Inference in Constrained Linear Regression

by

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Abstract

Regression analyses constitutes an important part of the statistical inference and has great applications in many areas. In some applications, we strongly believe that the regression function changes monotonically with some or all of the predictor variables in a region of interest. Deriving analyses under such constraints will be an enormous task. In this work, the restricted prediction interval for the mean of the regression function is constructed when two predictors are present. I use a modified likelihood ratio test (LRT) to construct prediction intervals.

Keywords: Least favorable distribution, Restricted prediction interval, Chi-bar-square distribution, Likelihood ratio test

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Chapter 1

Introduction

Techniques of statistical inference under order restrictions have been used in many applications. Regression analysis constitutes a large part of them. In many applications, experimenters believe that the regression function varies monotonically with the predictor variables in some region of interest. Usually the restricted regression analysis will consider the null hypothesis of the type $R\beta = r$ versus $R\beta \geqslant r$, $R\beta \neq r$, for some matrix R, vector r when $X \sim N(\theta, V)$ and V is arbitrary. I do not have to consider the case for a general V separately in this thesis because the inference problem can be restated in terms of the identity covariance matrix. (Silvapulle and Sen (2004))

And linear regression analysis is simple and efficient. Techniques of linear regression have been used for many areas and for a long time. However constrained regression analysis is more suitable and reasonable for the reality. If we use general linear regression inference techniques, we would not be able to get the benefit of our assumptions. In the fields such as economics and aerospace, constrained linear regression analysis might give more precise predictions which are important than regular one. Mukerjee and Tu (1995) already discussed constrained simple linear regression on a single variable. Commonly higher dimensional constrained inference is needed which is more practical. Peiris and Bhattacharya (2016) has developed the techniques for point and interval estimators for model parameters and the mean response for two predictor variables model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$ with sign constraints on slope parameters β_1 and β_2. Confidence intervals reflect the goodness of fitting and prediction intervals tell us that under certain probability the future observations will fall into the estimated intervals. In this thesis, I develop the formulas for the prediction intervals for the two predictor variables model with such constraints in slope parameters.

1.1 General Linear Regression

Regression analysis is a statistical process to estimate the relation between two or more variables. Regression analysis techniques are often used to help understanding how the response variables change under the variation of the predictor variables. Regression analysis has three main purposes: 1.describle the relation between two or more variables, 2.use predictor variables to control response variables, and 3.use statistical relation to make predictions. Further, they are widely used to make forecasting in many areas, like biology, business, and data science. A few examples of applications are:

1. The length of patient stay in a hospital in days can be predicted by utilizing the relationship between patient's age in years and the time in the hospital.

2. The patient's blood pressure can be predicted by utilizing the relationship between the blood pressure and body weight.

1.1.1 Model and Assumptions

I consider a general regression model where there are several predictor variables and the regression function is linear. The model can be stated as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_p X_{pi} + \epsilon_i, \qquad i = 1, 2, ..., n, \tag{1.1}$$

where Y_i is the repsonse for the i^{th} trial, β_0, β_1, $\beta_2 ... \beta_p$ are parameters, X_{1i}, X_{2i},...X_{pi} are the values of predictor variables in i^{th} trial, ϵ_i is a random error term and $\epsilon_i \sim N_n(0, \sigma^2)$ for all $i = 1, 2, ..., n$

1.1.2 Maximum Likelihood Estimator

In model (1.1), let p=2, then $\epsilon = Y_i - (\beta_0 + \beta_1 X_{1i} + \beta_1 X_{2i}) \sim N(0, \sigma^2)$,

$$f_{Y_i} = \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{1}{2}\left(\frac{Y_i - (\beta_0 + \beta_1 X_{1i} + \beta_1 X_{2i})}{\sigma}\right)^2\right].$$

Then likelihood function is

$$\begin{aligned} L(\beta_0, \beta_1, \beta_2, \sigma^2) =& \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left[-\frac{1}{2\sigma^2}(Y_i - \beta_0 - \beta_1 X_{1i} - \beta_1 X_{2i})^2\right] \\ =& \frac{1}{(2\pi\sigma^2)^{n/2}} exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_{1i} - \beta_1 X_{2i})^2\right] \end{aligned}.$$

To obtain the values of $\hat{\beta_0}$, $\hat{\beta_1}$, $\hat{\beta_2}$, and $\hat{\sigma^2}$ that maximize the likelihood functon $L(\beta_0, \beta_1, \beta_2, \sigma^2)$, we let corresponding first derivatives equal zero and solve those simultaneous equations for MLEs. Then the MLEs of β_0, β_1, and β_2 also can be obtained in matrix form as

$$\hat{\beta} = (X'X)^{-1}X'Y,$$

where $Y_{n\times 1} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$,and $X_{n\times 3} = \begin{pmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} \end{pmatrix}$.

1.1.3 Confidence and Prediction Intervals

Confidence Interval

One of linear regression analysis objects is to estimate the mean response $E(Y)$. Consider a study of the relationship between patient's blood pressure (Y) and body weight (X). The mean blood pressure at high and medium levels of body weight may be one of the purposes of analyzing the effect of overweight.

Let X_h as the level of X for which we wish to estimate the mean repsonse $E(Y_h)$. Then the point estimator of $E(Y_h)$ is

$$\hat{Y_h} = X_h\hat{\beta},$$

and $\hat{Y_h}$ follows normal distribution with mean $E(\hat{Y_h}) = E(Y_h)$ and variance $\sigma^2(\hat{Y_h}) = \sigma^2(\frac{1}{n} + \frac{(X_h-\bar{X})^2}{\sum(X_{i\cdot}-\bar{X})^2})$. When σ is known,

$$\frac{\hat{Y_h} - E(Y_h)}{\sigma(\hat{Y_h})} \sim N(0,1),$$

where N(0,1) is the standard normal distribution. Therefore $100(1-\alpha)\%$ condifence interval can be known as,

$$\hat{Y_h} \pm Z(1-\alpha/2)\sigma(\hat{Y_h}),$$

where $Z(1-\alpha/2)$ is the $(1-\alpha/2)$ 100 percentile of standard normal distribution.

Usually σ is unknown, we replace $\sigma^2(\hat{Y_h})$ with the estimated variance $s^2(\hat{Y_h}) =$

$MSE(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_{i\cdot} - \bar{X})^2})$, and we have

$$\frac{\hat{Y}_h - E(Y_h)}{s(\hat{Y}_h)} \sim t_v,$$

where t_ν denotes the t-distribution with ν degree s of freedom. Therefore $100(1-\alpha)\%$ condifence interval is

$$\hat{Y}_h \pm t_{(1-\alpha/2;v)} s(\hat{Y}_h),$$

where $t_{(1-\alpha/2;v)}$ is the $100(1-\alpha)$percentile of t distribution with v degrees of freedom.

Prediction Interval

Now we consider the prediction of a new observation Y corresponding to a given level X of the predictor variable. When σ is known,

$$\frac{\hat{Y}_{h(new)} - \hat{Y}_h}{\sigma\{\text{pred}\}} \sim N(0, 1),$$

where $\sigma^2\{\text{pred}\} = \sigma^2 + \sigma^2\{\hat{Y}_h\} = \sigma^2(1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_{i\cdot} - \bar{X})^2})$. So prediction interval can be obtained as

$$\hat{Y}_h \pm Z(1 - \alpha/2)\sigma\{\text{pred}\},$$

When σ is unknown,

$$\frac{\hat{Y}_{h(new)} - \hat{Y}_h}{s\{\text{pred}\}} \sim t_v,$$

where $s^2\{\text{pred}\} = MSE + s^2\{\hat{Y}_h\} = MSE(1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_{i\cdot} - \bar{X})^2})$. So prediction interval can be obtained as

$$\hat{Y}_h \pm t(1 - \alpha/2; v)s\{\text{pred}\}$$

1.2 Constrained Statistical Inference

Statistical inference has been used in many fields. The needs of developing for modeling and analysis of observational or experimental data in constrained enviroments are growing. In many applications, it is reasonable to assume that there are some constraints in our statistical models which means we have more information about our model parameter space. So the models will become more efficient than those wherein constraints are ignored if we properly incorporate those information.

1.2.1 The Basics

First we consider the observations X $\overset{iid}{\sim} N(\theta, V)$. In order-restricted regression analysis, it is more common to consider inference under null hyphothesis of type $R\beta = r$ versus $R\beta \geqslant r$, $R\beta \neq r$, for some matrix R, vector r when $X \overset{iid}{\sim} N(\theta, V)$. We should consider, i V is a known positive definite matrix, ii $V = \sigma^2 U$ where U is a known positive definite matrix and σ is unknown, and iii V is unknown. (Silvapulle and Sen (2004))

The following are some common terms in restricted inference,

Convex: A set $A \subset \mathbb{R}^P$ is said to be convex if and only if $\{\lambda x + (1-\lambda)y\} \in A$ where $x, y \in A$ and $0 < \lambda < 1$.

Cone: A set $A \subset \mathbb{R}^P$ is said to be a cone with vector x_0 if and only if $x_0 + k(x - x_0) \in A$ for every $x \in A$ and $k \geqslant 0$. Further if the vertex x_0 is the origin O, then A is a cone simply.

Fenchel Dual (or negative dual) Cone: $C^0 = \{\alpha : \alpha^T\theta \leqslant 0 \text{ for every } \theta \in C\}$ is called the dual cone of C with respect to the inner product. It can be shown that the boundaries of C^0 are the perpendiculars to the boundaries of C.

Maximum likelihood estimation:

If $X = (X_1, X_2)' \sim N(\theta, I)$, where I is the 2×2 identity matrix and $\theta = (\theta_1, \theta_2)'$. Then for a single observation X, the *kernel* $l(\theta)$ of the loglikelihood is given by

$$-2l(\theta) = \{(X_1 - \theta_1)^2 + (X_2 - \theta_2)^2\} = ||X - \theta||^2,$$

So in my work, I only use the *kernel* of the likelihood function to discuss our model.

1.2.2 Likelihood Ratio Test

Here is an simple example for likelihood ratio test.

Let $X = (X_1, X_2)' \sim N(\theta, I)$, where I is the 2×2 identity matrix and $\theta = (\theta_1, \theta_2)'$. Consider the likelihood ratio test of $H_0 : \theta_1 = \theta_2 = 0 \quad \text{vs} \quad H_1 : \theta_1 \geqslant 0, \theta_2 \geqslant 0$,

$$LRT = ||X||^2 - ||X - \theta^*||^2,$$

where $\theta^* \in \{(\theta_1, \theta_2) | \theta_1 \geqslant 0, \theta_2 \geqslant 0\}$

Then

$$\begin{aligned} Pr(LRT \leqslant c) &= \sum_{i=1}^{4} Pr(LRT \leqslant c \text{ and } X \in Q_i) \\ &= \sum_{i=1}^{4} Pr(LRT \leqslant c | X \in Q_i) Pr(X \in Q_i) \end{aligned}$$

where $Q_1 = \{\theta_1, \theta_2 : \theta_1 > 0, \quad \theta_2 > 0\}$, $Q_2 = \{\theta_1, \theta_2 : \theta_1 < 0, \quad \theta_2 > 0\}$, $Q_3 = \{\theta_1, \theta_2 : \theta_1 < 0, \quad \theta_2 < 0\}$, $Q_4 = \{\theta_1, \theta_2 : \theta_1 > 0, \quad \theta_2 < 0\}$.

The null distribution of the LRT is the weighted sum of chi-square distributions, known as the chi-bar-square distribution. I use the similar method in the following chapter for hypothesis tests.

Chapter 2

First Order Model with Two Variables

2.1 Model and Assumptions

Consider the normal linear regression model with two predictor variables

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i \qquad i = 1, 2, ..., n, \tag{2.1}$$

or

$$Y = X\beta + \epsilon,$$

where

$$Y_{n\times 1} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad X_{n\times 3} = \begin{pmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} \end{pmatrix}, \quad \beta_{3\times 1} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}, \quad \epsilon_{n\times 1} = \begin{pmatrix} \epsilon_1 \\ \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix},$$

and $\{\epsilon_i\}$ are iid $N(0, \sigma^2)$. Let $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ be the unrestricted maximum likelihood estimaros (MLEs) of β_0, β_1 and β_2 respectively. Let ${S_{X_1}}^2 = \sum X_{1i}^2$, ${S_{X_2}}^2 = \sum X_{2i}^2$

and $S^2 = \sum(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2/v$ where $v = n-3$. Then we assume $\sum X_{1i} = 0$, $\sum X_{2i} = 0$ and $\sum X_{1i}X_{2i} = 0$ to simplify our model. Then,

$$cov(\hat{\beta}) = \sigma^2 \begin{pmatrix} n & \sum_{i=1}^n X_{1i} & \sum_{i=1}^n X_{2i} \\ \sum_{i=1}^n X_{1i} & \sum_{i=1}^n X_{1i}^2 & \sum_{i=1}^n X_{1i}X_{2i} \\ \sum_{i=1}^n X_{2i} & \sum_{i=1}^n X_{1i}X_{2i} & \sum_{i=1}^n X_{2i}^2 \end{pmatrix} = \begin{pmatrix} n & 0 & 0 \\ 0 & {S_{X_1}}^2 & 0 \\ 0 & 0 & {S_{X_2}}^2 \end{pmatrix},$$

and so $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are independent.

