Project Number: MA-WJM-8401

Graphons: A New Model for Large Networks

A Major Qualifying Project

submitted to the Faculty of

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfillment of the requirements for the

Degree in Bachelor of Science by

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April 30, 2015

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Abstract

Over the past decade László Lovász has led the development of a new language for talking about large graphs and networks. It opens up new doors for analyzing large graphs and has many connections with classical graph theory results such as the Szemerédi Regularity Lemma. In this project I sought to understand and expand on some of the major concepts expressed in the new language. I prove several theorems concerning homomorphism densities and also look into graphons, the limit objects of Cauchy sequences of graphs. I study them in general and then look at the specific ones that are derived from variations of the Sierpiński carpet fractal.

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Chapter 1

Introduction

This Major Qualifying Project looks at a new language of representing and studying large graphs which was primarily developed by László Lovász. He and several other people have worked out a new set of tools for extremal graph theory over the past decade. As an attempt to summarize the findings, Lovász published a book called *Large Networks and Graph Limits* in 2012 [14]. In this report, we examine the theory primarily as it is expressed in this book and apply it to families of graphs derived from the Sierpiński carpet, a very well-known fractal [21].

Numerous famous results from classical graph theory can be expressed beautifully in this new language that Lovász developed. Theorems that take paragraphs to state in the language of classical graph theory can be stated in a simple sentence. The theory also provides useful ways to approximate large graphs. It lets us study the underlying structure of a graph even when the graph is too large to study by classical means. These ideas have taken off in recent years with research on how this theory can be applied to understand large graphs in a variety of applications.

1.1 Examples of large graphs

A graph, or network, is a set of points (also called nodes or vertices) and a set of edges which connect some of the points. Graphs can be used to represent relations between discrete items such as people, cities, atoms, websites, research papers, airports etc. For instance consider a group of five people. The friendships among them can be represented as a graph on five points. Each point represents one person. Any two points are connected with an edge if the two people they represent are friends. Suppose Sam knows Amy and Jamie and suppose Hunter is friends with Jesse while all other pairs of people are strangers to each other. The graph representing these friendships is pictured in Figure 1.1.

This pictorial representation allows us to immediately answer some questions

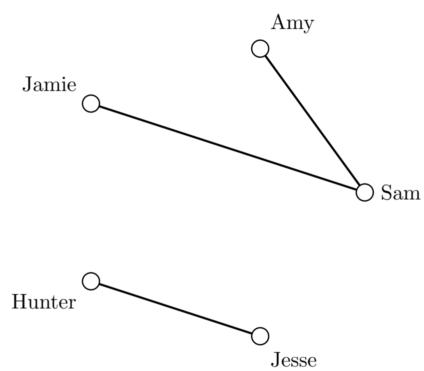


Figure 1.1: A friendship graph

that we might have. For instance, cliques of friends, reachability (can one pass a message through friends to someone else), number of mutual friends, and concepts such as popularity and loneliness are visible in graphs of this type. By studying it, one can also make assumptions about other characteristics. For instance, it may be possible to determine the genders of these friends. The graph gives us a way to very quickly answer some questions about the group of friends and a way to analyze it more deeply.

This is a very straightforward graph and there are many much more complicated networks that are studied and analyzed via graph theory. Some examples include road networks, airline route maps, the friendship graph of everyone in the world, bonds between atoms in crystals, genes in a person's genetic code, links between web pages, etc. These graphs are much larger and contain more information than the friendship graph in Figure 1.1 which is what we call a simple graph.

Now let us pause and make this precise. A *simple graph*, G, consists of a set of vertices, $V(G)$, and an edge set, $E(G)$, of unordered pairs of distinct vertices. The cardinalities of these two sets will be denoted by $v(G)$ and $e(G)$ respectively. A graph is *finite* if $v(G) < \infty$. We will find it convenient to switch between two different viewpoints of edges. Sometimes it will be convenient to refer to the adjacency relation rather than to edges in the edge set. If $\{u, v\} \in E(G)$ then we say that u and v are adjacent and write $u \sim v$. The *degree* of vertex v, written $\mathsf{deg}(v)$, is the number of vertices adjacent to it. The *codegree* of vertices v and u, written $\mathsf{codeg}(v, u)$, is the number of vertices adjacent to both v and u.

There are several special simple graphs which we will refer to frequently. The complete graph K_n has n vertices, $v(K_n) = n$, and every vertex is adjacent to every other vertex, $E(G) = \{\{u, v\} : u, v \in V(G)\}$. Complete bipartite graphs, $K_{m,n}$, have $m + n$ vertices divided into two partite sets of sizes m and n. Each vertex is adjacent to every vertex in the other set and to no vertex in its own set. This idea can be extended to complete k-partite graphs. The vertices of a complete k-partite graph, $K_{t_1,t_2,\ldots,t_k}$, are divided into k partite sets, $t_1, t_2, \ldots, t_k$.

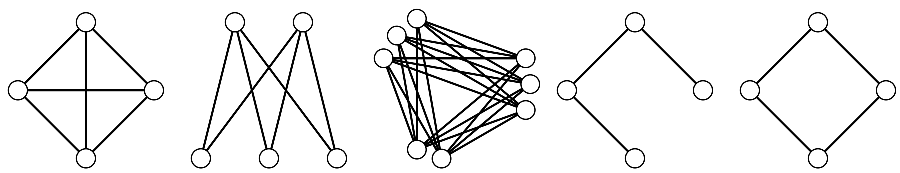
Vertices $u \in t_i$ and $v \in t_j$ are adjacent if $i \neq j$. The path, P_n has vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$ and edge set

$$E(P_n) = \{\{v_i, v_j\} \in V(P_n) \times V(P_n) : |i - j| = 1\}.$$

The last special type of graphs that we define is the cycle C_n. This graph has the same vertex set as P_n and almost the same edge set:

$$E(C_n) = \{\{v_i, v_j\} \in V(P_n) \times V(P_n) : i \pm 1 \equiv j \mod n\}.$$

Examples of these graphs are shown in Figure 1.2.



(a) K_4 (b) $K_{2,3}$ (c) $K_{2,3,3}$ (d) P_4 (e) C_4

Figure 1.2: Example graphs

Simple graphs can model many situations, but when they become more complex, the graphs modeling them need to be more nuanced. For instance, simple graphs can represent many road networks well. However if there is a one way street, or if we want to keep track of the number of lanes on a road, then the graph modeling this must contain more information. One way to do this is to use *digraphs* (graphs with directed edges). This allows node A to be joined to node B but at the same time node B is not joined to node A. In our friendship graph directed edges could be used to represent one sided relationships. For instance Alex, Sam, Jamie, Hunter and Jesse could all know Taylor Swift but she might only know Sam.

Other ways to capture more information using graphs include weighting the edges and/or nodes. A weight is a function from the vertex or edge set to the real numbers. For instance each edge in a friendship graph could have a weight corresponding to the strength of the friendship represented, or the weight could be equal to the number of years that the people had been friends. Node weights could potentially represent the number of years a person has been in the area.

Directed edges and weights are two ways for the graph to store more information. The type of graph used will be dependent on what it models and what information is desired. This report focuses on how Lovász's tools work on simple graphs. Many of the concepts have natural extensions to digraphs and weighted graphs, but we will keep it simple for the most part.

Our original friendship graph sheds light on a small group of friends. One might wonder what we could learn from looking at the friendship graph of everyone in a college or city or country or the world. Is everyone connected through a chain of friends to someone else? Or are there separate components that are not joined to one another in any way? What is the average degree of separation? What is the size of the largest clique of friends?

Such friendship graphs are frequently studied. Perhaps the most well-known one is generated by looking at all Facebook profiles. Each profile corresponds to a single node. Any two nodes are connected by an edge if the profiles they represent are friends. The resulting graph is called the *Facebook Social Graph*.

The Facebook Social Graph is the largest online social graph in the world [25]. As of 2011 it had 721 million active users with 68.7 billion friendship edges. Figure 1.3 shows a small subgraph that was generated by looking at the friends of a single Facebook user. Within this graph we can clearly see different groups of friends within which most people know each other and that there a few people who span the friend groups. We can also easily pick out the outcasts from this person's friend groups.

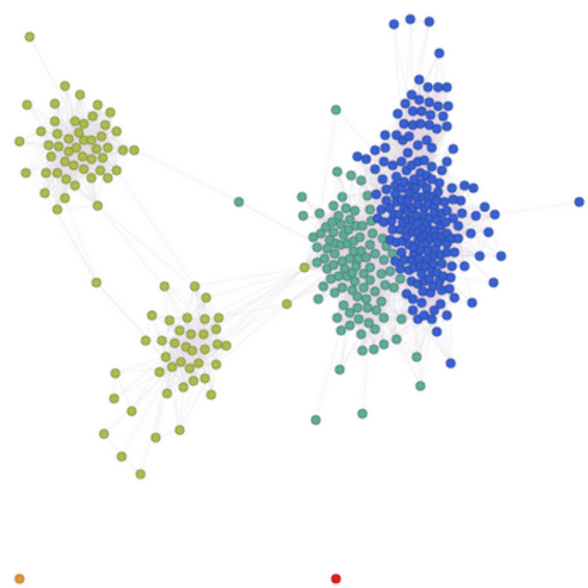


Figure 1.3: The friendship graph among the friends of a single Facebook user. This was generated using Wolfram Alpha computational knowledge engine.

The entire Facebook Social Graph is far too large to picture and is not even constant. New friendships are constantly being created, old friends are being unfriended, new profiles are made and old ones are deleted. The enormous size and changing nature make it hard to analyze. However there are tools to do this. In 2011 Ugander, Karrer, Backstrom and Marlow analyzed the graph and answered some of the questions we raised [25].

How many friends do users have? Figure 1.4 shows the fraction of users that have any given number of friends on a log-log scale. We can see that most people have few friends but that some have up to 5000 friends (this is a maximum imposed by Facebook).

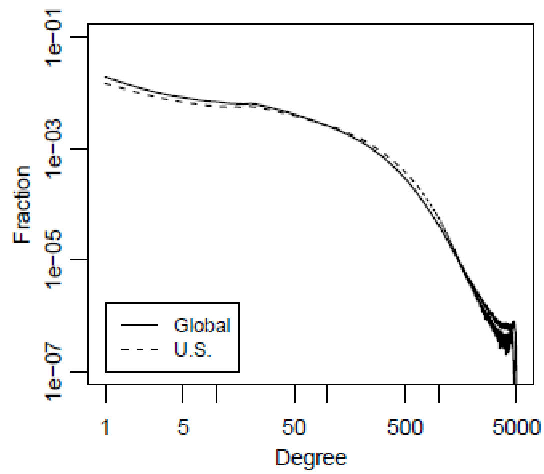


Figure 1.4: Fraction of vertices in the Facebook Social Graph with given degree. Taken from [25].

Is everyone connected to everyone else through a path of friends of friends? Almost, it turns out that the graph has a component consisting of 99.91% of the vertices that are all connected with each other in this way [25]. The next largest component has around 2000 vertices which is less than 0.0003% of the total number of vertices.

What is the average degree of separation between any two people? Figure 1.5 shows the fraction of pairs that are within so many friends of friends of each other. It is interesting to note that virtually everyone is within six friends of any other person.

The Facebook Social Graph is the object of much study and analysis from a multitude of angles. Facebook (the corporation) analyzes it to recommend friends and decide which stories certain people will find interesting enough to put on their news feed. Advertisers can use the graph's structure to target certain profiles. Researchers look to it to learn patterns in communication [6, 26], study social networks of certain communities [24], or to research techniques for studying large networks [25]. Such research also applies to other social networking sites. Linked In, Twitter, My Space, Google +, etc. all analyze huge constantly changing graphs. In order to upgrade services they must continue to find better ways to analyze these networks.

The tools analyzed in this report add to the methods available for analysis. Studying the full Facebook Social Graph has many difficulties. Besides being

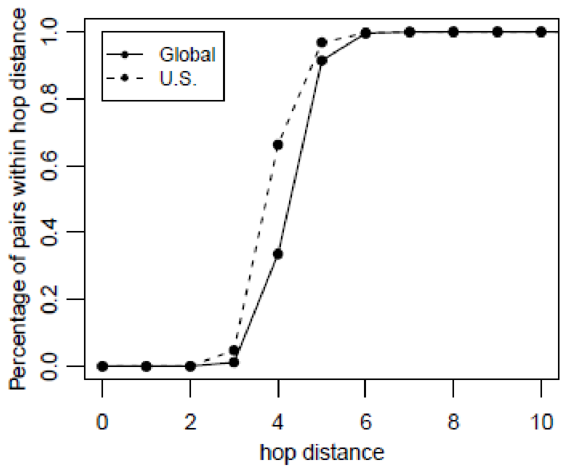


Figure 1.5: The number of hops required to go between pairs of people in the Facebook Social Graph.Taken from [25].

enormous and requiring massive amounts of computing power, it is constantly changing. The theory developed by Lovász and others smooths out large graphs and lets us look at an approximation instead. A good approximation is especially crucial when we look at networks that are constantly changing such as social networks. Even if we capture the graph from an instant in time it will simply be an approximation by the time we are done studying it. We want an approximation that captures as much of the information as possible while also making computations easier.

1.2 Homomorphism densities

As we have seen, large graphs are an increasingly important area of study. They occur in many areas with a number of important applications and we therefore require effective techniques to understand them. Many graphs are so large that it is impossible to store or even define them in the traditional sense. Even if the graph can be stored, it may be constantly changing like the Facebook Social Graph.

These problems suggest that it would be convenient to be able to approximate large graphs with smaller ones or to find a different way to represent them which is easier to analyze. In order to approximate graphs, we require a way to say how similar two graphs are. This is made difficult because the two graphs could be very different in some respects and yet very similar in others.

To see this consider the complete bipartite graphs $K_{2,1}$, $K_{10,5}$ and $K_{200,100}$. The complete bipartite graph $K_{m,n}$ has $m+n$ vertices partitioned into two sets of size m and n. Two vertices are adjacent if they are in different sets and not adjacent if they are in the same set. Graphs $K_{2,1}$ and $K_{10,5}$ are shown in Figure 1.6.

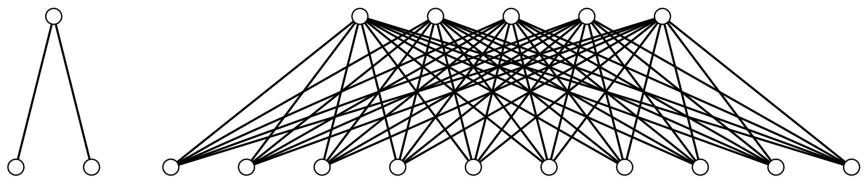


Figure 1.6: Graphs $K_{2,1}$ and $K_{10,5}$.

These graphs clearly share some similarities but are also quite different. Of course, $K_{2,1}$ has only three vertices while $K_{10,5}$ has fifteen and $K_{200,100}$ has 300. However it turns out that they are very similar in some key ways. For instance, if you chose any two vertices in either graph, the probability that they will be adjacent is identical for all three graphs. In fact, $K_{2,1}$, $K_{10,5}$ and $K_{100,200}$ are so similar that this can be extended.

