lambda_i)\zeta_i Z_{-i} \epsilon^{-2} \geq \lambda_i A_i^{1-\gamma_i} \epsilon^{-\gamma_i}.$$

Hence, the results are the same. Therefore,

$$\|l - l_{\Delta\gamma}\|_{2,\phi} \leq K_1(\gamma) \Delta^{1-\varepsilon} \ln \Delta,$$

for any $\varepsilon \in (0, 1/2)$. Letting $\varepsilon$ go to 0, we have

$$\|l - l_{\Delta\gamma}\|_{2,\phi} \leq K_1(\gamma) \Delta |\ln \Delta|,$$

□

**Lemma 31.** *For* $\min_i \gamma_i > 1$*, denote the perturbation in* $\zeta$ *as* $\Delta\zeta$*, and the according Pareto optimal Nash equilibrium as* $l_{\Delta\zeta}$*. Let* $\Delta = \|\Delta\zeta\|_2$*. Then, there exists some function* $K_2$ *depending on* $\zeta$ *such that*

$$\|l - l_{\Delta\zeta}\|_{2,\phi} \leq K_2(\zeta)\Delta|\ln \Delta|.$$

*Proof.* The proof is similar to the proof of Lemma 30. The difference is that $\left|\frac{b_i}{a_i}\right| \leq \frac{Z_{-i}}{\zeta_i}$. Note that

$$\left\{\frac{b_i}{a_i} \geq \epsilon^{-1}\right\} \subset \left\{Z_{-i} \geq \zeta_i\epsilon^{-1}\right\} \subset \left\{l_j(x) \geq \zeta_i\epsilon^{-1}\right\},$$

for some $j \neq i$ since if $Z_{-i} \geq \zeta_i\epsilon^{-1}$, at least one of the $l_j(x) \geq \zeta_i\epsilon^{-1}$ due to the fact that $Z_{-i}$ is the average of $l_j$ for $j \neq i$. Therefore, following the same proof in Lemma 30,

$$\|l - l_{\Delta\zeta}\|_{2,\phi} \leq K_2(\zeta)\Delta|\ln \Delta|,$$

□

**Lemma 32.** *For* $\min_i \gamma_i > 1$*, denote the perturbation in* $\lambda$ *as* $\Delta\lambda$*, and the according Pareto optimal Nash equilibrium as* $l_{\Delta\lambda}$*. Let* $\Delta = \|\Delta\lambda\|_2$*. Then, there exists some function* $K_3$ *depending on* $\lambda$ *such that*

$$\|l - l_{\Delta\lambda}\|_{2,\phi} \leq K_2(\lambda)\Delta|\ln \Delta|.$$

*Proof.* The proof is similar to the proof of Lemma 30. If $\gamma_i < 2$,

$$\left|\frac{b_i}{a_i}\right| \leq \frac{Z_{-i}}{1-\lambda_i} + \frac{(1-\tau_c)(A_i+B_i)}{(1-\lambda_i)\zeta_i}(x_{i3}^*)^2\left((1-\tau_c)(A_i+B_i)x_{i3}^* + \tau_c D_i - (1-\tau_c)B_i\right)^{-\gamma_i}.$$

Otherwise,

$$\left|\frac{b_i}{a_i}\right| \leq \frac{Z_{-i}}{1-\lambda_i} + \frac{(1-\tau_c)(A_i+B_i)}{(1-\lambda_i)\zeta_i}(x_{i1}^*)^{2-\gamma_i},$$

for some constant $K_i$. For $\gamma_i < 2$, $\left\{\frac{b_i}{a_i} \geq \epsilon^{-1}\right\}$ is a subset of

$$\begin{aligned}
&\left\{\frac{Z_{-i}}{1-\lambda_i} + K_i(x_{i3}^*)^{2-\gamma_i} \geq \epsilon^{-1}\right\} \subset \left\{\frac{Z_{-i}}{1-\lambda_i} \geq \frac{\epsilon^{-1}}{2}\right\} \bigcup \left\{K_i l_i^{2-\gamma_i}(x) \geq \frac{\epsilon^{-1}}{2}\right\} \\
&\subset \left\{l_j(x) \geq \frac{1-\lambda_i}{2}\epsilon^{-1}\right\} \bigcup \left\{l_i(x) \geq \left(\frac{1}{2K_i}\right)^{\frac{1}{2-\gamma_i}} \epsilon^{-\frac{1}{2-\gamma_i}}\right\},
\end{aligned}$$

for some $j \neq i$. Similarly, for $\gamma_i \geq 2$, $\left\{\left|\frac{b_i}{a_i}\right| \geq \epsilon^{-1}\right\}$ is a subset of

$$\left\{l_j(x) \geq \frac{1-\lambda_i}{2}\epsilon^{-1}\right\} \bigcup \left\{l_i(x) \leq (2K_i)^{\frac{1}{\gamma_i-2}} \epsilon^{\frac{1}{\gamma_i-2}}\right\} \subset \left\{l_j(x) \geq \frac{1-\lambda_i}{2}\epsilon^{-1}\right\} \bigcup \left\{G_i e^{-\theta x} \geq \lambda_i A_i^{1-\gamma_i}\epsilon^{-\frac{\gamma_i}{\gamma_i-2}}\right\},$$

where the last one has the same reason in the proof of Lemma 30. Hence, the same result can be achieve through the process in the the proof of Lemma 30,

$$\|l - l_{\Delta\lambda}\|_{2,\phi} \leq K_3(\lambda)\Delta|\ln \Delta|.$$

□

*Proof of Theorem 17:* Based on Lemma 30, 31 and 32, there exist a function $K_4$ depending on $\gamma$, $\zeta$ and $\lambda$ such that

$$\|l - l_\Delta\|_{2,\phi} \leq K_4(\gamma, \zeta, \lambda)\Delta|\ln \Delta|,$$

where $l_\Delta$ is the Pareto optimal solution to $l_\Delta = Pl_\Delta$ with $(\gamma + \Delta\gamma, \zeta + \Delta\zeta, \lambda + \Delta\lambda)$. Hence,

$$\begin{aligned}
&\left|\hat{l}_i(t,x) - \hat{l}_{i\Delta}(t,x)\right| \leq K_5(t,\gamma,\zeta,\lambda)\int_{\mathbb{R}} e^{-\frac{(y-x)^2}{2(T-t)}} \left|l_i(y-\theta T) - l_{i\Delta}(y-\theta T)\right| dy \\
=&K_5(t,\gamma,\zeta,\lambda)\int_{\mathbb{R}} e^{-\frac{(y-x)^2}{2(T-t)}} e^{\frac{(y-\theta T)^2}{4T}} e^{-\frac{(y-\theta T)^2}{4T}} \left|l_i(y-\theta T) - l_{i\Delta}(y-\theta T)\right| dy \\
\leq&K_5(t,\gamma,\zeta,\lambda)\left(\int_{\mathbb{R}} \exp\left(-\frac{y^2 - 2(2x-\theta T)y + x^2}{4(T-t)}\right) dy\right) \|l - l_\Delta\|_{2,\phi} \\
\leq&K_5(t,\gamma,\zeta,\lambda) e^{\frac{3\left(x+\frac{2}{3}\theta T\right)^2}{4(T-t)}} \Delta|\ln \Delta|,
\end{aligned}$$

where $K_5$ is a function of $t, \gamma, \zeta, \lambda$, and may vary from line to line. Denote the coefficient in the last line above as $\tilde{K}_1(t, x, \gamma, \zeta, \lambda)$. Similarly,

$$\begin{aligned}
&\left|(\hat{l}_i)_x(t,x) - (\hat{l}_{i\Delta})_x(t,x)\right| \leq K_6(t,\gamma,\zeta,\lambda)\int_{\mathbb{R}} \frac{y-x}{T-t} e^{-\frac{(y-x)^2}{2(T-t)}} \left|l_i(y-\theta T) - l_{i\Delta}(y-\theta T)\right| dy \\
\leq&K_6(t,\gamma,\zeta,\lambda)\left(\int_{\mathbb{R}} \left(\frac{y-x}{T-t}\right)^2 \exp\left(-\frac{y^2 - 2(2x-\theta T)y + x^2}{4(T-t)}\right) dy\right) \|l - l_\Delta\|_{2,\phi} \\
\leq&K_6(t,\gamma,\zeta,\lambda)\left((x+\theta T)^2 + \sqrt{2(T-t)}\right) e^{\frac{3\left(x+\frac{2}{3}\theta T\right)^2}{4(T-t)}} \|l - l_\Delta\|_{2,\phi},
\end{aligned}$$

where $K_6$ is a function of $t, \gamma, \zeta, \lambda$, and may vary from line to line. Denote the coefficient above as $\tilde{K}_2(t, x, \gamma, \zeta, \lambda)$. Therefore,

$$|\pi_{it} - \pi_{i\Delta t}| \leq \left|\frac{(\hat{l}_i)_x(t, W_t + \theta t) + K_2(t, W_t + \theta t, \gamma, \zeta, \lambda)\Delta|\ln \Delta|}{\sigma\left(\hat{l}_i(t, W_t + \theta t) - K_1(t, W_t + \theta t, \gamma, \zeta, \lambda)\Delta|\ln \Delta|\right)} - \frac{(\hat{l}_i)_x(t, W_t + \theta t)}{\sigma \hat{l}_i(t, W_t + \theta t)}\right| \leq K(t, W_t, \gamma, \zeta, \lambda)\Delta|\ln \Delta|.$$

Combining all players' optimal strategies, we get the required result. $\square$

# Chapter 4

# Model III: LQG Mean Field Games with Common Noise

## 4.1 Markov Chain as Common Noise

### 4.1.1 Model

Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t : 0 \le t \le T\}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which $W$ and $B$ are two independent standard Brownian motions, and $Y$ is a continuous time Markov chain (CTMC) independent of $(W, B)$ taking values in $\{0, 1\}$ with a generator

$$Q = \begin{bmatrix} -\gamma_0 & \gamma_0 \\ \gamma_1 & -\gamma_1 \end{bmatrix}, \tag{4.1.1}$$

for some $\gamma_0 \ge 0, \gamma_1 \ge 0$. The Brownian motion $B$ does not play any role in MFG problem formulation until the convergence proof of the $N$-player game to MFGs.

In this paper, we formulate the $N$-player game in the completed filtered probability space

$$(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{F}^{(N)} := \{\mathcal{F}_t^{(N)} : 0 \le t \le T\}, \mathbb{P}^{(N)}), \tag{4.1.2}$$

and $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the same generator given by (4.1.1) and $W^{(N)} = (W_i^{(N)} : i = 1, \ldots, N)$ is a $N$-dimensional standard Brownian motion. We assume $Y^{(N)}$ and $W^{(N)}$ are independent of each other.

For better clarity, we use the superscript $(N)$ for a random variable to emphasize the probability space $\Omega^{(N)}$ it belongs to. For example, Proposition 6 denotes a random variable in $\Omega^{(N)}$ by $X^{(N)}$, while its distribution copy in $\Omega$ by $Z^N$, but not by $Z^{(N)}$.

Given a random measure flow $m : (0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, consider a generic player who wants to minimize the expected accumulated cost on $[0, T]$:

$$J(y, x, \alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}\alpha_s^2 + F(Y_s, X_s, m_s)ds + G(Y_T, X_T, m_T)\Big| Y_0 = y, X_0 = x\right] \tag{4.1.3}$$

with some given cost functions $F, G : \{0, 1\} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ and underlying random processes $(Y, X) : [0, T] \times \Omega \mapsto \{0, 1\} \times \mathbb{R}$. Among three processes $(Y, X, m)$, the generic player can control the process $X$ via $\alpha$ in the form of

$$X_t = X_0 + \int_0^t \alpha_s ds + W_t, \tag{4.1.4}$$

for all $t \in [0, T]$. We assume that the initial state $X_0$ is independent of $Y$. The process $Y$ of (4.1.1) represents the common noise and $m = (m_t)_{0 \le t \le T}$ is a given random density flow normalized up to total mass one.

The objective of the control problem for the generic player is to find its optimal control $\hat{\alpha} \in \mathcal{A} := L^4_{\mathbb{F}}$ to minimize the total cost, i.e.

$$V(y, x) = J(y, x, \hat{\alpha}) \le J(y, x, \alpha), \tag{4.1.5}$$

for all $\alpha \in \mathcal{A}$. Associated to the optimal control $\hat{\alpha}$, we denote the optimal path by $\hat{X} = (\hat{X}_t)_{0 \le t \le T}$. The Nash equilibrium is defined in Definition 5. Denote the value function of the control problem associated to the equilibrium measure $\hat{m}$ as MFG value function by

$$U(m_0, y, x) = V[\hat{m}](y, x). \tag{4.1.6}$$

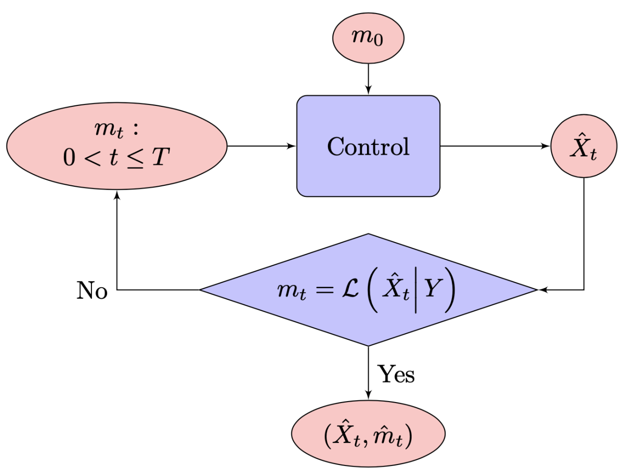



Figure 4.1: MFGs diagram.

The flowchart of MFGs diagram is given in Figure 4.1. It is noted from the optimality condition (4.1.5) and the fixed point condition (1.2.1) that

$$J[\hat{m}](y, x, \hat{\alpha}) \le J[\hat{m}](y, x, \alpha),$$

holds for any $\alpha$ for the equilibrium measure $\hat{m}$ and its associated equilibrium control $\hat{\alpha}$, while it is not

$$J[\hat{m}](y, x, \hat{\alpha}) \le J[m](y, x, \alpha),$$

for any $\alpha, m$. Otherwise this problem turns into a McKean-Vlasov control problem discussed in Nguyen et al. (2020).

The discrete counterpart of MFGs is $N$-player game, which is formulated below in the probability space $\Omega^{(N)}$. Recall that, $W^{(N)}_{it}$ and $W^{(N)}_{jt}$ are independent Brownian motions for $j \neq i$ and the common noise $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the generator given by (4.1.1). Let the player $i$ follow the dynamic, for $i = 1, 2, \ldots, N$,

$$dX^{(N)}_{it} = \alpha^{(N)}_{it} dt + dW^{(N)}_{it}, \quad X^{(N)}_{i0} = x^N_i. \tag{4.1.7}$$

In the above, the initial state is denoted by $x^N_i$ instead of $x^{(N)}_i$, since $x^N_i$ is independent of the choice of the sample $\omega^{(N)} \in \Omega^{(N)}$ as a constant.

The cost function for player $i$ associated to the control $\alpha^{(N)} = (\alpha^{(N)}_i : i = 1, \ldots, N) \in \mathcal{A}^N$ is

$$\begin{aligned} J^N_i(y, x^N, \alpha^{(N)}) = \mathbb{E}\bigg[\int_0^T \left(\frac{1}{2}|\alpha^{(N)}_{it}|^2 + F(Y^{(N)}_t, X^{(N)}_{it}, \rho(X^{(N)}_t))\right) dt + \\ G(Y^{(N)}_T, X^{(N)}_{iT}, \rho(X^{(N)}_T))\Big| X^{(N)}_0 = x^N, Y^{(N)}_0 = y\bigg], \end{aligned} \tag{4.1.8}$$

where $x^N = (x^N_1, x^N_2, \ldots, x^N_N) \in \mathbb{R}^N$ is the initial state for $N$ players and

$$\rho(x^N) = \frac{1}{N}\sum_{i=1}^N \delta_{x^N_i} \tag{4.1.9}$$

is the empirical measure of a vector $x^N$ with Dirac measure $\delta$.

### 4.1.2 Main Result with Quadratic Cost Structures

We consider the following two functions $F, G : \{0, 1\} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ in the cost functional (4.1.3):

$$F(y, x, m) = h(y) \int_{\mathbb{R}} (x - z)^2 m(dz), \tag{4.1.10}$$

and

$$G(y, x, m) = g(y) \int_{\mathbb{R}} (x - z)^2 m(dz), \tag{4.1.11}$$

for some $h, g : \{0, 1\} \mapsto \mathbb{R}^+$. In this case, the $F$ and $G$ terms in (4.1.8) of the $N$-player game can be written by

$$F(Y^{(N)}_t, X^{(N)}_{it}, \rho(X^{(N)}_t)) = \frac{h(Y^{(N)}_t)}{N}\sum_{j=1}^N (X^{(N)}_{it} - X^{(N)}_{jt})^2,$$

and

$$G(Y^{(N)}_T, X^{(N)}_{iT}, \rho(X^{(N)}_T)) = \frac{g(Y^{(N)}_T)}{N}\sum_{j=1}^N (X^{(N)}_{iT} - X^{(N)}_{jT})^2,$$

respectively.

First, we note that $F$ and $G$ possess the quadratic structures in $x$. Secondly, the coefficients $h(y)$ and $g(y)$ provide the sensitivity to the mean field effects, which depend on the current CTMC state. For another remark, let us consider the scenario where sensitivities are invariant, say

$$h(0) = h(1) = h, \; g(0) = g(1) = 0.$$

Then the cost function and hence the entire problem is free from the common noise. Interestingly, as shown in the Appendix 4.3.1, there is no global solution for MFGs when $h < 0$, while there is global solution when $h > 0$. Therefore, we require positive values for all sensitivities for simplicity. It is of course an interesting problem to investigate the explosion when some sensitivities take negative.

Wrapping up the above discussions, we impose the following assumptions:

(A1) The cost functions are given by (4.1.10)-(4.1.11) with $h, g > 0$; The initial $X_0$ of MFGs satisfies $\mathbb{E}[X_0^2] < \infty$.