Further

$$\begin{aligned} cov(\hat{\beta}, Y - X\hat{\beta}) =& cov\left((X'X)^{-1}X'Y,\ Y - X(X'X)^{-1}X'Y\right) \\ =& cov\left((X'X)^{-1}X'Y,\ (I_n - X(X'X)^{-1}X')Y\right) \\ =& (X'X)^{-1}X'(\sigma^2 I)(I_n - X(X'X)^{-1}X') \\ =& \sigma^2\left((X'X)^{-1}X' - (X'X)^{-1}X'X(X'X)^{-1}X'\right) = 0. \end{aligned}$$

So $\hat{\beta}$ and $Y - X\hat{\beta}$ are independent. Then $\hat{\beta}$ and $S^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})/v$ are independent. Thus $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$ and S^2 are mutually independent. Following the properties of multivariate normal distribution, $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$ and S^2 have normal distributions. They are unbiased estimators for β_0, β_1, β_2 and σ^2. Hence

$$\hat{\beta}_0 \sim N(\beta_0, \sigma^2/n), \qquad \hat{\beta}_1 \sim N(\beta_1, \sigma^2/{S_{X_1}}^2),$$

$$\hat{\beta}_2 \sim N(\beta_2, \sigma^2/{S_{X_2}}^2), \qquad vS^2/\sigma^2 \sim \chi^2_v,$$

where $v = n-3$.

We consider the sign constraints for β_1 and β_2. First I consider,

$$\beta_1 \geqslant 0 \quad \text{and} \quad \beta_2 \geqslant 0 \tag{2.2}$$

We can always make transformations of predictor variables for other constraints of β. The restricted MLEs of β_0, β_1, and β_2 under the constraint (2.2) are given by

$$\beta_0^* = \hat{\beta}_0 \ , \qquad \beta_1^* = \hat{\beta}_1^{\ +} = max\{\hat{\beta}_1, \, 0\} \ , \qquad \beta_2^* = \hat{\beta}_2^{\ +} = max\{\hat{\beta}_2, \, 0\},$$

which is obvious and reasonable.

2.2 Inferences for $\beta_0 + \beta_1 X_{01} + \beta_2 X_{02}$

We consider inferences for the mean response $E(Y) = \beta_0 + \beta_1 X_{01} + \beta_2 X_{02}$, for given point (X_{01}, X_{02}). For example, we already known the length of patient stay in a hospital in days (Y) depends on the patient's age in years (X_{01}) and the infection risk (X_{02}). Given a patient's age and the infection risk, we want to know how long the patient will stay in hospital.

Peiris and Bhattacharya (2016) have already proposed formulas for the confidence interval for different signes of X_{01} and X_{02}.

Chapter 3

Inference for $\beta_0 + \beta_1 X_{01} + \beta_2 X_{02}$

Mentioned by Cox and Hinkley (1972), inverting one-sdied tests for ”two-sample” problems can derive the correct $(1-\alpha)$-coefficient prediction intervals for a new observation Y. Hence I derive the prediction intervals fro Y by inverting one-sided tests that test whether $\mu = E[Y|(X_{01}, X_{02})]$ excceds or is exceeded by $\beta_0 + \beta_1 X_{01} + \beta_2 X_{02}$, where the estimate of μ is to be obtained from the future observation Y and the estimate of $\beta_0 + \beta_1 X_{01} + \beta_2 X_{02}$ is obtained from the observations in the past.

Here there are four possible cases based on the signs of X_{01} and X_{02}.

3.1 Test when $X_{01} > 0$ and $X_{02} > 0$

First we consider the hypothesis,

$$\begin{aligned} G_{0L} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \leqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0 \\ G_{0U} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \geqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0 \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0 \end{aligned} \tag{3.1}$$

Define a $(1-\alpha)$-coefficient prediction interval (L_P, U_P) for Y, where

$$L_P = min\{Y|G_{0L} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0L}\},$$

$$U_P = max\{Y|G_{0U} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}\}.$$

Then we use the transformation from β to r. Let $r_0 = \beta_0/\sqrt{1+1/n}$, $r_1 = S_{X_1}\beta_1$, $r_2 = S_{X_2}\beta_2$, then $\hat{r} = (\hat{r_0}, \hat{r_1}, \hat{r_2})' = (\frac{\hat{\beta_0}}{\sqrt{1+1/n}}, S_{X_1}\hat{\beta_1}, S_{X_2}\hat{\beta_2})' \sim N_3(r, \sigma^2 I)$ where $\hat{r}$ is the unrestricted MLE of r. Let $b_1 = \frac{\mu S_{X_2}}{X_{02}}$, $c_1 = \frac{S_{X_2}}{X_{02}}\sqrt{1+1/n} > 0$, $d_1 = \frac{X_{01}S_{X_2}}{X_{02}S_{X_1}} > 0$. Then

$$\begin{aligned} & r_0\sqrt{1+1/n} + \frac{r_1 X_{01}}{S_{X_1}} + \frac{r_2 X_{02}}{S_{X_2}} \leqslant \mu, \\ \Rightarrow \quad & c_1 r_0 + d_1 r_1 + r_2 \leqslant b_1, \\ \Rightarrow \quad & r_2 \leqslant b_1 - c_1 r_0 - d_1 r_1. \end{aligned}$$

Hence our hypothesis can be restated in terms of r,

$$\begin{aligned} & G_{01} : 0 \leqslant r_2 \leqslant b_1 - c_1 r_0 - d_1 r_1, \quad r_1 \geqslant 0, \\ & G_{11} : r_1 \geqslant 0, \, r_2 \geqslant 0, \end{aligned} \tag{3.2}$$

and test G_{01} against $G_a = G_{11} - G_{01}$. Suppose we use the same notation G_{01} to denote the null hypothesis region. Here note that G_{01} is a polyhedral cone with vertex $L = (b_1/c_1, 0, 0)$. If we shift G_{01} along the r_0 axis to the origin, we obtain a shifted cone K. K is a closed convex cone bounded by three hyperplanes $\{c_1 r_0 + d_1 r_1 + r_2 = 0, r_1 \geqslant 0, r_2 \geqslant 0\}$, $\{r_0 \leqslant 0, 0 \leqslant r_1 \leqslant -\frac{c_1 r_0}{d_1}, r_2 = 0\}$, $\{r_0 \leqslant 0, r_1 = 0, 0 \leqslant r_2 \leqslant -c_1 r_0\}$. Then $G_{01} = K + L$, bounded by $\{c_1 r_0 + d_1 r_1 + r_2 = b_1, r_1 \geqslant 0, r_2 \geqslant 0\}$, $\{r_0 \leqslant \frac{b_1}{c_1}, c_1 r_0 + d_1 r_1 \leqslant b_1, r_2 = 0\}$, $\{r_0 \leqslant \frac{b_1}{c_1}, r_1 = 0, c_1 r_0 + r_2 \leqslant b_1\}$. Let $G_{01}^* = K^* + L$, where K^* is the dual cone of K. It can be shown that the boundaries of K^* are the perpendiculars to the boundaries of K. So the Fenchel dual cone K^* is bounded by three hyperplance $\{r_0 \geqslant 0, r_1 \leqslant \frac{d_1}{c_1}r_0, r_2 = \frac{1}{c_1}r_0\}$, $\{r_0 \geqslant 0, r_1 = \frac{d_1}{c_1}r_0, r_2 \leqslant \frac{1}{c_1}r_0\}$, $\{r_0 = 0, r_1 \leqslant \frac{d_1}{c_1}r_0, r_2 \leqslant \frac{1}{c_1}r_0\}$. Now let $\hat{r} \sim N_3(r, \sigma^2 I)$, where $\hat{r}$ is the unrestricted MLE of r. Hence note that the restricted MLE of r in G_{11} is $r^* = (r_0^*, r_1^*, r_2^*)' = (\hat{r_0}, \hat{r_1}^+, \hat{r_2}^+)'$, and r^* is the equal weight projection of $\hat{r}$ onto parameter space G_{11}.

Let $\bar{r}$ be the MLE of r under G_{01} and $\bar{r}$ is the equal weight prejection of $\hat{r}$ onto G_{01}. When σ is known, the likelihood ratio test(LRT) rejects G_{01} for large values of the test statistic

$$\bar{\chi_{01}}^2 = -2log\Lambda = (||\hat{r} - \bar{r}||^2 - ||\hat{r} - r^*||^2)/\sigma^2, \tag{3.3}$$

where Λ is the kernel of LRT statistic.

Figure 3.4, we consider several different cases when $\hat{r}$ located in several different regions. With the boundaries of G_{01} and G_{01}^*, we consider the whole region as an union of 13 disjoint regions.

Depending on the signs of $\hat{r_1}$ and $\hat{r_2}$, we discuss the test statistic $\bar{\chi_{01}}^2$ in each area separately. First when $\hat{r_1} < 0$ and $\hat{r_2} < 0$, we partition the region $\{(r_0, r_1, r_2) : r_1 < 0, r_2 < 0\}$ into $S_1 = \{(r_0, r_1, r_2) : r_0 < \frac{b_1}{c_1}, r_1 < 0, r_2 < 0\}$ and $S_2 = \{(r_0, r_1, r_2) : r_0 \geqslant \frac{b_1}{c_1}, r_1 < 0, r_2 < 0\}$. When $\hat{r} \in S_1$, $r^* = \bar{r} = (\hat{r_0}, 0, 0)$. So $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 = 0$, so S_1 is inside the accept region.

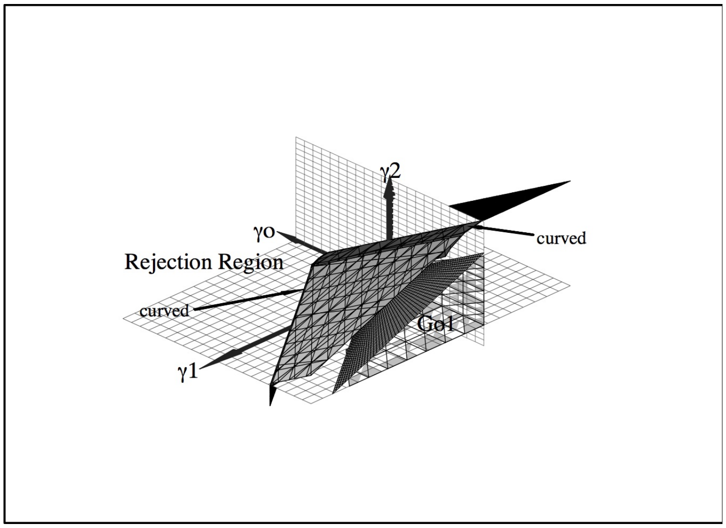

Figure 3.1: The region G_{01} and boundary of the rejection region

When $\hat{r} \in S_2$, $r^* = (\hat{r_0}, 0, 0)$ and $\bar{r} = (\frac{b_1}{c_1}, 0, 0)$. So $\bar{\chi_{01}}^2 = (\hat{r_0} - \frac{b_1}{c_1})^2/\sigma^2$ which has chi-square distribution with 1 degree of freedom. Furthere the boundary of rejection region is $r_0 = \frac{b_1}{c_1} + C_\alpha \sigma$.

When $\hat{r_1} < 0$ and $\hat{r_2} \geqslant 0$, we can define regions $S_3 = \{(r_0, r_1, r_2) : r_0 < \frac{b_1}{c_1} - \frac{1}{c_1} r_2, r_1 < 0, r_2 \geqslant 0\}$, $S_4 = \{(r_0, r_1, r_2) : r_1 < 0, r_2 \geqslant max\{b_1 - c_1 r_0, \frac{1}{c_1} r_0 - \frac{b_1}{c_1^2}\}\}$, and $S_5 = \{(r_0, r_1, r_2) : r_0 > c_1 r_2 - \frac{b_1}{c_1}, r_1 < 0, r_2 \geqslant 0\}$ such that the disjoint union of S_3, S_4, and S_5 is the region $\{(r_0, r_1, r_2) : r_1 < 0, r_2 \geqslant 0\}$.