The general idea involves comparing samples of subgraphs from the two graphs. Choose any n vertices from a source graph, G_1, and consider the graph, G_1', formed by the chosen vertices and all edges between them that were in G. We can consider graphs G_1 and G_2 similar or "close" if the subgraphs, G_1' and G_2', obtained in this manner are likely to be similar.

Let us make this idea more precise. Consider two finite simple graphs F and G with vertex sets $V(F)$ and $V(G)$ respectively. A *graph homomorphism* from F to G is a mapping (not necessarily one-to-one) $\phi : V(F) \to V(G)$ satisfying the condition that if a and b are adjacent in F then $\phi(a)$ and $\phi(b)$ are adjacent in G. We write: $a \sim b$ implies $\phi(a) \sim \phi(b)$.

For example, consider the graphs F and G in Figure 1.7 and the mapping $\phi : V(F) \to V(G)$ defined as follows.

$$\phi(f_1) := g_1 \;\; \phi(f_2) := g_2 \;\; \phi(f_3) := g_3 \;\; \phi(f_4) := g_2$$

Graph F has four edges namely $\{f_1, f_2\}$, $\{f_2, f_3\}$, $\{f_1, f_3\}$ and $\{f_1, f_4\}$. For ϕ to be a graph homomorphism $\{\phi(f_1), \phi(f_2)\}$, $\{\phi(f_2), \phi(f_3)\}$, $\{\phi(f_1), \phi(f_3)\}$ and $\{\phi(f_1), \phi(f_4)\}$ must all be edges in G. These clearly are edges in G, so ϕ is a homomorphism. In this report, "homomorphism" always refers to a graph homomorphism.

Now we define $\mathsf{hom}(F, G)$ as the number of homomorphisms from F to G. This number is very different for graphs with different numbers of vertices so we normalize it. We define the *homomorphism density* of F in G as

$$t(F, G) := \frac{\mathsf{hom}(F, G)}{v(G)^{v(F)}}. \tag{1.1}$$

Note that $v(G)^{v(F)}$ is the total number of functions from $V(F)$ to $V(G)$. So $t(F, G)$ is the probability that a random mapping from $V(F)$ to $V(G)$ is a homomorphism.

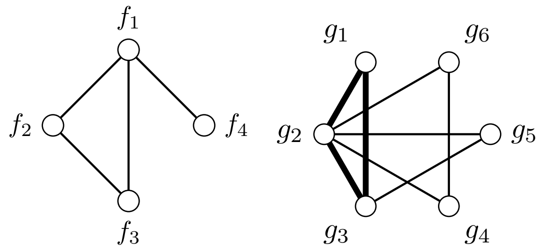


Figure 1.7: Graphs F and G.

The homomorphism densities, $t(F, G)$, of a graph G contain much information about the graph. If two graphs have similar homomorphism densities then in many ways they can be viewed as similar graphs. It turns out that $K_{2,1}$, $K_{10,5}$ and $K_{100,200}$ have identical homomorphism densities. That is, for all finite simple graphs F, $t(F, K_{2,1}) = t(F, K_{10,5}) = t(F, K_{100,200})$. We prove this on page 23. This is why $K_{2,1}$, $K_{10,5}$ and $K_{100,200}$ can be considered "close".

In addition to being closely related to a notion of the distance between graphs, homomorphism densities are a convenient language in some cases. They allow some extremal graph problems to be expressed algebraically. Also, some classical results in extremal graph theory can be expressed very simply. For instance Turáns Theorem [2], which is a basic theorem taught in introductory graph theory classes, can be expressed in a single line, while a typical statement of it might take several lines.

1.3 Graphons

One way to approximate large graphs is to build a sequence of graphs of increasing sizes that get close to what we want to study. As the number of vertices goes to infinity, we view the graph as a continuous structure instead of a discrete one.

This is similar to how we might approximate the structure of a sheet of metal. In reality the metal is a discrete structure made of millions of atoms with bonds between them. However we frequently ignore the discrete nature and view it as a continuous object with certain properties. We can do a similar approximation with graphs. In order to do this we need a couple of tools: we need a notion of the distance between graphs so we know when they are close and we need a way to represent graphs as continuous objects.

We have already discussed how homomorphism densities can be used to judge how similar graphs are. However we need something more precise. The mathematical concept of a *metric* is a formalization of the concept of distance. For now it suffices to say that a metric assigns a positive real number to be the distance between two different objects. If the distance is 0, we consider the objects equal.

Different objects require different metrics and frequently there are multiple meaningful choices of metric for any given class of objects. Accordingly, there are a few possible options for metrics on graphs. Lovász uses what is called the *cut metric*. This is the metric most closely related to homomorphism densities.

Once we have this precise definition of the distance between graphs in hand, we can look at sequences of graphs where the graphs get closer and closer to each other. We call such a sequence *Cauchy*. The idea of a Cauchy sequence is most familiar in the context of real numbers. For example, a Cauchy sequence of rational numbers is the sequence $\{x_n\}$ defined by

$$x_n = \sum_{k=0}^{n} \frac{\pi_k}{10^k}.$$

Here π_k is the k^{th} digit of π after the decimal point and π_0 is 3. As n gets larger, the distance between x_n and x_m for $m > n$ gets smaller and smaller. In fact, with this sequence the terms don't just get closer to each other, they also get closer to a specific limit namely π. Even though x_n never equals π it can be made as close as we would like by making n large enough. We say that the sequence *converges* to the *limit* π.

We would like to be able to say the same thing about graphs. However when we look at sequences of graphs where the number of vertices goes to infinity, we need something other than a graph to be the limit of the sequence. It seems odd to want a sequence of graphs to converge to something other than a graph. However this is exactly what happened with our earlier sequence. The x_n are all rational numbers but they converge to an irrational number. We need something analogous to an irrational number to be the limit objects for our sequences of graphs.

Lovász showed that objects called *graphons* are the limits of Cauchy sequences of graphs. The name comes from **graph** functi**ons**. Graphons are Lebesgue measurable functions on the unit square. For our purposes Lebesgue measurable means that we can integrate the functions. The graphon corresponding to a graph is closely related to the graph's *adjacency matrix*. The adjacency matrix of a graph can be constructed by labeling the graph's vertices $1, 2, ..., n$. Then the adjacency matrix is an $n \times n$ matrix of 0's and 1's. The entry in the i^{th} row and j^{th} column is 1 if the vertices i and j are adjacent and 0 if they are not adjacent. For example, consider the graph G in Figure 1.8.

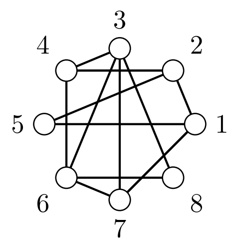


Figure 1.8: G

Its adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The graphon corresponding to G can be viewed as making the adjacency matrix a step function on the unit square. The function is 0 where there is a 0 in the adjacency matrix and the function is 1 where there is a 1 in the adjacency matrix. The graphon for G is shown in Figure 1.9. Where the function is 1, the figure is colored black and where the function is 0 the figure is left white. The origin is shown in the upper left corner to match the typical labeling in the adjacency matrix.

This graphon can be viewed as the graph G similarly to how the real number $0.250\bar{0}$ can be interpreted as the rational number $\frac{1}{4}$. However there are graphons that cannot be treated as a graph but are the limit of a sequence of graphs. Figure 1.10 shows two graphs (along with their adjacency matrices and graphons) in a convergent sequence of graphs. The limit graphon is pictured in Figure 1.11. The limit graphon does not correspond to any finite graph.

Lovász showed that every Cauchy sequence of graphs converges to a graphon and there is a surprising converse. Every graphon is the limit of a Cauchy sequence of graphs. So if we have a sequence of graphs that models some phenomena that we would like to study, we can study the limit graphon.

For instance, consider a sequence of random graphs $\{R_n\}$ where each graph has order n and any two vertices are adjacent with probability $\frac{1}{2}$. Notice that the graph G from Figure 1.8 could be taken as R_8. As n approaches infinity the sequence of graphs grows closer and closer to the constant graphon $W(x, y) = \frac{1}{2}$.

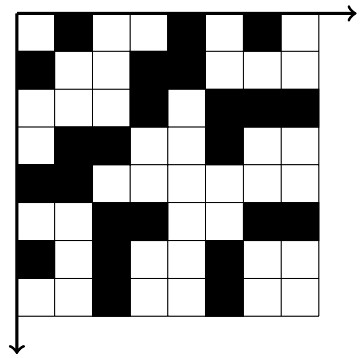


Figure 1.9: Graphon corresponding to G

This is a case where the graphon smooths out the minute details while preserving major properties (such as homomorphism densities) of the graphs in the sequence. It is quite incredible that we can study functions in the place of graphs. This allows us to apply tools from analysis to graph theory. Analysis is a much older branch of mathematics than graph theory. Mathematicians have been developing theorems in analysis for hundreds more years than in graph theory. Graphons bridge the gap between these two separate areas of mathematics allowing tools from one to be used to address questions in the other.

Another benefit of graphons is that they are a beautiful and natural language with which to talk about certain results. Perhaps this is most noticeable in regards to the Szemerédi Regularity Lemma [7, 23]. This is one of the greatest results in graph theory and it has many connections with graphons. An easy way to see how natural it is to use graphons when talking about the Regularity Lemma is just to look at how it is stated. In classical graph theory language, it can take a paragraph just to state the Regularity Lemma precisely. Using the new language the lemma is reduced to the statement that a certain space is compact.

Informally, the Regularity Lemma says that the vertices of every graph can be partitioned into a number of sets of nearly equal size such that the edges between any two of these sets behave in a random-like fashion within some prespecified error. A graph where the subgraphs between partition classes are quasirandom is called a *multipartite quasirandom graph.* The Regularity Lemma essentially says that any graph can be approximated by a multipartite quasirandom graph. This is one of the main tools used to approximate large graphs.

Recall that our motivation for introducing graphons was to approximate large graphs. It turns out that the Regularity Lemma can be extended to graphons. Approximating graphs with multipartite quasirandom graphs in terms of graphons is the same thing as approximating measurable functions with step functions. This is a classic tool in analysis and is one of the ways that graphons bridge the gap between graph theory and analysis.

Saying that a graph can be broken down into structured components that be-

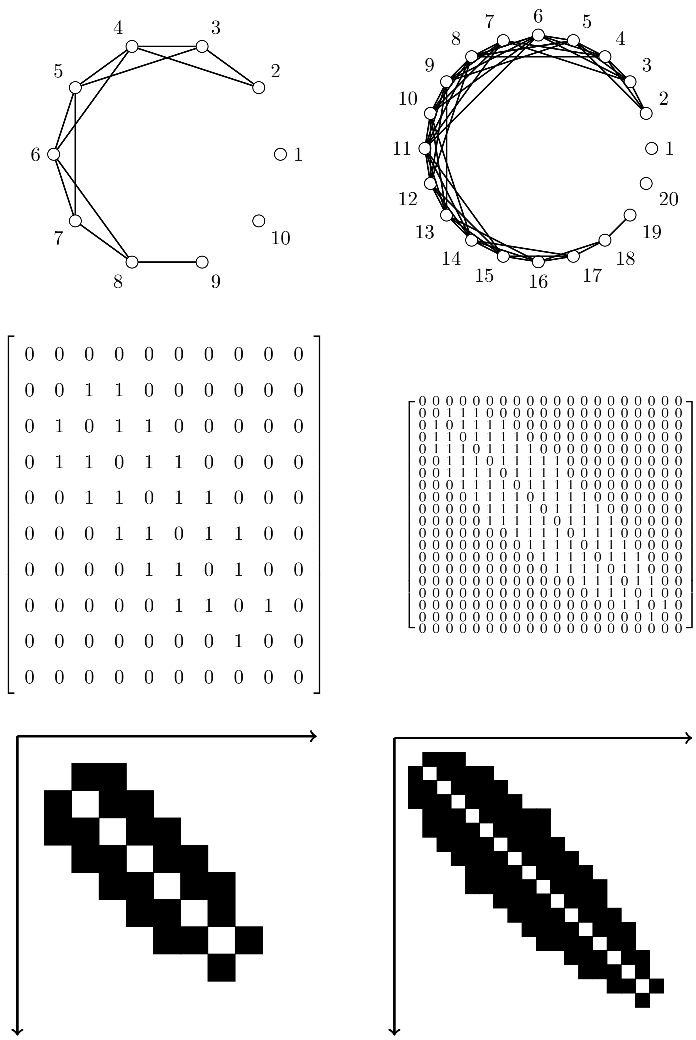


Figure 1.10: Two graphs from a convergent sequence along with their adjacency matrices and graphons

have randomly with each other is a remarkable connection between randomness and structure. This was a huge contribution to the mathematical community's understanding of large graphs and has many applications in extremal graph theory, graph property testing, machine learning, etc. It was used to prove the Green-Tao theorem which is arguably one of the most important discoveries in number theory in recent years [8]. In recognition of the impact of the Regularity Lemma, Endré Szemerédi was awarded the Abel prize in 2012. The Abel prize is frequently described as the Nobel prize for mathematicians. It is no small thing that the theory we consider here has many connections with the Regular-

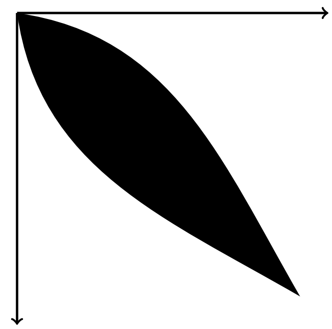


Figure 1.11: The limit graphon of the sequence in Figure 1.10

ity Lemma. Rather it points to how big of an impact this theory can have on mathematics.

1.4 Brief Overview

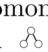
The theory developed in [14] is still new and not widely understood. In this report, I go over parts of the theory and apply it to specific examples. The first part of this report is on homomorphism densities and their properties. I rewrite classical results from graph theory in the language of homomorphism densities, derive some properties with combinatorics, and collect other properties that Lovász claims in [14]. After looking at homomorphism densities in graphs via combinatorics, I introduce graphons and look at how analysis gives the same results. I then look at specific sequences of graphons and their corresponding graphs which are derived from variations on the Sierpiński carpet fractal. I use the tools developed to answer graph theoretic questions about these graphs and their limits.

Chapter 2

Homomorphism densities

A good way to compare graphs or understand their properties is to look at the density of other graphs in them. This method works well when the graphs are very large or when two graphs are of different sizes but still similar. K_{1000} and K_{1001} are very similar graphs however it is not possible to talk about isomorphisms from one to the other since they have different vertex sets. Their similarity can be shown by the densities of other graphs in them or by their densities in other graphs.

This notion of how similar or close graphs are is necessary to study how well a manageably sized graph can approximate a large graph or when looking at a convergent sequence of graphs. Without a measure of distance (or a topology, at least), convergence has no meaning. Lovász measures distance between graphs using a metric called the cut metric. One of the main results of the developed theory is that there is a very close relationship between homomorphism densities and the cut metric. In fact, homomorphism densities can be used to determine precisely which sequences of graphs are Cauchy in the cut metric.