(A2) In addition to (A1), as $N \to \infty$, the initial $\rho(x^N)$ of the $N$-player game is weakly convergent to the initial $\mathcal{L}(X_0)$ of MFGs.

Our objective of this paper is to understand the Nash equilibrium of MFGs and its connection to the $N$-player game equilibrium:

(P1) With Assumption (A1), characterize the MFG equilibrium path $\hat{X}$ and the value function $U$, as well as associated equilibrium measure $\hat{m}$ along the Definition 5;

(P2) With Assumption (A2), prove the convergence of $\hat{X}_{ut}^{(N)}$ from the $N$-player game in Definition 7 to $\hat{X}_t$ from MFGs in Definition 5.

For our first main result, we present the Riccati system for $(a_y, b_y, c_y, k_y : y = 0, 1)$:

$$\begin{cases} a_y' - 2a_y^2 - \gamma_y a_y + \gamma_y a_{1-y} + h_y = 0, \\ b_y' - 4a_y b_y - \gamma_y b_y + \gamma_y b_{1-y} + h_y = 0, \\ c_y' + a_y + b_y - \gamma_y c_y + \gamma_y c_{1-y} = 0, \\ k_y' - 2a_y^2 + 4a_y b_y - \gamma_y k_y + \gamma_y k_{1-y} = 0, \\ a_y(T) = b_y(T) = g_y \ , \ c_y(T) = k_y(T) = 0, \end{cases} \tag{4.1.12}$$

where $h_y = h(y)$, $g_y = g(y)$ for $y = 0, 1$.

**Theorem 33** (MFG)**.** *Under (A1), there exists a unique solution $(a_y, b_y, c_y, k_y : y = 0, 1)$ for the Riccati system (4.1.12). With these solutions, the MFG equilibrium path $\hat{X} = \hat{X}[\hat{m}]$ is given by*

$$d\hat{X}_t = 2a_{Y_t}(t)(\mathbb{E}[X_0] - \hat{X}_t)dt + dW_t, \quad \hat{X}_0 = X_0, \tag{4.1.13}$$

*with equilibrium control*

$$\hat{\alpha}_t = 2a_{Y_t}(t)(\mathbb{E}[X_0] - \hat{X}_t). \tag{4.1.14}$$

*Moreover, the value function $U$ is*

$$U(m_0, y, x) = a_y(0)x^2 - 2a_y(0)x[m_0]_1 + k_y(0)[m_0]_1^2 + b_y(0)[m_0]_2 + c_y(0), \quad y = 0, 1.$$

**Theorem 34** (Convergence)**.** *Under Assumption (A2), $(\hat{X}_{ut}^{(N)}, Y_t^{(N)})$ of the $N$-player game converges in distribution to the MFG equilibrium $(\hat{X}_t, Y_t)$ for any $t \in (0, T]$.*

#### 4.1.2.1 Remarks on the main results

The Nash equilibrium of the $N$-player game can be considered as $N$-coupled stochastic control problem. With the presence of the quadratic cost, the problem can be solved in the framework of LQG via Riccati system (4.1.27) in the subsequent part. However, the number of unknowns in this Riccati system is in the order of $O(N^3)$, which means the complexity of the solution has a polynomial growth in the number of players $N$.

To reduce the complexity, one can solve MFGs instead of solving the huge Riccati system. In our case, the MFG equilibrium control (4.2.14) of Theorem 33 suggests that a player in the $N$-player game shall steer towards the population center

$$\bar{x}^N = \frac{1}{N} \sum_{i=1}^{N} x_i^N \approx \mathbb{E}[X_0]$$

at a velocity proportional to her distance to the center $\bar{x}^N - X_{it}^{(N)}$ with the proportionality $2a_{Y_t}(t)$ simply determined from an ODE system of two equations:

$$a_y' - 2a_y^2 - \gamma_y a_y + \gamma_y a_{1-y} + h_y = 0, \ a_y(T) = g_y, \ y = 0, 1.$$

The above fact is exactly the essence of the MFG as its presence as the asymptotic version of $N$-player game and it can be demonstrated by numerical computations, see Section 4.1.5. Theorem 34 provides the theoretical justification for this phenomenon: the generic player $\hat{X}_{ut}^{(N)}$ from the $N$-player game behaves similarly to the generic player $\hat{X}_t$ from MFGs, which confirms why one can use MFG strategy in the $N$-player game for its approximation. To prove the convergence in distribution, we construct $Z_t^N$ in $\Omega$ such that $\mathcal{L}(Z_t^N, Y) = \mathcal{L}(\hat{X}_{ut}^{(N)}, Y^{(N)})$ and then prove the almost sure convergence $Z_t^N \to \hat{X}_t$. This procedure is called embedding and it is not a trivial matter. To see this, since $\Omega^{(N)}$ accommodating $N$-dimensional Brownian motion $W^{(N)}$ is much richer than $\Omega$ having only 2-dimensional Brownian motion $(W, B)$, it is in general impossible to replicate the distribution of any random variable from $\Omega^{(N)}$ to $\Omega$. The reason having such embedding is exactly due to the dimension-invariant feature of the mean field terms at equilibrium, see more details in the proof of Lemma 41. The crucial observation towards this decomposition is the pattern of $N \times N$ matrix $A_{iy}$ described in Table 4.1 and equation (4.1.29).

### 4.1.3 Riccati system for MFGs

This section is devoted to the proof of the first main result Theorem 33 on the MFG solution. First, we outline the scheme based on the Markovian structure of the equilibrium by reformulating the MFG problem. Next, we solve the underlying control problem and provide the corresponding Riccati system. Finally, Theorem 33 is proven by checking the fixed point condition of MFG problem.

By Definition 7, to solve for the equilibrium measure, one shall search the infinite dimensional space of the random measure flows $m : (0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, until a measure flow satisfies the fixed point condition $m_t = \mathcal{L}(\hat{X}_t | Y)$ for any $t \in (0, T]$, see Figure 4.1, which is equivalent to check the following infinitely many conditions: for any $k \in \mathbb{N}^+$, if they exist,

$$[m_t]_k = \mathbb{E}\left[ \hat{X}_t^k \middle| Y \right].$$

The first observation is that the cost functions $F$ and $G$ in (4.1.10)-(4.1.11) are dependent on the measure $m$ only via the first two moments:

$$F(y,x,m) = h(y)(x^2 - 2x[m]_1 + [m]_2),$$
$$G(y,x,m) = g(y)(x^2 - 2x[m]_1 + [m]_2).$$

Therefore, the underlying stochastic control problem for MFGs can be entirely determined by the input given by $\mathbb{R}^2$ valued random process $\mu_t = [m_t]_1$ and $\nu_t = [m_t]_2$, which implies that the fixed point condition can be effectively reduced to check two conditions only:

$$\mu_t = \mathbb{E}\left[\hat{X}_t \middle| Y\right], \ \nu_t = \mathbb{E}\left[\hat{X}_t^2 \middle| Y\right].$$

This observation effectively reduces our search from the space of random measure-valued processes $m : (0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$ to the space of $\mathbb{R}^2$-valued random processes $(\mu, \nu) : (0,T] \times \Omega \mapsto \mathbb{R}^2$.

Note that, if underlying MFGs have no common noise $Y$, then $(\mu,\nu)$ is a deterministic mapping $[0,T] \mapsto \mathbb{R}^2$ and the above observation is enough to reduce the original infinite dimensional MFGs into a finite dimensional system. However, the following example shows that this is not the case for MFGs with a common noise and it becomes the main drawback to characterize MFGs via a finite dimensional system.

To illustrate, we consider the following uncontrolled mean field dynamics: Let the mean field term $\mu_t := \mathbb{E}\left[\hat{X}_t \middle| Y\right]$, where the underlying dynamic is given by

$$d\hat{X}_t = -\mu_t Y_t dt + dW_t.$$

- $\mu_t$ is path dependent on $Y$, i.e.

  $$\mu_t = \mu_0 \exp\Big\{ - \int_0^t Y_s ds \Big\}.$$

  This implies that no finite dimensional system is possible to characterize the process $\mu_t$, since the $(t, Y) \mapsto \mu_t$ is a function on an infinite dimensional domain.

- $\mu_t$ is *Markovian*, i.e.

  $$d\mu_t = -Y_t \mu_t dt.$$

  It might be possible to characterize $\mu_t$ via a function $(t, Y_t, \mu_t) \mapsto \frac{d\mu_t}{dt}$ on a finite dimensional domain.

To solidify the above idea, we need to postulate the Markovian structure for the first and second moment of the MFG equilibrium. More precisely, our search for the equilibrium will be confined to the space $\mathcal{M}$ of measure flows whose first and second moment exhibits Markovian structure.

**Definition 35.** *The space $\mathcal{M}$ is the collection of all $\mathcal{F}_t^Y$-adapted measure flows $m : [0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, whose first moment $[m_t]_1 := \mu_t$ and second moment $[m_t]_2 := \nu_t$ satisfy*

$$\begin{aligned} \mu_t &= \mu_0 + \int_0^t \left(w_0(Y_s,s)\mu_s + w_1(Y_s,s)\right) ds, \\ \nu_t &= \nu_0 + \int_0^t \big(w_2(Y_s,s)\mu_s + w_3(Y_s,s)\nu_s + w_4(Y_s,s)\mu_s^2 + w_5(Y_s,s)\big) ds, \end{aligned} \tag{4.1.15}$$

*for all $t \in [0,T]$ and some smooth deterministic functions $(w_i : i = 0,1,\dots,5)$.*

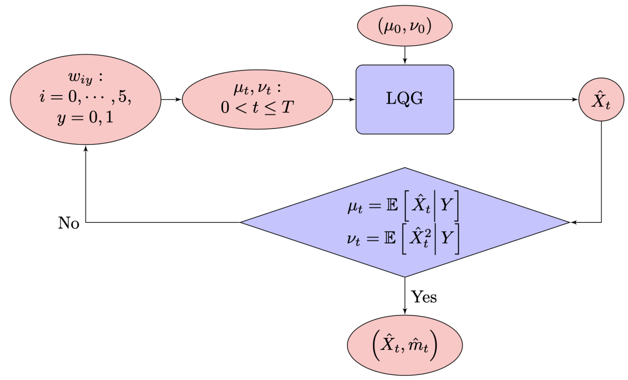



Figure 4.2: Equivalent MFGs diagram with $\mu_0 = [m_0]_1$ and $\nu_0 = [m_0]_2$.

The flowchart for our equilibrium is depicted in Figure 4.2.

Next, we give the derivation of the Riccati system for generic player's LQG problem with a given measure flow $m \in \mathcal{M}$. The advantage of the generic player's control problem is that its optimal path can be characterized via the following classical stochastic control problem:

- (P3) Given smooth functions $w = (w_i : i = 0, 1, \dots, 5)$, find the optimal value $\bar{V} = \bar{V}[w]$

$$\begin{aligned}\bar{V}(y, x, t, \bar{\mu}, \bar{v}) \quad &= \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\alpha_s^2 + \bar{F}(Y_s, X_s, \mu_s, \nu_s)\right) ds\right. \\ &\quad \left. + \bar{G}(Y_T, X_T, \mu_T, \nu_T) \middle| Y_t = y, X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu}\right]\end{aligned}$$

  underlying $\mathbb{R}^4$-valued processes $(Y, X, \mu, \nu)$ defined through (4.1.1)-(4.1.4)-(4.1.15) with the finite dimensional cost functions: $\bar{F}, \bar{G} : \mathbb{R}^4 \mapsto \mathbb{R}$ given by

$$\bar{F}(y, x, \bar{\mu}, \bar{\nu}) = h(y)(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

$$\bar{G}(y, x, \bar{\mu}, \bar{\nu}) = g(y)(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

  where $\bar{\mu}, \bar{\nu}$ are scalars, while $\mu, \nu$ are used as processes.

**Lemma 36.** *Given $m \in \mathcal{M}$ associated with $w = (w_i : i = 0, \dots, 5)$, the player's value (4.1.5) under assumption (A1) is*

$$U[m_0](y, x) = \bar{V}(y, x, 0, [m_0]_1, [m_0]_2),$$

*and the optimal control has a feedback form*

$$\hat{\alpha}_t = \bar{\alpha}(Y_t, X_t, t, \mu_t, \nu_t)$$

*underlying the processes $(Y, X, \mu, \nu)$ defined through (4.1.1)-(4.1.4)-(4.1.15), whenever there exists a feedback optimal control $\bar{\alpha}$ for the problem (P3).*

*Proof.* This is due to the quadratic cost structure in (4.1.10)-(4.1.11) $\square$

For the simplicity of notations, for each $i \in \{0, 1, 2, 3, 4, 5\}$ and $y \in \{0, 1\}$, denote function $v(y, x, t, \bar{\mu}, \bar{\nu})$ as $v_y$, and denote $w_i(y, t)$ as $w_{iy}$. We apply similar notations for other functions whenever they have a variable $y \in \{0,1\}$. Formally, under enough regularity conditions, the value function $\bar{V}$ defined in (P3) is the solution $v$ of the following coupled HJBs

$$\begin{cases} \partial_t v_0 + \frac{1}{2}\partial_{xx} v_0 - \frac{1}{2}(\partial_x v_0)^2 + \partial_\mu v_0 \left(w_{00}\bar{\mu} + w_{10}\right) + \partial_\nu v_0 \big(w_{20}\bar{\mu} + w_{30}\bar{\nu} \\ \qquad\qquad + w_{40}\bar{\mu}^2 + w_{50}\big) - \gamma_0 v_0 + \gamma_0 v_1 + \bar{F}_0 = 0, \\ \partial_t v_1 + \frac{1}{2}\partial_{xx} v_1 - \frac{1}{2}(\partial_x v_1)^2 + \partial_\mu v_1 \left(w_{01}\bar{\mu} + w_{11}\right) + \partial_\nu v_1 \big(w_{21}\bar{\mu} + w_{31}\bar{\nu} \\ \qquad\qquad + w_{41}\bar{\mu}^2 + w_{51}\big) - \gamma_1 v_1 + \gamma_1 v_0 + \bar{F}_1 = 0, \\ v_y(x, T, \mu_T, \nu_T) = \bar{G}_y(x, \mu_T, \nu_T), \ y = 0, 1. \end{cases} \tag{4.1.16}$$

Furthermore, the optimal control has to admit the feedback form of

$$\hat{\alpha}(t) = -\partial_x v(Y_t, \hat{X}_t, t, \mu_t, \nu_t). \tag{4.1.17}$$

Denote

$$\mathcal{S} = \left\{v \in L^\infty : \|\partial_{xx} v\|_\infty + \|\partial_t v\|_\infty < \infty, \|\partial_\mu v\|_\infty < \infty, \|\partial_\nu v\|_\infty < \infty\right\}.$$

**Lemma 37.** *Consider the control problem (P3) with some given smooth $w$.*

1. *(Verification theorem) Suppose there exists a solution $v \in \mathcal{S}$ of (4.1.16). Then, $v_y(x, t, \bar{\mu}, \bar{\nu}) = \bar{V}(y, x, t, \bar{\mu}, \bar{\nu})$ holds, and an optimal control is provided by (4.1.17).*

2. *Suppose that the value function $\bar{V}$ belongs to $\mathcal{S}$, and then $\bar{V}_y(x, t, \bar{\mu}, \bar{\nu}) := \bar{V}(y, x, t, \bar{\mu}, \bar{\nu})$ solves HJB equation (4.1.16). Moreover, $\hat{\alpha}$ of (4.1.17) is an optimal control.*

The costs $\bar{F}$ and $\bar{G}$ of (P3) are quadratic functions in $(x, \bar{\mu}, \bar{\nu})$, while the drift function of the process $\nu$ of (4.1.15) is not linear in $(x, \bar{\mu}, \bar{\nu})$. Therefore, the control problem (P3) does not fall into the standard LQG control framework. Nevertheless, similar to the LQG solution, we guess the value function as a quadratic function in the form of

$$v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t), \ y = 0, 1. \tag{4.1.18}$$

With the above setup, for $t \in [0, T]$, the optimal control is

$$\hat{\alpha}_t = -\partial_x v(Y_t, \hat{X}_t, t, \mu_t, \nu_t) = -2a_{Y_t}(t)\hat{X}_t - d_{Y_t}(t) - f_{Y_t}(t)\mu_t, \tag{4.1.19}$$

and the optimal path $\hat{X}$ is

$$d\hat{X}_t = \left(-2a_{Y_t}(t)\hat{X}_t - d_{Y_t}(t) - f_{Y_t}(t)\mu_t\right) dt + dW_t. \tag{4.1.20}$$

Denote the following ODE systems for $y, z = 0, 1$ and $y \neq z$,

$$\begin{cases} a_y' - 2a_y^2 - \gamma_y a_y + \gamma_y a_z + h_y = 0, \\ d_y' - 2a_y d_y + f_y w_{1y} - \gamma_y d_y + \gamma_y d_z = 0, \\ e_y' - d_y f_y + 2k_y w_{1y} + e_y w_{0y} + b_y w_{2y} - \gamma_y e_y + \gamma_y e_z = 0, \\ f_y' - 2a_y f_y + f_y w_{0y} - \gamma_y f_y + \gamma_y f_z - 2h_y = 0, \\ k_y' - \frac{1}{2}f_y^2 + 2k_y w_{0y} + b_y w_{4y} - \gamma_y k_y + \gamma_y k_z = 0, \\ b_y' + b_y w_{3y} - \gamma_y b_y + \gamma_y b_z + h_y = 0, \\ c_y' + a_y - \frac{1}{2}d_y^2 + e_y w_{1y} + b_y w_{5y} - \gamma_y c_y + \gamma_y c_z = 0, \end{cases} \tag{4.1.21}$$

with terminal conditions

$$a_y(T) = g_y, \ b_y(T) = g_y, \ c_y(T) = 0, \ d_y(T) = 0, \ e_y(T) = 0, \ f_y(T) = -2g_y, \ k_y(T) = 0. \tag{4.1.22}$$

**Lemma 38.** *Suppose there exists a unique solution* $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0, 1)$ *to the ODE system* (4.1.21)-(4.2.10) *on* $[0, T]$*. Then the value function of (P3) is*

$$\begin{aligned}\bar{V}(y, x, t, \bar{\mu}, \bar{\nu}) &= v_y(x, t, \bar{\mu}, \bar{\nu}) \\ =& a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t)\end{aligned} \tag{4.1.23}$$

*for* $y = 0, 1$ *and the optimal control and optimal path are given by* (4.1.19) *and* (4.1.20)*, respectively.*

Going back to the ODE system (4.1.21), there are 14 (7 pairs) equations, while we have total 26 deterministic functions of $[0, T] \times \mathbb{R}$ to be determined to characterize MFGs. Those are

$$(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0, 1) \text{ and } (w_{iy} : i = 0, \dots 5,\ y = 0, 1).$$

In this below, we identify the missing 12 equations by checking the fixed point condition:

$$\mu_s = \mathbb{E}\left[\hat{X}_s \middle| Y\right],\ \nu_s = \mathbb{E}\left[\hat{X}_s^2 \middle| Y\right], \quad \forall s \in [0, T], \tag{4.1.24}$$

where $\mu$ and $\nu$ are two auxiliary processes $(\mu, \nu)[w]$ defined in (4.1.15), see Figure 4.2. This leads to a complete characterization of the equilibrium for MFGs (P1).