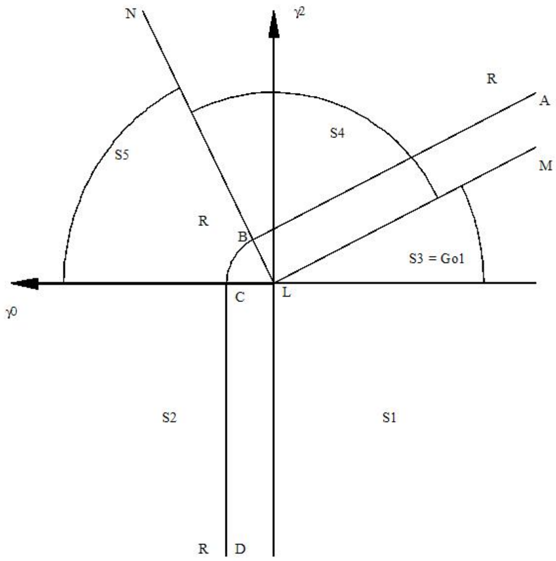

Figure 3.2: 2-Dimensional illustration of S_3, S_4 and S_5

When $\hat{r} \in S_3$, $r^* = (\hat{r_0}, 0, \hat{r_2})$ and $\bar{r} = (\hat{r_0}, 0, \hat{r_2})$.So $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 = 0$. So S_3 is inside the accept region. When $\hat{r} \in S_4$, $r^* = (\hat{r_0}, 0, \hat{r_2})$ and $\bar{r} = (\hat{r_0}, 0, (\hat{r_2} \cdot u)u)$ where u is a unit vector along the line $\{c_1 r_0 + r_2 = b_1, r_1 = 0\}$. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 > C_\alpha^2$ which belongs to chi-square distribution with 1 degree of freedom.Then the boundary of the rejection region is $c_1 r_0 + r_2 = b_1 + \sqrt{1 + c_1^2} C_\alpha \sigma$. When $\hat{r} \in S_5$, $r^* = (\hat{r_0}, 0, \hat{r_2})$ and $\bar{r} = (\frac{b_1}{c_1}, 0, 0)$. $\bar{\chi_{01}}^2 = ((\hat{r_0} - \frac{b_1}{c_1})^2 + \hat{r_2}^2)/\sigma^2 > C_\alpha^2$ which has chi-square distribution with 2 degree of freedom. Then a part of the boundary of the rejection region is $(r_0 - \frac{b_1}{c_1})^2 + r_2^2 = C_\alpha^2 \sigma^2$.

And when $\hat{r_1} \geqslant 0$ and $\hat{r_2} < 0$, we can define regions $S_6 = \{(r_0, r_1, r_2) : 0 \leqslant r_1 < \frac{b_1}{d_1} - \frac{c_1}{d_1} r_0, r_2 < 0\}$, $S_7 = \{(r_0, r_1, r_2) : r_1 \geqslant max\{\frac{b_1}{d_1} - \frac{c_1}{d_1} r_0, \frac{d_1}{c_1} r_0 - \frac{b_1 d_1}{c_1^2}\}, r_2 < 0\}$, and $S_8 = \{(r_0, r_1, r_2) : 0 \leqslant r_1 < \frac{d_1}{c_1} r_0 - \frac{b_1 d_1}{c_1^2}, r_2 < 0\}$ such that the disjoint union of S_6, S_7 and S_8 is the region $\{(r_0, r_1, r_2) : r_1 \geqslant 0, r_2 < 0\}$.

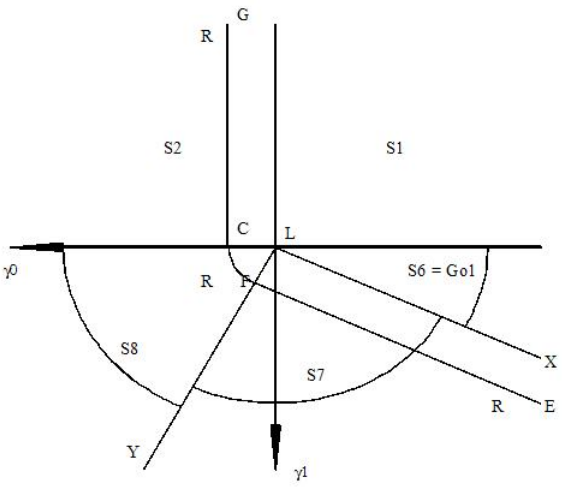

Figure 3.3: 2-Dimensional illustration of S_6, S_7 and S_8

When $\hat{r} \in S_6$, $r^* = (\hat{r_0}, \hat{r_1}, 0)$ and $\bar{r} = (\hat{r_0}, \hat{r_1}, 0)$.So $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 = 0$. So S_6 is inside the accept region. When $\hat{r} \in S_7$, $r^* = (\hat{r_0}, \hat{r_1}, 0)$ and $\bar{r} = ((\hat{r_0}, \hat{r_1}, 0) \cdot v)v$ where v is a unit vector along the line $\{c_1 r_0 + d_1 r_1 = b_1, r_2 = 0\}$. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 > C_\alpha^2$, which has chi-square distribution with 1 degree of freedom.Then the boundary of the rejection region is $c_1 r_0 + d_1 r_1 = b_1 + \sqrt{c_1^2 + d_1^2} C_\alpha \sigma$. When $\hat{r} \in S_8$, $r^* = (\hat{r_0}, \hat{r_1}, 0)$ and $\bar{r} = (\frac{b_1}{c_1}, 0, 0)$. $\bar{\chi_{01}}^2 = ((\hat{r_0} - \frac{b_1}{c_1})^2 + \hat{r_1}^2)/\sigma^2 > C_\alpha^2$ which has chi-square distribution with 2 degree of freedom. Then a part of the boundary of the rejection region is $(r_0 - \frac{b_1}{c_1})^2 + r_1^2 = C_\alpha^2 \sigma^2$.

And when $\hat{r_1} \geqslant 0$ and $\hat{r_2} \geqslant 0$, we can define regions $S_9 = \{(r_0, r_1, r_2) : c_1 r_0 + d_1 r_1 + r_2 \leqslant b_1, 0 \leqslant r_1, 0 \leqslant r_2\}$, $S_{10} = \{(r_0, r_1, r_2) : 0 \leqslant r_1 \leqslant \frac{c_1 d_1}{1+c_1^2} r_0 + \frac{d_1}{1+c_1^2}, r_2 \geqslant \frac{1}{c_1} r_0 - \frac{b_1}{c_1^2}\}$, $S_{11} = \{(r_0, r_1, r_2) : r_1 \geqslant \frac{d_1}{c_1} r_0 - \frac{b_1 d_1}{c_1^2}, 0 \leqslant r_2 \leqslant \frac{c_1}{c_1^2 + d_1^2} r_0 + \frac{d_1}{c_1^2 + d_1^2} r_1 - \frac{b_1}{c_1^2 + d_1^2}\}$, $S_{13} = \{(r_0, r_1, r_2) : 0 \leqslant r_1 < \frac{d_1}{c_1} r_0 - \frac{b_1 d_1}{c_1^2}, 0 \leqslant r_2 \leqslant \frac{1}{c_1} r_0 - \frac{b_1 d_1}{c_1^2}\}$, $S_{12} = \{(r_0, r_1, r_2) : r_1 \geqslant 0, r_2 \geqslant 0\} - S_9 \cup S_{10} \cup S_{11} \cup S_{13}$, such that the disjoint union of S_9, S_{10}, S_{11}, S_{12}, and S_{13} is the region $\{(r_0, r_1, r_2) : r_1 \geqslant 0, r_2 \geqslant 0\}$.

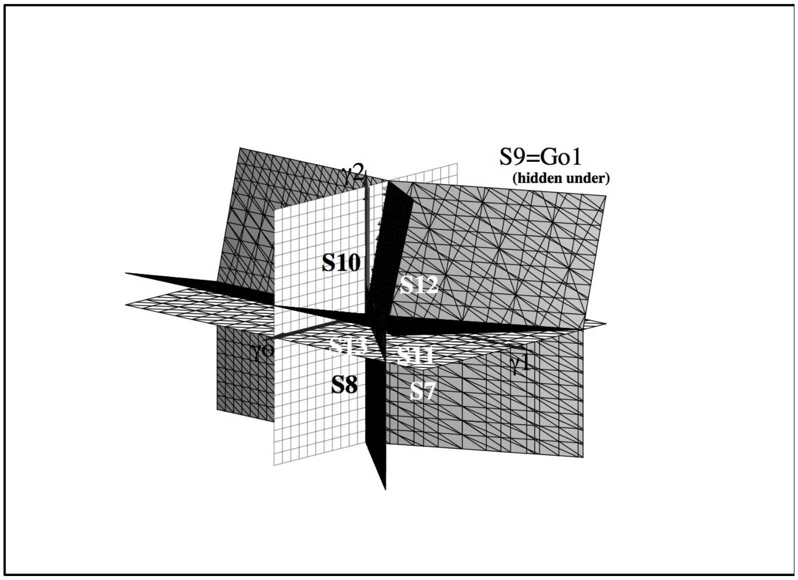

Figure 3.4: 3-Dimensional view of disjoint regions S_9, S_{10}, S_{11}, S_{12} and S_{13}

When $\hat{r} \in S_9$, $r^* = (\hat{r_0}, \hat{r_1}, \hat{r_2})$ and $\bar{r} = (\hat{r_0}, \hat{r_1}, \hat{r_2})$. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 = 0$. So S_9 is inside the accept region. When $\hat{r} \in S_{10}$, $r^* = (\hat{r_0}, \hat{r_1}, \hat{r_2})$ and $\bar{r} = ((\hat{r_0}, \hat{r_1}, \hat{r_2}) \cdot u)u$ where u is a unit vector along the line $\{c_1 r_0 + r_2 = b_1, r_1 = 0\}$. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 > C_\alpha^2$, which has to chi-square distribution with 2 degree of freedom. The boundary of the rejection region is a part of a rotated cylinder, $r_1^2 + [\frac{1}{\sqrt{1+c_1^2}} r_2 + \frac{c_1}{\sqrt{1+c_1^2}} (r_0 - \frac{b_1}{c_1})]^2 = C_\alpha^2 \sigma^2$. When $\hat{r} \in S_{11}$, $r^* = (\hat{r_0}, \hat{r_1}, \hat{r_2})$ and $\bar{r} = ((\hat{r_0}, \hat{r_1}, \hat{r_2}) \cdot v)v$ where v is a unit vector along the line $\{c_1 r_0 + d_1 r_1 = b_1, r_2 = 0\}$. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 > C_\alpha^2$ which belongs to chi-square distribution with 2 degree of freedom. The boundary of the rejection region is a part of a rotated cylinder: $r_2^2 + [\frac{d_1}{\sqrt{c_1^2+d_1^2}} r_2 + \frac{c_1}{\sqrt{c_1^2+d_1^2}} (r_0 - \frac{b_1}{c_1})]^2 = C_\alpha^2 \sigma^2$. When $\hat{r} \in S_{12}$, $r^* = (\hat{r_0}, \hat{r_1}, \hat{r_2})$ and $\bar{r} = ((\hat{r_0}, \hat{r_1}, \hat{r_2}) \cdot w)w$ where w is a unit vector along the line LB. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 > C_\alpha^2$ which belongs to chi-square distribution with 1 degree of freedom. The boundary of the rejection region is hyperplane above G_{01}, which is $c_1 r_0 + d_1 r_1 + r_2 = b_1 + \sqrt{1 + c_1^2 + d_1^2} C_\alpha \sigma$. When $\hat{r} \in S_{13}$, $r^* = (\hat{r_0}, \hat{r_1}, \hat{r_2})$ and $\bar{r} = (\frac{b_1}{c_1}, 0, 0)$. $\bar{\chi_{01}}^2 = ||r^* - \bar{r}||^2/\sigma^2 = [(\hat{r_0} - \frac{b_1}{c_1})^2 + (\hat{r_1})^2 + (\hat{r_2})^2]/\sigma^2 > C_\alpha^2$ which belongs to chi-square distribution with 3 degree of freedom. The boundary of the rejection region is a part of sphere surface$(\hat{r_0} - \frac{b_1}{c_1})^2 + \hat{r_1}^2 + \hat{r_2}^2 = C_\alpha^2\sigma^2$. The least favorable null value of $\bar{\chi_{01}}^2$ is attained at $r = L = (\frac{b_1}{c_1}, 0, 0)$ and

$$\sup_{r \in G_{01}} Pr_r\{\hat{r} : ||\bar{r} - r^*||^2 \geqslant C_\alpha^2\sigma^2\} = Pr_L\{||\bar{r} - r^*|| \geqslant C_\alpha\sigma\}. \tag{3.4}$$

THe proof of above result is given by Peiris and Bhattacharya (2016). When $\hat{r}$ is attained the least favorable null value, the distribution of LRT $\bar{\chi}_{01}^2$ is given by folloing formula. (See Peiris and Bhattacharya (2016) for the proof and more details.)