Recall from Chapter 1 that $\mathsf{hom}(F, G)$ is the number of graph homomorphisms from a graph F to a graph G (see page 8). The set of all homomorphisms from F to G is $\mathsf{Hom}(F, G)$. Typically we are more interested in the homomorphism density defined in Equation (1.1). We will usually be interested in homomorphism densities for a small F and a large dense G. For very small F and general fixed G, we will abbreviate the notation and use a pictogram of F for the homomorphism density. For instance  is defined as shorthand for $t(K_2, G)$ and  is defined as $t(P_3, G)$.

2.1 Homomorphism density propositions

In this section, we collect propositions concerning homomorphism densities and relations between different homomorphism densities in any given graph G. Some are restatements of classical results of graph theory, such as Turán's theorem [2], in the language of homomorphism densities. We prove some of the propositions and some are taken, without proof, from [14]. At the end of this section all of them are collected in Table 2.1.

There are some properties of homomorphism densities that we will use in this chapter and begin by presenting them as lemmas. The first allows us to linearize homomorphism density expressions. The product of the homomorphism densities of graphs F_1 and F_2 into a graph G is equal to the homomorphism density of the disjoint union of F_1 and F_2 into G. The disjoint union of graphs F and G, denoted by $F \cup G$ or by FG, has vertex set $V(F) \cup V(G)$ and edge set $E(F) \cup E(G)$.

Lemma 2.1. $t(F_1, G)t(F_2, G) = t(F_1F_2, G)$

Proof. Let F_1, F_2 and G be finite simple graphs. Each $\phi \in \mathsf{Hom}(F_1F_2, G)$ can be written as (ψ, σ) for some $\psi \in \mathsf{Hom}(F_1, G)$ and $\sigma \in \mathsf{Hom}(F_2, G)$. Also each choice of ψ and σ determine a unique ϕ so

$$\mathsf{hom}(F_1F_2, G) = \mathsf{hom}(F_1, G)\,\mathsf{hom}(F_2, G).$$

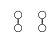
This means that

$$\begin{aligned} t(F_1, G)t(F_2, G) &= \frac{\mathsf{hom}(F_1, G)}{v(G)^{v(F_1)}} \frac{\mathsf{hom}(F_2, G)}{v(G)^{v(F_2)}} \\ &= \frac{\mathsf{hom}(F_1F_2, G)}{v(G)^{v(F_1)+v(F_2)}}. \end{aligned}$$

Therefore

$$t(F_1, G)t(F_2, G) = t(F_1F_2, G). \tag{2.1}$$

□

This means, for example, that the pictogram  can be interpreted as $t(K_2, G)^2$ or $t(K_2K_2, G)$.

The second property shows how the homomorphism density of K_2 in a graph G is related to the number of edges and vertices in G.

Lemma 2.2. $t(K_2, G) = \frac{2e(G)}{v(G)^2}$

Proof. There are two homomorphisms from K_2 to any edge in G. So the homomorphism density of K_2 in G is

$$t(K_2, G) = \frac{\mathsf{hom}(K_2, G)}{v(G)^2} = \frac{2e(G)}{v(G)^2}.$$

□

Note that $t(K_2, G)$ is almost equal to the fraction of all edges in the complete graph on $v(G)$ vertices that appear as edges in G, namely $\frac{e(G)}{\binom{v(G)}{2}}$.

The codomain of $t(K_2, G)$ is clearly a subset of $\mathbb{Q} \cap [0, 1)$. It turns out that these two sets are equal. We can construct a graph of at least size n with any desired K_2 density. Let the density $\frac{p}{q} \in \mathbb{Q} \cap [0,1)$ be given. Choose a positive integer m such that $m \geq \max\left\{\frac{n}{2q}, \frac{1}{2(q-p)}\right\}$. Then set $v(G) = 2qm$ and $e(G) = 2pqm^2$. It is possible to do this because the only constraint on the number of edges is $e(G) \leq \binom{v(G)}{2}$. Then $t(K_2, G) = \frac{2e(g)}{v(G)^2} = \frac{4pqm^2}{(2qm)^2} = \frac{p}{q}$.

We begin by looking at Turán's theorem. It says that there is a unique graph of order n that has the most edges while not containing the complete graph, K_{k+1}, as a subgraph. This graph is called the *Turán graph*, $T_{n,k}$. It is defined to be the complete k-partite graph $K_{t_1,\ldots t_k}$ on n vertices where the vertices are split as evenly as possible into k partite sets. That is $t_i = \left\lceil \frac{n}{k} \right\rceil$ or $\left\lfloor \frac{n}{k} \right\rfloor$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k} t_i = n$. For example $T_{8,3}$ is $K_{3,3,2}$ which is shown in Figure 2.1. Turán's theorem guarantees that this is the unique graph (up to isomorphism) on eight vertices with the maximum number of edges such that K_4 is not a subgraph.

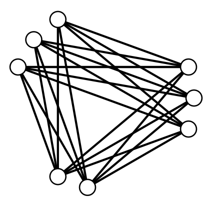


Figure 2.1: The Turán graph $T_{8,3}$

Turán graphs have at most

$$\frac{v^2}{2}\left(1 - \frac{1}{k}\right)$$

edges. This implies that if, for some graph G, we have

$$e(G) > \frac{v^2}{2}\left(1 - \frac{1}{k}\right), \tag{2.2}$$

then G must contain K_{k+1}, i.e. $t(K_{k+1}, G) > 0$. Translating this into homomorphism density notation gives the following proposition, which we prove assuming Turán's Theorem.

Proposition 2.3. *Given a graph G, if $t(K_2, G) > 1 - \frac{1}{k}$, then $t(K_{k+1}, G) > 0$.*

Proof. Assume that $t(K_2, G) > 1 - \frac{1}{k}$ for a graph G. Recall from Lemma 2.2 that the K_2 density is related to the number of edges via

$$t(K_2, G) = \frac{2e(G)}{v^2}.$$

This gives that

$$\frac{2e(G)}{v^2} > 1 - \frac{1}{k}.$$

Rearranging gives

$$e(G) > \frac{v^2}{2}\left(1 - \frac{1}{k}\right).$$

So Turán's theorem guarantees that $t(K_{k+1}, G) > 0$. $\square$

Turán's theorem gives us that if a graph, G, has enough edges (the K_2 density is large enough), then the density of K_n is nonzero. Now we ask how the homomorphism densities of K_n and K_{n-1} compare in dense graphs of large order. We find a relation between the two densities that is approached as the order of G approaches infinity. We call this an asymptotic bound.

Definition 2.4. *For finite simple graphs F_1 and F_2 we say that $t(F_1, G)$ is* asymptotically greater than or equal to $t(F_2, G)$ *if for all $\epsilon > 0$ there exists a v such that $t(F_1, G) \geq t(F_2, G) - \epsilon$ for all graphs G with $v(G) \geq v$. This is denoted by $t(F_1, G) \gtrsim t(F_2, G)$.*

If the homomorphism density of K_{n-1} is close to 1, then there is a lower bound, L, such that the homomorphism density of K_n is asymptotically greater than or equal to L. When the homomorphism density of K_{n-1} is not close to 1, the bound is very loose.

Proposition 2.5. *If $t(K_{n-1}, G) = 1 - \epsilon$ then $t(K_n, G) \gtrsim 1 - n\epsilon$.*

Proof. Let a graph G with v vertices be almost complete so that it has $\binom{v}{n-1} - c$ copies of K_{n-1}. We want to look at how many copies of K_n it contains. Every K_{n-1} that is not in G is a part of $v - (n-1)$ copies of K_n in K_v. Since there are $n!$ homomorphisms from K_n to itself, there are no more than $c(v - n + 1)n!$ more homomorphisms of K_n in K_v than in G, i.e.

$$\mathsf{hom}(K_n, G) \geq \mathsf{hom}(K_n, K_v) - n!c(v - n + 1). \tag{2.3}$$

Since c is the number of copies of K_{n-1} missing from G we can write

$$\begin{aligned} c &= \binom{v}{n-1} - \frac{\mathsf{hom}(K_{n-1}, G)}{(n-1)!} \\ &= \frac{v!}{(n-1)!(v-n+1)!} - \frac{\mathsf{hom}(K_{n-1}, G)}{(n-1)!}. \end{aligned}$$

Substituting this value of c into (2.3) gives

$$\begin{aligned} \mathsf{hom}(K_n, G) &\geq \binom{v}{n} n! - n!(v-n+1)\left(\frac{v!}{(n-1)!(v-n+1)!} - \frac{\mathsf{hom}(K_{n-1}, G)}{(n-1)!}\right) \\ &= \frac{v!}{(v-n)!} - \frac{v!n}{(v-n)!} + n(v-n+1)\,\mathsf{hom}(K_{n-1}, G) \\ &= -\frac{v!(n-1)}{(v-n)!} + n(v-n+1)\,\mathsf{hom}(K_{n-1}, G). \end{aligned}$$

Dividing by v^n gives

$$t(K_n, G) \geq -\frac{v!(n-1)}{v^n(v-n)!} + \frac{n(v-n+1)}{v} t(K_{n-1}, G).$$

Taking the limit as $v \to \infty$ gives

$$t(K_n, G) \gtrsim 1 - n + nt(K_{n-1}, G).$$

Now set $t(K_{n-1}, G) = 1 - \epsilon$. Then

$$\begin{aligned} t(K_n, G) &\gtrsim 1 - n + n(1 - \epsilon) \\ &= 1 - n\epsilon. \end{aligned}$$

□

The next few propositions have to do with the number of homomorphisms from P_3 into some graph G so we first introduce the following lemma.

Lemma 2.6. $\mathsf{hom}(P_3, G) = \sum_{v \in V(G)} \mathsf{deg}(v)^2$

Proof. We refer to the vertex labeling in Figure 2.2. If b is mapped to v in G, then a and c must be mapped to any of the vertices adjacent to b. Thus there are $\mathsf{deg}(v)^2$ homomorphisms that map b to v. Since b can be mapped to any vertex in G, the total number of homomorphisms from P_3 to G is $\sum_{v \in V(G)} \mathsf{deg}(v)^2$.

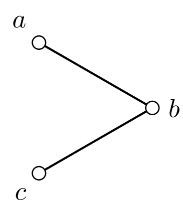


Figure 2.2: Vertex labeling for P_3

□

Next we look at how the K_2 and P_3 homomorphism densities are related in graphs that have fewer edges than vertices.

Proposition 2.7. *If a graph G has fewer edges than vertices, then $t(K_2, G) \geq 2t(P_3, G)$.*

Proof. Given a fixed number, e, of edges, $G = K_{1,e}$ maximizes $\mathsf{hom}(P_3, G)$ because every edge is incident to every other edge. So the image of P_3 can be made of any two edges in G. For any other graph when $e > 3$ it is impossible for every edge to be incident to every other edge. This, along with Lemma 2.6 gives us

$$\begin{aligned}
\mathsf{hom}(P_3, G) &\leq \mathsf{hom}(P_3, K_{1,e}) \\
&= \sum_{v \in K_{1,e}} \mathsf{deg}(v)^2 \\
&= e(e-1) \\
&= \frac{1}{2}\,\mathsf{hom}(K_2, G)(e+1).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\frac{\mathsf{hom}(P_3, G)}{v^3} &\leq \frac{\mathsf{hom}(K_2, G)(e+1)}{2v^3} \\
t(P_3, G) &\leq t(K_2, G)\frac{e+1}{2v}.
\end{aligned}$$

Restricting ourselves to graphs where $e \leq v-1$ lets us simplify this further since $\frac{e+1}{2v} \leq \frac{v}{2v}$. In such a case we have

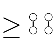
$$\begin{aligned}
t(P_3, G) &\leq t(K_2, G)\frac{1}{2} \\
2t(P_3, G) &\leq t(K_2, G).
\end{aligned}$$

$\square$

Notice that equality is achieved ($2t(P_3, G) = t(K_2, G)$) when $G = K_{1,e}$. Then $t(P_3, G) = t(K_2, G)\frac{e+1}{2v}$ and $e = v - 1$.

A large part of proving Proposition 2.7 was counting the homomorphisms of P_3 into $K_{1,e}$. The next proposition involves more combinatorics. We use the Cauchy-Schwarz inequality to compare the homomorphism densities of P_3 and two disjoint edges. Recall the pictogram notation for $t(F, G)$ given on page 15.

Proposition 2.8.

 $\geq$ 

Proof. By Cauchy-Schwarz we have

$$\left(\sum_{a \in V(G)} \mathsf{deg}^2(a)\right) v(G) \geq \left(\sum_{a \in V(G)} \mathsf{deg}(a)\right)^2.$$

Using Lemma 2.6 and $\sum_{a \in V(G)} \mathsf{deg}(a) = 2e(G)$ gives us

$$\begin{aligned}
\mathsf{hom}(P_3, G)v(G) &\geq (2e(G))^2 \\
\frac{\mathsf{hom}(P_3, G)}{v(G)^3} &\geq \left(\frac{2e(G)}{v(G)^2}\right)^2.
\end{aligned}$$

Then by Lemma 2.2 we have

$$t(P_3, G) \geq t(K_2, G)t(K_2, G) = t(K_2K_2, G).$$

$\square$

Notice that P_3 is the same as $K_{1,2}$. Proposition 2.8 can be generalized to say that the $K_{1,n}$ homomorphism density is at least as big as the homomorphism density of n copies of K_2. To show this we use Jensen's inequality [10] instead of the Cauchy-Schwarz inequality. Jensen's inequality states that if f is a convex function on the interval $[m, n]$, $\{x_i\}_{i=1}^{k} \in [m, n]$, and $\{w_i\}_{i=1}^{k}$ are positive weights, then

$$f\left(\frac{\sum_{i=1}^{k} w_i x_i}{\sum_{i=1}^{k} w_i}\right) \leq \frac{\sum_{i=1}^{k} w_i f(x_i)}{\sum_{i=1}^{k} w_i}.$$

Using this, we can generalize the previous proposition.

Proposition 2.9.

$$t(K_{1,n}, G) \geq \overset{\circ}{\underset{\circ}{|}}^{\,n}$$

Proof. Let b be the unique vertex of degree n in $K_{1,n}$ and let a be the image of b is G. To count $\mathsf{hom}(K_{1,n}, G)$ we look at the possible homomorphisms given a. Each of the n vertices adjacent to b in $K_{1,n}$ can be mapped to any vertex adjacent to a giving $\mathsf{deg}(a)^n$ homomorphisms. Since a could be any vertex in G the total number of homomorphisms is

$$\mathsf{hom}(K_{1,n}, G) = \sum_{a \in V(G)} \mathsf{deg}(a)^n.$$

We can use Jensen's inequality now because $f(x) = x^n$ is convex on $(0, \infty)$ for $n > 1$ so if $w_i = 1$ for $1 \leq i \leq k$ we have

$$\begin{aligned}
\frac{\sum_{a \in V(G)} \mathsf{deg}(a)^n}{v(G)} &\geq \left(\frac{\sum_{a \in V(G)} \mathsf{deg}(a)}{v(G)}\right)^n \\
\frac{\sum_{a \in V(G)} \mathsf{deg}(a)^n}{v(G)^{n+1}} &\geq \left(\frac{\sum_{a \in V(G)} \mathsf{deg}(a)}{v(G)^2}\right)^n \\
\frac{\mathsf{hom}(K_{1,n}, G)}{v(G)^{n+1}} &\geq \left(\frac{\mathsf{hom}(K_2, G)}{v(G)^2}\right)^n \\
t(K_{1,n}, G) &\geq t(K_2^n, G).
\end{aligned}$$

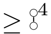
$\square$

This next proof also utilizes Cauchy-Schwarz and combinatorics. We use one of the fundamental building blocks of combinatorics, which is to count $\mathsf{hom}(P_3, G)$ in two different ways and setting the results equal to each other. One of these ways is to sum up the codegrees of any two vertices. For this, we need the following definition.