Note that based on the dynamic of the optimal $\hat{X}$ defined in (4.1.20), the fixed point condition (4.1.24) implies that the first moment $\hat{\mu}_s := \mathbb{E}\left[\hat{X}_s \middle| Y\right]$ and the second moment $\hat{\nu}_s := \mathbb{E}\left[\hat{X}_s^2 \middle| Y\right]$ of the optimal path conditioned on $Y$ satisfy

$$\begin{cases}\hat{\mu}_s = \bar{\mu} + \displaystyle\int_t^s \left(-\left(2a_{Y_r}(r) + f_{Y_r}(r)\right)\hat{\mu}_r - d_{Y_r}(r)\right) dr, \\ \hat{\nu}_s = \bar{\nu} + \displaystyle\int_t^s \left(1 - 4a_{Y_r}(r)\hat{\nu}_r - 2d_{Y_r}(r)\hat{\mu}_r - 2f_{Y_r}(r)\hat{\mu}_r^2\right) dr,\end{cases} \tag{4.1.25}$$

for $s \geq t$. Note that under the optimal control in (4.1.19), comparing the terms in (4.1.15) and (4.1.25), we obtain another 12 equations:

$$w_{0y} = -2a_y - f_y,\ w_{1y} = -d_y,\ w_{2y} = -2d_y,\ w_{3y} = -4a_y,\ w_{4y} = -2f_y,\ w_{5y} = 1. \tag{4.1.26}$$

Using further algebraic structures, one can reduce the ODE system of 26 equations composed by (4.1.21) and (4.1.26) into a system of 8 equations of the form (4.1.12) for the MFG characterization in Theorem 33.

**Proof of Theorem 33.** Since $a_y$ $(y = 0, 1)$ has the same expressions as (4.1.12), its existence, uniqueness and boundedness are shown in Lemma 53. Given $a_y$ $(y = 0, 1)$ and smooth bounded $w$'s, $(b_y, d_y, e_y, f_y : y = 0, 1)$ is a coupled linear system, and their existence, uniqueness and boundedness is shown by Theorem 12.1 in Antsaklis and Michel (2006). Similarly, given $(b_y, d_y, f_y : y = 0, 1)$, $(k_y, c_y : y = 0, 1)$ is a linear system, and their existence and uniqueness is also guaranteed by Theorem 12.1 in Antsaklis and Michel (2006).

The ODE system (4.1.21) can be rewritten by

$$\begin{cases} a_y' - 2a_y^2 - \gamma_y a_y + \gamma_y a_z + h_y = 0, \\ d_y' - 2a_y d_y - f_y d_y - \gamma_y d_y + \gamma_y d_z = 0, \\ e_y' - d_y f_y - 2k_y d_y - e_y(2a_y + f_y) - 2b_y d_y - \gamma_y e_y + \gamma_y e_z = 0, \\ f_y' - 2a_y f_y - f_y(2a_y + f_y) - \gamma_y f_y + \gamma_y f_z - 2h_y = 0, \\ k_y' - \frac{1}{2}f_y^2 - 2k_y(2a_y + f_y) - 2b_y f_y - \gamma_y k_y + \gamma_y k_z = 0, \\ b_y' - 4a_y b_y - \gamma_y b_y + \gamma_y b_z + h_y = 0, \\ c_y' + a_y - \frac{1}{2}d_y^2 - e_y d_y + b_y - \gamma_y c_y + \gamma_y c_z = 0,\end{cases}$$

with the terminal conditions

$$a_y(T) = g_y,\ b_y(T) = g_y,\ c_y(T) = 0,\ d_y(T) = 0,\ e_y(T) = 0,\ f_y(T) = -2g_y,\ k_y(T) = 0.$$

Since $b_y$ $(y = 0, 1)$ has the same expressions as (4.1.12), its existence, uniqueness and boundedness are shown in Lemma 53. Meanwhile, with the given $(a_y, b_y : y = 0, 1)$, we denote $l_y = 2a_y + f_y$, and then

$$l_y' - l_y^2 - \gamma_y l_y + \gamma_y l_z = 0\ ,\ l_y(T) = 0.$$

By Lemma 51 and Lemma 52 in Appendix, there exists a unique solution for $l_y$ $(y = 0, 1)$, which is $l_y = 0, y = 0, 1$. This gives $f_y = -2a_y$ and $d_y' - \gamma_y d_y + \gamma_y d_z = 0$, which implies $d_y = 0, y = 0, 1$. Then, the equation for $e_y$ can be simplified as $e_y' - \gamma_y e_y + \gamma_y e_z = 0$, which indicates that $e_y = 0, y = 0, 1$. For $k_y, c_y$, with the given of $(a_y, b_y, f_y : y = 0, 1)$, we have

$$\begin{aligned}
&k_y' - \gamma_y k_y + \gamma_y k_z - 2a_y^2 + 4a_y b_y = 0\ ,\ k_y(T) = 0,\\
&c_y' + a_y - \frac{1}{2} d_y^2 - e_y d_y + b_y - \gamma_y c_y + \gamma_y c_z = 0.
\end{aligned}$$

The existence and uniqueness of the solution for $k_y, c_y$ $(y = 0, 1)$ are yielded by Theorem 12.1 in Antsaklis and Michel (2006).

Note that in this case, since $2a_y + f_y = 0$ and $d_y = 0$ for $y = 0, 1$, from (4.1.25) we have $\hat{\mu}_s = \bar{\mu}$ for all $s \in [t, T]$. Then

$$\hat{\nu}_s = \bar{\nu} + \int_t^s \left(1 + 4a_{Y_r}(r)\bar{\mu}^2 - 4a_{Y_r}(r)\hat{\nu}_r\right) dr.$$

Plugging $d_y = 0$ for $y = 0, 1$ and $\hat{\mu}_s = \bar{\mu}$ back to (4.1.19), we obtain the optimal control by

$$\hat{\alpha}_s = 2a_{Y_s}(s)(\bar{\mu} - \hat{X}_s).$$

Since we have $d_y = 0$ for $y = 0, 1$ and $\mu_s = \bar{\mu}$ for $s \in [t, T]$, the value function can be simplified from (4.1.18) to

$$v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t)x^2 - 2a_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t).$$

By the equivalence Lemma 36, it yields the value function $U$ of Theorem 33 . Moreover, since $f_y = -2a_y$ and $k_y \neq 0$, the ODE system (4.1.21) together with (4.1.26) can be reduced into (4.1.12). From the Lemma 53, the existence and uniqueness of $(a_y, b_y, c_y, k_y : y = 0, 1)$ in (4.1.12) is guaranteed. $\square$

### 4.1.4 The $N$-player Game and its Convergence to MFGs

In this section, we show the convergence of the $N$-player game to MFGs. To simplify the presentation, we omit the superscript $(N)$ for the processes in the probability space $\Omega^{(N)}$, whenever there is no confusion. First, we solve the $N$-player game, which provides a Riccati system consisting of $O(N^3)$ equations. Then the corresponding Riccati system is reduced into an ODE system whose dimension is independent to $N$. This becomes the key building block for the proof of the convergence.

The $N$-player game is indeed an $N$-coupled stochastic LQG problem by its very own definition. Therefore, the solution can be derived via Riccati system given below: For $i = 1, 2, \ldots, N$, $y = 0, 1$,

$$
\begin{cases}
A'_{iy} - 2A_{iy}^\top e_i e_i^\top A_{iy} - 4\sum_{j\neq i} A_{jy}^\top e_j e_j^\top A_{iy} - \gamma_y A_{iy} + \gamma_y A_{i(1-y)} \\
\qquad\qquad\qquad\qquad\qquad + \dfrac{h_y}{N}\sum_{j\neq i}(e_i - e_j)(e_i - e_j)^\top = 0, \\
B'_{iy} - 2A_{iy}^\top e_i e_i^\top B_{iy} - 2\sum_{j\neq i}\Big(A_{iy}^\top e_j e_j^\top B_{jy} + A_{jy}^\top e_j e_j^\top B_{iy}\Big) - \gamma_y B_{iy} + \gamma_y B_{i(1-y)} = 0, \\
C'_{iy} - \dfrac{1}{2}B_{iy}^\top e_i e_i^\top B_{iy} - \sum_{j\neq i} B_{jy}^\top e_j e_j^\top B_{iy} + tr(A_{iy}) - \gamma_y C_{iy} + \gamma_y C_{i(1-y)} = 0, \\
A_{iy}(T) = \dfrac{g_y}{N}\Lambda_i \ , \ B_{iy}(T) = C_{iy}(T) = 0,
\end{cases}
\tag{4.1.27}
$$

where the solutions consist of $N \times N$ symmetric matrices $A_{iy}$'s, $N$-dimensional vectors $B_{iy}$'s, and $C_{iy} \in \mathbb{R}$. In the above, $\Lambda_i$'s are $N \times N$ matrices with diagonal 1 except $(\Lambda_i)_{ii} = N - 1$, $(\Lambda_i)_{ij} = (\Lambda_i)_{ji} = -1$ for any $j \neq i$ and the rest entries as 0, and $e_i$'s are the $N$-dimensional natural basis.

**Lemma 39.** *Suppose $(A_{iy}, B_{iy}, C_{iy} : i = 1, 2, \ldots, N,\ y = 0, 1)$ is the solution of (4.1.27). Then, the value functions of $N$-player game defined by (1.2.2) is*

$$
V_i(y, x^N) = (x^N)^\top A_{iy}(0) x^N + (x^N)^\top B_{iy}(0) + C_{iy}(0), \quad i = 1, \ldots, N.
$$

*Moreover, the path and the control under the equilibrium are*

$$
d\hat{X}_{it} = \Big(-2(A_{iY_t})_i^\top \hat{X}_t - (B_{iY_t})_i\Big)\, dt + dW_{it}, \quad i = 1, \ldots, N, \tag{4.1.28}
$$

*and*

$$
\hat{\alpha}_{it} = -2(A_{iY_t})_i^\top \hat{X}_t - (B_{iY_t})_i,
$$

*where $(A)_i$ denotes the $i$-th column of matrix $A$, $(B)_i$ denotes the $i$-th entry of vector $B$ and $\hat{X}_t = \begin{bmatrix}\hat{X}_{1t} & \hat{X}_{2t} & \cdots & \hat{X}_{Nt}\end{bmatrix}^\top$.*

Our objective is the convergence of $(\hat{X}^{(N)}_{ut}, Y^{(N)}_t)$ generated by (4.1.28) relying on the solution of Riccati system (4.1.27) to the $(\hat{X}_t, Y_t)$ of (4.2.13). Note that $\hat{X}_t$ relies only on two functions $(a_y : y = 0, 1)$ while $\rho(\hat{X}^{(N)}_t)$ depends on $O(N^3)$ functions from $(A_{iy} : i = 1, 2, \ldots, N,\ y = 0, 1)$, which can be solved from a huge Riccati system. Therefore, it is almost a hopeless task to see the connection between these two processes without gaining further insight on the structure of Riccati system (4.1.27).

To proceed, let us first observe some hidden patterns from a numerical result for the solution of Riccati (4.1.27). Table 4.1 shows the $A_{20}$ at $t = 1$ for $N = 5$ with same parameters as in figure 4.3 and figure 4.4 in Section 4.1.5. Inspired by the numerical example of $A_{iy}$ in Table 4.1, we may want to believe a pattern

$$
(A_{iy})_{pq} = \begin{cases}
a_{1y}(t), & \text{if } p = q = i, \\
a_{2y}(t), & \text{if } p = q \neq i, \\
a_{3y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\
a_{4y}(t), & \text{otherwise.}
\end{cases}
\tag{4.1.29}
$$

|  |  |  |  |  |
|---|---|---|---|---|
| 0.1319 | -0.1924 | 0.0202 | 0.0202 | 0.0202 |
| -0.1924 | 0.7696 | -0.1924 | -0.1924 | -0.1924 |
| 0.0202 | -0.1924 | 0.1319 | 0.0202 | 0.0202 |
| 0.0202 | -0.1924 | 0.0202 | 0.1319 | 0.0202 |
| 0.0202 | -0.1924 | 0.0202 | 0.0202 | 0.1319 |

Table 4.1: $A_{20}(1)$ for $N = 5$

The next result justifies the above pattern: the $N^2$ entries of the matrix $A_{iy}$ can be embedded to a 4-dimensional vector space no matter how big $N$ is.

**Lemma 40.** *There exists a unique solution $(a_{1y}^N, a_{2y}^N)$ from the ODE system(4.1.30).*

$$\begin{cases} a'_{1y} - \frac{2(N+1)}{N-1} a_{1y}^2 - \gamma_y a_{1y} + \gamma_y a_{1(1-y)} + \frac{N-1}{N} h_y = 0, \\ a'_{2y} + \frac{2}{(N-1)^2} a_{1y}^2 - \frac{4N}{N-1} a_{1y} a_{2y} - \gamma_y a_{2y} + \gamma_y a_{2(1-y)} + \frac{h_y}{N} = 0, \\ a_{1y}(T) = \frac{N-1}{N} g_y \ , \ a_{2y}(T) = \frac{g_y}{N} \end{cases} \tag{4.1.30}$$

*for $y = 0, 1$. Moreover,*

$$(A_{iy})_{pq} = \begin{cases} a_{1y}(t), & \text{if } p = q = i, \\ a_{2y}(t), & \text{if } p = q \neq i, \\ -\frac{1}{N-1} a_{1y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ \frac{1}{(N-1)(N-2)} a_{1y}(t) - \frac{1}{N-2} a_{2y}(t), & \text{otherwise.} \end{cases} \tag{4.1.31}$$

*The path and the control of player $i$ under the equilibrium are*

$$d\hat{X}_{it}^{(N)} = -2a^N_{1Y_t^{(N)}}(t) \left( \hat{X}_{it}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^{N} \hat{X}_{jt}^{(N)} \right) dt + dW_{it}^{(N)}, \quad i = 1, \dots, N, \tag{4.1.32}$$

*and*

$$\hat{\alpha}_{it}^{(N)} = -2a^N_{1Y_t^{(N)}}(t) \left( \hat{X}_{it}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^{N} \hat{X}_{jt}^{(N)} \right).$$

The key is to provide an explicit embedding of $(\hat{X}_{ut}^{(N)}, Y_t^{(N)})$ to the same probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ when we prove the convergence in distribution of $(\hat{X}_{ut}^{(N)}, Y_t^{(N)})$ generated by (4.1.32) in the sample space $\Omega^{(N)}$ towards the $(\hat{X}_t, Y_t)$ of (4.2.13) in $\Omega$ . Note that, no matter how large $N$ is, our objective is to copy the distribution of $\hat{X}_t^{(N)}$ from $\Omega^{(N)}$ having $N$-dimensional Brownian motion $W^{(N)}$ to a smaller space $\Omega$ having only two Brownian motions $W$ and $B$. In general, the space of random processes generated by $N$-dimensional Brownian motion is much richer than the one generated by 2-dimensional Brownian motion whenever $N > 2$. However, it is possible for our case to copy the distribution $(\hat{X}_{ut}^{(N)}, Y_t^{(N)})$ due to the nature of the mean field effect. Next we present the coupling result.

**Lemma 41.** *Let $Z^N$ be the solution of*

$$Z_t^N = x_u^N - \int_0^t 2\hat{a}_{1Y_s}^N(s) \left( Z_s^N - (\bar{x}^N + \frac{\sqrt{N-1}}{N} B_s + \frac{1}{N} W_s) \right) ds + W_t, \tag{4.1.33}$$

*where $u$ is the random variable uniformly distributed on set $\{1, 2, \ldots, N\}$, $W$ and $B$ are Brownian motions on the $(\Omega, \mathcal{F}_T, \mathbb{P})$ defined in Section 4.1.1, and*

$$\hat{a}_{1y}^N = \frac{N}{N-1} a_{1y}^N,$$

*where $a_{1y}^N$ is from the ODE system(4.1.30). Then, $(Z_t^N, Y_t)$ in $(\Omega, \mathcal{F}_T, \mathbb{P})$ has the same distribution as $(\hat{X}_{ut}^{(N)}, Y_t^{(N)})$ in $(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)})$.*

Now we turn to the proof of the convergence.