The least favorable null distribution of LRT is

$$Pr(LRT \leqslant t|\hat{r} = L) = \sum_{i=0}^{3} w_i P({\chi_i}^2 \leqslant t),$$

where

$$\begin{aligned}
w_0 &= (4\pi)^{-1}(cos^{-1}\frac{1}{\sqrt{1+c_1^2}} + cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}} + cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}} \\
&\qquad + cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}}), \\
w_1 &= (4\pi)^{-1}(\frac{3}{2}\pi + cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}}), \\
w_2 &= (4\pi)^{-1}(\pi + cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}} + cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}} \\
&\qquad - cos^{-1}\frac{1}{\sqrt{1+c_1^2}} - cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}}), \\
w_3 &= (4\pi)^{-1}(\frac{3}{2}\pi - cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}} - cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}} - cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}} \\
&\qquad - cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}} - cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}}).
\end{aligned} \tag{3.5}$$

And the prediction upper bound is

$$U_P = max\{Y|G_{0U} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}\}.$$

Use the transformation from β to r. Let $r_0 = \beta_0/\sqrt{1+1/n}$, $r_1 = S_{X_1}\beta_1$, $r_2 = S_{X_2}\beta_2$ then $\hat{r} = (\hat{r_0}, \hat{r_1}, \hat{r_2})' = (\beta_0/\sqrt{1+1/n}, S_{X_1}\hat{\beta_1}, S_{X_2}\hat{\beta_2})' \sim N_3(r, \sigma^2 I)$ where $\hat{r}$ is the unrestricted MLE of r. Let $b_1' = \frac{\mu S_{X_2}}{X_{02}}$, $\quad c_1 = \frac{S_{X_2}}{X_{02}}\sqrt{1+1/n} > 0$, $\quad d_1 = \frac{X_{01}S_{X_2}}{X_{02}S_{X_1}} > 0$.

$$\begin{aligned} & r_0\sqrt{1+1/n} + \frac{r_1 X_{01}}{S_{X_1}} + \frac{r_2 X_{02}}{S_{X_2}} \geqslant \mu, \\ \Rightarrow \quad & c_1 r_0 + d_1 r_1 + r_2 \geqslant b_1', \\ \Rightarrow \quad & r_2 \geqslant b_1' - c_1 r_0 - d_1 r_1, \end{aligned}$$

Hence our hypothesis is

$$\begin{aligned} & H_{01} : r_2 \geqslant b_1' - c_1 r_0 - d_1 r_1, \quad r_1 \geqslant 0,\, r_2 \geqslant 0, \\ & H_{11} : r_1 \geqslant 0 \quad r_2 \geqslant 0, \end{aligned} \tag{3.6}$$

and test H_{01} against $H_a = H_{11} - H_{01}$. Similarly, to illustrate the construction of rejection region, we need the boundaries of H_{01} in the r form. Let K' be the shifted cone of H_{01}, and $K^{*\prime}$ be the dual cone of K'. And $H_{01} = K' + L'$ and $H_{01}' = K^{*\prime} + L'$, where $L' = (b_1'/c_1, 0, 0)$. Then we can get the 6 regions divided by the boundaries of K' and $K^{*\prime}$.

Now let $\hat{r} \sim N_3(r, \sigma^2 I)$, where $\hat{r}$ si the unrestricted MLE of r. And the restricted MLE of β is $\beta^* = (\beta_0^*, \beta_1^*, \beta_2^*)'$ in section 2.1 . Hence we can define the restricted MLE of r is $r^* = (r_0^*, r_1^*, r_2^*)' = (\hat{r_0}, \hat{r_1}^+, \hat{r_2}^+)'$, and r^* is the equal weight projection of $\hat{r}$ onto parameter space H_{11}. Let $\bar{r}$ be the MLE of r under H_{01} and $\bar{r}$ is the equal weight prejection of $\hat{r}$ onto H_{01}. When σ is known, the likelihood ratio test(LRT) rejects H_{01} for large values of the test statistic is

$$\bar{\chi_{01}}^2 = -2log\Lambda = (||\hat{r} - \bar{r}||^2 - ||\hat{r} - r^*||^2)/\sigma^2, \tag{3.7}$$

where Λ is the kernel of LRT statistic.

According to the discussion in Peiris and Bhattacharya (2016), the least favorable null value of LRT(2.8) is attained at $\lim\limits_{t\to\infty, s\to\infty}(b_1'/c_1 - s - c_1t, c_1t, c_1s)$ and

$$\begin{aligned}\sup_{r\in H_{01}} Pr_r\{\hat{r} : ||\hat{r}-\bar{r}||^2 - ||\hat{r}-r^*||^2 \geqslant D_\alpha^2\sigma^2\} \\ = \lim_{t\to\infty, s\to\infty} Pr_{(b_1'/c_1 - s - c_1t, c_1t, c_1s)}\{\bar{\chi_{01}}^2 > D_\alpha^2\sigma^2\}\end{aligned} \tag{3.8}$$

Also, the null critical value is $D_\alpha^2 = \chi_{1,\alpha}^2$.

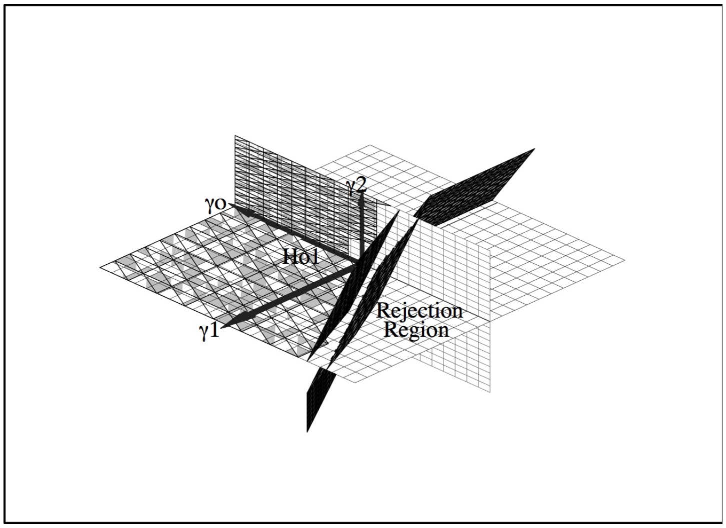

Figure 3.5: H_{01} and Rejection region

Shown in figure 3.5, we consider the whole region as an union of several disjoint areas. Depending on the signs of $\hat{r_1}$ and $\hat{r_2}$, I discuss the test statistic $\bar{\chi}_{01}^2$ in each area separately. However I can not obtain the exact rejection region to get confidence and prediction intervals. Hence we need to modify our likelihood ratio test under same power.

Consider the hypothesis without the restrictions $r_1 \geqslant 0$ and $r_2 \geqslant 0$

$$\begin{aligned} H_{01}^{**} &: c_1 r_0 + d_1 r_1 + r_2 \geqslant b_1', \\ H_{11}^{**} &: c_1 r_0 + d_1 r_1 + r_2 < b_1'. \end{aligned} \tag{3.9}$$

Then the null hypothesis is excatly same as the unrestricted case. LRT rejects H_{01}^{**} for small values $\chi_{03} = \frac{c_1 \hat{r_0} + d_1 \hat{r_1} + \hat{r_2} - b_1'}{\sqrt{1+c_1^2+d_1^2}\sigma}$. So the rejection region of LRT is $\{\hat{r} : \chi_{03} < -X_\alpha \sigma\}$ Here rejection region for the unrestricted LRT contained that for the restricted LRT. So the unrestricted LRT is more powerful than the restricted LRT. But this creates a philosophical dilemma in some cases. In some cases, we will reject H_{01} under unrestricted LRT but will not reject it under restricted LRT. So we need to modify LRT. Then consider following four regions. We use a similar idea as that for two dimensional model $EY = \beta_0 + \beta_1 X_1$ discussed in Mukerjee and Tu (1995). We consider four regions in $\mathbb{R}^3$: S_1, S_2, S_3, and S_4, where $S_2 = \{r : r_1 \leqslant -\frac{1}{d_1}\sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma, r_2 \geqslant 0\}$, $S_3 = \{r : r_1 \geqslant 0, r_2 \leqslant -\sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma\}$, $S_4 = \{r : r_1 < 0, r_2 < min\{0, -d_1 r_1 - \sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma\}\}$, and $S_1 = \mathbb{R}^3 - S_2 \cup S_3 \cup S_4$. The boundary of the H_{01} which is $c_1 r_0 + d_1 r_1 + r_2 = b_1'$ meets the hyperplane $\{r_2 = 0\}$ on the line $\{r_2 = 0, c_1 r_0 + d_1 r_1 = b_1'\}$ and the hyperplane $\{r_1 = 0\}$ on the line $\{r_1 = 0, c_1 r_0 + r_2 = b_1'\}$. Hyperplane $c_1 r_0 + d_1 r_1 + r_2 = b_1' - \sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$ and hyperplane $c_1 r_0 + d_1 r_1 = b_1'$ intersect on the line $r_2 = -\sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$. Hyperplane $c_1 r_0 + d_1 r_1 + r_2 = b_1' - \sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$ and hyperplane $c_1 r_0 + r_2 = b_1'$ intersect on the hyperplane $r_1 = -\frac{1}{d_1}\sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$.

To keep the same rejection level α, we modify LRT as follows, when $\hat{r} \in S_1$, we use the same boundary of the rejection region of the unrestricted case, which is $c_1 r_0 + d_1 r_1 + r_2 = b_1' - \sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$. When $\hat{r} \in S_2$, we already know the intersection of hyperplane $c_1 r_0 + d_1 r_1 + r_2 = b_1' - \sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$ and S2's boundary $r_1 = -\frac{1}{d_1}\sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma$ is the plane $\{c_1 r_0 + r_2 = b_1', r_1 = -\frac{1}{d_1}\sqrt{1+c_1^2+d_1^2} Z_\alpha \sigma\}$.

So let $c_1 r_0 + r_2 = b_1'$ as a part of the boundaries of rejection region in S_2. Similarly when $\hat{r} \in S_4$, we let $c_1 r_0 + d_1 r_1 = b_1'$ as a part of boundaries of rejection region in S_4. when $\hat{r} \in S_3$, we let $c_1 r_0 = b_1'$ as a part of boundaries of rejection region in S_3.

3.2 Test when $X_{01} < 0$ and $X_{02} < 0$

We consider the hypothesis,

$$\begin{aligned} G_{0L} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \leqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ G_{0U} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \geqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0. \end{aligned} \tag{3.10}$$

Define a $(1-\alpha)$-coefficient prediction interval (L_P, U_P) for Y, where

$$L_P = min\{Y | G_{0L} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0L}\},$$

$$U_P = max\{Y | G_{0U} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}\}.$$

Then we can make a tranformation, where $X_{01}^* = -X_{01} > 0$, $X_{02}^* = -X_{02} > 0$, $\beta_0^* = -\beta_0$, and $\mu^* = -\mu$. Then the new hypothesis will be,

$$\begin{aligned} G_{0L}^* &: \beta_0^* + \beta_1 X_{01}^* + \beta_2 X_{02}^* \geqslant \mu^* \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ G_{0U} &: \beta_0^* + \beta_1 X_{01}^* + \beta_2 X_{02}^* \leqslant \mu^* \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0. \end{aligned} \tag{3.11}$$

Define a $(1-\alpha)$-coefficient prediction interval (L_P, U_P) for Y, where

$$L_P^* = min\{Y|G_{0L}^* \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0L}^*\} = U_P,$$

$$U_P^* = max\{Y|G_{0U}^* \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}^*\} = L_P,$$

where L_P and U_P are in section 3.1. Hence the formulas for rejection region and prediciton intervals can be obtained using the symetric property and are shown in next chapter.

3.3 Test when $X_{01} > 0$ and $X_{02} < 0$

Then we consider the hypothesis,

$$\begin{aligned} G_{0L} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \leqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ G_{0U} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \geqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0. \end{aligned} \tag{3.12}$$

Define a $(1-\alpha)$-coefficient prediction interval (L_P, U_P) for Y, where

$$L_P = min\{Y|G_{0L} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0L}\},$$

$$U_P = max\{Y|G_{0U} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}\}.$$

We use the transformation from β to r. Let $r_0 = \beta_0/\sqrt{1+1/n}$, $r_1 = S_{X_1}\beta_1$, $r_2 = S_{X_2}\beta_2$, then $\hat{r} = (\hat{r_0}, \hat{r_1}, \hat{r_2})' = (\hat{\beta}_0/\sqrt{1+1/n}, S_{X_1}\hat{\beta}_1, S_{X_2}\hat{\beta}_2)' \sim N_3(r, \sigma^2 I)$, where $\hat{r}$ is the unrestricted MLE of r.