Definition 2.10. *The* codegree *of any two vertices a and b in a graph G, written $\mathsf{codeg}(a, b)$, is the number of vertices that are adjacent to both a and b.*

Now we can present the Proposition.

Proposition 2.11.

 $\geq$  ^{4}

Proof. Let the images $a, b \in V(G)$ of two non-adjacent vertices of C_4 be given. The images of the other two vertices of C_4 under a homomorphism can be any vertices adjacent to both a and b so there are $\mathsf{codeg}(a,b)^2$ homomorphisms for any given a and b. Therefore the total number of homomorphisms is

$$\begin{aligned} \mathsf{hom}(C_4, G) &= \sum_{a,b \in V(G)} \mathsf{codeg}(a,b)^2 \\ &\geq \frac{\left(\sum_{a,b\in V(G)} \mathsf{codeg}(a,b)\right)^2}{v(G)^2} \end{aligned} \tag{2.4}$$

by Cauchy-Schwarz. The next step is to relate the sum of the codegrees in a graph to the sum of the degrees. This way we can compare it to the homomorphism density of K_2. We do this by counting $\mathsf{hom}(P_3, G)$ two different ways. The first way is to look at the image of the middle vertex of P_3 as in Lemma 2.6 giving

$$\mathsf{hom}(P_3, G) = \sum_{a \in V(G)} \mathsf{deg}(a)^2.$$

The second way to count $\mathsf{hom}(P_3, G)$ is to look at the images of the two end vertices. Given that these are mapped to $a, b \in V(G)$ the middle vertex can be mapped to any vertices adjacent to both a and b. Note that it is possible for a to equal b. For any given a and b there are $\mathsf{codeg}(a,b)$ possible homomorphisms so we have

$$\mathsf{hom}(P_3, G) = \sum_{a,b \in V(G)} \mathsf{codeg}(a,b).$$

Equating the results of both ways of counting and then applying Cauchy-Schwarz gives

$$\begin{aligned} \sum_{a,b \in V(G)} \mathsf{codeg}(a,b) &= \sum_{a \in V(G)} \mathsf{deg}(a)^2 \\ &\geq \frac{\left(\sum_{a\in V(G)} \mathsf{deg}(a)\right)^2}{v(G)}. \end{aligned}$$

Substituting this into (2.4) gives

$$\begin{aligned}
\mathsf{hom}(C_4, G) &\geq \frac{\left(\sum_{a\in V(G)} \mathsf{deg}(a)\right)^4}{v(G)^4};\\
t(C_4, G) &\geq \frac{\left(\sum_{a\in V(G)} \mathsf{deg}(a)\right)^4}{v(G)^8}\\
&= \left(\frac{\sum_{a\in V(G)} \mathsf{deg}(a)}{v(G)^2}\right)^4\\
&= t(K_2K_2K_2K_2, G).
\end{aligned}$$

□

We have now seen several inequalities between homomorphism densities that have been proved using Cauchy-Schwarz. It is reasonable to ask how far Cauchy-Schwarz can take us. The answer is quite far. Lovász and Szegedy [18] showed that if arbitrarily small error is allowed, then Cauchy-Schwarz can be used to prove all linear inequalities between homomorphism densities. Of course sometimes it must be applied multiple times and in very non-trivial ways.

Any finite graph G can be extended to a sequence of graphs of increasing order where each graph in the sequence has identical homomorphism densities. This sequence of graphs is $G \bowtie K_m$. The graph product $G \bowtie H$ defines a graph F with vertex set $V(G)\times V(H)$. Any two vertices, $(g_1, h_1), (g_2, h_2) \in V(F)$, are adjacent if and only if ($g_1 \sim g_2$ and $h_1 \sim h_2$) or ($g_1 \sim g_2$ and $h_1 = h_2$).

To visualize $G \bowtie K_m$ replace each vertex in G with a group of m vertices. Any two of these vertices are adjacent if and only if they are in groups that replaced adjacent vertices in G. Figure 2.3 shows K_3, $K_3 \bowtie K_2$ and $K_3 \bowtie K_3$. Note that these are the Turán graphs for K_3 on 3, 6 and 9 vertices.

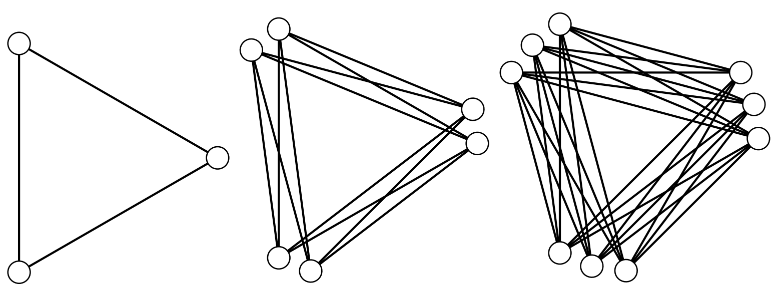


Figure 2.3: Extensions of K_3

Proposition 2.12.

$$t(F, G) = t(F, G \bowtie K_m)$$

Proof. Showing that $G \bowtie K_m$ has identical homomorphism densities as G is equivalent to showing that

$$\frac{\mathsf{hom}(F, G \bowtie K_m)}{(v(G)m)^{v(F)}} = \frac{\mathsf{hom}(F, G)}{v(G)^{v(F)}}.$$

So we want to show that $\mathsf{hom}(F, G \bowtie K_m) = m^{v(F)} \mathsf{hom}(F, G)$.

Consider any homomorphism $\phi : F \to G$. It defines $m^{v(F)}$ homomorphisms $\phi' : F \to G \bowtie K_m$. Every $v \in V(F)$ can be mapped to $(\phi(v), i) \in V(G \bowtie K_m)$ for $1 \leq i \leq m$. This preserves adjacency by definition of $G \bowtie K_m$. So we have $\mathsf{hom}(F, G \bowtie K_m) \geq m^{v(F)} \mathsf{hom}(F, G)$.

Now we want every element in $\mathsf{Hom}(F, G \bowtie K_m)$ to be of this form. Define the projection map $\pi : V(G \bowtie K_m) \to V(G)$ such that $\pi(v, i) = v$ for all vertices v in G and i in (K_m). Then π preserves adjacency. So for all $\phi' \in \mathsf{Hom}(F, G \bowtie K_m)$, we have $\pi \circ \phi' \in \mathsf{Hom}(F, G)$. In this sense, each homomorphism from F to $G \bowtie K_m$ is an expansion of a homomorphism from F to G. □

The last proposition we prove compares the homomorphism densities of two graphs where one is a subgraph of the other.

Proposition 2.13. *If A is a subgraph of B, then $t(B, G) \leq t(A, G)$ for all graphs G.*

Proof. Consider the graphs A, B and G where A is a subgraph of B. Set $n = v(A)$ and $k = v(B) - v(A)$. We begin by proving

$$\mathsf{hom}(B, G) \leq \mathsf{hom}(A, G)v(G)^k. \tag{2.5}$$

Let ψ be a one-to-one homomorphism from A to B i.e. for all adjacent vertices a and b in $V(A)$ we have that $\psi(a)$ and $\psi(b)$ are also adjacent. Let $\mathsf{Hom}(A, G)$ be the set of homomorphisms from A to G.

$$\mathsf{Hom}(A, G) := \{\alpha : V(A) \to V(G) : \alpha(a) \sim \alpha(b) \ \forall a, b \in V(A) \text{ where } a \sim b\}.$$

Let $\mathcal{B}$ be the set of functions from $V(B)$ to $V(G)$ that preserve adjacency on the subgraph A.

$$\mathcal{B} := \{\sigma : V(B) \to V(G) : \exists \alpha \in \mathsf{Hom}(A, G) \text{ s.t. } \sigma(\psi(a)) = \alpha(a) \ \forall a \in A\}$$

We claim that $|\mathcal{B}| = \mathsf{hom}(A, G)v(G)^k$. Label the vertices of A as a_1 through a_n. Let $b_1, ..., b_k$ be the vertices of B that are not in the codomain of ψ. Any $\sigma \in \mathcal{B}$ is described by $(\sigma(b_1), \sigma(b_2), ..., \sigma(b_{k+n}))$. This can be rewritten as

$$(\sigma(b_1), ..., \sigma(b_k), \sigma(b_{k+1}), ..., \sigma(b_{k+n})) \equiv (\sigma(b_1), ..., \sigma(b_k), \alpha).$$

By definition there are $\mathsf{hom}(A, G)$ functions α and there are $v(G)$ possibilities for each $\sigma(b_i)$ where $1 \leq i \leq k$. Therefore $|\mathcal{B}| = \mathsf{hom}(A, G)v(G)^k$.

We next claim that $\mathsf{Hom}(B, G) \subseteq \mathcal{B}$. Let $\beta \in \mathsf{Hom}(B, G)$ be given. $\forall a, b \in V(A)$ If $\psi(a) \sim \psi(b)$ then $\beta(\psi(a)) \sim \beta(\psi(b))$ since β is a homomorphism.

Therefore $\exists \alpha \in \mathsf{Hom}(A, G)$ s.t. $\alpha(a) = \beta(\psi(a))\ \forall a \in A$. This means that $\beta \in \mathcal{B}$ and so $\mathsf{Hom}(B, G) \subseteq \mathcal{B}$.

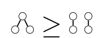
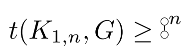
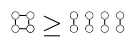
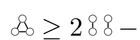
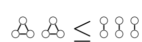
Now that we have (2.5), simply divide both sides by $v(G)^{v(B)}$ to get

$$\frac{\mathsf{hom}(B,G)}{v(G)^{v(B)}} \le \frac{\mathsf{hom}(A,G)}{v(G)^{v(A)}};$$

$$t(B,G) \le t(A,G).$$

□

In summary, Table 2.1 lists all of the above propositions as well as some taken without proof from [14].

Reference	Proposition	
p. 17	$t(K_2, G) > 1 - \frac{1}{k} \Rightarrow t(K_{k+1}, G) > 0$	Prop. 2.3
p. 18	$t(K_{m-1}, G) = 1 - \epsilon \Rightarrow t(K_m, G) \gtrsim 1 - m\epsilon$	Prop. 2.5
p. 19	$e(G) < v(G) \Rightarrow t(K_2, G) \ge 2t(P_3, G)$	Prop. 2.7
p. 20		Prop. 2.8
p. 21		Prop. 2.9
p. 22		Prop. 2.11
p. 23	$t(F, G) = t(F, G \bowtie K_m)$	Prop. 2.12
p. 24	A subgraph of $B \Rightarrow t(B, G) \le t(A, G)$	Prop. 2.13
[14, p. 27]		(2.6)
[14, p. 28]		(2.7)
[14, p. 28]	$t(P_k, G) \ge t(K_2^{k-1}, G)$	(2.8)
[14, p. 68]	$\mathsf{hom}(F, G_1) = \mathsf{hom}(F, G_2)\ \forall F \implies G_1 \cong G_2$	(2.9)
[14, p. 68]	$\mathsf{hom}(G_1, F) = \mathsf{hom}(G_2, F)\ \forall F \implies G_1 \cong G_2$	(2.10)
[14, p. 293]	$t(K'_{r+1}, G) \ge \frac{t(K_r K_r, G)}{t(K_{r-1}, G)}$ 1	(2.11)

[14, p. 293]	$t(K_{r+1},G) - t(K_r,G) \leq r(t(K_{r+1},G) - t(K'_{r+1},G))$ [1]	(2.12)
[14, p. 293]	$r\frac{t(K_r,G)}{t(K_{r-1},G)} \leq (r-1)\frac{t(K_{r+1},G)}{t(K_r,G)} + 1$	(2.13)

Table 2.1: Homomorphism density propositions

2.2 Linear programming with homomorphism densities

One reasonable question to ask is if linear programming methods, like the simplex method, can be used to solve extremal graph theory problems having to do with homomorphism densities. This seems reasonable given that many properties of graphs can be written as inequalities on homomorphism densities. A fair sample have already been listed in Table 2.1. These inequalities could either be used as constraints in a linear programming problem (LP) or used to simplify variables.

For example, we can ask if there is a graph G that maximizes the objective function $t(P_3,G)+t(K_3,G)$ subject to the constraints $t(P_3,G)+2t(K_3,G) \leq \frac{8}{9}$ and $2t(P_3,G)+t(K_3,G) \leq \frac{10}{9}$.

Linear programming tells us that the solution to

$$\begin{aligned} &\max x_1 + x_2 \\ &\text{s.t. } x_1 + 2x_2 \leq \frac{8}{9} \\ &\quad\;\; 2x_1 + x_2 \leq \frac{10}{9} \end{aligned}$$

is $x_1 = \frac{4}{9}$ and $x_2 = \frac{2}{9}$. So if there is a G where $t(P_3,G) = \frac{4}{9}$ and $t(K_3,G) = \frac{2}{9}$ then that G maximizes the objective function. In this case we can see that $G = K_3 \bowtie K_m$ is such a graph for any m. From above, we need only verify this for K_3.

$$\mathsf{hom}(P_3,K_3) = \sum_{a \in V(K_3)} \mathsf{deg}(a)^2 = 3 \cdot 2^2 = 12$$

$$t(P_3,K_3) = \frac{12}{3^3} = \frac{4}{9}$$

To count the homomorphisms from K_3 to K_3, let the vertices of K_3 be labeled a, b and c. a can be mapped to any of the 3 vertices then b has 2 possibilities so $\mathsf{hom}(K_3,K_3) = 6$. Therefore $t(K_3,K_3) = \frac{6}{3^3} = \frac{2}{9}$.

At first, the goal of this MQP was to use linear programming with homomorphism density inequalities to solve extremal graph theory problems such as the

[1] K'_r denotes the graph obtained by deleting an edge from K_r.

above example. However there are two problems with this idea. One is that determining whether a given inequality holds for all graphs G is an algorithmically undecidable problem [14]. The other problem is that of the constraints imposed by properties of homomorphism densities are frequently nonlinear. This can prevent the feasible region of the LP from being convex which renders linear programming methods ineffective. An example of a non-convex feasible region can be found on page 26 of [14]. Both of these problems severely restrict when linear programming could be applied.