*Proof of Theorem 34.* We define

$$\mathcal{E}_t(b) = \exp\left\{ \int_0^t b_s ds \right\}.$$

and

$$G_t(x, b, W) = \mathcal{E}_t(-b)x + \mathcal{E}_t(-b) \int_0^t \mathcal{E}_s(b) dW_s.$$

Then, we can rewrite the process $Z^N$ of (4.1.33) by

$$Z_t^N = G_t(x_u^N, 2\hat{a}_1^N(Y_\cdot, \cdot), W)$$

and write $\hat{X}$ by

$$\hat{X}_t = G_t(X_0, 2a(Y_\cdot, \cdot), W)$$

1. We recall that the term $\bar{x}^N = \frac{1}{N}\sum_{i=1}^N x_i^N$ is a deterministic real number. Since $\rho(x^N)$ is weakly convergent to the law of $X_0$ by (A2), one can have

$$\langle \phi, \rho(x^N) \rangle \to \mathbb{E}[\phi(X_0)],$$

   for all test functions $\phi$. If we use $\phi(x) = x$, then it yields $\bar{x}^N \to \mathbb{E}[X_0]$. Hence, we have

$$\bar{x}^N + \frac{\sqrt{N-1}}{N} B_t + \frac{1}{N} W_t \to \mathbb{E}[X_0] \text{ almost surely.}$$

2. Note that from (4.1.30), the convergence $a_{1y}^N \to a_y$ holds in $L^\infty[0,T]$, where $a_y$ is the solution from (4.1.12). Therefore, we have the almost sure convergence

$$\lim_{N\to\infty} \|\hat{a}_1^N(Y_\cdot, \cdot) - a(Y_\cdot, \cdot)\|_\infty = 0, \text{ almost surely .}$$

   By (A2), $x_u^N$ converges to $X_0$ in distribution. By Skorohod representation theorem, we can have a replication $\bar{x}_u^N$ and $\bar{X}_0$ in the same probability space with almost sure convergence. For the simplicity of notation, we assume that

$$x_u^N \to X_0 \text{ almost surely .}$$

   Finally, since $G_t$ is continuous on $\mathbb{R} \times L^\infty \times L^\infty$, we have

$$G_t(x_u^N, 2\hat{a}_1^N(Y_\cdot, \cdot), W) \to G_t(X_0, 2a(Y_\cdot, \cdot), W), \text{ almost surely .}$$

Therefore, we conclude that $Z_t^N \to \hat{X}_t$ almost surely. Combine with Lemma 41 and Proposition 6, we conclude the desired weak convergence. □

### 4.1.5 Numerical Examples

For MFGs, we have derived a 8 dimensional Riccati ODE system (4.1.12) to determine the parameter functions

$$(a_y, b_y, c_y, k_y : y = 0, 1)$$

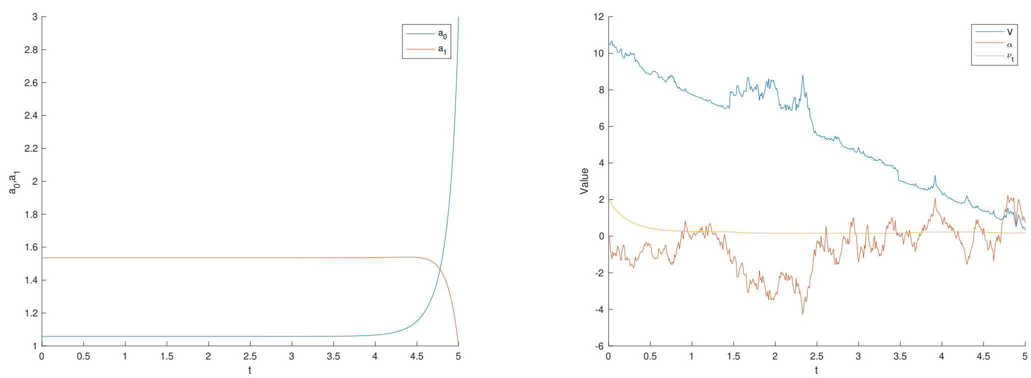
needed for the characterization of the equilibrium and the value function. Meanwhile, we also show the solvability of the Riccati ODE system in Section 4.1.3.

As mentioned earlier, different from the MFG characterization with the common noise, the derived Riccati system is essentially finite dimensional. In this subsection, we present an numerical experiment and show some numerical results for solving Riccati system to demonstrate its computational advantages.

For the illustration purpose, assume the finite time horizon is given with $T = 5$ and that the coefficients of the dynamic equation are listed below

$$\gamma_0 = 0.5,\ \gamma_1 = 0.6,\ h_0 = 2,\ h_1 = 5,\ g_0 = 3,\ g_1 = 1,\ \mu_0 = 0,\ \nu_0 = 2.$$

Firstly, using Euler's forward difference method with the step size $\delta = 10^{-2}$, we can obtain trajectories of $(a_y, b_y, c_y : y = 0, 1)$, which is the solution of ODE system (4.1.12). Next, using the trajectories of the parameter functions and Markov chain $Y_t$, we can achieve the simulations for $\hat{\alpha}_t$ and $\hat{X}_t$.




(a) $a_y(t)$

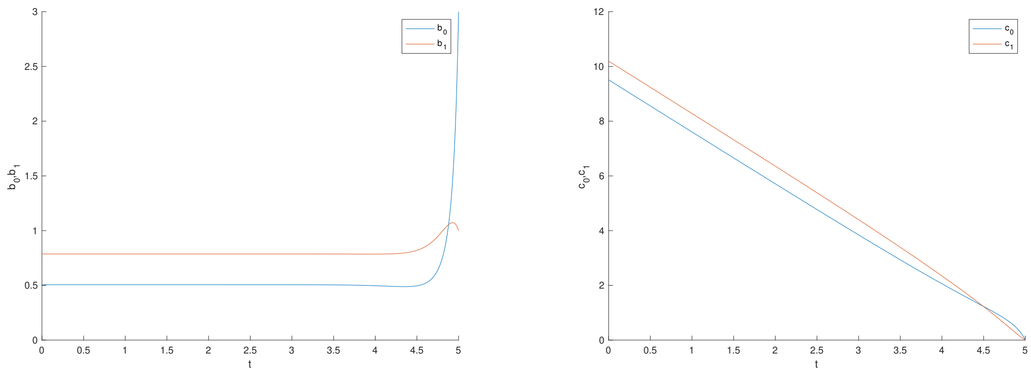
(b) Value function $V$, optimal control $\alpha$, and conditional second moment $\nu$

Figure 4.3: Simulations for $a_y, V, \alpha$ and $\nu$.

As shown in figure 4.3, people tend to centralize since the conditional second moment of the population density $\nu_t$ is always decreasing.

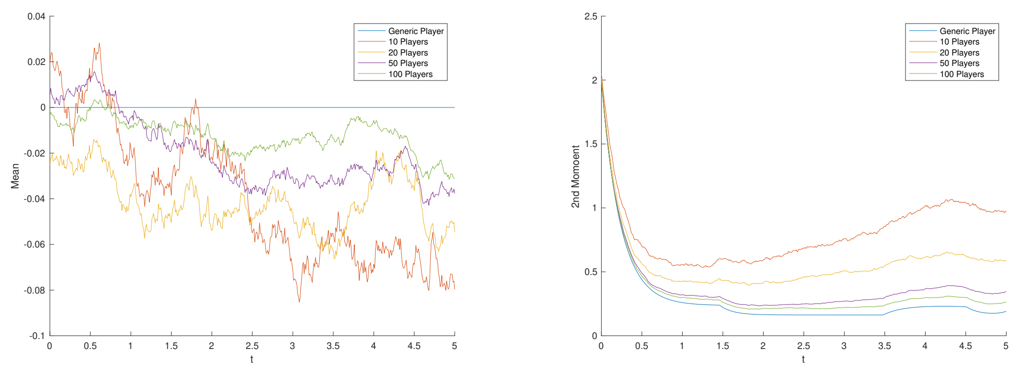
In section 4.1.4, we showed that the generic player's path for $N$-player game is convergent to the generic player's path for MFGs. In this subsection, we demonstrate the convergence of the conditional first moment, conditional second moment and the value functions of the $N$-player game to the corresponding terms of the generic player in Mean Field Game setup by using some numerical examples.

The following figures show the value functions, $\mu^{(N)}$ and $\nu^{(N)}$ under $N \in \{10, 20, 50, 100\}$ with the same parameters' settings as in figure 4.3 and figure 4.4. We can clearly see the convergence to the solution of the generic player.




(a) $b_y(t)$ (b) $c_y(t)$

Figure 4.4: Simulations for $b_y$ and $c_y$.




(a) $\mu_t$: conditional mean of the population density (b) $\nu_t$: conditional 2nd moment of the population density

Figure 4.5: Simulations for $\mu_t$ and $\nu_t$.

## 4.2 Brownian Motion as Common Noise

### 4.2.1 Model

The settings are similar to those in Section 4.1.1 except that the common noise is an independent Brownian motion instead of the continuous time Markov Chain. Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t : 0 \le t \le T\}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which $W$, $\tilde{W}$ and $B$ are three independent standard Brownian motions. $(W_t)_{0\le t\le T}$ is called individual or idiosyncratic noise and $(\tilde{W}_t)_{0\le t\le T}$ is called common noise. Denote $\tilde{\mathcal{F}}_t^s$ as the filtration generated by $(\tilde{W}_r - \tilde{W}_s :\ s \le r \le t)$ and $\tilde{\mathcal{F}}_t := \tilde{\mathcal{F}}_t^0$.

The $N$-player game is in the same complete filtered space defined in (4.1.2) generated by $W^{(N)} = (W_i^{(N)} : i = 1, \cdots, N)$ and $\tilde{W}$. All Brownian motions are independent. The accumulated cost

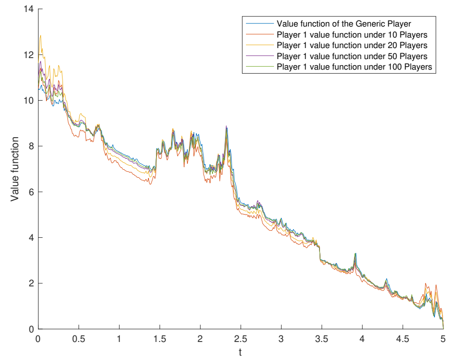



Figure 4.6: Simulation of player 1's optimal value function $V$.

function is the same as (4.1.3) with players control the process $X$ via $\alpha$ following

$$X_t = \xi + \int_0^t \alpha_s ds + W_t + \tilde{W}_t, \tag{4.2.1}$$

for all $t \in [0, T]$ and $\mathbb{E}\left[\xi^2\right] < \infty$. Given a random measure flow $m : (0, T] \to \mathcal{P}_2(\mathbb{R})$, the generic player wants to minimize the expected accumulated cost on $[0, T]$:

$$J(x, \alpha) = \mathbb{E}\left[ \int_0^T \frac{1}{2}\alpha_s^2 + F(X_s, m_s)\, ds + G(X_T, m_T) \middle| X_0 = x \right], \tag{4.2.2}$$

with

$$F(x, m) = k_f \int_{\mathbb{R}} (x - z)^2 m(dz) \ , \ G(x, m) = k_g \int_{\mathbb{R}} (x - z)^2 m(dz),$$

for some $k_f, k_h > 0$. Similarly, for the according $N$-player game, player $i$ follows

$$dX_{it}^{(N)} = \alpha_{it}^{(N)} dt + dW_t + d\tilde{W}_t, \tag{4.2.3}$$

and the cost function is

$$J(x, \alpha) = \mathbb{E}\left[ \int_0^T \frac{1}{2}\alpha_s^2 + F(X_s, \rho(X_t^{(N)}))\, ds + G(X_T, \rho(X_T^{(N)}) \middle| X_0^{(N)} = x^N \right], \tag{4.2.4}$$

where $\rho(x^N)$ is defined in (4.1.9).

### 4.2.2 Main Results for MFGs

Similar to the analysis in Section 4.1, instead of directly working on the infinite dimensional problem defined in (4.2.2), we base our calculations on its alternative version with $F$ and $G$ are represented as function of the first and second moment of the measure $m$, denoted as $\bar{\mu}$ and $\bar{\nu}$:

$$\bar{F}(x, \bar{\mu}, \bar{\nu}) = k_f(x^2 - 2x\bar{\mu} + \bar{\nu}) \ , \ \bar{G}(x, \bar{\mu}, \bar{\nu}) = k_g(x^2 - 2x\bar{\mu} + \bar{\nu}).$$

The control problem becomes similar to (P3) defined in Section 4.1 except that

$$\bar{V}(x,t,\bar{\mu},\bar{\nu}) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[ \int_t^T \frac{1}{2}\alpha_s^2 ds + \bar{F}(X_s,\mu_s,\nu_s)ds + \bar{G}(X_T,\mu_T,\nu_T) \middle| X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu} \right].$$

Thanks to the Markovian structure of $\bar{V}$, we can assume the value function based on the LQG nature of the problem as

$$\bar{V}(x,t,\bar{\mu},\bar{\nu}) = a(t)x^2 + e(t)x + b(t)\bar{\mu}^2 + f(t)\bar{\mu} + g(t)x\bar{\mu} + c(t)\bar{\nu} + d(t). \tag{4.2.5}$$

**Definition 42.** *The space $\mathcal{M}$ is the collection of all $\tilde{\mathcal{F}}_t$-adapted measure flows $m : [0,T]\times\Omega \mapsto \mathcal{P}_2(\mathbb{R})$, whose first moment $[m_t]_1 := \mu_t$ and second moment $[m_t]_2 := \nu_t$ satisfy*

$$\begin{aligned} \mu_t &= \mu_0 + \int_0^t \left(w_1(s)\mu_s + w_2(s)\right)\, ds + \tilde{W}_t, \\ \nu_t &= \nu_0 + \int_0^t \left(w_3(s)\mu_s + w_4(s)\nu_s + w_5(s)\mu_s^2 + w_6(s)\right)\, ds + 2\int_0^t \mu_s \, d\tilde{W}_s, \end{aligned} \tag{4.2.6}$$

*for all $t \in [0,T]$ and some smooth deterministic functions $(w_i : i = 1,2,\ldots,6)$.*

Under enough regularity conditions, the value function $\bar{V}$ defined in (4.2.5) is the solution $v$ of the following coupled HJBs

$$\begin{cases} \partial_t v - \frac{1}{2}(\partial_x v)^2 + (w_1\bar{\mu} + w_2)\,\partial_{\bar{\mu}} v + \left(w_3\bar{\mu} + w_4\bar{\nu} + w_5\bar{\mu}^2 + w_6\right)\partial_{\bar{\nu}} v + \partial_{xx} v + \frac{1}{2}\partial_{\bar{\mu}\bar{\mu}} v \\ \qquad\qquad + \partial_{x\bar{\mu}} v + 2\bar{\mu}^2 \partial_{\bar{\nu}\bar{\nu}} v + 2\bar{\mu}\partial_{\bar{\mu}\bar{\nu}} v + 2\bar{\mu}\partial_{x\bar{\nu}} v + k(x^2 - 2\bar{\mu}x + \bar{\nu}) = 0, \\ v(T,x,\mu_T,\nu_T) = k_g(x^2 - 2x\mu_T + \nu_T). \end{cases} \tag{4.2.7}$$

Furthermore, the optimal control has to admit the feedback form of

$$\hat{\alpha}(t) = -\partial_x v\left(t, \hat{X}_t, \mu_t, \nu_t\right). \tag{4.2.8}$$

The proof of the verification theorem is similar to that of Lemma 37, which is omitted here. After plugging in the expression of (4.2.5) into the HJB equation, we have

$$\begin{cases} a' - 2a^2 + k_f = 0, \\ e' - 2ae + w_2 g = 0, \\ f' - eg + w_1 f + 2bw_2 + cw_3 = 0, \\ g' - 2ag + w_1 g - 2k_f = 0, \\ b' - \frac{1}{2}g^2 + 2bw_1 + cw_5 = 0, \\ c' + cw_4 + k_f = 0, \\ d' + 2a + g + b - \frac{1}{2}e^2 + fw_2 + cw_6 = 0, \end{cases} \tag{4.2.9}$$

with terminal conditions

$$a(T) = c(T) = k_g\ ,\ g(T) = -2k_g\ ,\ b(T) = d(T) = e(T) = f(T) = 0. \tag{4.2.10}$$

Meanwhile, based on the fixed point condition illustrated in figure 4.2, the first moment $\hat{\mu}_s := \mathbb{E}\left[\hat{X}_s \middle| \tilde{\mathcal{F}}_t\right]$ and the second moment $\hat{\nu}_s := \mathbb{E}\left[\hat{X}_s^2 \middle| \tilde{\mathcal{F}}_t\right]$ of the optimal path conditioned on $\tilde{\mathcal{F}}_t$ satisfy

$$\begin{cases} \hat{\mu}_s = \bar{\mu} + \int_t^s \left(\left(-2a(r) - g(r)\right)\hat{\mu}_r - e(r)\right)\, dr + \tilde{W}_s, \\ \hat{\nu}_s = \bar{\nu} + \int_t^s \left(2 - 4a(r)\hat{\nu}_r - 2e(r)\hat{\mu}_r - 2g(r)\hat{\mu}_r^2\right)\, dr + \int_t^s 2\hat{\mu}_r\, d\tilde{W}_r, \end{cases} \tag{4.2.11}$$

for $s \geq t$. Hence, we obtain another 6 equations for any $t \in [0, T]$,

$$w_1 = -2a - g,\ w_2 = -e,\ w_3 = -2e,\ w_4 = -4a,\ w_5 = -2g,\ w_6 = 2. \tag{4.2.12}$$

**Theorem 43** (MFG)**.** *The MFG equilibrium path $\hat{X} = \hat{X}[\hat{m}]$ is given by*

$$d\hat{X}_t = 2a(t)\left(\mathbb{E}[\xi] + \tilde{W}_t - \hat{X}_t\right)\, dt + dW_t + d\tilde{W}_t, \quad \hat{X}_0 = \xi, \tag{4.2.13}$$

*with equilibrium control*

$$\hat{\alpha}_t = 2a(t)\left(\mathbb{E}[\xi] + \tilde{W}_t - \hat{X}_t\right). \tag{4.2.14}$$

*Moreover, the value function $U$ is*

$$U(m_0, x) = a(0)x^2 - 2a(0)[m_0]_1 x + b(0)[m_0]_1^2 + c(0)[m_0]_2 + d(0),$$

*where $A = \frac{\sqrt{\frac{k_f}{2}} - k_g}{\sqrt{\frac{k_f}{2}} + k_g}$ and*

$$\begin{cases} a(t) = \sqrt{\frac{k_f}{2}}\frac{1 - Ae^{-2\sqrt{2k_f}(T-t)}}{1 + Ae^{-2\sqrt{2k_f}(T-t)}}, \\ b(t) = \int_t^T \left(4a(s)c(s) - 2a^2(s)\right) ds, \\ c(t) = k_g + k_f \int_t^T e^{\int_t^s -4a(r)dr} ds, \\ d(t) = \int_t^T \left(b(s) + 2c(s)\right) ds. \end{cases} \tag{4.2.15}$$

### 4.2.3 The $N$-player Game and its Convergence to MFGs

Our first result is similar to Theorem 34 except slight difference in the expressions of $a_1^N$ and $a_2^N$ due to the lack of different statuses.

$$\begin{cases} (a_1^N)' - \frac{2(N+1)}{N-1}(a_1^N)^2 + \frac{N-1}{N}k_f = 0, \\ (a_2^N)' + \frac{2}{(N-1)^2}(a_1^N)^2 - \frac{4N}{N-1}a_1^N a_2^N + \frac{k_f}{N} = 0, \\ a_1^N(T) = \frac{N-1}{N}k_g\ ,\ a_2^N(T) = \frac{k_g}{N}. \end{cases} \tag{4.2.16}$$

Meanwhile, the path and the control of player $i$ under the equilibrium are

$$d\hat{X}_{it}^{(N)} = -2a_1^N(t)\left(\hat{X}_{it}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^N \hat{X}_{jt}^{(N)}\right) dt + dW_{it}^{(N)} + d\tilde{W}_t, \tag{4.2.17}$$

for $i = 1, \cdots, N$, and

$$\hat{\alpha}_{it}^{(N)} = -2a_1^N(t)\left(\hat{X}_{it}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^N \hat{X}_{jt}^{(N)}\right).$$

Hence, without the detailed proof, we present the following theorem of the convergence.