And $b_2 = \frac{\mu S_{X_2}}{X_{02}}$, $\quad c_2 = \frac{S_{X_2}}{X_{02}}\sqrt{1+1/n} < 0$, $\quad d_2 = \frac{X_{01}S_{X_2}}{X_{02}S_{X_1}} < 0$,

$$\begin{aligned}
& r_0\sqrt{1+1/n} + \frac{r_1 X_{01}}{S_{X_1}} + \frac{r_2 X_{02}}{S_{X_2}} \leqslant \mu \\
\Rightarrow \quad & c_2 r_0 + d_2 r_1 + r_2 \geqslant b_2 \\
\Rightarrow \quad & r_2 \geqslant b_2 - c_2 r_0 - d_2 r_1
\end{aligned}$$

Hence our hypothesis in terms of r is

$$\begin{aligned}
& G_{03} : r_2 \geqslant b_2 - c_2 r_0 - d_2 r_1, \quad r_1 \geqslant 0,\, r_2 \geqslant 0, \\
& G_{13} : r_1 \geqslant 0 \quad r_2 \geqslant 0.
\end{aligned} \tag{3.13}$$

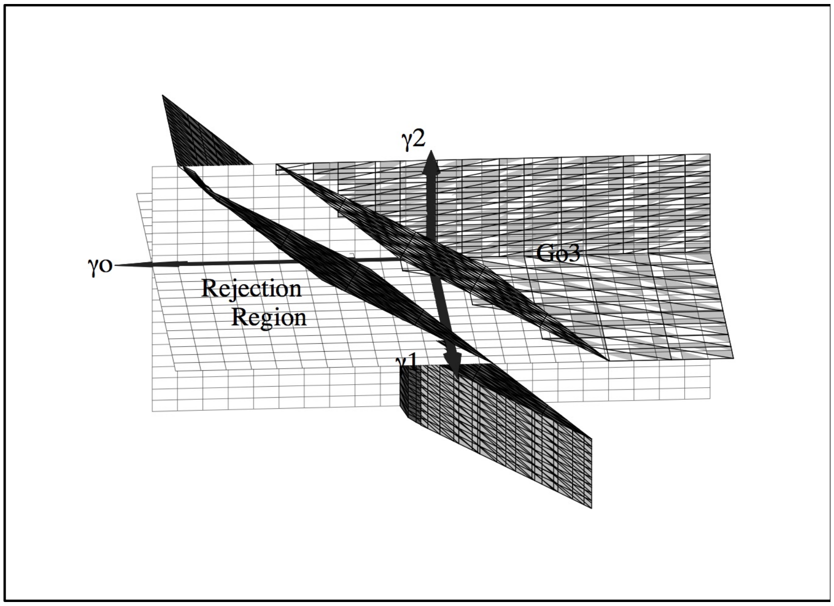

Figure 3.6: G_{03} and Rejection region

Suppose we use the same notation G_{03} to denote the null hypothesis region. Here note that G_{03} is a polyhedral cone with vertex $L = (b_1/c_1, 0, 0)$. If we shift G_{03} along the r_0 axis to the origin, we obtain a shifted cone K. K is the closed convex cone bounded by three hyperplanes $\{c_2r_0 + d_2r_1 + r_2 = 0, r_1 \geqslant 0, r_2 \geqslant 0\}$, $\{r_1 = 0, c_2r_0 + d_2r_1 + r_2 \leqslant 0, r_2 \leqslant 0\}$, $\{r_2 = 0, c_2r_0 + d_2r_1 + r_2 \geqslant 0, r_1 \geqslant 0\}$. Then $G_{03} = K + L$, bounded by $\{c_2r_0 + d_2r_1 + r_2 = b_2, r_1 \geqslant 0, r_2 \geqslant b_2\}$, $\{r_1 = 0, c_2r_0 + d_2r_1 + r_2 \leqslant b_2, r_2 \leqslant 0\}$, $\{r_2 = 0, c_2r_0 + d_2r_1 + r_2 \geqslant b_2, r_1 \geqslant 0\}$.

Recall the definition of dual cone, let $G^*_{03} = K^* + L$, where K^* is the dual cone of K. It can be shown that the boundaries of K^* are the perpendiculars to the boundaries of K. So the Fenchel dual cone G^*_{03} is bounded by three hyperplance $\{r_0 - \frac{c_2}{d_2}r_1 \leqslant \frac{b_2}{c_2}, r_0 \leqslant \frac{b_2}{c_2}, r_0 - c_2r_2 = \frac{b_2}{c_2}\}$, $\{r_0 - \frac{c_2}{d_2}r_1 = \frac{b_2}{c_2}, r_0 \leqslant \frac{b_2}{c_2}, r_0 - c_2r_2 \leqslant \frac{b_2}{c_2}\}$, $\{r_0 - \frac{c_2}{d_2}r_1 \leqslant \frac{b_2}{c_2}, r_0 = \frac{b_2}{c_2}, r_0 - c_2r_2 \leqslant \frac{b_2}{c_2}\}$. Now let $\hat{r} \sim N_3(r, \sigma^2 I)$, where $\hat{r}$ si the unrestricted MLE of r. And the restricted MLE of β is $\beta^* = (\beta^*_0, \beta^*_1, \beta^*_2)'$ in section 2.1 . Hence we can define the restricted MLE of r is $r^* = (r^*_0, r^*_1, r^*_2)' = (\hat{r_0}, \hat{r_1}^+, \hat{r_2}^+)'$, and r^* is the equal weight projection of $\hat{r}$ onto parameter space G_{13}.

Let $\bar{r}$ be the MLE of r under G_{03} and $\bar{r}$ is the equal weight prejection of $\hat{r}$ onto G_{03}. When σ is known, the likelihood ratio test(LRT) rejects G_{03} for large values of the test statisti is

$$\bar{\chi_{03}}^2 = -2log\Lambda = (||\hat{r} - \bar{r}||^2 - ||\hat{r} - r^*||^2)/\sigma^2, \tag{3.14}$$

where Λ is the kernel of LRT statistic.

So the rejection region with a level α is $\{(||\hat{r} - \bar{r}||^2 - ||\hat{r} - r^*||^2) > E^2_\alpha \sigma^2\}$, where E_α is the critical value. Similar as previous section, when $r_2 \geqslant 0$, the rejection region is $\{||\hat{r} - \bar{r}||^2 > E^2_\alpha \sigma^2\}$. We can obtain boundaries of the rejection region $\{c_2r_0 + r_2 \leqslant b_2 - \sqrt{1 + c_2^2}F_\alpha \sigma\ , r_1 < 0\}$, $\{r_1^2 + (\frac{1}{\sqrt{1+c_2^2}}r_2 + \frac{c_2}{\sqrt{1+c_2^2}}(r_0 - \frac{b_2}{c_2}))^2 = F^2_\alpha \sigma^2\ , 0 \leqslant r_1 \leqslant \frac{c_2d_2}{1+c_2^2}r_0 + \frac{d_2}{1+c_2^2}r_2 - \frac{b_2d_2}{1+c_2^2}\}$, $\{c_2r_0 + d_2r_1 + r_2 = b_2 - \sqrt{1 + c_2^2 + d_2^2}F_\alpha \sigma\ , r_1 \geqslant max\{0, \frac{c_2d_2}{1+c_2^2}r_0 + \frac{d_2}{1+c_2^2}r_2 - \frac{b_2d_2}{1+c_2^2}\}\}$.

But when $r_2 < 0$, the rejection region has a complicated formulas and it is hard to illstruate them with figures. We propose a new modified rejection region which is similar as previous section.

The least favorable null value and the least favorable distribution is given in Peiris and Bhattacharya (2016). So the least favorable null value of $\bar{\chi}_{03}^2$ is attained at infinity with $\lim\limits_{r_0 \to \infty}(r_0, 0, b_2 - c_2 r_0)$ and

$$\sup_{r \in G_{03}} Pr_r\{\hat{r} : (||\hat{r} - \bar{r}||^2 - ||\hat{r} - r^*||^2)/\sigma^2 \geqslant E_\alpha^2\} = \lim_{r_0 \to \infty} Pr_{(r_0, 0, b_2 - c_2 r_0)}\{\bar{\chi_{03}}^2 > E_\alpha^2 \sigma^2\}.$$

The least favorable null distribution of LRT is,

$$\sup_{r \in G_{03}} Pr(LRT \geqslant c) = (\frac{1}{4} + \frac{\theta_1}{2\pi})P(\chi_0^2 \geqslant c) + \frac{1}{2}P(\chi_1^2 \geqslant c) + (\frac{1}{4} - \frac{\theta_1}{2\pi})P(\chi_2^2 \geqslant c),$$

where θ_1 is the anlge between hyperplane $C_2 r_0 + d_2 r_1 + r_2 = b_2$ and hyperplane $r_1 = 0$

To obtain the modified LRT, first we consider hypothesis (2.13) without the restriction $r_2 \geqslant 0$,

$$M_{02} : r_2 \geqslant b_2 - c_2 r_0 - d_2 r_1 \quad r_1 \geqslant 0 \quad \text{against} \quad M_{12} : r_1 \geqslant 0.$$

So the new LRT rejects M_{02} for large values is,

$$\bar{\chi_{03}}^2 = (||\hat{r} - \bar{\bar{r}}||^2 - ||\hat{r} - r^{**}||^2)/\sigma^2,$$

where $\bar{\bar{r}}$ is the MLE under M_{02} and r^{**} is the MLE under M_{12} .

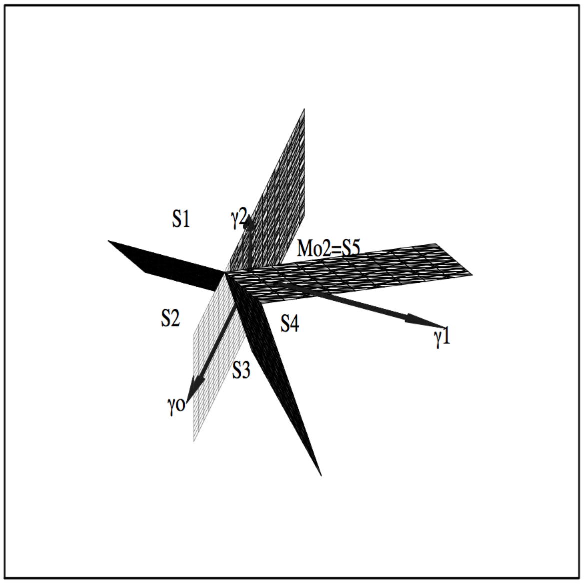

Figure 3.7: M_{02} and Rejection region and dual cone M_{02}^*

We can have the projection of those regions' boundaries to the hyperplane which the intersection line of regions is orthogonal to. Then the discussion for rejection region is similar to two predictor variables model case (Mukerjee and Tu (1995)). Hence when $r_1 < 0$, divide the region into two parts S_1 and S_2, where $S_1 = \{r : r_1 < 0, c_2r_0 + r_2 \geqslant b_2\}$, $S_2 = \{r : r_1 < 0, c_2r_0 + r_2 < b_2\}$, and obtain the center axis which is the intersection line of five regions, where $\{c_2r_0 + r - 2 = b_2,\ r_1 = 0\}$ When $\hat{r} \in S_1$, $\bar{\chi_{03}}^2 = ||r^{**} - \bar{\bar{r}}||^2 = 0$ where $\bar{\bar{r}} = r^{**} = (\hat{r_0}, 0, \hat{r_2})$. So S_1 is in the acceptance region.

When $\hat{r} \in S_2$, $\bar{\chi_{03}}^2 = ||r^{**} - \bar{\bar{r}}||^2 = ||(\hat{r_0}, 0, \hat{r_2}) - ((\hat{r_0}, 0, \hat{r_2}) \cdot u)u||^2 \geqslant F_\alpha^2\sigma^2$ where $\bar{\bar{r}} = ((\hat{r_0}, 0, \hat{r_2}) \cdot u)u$ and $r^{**} = (\hat{r_0}, 0, \hat{r_2})$ It has chi-square distribution with 1 degree of freedom. The boundary of rejection region is a hyperplane parallely above the M_{02} and has $F_\alpha\sigma$ distance to the hyperplane $c_2r_0 + r_2 = b_2$.

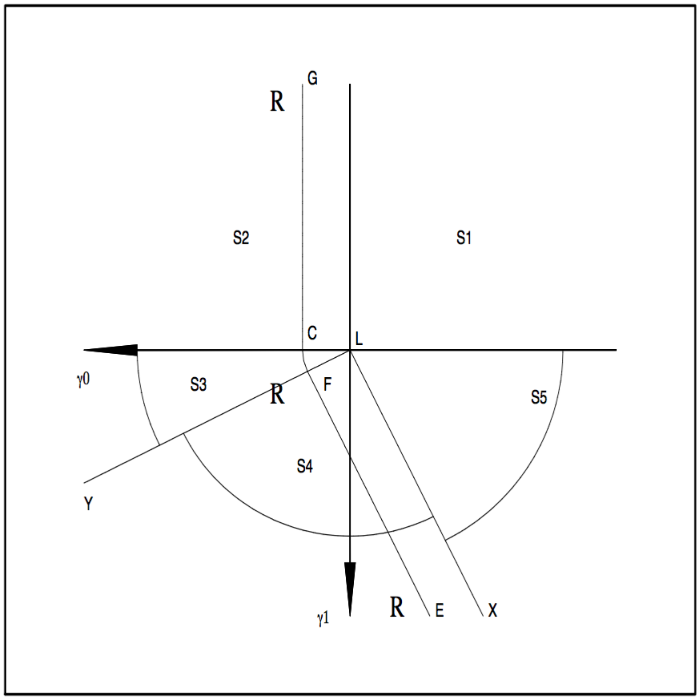

Figure 3.8: 2-Dimensional view of M_{02} and Rejection region

when $r_1 \geqslant 0$, the hyperplane $c_2 r_0 + d_2 r_1 + r_2 = b_2$ and $c_2 d_2 r_0 - (1 + c_2^2) r_1 + d_2 r_2 = b_2 d_2$ divide the region into three subregions S_3, S_4, and S_5. When $\hat{r} \in S_3$, $\bar{\chi_{03}}^2 = ||r^{**} - \bar{\bar{r}}||^2$ where $\bar{\bar{r}} = ((\hat{r_0}, \hat{r_1}, \hat{r_2}) \cdot u)u$ and $r^{**} = (\hat{r_0}, \hat{r_1}, \hat{r_2})$, where u is a unit vector along the center axis. It has chi-square distribution with 2 degree of freedom. The boundary of rejection region is a part of cylinder (center axis is the axis and radius is $F_\alpha \sigma$). When $\hat{r} \in S_4$, $\bar{\chi_{03}}^2 = ||r^{**} - \bar{\bar{r}}||^2$ where $\bar{\bar{r}} = ((\hat{r_0}, \hat{r_1}, \hat{r_2}) \cdot w)w$ and $r^{**} = (\hat{r_0}, \hat{r_1}, \hat{r_2})$, where w is a unit vector along the projection of $\hat{r}$ onto hyperplane $c_2 r_0 + d_2 r_1 + r_2 = b_2$. It has chi-square distribution with 1 degree of freedom. The boundary of rejection region is above the hyperplane $c_2 r_0 + d_2 r_1 + r_2 = b_2$ with distance $F_\alpha \sigma$. When $\hat{r} \in S_5$, $\bar{\chi_{03}}^2 = ||r^{**} - \bar{\bar{r}}||^2 = 0$ where $\bar{\bar{r}} = r^{**} = (\hat{r_0}, r \hat{-} 1, \hat{r_2})$. So S_5 is in the acceptance region.