Chapter 3

Graphons

When studying a large graph, we would like to smooth out the discreteness inherent in graphs. We would like to study some continuous object which contains the graph's properties that come from its discrete structure. Also, there are some graphs we may wish to study that are constantly changing such as the internet graph or the Facebook Social Graph. Since they are not constant but tend to retain a similar structure, a continuous representation would be a good object to study instead. It would retain the major properties and gloss over any small changes in the discrete nature.

In Chapter 1 we showed how graphons were introduced for precisely this purpose: these are typically piecewise continuous functions on the unit square that can represent graphs or infinite sequences of graphs. The goal of this chapter is to introduce graphons more formally and look at what it means for a sequence of graphs to converge to a graphon.

3.1 Graphons as generalizations of graphs

We begin by defining the linear space of *kernels*:

$$\mathcal{W} := \{W : [0,1]^2 \to \mathbb{R} : W \text{ measurable, bounded and symmetric}\}.$$

We also define the subspace

$$\mathcal{W}_0 := \{W \in \mathcal{W} : 0 \leq W \leq 1\}$$

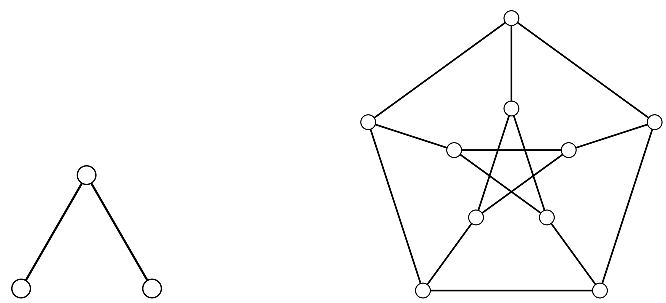
and call any $W \in \mathcal{W}_0$ a *graphon*. Note that $\mathcal{W}$ is the set of symmetric functions in $L_\infty([0,1]^2)$. Although these spaces are defined on the same set of functions, the notation $L_\infty([0,1]^2)$ implies a metric defined in terms of the L_∞ norm, whereas $\mathcal{W}$ is endowed with Lovász's cut metric $\delta_\square$.

Intuitively, a graphon can be thought of as a pixelation of the adjacency matrix of the graph it represents (see Figure 3.1). Formally, any graph G

can be represented as a graphon as follows. For convenience, assume $V(G) = \{1, 2, ..., v(G)\}$. For $x \in \left[\frac{i-1}{v(G)}, \frac{i}{v(G)}\right)$ and $y \in \left[\frac{j-1}{v(G)}, \frac{j}{v(G)}\right)$ we define

$$W_G(x, y) := \begin{cases} 1 & \{i, j\} \in E(G); \\ 0 & \{i, j\} \notin E(G). \end{cases}$$

Figure 3.1 shows P_3 and the Petersen graph (PG), their adjacency matrices, and their graphons, W_{P_3} and W_{PG}.



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

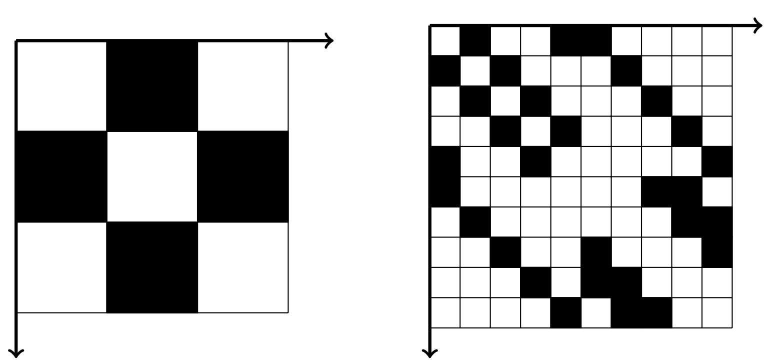


Figure 3.1: Example showing two graphs, P_3 and the Petersen graph (PG), with their adjacency matrices and graphons, W_{P_3} and W_{PG}.

Typically when going from discrete to continuous objects, counting and sums become integrals. Homomorphism densities are no exception. Let W be a kernel and let F be a multigraph (without loops) with vertex set V and edge set E. Then the homomorphism density of F into W is defined as

$$t(F,W) := \int_{[0,1]^{|V|}} \prod_{ij\in E} W(x_i,x_j) \prod_{i\in V} dx_i.$$

This is analogous to the definition of homomorphism densities for graphs. It can be shown that $t(F,G) = t(F,W_G)$ for all graphs F and G [14, p. 116]. We do not prove this here but give a couple of examples.

Example 3.1. *As an example we will calculate $t(K_2,W_{P_3})$ expecting it to equal $t(K_2,P_3)$.*

$$\begin{aligned}
t(K_2,W_{P_3}) = &\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} W(x_1,x_2)dx_1dx_2 + \int_0^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} W(x_1,x_2)dx_1dx_2 \\
&+ \int_0^{\frac{1}{3}}\int_{\frac{2}{3}}^{1} W(x_1,x_2)dx_1dx_2 + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_0^{\frac{1}{3}} W(x_1,x_2)dx_1dx_2 \\
&+ \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} W(x_1,x_2)dx_1dx_2 + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1} W(x_1,x_2)dx_1dx_2 \\
&+ \int_{\frac{2}{3}}^{1}\int_0^{\frac{1}{3}} W(x_1,x_2)dx_1dx_2 + \int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}} W(x_1,x_2)dx_1dx_2 \\
&+ \int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1} W(x_1,x_2)dx_1dx_2 \\
= &\int_0^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} dx_1dx_2 + \int_{\frac{1}{3}}^{\frac{2}{3}}\int_0^{\frac{1}{3}} dx_1dx_2 \\
&+ \int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1} dx_1dx_2 + \int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}} dx_1dx_2 \\
= &\frac{4}{9}
\end{aligned}$$

It is easy to verify that $t(K_2,P_3) = t(K_2,W_{P_3})$ since

$$t(K_2,P_3) = \frac{2e(P_3)}{v(P_3)^{v(K_2)}} = \frac{2\cdot 2}{3^2} = \frac{4}{9}.$$

As another example, we will show that the K_2 homomorphism density is the same in half-graphs and in the graphons corresponding to half-graphs. The half-graph H_n is defined on the vertex set $V(H_n) = \{1,2,...,n,1',2',...,n'\}$. The edge $\{i,j'\}$ is present if $i \leq j$. The graph H_4 is shown in Figure 3.2.

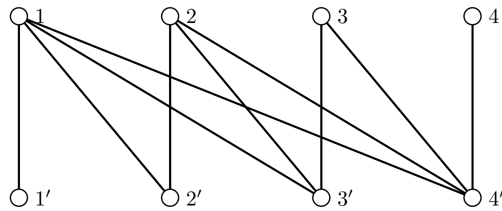


Figure 3.2: H_4

Example 3.2. *We'll show that* $t(K_2, H_n) = t(K_2, W_{H_n})$

$$\begin{aligned} t(K_2, H_n) &= \frac{2e(H_n)}{v(H_n)^{v(K_2)}} \\ &= \frac{2\frac{n(n+1)}{2})}{(2n)^2} \\ &= \frac{n+1}{4n}. \end{aligned}$$

$$\begin{aligned} t(K_2, W_{H_n}) &= \sum_{i=1}^{2n}\sum_{j=1}^{2n}\int_{\frac{i-1}{2n}}^{\frac{i}{2n}}\int_{\frac{j-1}{2n}}^{\frac{j}{2n}} W(x_1,x_2)dx_1dx_2 \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\int_{\frac{i-1}{2n}}^{\frac{i}{2n}}\int_{\frac{j-1}{2n}}^{\frac{j}{2n}} W(x_1,x_2)dx_1dx_2 \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\int_{\frac{i-1}{2n}}^{\frac{i}{2n}}\int_{\frac{j-1}{2n}}^{\frac{j}{2n}} dx_1dx_2 \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\left(\frac{i}{2n}-\frac{i-1}{2n}\right)\left(\frac{j}{2n}-\frac{j-1}{2n}\right) \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\frac{1}{4n^2} \\ &= \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n+1-i} 1 \\ &= \frac{1}{2n^2}\sum_{i=1}^{n} n+1-i \\ &= \frac{1}{2n^2}\left((n+1)n - \frac{(n+1)n}{2}\right) \\ &= \frac{n+1}{4n}. \end{aligned}$$

Note that $\frac{n+1}{4n} \to \frac{1}{4}$ as $n \to \infty$ and therefore $t(K_2, H_n) \to \frac{1}{4}$. In other words, the homomorphism densities of K_2 in H_n converge as n goes to ∞. This begins to touch on what we mean by convergent sequences of graphs.

3.2 Graphons as limit objects of graph sequences

We have said that graphons are the limit objects of sequences of graphs, but have yet to say what it means for a sequence of graphs, $\{G_n\}$, with $v(G_n) \to \infty$, to converge. It will end up having to do with homomorphism densities and the cut metric, but perhaps the most intuitive way to define convergence is to look at what happens when we sample k nodes from graphs in the sequence. We say that the sequence of graphs converges if this behavior converges.

To make this precise, let F be a finite simple graph on k vertices. Then $t_{ind}(F, G_n)$ is the probability that, if k distinct vertices are chosen at random from G_n, the induced subgraph formed by them is isomorphic to F. For example, $t_{ind}(P_3, P_3) = 1$ since we must choose all 3 vertices of P_3 and those three vertices form P_3. Also, $t_{ind}(K_2, P_3) = \frac{2}{3}$ since there are three ways to choose two vertices from P_3 and two of these choices form K_2.

Definition 3.3. *A sequence of graphs, $\{G_n\}$, is convergent if $t_{ind}(F, G_n)$ converges for all finite simple F.*

Intuitively this means we define the sequence as convergent if, as n approaches infinity, we can barely tell the difference between the G_n by looking at samples of their vertices. Without proof, we state Lovász's result that this is the same as saying that the homomorphism densities converge.

Theorem 3.4. *[14, p. 173] A sequence $\{G_n\}$ of simple graphs with $v(G) \to \infty$ is convergent if and only if $t(F, G_n)$ is convergent for every finite graph F.*

We return to the sequence of half-graphs. We noted that $t(K_2, H_n)$ converges. However to show H_n is a convergent sequence of graphs, we need $t(F, H_n)$ to converge for all F. Calculating $t(P_3, W_{H_n})$ is significantly more complex than calculating $t(K_2, W_{H_n})$ using the method of the previous example. Calculating $t(F, W_{H_n})$ for arbitrary F is even more daunting. We therefore defer to other methods to prove convergence and introduce the cut distance.

When talking about convergent sequences, it is convenient to have a notion of distance. We use the cut distance, $\delta_\square$, because convergent sequences will be Cauchy under it. The reader is referred to Chapter 8 of [14] for the definition of the cut distance on graphs. It is not simple to say the least. We will skip straight to the analogous definition on kernels which is easier to work with. We can do this because $\delta_\square(H, H') = \delta_\square(W_H, W_{H'})$ [14, p. 132].

The cut norm on $\mathcal{W}$, the linear space of kernels, is defined as

$$||W||_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} W(x, y) dxdy \right|.$$

where the supremum is taken over all measurable subsets S and T. We know the sup exists because $S \times T$ and $W(x, y)$ are bounded.

The cut metric is defined as

$$d_\square(U, W) = \|U - W\|_\square .$$

We want to generalize this. Say graphs G and H are isomorphic but are labeled differently so that $W_G \neq W_H$. Then $d_\square(G, H) \neq 0$. But we would like the distance between graphons that represent isomorphic graphs to be 0. To fix this we introduce invertible measure preserving maps to essentially relabel the vertices. Let $S_{[0,1]}$ define the set of all invertible measure preserving maps $\phi : [0, 1] \to [0, 1]$. Let $W^\phi(x, y) = W(\phi(x), \phi(y))$. Then we define the cut distance as

$$\delta_\square(U, W) = \inf_{\phi \in S_{[0,1]}} d_\square(U, W^\phi).$$

Intuitively this says that the distance between W_G and W_H is the norm of their difference after the vertices of H have been relabeled to minimize the distance.

As promised, though without proof, we can equivalently define sequences of graphs to be convergent based on the cut distance and homomorphism densities.

Theorem 3.5. *A sequence $\{G_n\}$ of simple graphs with $v(G) \to \infty$ is convergent if and only if it is a Cauchy sequence in the metric $\delta_\square$ [14, p. 174].*

Furthermore, the next two theorems guarantee that a graphon is the limit to every convergent sequence of graphs.

Theorem 3.6. *For any convergent sequence $\{G_n\}$ of simple graphs there exists a graphon W such that $t(F, G_n) \to t(F, W)$ for every simple graph F. We say W is the limit of the graph sequence and write $G_n \to W$ [14, p. 180].*

Theorem 3.7. *For a sequence $\{G_n\}$ of graphs with $v(G_n) \to \infty$ and graphon W, we have $G_n \to W$ if and only if $\delta_\square(W_{G_n}, W) \to 0$ [14, p. 181].*

Note that the limit graphon to a sequence of graphs is not unique. If W is the limit graphon of a graph sequence and W' is a graphon such that $\delta_\square(W, W') = 0$ then W' is also the limit to the graph sequence. Because of this, we will sometimes find it convenient to refer to equivalence classes of graphons where two graphons, W and W', are in the same equivalence class if $\delta_\square(W, W') = 0$.

We now present several examples of convergent graph sequences and use different methods to show convergence based on which is easiest in a given case. Sometimes it is easier to look at the cut distance and sometimes calculating homomorphisms densities is easier.