**Theorem 44** (Convergence of the processes)**.** *Under Assumption (A2) defined in Section 4.1, $\hat{X}_{ut}^{(N)}$ of the $N$-player game converges in distribution to the MFG equilibrium $\hat{X}_t$ for any $t \in (0,T]$.*

Meanwhile, the critical part of the convergence is that the empirical distribution of the $N$-player game converges to the equilibrium measure of MFGs. This gives the last building brick to show that the MFGs are a good approximation of the complicated $N$-player game. However, it is hard to directly work on the convergence. Instead, we show that the $p$-th moments of the distribution converges.

Denote $\bar{X}_{pt}^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{X}_{it}^{(N)}\right)^p$ and $\hat{a}_1^N = \frac{N}{N-1}a_1^N$. By Itô's formula, the dynamic of $\bar{X}_{pt}^{(N)}$ follows

$$\begin{aligned}\bar{X}_{pt}^{(N)} =& \frac{1}{N}\sum_{i=1}^{N}\left(x_i^N\right)^p - \int_0^t 2\hat{a}_1^N(s)p\bar{X}_{ps}^{(N)}ds + \int_0^t 2\hat{a}_1^N(s)p\left(\bar{x}^N + \frac{1}{\sqrt{N}}B_s + \tilde{W}_s\right)\bar{X}_{(p-1)s}^{(N)}ds \\ &+ \int_0^t \frac{p}{N}\sum_{i=1}^{N}\left(\hat{X}_{is}^{(N)}\right)^{p-1}dW_{is}^{(N)} + \int_0^t p\bar{X}_{(p-1)s}^{(N)}d\tilde{W}_s + \int_0^t p(p-1)\bar{X}_{(p-2)s}^{(N)}ds,\end{aligned}$$

or equivalently,

$$\begin{aligned}\bar{X}_{pt}^{(N)} =& \mathcal{E}_t\left(-2p\hat{a}_1^N\right)\left(\frac{1}{N}\sum_{i=1}^{N}\left(x_i^N\right)^p + \int_0^t 2p\hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right)\left(\bar{x}^N + \frac{1}{\sqrt{N}}B_s + \tilde{W}_s\right)\bar{X}_{(p-1)s}^{(N)}ds\right. \\ &+ \int_0^t p\mathcal{E}_s\left(2p\hat{a}_1^N\right)\frac{1}{N}\sum_{i=1}^{N}\left(\hat{X}_{is}^{(N)}\right)^{p-1}dW_{is}^{(N)} + \int_0^t p\mathcal{E}_s\left(2p\hat{a}_1^N\right)\bar{X}_{(p-1)s}^{(N)}d\tilde{W}_s \\ &\left.+ \int_0^t p(p-1)\mathcal{E}_s\left(2p\hat{a}_1^N\right)\bar{X}_{(p-2)s}^{(N)}ds\right), \qquad (4.2.18)\end{aligned}$$

where $\mathcal{E}_t(a) = \exp\left(\int_0^t a(s)ds\right)$.

**Theorem 45** (Convergence of the empirical measure)**.** *Under Assumption (A2) defined in Section 4.1, the random empirical measure*

$$\rho\left(\hat{X}_t^{(N)}\right) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\hat{X}_{it}^{(N)}}$$

*from the $N$-player game converges almost surely to the MFG equilibrium measure $\hat{m}_t = \mathcal{L}\left(\hat{X}_t|\tilde{\mathcal{F}}_t\right)$ for any $t \in (0,T]$.*

Before the proof of Theorem 45, let us first look at one numerical example of the simulated density of MFGs and $N$-player game. The parameters that we use are listed in the following:

$$T = 5\ ,\ k_f = 2\ ,\ k_g = 3\ ,\ \mu_0 = 0\ ,\ \nu_0 = 1.$$

It is easy to see in figure 4.7 that the distribution of $N$-player game converges to that of MFGs.

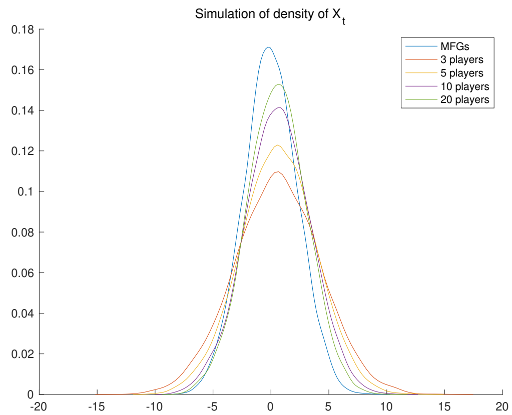



Figure 4.7: Simulation of Player 1's optimal path for MFGs and $N$-player game with $N = 3, 5, 10, 20$.

*The proof of Theorem 45:* Note that

$$\mathbb{E}\left[\hat{X}_t^p\middle|\tilde{\mathcal{F}}_t\right] = \mathbb{E}\left[x_0^p\right] - \int_0^t 2pa(s)\left(\mathbb{E}\left[\left(\hat{X}_s\right)^p\middle|\tilde{\mathcal{F}}_s\right] - \left(\mu_0 + \tilde{W}_s\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right)ds + \int_0^t p\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]d\tilde{W}_s + \int_0^t p(p-1)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-2}\middle|\tilde{\mathcal{F}}_s\right]ds.$$

or equivalently,

$$\mathbb{E}\left[\hat{X}_t^p\middle|\tilde{\mathcal{F}}_t\right] = \mathcal{E}_t(-2pa)\left(\mathbb{E}\left[x_0^p\right] + \int_0^t 2pa(s)\mathcal{E}_s(2pa)\left(\mu_0 + \tilde{W}_s\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]ds + \int_0^t p\mathcal{E}_s(2pa)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]d\tilde{W}_s + \int_0^t p(p-1)\mathcal{E}_s(2pa)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-2}\middle|\tilde{\mathcal{F}}_s\right]ds\right). \tag{4.2.19}$$

We want to prove by induction that the dynamic (4.2.18) of $\bar{X}_{pt}^N$ converges to (4.2.19) of $\mathbb{E}\left[\left(\hat{X}_t\right)^p\middle|\tilde{\mathcal{F}}_t\right]$. First, when $p = 0$, $\bar{X}_{pt}^{(N)} = 1 = \mathbb{E}\left[\left(\hat{X}_t\right)^0\middle|\tilde{\mathcal{F}}_t\right]$. For $p = 1$,

$$\bar{X}_{pt}^{(N)} = \bar{X}_t^{(N)} = \bar{x}^N + \frac{1}{\sqrt{N}}B_t + \tilde{W}_t \to \mu_0 + \tilde{W}_t = \mathbb{E}\left[\hat{X}_t\middle|\tilde{\mathcal{F}}_t\right] \ a.s.,$$

by the Law of large number. Now assume that the statement is true for any $p' \leq p-1$, and for the $p$-th moment, we have

1. $\hat{a}_1^N \to a$ uniformly.
2. $\frac{1}{N}\sum_{i=1}^N \left(x_i^N\right)^p \to \mathbb{E}\left[(\xi)^p\right]$ for $p \geq 1$ by Assumption (A2).
3. $\lim_{N\to\infty}\int_0^t p\mathcal{E}_s\left(2p\hat{a}_1^N\right)\frac{1}{N}\sum_{i=1}^N\left(\hat{X}_{is}^{(N)}\right)^{p-1}dW_{is}^{(N)} = 0.$

Denote $Y_{it} = \int_0^t p\mathcal{E}_s(2p\hat{a}_1^N)\left(\hat{X}_{is}^{(N)}\right)^{p-1} dW_{is}^{(N)}$. Due to the $L^1$-boundedness of $\left(\hat{X}_{it}^{(N)}\right)^{2p-2}$ shown in Lemma 55, $Y_{it}$'s are martingale. Meanwhile, by the independence of $W_{it}$'s,

$$\mathbb{E}\left[Y_{it}Y_{jt}Y_{mt}Y_{nt}\right] = \begin{cases} \mathbb{E}\left[(Y_{it})^4\right], & i=j=m=n, \\ \mathbb{E}\left[(Y_{it})^2(Y_{mt})^2\right], & i=j\neq m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Meanwhile, from the Itô's formula and Hölder's inequality, we have

$$\begin{aligned} \mathbb{E}\left[(Y_{it})^4\right] =& \mathbb{E}\left[\int_0^t 6\,(Y_{is})^2\left(p\mathcal{E}_s(2p\hat{a}_1^N)\left(\hat{X}_{is}^{(N)}\right)^{p-1}\right)^2 ds\right] \\ \leq& \int_0^t 6\mathbb{E}\left[(Y_{is})^4\right]^{\frac{1}{2}}\mathbb{E}\left[\left(p\mathcal{E}_s(2p\hat{a}_1^N)\left(\hat{X}_{is}^{(N)}\right)^{p-1}\right)^4\right]^{\frac{1}{2}} ds \\ \leq& \int_0^t 3\left(1+\mathbb{E}\left[(Y_{is})^4\right]\right)p^2\mathcal{E}_s^2(2p\hat{a}_1^N)\sqrt{C(T,4p-4)}ds, \end{aligned}$$

where $C$ is function defined in Lemma 55. By Grönwall's equality, $\mathbb{E}\left[(Y_{it})^4\right]$ is bounded. Denote that $\mathbb{E}\left[(Y_{it})^4\right] \leq C_3(T,p)$. Then

$$\mathbb{E}\left[\left(\sum_{i=1}^N Y_{it}\right)^4\right] = \sum_{i,j=1}^N \mathbb{E}\left[(Y_{it})^2(Y_{jt})^2\right] \leq \sum_{i,j}^N \mathbb{E}\left[(Y_{it})^4\right]^{\frac{1}{2}}\mathbb{E}\left[(Y_{jt})^4\right]^{\frac{1}{2}} \leq N^2 C_3(T,p).$$

Therefore, by Chebyshev's inequality, for any $\varepsilon > 0$,

$$\sum_{N=1}^\infty \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N Y_{it} > \varepsilon\right] \leq \sum_{N=1}^\infty \frac{1}{N^4\varepsilon^4}\mathbb{E}\left[\left(\sum_{i=1}^N Y_{it}\right)^4\right] \leq \sum_{N=1}^\infty \frac{1}{N^2\varepsilon^4}C_3(T,p) < \infty.$$

Applying the Borel-Cantelli Lemma, we obtain that $\mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N Y_{it} > \varepsilon \text{ i.o.}\right] = 0$, which gives the result.

4. From the dominant convergent theorem,

$$\begin{aligned} \lim_{N\to\infty}\Bigg|\int_0^t & \hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right)\left(\bar{x}^N + \frac{1}{\sqrt{N}}B_s + \tilde{W}_s\right)\bar{X}_{(p-1)s}^{(N)} \\ & - a(s)\mathcal{E}_s\left(2pa\right)\left(\mu_0 + \tilde{W}_s\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]ds\Bigg| = 0. \end{aligned}$$

This is from the facts that $\hat{a}_1^N \to a$, $\bar{x}^N \to \mu_0$, $\bar{X}_{(p-1)t}^{(N)} \to \mathbb{E}\left[\left(\hat{X}_t\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]$, and for a fixed $\epsilon$, there exists a $N_0$ such that for all $N > N_0$,

$$\left|\hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right) - a(s)\mathcal{E}_s\left(2pa\right)\right| < \epsilon\ ,\ \left|\bar{x}^N - \mu_0\right| \leq \epsilon\ ,\ \left|\bar{X}_{(p-1)t}^{(N)} - \mathbb{E}\left[\left(\hat{X}_t\right)^{p-1}\middle|\tilde{\mathcal{F}}_t\right]\right| \leq \epsilon.$$

Then,

$$
\begin{aligned}
&\left|\hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right)\left(\bar{x}^N+\frac{1}{\sqrt{N}}B_s+\tilde{W}_s\right)\bar{X}^{(N)}_{(p-1)s}-a(s)\mathcal{E}_s\left(2pa\right)\left(\mu_0+\tilde{W}_s\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right|\\
\leq&\hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right)\left|\left(\bar{x}^N+\frac{1}{\sqrt{N}}B_s+\tilde{W}_s\right)\bar{X}^{(N)}_{(p-1)s}-\left(\mu_0+\tilde{W}_s\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right|+\\
&\left|\hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right)-a(s)\mathcal{E}_s\left(2pa\right)\right|\left|\left(\mu_0+\tilde{W}_s\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right|\\
\leq&\hat{a}_1^N(s)\mathcal{E}_s\left(2p\hat{a}_1^N\right)\left((\mu_0+\epsilon)\epsilon+\epsilon\left|\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right|+\left|\frac{1}{\sqrt{N}}B_s\right|\left(\left|\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right|+\epsilon\right)+\epsilon\left|\tilde{W}_s\right|\right)\\
&+\epsilon\left|\left(\mu_0+\left|\tilde{W}_s\right|\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]\right|,
\end{aligned}
$$

which is $L^1$-bounded.

5. With similar argument as in 4, we have

$$
\begin{aligned}
&\lim_{N\to\infty}\left|\int_0^t\mathcal{E}_s\left(2p\hat{a}_1^N\right)\bar{X}^{(N)}_{(p-1)t}-\mathcal{E}_s\left(2pa\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-1}\middle|\tilde{\mathcal{F}}_s\right]d\tilde{W}_s\right|=0,\\
&\lim_{N\to\infty}\left|\int_0^t\mathcal{E}_s\left(2p\hat{a}_1^N\right)\bar{X}^{(N)}_{(p-2)t}-\mathcal{E}_s\left(2pa\right)\mathbb{E}\left[\left(\hat{X}_s\right)^{p-2}\middle|\tilde{\mathcal{F}}_s\right]ds\right|=0.
\end{aligned}
$$

With above arguments, if we take the limit of right hand side of (4.2.18), and it converges almost surely to (4.2.19). Therefore, by induction, we conclude the required result. □

## 4.3 Appendix

### 4.3.1 Some explicit solutions on LQG-MFGs

In this part, we only provide explicit solutions to some LQG-MFGs without the common noise. The methodology could be the utilization of the standard Stochastic Maximum Principle or Dynamic Programming approach, and all proofs will be omitted.

Suppose the position of a generic player $X_t$ follows

$$dX_t=\alpha_t dt+\sigma dW_t,\ X_0\sim\mathcal{N}(0,1).$$

The goal of the generic player is to minimize the running cost

$$\inf_{\alpha\in\mathcal{A}}\mathbb{E}\left[\int_0^T\left(\frac{1}{2}\alpha_t^2+h\int_{\mathbb{R}}(X_t-y)^2m(t,dy)\right)dt\right],$$

subject to

$$m_t=\mathcal{L}aw(X_t),\quad\forall t\in[0,T],$$

where $h\in\mathbb{R}$ is a constant.

Denote

$$V(x,t)=\inf_{\alpha}\mathbb{E}\left[\int_t^T\left(\frac{1}{2}\alpha_s^2+h\int_{\mathbb{R}}(X_s-y)^2m(s,dy)\right)ds\middle|X_t=x\right].$$

Note that the model can be characterized by Hamilton-Jacobian-Bellman equation coupled by Fokker-Planck-Kolmogorov equation:

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2 \partial_{xx} V - \frac{1}{2}(\partial_x V)^2 + F(x,m) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ \partial_t m - \frac{1}{2}\sigma^2 \partial_{xx} m - \partial_x (m \partial_x V) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ m_0 \sim \mathcal{N}(0,1), V(x,T) = 0, & x \in \mathbb{R}, \end{cases}$$

where $F(x,m) = h \int_{\mathbb{R}} (x-y)^2 m(t,dy)$.

The monotonicity condition on the source term $F$ in the variable $m$ plays crucial role for the uniqueness of the MFG system. A monotone function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be increasing if it satisfies $(f(x_1) - f(x_2))(x_1 - x_2) \geq 0$ , and decreasing if $-f$ is increasing. This definition can be generalized to an infinite dimensional function $F(x,m)$.

**Definition 46.** *The real function $F$ on $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$ is said to be monotone, if, for all $m \in \mathcal{P}_2(\mathbb{R})$, the mapping $\mathbb{R} \ni x \mapsto F(x,m)$ is at most of quadratic growth, and for all $m_1$, $m_2$ it satisfies*

$$\int_{\mathbb{R}} (F(x,m_1) - F(x,m_2))\, d(m_1 - m_2)(x) \geq 0.$$

*$F$ is said to be anti-monotone, if $(-F)$ is monotone.*

According to Cardaliaguet (2010), if $F$ is monotone, then MFGs have at most one solution. Interestingly, the monotonicity of $F$ is dependent on the sign of $h$.