Using the same argument as in the previous section, I propose a modified LRT for (3.10), keep the same boundary of LRT for (3.10) when $r_2 \geqslant -\sqrt{1+c_2^2}E_\alpha\sigma$. Note that the hyperplane $c_2r_0 + r_2 = b_2 - \sqrt{1+c_2^2}E_\alpha\sigma$ and $r_0 = \frac{b_2}{c_2}$ intersected at $r_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$. When $r_2 < -\sqrt{1+c_2^2}E_\alpha\sigma$, modify the rejection region with "cut-off". I propose a hyperplane $r_0 = \frac{b_2}{c_2}$ as the part boundary of the rejection region. And I propose a curved plane which is parallel to r_2 axis and a hyperplane which is $\{c_2r_0 + d_2r_1 = b_2 - (\sqrt{1+c_2^2+d_2^2} - \sqrt{1+c_2^2})E_\alpha\sigma\}$

We consider another hypothesis G_{0U} agiainst $G_a = G_1 - G_{0U}$ because

$$U_P = max\{Y|G_{0U} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}\}.$$

We use the transformation from β to r. Let $r_0 = \beta_0/\sqrt{1+1/n}$, $r_1 = S_{X_1}\beta_1$, $r_2 = S_{X_2}\beta_2$, then $\hat{r} = (\hat{r_0}, \hat{r_1}, \hat{r_2})' = (\hat{\beta_0}/\sqrt{1+1/n}, S_{X_1}\hat{\beta_1}, S_{X_2}\hat{\beta_2})' \sim N_3(r, \sigma^2 I)$ where $\hat{r}$ is the unrestricted MLE of r. And $b_2 = \frac{\mu S_{X_2}}{X_{02}}$, $\quad c_2 = \frac{S_{X_2}}{X_{02}}\sqrt{1+1/n} < 0$, $\quad d_2 = \frac{X_{01}S_{X_2}}{X_{02}S_{X_1}} < 0$,

$$\begin{aligned} & r_0\sqrt{1+1/n} + \frac{r_1X_{01}}{S_{X_1}} + \frac{r_2X_{02}}{S_{X_2}} \geqslant \mu, \\ \Rightarrow \quad & c_2r_0 + d_2r_1 + r_2 \leqslant b_2, \\ \Rightarrow \quad & r_2 \leqslant b_2 - c_2r_0 - d_2r_1. \end{aligned}$$

Hence our hypothesis in terms of r is

$$\begin{aligned} & H_{03} : 0 \leqslant r_2 \leqslant b_2 - c_2r_0 - d_2r_1, \quad r_1 \geqslant 0, \\ & H_{13} : r_1 \geqslant 0 \quad r_2 \geqslant 0. \end{aligned} \tag{3.15}$$

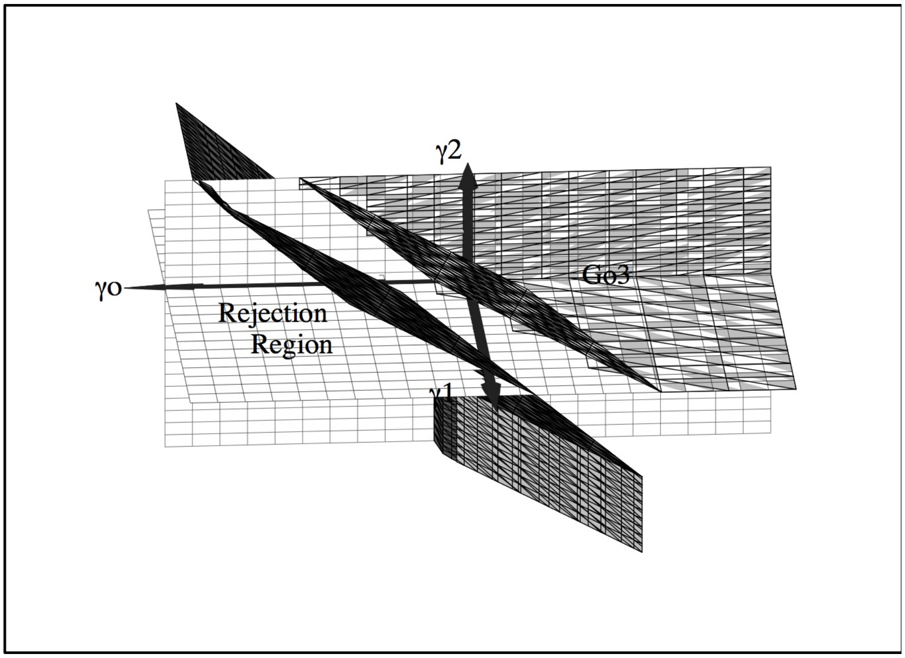

Figure 3.9: H_{03} and Rejection region

Here I note that the null region H_{03} is a mirror image of the null region G_{03} in the previous section.

Considering hypothesis without the restriction $r_1 \geqslant 0$, keep the boudary of rejection region which is same as in (3.12) when $r_1 \geqslant \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha \sigma$. Then I propose a modified LRT when $r_1 < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha \sigma$.

The least favorable null value of $\bar{\chi}_{03}{}^2$ is attained at infinity with $\lim\limits_{r_0 \to \infty}(r_0, \frac{b_2' - c_2 r_0}{d_2}, 0)$ and

$$\sup_{r \in H_{03}} Pr_r\{\hat{r} : (||\hat{r} - \bar{r}||^2 - ||\hat{r} - r^*||^2)/\sigma^2 \geqslant K_\alpha^2\} = \lim_{r_0 \to -\infty} Pr_{(r_0, \frac{b_2' - c_2 r_0}{d_2}, 0)}\{\bar{\chi}_{03}{}^2 > K_\alpha^2 \sigma^2\}$$

The least favorable distribution of LRT is,

$$Pr(LRT \leqslant c) = (\frac{1}{4} + \frac{\theta_2}{2\pi})P(\chi_0^2 \leqslant c) + \frac{1}{2}P(\chi_1^2 \leqslant c) + (\frac{1}{4} - \frac{\theta_2}{2\pi})P(\chi_2^2 \leqslant c),$$

where θ_2 is the anlge between hyperplane $C_2r_0 + d_2r_1 + r_2 = b_2'$ and hyperplane $r_2 = 0$ (See more details in Peiris and Bhattacharya (2016)).

3.4 Test when $X_{01} < 0$ and $X_{02} > 0$

Then we consider the hypothesis,

$$\begin{aligned} G_{0L} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \leqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ G_{0U} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \geqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0. \end{aligned} \tag{3.16}$$

Define a $(1-\alpha)$-coefficient prediction interval (L_P, U_P) for Y, where

$$L_P = min\{Y|G_{0L} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0L}\},$$

$$U_P = max\{Y|G_{0U} \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}\}.$$

Then we can make a tranformation, where $X_{01}^* = -X_{01} > 0$, $X_{02}^* = -X_{02} < 0$, $\beta_0^* = -\beta_0$, and $\mu^* = -\mu$. Then the new hypothesis will be,

$$\begin{aligned} G_{0L}^* &: \beta_0^* + \beta_1 X_{01}^* + \beta_2 X_{02}^* \geqslant \mu^* \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ G_{0U} &: \beta_0^* + \beta_1 X_{01}^* + \beta_2 X_{02}^* \leqslant \mu^* \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0, \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0. \end{aligned} \tag{3.17}$$

Define a $(1-\alpha)$-coefficient prediction interval (L_P, U_P) for Y, where

$$L_P^* = min\{Y|G_{0L}^* \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0L}^*\} = U_P,$$

$$U_P^* = max\{Y|G_{0U}^* \text{ is accepted at level } \alpha/2 \text{ against } G_a = G_1 - G_{0U}^*\} = L_P.$$

where L_P and U_P are in section 3.3. Hence the formulas for rejection region and prediciton intervals can be obtained using symetric properties of these cases as shown in next chapter.

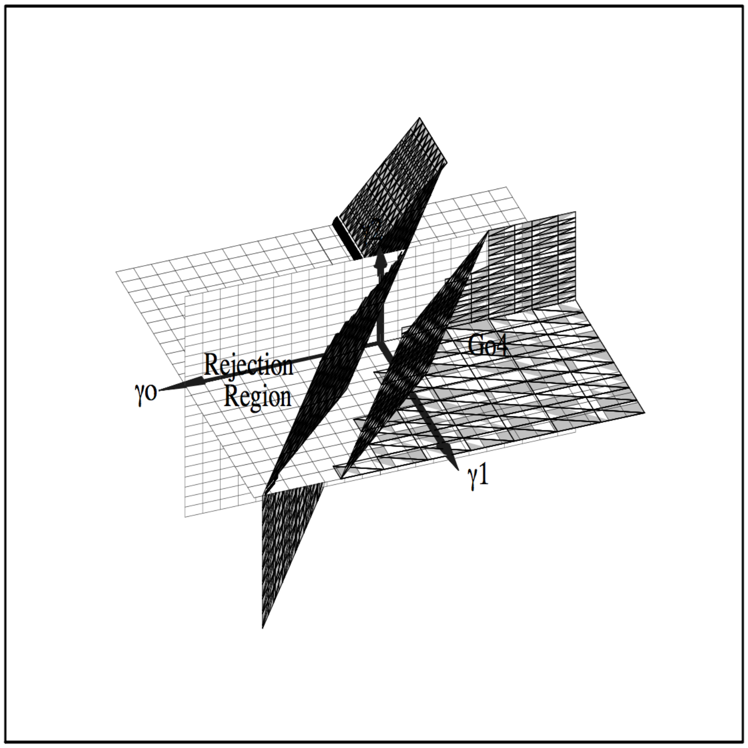

Figure 3.10: G_{04} and Rejection region

Chapter 4

Formulas for The Rejection Region

4.1 when $X_{01} > 0$ and $X_{02} > 0$

For hypothesis (3.2)

$$
\begin{aligned}
&G_{01} : 0 \leqslant r_2 \leqslant b_1 - c_1 r_0 - d_1 r_1, \quad r_1 \geqslant 0,\\
&G_{11} : r_1 \geqslant 0, r_2 \geqslant 0.
\end{aligned}
$$

The rejection region of r form,

$$1.\{\hat{r_0} \geqslant \frac{b_1}{c_1} + C_{\alpha/2}\sigma, \quad \hat{r_1} < 0, \quad \hat{r_2} < 0\},$$

$$2.\{(\hat{r_0} - \frac{b_1}{c_1})^2 + \hat{r_2}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad \hat{r_1} < 0, \quad 0 \leqslant \hat{r_2} < \frac{1}{c_1}\hat{r_0} - \frac{b_1}{c_1^2}\},$$

$$3.\{(\hat{r_0} - \frac{b_1}{c_1})^2 + \hat{r_1}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad 0 \leqslant \hat{r_1} < \frac{d_1}{c_1}\hat{r_0} - \frac{b_1 d_1}{c_1^2}, \quad \hat{r_2} < 0\},$$

$$4.\{(\hat{r_0} - \frac{b_1}{c_1})^2 + \hat{r_1}^2 + \hat{r_2}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad 0 \leqslant \hat{r_1} < \frac{d_1}{c_1}\hat{r_0} - \frac{b_1 d_1}{c_1^2}, \quad 0 \leqslant \hat{r_2} < \frac{1}{c_1}\hat{r_0} - \frac{b_1}{c_1^2}\},$$

$$5.\{c_1\hat{r_0} + \hat{r_2} \geqslant b_1 + \sqrt{1 + c_1^2}C_{\alpha/2}\sigma, \quad \hat{r_1} < 0, \quad \hat{r_2} \geqslant \frac{1}{c_1}\hat{r_0} - \frac{b_1}{c_1^2}\},$$

$$6.\{c_1\hat{r_0} + d_1\hat{r_1} - b_1 \geqslant \sqrt{c_1^2 + d_1^2}C_{\alpha/2}\sigma, \quad \hat{r_1} \geqslant \frac{d_1}{c_1}\hat{r_0} - \frac{b_1 d_1}{c_1^2}, \quad \hat{r_2} < 0\},$$

$$7.\{\hat{r_1}^2 + (\frac{1}{\sqrt{1 + c_1^2}}\hat{r_2} + \frac{c_1}{\sqrt{1 + c_1^2}}(\hat{r_0} - \frac{b_1}{c_1}))^2 \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{r_1} < \frac{c_1 d_1}{1 + c_1^2}\hat{r_0} + \frac{d_1}{1 + c_1^2}\hat{r_2} - \frac{b_1 d_1}{1 + c_1^2}, \quad \hat{r_2} \geqslant \frac{1}{c_1}\hat{r_0} - \frac{b_1}{c_1^2}\},$$

$$8.\{\hat{r_2}^2 + (\frac{d_1}{\sqrt{c_1^2 + d_1^2}}\hat{r_1} + \frac{c_1}{\sqrt{c_1^2 + d_1^2}}(\hat{r_0} - \frac{b_1}{c_1}))^2 \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{r_2} < \frac{c_1 d_1}{c_1^2 + d_1^2}\hat{r_0} + \frac{d_1}{c_1^2 + d_1^2}\hat{r_2} - \frac{b_1}{c_1^2 + d_1^2}, \quad \hat{r_1} \geqslant \frac{d_1}{c_1}\hat{r_0} - \frac{b_1 d_1}{c_1^2}\},$$

$$9.\{c_1\hat{r_0} + d_1\hat{r_1} + \hat{r_2} - b_1 \geqslant C_{\alpha/2}\sigma\sqrt{1 + c_1^2 + d_1^2},$$

$$\hat{r_1} \geqslant \max\{0, \frac{c_1 d_1}{1 + c_1^2}\hat{r_0} + \frac{d_1}{1 + c_1^2}\hat{r_2} - \frac{b_1 d_1}{1 + c_1^2}\},$$

$$\hat{r_2} \geqslant \max\{0, \frac{c_1 d_1}{c_1^2 + d_1^2}\hat{r_0} + \frac{d_1}{c_1^2 + d_1^2}\hat{r_2} - \frac{b_1}{c_1^2 + d_1^2}\}\}.$$