Example 3.8. *We will show that the sequence of half-graphs, $\{H_n\}$, converges to W where*

$$W(x, y) := \begin{cases} 1 & y \geq x + \frac{1}{2} \text{ or } y \leq x - \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}\delta_\square(W_{H_n}, W) &= \inf_{\phi\in S_{[0,1]}} d_\square(W_{H_n}, W^\phi)\\ &= \inf_{\phi\in S_{[0,1]}} \sup_{S,T\subseteq[0,1]} \left|\int_{S\times T} W_{H_n}(x,y) - W(\phi(x),\phi(y))dxdy\right|\\ &\le \sup_{S,T\subseteq[0,1]} \left|\int_{S\times T} W_{H_n}(x,y) - W(x,y)dxdy\right|.\end{aligned}$$

Note that $W_{H_n} \ge W$*. Define the region* $A := \{(x,y) \in [0,1]^2 : W_{H_n}(x,y) > W(x,y)\}$*.* A *maximizes the integral*

$$\int_A W_{H_n}(x,y) - W(x,y)dxdy.$$

Therefore we have

$$\begin{aligned}\sup_{S,T\subseteq[0,1]} &\left|\int_{S\times T} W_{H_n}(x,y) - W(x,y)dxdy\right|\\ &\le \left|\int_A W_{H_n}(x,y) - W(x,y)dxdy\right|\\ &= \left|\int_A dxdy\right|\\ &= n\left(\frac{1}{2n}\right)^2\\ &= \frac{1}{4n}.\end{aligned}$$

Since $\delta_\square(W_{H_n}, W) = \frac{1}{4n} \to 0$ *as* $n \to \infty$*, the sequence of graphs* $\{H_n\}$ *converges to the graphon* W*.*

Example 3.9. *A simple threshold graph* G_n *is defined on the set of vertices* $\{1, 2, ..., n\}$*and* $E(G) := \{(i,j) : i + j \le n, i \ne j\}$*. Define the graphon*

$$W(x,y) := \begin{cases} 1 & x + y \le 1; \\ 0 & \text{otherwise.}\end{cases}$$

Figure 3.3 shows the graphons W_{G_3}*,* W_{G_4}*,* W_{G_5} *and* W*. We show that* $G_n \to W$*.*

$$\begin{aligned}\delta_\square(W_{G_n}, W) &\le \sup_{S,T\subseteq[0,1]} \left|\int_{S\times T} W_{G_n}(x,y) - W(x,y)dxdy\right|\\ &\le \sup_{S,T\subseteq[0,1]} \int_{S\times T} \left|W_{G_n}(x,y) - W(x,y)\right| dxdy.\end{aligned}$$

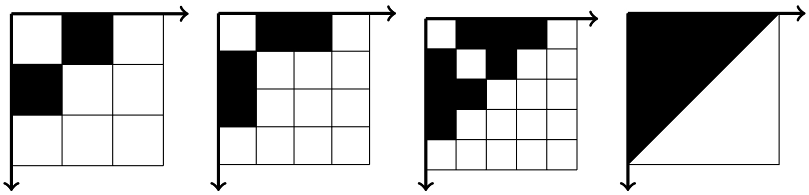


Figure 3.3: Threshold graphons W_{G_3}, W_{G_4}, W_{G_5} and W

Define A as the subset of $[0,1]^2$ where $W_{G_n}(x,y) \neq W(x,y)$. Then we have

$$\sup_{S,T\subseteq[0,1]} \int_{S\times T} |W_{G_n}(x,y) - W(x,y)|\, dxdy = \int_A dxdy$$
$$= \left(\frac{n}{2} + \lfloor\frac{n}{2}\rfloor\right)\frac{1}{n^2}$$
$$\leq \frac{1}{n}.$$

Therefore $G_n \to W \in \mathcal{W}$ since $\delta_\square(W_{G_n}, W) \to 0$.

The next two examples have to do with quasirandom graph sequences. These are sequences of graphs $\{G_n\}$ where $v(G_n)$ approaches infinity that share many properties with random graphs. Here we use the random graph model developed by Erdös and Rényi [3] and Gilbert [4]. Given a natural number n and a real number p where $0 \leq p \leq 1$, the random graph $\mathbb{G}(n,p)$ is generated by taking n nodes and connecting any two of them with probability p.

A quasirandom graph sequence has many properties that we would expect from a sequence of random graphs, $\mathbb{G}(n,p)$, for a given p as n approaches infinity. It turns out that there are many equivalent ways to describe quasirandom graph sequences. Here is one definition that we will find convenient.

Definition 3.10. *Let p be a real number between 0 and 1 and let $\{G_n\}$ be a sequence of graphs with increasing number of vertices. For simplicity, let us say that $v(G_n) = n$. Then $\{G_n\}$ is quasirandom with density p if for every fixed graph F, the number of homomorphisms of F into G_n is asymptotically $p^{e(F)}n^{v(F)}$.*

Note that $G_n = \mathbb{G}(n,p)$ defines a quasirandom graph sequence. In the next example we find the limit of a quasirandom graph sequence.

Example 3.11. *Let $\{G_n\}$ be a quasirandom graph sequence with $v(G_n) \to \infty$. Then there exists p in the interval $(0,1)$ such that*

$$\lim_{n\to\infty} t(F,G_n) = p^{e(F)}$$

for every fixed graph F [14, p. 9]. Define the graphon

$$W(x,y) := p.$$

We show that $G_n \to W$.

Let a simple graph $F = (V, E)$ be given. Then

$$
\begin{aligned}
t(F, W) &= \int_{[0,1]^v} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i \\
&= \int_{[0,1]^v} p^{e(F)} \prod_{i \in V} dx_i \\
&= p^{e(F)}.
\end{aligned}
$$

Therefore $t(F, G_n) \to t(F, W)$. By Theorem 3.6, this means that $G_n \to W$.

The idea of quasirandom graph sequences can be modified to bipartite quasirandom graph sequences and multipartite quasirandom graph sequences. For a bipartite graph G, we can think of it having an "upper" and "lower" bipartition class denoted $U(G)$ and $L(G)$ respectively. For two simple, unweighted bipartite graphs F and G let $\mathsf{hom}'(F, G)$ be the number of homomorphisms that map $U(F)$ to $U(G)$ and $L(F)$ to $L(G)$.

Definition 3.12. *A sequence of bipartite graphs, $\{G_n\}$, is bipartite quasirandom with density $p \in [0, 1]$ if for every simple bipartite graph F*

$$
\frac{\mathsf{hom}'(F, G_n)}{|U(G_n)|^{|U(F)|} |L(G_n)|^{|L(F)|}} \to p^{|E(F)|} \qquad (n \to \infty).
$$

Multipartite quasirandom graph sequences are defined on a weighted template graph. Let the template graph H have q vertices with weights $\alpha_i > 0$ where $\sum \alpha_i = 1$ and edge weights $\beta_{ij} \in [0, 1]$. Note that H can be viewed as the weighted complete graph K_q since some of the edge weights can be 0. Then we can define $\{G_n\}$ as a multipartite quasirandom graph sequence with template graph H where $v(G_n) = n$. This means that $V(G_n)$ can be partitioned into q sets $V_1, V_2, ..., V_q$ with $v(V_i) = \alpha_i v(G_n) + o(n)$. The Landau symbol $o(n)$ denotes a function $f(n)$ where $\frac{f(n)}{n}$ approaches 0 as n approaches infinity. For every $i \leq q$, $G_n[V_i]$ forms a quasirandom graph sequence. Also, for every $i \neq j$, $G_n[V_i, V_j]$ forms a quasirandom bipartite graph sequence with density β_{ij}.

Example 3.13. *Let $\{G_n\}$ be a multipartite quasirandom graph sequence with template graph H. Then $\{G_n\}$ converges to the graphon W_H. This is because $\delta_\square(W_{G_n}, W_H) \to 0$ and so Theorem 3.7 says that the sequence converges.*

This example sheds light on the connection between this theory of graph limits and Szemerédi's Regularity Lemma [23]. As stated in Chapter 1, this is a major result in graph theory about approximating large graphs with multipartite quasirandom graphs. It has many applications and was the reason Endre Szemerédi won the Abel prize in 2012.

From this example, we see that a multipartite quasirandom graph is essentially a step function. At least it can be made as close as we like by letting the number of vertices be large enough. The Regularity Lemma informally says that we can approximate any graph with a multipartite quasirandom graph. In the language of graphons, this means that we can approximate a measurable function (the graphon corresponding to the large graph) with a step function (a graphon close to the multipartite quasirandom approximation of the graph). So the Regularity Lemma is, in a sense, the graph equivalent of approximating measurable functions with step functions in analysis.

The connections between the Regularity Lemma and graphons run deeper than this. The Regularity Lemma can essentially be reduced to the statement that a certain space is compact [14, p. 149]. This is worthy of much more study but we will say no more about it.

Chapter 4

Sierpiński carpet graphons

In this chapter we examine, in detail, sequences of graphons that are derived from the Sierpiński carpet fractal. We define graphs associated with each iteration of the fractal and use the theorems and methods developed in Chapter 3 to find properties of the graphs. The graphons from the original Sierpiński carpet converge to the equivalence class containing the null graph on infinitely many vertices. To make the sequence more interesting we modify the original fractal and study the modified version.

4.1 The original Sierpiński carpet

Sierpiński's carpet is defined as follows: Divide a square into nine equal squares in a 3x3 grid and remove the middle square. Then apply the same process ad infinitum to the eight remaining squares. The fourth iteration is shown in Figure 4.1.

We define a graphon corresponding to each iteration of Sierpiński's carpet as follows. For $x, y \in [0,1]$ let $0.x_1x_2x_3...$ and $0.y_1y_2y_3...$ be their respective base 3 expansions. Then we define the graphon

$$S_k(x,y) := \begin{cases} 0 & 0.x_1x_2...x_k = 0.y_1y_2...y_k \text{ or } x_i = y_i = 1 \text{ for some } 1 \leq i \leq k; \\ 1 & \text{otherwise.} \end{cases}$$

Figure 4.2 illustrates S_4.

For each graphon S_k we can define a graph G_k such that $W_{G_k} = S_k$. G_k has vertex set

$$V(G_k) = \{1, 2, ..., 3^k\}$$

and edge set

$$E(G_k) = \{(i,j) \in V(G_k)^2 : S_k(\frac{i}{3^k}, \frac{j}{3^k}) = 1\}.$$

Note that the graphon $S_k(x,y)$ is defined to be 0 when $0.x_1x_2...x_k = 0.y_1y_2...y_k$ to prevent loops in the graph G_k.

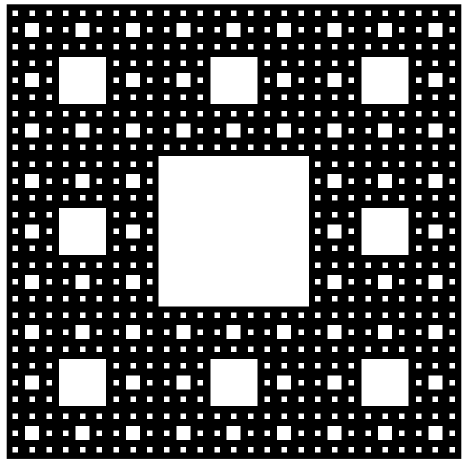


Figure 4.1: Sierpiński's carpet (four iterations)

Now to see if $\{G_k\}$ converges, we calculate the homomorphism density of K_2.

$$
\begin{aligned}
t(K_2, S_k) &= \int_{[0,1]^2} S_k(x_1, x_2) dx_1 dx_2 \\
&= \left(\frac{8}{9}\right)^k - \frac{1}{k}\left(\frac{2}{3}\right)^k
\end{aligned}
$$

Therefore

$$\lim_{k\to\infty} t(K_2, S_k) = 0.$$

Since K_2 is a subgraph of all nonempty graphs F, we have $t(K_2, S_k) \geq t(F, S_k)$ for every graph F by Proposition 2.13. Therefore

$$\lim_{i\to\infty} t(F, S_k) = 0.$$

Since $t(F, S_k)$ converges for all F, $t(F, W_{G_k})$ converges for all F and therefore $\{G_k\}$ converges.

In fact, the sequence converges to a multitude of different graphons. Recall that $\{G_k\}$ converges to a graphon W if $t(F, G_n)$ converges to $t(F, W)$ for every finite simple graph F. Since $t(F, G_n)$ always converges to zero, any graphon with measure zero can be taken as the limit graphon.

4.2 Generalized Sierpiński carpet

We now generalize this result. Let a and b be in the interval $(0, 1)$ such that $a + b < 1$. Partition $[0, 1]$ into three intervals, I_0, I_1, I_2, such that $I_0 := [0, a)$,

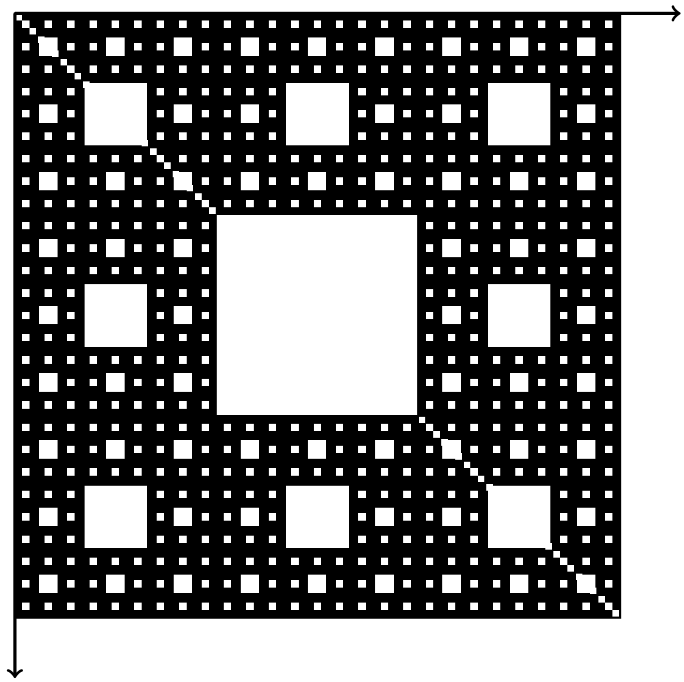


Figure 4.2: Sierpinski Carpet Graphon S_4

$I_1 := [a, a+b)$ and $I_2 := [a+b, 1]$. Define the graphon

$$W_1(x,y) := \begin{cases} 0 & x, y \in I_1; \\ 1 & \text{otherwise.} \end{cases}$$

Partition each I_i into three intervals, I_{i0}, I_{i1}, I_{i2}, where for $I_i = [m, n)$ define

$$\begin{aligned} I_{i0} &:= [m, m + a(n-m)) \\ I_{i1} &:= [m + a(n-m), m + (a+b)(n-m)) \\ I_{i2} &:= [m + (a+b)(n-m), n). \end{aligned}$$

For $I_i = [m, 1]$ define

$$\begin{aligned} I_{i0} &:= [m, m + a(1-m)) \\ I_{i1} &:= [m + a(1-m), m + (a+b)(1-m)) \\ I_{i2} &:= [m + (a+b)(1-m), 1]. \end{aligned}$$

Then for $x \in I_{x_1,x_2}$ and $y \in I_{y_1,y_2}$ define

$$W_2(x,y) := \begin{cases} 0 & x_i = y_i = 1 \text{ for any } i \leq 2; \\ 1 & \text{otherwise.} \end{cases}$$

Continue to define each iteration W_k for all positive integers k as follows. Partition each $I_{i_1,i_2,...,i_{k-1}} = [m, n)$ into three intervals in the same way. For instance $I_{i_1,i_2,...,i_{k-1},1} = [m, m + a(n - m))$. Then for $x \in I_{x_1,x_2,...,x_k}$ and $I_{y_1,y_2,...,y_k}$ define the graphon

$$W_k(x, y) := \begin{cases} 0 & x_i = y_i = 1 \text{ for any } i \leq k; \\ 1 & \text{otherwise.} \end{cases}$$

In Sierpiński's carpet the "middle ninth" of each remaining square is removed each iteration. In this generalization a rectangle is removed from each remaining rectangle. If the remaining rectangle has area A, then the rectangle to be removed has area b^2A. Therefore

$$\begin{aligned} t(K_2, W_k) &= \int_{[0,1]^2} W_k(x_1, x_2)dx_1dx_2 \\ &= (1 - b^2)^k. \end{aligned}$$

Taking the limit as k approaches infinity gives

$$\lim_{k\to\infty} t(K_2, W_k) = 0.$$

Therefore $\{W_k\}$ converges to the 0 graphon.