**Lemma 47.** *$F(x,m) = h \int_{\mathbb{R}} (x-y)^2 m(t,dy)$ is monotone if $h < 0$, and anti-monotone if $h > 0$.*

A natural question is that, how the MFG system behaves differently to the monotonicity of $F$?

#### 4.3.1.1 Case I: $h > 0$

**Lemma 48.** *For $h > 0$, there exists a solution (may not be unique) to the MFG system in the form of $V(x,t) = f_1(t)x^2 + f_3(t)$ and $m(t) \sim \mathcal{N}(0, \gamma(t))$, where*

$$f_1(t) = \sqrt{\frac{h}{2}} \frac{1 - e^{-2\sqrt{2h}(T-t)}}{1 + e^{-2\sqrt{2h}(T-t)}} \ , \ \gamma(t) = e^{-\int_0^t 4 f_1(s) ds} \left(1 + \int_0^t \sigma^2 e^{\int_0^s 4 f_1(u) du} ds\right),$$
$$f_3(t) = \int_t^T (\sigma^2 f_1(s) + h\gamma(s)) ds.$$

#### 4.3.1.2 Case II: $h < 0$

**Lemma 49.** *For $h < 0$, there exists a unique solution in $[t_0, T]$ to the MFG system in the form of $V(x,t) = g_1(t)x^2 + g_3(t)$ and $m(t) \sim \mathcal{N}(0, \lambda(t))$, where*

$$g_1(t) = -\sqrt{-\frac{h}{2}} \tan\left(\sqrt{-2h}(T-t)\right) \ , \ \lambda(t) = e^{-\int_0^t 4 g_1(s) ds} \left(1 + \int_0^t \sigma^2 e^{\int_0^s 4 g_1(u) du} ds\right),$$
$$g_3(t) = \int_t^T (\sigma^2 g_1(s) + h\lambda(s)) ds \ , \ t_0 = max\left(0, T - \frac{1}{\sqrt{-2h}} \frac{\pi}{2}\right).$$

#### 4.3.1.3 Remark

When $h > 0$, the cost is anti-monotone, and there exists at least one global solution. When $h < 0$, the cost is monotone, and there exists at most one solution. Unfortunately, this solution lives in a short period of time. Lemma 49 coincides with the notes in Section 3.8 of Carmona et al. (2018) saying that due to the opposite time evolution of the system of HJB-FPK, the existence of the solution may exist for only a short period of time.

### 4.3.2 Dynkin's formula for a regime-switching diffusion with a quadratic function

Since the running cost (4.1.10) has a quadratic growth in the state variable, the value function $V[\hat{m}](y, x, t)$ is expected to possess similar growth. Next, we present a version of Dynkin's formula for the functions of quadratic growth, which is sufficient for our purpose. Throughout this subsection, we will use $K$ in various places as a generic constant which varies from line to line. The notions of this subsection is independent to other parts of the paper.

**Lemma 50.** *Let $X$ be the solution of*

$$dX_t = \alpha_t dt + \sigma_t dW_t,$$

*where $X$, $\alpha$ and $\sigma$ are bounded and take value in $\mathbb{R}^3$. $Y$ is CTMC with a generator*

$$Y \sim Q = (q_{ij})_{i,j=1,2,\ldots,n}.$$

*If $X_0 \in L^4$, $\alpha \in L^4_{\mathbb{F}}$ and $f : \mathbb{R}^3 \mapsto \mathbb{R}$ satisfies $\|\Delta f\|_\infty + \|\partial_t f\|_\infty < \infty$, then the following identity holds for all $t \in [0, T]$:*

$$\mathbb{E}\left[f(Y_t, X_t, t)\right] = \mathbb{E}\left[f(Y_0, X_0, 0)\right] + \mathbb{E}\left[\int_0^t (\partial_t + \mathcal{L} + \mathcal{Q}) f(Y_s, X_s, s) ds\right],$$

*where*

$$\mathcal{L} f(y, x, s) = \left(\frac{1}{2} Tr(\sigma_s \sigma_s^\top \Delta) + \alpha_s \cdot \nabla_x\right) f(y, x, s)$$

*and*

$$\mathcal{Q} f(y, x, s) = \sum_{i=1}^n q_{y,i} f(i, x, s).$$

*Proof.* It's enough to show that the local martingale defined by Itô's formula

$$M_t^f = f(Y_t, X_t, t) - f(Y_0, X_0, 0) - \int_0^t (\partial_t + \mathcal{L} + \mathcal{Q}) f(Y_s, X_s, s) ds \tag{4.3.1}$$

is uniformly integrable, hence is a true martingale.

First, $X_t$ is $L^4$ bounded uniformly in $t$ from the following inequality due to our assumptions on $X_0$ and $\alpha$:

$$\sup_{t \in [0,T]} \mathbb{E}\left[\|X_t\|^4\right] \leq K \mathbb{E}\left[\|X_0\|^4 + \int_0^T \|\alpha_s\|^4 ds + \int_0^T \|\sigma_s W_s\|^4 ds\right] \leq K,$$

where $K$ is a generic constant which varies from line to line.

On the other hand, since $\Delta f$ is uniformly bounded, $f$ is at most quadratic growth, i.e.

$$|f(x)| \leq K(x^2+1), \forall x \text{ for some large } K.$$

Hence, we conclude that $f(Y_t, X_t, t)$ is uniformly bounded in $L^2$ in $t$ from the fact

$$\sup_{t\in[0,T]} \mathbb{E}\left[f^2(Y_t, X_t, t)\right] \leq K \sup_{t\in[0,T]} \mathbb{E}\left[\|X_t\|^4\right] + K \leq K.$$

The uniform $L^2$-boundedness of $\int_0^t \partial_t f(Y_s, X_s, s)ds$ follows from our assumption on $\partial_t f$. Similarly, since $\mathcal{Q}f$ has a quadratic growth uniformly in $y$ and $t$,

$$\left\{\int_0^t \mathcal{Q}f(Y_s, X_s, s)ds : 0 \leq t \leq T\right\}$$

is $L^2$ bounded. At last, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \mathcal{L}f(Y_s, X_s, s)ds\right)^2\right] \\
&\leq K\mathbb{E}\left[\int_0^t (\alpha_s \cdot \nabla f + \frac{1}{2}\mathrm{Tr}(\sigma_s\sigma_s^\top \Delta f))^2(Y_s, X_s, s)ds\right] \\
&\leq K\mathbb{E}\left[\int_0^t \|\alpha_s\|^2\|\nabla f\|^2(Y_s, X_s, s)ds\right] + K\mathbb{E}\left[\int_0^t \frac{1}{4}\|\mathrm{Tr}(\sigma_s\sigma_s^\top \Delta f)\|^2(Y_s, X_s, s)ds\right] \\
&\leq K\mathbb{E}\left[\int_0^t \|\alpha_s\|^4 ds\right] + K\mathbb{E}\left[\int_0^t |\nabla f|^4(Y_s, X_s, s)ds\right] + K\mathbb{E}\left[\int_0^t \frac{1}{4}\|\mathrm{Tr}\Delta f\|^2(Y_s, X_s, s)ds\right].
\end{aligned}$$

Since $\nabla f$ is linear growth in $x$, the second term $\sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t \|\nabla f\|^4(Y_s, X_s, s)ds\right]$ is finite. Together with assumptions on $\Delta f$ and $\alpha$, we have uniform $L^2$-boundedness of $\int_0^t \mathcal{L}f(Y_s, X_s, s)ds$.

As a result, each term of the right hand side of (4.3.1) is uniformly $L^2$-bounded in $t$, and thus $M_t^f$ belongs to $L^2_{\mathbb{F}}$ and this implies the uniformly integrability. □

### 4.3.3 Proof of the existence and uniqueness of the ODE system

Consider the following ODE system

$$\begin{cases} a_0' - Ca_0^2 - \gamma_0(a_0 - a_1) + h_0 = 0, \\ a_1' - Ca_1^2 + \gamma_1(a_0 - a_1) + h_1 = 0, \\ a_0(T) = g_0 \ , \ a_1(T) = g_1, \end{cases} \tag{4.3.2}$$

where $C, h_i, g_i, \gamma_i$ $(i = 0, 1)$ are in $\mathbb{R}^+$. We need to show the existence and uniqueness of the solution to (4.3.2). Define $T_0^{(N)}$ and $T_1^{(N)}$ as

$$\begin{aligned}
T_0^{(N)}[a](t) &= \left[\left(g_0 + \int_t^T \left(h_0 - Ca_0^2(s) - \gamma_0(a_0(s) - a_1(s))\right) ds\right) \wedge N\right] \vee 0, \\
T_1^{(N)}[a](t) &= \left[\left(g_1 + \int_t^T \left(h_1 - Ca_1^2(s) - \gamma_1(a_1(s) - a_0(s))\right) ds\right) \wedge N\right] \vee 0,
\end{aligned}$$

where $a = \begin{bmatrix} a_0 & a_1 \end{bmatrix}^\top$. Let $D = \{f \in C([0,T]) : 0 \leq \sup_{t\in[0,T]} f(t) \leq N\}$. Note that $T_y^{(N)}(y = 0, 1)$ maps $D^2$ to $D^2$.

**Lemma 51.** *For fixed $N$, there exists a unique solution in $C([0,T])$ to*

$$a = \begin{bmatrix} T_0^{(N)}[a] \\ T_1^{(N)}[a] \end{bmatrix}. \tag{4.3.3}$$

*Proof.* Denote the norm $\|f\|_k = \left\|e^{kt}\max_y|f_y|\right\|_\infty$, where $k$ needs to be determined later and $f$ is a two dimensional vector with entry of $f_y, y=0,1$, which is equivalent to the infinite norm. Define the iteration rule $a_y^{(n+1)} = T_y^{(N)}\left[a_0^{(n)}, a_1^{(n)}\right]$ for $y=0,1$. Note that

$$\begin{aligned}
&\left\|e^{kt}\left(a_0^{(n+1)}(t) - a_0^{(n)}(t)\right)\right\|_\infty \\
\leq& \sup_{t\in[0,T]} e^{kt}\int_t^T C\left|\left(a_0^{(n)}(s)\right)^2 - \left(a_0^{(n-1)}(s)\right)^2\right| + \\
&\gamma_0\left(\left|a_0^{(n)}(s) - a_0^{(n-1)}(s)\right| + \left|a_1^{(n)}(s) - a_1^{(n-1)}(s)\right|\right) ds \\
\leq& \sup_{t\in[0,T]} e^{kt}\int_t^T e^{-ks}\left(2CN + 2\gamma_0\right)\left\|a^{(n)} - a^{(n-1)}\right\|_k ds \\
\leq& \frac{2CN + 2\gamma_0}{k}\left\|a^{(n)} - a^{(n-1)}\right\|_k.
\end{aligned}$$

Similarly, we have

$$\left\|e^{kt}\left(a_1^{(n+1)}(t) - a_1^{(n)}(t)\right)\right\|_\infty \leq \frac{2CN + 2\gamma_1}{k}\left\|a^{(n)} - a^{(n-1)}\right\|_k.$$

Choose $k > 2CN + 2\max\{\gamma_0, \gamma_1\}$, then

$$\left\|a^{(n+1)} - a^{(n)}\right\|_k \leq \frac{2CN + 2\max\{\gamma_0,\gamma_1\}}{k}\left\|a^{(n)} - a^{(n-1)}\right\|_k,$$

which gives us a contraction mapping from $D^2$ to $D^2$. Hence, by the Banach fixed point theorem, there exists a unique solution to (4.3.3). ☐

Next, we want to show that for large enough $N$, the solution to (4.3.3) is also the solution to (4.3.2).

**Lemma 52.** *For $N \geq e^{(\gamma_0+\gamma_1)T}\left((h_0+h_1)T + (g_0+g_1)\right)$, the solution $a^{(N)}$ to (4.3.3) satisfies the inequalities*

$$0 \leq g_i + \int_t^T \left(h_i - 2\left(a_i^{(N)}(s)\right)^2 - \gamma_i\left(a_i^{(N)}(s) - a_j^{(N)}(s)\right)\right) ds \leq N \tag{4.3.4}$$

*for all $t\in[0,T]$, where $i,j\in\{0,1\}$ and $i\neq j$.*

*Proof.* For simplicity of notations, $a_i$ is used instead of $a_i^{(N)}$ for $i=0,1$ if there is no confusion.

First, for $i=0,1$, we prove the positiveness of $a_i$ by contradiction. Suppose $a_i$ ($i=0,1$) are not positive functions on $[0,T]$. Since $a_0$ is continuous and $a_0(T) = g_0 > 0$, there exists some $\tau\in[0,T]$ as the closest time to $T$ such that $a_0(\tau)=0$. Note that finding such a $\tau$ is possible. Let $t_n\in[0,T]$ be a non-decreasing sequence such that $a_0(t_n)=0$, there exists some $\tau$ such that

$t_n \to \tau < T$ as $n \to \infty$ since $a_0$ is continuous and $a_0(T) = g_0 > 0$. By the continuity of $a_0$, we have $a_0(\tau) = 0$, which gives the desirable point $\tau$. Then for all $t \in (\tau, T]$, $a_0(t) > 0$ and it implies that $a_0'(\tau) > 0$. In this case, plugging $t = \tau$ to (4.3.2), we have $a_0'(\tau) = -h_0 - \gamma_0 a_1(\tau) > 0$, which yields $a_1(\tau) < 0$. Since $a_1$ is continuous on $[0, T]$ and $a_1(T) = g_1 > 0$, from the intermediate value theorem, there exists some $\hat{\tau} \in (\tau, T)$ such that $a_1(\hat{\tau}) = 0$ and $a_1'(\hat{\tau}) > 0$. However, this indicates that $a_1'(\hat{\tau}) = -h_1 - \gamma_1 a_0(\hat{\tau}) > 0$ by plugging $t = \hat{\tau}$ back to (4.3.2), and it implies $a_0(\hat{\tau}) < 0$, which contradicts with the fact that $a_0(t) > 0$ for all $t \in (\tau, T]$. Thus the positiveness of $a_0$ and $a_1$ on $[0, T]$ is obtained.

Next, we prove the upper bound for the integral in (4.3.4). Note that for all $t \in [0, T]$,

$$
\begin{aligned}
&(a_0 + a_1)'(T - t) \\
=&(h_0 + h_1) - C(a_0^2 + a_1^2)(T - t) - (\gamma_0 - \gamma_1)a_0(T - t) + (\gamma_0 - \gamma_1)a_1(T - t) \\
\leq&(h_0 + h_1) + (\gamma_0 + \gamma_1)(a_0 + a_1)(T - t),
\end{aligned}
$$

with $(a_0 + a_1)(T) = g_0 + g_1$. By Grönwall's inequality,

$$
(a_0 + a_1)(T - t) \leq e^{(\gamma_0 + \gamma_1)T}((g_0 + g_1) + (h_0 + h_1)T), \quad \forall t \in [0, T].
$$

Hence $a_i(t) \leq e^{(\gamma_0 + \gamma_1)T}((g_0 + g_1) + (h_0 + h_1)T)$ for all $t \in [0, T], i = 0, 1$. Hence, when $N \geq e^{(\gamma_0 + \gamma_1)T}((g_0 + g_1) + (h_0 + h_1)T)$, (4.3.4) holds. □

**Lemma 53.** *With the given of $h_y, g_y \in \mathbb{R}^+$, $y = 0, 1$, there exists a unique solution to the Riccati system* (4.1.12).

*Proof.* The existence, uniqueness and boundedness of the solution to $a_y$ $(y = 0, 1)$ are shown in Lemma 51 and Lemma 52. Given $(a_y : y = 0, 1)$, the coefficient functions $b_y$ $(y = 0, 1)$ form a linear ordinary differential equation system. Their existence and uniqueness are guaranteed by Theorem 12.1 in Antsaklis and Michel (2006). Similarly, with the given of $(a_y, b_y : y = 0, 1)$, the coefficient functions $c_y, k_y$ $(y = 0, 1)$ also form a linear ordinary differential equation system. Applying the Theorem 12.1 in Antsaklis and Michel (2006), we can obtain the existence and uniqueness of $c_y, k_y$ $(y = 0, 1)$. □

### 4.3.4 Multidimensional Problem

In this subsection we consider the multidimensional problem, which is a straightforward extension of the previous one-dimensional setup. The same type of Ricatti system to characterize the equilibrium and the value function is obtained, and we have a similar result as the Theorem 33.