Then we have the transformation from r to β. Let $r_0 = \beta_0/\sqrt{1 + 1/n}$, $r_1 = S_{X_1}\beta_1$, $r_2 = S_{X_2}\beta_2$, then $\hat{r} = (\hat{r_0}, \hat{r_1}, \hat{r_2})' = (\frac{\hat{\beta_0}}{\sqrt{1+1/n}}, S_{X_1}\hat{\beta_1}, S_{X_2}\hat{\beta_2})' \sim N_3(r, \sigma^2 I)$ where $\hat{r}$ is the unrestricted MLE of r. And $b_1 = \frac{\mu S_{X_2}}{X_{02}}$, $c_1 = \frac{S_{X_2}}{X_{02}}\sqrt{1 + 1/n} > 0$, $d_1 = \frac{X_{01} S_{X_2}}{X_{02} S_{X_1}} > 0$.

Then we transform to the original variables,

rejection region of $\hat{\beta}$ form,

$$1.\{\hat{\beta}_0 \geqslant \mu + C_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}}, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < 0\},$$

$$2.\{\frac{1}{1+1/n}(\hat{\beta}_0-\mu)^2 + {S_{X_2}}^2{\hat{\beta}_2}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad \hat{\beta}_1 < 0, \quad 0 \leqslant \hat{\beta}_2 < \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$3.\{\frac{1}{1+1/n}(\hat{\beta}_0-\mu)^2 + {S_{X_1}}^2{\hat{\beta}_1}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad 0 \leqslant \hat{\beta}_1 < \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \quad \hat{\beta}_2 < 0\},$$

$$4.\{\frac{1}{1+1/n}(\hat{\beta}_0-\mu)^2 + {S_{X_1}}^2{\hat{\beta}_1}^2 + {S_{X_2}}^2{\hat{\beta}_2}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad 0 \leqslant \hat{\beta}_1 < \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu),$$

$$0 \leqslant \hat{\beta}_2 < \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$5.\{\hat{\beta}_0 + \hat{\beta}_2X_{02} > \mu + \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}C_{\alpha/2}\sigma, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 \geqslant \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$6.\{\hat{\beta}_0 + \hat{\beta}_1X_{01} > \mu + \sqrt{\frac{X_{01}^2}{{S_{X_1}}^2}+1+\frac{1}{n}}C_{\alpha/2}\sigma, \quad \hat{\beta}_1 \geqslant \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \quad \hat{\beta}_2 < 0\},$$

$$7.\{{S_{X_1}}^2{\hat{\beta}_1}^2 + (\hat{\beta}_0+\hat{\beta}_2X_{02}-\mu)^2\frac{1}{1+\frac{1}{n}+\frac{X_{02}^2}{{S_{X_2}}^2}} \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{\beta}_1 < \frac{X_{01}}{{S_{X_1}}^2}\frac{1}{(\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_2X_{02}-\mu), \quad \hat{\beta}_2 \geqslant \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$8.\{{S_{X_2}}^2{\hat{\beta}_2}^2 + (\hat{\beta}_0+\hat{\beta}_1X_{01}-l)^2\frac{1}{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}} \geqslant C_{\alpha/2}^2\sigma^2,$$

$$\hat{\beta}_1 \geqslant \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \quad 0 \leqslant \hat{\beta}_2 < \frac{X_{02}}{{S_{X_2}}^2}\frac{1}{(\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_1X_{01}-\mu)\},$$

$$9.\{\hat{\beta}_0+\hat{\beta}_1X_{01}+\hat{\beta}_2X_{02} > \mu + C_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}},$$

$$\hat{\beta}_1 \geqslant max\{0, \frac{X_{01}}{{S_{X_1}}^2}\frac{1}{(\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_2X_{02}-\mu)\},$$

$$\hat{\beta}_2 \geqslant max\{0, \frac{X_{02}}{{S_{X_2}}^2}\frac{1}{(\frac{X_{01}^2}{{S_{X_1}}^2}+1+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_1X_{01}-\mu)\}\}.$$

For hypothesis (3.6)

$$
\begin{aligned}
&H_{01} : r_2 \geqslant b'_1 - c_1 r_0 - d_1 r_1, \quad r_1 \geqslant 0,\, r_2 \geqslant 0 \\
&H_{11} : r_1 \geqslant 0 \quad r_2 \geqslant 0
\end{aligned}
$$

rejection region in terms of $\hat{r}$,

1. $\{c_1\hat{r_0} + \hat{r_2} \leqslant b'_1, \quad \hat{r_1} \leqslant \frac{-1}{d_1}\sqrt{1 + c_1^2 + d_1^2}Z_{\alpha/2}\sigma, \quad \hat{r_2} \geqslant 0\}$,
2. $\{c_1\hat{r_0} + d_1\hat{r_1} \leqslant b'_1, \quad \hat{r_1} \geqslant 0, \quad \hat{r_2} \leqslant -\sqrt{1 + c_1^2 + d_1^2}Z_{\alpha/2}\sigma\}$,
3. $\{\hat{r_0} < \frac{b'_1}{c_1}, \quad \hat{r_1} \leqslant 0, \quad \hat{r_2} \leqslant min\{0, -d_1\hat{r_1} - \sqrt{1 + c_1^2 + d_1^2}Z_{\alpha/2}\sigma\}$,
4. $\{\hat{r_0} + d_1\hat{r_1} + \hat{r_2} \leqslant b'_1 - \sqrt{1 + c_1^2 + d_1^2}Z_{\alpha/2}\sigma, \quad \text{otherwise}\}$.

Then we transform to the original variables,

rejection region in terms of $\hat{\beta}$,

1. $\{\hat{\beta}_0 + \hat{\beta}_2 X_{02} \leqslant \mu, \quad \hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}Z_{\alpha/2}\sigma, \quad \hat{\beta}_2 \geqslant 0\}$,
2. $\{\hat{\beta}_0 + \hat{\beta}_1 X_{01} \leqslant \mu, \quad \hat{\beta}_1 \geqslant 0, \quad \hat{\beta}_2 < -\frac{1}{X_{02}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}Z_{\alpha/2}\sigma\}$,
3. $\{\hat{\beta}_0 \leqslant \mu, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < min\{0, -\frac{1}{X_{02}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}Z_{\alpha/2}\sigma\}\}$,
4. $\{\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} \leqslant \mu - \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}Z_{\alpha/2}\sigma, \quad \text{otherwise}\}$.

4.2 when $X_{01} < 0$ and $X_{02} < 0$

rejection region lower bound in section 4.1.

1. $\{\hat{\beta_0}^* + \hat{\beta}_2 X_{02}^* \leqslant \mu^*, \quad \hat{\beta}_1 < -\frac{1}{X_{01}^*}\sqrt{1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X_1}}^2} + \frac{{X_{02}^*}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma, \quad \hat{\beta}_2 \geqslant 0\}$
2. $\{\hat{\beta_0}^* + \hat{\beta}_1 X_{01}^* \leqslant \mu^*, \quad \hat{\beta}_1 \geqslant 0, \quad \hat{\beta}_2 < -\frac{1}{X_{02}^*}\sqrt{1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X_1}}^2} + \frac{{X_{02}^*}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma\}$
3. $\{\hat{\beta_0}^* \leqslant \mu^*, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < min\{0, -\frac{1}{X_{02}^*}\sqrt{1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X_1}}^2} + \frac{{X_{02}^*}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma\}\}$
4. $\{\hat{\beta_0}^* + \hat{\beta}_1 X_{01}^* + \hat{\beta}_2 X_{02}^* \leqslant \mu^* - \sqrt{1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X_1}}^2} + \frac{{X_{02}^*}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma, \quad \text{otherwise}\}$

Tranform to $X_{01} < 0$ and $X_{02} < 0$ case,

$$\Rightarrow 1.\ \{\hat{\beta}_0 + \hat{\beta}_2 X_{02} \geqslant \mu, \quad \hat{\beta}_1 < \frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma, \quad \hat{\beta}_2 \geqslant 0\},$$

$$2.\ \{\hat{\beta}_0 + \hat{\beta}_1 X_{01} \geqslant \mu, \quad \hat{\beta}_1 \geqslant 0, \quad \hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma\},$$

$$3.\ \{\hat{\beta}_0 \geqslant \mu, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < min\{0, \frac{1}{X_{02}}\sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma\}\},$$

$$4.\ \{\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} \geqslant \mu + \sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} Z_{\alpha/2}\sigma, \quad \text{otherwise}\}.$$

rejection region upper bound in section 4.1,

$$1.\{\hat{\beta_0}^* \geqslant \mu^* + C_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}}, \quad \hat{\beta_1} < 0, \quad \hat{\beta_2} < 0\},$$

$$2.\{\frac{1}{1+1/n}(\hat{\beta_0}^* - \mu^*)^2 + S_{X_2}{}^2\hat{\beta_2}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad \hat{\beta_1} < 0, 0 \leqslant \hat{\beta_2} < \frac{\frac{1}{1+1/n}X_{02}^*}{S_{X_2}{}^2}(\hat{\beta_0}^* - \mu^*)\},$$

$$3.\{\frac{1}{1+1/n}(\hat{\beta_0}^* - \mu^*)^2 + S_{X_1}{}^2\hat{\beta_1}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad 0 \leqslant \hat{\beta_1} < \frac{\frac{1}{1+1/n}X_{01}^*}{S_{X_1}{}^2}(\hat{\beta_0}^* - \mu^*), \hat{\beta_2} < 0\},$$

$$4.\{\frac{1}{1+1/n}(\hat{\beta_0}^* - \mu^*)^2 + S_{X_1}{}^2\hat{\beta_1}^2 + S_{X_2}{}^2\hat{\beta_2}^2 \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{\beta_1} < \frac{\frac{1}{1+1/n}X_{01}^*}{S_{X_1}{}^2}(\hat{\beta_0}^* - \mu^*), 0 \leqslant \hat{\beta_2} < \frac{\frac{1}{1+1/n}X_{02}^*}{S_{X_2}{}^2}(\hat{\beta_0}^* - \mu^*)\},$$

$$5.\{\hat{\beta_0}^* + \hat{\beta_2}X_{02}^* \geqslant \mu^* + \sqrt{\frac{X_{02}^*{}^2}{S_{X_2}{}^2} + 1 + \frac{1}{n}}C_{\alpha/2}\sigma, \quad \hat{\beta_1} < 0, \quad \hat{\beta_2} \geqslant \frac{\frac{1}{1+1/n}X_{02}^*}{S_{X_2}{}^2}(\hat{\beta_0}^* - \mu^*)\},$$

$$6.\{\hat{\beta_0}^* + \hat{\beta_1}X_{01}^* \geqslant \mu^* + \sqrt{\frac{X_{01}^*{}^2}{S_{X_1}{}^2} + 1 + \frac{1}{n}}C_{\alpha/2}\sigma, \quad \hat{\beta_1} \geqslant \frac{\frac{1}{1+1/n}X_{01}^*}{S_{X_1}{}^2}(\hat{\beta_0}^* - \mu^*), \quad \hat{\beta_2} < 0\},$$

$$7.\{S_{X_1}{}^2\hat{\beta_1}^2 + (\hat{\beta_0}^* + \hat{\beta_2}X_{02}^* - \mu^*)^2\frac{1}{1+\frac{1}{n}+\frac{X_{02}^*{}^2}{S_{X_2}{}^2}} \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{\beta_1} < \frac{X_{01}^*}{S_{X_1}{}^2}\frac{1}{(\frac{X_{02}^*{}^2}{S_{X_2}{}^2}+1+\frac{1}{n})}(\hat{\beta_0}^* + \hat{\beta_2}X_{02}^* - \mu^*), \quad \hat{\beta_2} \geqslant \frac{\frac{1}{1+1/n}X_{02}^*}{S_{X_2}{}^2}(\hat{\beta_0}^* - \mu^*)\},$$

$$8.\{S_{X_2}{}^2\hat{\beta_2}^2 + (\hat{\beta_0}^* + \hat{\beta_1}X_{01}^* - l)^2\frac{1}{1+\frac{1}{n}+\frac{X_{01}^*{}^2}{S_{X_1}{}^2}} \geqslant C_{\alpha/2}^2\sigma^2,$$

$$\hat{\beta_1} \geqslant \frac{\frac{1}{1+1/n}X_{01}^*}{S_{X_1}{}^2}(\hat{\beta_0}^* - \mu^*), \quad 0 \leqslant \hat{\beta_2} < \frac{X_{02}^*}{S_{X_2}{}^2}\frac{1}{(\frac{X_{01}^*{}^2}{S_{X_1}{}^2}+\frac{1}{n})}(\hat{\beta_0}^* + \hat{\beta_1}X_{01}^* - \mu^*)\},$$

$$9.\{\hat{\beta_0}^* + \hat{\beta_1}X_{01}^* + \hat{\beta_2}X_{02}^* \geqslant \mu^* + C_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}+\frac{X_{01}^*{}^2}{S_{X_1}{}^2}+\frac{X_{02}^*{}^2}{S_{X_2}{}^2}},$$

$$\hat{\beta_1} \geqslant max\{0, \frac{X_{01}^*}{S_{X_1}{}^2}\frac{1}{(\frac{X_{02}^*{}^2}{S_{X_2}{}^2}+1+\frac{1}{n})}(\hat{\beta_0}^* + \hat{\beta_2}X_{02}^* - \mu^*)\},$$

$$\hat{\beta_2} \geqslant max\{0, \frac{X_{02}^*}{S_{X_2}{}^2}\frac{1}{(\frac{X_{01}^*{}^2}{S_{X_1}{}^2}+1+\frac{1}{n})}(\hat{\beta_0}^* + \hat{\beta_1}X_{01}^* - \mu^*)\}\}.$$