Note that this sequence of graphons cannot represent a sequence of graphs that is as simple as the sequence of graphs for Sierpiński's carpet. First, the graphs contain loops. Second, the uneven partitions of $[0, 1]$ do not easily divide up into equal intervals. One solution is to define the graphs on a weighted vertex set. For instance, let G_k have vertex set $V(G_k) = \{1, 2, ..., 3^k\}$. Define a bijective mapping $\phi : V(G_k) \to \{I_{i_1,i_2,...,i_k}\}$. Let vertex n have weight equal to the length of $\phi(n)$. Then $W_{G_k} = W_k$.

The presence of loops does not change the sequence very much. The limit of the homomorphism density of any finite simple graph F is identical for a sequence of graphs with loops and for the sequence of the same graphs with the loops removed.

Theorem 4.1. *Let $\{G_n\}$ be a sequence of finite simple graphs containing loops with increasing number of vertices. Define G'_n as the graph G_n without loops. Then for any finite simple graph F*

$$\lim_{n\to\infty} t(F, G_n) = \lim_{n\to\infty} t(F, G'_n)$$

Proof. Define the graphons $W_n := W_{G_n}$ and $W'_n := W_{G'_n}$. So we have $t(F, W_n) = t(F, G_n)$ and $t(F, W'_n) = t(F, G'_n)$. It is sufficient to show that

$$\lim_{n\to\infty} t(F, W_n) - t(F, W'_n) = 0.$$

Let $v_n := |V(G_n)| = |V(G'_n)|$ and let $v := |V(F)|$. Define the intervals $I_k := \left[\frac{k-1}{v_n}, \frac{k}{v_n}\right)$ for $k = 1, ..., v_n$. Recall that on $I_i \times I_j$

$$W_n = \begin{cases} 1 & (i, j) \in E(G_n); \\ 0 & \text{otherwise;} \end{cases} \qquad \text{and} \qquad W'_n = \begin{cases} 1 & (i, j) \in E(G'_n); \\ 0 & \text{otherwise.} \end{cases}$$

We use these intervals to break up the integral for $t(F,W_n) - t(F,W_n')$ into integrals of constants.

$$t(F,W_n) - t(F,W_n')$$

$$= \int_{[0,1]^v} \left[\prod_{ij\in E(F)} W_n(x_i,x_j) - \prod_{ij\in E(F)} W_n'(x_i,x_j) \right] \prod_{i\in V(F)} dx_i$$

$$= \sum_{J_1,..,J_v \in \{I_1,..,I_{v_n}\}} \int_{J_1} \cdots \int_{J_v} \left[\prod_{ij\in E(F)} W_n(x_i,x_j) - \prod_{ij\in E(F)} W_n'(x_i,x_j) \right] \prod_{i\in V(F)} dx_i$$

If $J_1, ..., J_v$ are all distinct then

$$\int_{J_1} \cdots \int_{J_v} \left[\prod_{ij\in E(F)} W_n(x_i,x_j) - \prod_{ij\in E(F)} W_n'(x_i,x_j) \right] \prod_{i\in V(F)} dx_i = 0.$$

There are $\frac{v_n!}{(v_n-v)!}$ permutations such that $J_1, ..., J_v$ are all distinct. So there are $v_n^v - \frac{v_n!}{(v_n-v)!}$ integrals left to consider.

Since G_n contains every edge in G_n', then $W_n(x_1,x_2) \geq W_n'(x_1,x_2) \geq 0$ for all x_1 and x_2 in $[0,1]$. Therefore for any $J_1, ..., J_v$

$$\int_{J_1} \cdots \int_{J_v} \left[\prod_{ij\in E(F)} W_n(x_i,x_j) - \prod_{ij\in E(F)} W_n'(x_i,x_j) \right] \prod_{i\in V(F)} dx_i$$

$$\leq \int_{J_1} \cdots \int_{J_v} \prod_{i\in V(F)} dx_i$$
$$= \left(\frac{1}{v_n}\right)^v.$$

This means that

$$\begin{aligned} t(F,W_n) - t(F,W_n') &\leq \left(v_n^v - \frac{v_n!}{(v_n-v)!}\right)\left(\frac{1}{v_n}\right)^v \\ &= 1 - \frac{v_n!}{(v_n-v)!v_n^v} \\ &= 1 - \left(\frac{v_n-1}{v_n}\right)\left(\frac{v_n-2}{v_n}\right)\cdots\left(\frac{v_n-v+1}{v_n}\right) \end{aligned}$$

For any fixed k,

$$\lim_{n\to\infty} \frac{n-k}{n} = 1.$$

So

$$\lim_{n\to\infty} t(F,W_n) - t(F,W_n') = \lim_{n\to\infty} 1 - \left(\frac{v_n - 1}{v_n}\right)\left(\frac{v_n - 2}{v_n}\right) ... \left(\frac{v_n - v + 1}{v_n}\right)$$
$$= 1 - 1$$
$$= 0.$$

□

In the next sections we will find it easier to work with graphons that correspond to graphs with loops. Since we are primarily interested in the limit graphon and the presence of loops does not change the homomorphism densities in the limit graphon, we can also consider the limit graphon as the limit of the sequence of graphs with the loops removed.

4.3 A modified Sierpiński carpet

In both the original fractal and the generalized version, the associated graphs converge to any graphon with measure zero. We now modify Sierpiński's carpet so that the limit graphon has non-zero edge density. For $x, y \in [0,1]$ let $0.x_1x_2x_3...$ and $0.y_1y_2y_3...$ be their respective base 3 expansions. Then we define the graphon

$$W_k(x,y) := \begin{cases} 0 & x_i = y_i = 1 \text{ and } x_j, y_j \neq 1 \ \forall j < i \leq k; \\ 1 & \text{otherwise.} \end{cases}$$

W_4 is shown in Figure 4.3.

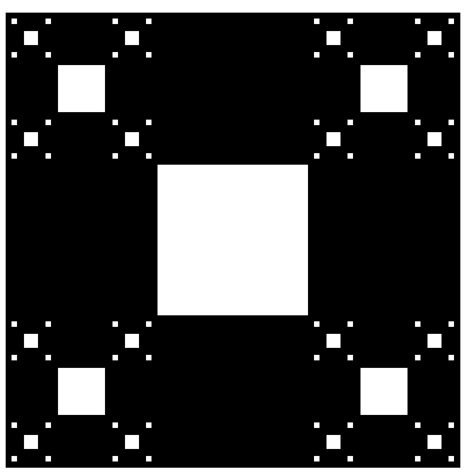


Figure 4.3: Modified Sierpiński's carpet (fourth iteration)

In this sequence of graphons the homomorphism density of K_2 does not converge to zero. Let S be the limit graphon as $k \to \infty$. Then notice that for $x_1, x_2 \in \left[0, \frac{1}{3}\right]$

$$S(x_1, x_2) = S(3x_1, 3x_2).$$

By the symmetry of Sierpiński's carpet we have

$$\begin{aligned}
&\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} S(x_1, x_2) dx_1 dx_2 \\
&= \int_0^{\frac{1}{3}} \int_{\frac{2}{3}}^1 S(x_1, x_2) dx_1 dx_2 \\
&= \int_{\frac{2}{3}}^1 \int_0^{\frac{1}{3}} S(x_1, x_2) dx_1 dx_2 \\
&= \int_{\frac{2}{3}}^1 \int_{\frac{2}{3}}^1 S(x_1, x_2) dx_1 dx_2.
\end{aligned}$$

Also note that

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} S(x_1, x_2) dx_1 dx_2 = 0$$

and that

$$\begin{aligned}
&\int_{\frac{1}{3}}^{\frac{2}{3}} \int_0^{\frac{1}{3}} S(x_1, x_2) dx_1 dx_2 \\
&= \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{2}{3}}^1 S(x_1, x_2) dx_1 dx_2 \\
&= \int_{\frac{2}{3}}^1 \int_{\frac{1}{3}}^{\frac{2}{3}} S(x_1, x_2) dx_1 dx_2 \\
&= \int_0^{\frac{1}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} S(x_1, x_2) dx_1 dx_2 = \frac{1}{9}.
\end{aligned}$$

Therefore

$$\begin{aligned}
t(K_2, S) &= \int_{[0,1]^2} S(x_1, x_2) dx_1 dx_2 \\
&= 4 \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} S(x_1, x_2) dx_1 dx_2 + \frac{4}{9}.
\end{aligned}$$

We use the change of variables $y_1 = 3x_1$ and $y_2 = 3x_2$ to get

$$\begin{aligned}
\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}} S(x_1,x_2)dx_1dx_2 &= \int_0^1\int_0^1 S(y_1,y_2)\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} dy_1dy_2 \\
&= \int_0^1\int_0^1 S(y_1,y_2)\begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{vmatrix} dy_1dy_2 \\
&= \frac{1}{9}\int_0^1\int_0^1 S(y_1,y_2)dy_1dy_2 \\
&= \frac{1}{9}t(K_2,S).
\end{aligned}$$

Therefore

$$\begin{aligned}
t(K_2,S) &= \frac{4}{9}t(K_2,S) + \frac{4}{9}; \\
t(K_2,S) &= \frac{4}{5}.
\end{aligned}$$

Since the homomorphism density of K_2 is non-zero, the homomorphism density of other graphs can also be non-zero. We now calculate $t(P_3,S)$. First define the intervals

$$I_0 := \left[0,\frac{1}{3}\right], \qquad I_1 := \left[\frac{1}{3},\frac{2}{3}\right] \text{ and } I_2 := \left[\frac{2}{3},1\right].$$

Also define, for $A,B,C \in \{I_0,I_1,I_2\}$,

$$f(A,B,C) := \int_A\int_B\int_C S(x_1,x_2)S(x_2,x_3)dx_1dx_2dx_3.$$

Then

$$t(P_3,S) = \sum_{A,B,C\in\{I_0,I_1,I_2\}} f(A,B,C).$$

The values of $f(A,B,C)$ for $A,B,C \in \{I_0,I_1,I_2\}$ are calculated in Table 4.1. Because S is highly symmetric, many permutations of I_0, I_1 and I_2 map to the same value of $f(A,B,C)$. Table 4.1 partitions all permutations of I_0, I_1 and I_2 into five groups. Each group has n permutations and each of those permutations gives the same value for $f(A,B,C)$.

Therefore

$$t(P_3,S) = \frac{6}{27} + \frac{32}{135} + \frac{8}{27}t(P_3,S)$$

and

$$t(P_3,S) = \frac{62}{135}\frac{27}{19} = \frac{62}{95}.$$

Permutation		n	$f(A,B,C)$
$B = I_1$	$A = I_1$ or $C = I_1$	3	0
	$A \neq I_1$ and $C \neq I_1$	4	$\frac{1}{27}$
$B \neq I_1$	$A = C = I_1$	2	$\frac{1}{27}$
	$A = I_1$ xor $C = I_1$	8	$\frac{1}{27}t(K_2, S) = \frac{4}{135}$
	$A \neq I_1$ and $C \neq I_1$	8	$\frac{1}{27}t(P_3, S)$

Table 4.1: Calculating $f(A, B, C)$ for $A, B, C \in \{I_0, I_1, I_2\}$.

4.4 Another Sierpiński modification (S^β)

Here is yet another modification of Sierpiński's carpet. Instead of the entire middle ninth being removed, just a fraction of the middle ninth is removed. Let $\beta \in \mathbb{Q} \cap (0, \frac{1}{3})$ be given. Then we define the modified carpet, S^β, by iteratively removing $\left[\frac{1}{2} - \frac{\beta}{2}, \frac{1}{2} + \frac{\beta}{2}\right] \times \left[\frac{1}{2} - \frac{\beta}{2}, \frac{1}{2} + \frac{\beta}{2}\right]$ from the middle square. Call the n^{th} iteration S_n^β.

To make this precise let $0.x_1x_2x_3...$ and $0.y_1y_2y_3...$ be the base three expansions of x and y respectively. Then $S_n^\beta(x, y) := 0$ if for some $1 \leq i \leq n$ these three conditions hold:

(i) $x_i = y_i = 1$,

(ii) $\left|0.x_1x_2...x_i + \frac{1}{2\cdot 3^i} - x\right| \leq \frac{\beta}{2\cdot 3^{i-1}}$ and

(iii) $\left|0.y_1y_2...y_i + \frac{1}{2\cdot 3^i} - y\right| \leq \frac{\beta}{2\cdot 3^{i-1}}$.

Otherwise $S_n^\beta(x, y) := 1$.

We also define a corresponding sequence of graphs, $\{G_n\}$, that converge to S^β. Let $\beta = \frac{p}{q}$. We define G_n on vertex set $V(G_n) = \{1, 2, ..., 3^n q\}$ and with edge set $E(G_n) = \left\{\{i, j\} \in V(G_n)^2 : S_n^\beta(\frac{i}{3^n q}, \frac{j}{3^n q}) = 1\right\}$. Notice that since S_n^β is constant on the intervals $\left[\frac{i-1}{3^n q}, \frac{i}{3^n q}\right] \times \left[\frac{j-1}{3^n q}, \frac{j}{3^n q}\right]$ for all $1 \leq i, j \leq 3^n q$, we have that $S_n^\beta = W_{G_n}$.

For this modification we find $t(K_2, S^\beta)$ and estimate $t(P_3, S^\beta)$. Note that

$$t(K_2, S^\beta) = \int_{[0,1]^2} S^\beta(x_1, x_2) dx_1 dx_2$$
$$= \frac{1}{9} - \beta^2 + \frac{8}{9} \cdot t(K_2, S^\beta).$$

Therefore

$$t(K_2, S^\beta) = 1 - 9\beta^2. \tag{4.1}$$

Under this construction, we have control over the edge density. By varying β between 0 and $\frac{1}{3}$ we can vary the edge density between 0 and 1.

Now we find $t(P_3, S^\beta)$. Define $I_0 := [0,1/3], I_1 := [1/3,2/3], I_2 := [2/3,1]$ as we did above.

$$\begin{aligned}
t(P_3, S^\beta) &= \int_0^1 \int_0^1 \int_0^1 S^\beta(x_1,x_2)S^\beta(x_2,x_3)dx_1dx_2dx_3 \\
&= \sum_{i,j,k\in\{1,2,3\}} \int_{I_i}\int_{I_j}\int_{I_k} S^\beta(x_1,x_2)S^\beta(x_2,x_3)dx_1dx_2dx_3.
\end{aligned}$$

On each square $I_i \times I_j$, except for $I_i = I_j = I_1$, we have that S^β is just a rescaling of S^β on $[0,1]^3$ i.e. $S^\beta(x,y) = S^\beta(3(x-\frac{i}{3}), 3(y-\frac{j}{3}))$. Therefore as long as no more than one I_i is equal to I_1 we have

$$\begin{aligned}
&\int_{I_i}\int_{I_j}\int_{I_k} S^\beta(x_1,x_2)S^\beta(x_2,x_3)dx_1dx_2dx_3 \\
=&\frac{1}{27}\int_0^1\int_0^1\int_0^1 S^\beta(x_1,x_2)S^\beta(x_2,x_3)dx_1dx_2dx_3. \\
=&\frac{1}{27}t(P_3,S^\beta).
\end{aligned}$$

There are 22 possibilities for i,j,k that do not include $I_i = I_j = I_1$ or $I_j = I_k = I_1$. For $I_j \neq I_1$ there are two possibilities for I_j and three possibilities for both I_i and I_k. For $I_j = I_1$ there are two possibilities for both I_i and I_k. Therefore there are $2\cdot 3^2 + 2^2 = 22$ such possibilities.