Suppose that $X_t$, $W_t$ and $\alpha_t$ take values in $\mathbb{R}^d$, and all components of $W_t$ are independent. Suppose that the dynamic of the generic player is given by

$$
dX_t = \alpha_t dt + dW_t.
$$

Consider the cost function

$$
\begin{aligned}
&J[m](y,x,t,\bar\mu,\bar\nu)\\
=&\mathbb{E}\left[\int_t^T\left(\frac{1}{2}\|\alpha_s\|_2^2+h(Y_s)\int_{\mathbb{R}^d}\|X_s-z\|_2^2m(dz)\right)ds+\right.\\
&\left.g(Y_T)\int_{\mathbb{R}^d}\|X_T-z\|_2^2m(dz)\right|X_t=x,Y_t=y,\mu_t=\bar\mu,\nu_t=\bar\nu\Bigg]\\
=&\mathbb{E}\left[\int_t^T\left(\frac{1}{2}\alpha_s^\top\alpha_s+h(Y_s)\left(X_s^\top X_s-2\mu_s^\top X_s+\nu_s\cdot\mathbb{1}\right)\right)ds+\right.\\
&\left.g(Y_T)\left(X_T^\top X_T-2\mu_T^\top X_T+\nu_T\cdot\mathbb{1}\right)\right|X_t=x,Y_t=y,\mu_t=\bar\mu,\nu_t=\bar\nu\Bigg],
\end{aligned}
$$

where $m$ is the joint density function in $\mathbb{R}^d$, and $\mu,\nu$ take value in $\mathbb{R}^d$. For $y=0,1$, define

$$
\begin{cases}
a_y'-2a_y^2-\gamma_y a_y+\gamma_y a_{1-y}+h_y=0,\\
b_y'-4a_yb_{1-y}-\gamma_y b_y+\gamma_y b_z+h_y=0,\\
c_y'+da_y+db_y-\gamma_y c_y+\gamma_y c_{1-y}=0,\\
k_y'-2a_y^2+4a_yb_y-\gamma_y k_y+\gamma_y k_{1-y}=0,\\
a_y(T)=g_y\ ,\ b_y(T)=g_y\ ,\ c_y(T)=0.
\end{cases}
\tag{4.3.5}
$$

**Theorem 54** (Verification theorem for MFGs)**.** *There exists a unique solution $(a_y,b_y,c_y,k_y:y=0,1)$ for the Riccati system (4.3.5). With these solutions, for $t\in[0,T]$, the MFG equilibrium path follows $\hat X=\hat X[\hat m]$ is given by*

$$
d\hat X_t=2a_{Y_t}(t)(\mathbb{E}[X_0]-\hat X_t)dt+dW_t,\quad \hat X_0=X_0,
$$

*with equilibrium control $\hat\alpha_t=2a_{Y_t}(t)(\mathbb{E}[X_0]-\hat X_t)$. Moreover, the value function $U$ is*

$$
U(m_0,y,x)=a_y(0)x^\top x-2a_y(0)x^\top[m_0]_1+k_y(0)[m_0]_1^\top[m_0]_1+b_y(0)[m_0]_2^\top\mathbb{1}+c_y(0)
$$

*for $y=0,1$.*

The proof is similar to the one-dimensional problem, and we don't show the details here.

### 4.3.5 Proofs of Lemmas and Theorems

*The proof of Lemma 37:* 1. We first prove the verification theorem. Since $v\in\mathcal{S}$, for any admissible $\alpha\in L^4_{\mathbb{F}}$, the process $X^\alpha$ is well defined and one can use Dynkin's formula given by Lemma 50 to write

$$
\mathbb{E}\left[v(Y_T,X_T,T,\mu_T,\nu_T)\right]=v(y,x,t,\bar\mu,\bar\nu)+\mathbb{E}\left[\int_t^T\mathcal{G}^{\alpha(s)}v(Y_s,X_s,s,\mu_s,\nu_s)ds\right],
$$

where

$$
\begin{aligned}
\mathcal{G}^a f(y,x,s,\bar\mu,\bar\nu)=\bigg(&\partial_t+a\partial_x+\frac{1}{2}\partial_{xx}+\mathcal{Q}+\left(w_{0y}\bar\mu+w_{1y}\right)\partial_{\bar\mu}+\\
&\left(w_{2y}\bar\mu+w_{3y}\bar\nu+w_{4y}\bar\mu^2+w_{5y}\right)\partial_{\bar\nu}\bigg)f(y,x,s,\bar\mu,\bar\nu).
\end{aligned}
$$

Note that HJB actually implies that

$$\inf_a \left\{ \mathcal{G}^a v + \frac{1}{2} a^2 \right\} = -\bar{F},$$

which again implies

$$-\mathcal{G}^a v \leq \frac{1}{2} a^2 + \bar{F}.$$

Hence, we obtain that for all $\alpha \in L^4_{\mathbb{F}}$,

$$\begin{aligned}
& v(y, x, t, \bar{\mu}, \bar{\nu}) \\
=& \mathbb{E}\left[ \int_t^T -\mathcal{G}^{\alpha(s)} v(Y_s, X_s, s, \mu_s, \nu_s) ds \right] + \mathbb{E}\left[ v(Y_T, X_T, T, \mu_T, \nu_T) \right] \\
\leq& \mathbb{E}\left[ \int_t^T \left( \frac{1}{2}\alpha^2(s) + \bar{F}(Y_s, X_s, \mu_s, \nu_s) \right) ds \right] + \mathbb{E}\left[ \bar{G}(Y_T, X_T, \mu_T, \nu_T) \right] \\
=& J(y, x, t, \alpha, \bar{\mu}, \bar{\nu}).
\end{aligned}$$

In the above, if $\alpha$ is replaced by $\hat{\alpha}$ given by the feedback form (4.1.17), then since $\partial_x v$ is Lipschitz continuous in $x$, there exists corresponding optimal path $\hat{X} \in L^4_{\mathbb{F}}$. Thus, $\hat{\alpha}$ is also in $L^4_{\mathbb{F}}$. One can repeat all above steps by replacing $X$ and $\alpha$ by $\hat{X}$ and $\hat{\alpha}$, and $\leq$ sign by $=$ sign to conclude that $v$ is indeed the optimal value.

2. The opposite direction of the verification theorem follows by taking $\theta \to t$ for the dynamic programing principle, for all stopping time $\theta \in [t, T]$,

$$\begin{aligned}
& \bar{V}(y, x, t, \bar{\mu}, \bar{\nu}) \\
=& \mathbb{E}\left[ \int_t^\theta \frac{1}{2}\alpha_s^2 + \bar{F}(Y_s, X_s, \mu_s, \nu_s) ds + \bar{V}(Y_\theta, X_\theta, \theta, \mu_\theta, \nu_\theta) \middle| X_t = x, Y_t = y, \mu_t = \bar{\mu}, \nu_t = \bar{\nu} \right],
\end{aligned}$$

which is valid under our regularity assumptions on all the partial derivatives.

□

*The proof of Lemma 38.* With the form of value function $v_y$ given in (4.1.18) and the first and second moment of the conditional population density given in (4.1.15), we have

$$\begin{aligned}
\partial_t v_y &= a_y'(t)x^2 + d_y'(t)x + e_y'(t)\bar{\mu} + f_y'(t)x\bar{\mu} + k_y'(t)\bar{\mu}^2 + b_y'(t)\bar{\nu} + c_y'(t), \\
\partial_x v_y &= 2x a_y(t) + d_y(t) + f_y(t)\bar{\mu}, \\
\partial_{xx} v_y &= 2a_y(t), \\
\partial_{\bar{\mu}} v_y &= e_y(t) + f_y(t)x + 2k_y(t)\bar{\mu}, \\
\partial_{\bar{\nu}} v_y &= b_y(t),
\end{aligned}$$

for $y = 0, 1$. Plugging them back to the coupled HJBs in (4.1.16), we get a system of ODEs in (4.1.21) by equating $x$, $\bar{\mu}$, $\bar{\nu}$-like terms in each equation.

Therefore, any solution $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0, 1)$ of ODE system (4.1.21) leads to the solution of HJB (4.1.16) in the form of the quadratic function given by (4.1.23). Since the $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0, 1)$ are differentiable functions on the closed set $[0, T]$, they are also bounded, and the regularity condition $\|\partial_{xx} v\|_\infty + \|\partial_t v\|_\infty + \|\partial_\mu v\|_\infty + \|\partial_\nu v\|_\infty < \infty$ is valid. Finally, we invoke the verification theorem given by Lemma 37 to conclude the desired result. □

*The proof of Lemma 39:* It is standard that, under the enough regularities, the players' value function $V(y, x^N) = (V_1, \dots, V_N)(y, x^N)$ can be lifted to the solution $v_{iy}(x^N, t)$ of the following system of HJB equation, for $i = 1, \dots, N$,

$$\begin{cases} \partial_t v_{i0} - \frac{1}{2}(\partial_i v_{i0})^2 - \sum_{j \neq i} \partial_j v_{j0} \partial_j v_{i0} + \frac{1}{2}\Delta v_{i0} - \gamma_0 v_{i0} + \gamma_0 v_{i1} + \frac{h_0}{N} \sum_{j \neq i} \left( (e_i - e_j)^\top x^N \right)^2 = 0, \\ \partial_t v_{i1} - \frac{1}{2}(\partial_i v_{i1})^2 - \sum_{j \neq i} \partial_j v_{j1} \partial_j v_{i1} + \frac{1}{2}\Delta v_{i1} - \gamma_1 v_{i1} + \gamma_1 v_{i0} + \frac{h_1}{N} \sum_{j \neq i} \left( (e_i - e_j)^\top x^N \right)^2 = 0, \\ v_{iy}(x^N, T) = \frac{g_y}{N} \sum_{j \neq i} \left( (e_i - e_j)^\top x^N \right)^2. \end{cases} \tag{4.3.6}$$

Then, the value functions $V$ of $N$-player game defined by (1.2.2) is $V_i(y, x^N) = v_{iy}(x^N, 0)$ for all $i = 1, \dots, N$. Moreover, the path and the control under the equilibrium are

$$d\hat{X}_{it} = -\partial_i v_{iY_t}(\hat{X}_t, t)dt + dW_{it}, \quad i = 1, \dots, N,$$

and

$$\hat{\alpha}_{it} = -\partial_i v_{iY_t}(\hat{X}_t, t).$$

The proof is the application of Dynkin's formula and the details are omitted here. Due to its LQG structure, the value function leads to a quadratic function of the form

$$v_{iy}(x^N, t) = (x^N)^\top A_{iy}(t) x^N + (x^N)^\top B_{iy}(t) + C_{iy}(t).$$

For each $i = 1, 2, \dots, N$, after plugging $V_{iy}$ into (4.3.6), and matching the coefficient of variables, we get the desired results. $\square$

*The proof of Lemma 40.* It is obvious to see that $B_{iy} = 0$ for all time $t \in [0, T]$. Note that in this case, for $i = 1, 2, \dots, N$, the optimal control is given by

$$\hat{\alpha}_i = -2 \sum_{j=1}^{N} (A_{iY_t^{(N)}})_{ij} \hat{X}_{jt}^{(N)} = -2 \left( A_{iY_t^{(N)}} \right)_i^\top \hat{X}_t^{(N)}.$$

Plugging the pattern (4.1.29) into the differential equation of $A_{iy}$, we have

$$\begin{aligned} & a'_{1y} - 2a_{1y}^2 - 4(N-1)a_{3y}^2 - \gamma_y a_{1y} + \gamma_y a_{1(1-y)} + \frac{N-1}{N} h_y = 0, \\ & a'_{2y} - 2a_{3y}^2 - 4a_{1y}a_{2y} - 4(N-2)a_{3y}a_{4y} - \gamma_y a_{2y} + \gamma_y a_{2(1-y)} + \frac{h_y}{N} = 0, \\ & a'_{3y} - 2a_{1y}a_{3y} - 4a_{1y}a_{3y} - 4(N-2)a_{3y}^2 - \gamma_y a_{3y} + \gamma_y a_{3(1-y)} - \frac{h_y}{N} = 0, \\ & a'_{3y} - 2a_{1y}a_{3y} - 4a_{2y}a_{3y} - 4(N-2)a_{3y}a_{4y} - \gamma_y a_{3y} + \gamma_y a_{3(1-y)} - \frac{h_y}{N} = 0, \\ & a'_{4y} - 2a_{3y}^2 - 4a_{2y}a_{3y} - 4a_{1y}a_{4y} - 4(N-3)a_{3y}a_{4y} - \gamma_y a_{4y} + \gamma_y a_{4(1-y)} = 0, \end{aligned}$$

which gives $a_{1y}+(N-2)a_{3y}=a_{2y}+(N-2)a_{4y}$ since two expressions for $a_{3y}$ should be equal. This implies that $(a_{1y}+(N-2)a_{3y})'=(a_{2y}+(N-2)a_{4y})'$ or

$$\begin{aligned}
&2a_{1y}^2+2(N-2)a_{1y}a_{3y}+4(N-1)a_{3y}^2+4(N-2)a_{2y}a_{3y}+4(N-2)^2a_{3y}a_{4y}\\
&+\gamma_y\left(a_{1y}+(N-2)a_{3y}\right)-\gamma_y\left(a_{1(1-y)}+(N-2)a_{3(1-y)}\right)-\frac{h_y}{N}\\
=&2(N-1)a_{3y}^2+4a_{1y}a_{2y}+4(N-2)a_{2y}a_{3y}+4(N-2)a_{3y}a_{4y}+4(N-2)a_{1y}a_{4y}\\
&+4(N-2)(N-3)a_{3y}a_{4y}+\gamma_y\left(a_{2y}+(N-2)a_{4y}\right)-\gamma_y\left(a_{2(1-y)}+(N-2)a_{4(1-y)}\right)-\frac{h_y}{N}.
\end{aligned}$$

After combining terms and substituting $a_{2y}+(N-2)a_{4y}$ with $a_{1y}+(N-2)a_{3y}$, we get $a_{1y}^2+(N-2)a_{1y}a_{3y}-(N-1)a_{3y}^2=0$, which yields $a_{3y}=a_{1y}$ or $a_{3y}=-\frac{1}{N-1}a_{1y}$. Note that $a_{3y}\neq a_{1y}$ due to their different differential equations. Hence, we can conclude that $a_{3y}=-\frac{1}{N-1}a_{1y}$. In conclusion, for $i=1,2,\dots,N$, $A_{iy}$ $(y=0,1)$ has the expression of (4.1.31).

The existence and uniqueness of (4.1.27) is equivalent to the existence and uniqueness of (4.1.30). For $a_{1y}$, the existence and uniqueness can be deduced from Lemma 51 and 52. Given $a_{1y}$'s, $a_{2y}$'s are linear equations, thus their existence and uniqueness are guaranteed by Theorem 12.1 in Antsaklis and Michel (2006). Together with previous discussions, we conclude the results. □

*The proof of Lemma 41:* Continued from the Lemma 40, player $i$'s path in the $N$-player game follows

$$\hat{X}_{it}^{(N)}=x_i^N-\int_0^t 2a^N_{1Y_s^{(N)}}\left(\hat{X}_{is}^{(N)}-\frac{1}{N-1}\sum_{j\neq i}^N\hat{X}_{js}^{(N)}\right)ds+W_{it}^{(N)}.$$

With the notation

$$\bar{X}_s^{(N)}=\frac{1}{N}\sum_{i=1}^N\hat{X}_{is}^{(N)},$$

one can rewrite the path by

$$\hat{X}_{it}^{(N)}=x_i^N-\int_0^t 2\hat{a}^N_{1Y_s^{(N)}}\left(\hat{X}_{is}^{(N)}-\bar{X}_s^{(N)}\right)ds+W_{it}^{(N)}. \tag{4.3.7}$$

By adding up the above equations (4.3.7) indexed by $i=1$ to $N$, one can have

$$\bar{X}_t^{(N)}=\bar{x}^N+\frac{1}{N}\sum_{i=1}^N W_{it}^{(N)}=\bar{x}^N+\frac{\sqrt{N-1}}{N}\left(\sqrt{N-1}\bar{W}_{-it}^{(N)}\right)+\frac{1}{N}W_{it}^{(N)}, \tag{4.3.8}$$

where $\bar{W}_{-it}^{(N)}:=\frac{1}{N-1}\sum_{j\neq i}W_{jt}^{(N)}$.

Finally, to see the distribution of $Z_t^N$ in the space $\Omega$ identical distribution to $\hat{X}_{ut}^{(N)}$ in $\Omega^{(N)}$, we follow the following steps:

- Embed $Y^{(N)}$ from $\Omega^{(N)}$ to $Y$ from $\Omega$;
- Replace the index $i$ of (4.3.7) by uniform random variable $u$;
- Since $\sqrt{N-1}\bar{W}_{-ut}^{(N)}$ is a Borwnian motion independent of the term $W^{(N)}$ and $Y^{(N)}$, we replace $\sqrt{N-1}\bar{W}_{-ut}^{(N)}$ and $W_{ut}^{(N)}$ of (4.3.8) by $B$ and $W$, respectively;

- Substitute $\bar{X}^{(N)}$ of (4.3.7) by (4.3.8).

$\square$

*The proof of Theorem 43.* Given the smooth and bounded function $\{w_i : i = 1, 2, \dots, 6\}$, the functions $(a, b, c, d, e, f, g)$ in (4.2.9) is a coupled linear system, and their existence, uniqueness and boundedness is shown by Theorem 12.1 in Antsaklis and Michel (2006).

Plugging the 6 equations in (4.2.12) to the ODE system (4.2.9), which can be rewritten by

$$\begin{cases} a' - 2a^2 + k = 0, \\ e' - 2ae - eg = 0, \\ f' - eg - 2af - gf - 2be - 2ce = 0, \\ g' - 4ag - g^2 - 2k = 0, \\ b' - \frac{1}{2}g^2 - 4ab - 2bg - 2cg = 0, \\ c' - 4ac + k = 0, \\ d' + 2a + g + b - \frac{1}{2}e^2 - ef + 2c = 0, \end{cases}$$

with the terminal conditions

$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = g(T) = 0.$$

Let $l = 2a + g$, and then

$$l'(t) - l^2(t) = 0, \quad l(T) = 0,$$

which implies that $l(t) = 2a(t) + g(t) = 0$ for all $t \in [0, T]$. This gives $g = -2a$ and $e' = 0$. Thus $e(t) = 0$ for all $t \in [0, T]$ and then one can obtain $f' = 0$, which indicates that $f(t) = 0$ for all $t \in [0, T]$. Therefore the ODE system (4.2.9) can be simplified to

$$\begin{cases} a'(t) - 2a^2(t) + k = 0, \\ b'(t) - 2a^2(t) + 4a(t)c(t) = 0, \\ c'(t) - 4a(t)c(t) + k = 0, \\ d'(t) + b(t) + 2c(t) = 0, \end{cases}$$

which gives the expression of $(a, b, c, d)$ as shown in (4.2.15).

Note that in this case, since $2a + g = 0$ and $e = 0$ for all $t \in [0, T]$, from (4.2.11) we have $\hat{\mu}_s = \bar{\mu} + \tilde{W}_s$ for all $s \in [t, T]$. Then

$$\hat{\nu}_s = \bar{\nu} + \int_t^s \left(2 + 4a(r)\hat{\mu}_r^2 - 4a(r)\hat{\nu}_r\right) \, dr + \int_t^s 2\hat{\mu}_r \, d\tilde{W}_r.$$

Plugging $e = 0$ and $\hat{\mu}_s = \bar{\mu} + \tilde{W}_r$ back to (4.2.8), we obtain the optimal control as

$$\hat{\alpha}_s = 2a(s)(\bar{\mu} + \tilde{W}_s - \hat{X}_s).$$

Since $e = f = 0$ and $g = -2a$ for $s \in [t, T]$, the value function can be simplified to

$$v(t, x, \bar{\mu}, \bar{\nu}) = a(t)x^2 - 2a(t)x\bar{\mu} + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t).$$

Meanwhile, similar to the equivalence Lemma 36, we have

$$U(m_0, x) = v(0, x, [m_0]_1, [m_0]_2),$$

and it yields the value function $U$ of Theorem 43. $\square$

**Lemma 55.** *For arbitrary non-negative $p \in \mathbb{N}$, $i \in \{1, 2, \ldots, N\}$, and $t \in [0, T]$,*

$$\mathbb{E}\left[\left|\hat{X}_{it}^{(N)}\right|^p\right] \le C(T,p) < \infty,$$

*where $C(T,p)$ is a bounded function of terminal time $T$ and $p$.*

*Proof.* Note that for $p \ge 1$, from the Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\left[\left|\hat{X}_{it}^{(N)}\right|^p\right] =& \mathbb{E}\left[\left|x_i^N - \int_0^t 2\hat{a}_1^N(s)\hat{X}_{is}^{(N)}ds + \int_0^t 2\hat{a}_1^N(s)\left(\bar{x}^N + \frac{1}{\sqrt{N}}B_s + \tilde{W}_s\right)ds + W_{it}^{(N)} + \tilde{W}_t\right|^p\right] \\
\le& 6^{p-1}\Bigg\{\left|x_i^N + \bar{x}^N\int_0^T 2\hat{a}_1^N(s)ds\right|^p + T^{p-1}\int_0^t 2^p\left(\hat{a}_1^N(s)\right)^p\mathbb{E}\left[\left|\hat{X}_{is}^{(N)}\right|^p\right]ds + \\
& T^{p-1}\int_0^T 2^p\left(\hat{a}_1^N(s)\right)^p\left(\mathbb{E}\left[\left|\frac{1}{\sqrt{N}}B_s\right|^p\right] + \mathbb{E}\left[\left|\tilde{W}_s\right|^p\right]\right)ds + \sup_{t\in[0,T]}\left(\mathbb{E}\left[\left|W_{it}^{(N)}\right|^p\right] + \mathbb{E}\left[\left|\tilde{W}_t\right|^p\right]\right)\Bigg\} \\
=& C_1(p,T) + \int_0^t C_2(p,T)\mathbb{E}\left[\left|\hat{X}_{is}^{(N)}\right|^p\right]ds,
\end{aligned}$$

where $C_1, C_2$ are bounded functions of $p$ and $T$. By Grönwall's inequality,

$$\mathbb{E}\left[\left|\hat{X}_{it}^{(N)}\right|^p\right] \le C_1(p,T)e^{C_2(p,T)T} := C(p,T),$$

which gives the desired result. $\square$

# Chapter 5

# Conclusions and Future Work

## 5.1 Conclusions

We study three stochastic control problems among $N$ players and under the mean-field settings. The main objective is to discover how players behave under interactions with the population and how interplay distorts their optimal controls. To achieve that, certain types of the Nash equilibrium are established for each model, and the optimal controls are compared with the standard model without interactions.

Chapter 2 establishes a constant Nash equilibrium among $N$ mutual funds competing for investment flows based on relative performance. We compare the after-fee Sharpe ratio and Beta coefficients with and without competition, namely Merton's model, and verify the existence of herd effect – funds may converge in optimal controls to avoid the possible loss caused by competition – in most cases, which hurts the performance of funds. However, if funds are extremely disadvantaged in the risk-aversion adjusted Sharpe ratios, they tend to be more aggressive and gamble their way up to get a decent chance of winning the game. This pushes them away from the market average and boosts their after-fee performance. The model is different from what is in Basak and Makarov (2015), Lacker and Zariphopoulou (2019) since its merges the competition into the dynamic of processes instead of at the terminal time.

Chapter 3 builds a principal-agent model where a representative policy maker is the principal who decides the optimal capital gain tax, while one or $N$ funds serve as agents who choose the optimal portfolios based on the tax rate given. From the funds' side, a (Pareto optimal) Nash equilibrium is deduced for both one and $N$ funds. The competition among funds cause less aggressive optimal portfolios in case there are potential losses brought by the game. Meanwhile, whether optimal portfolios become more aggressive or not due to different tax rates is inconclusive due to the influence of both income and substitution effects. Managers may tend to gamble more when it is closer to the terminal time due to their eagerness to make up for the loss caused by the high tax rate. Accordingly, policy makers with higher risk aversion incline to lower their optimal tax rate to mainly decrease the volatility of the market.

Chapter 4 solves LQG mean field games with two types of common noises: continuous time Markov chain and Brownian motion. We manage to transform this infinite dimensional problem caused by the path dependence feature into a finite dimension Riccati system. (Semi-)closed solutions are deduced for both cases. At the same time, we reduce the $\mathcal{O}(N^3)$ dimension of the counterpart $N$-player game to a Riccati system independent of $N$ and embed it into one specific probability space. The convergences of the dynamic process and its empirical measure are proved based on the simplification of the Riccati system of $N$-player game.

## 5.2 Future Work

In the model of Chapter 2, the influence of the relative performance to the fund's flow is symmetric. However, model like Basak and Makarov (2014) gives a discrete-time asymmetric influence of relative performance. It is hard for us to combine the asymmetry into continuous competition based on the fund flows, but it is worth further studies to see whether there is any distortion caused by the asymmetry.

Model in Chapter 3 shows a decreasing trend in optimal tax rate with respect to higher policy makers' risk aversions. Nevertheless, this may slightly differ from intuitions. Part of the reasons may be the policy makers' value function since they may have other concerns like employment rate when choosing the optimal tax rate. Meanwhile, the capital gain tax rate is relatively stable compared to the optimal portfolio from the funds' side, so there could be a time mismatch problem in the setting as well.

As mentioned in Section 4.3.1, the coefficient before the mean field term in the cumulative cost function must be positive to make sure the global existence of the Riccati equations. Despite that, the coefficient can be influenced by the relative variance of the population and be set to negative in certain situation. In detail, for some positive $\bar{\nu}$,

$$k_f = \begin{cases} k_1 < 0 & \text{, if } \nu < \bar{\nu}, \\ k_2 > 0 & \text{, if } \nu \geq \bar{\nu}, \end{cases}$$

where $\nu$ is the variance of the population. In theory, it could relieve the exploding problem when the coefficient is negative since before exploding, the coefficient has already been switched to positive.

# Bibliography


Ahuja, S. (2015), *Mean Field Games with Common Noise*, Stanford University.

Aivaliotis, G. and Palczewski, J. (2014), 'Investment strategies and compensation of a mean–variance optimizing fund manager', *European journal of operational research* **234**(2), 561–570.

Anthropelos, M., Geng, T. and Zariphopoulou, T. (2020), 'Competition in fund management and forward relative performance criteria', *Working paper, available at SSRN 3723229* .

Antsaklis, P. J. and Michel, A. N. (2006), *Linear systems*, Springer Science & Business Media.

Balcer, Y. and Judd, K. L. (1987), 'Effects of capital gains taxation on life-cycle investment and portfolio management', *The Journal of Finance* **42**(3), 743–758.

Bardi, M. and Priuli, F. S. (2013), Lqg mean-field games with ergodic cost, *in* '52nd IEEE Conference on Decision and Control', IEEE, pp. 2493–2498.

Basak, S. and Makarov, D. (2014), 'Strategic asset allocation in money management', *The Journal of Finance* **69**(1), 179–217.

Basak, S. and Makarov, D. (2015), 'Competition among portfolio managers and asset specialization', *Available at SSRN 1563567* .

Basak, S., Pavlova, A. and Shapiro, A. (2007), 'Optimal asset allocation and risk shifting in money management', *The Review of Financial Studies* **20**(5), 1583–1621.

Bichuch, M. and Sturm, S. (2014), 'Portfolio optimization under convex incentive schemes', *Finance and Stochastics* **18**(4), 873–915.

Bielagk, J., Lionnet, A. and Reis, G. D. (2017), 'Equilibrium pricing under relative performance concerns', *SIAM Journal on Financial Mathematics* **8**(1), 435–482.

Bogle, J. C. (2005), 'The mutual fund industry 60 years later: For better or worse?', *Financial Analysts Journal* **61**(1), 15–24.

Boyle, P., Garlappi, L., Uppal, R. and Wang, T. (2012), 'Keynes meets markowitz: The trade-off between familiarity and diversification', *Management Science* **58**(2), 253–272.

Brennan, M. J. (1975), 'The optimal number of securities in a risky asset portfolio when there are fixed costs of transacting: Theory and some empirical results', *Journal of Financial and Quantitative Analysis* **10**(3), 483–496.

Brown, S. J. and Goetzmann, W. N. (1997), 'Mutual fund styles', *Journal of Financial Economics* **43**(3), 373–399.

Browne, S. (2000), 'Stochastic differential portfolio games', *Journal of Applied Probability* **37**(1), 126–147.

Cardaliaguet, P. (2010), Notes on mean field games, Technical report, Technical report.

Carmona, R., Delarue, F. et al. (2018), *Probabilistic Theory of Mean Field Games with Applications I-II*, Springer.

Chetty, R. (2006), 'A new method of estimating risk aversion', *American Economic Review* **96**(5), 1821–1834.

Chevalier, J. and Ellison, G. (1997), 'Risk taking by mutual funds as a response to incentives', *Journal of Political Economy* **105**(6), 1167–1200.

Dal Forno, A. and Merlone, U. (2010), 'Incentives and individual motivation in supervised work groups', *European Journal of Operational Research* **207**(2), 878–885.

Espinosa, G.-E. and Touzi, N. (2015), 'Optimal investment under relative performance concerns', *Mathematical Finance* **25**(2), 221–257.

Falsetta, D., Rupert, T. J. and Wright, A. M. (2013), 'The effect of the timing and direction of capital gain tax changes on investment in risky assets', *The Accounting Review* **88**(2), 499–520.

Feldstein, M. S. (1969), 'The effects of taxation on risk taking', *Journal of Political Economy* **77**(5), 755–764.

Feldstein, M. and Yitzhaki, S. (1978), 'The effects of the capital gains tax on the selling and switching of common stock', *Journal of Public Economics* **9**(1), 17–36.

Feng, X., Huang, J. and Qiu, Z. (2019), 'Mixed social optima and nash equilibrium in linear-quadratic-gaussian mean-field system', *arXiv preprint arXiv:1911.01886* .

Firoozi, D., Jaimungal, S. and Caines, P. E. (2020), 'Convex analysis for lqg systems with applications to major-minor lqg mean-field game systems', *Systems & Control Letters* **142**, 104734.

Frei, C. and Dos Reis, G. (2011), 'A financial market with interacting investors: does an equilibrium exist?', *Mathematics and financial economics* **4**(3), 161–182.

Friend, I. and Blume, M. E. (1975), 'The demand for risky assets', *The American Economic Review* **65**(5), 900–922.

Gao, S., Caines, P. E. and Huang, M. (2020), 'Lqg graphon mean field games', *arXiv preprint arXiv:2004.00679* .

Golub, G. H. and Van Loan, C. F. (1996), *Matrix computations*, Johns Hopkins Studies in the Mathematical Sciences, third edn, Johns Hopkins University Press, Baltimore, MD.

Graham, J. R. (1999), 'Herding among investment newsletters: Theory and evidence', *The Journal of Finance* **54**(1), 237–268.

Grinblatt, M., Titman, S. and Wermers, R. (1995), 'Momentum investment strategies, portfolio performance, and herding: A study of mutual fund behavior', *The American Economic Review* pp. 1088–1105.

Gruber, M. (1996), 'Another puzzle: The growth in actively managed mutual funds', *The Journal of Finance* **51**(3), 783–810.

Han, J., Ma, G. and Yam, S. C. P. (2022), 'Relative performance evaluation for dynamic contracts in a large competitive market', *European Journal of Operational Research* .

Hanche-Olsen, H. and Holden, H. (2010), 'The kolmogorov–riesz compactness theorem', *Expositiones Mathematicae* **28**(4), 385–394.

Hardy, G. H., Littlewood, J. E. and Pólya, G. (1952), *Inequalities*, Cambridge university press.

Huang, J. and Huang, M. (2013), Mean field lqg games with model uncertainty, *in* '52nd IEEE Conference on Decision and Control', IEEE, pp. 3103–3108.

Huang, J., Li, X. and Wang, T. (2015), 'Mean-field linear-quadratic-gaussian (lqg) games for stochastic integral systems', *IEEE Transactions on Automatic Control* **61**(9), 2670–2675.

Huang, J., Wang, S. and Wu, Z. (2014), 'Mean field linear-quadratic-gaussian (lqg) games: major and minor players', *arXiv preprint arXiv:1403.3999* .

Huang, J., Wei, K. D. and Yan, H. (2007), 'Participation costs and the sensitivity of fund flows to past performance', *The Journal of Finance* **62**(3), 1273–1311.

Huang, M. (2009/10), 'Large-population LQG games involving a major player: the Nash certainty equivalence principle', *SIAM J. Control Optim.* **48**(5), 3318–3353.
**URL:** *https://doi.org/10.1137/080735370*

Huang, M., Caines, P. E. and Malhamé, R. P. (2012), 'Social optima in mean field lqg control: centralized and decentralized strategies', *IEEE Transactions on Automatic Control* **57**(7), 1736–1751.

Huang, M., Malhamé, R. P., Caines, P. E. et al. (2006), 'Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle', *Communications in Information & Systems* **6**(3), 221–252.

Ippolito, R. A. (1992), 'Consumer reaction to measures of poor quality: Evidence from the mutual fund industry', *The Journal of Law and Economics* **35**(1), 45–70.

Kempf, A. and Ruenzi, S. (2008), 'Tournaments in mutual-fund families', *The Review of Financial Studies* **21**(2), 1013–1036.

Lacker, D. and Soret, A. (2020), 'Many-player games of optimal consumption and investment under relative performance criteria', *Mathematics and Financial Economics* **14**(2), 263–281.

Lacker, D. and Zariphopoulou, T. (2019), 'Mean field and n-agent games for optimal investment under relative performance criteria', *Mathematical Finance* **29**(4), 1003–1038.

Lasry, J.-M. and Lions, P.-L. (2007), 'Mean field games', *Japanese journal of mathematics* **2**(1), 229–260.

Lioui, A. and Poncet, P. (2013), 'Optimal benchmarking for active portfolio managers', *European Journal of Operational Research* **226**(2), 268–276.

Liu, H. (2014), 'Solvency constraint, underdiversification, and idiosyncratic risks', *Journal of Financial and quantitative Analysis* **49**(2), 409–430.

Maug, E. and Naik, N. (2011), 'Herding and delegated portfolio management: The impact of relative performance evaluation on asset allocation', *The Quarterly Journal of Finance* **1**(02), 265–292.

Nguyen, S. L. and Huang, M. (2012), Mean field lqg games with mass behavior responsive to a major player, *in* '2012 IEEE 51st IEEE Conference on Decision and Control (CDC)', IEEE, pp. 5792–5797.

Nguyen, S. L., Nguyen, D. T. and Yin, G. (2020), 'A stochastic maximum principle for switching diffusions using conditional mean-fields with applications to control problems', *ESAIM: Control, Optimisation and Calculus of Variations* **26**, 69.

Ou-Yang, H. (2003), 'Optimal contracts in a continuous-time delegated portfolio management problem', *The Review of Financial Studies* **16**(1), 173–208.

Palomino, F. (2005), 'Relative performance objectives in financial markets', *Journal of Financial Intermediation* **14**(3), 351–375.

Patel, J., Zeckhauser, R. and Hendricks, D. (1991), 'The rationality struggle: Illustrations from financial markets', *The American Economic Review* **81**(2), 232–236.

Scharfstein, D. S. and Stein, J. C. (1990), 'Herd behavior and investment', *The American Economic Review* pp. 465–479.

Seifried, F. T. (2010), 'Optimal investment with deferred capital gains taxes', *Mathematical Methods of Operations Research* **71**(1), 181–199.

Siemsen, E., Balasubramanian, S. and Roth, A. V. (2007), 'Incentives that induce task-related effort, helping, and knowledge sharing in workgroups', *Management science* **53**(10), 1533–1550.

Sirri, E. R. and Tufano, P. (1998), 'Costly search and mutual fund flows', *The Journal of Finance* **53**(5), 1589–1622.

Stiglitz, J. E. (1975), 'The effects of income, wealth, and capital gains taxation on risk-taking', *Stochastic Optimization Models in Finance* pp. 291–311.

Szpiro, G. G. (1986), 'Measuring risk aversion: an alternative approach', *The review of economics and statistics* pp. 156–159.

Taylor, J. (2003), 'Risk-taking behavior in mutual fund tournaments', *Journal of Economic Behavior & Organization* **50**(3), 373–383.

Tchuendom, R. F. (2018), 'Uniqueness for linear-quadratic mean field games with common noise', *Dynamic Games and Applications* **8**(1), 199–210.

Uppal, R. and Wang, T. (2003), 'Model misspecification and underdiversification', *The Journal of Finance* **58**(6), 2465–2486.

Van Nieuwerburgh, S. and Veldkamp, L. (2009), 'Information immobility and the home bias puzzle', *The Journal of Finance* **64**(3), 1187–1215.

Vissing-Jørgensen, A. and Attanasio, O. P. (2003), 'Stock-market participation, intertemporal substitution, and risk-aversion', *American Economic Review* **93**(2), 383–391.

Yong, J. and Zhou, X. Y. (1999), *Stochastic controls: Hamiltonian systems and HJB equations*, Vol. 43, Springer Science & Business Media.

Yost, B. P. (2018), 'Locked-in: The effect of ceos' capital gains taxes on corporate risk-taking', *The Accounting Review* **93**(5), 325–358.