For the case $I_i = I_j = I_k = I_1$, we split I_1 in two. We set $B := [\frac{1}{3}, \frac{1}{2} - \frac{\beta}{2}] \cup [\frac{1}{2} + \frac{\beta}{2}, \frac{2}{3}]$ and set $C := [\frac{1}{2} - \frac{\beta}{2}, \frac{1}{2} + \frac{\beta}{2}]$ and let B and C have lengths b and c respectively. Then we have

$$\begin{aligned}
&\int_{I_1}\int_{I_1}\int_{I_1} S^\beta(x_1,x_2)S^\beta(x_2,x_3)dx_1dx_2dx_3 \\
=&\int_B\int_B\int_B + \int_B\int_B\int_C + \int_B\int_C\int_B + \int_C\int_B\int_B + \int_C\int_B\int_C \\
=&\frac{1}{(\frac{1}{3}-\beta)^3} + \frac{3}{(\frac{1}{3}-\beta)^2\beta} + \frac{1}{(\frac{1}{3}-\beta)\beta^2} \\
=&\frac{-27\beta^2+9\beta+3}{\beta^2(1-3\beta)^3}.
\end{aligned}$$

The last four integrals have the same value so we only calculate one of them.

Take

$$\int_{I_0}\int_{I_1}\int_{I_1} S^\beta(x_1,x_2)S^\beta(x_2,x_3)dx_1dx_2dx_3$$
$$=\int_{I_0}\int_B\int_B + \int_{I_0}\int_B\int_C + \int_{I_0}\int_C\int_B + \int_{I_0}\int_C\int_C$$
$$=b\int_{I_0}\int_B S^\beta(x_2,x_3)dx_2dx_3 + c\int_{I_0}\int_B S^\beta(x_2,x_3)dx_2dx_3 + b\int_{I_0}\int_C S^\beta(x_2,x_3)dx_2dx_3$$
$$=b\int_{I_0}\int_{I_1} S^\beta(x_2,x_3)dx_2dx_3 + c\int_{I_0}\int_B S^\beta(x_2,x_3)dx_2dx_3$$

The first of these two integrals can be calculated as follows

$$\begin{aligned}\int_{I_0}\int_{I_1} S^\beta(x_2,x_3)dx_2dx_3 &= \frac{1}{9}t(K_2,S^\beta)\\ &= \frac{1}{9}-\beta^2.\end{aligned}$$

However we can only find upper and lower bounds for the second integral. Since $0 \le S^\beta(x_2,x_3) \le 1$, we have

$$0 \le \int_{I_0}\int_B S^\beta(x_2,x_3)dx_2dx_3 \le \frac{1}{3}(\frac{1}{3}-\beta).$$

Putting everything together we have

$$t(P_3,S^\beta) = \frac{22}{27}t(P_3,S^\beta) + \frac{-27\beta^2+9\beta+3}{\beta^2(1-3\beta)^3} + \frac{1}{9} - \beta^2 + \int_{I_0}\int_B S^\beta(x_2,x_3)dx_2dx_3$$

Which gives us

$$t(P_3,S^\beta) \ge \frac{27}{5}\left(\frac{-27\beta^2+9\beta+3}{\beta^2(1-3\beta)^3} + \frac{1}{9} - \beta^2\right); \tag{4.2}$$
$$t(P_3,S^\beta) \le \frac{27}{5}\left(\frac{-27\beta^2+9\beta+3}{\beta^2(1-3\beta)^3} + \frac{1}{9} - \beta^2 + \frac{\beta}{3}(\frac{1}{3}-\beta)\right). \tag{4.3}$$

4.5 Fixed $\beta = \frac{1}{9}$ in S^β graphons

Letting β vary between 0 and $\frac{1}{3}$ clearly introduces complications since even calculating $t(P_3,S^\beta)$ is nontrivial. Setting β equal to $\frac{1}{9}$ makes the series of graphons much more manageable. In this section we study this sequence of graphons and the graphs they represent.

The graphons $S_n^{\frac{1}{9}}$ are defined as before on page 46. Substituting $\frac{1}{9}$ for β gives the following definition.

Definition 4.2. *Let $0.x_1x_2x_3...$ and $0.y_1y_2y_3...$ be the base three expansions of x and y respectively. Then $S_n^{\frac{1}{9}}(x,y) = 0$ if for some $1 \le i \le n$ these three conditions hold:*

(i) $x_i = y_i = 1$,

(ii) $\left|0.x_1x_2...x_i + \frac{1}{2\cdot 3^i} - x\right| \leq \frac{1}{2\cdot 3^{i+1}}$ *and*

(iii) $\left|0.y_1y_2...y_i + \frac{1}{2\cdot 3^i} - y\right| \leq \frac{1}{2\cdot 3^{i+1}}$.

Otherwise $S_n^{\frac{1}{9}}(x,y) = 1$.

With fixed $\beta = \frac{1}{9}$ we can also easily define a corresponding sequence of graphs $\{G_n\}$. By corresponding we mean that $S_n^{\frac{1}{9}} = W_{G_n}$.

Definition 4.3. *For each* $S_n^{\frac{1}{9}}$ *the graph* G_n *is defined to have the following vertex and edge sets.*

$$V(G_n) := \{0, 1, 2, ..., 3^{n+1} - 1\}.$$

$$E(G_n) := \{(i,j) \in V(G_n)^2 : S_n^{\frac{1}{9}}\left(\frac{i}{3^{n+1}}, \frac{j}{3^{n+1}}\right) = 1\}.$$

The first three graphons in this sequence are shown in Figure 4.4.

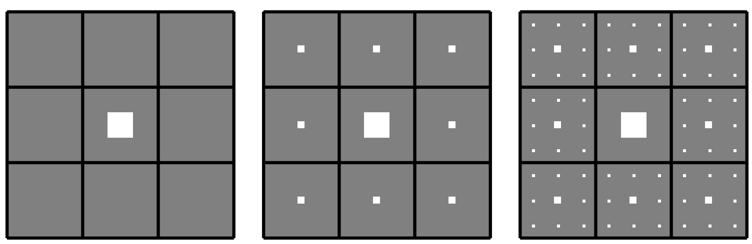


Figure 4.4: Graphons $S_1^{\frac{1}{9}}$, $S_2^{\frac{1}{9}}$ and $S_3^{\frac{1}{9}}$

The first simplification we see from fixing $\beta = \frac{1}{9}$ is the definition of $S_n^{\frac{1}{9}}$.

Proposition 4.4.

$$S_n^{\frac{1}{9}}(x,y) = \begin{cases} 0 & x_i = x_{i+1} = y_i = y_{i+1} = 1 \textit{ for some } 1 \leq i \leq n; \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Denote the two different graphon definitions by $\bar{S}_n^{\frac{1}{9}}(x,y)$ and $\hat{S}_n^{\frac{1}{9}}(x,y)$. Then $\bar{S}_n^{\frac{1}{9}}(x,y) := 0$ if for some $1 \leq i \leq n$ these three conditions hold:

(i) $x_i = y_i = 1$,

(ii) $\left|0.x_1x_2...x_i + \frac{1}{2\cdot 3^i} - x\right| \leq \frac{1}{2\cdot 3^{i+1}}$ and

(iii) $\left|0.y_1y_2...y_i + \frac{1}{2\cdot 3^i} - y\right| \leq \frac{1}{2\cdot 3^{i+1}}$.

Otherwise $\bar{S}_n^{\frac{1}{9}}(x,y) := 1$. And define

$$\hat{S}_n^{\frac{1}{9}}(x,y) := \begin{cases} 0 & x_i = x_{i+1} = y_i = y_{i+1} = 1 \text{ for some } 1 \leq i \leq n; \\ 1 & \text{otherwise.} \end{cases}$$

Let $x, y \in [0,1]$ be given with base three expansions $0.x_1x_2x_3...$ and $0.y_1y_2y_3...$ respectively. We want to show that $\hat{S}_n^{\frac{1}{9}}(x,y) = \bar{S}_n^{\frac{1}{9}}(x,y)$.

Assume $\hat{S}_n^{\frac{1}{9}}(x,y) = 0$. Then $x_i = x_{i+1} = y_i = y_{i+1} = 1$ for some $1 \leq i \leq n$. So condition (i) is given. Condition (ii) can be proven as follows.

$$\begin{aligned} |0.x_1x_2...x_i + \frac{1}{2 \cdot 3^i} - x| &= |0.\underbrace{0...0}_{i \text{ 0's}}1\bar{1} - 0.\underbrace{0...0}_{i \text{ 0's}}x_{i+1}x_{i+2}x_{i+3}...| \\ &= |0.\underbrace{0...0}_{i \text{ 0's}}1\bar{1} - 0.\underbrace{0...0}_{i \text{ 0's}}1x_{i+2}x_{i+3}...| \\ &\leq 0.\underbrace{0...0}_{i+1 \text{ 0's}}1\bar{1} \\ &= \frac{1}{2 \cdot 3^{i+1}} \end{aligned}$$

Condition (iii) can be proven identically.

Now assume $\bar{S}_n^{\frac{1}{9}}(x,y) = 0$. Then there is an $1 \leq i \leq n$ such that

(i) $x_i = y_i = 1$,

(ii) $\left|0.x_1x_2...x_i + \frac{1}{2 \cdot 3^i} - x\right| \leq \frac{1}{2 \cdot 3^{i+1}}$ and

(iii) $\left|0.y_1y_2...y_i + \frac{1}{2 \cdot 3^i} - y\right| \leq \frac{1}{2 \cdot 3^{i+1}}$.

Condition (ii) means that

$$|0.\underbrace{0...0}_{i \text{ 0's}}1\bar{1} - 0.\underbrace{0...0}_{i \text{ 0's}}x_{i+1}x_{i+2}...| \leq 0.\underbrace{0...0}_{i \text{ 0's}}01\bar{1}$$

$$0.\underbrace{0...0}_{i \text{ 0's}}1\bar{1} - 0.\underbrace{0...0}_{i \text{ 0's}}01\bar{1} \leq 0.\underbrace{0...0}_{i \text{ 0's}}x_{i+1}x_{i+2}... \leq 0.\underbrace{0...0}_{i \text{ 0's}}1\bar{1} + 0.\underbrace{0...0}_{i \text{ 0's}}01\bar{1}$$

$$0.\underbrace{0...0}_{i \text{ 0's}}1 \leq 0.\underbrace{0...0}_{i \text{ 0's}}x_{i+1}x_{i+2}... \leq 0.\underbrace{0...0}_{i \text{ 0's}}1\bar{2}$$

Therefore $x_{i+1} = 1$. Similarly, condition (iii) means that $y_{i+1} = 1$. And so $\hat{S}_n^{\frac{1}{9}}(x,y) = 0$.

Since $\hat{S}_n^{\frac{1}{9}}(x,y) = 0$ if and only if $\bar{S}_n^{\frac{1}{9}}(x,y) = 0$, then $\hat{S}_n^{\frac{1}{9}}(x,y) = 1$ if and only if $\bar{S}_n^{\frac{1}{9}}(x,y) = 1$. So $\hat{S}_n^{\frac{1}{9}}(x,y) = \bar{S}_n^{\frac{1}{9}}(x,y)$. □

Setting $\beta = \frac{1}{9}$ also makes calculating homomorphism densities easier or even possible. The homomorphism density of K_2 in $S^{\frac{1}{9}}$ can be easily calculated using equation (4.1),

$$t(K_2, S^{\frac{1}{9}}) = 1 - 9\beta^2 = \frac{8}{9}.$$

Calculating $t(P_3, S^{\frac{1}{9}})$ is involved but not particularly difficult. Recall that the intervals I_0, I_1 and I_2 are defined to be $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ respectively. They can be used to split up the integral into twenty-seven pieces:

$$\begin{aligned} t(P_3, S^{\frac{1}{9}}) &= \int_0^1 \int_0^1 \int_0^1 S^{\frac{1}{9}}(x_1, x_2) S^{\frac{1}{9}}(x_2, x_3) dx_1 dx_2 dx_3 \\ &= \sum_{i,j,k \in \{0,1,2\}} \int_{I_i} \int_{I_j} \int_{I_k} S^{\frac{1}{9}}(x_1, x_2) S^{\frac{1}{9}}(x_2, x_3) dx_1 dx_2 dx_3 \\ &= \sum_{i,j,k \in \{0,1,2\}} f(i,j,k) \end{aligned}$$

where

$$f(i,j,k) := \int_{I_i} \int_{I_j} \int_{I_k} S^{\frac{1}{9}}(x_1, x_2) S^{\frac{1}{9}}(x_2, x_3) dx_1 dx_2 dx_3.$$

These integrals can be partitioned into three sets where integrals in the same set have the same value. Table 4.2 shows the partition, the number of integrals in each set (n) and the value of the integrals in each set.

Set	n	$f(i,j,k)$
$i = j = k = 1$	1	$\frac{22}{9^3}$
$j = 1$ and $(i = 1$ xor $k = 1)$	4	$\frac{64}{3 \cdot 9^3}$
All other integrals	22	$\frac{1}{27} t(P_3, S^{\frac{1}{9}})$

Table 4.2: Partition of integrals to calculate $t(P_3, S^{\frac{1}{9}})$

Putting them all together gives

$$t(P_3, S^{\frac{1}{9}}) = \frac{22}{9^3} + \frac{256}{3 \cdot 9^3} + \frac{22}{27} t(P_3, S^{\frac{1}{9}}).$$

So

$$t(P_3, S^{\frac{1}{9}}) = \frac{322}{405}.$$

Chapter 5

Conclusions and ideas for further study

The theory of graph limits is an exciting new development in graph theory with connections to classical graph theory results. In this project I looked into aspects of the theory and specifically studied homomorphism densities and graphons.

I applied the theory to graphons derived from the Sierpiński carpet fractal. I created several sequences of graphs and graphons derived from the Sierpiński carpet fractal and used the tools developed in this theory to study them. I calculated the homomorphism densities of small graphs into the sequences, but much more could be done to study them. More homomorphism densities could be calculated and other questions could be answered: Are the graphs Hamiltonian? Do they have a perfect triangle matching?

Homomorphism densities helped determine when sequences of graphs converged. However I also studied them apart from sequences of graphs. I proved several properties of homomorphism densities and several inequalities between them. As with the Sierpiński carpet graphons there are many questions left to be asked and answered: How do homomorphism densities in G compare to those in G complement? Is there a sequence of graphs where $t(K_{1,n}, G)$ and $t(P_{n+1}, G)$ diverge? Where is $t(P_k, G) \geq t(K_2^{k-1}, G)$ a tight bound and where is it far off? What are the K_n densities in a quasirandom graph sequence?

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