Transform to $X_{01} < 0$ and $X_{02} < 0$ case,

$$\Rightarrow 1.\{\hat{\beta}_0 \leqslant \mu - C_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}}, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < 0\},$$

$$2.\{\frac{1}{1+1/n}(\hat{\beta}_0-\mu)^2 + {S_{X_2}}^2{\hat{\beta}_2}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad \hat{\beta}_1 < 0, 0 \leqslant \hat{\beta}_2 < \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$3.\{\frac{1}{1+1/n}(\hat{\beta}_0-\mu)^2 + {S_{X_1}}^2{\hat{\beta}_1}^2 \geqslant C_{\alpha/2}^2\sigma^2, \quad 0 \leqslant \hat{\beta}_1 < \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \hat{\beta}_2 < 0\},$$

$$4.\{\frac{1}{1+1/n}(\hat{\beta}_0-\mu)^2 + {S_{X_1}}^2{\hat{\beta}_1}^2 + {S_{X_2}}^2{\hat{\beta}_2}^2 \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{\beta}_1 < \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \quad 0 \leqslant \hat{\beta}_2 < \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$5.\{\hat{\beta}_0 + \hat{\beta}_2 X_{02} \leqslant \mu - \sqrt{\frac{{X_{02}}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}C_{\alpha/2}\sigma, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 \geqslant \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$6.\{\hat{\beta}_0 + \hat{\beta}_1 X_{01} \leqslant \mu - \sqrt{\frac{{X_{01}}^2}{{S_{X_1}}^2}+1+\frac{1}{n}}C_{\alpha/2}\sigma, \quad \hat{\beta}_1 \geqslant \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \quad \hat{\beta}_2 < 0\},$$

$$7.\{{S_{X_1}}^2{\hat{\beta}_1}^2 + (\hat{\beta}_0+\hat{\beta}_2X_{02}-\mu)^2\frac{1}{1+\frac{1}{n}+\frac{{X_{02}}^2}{{S_{X_2}}^2}} \geqslant C_{\alpha/2}^2\sigma^2,$$

$$0 \leqslant \hat{\beta}_1 < \frac{X_{01}}{{S_{X_1}}^2}\frac{1}{(\frac{{X_{02}}^2}{{S_{X_2}}^2}+1+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_2X_{02}-\mu), \quad \hat{\beta}_2 \geqslant \frac{\frac{1}{1+1/n}X_{02}}{{S_{X_2}}^2}(\hat{\beta}_0-\mu)\},$$

$$8.\{{S_{X_2}}^2{\hat{\beta}_2}^2 + (\hat{\beta}_0+\hat{\beta}_1X_{01}-\mu)^2\frac{1}{1+\frac{1}{n}+\frac{{X_{01}}^2}{{S_{X_1}}^2}} \geqslant C_{\alpha/2}^2\sigma^2,$$

$$\hat{\beta}_1 \geqslant \frac{\frac{1}{1+1/n}X_{01}}{{S_{X_1}}^2}(\hat{\beta}_0-\mu), \quad 0 \leqslant \hat{\beta}_2 < \frac{X_{02}}{{S_{X_2}}^2}\frac{1}{(\frac{{X_{01}}^2}{{S_{X_1}}^2}+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_1X_{01}-\mu)\},$$

$$9.\{\hat{\beta}_0+\hat{\beta}_1X_{01}+\hat{\beta}_2X_{02} \leqslant \mu - C_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}+\frac{{X_{01}}^2}{{S_{X_1}}^2}+\frac{{X_{02}}^2}{{S_{X_2}}^2}},$$

$$\hat{\beta}_1 \geqslant max\{0, \frac{X_{01}}{{S_{X_1}}^2}\frac{1}{(\frac{{X_{02}}^2}{{S_{X_2}}^2}+1+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_2X_{02}-\mu)\},$$

$$\hat{\beta}_2 \geqslant max\{0, \frac{X_{02}}{{S_{X_2}}^2}\frac{1}{(\frac{{X_{01}}^2}{{S_{X_1}}^2}+1+\frac{1}{n})}(\hat{\beta}_0+\hat{\beta}_1X_{01}-\mu)\}\}.$$

4.3 when $X_{01} > 0$ and $X_{02} < 0$

For hypothesis (3.13)

$$G_{03} : r_2 \geqslant b_2 - c_2 r_0 - d_2 r_1, \quad r_1 \geqslant 0, r_2 \geqslant 0$$

$$G_{13} : r_1 \geqslant 0 \quad r_2 \geqslant 0.$$

rejection region lower bound of r form,

1. $\{\hat{r}_0 > \frac{b_2}{c_2}, \quad \hat{r}_1 < 0, \quad \hat{r}_2 < -\sqrt{1+c_2^2}E_{\alpha/2}\sigma\}$
2. $\{(c_2\hat{r}_0 - (b_2 + \sqrt{1+c_2^2}E_{\alpha/2}\sigma))^2 + (1+c_2^2)\hat{r}_1{}^2 \geqslant (1+c_2^2)E_{\alpha/2}^2\sigma^2,$
   $0 \leqslant \hat{r}_1 < \frac{c_2 d_2}{1+c_2^2}\hat{r}_0 + \frac{d_2}{1+c_2^2}\hat{r}_2 - \frac{b_2 d_2}{1+c_2^2}, \quad \hat{r}_2 < -\sqrt{1+c_2^2}E_{\alpha/2}\sigma\}$
3. $\{c_2\hat{r}_0 + d_2\hat{r}_1 \leqslant b_2 - (\sqrt{1+c_2^2+d_2^2} - \sqrt{1+c_2^2})E_{\alpha/2}\sigma,$
   $\hat{r}_1 \geqslant max\{0, \frac{c_2 d_2}{1+c_2^2}\hat{r}_0 + \frac{d_2}{1+c_2^2}\hat{r}_2 - \frac{b_2 d_2}{1+c_2^2}\}, \quad \hat{r}_2 < -\sqrt{1+c_2^2}E_{\alpha/2}\sigma\}$
4. $\{c_2\hat{r}_0 + \hat{r}_2 \leqslant b_2 - \sqrt{1+c_2^2}, \quad \hat{r}_1 < 0, \quad \hat{r}_2 > -\sqrt{1+c_2^2}E_{\alpha/2}\sigma\}$
5. $\{\hat{r}_1{}^2 + (\frac{1}{\sqrt{1+c_2^2}}\hat{r}_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\hat{r}_0 - \frac{b_2}{c_2}))^2 \geqslant E_{\alpha/2}^2\sigma^2,$
   $0 < \hat{r}_1 \leqslant \frac{c_2 d_2}{1+c_2^2}\hat{r}_0 + \frac{d_2}{1+c_2^2}\hat{r}_2 - \frac{b_2 d_2}{1+c_2^2}, \quad \hat{r}_2 \geqslant -\sqrt{1+c_2^2}E_{\alpha/2}\sigma\}$
6. $\{c_2\hat{r}_0 + d_2\hat{r}_1 + \hat{r}_2 \leqslant b_2 - \sqrt{1+c_2^2+d_2^2}E_{\alpha/2}\sigma,$
   $\hat{r}_1 \geqslant max\{0, \frac{c_2 d_2}{1+c_2^2}\hat{r}_0 + \frac{d_2}{1+c_2^2}\hat{r}_2 - \frac{b_2 d_2}{1+c_2^2}\}, \quad \hat{r}_2 \geqslant -\sqrt{1+c_2^2}E_{\alpha/2}\sigma\}$

We use the transformation from r to β. Let $r_0 = \beta_0/\sqrt{1+1/n}$, $r_1 = S_{X_1}\beta_1$, $r_2 = S_{X_2}\beta_2$, $b_2 = \frac{\mu S_{X_2}}{X_{02}}$, $\quad c_2 = \frac{S_{X_2}}{X_{02}}\sqrt{1+1/n} < 0, \quad d_2 = \frac{X_{01}S_{X_2}}{X_{02}S_{X_1}} < 0$,

1. $\{\hat{\beta}_0 > \mu, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < \dfrac{1}{X_{02}}\sqrt{\dfrac{X_{02}^2}{{S_{X_2}}^2} + 1 + \dfrac{1}{n}}E_{\alpha/2}\sigma\}$,

2. $$\{(\hat{\beta}_0 - \mu - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma)^2 + (\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n})S_{X_1}^2{\hat{\beta}_1}^2$$
$$\geqslant (\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n})E_{\alpha/2}^2\sigma^2,$$
$$0 \leqslant \hat{\beta}_1 < (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}},$$
$$\hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma\},$$

3. $$\{\hat{\beta}_0 + X_{01}\hat{\beta}_1 \geqslant \mu + (\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{{X_{01}^*}^2}{{S_{X_1}}^2} + 1 + \frac{1}{n}} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}})E_{\alpha/2}\sigma,$$
$$\hat{\beta}_1 \geqslant (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}, \quad \hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma\},$$

4. $$\{\hat{\beta}_0 + \hat{\beta}_2X_{02} \geqslant \mu + \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma, \quad \hat{\beta}_1 < 0,$$
$$\hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma\},$$

5. $$\{(\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)^2\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}} + {S_{X_1}}^2{\hat{\beta}_1}^2 \geqslant E_{\alpha/2}^2\sigma^2,$$
$$0 \leqslant \hat{\beta}_1 < (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}, \hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma\},$$

6. $$\{\hat{\beta}_0 + \hat{\beta}_1X_{01} + \hat{\beta}_2X_{02} \geqslant \mu + \sqrt{\frac{{X_{01}^*}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma,$$
$$\hat{\beta}_1 \geqslant (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}, \quad \hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}E_{\alpha/2}\sigma\}.$$

For hypothesis (3.15)

$$H_{03} : 0 \leqslant r_2 \leqslant b_2 - c_2 r_0 - d_2 r_1, \quad r_1 \geqslant 0,$$

$$H_{13} : r_1 \geqslant 0 \quad r_2 \geqslant 0.$$

rejection region upper bound of r form,

1. $\{\hat{r_0} \leqslant \frac{b_2'}{c_2}, \quad \hat{r_1} < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma, \quad \hat{r_2} < 0\}$,
2. $\{(c_2\hat{r_0} - (b_2' - \sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma))^2 + (c_2^2 + d_2^2)\hat{r_2}^2 \geqslant (c_2^2 + d_2^2) K_{\alpha/2}^2 \sigma^2$,
   $\hat{r_1} < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma \quad 0 \leqslant \hat{r_2} < \frac{c_2}{c_2^2 + d_2^2}\hat{r_0} + \frac{d_2}{c_2^2 + d_2^2}\hat{r_2} - \frac{b_2'}{c_2^2 + d_2^2}\}$,
3. $\{c_2\hat{r_0} + \hat{r_2} \geqslant b_2' + (\sqrt{1 + c_2^2 + d_2^2} - \sqrt{c_2^2 + d_2^2}) K_{\alpha/2}\sigma$,
   $\hat{r_1} < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma, \quad \hat{r_2} \geqslant \frac{c_2}{c_2^2 + d_2^2}\hat{r_0} + \frac{d_2}{c_2^2 + d_2^2}\hat{r_2} - \frac{b_2'}{c_2^2 + d_2^2}\}$,
4. $\{c_2\hat{r_0} + d_2\hat{r_1} \geqslant b_2' + \sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma, \quad \hat{r_1} \leqslant \frac{1}{d_2}\sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma, \quad \hat{r_2} < 0\}$,
5. $\{\hat{r_2}^2 + (\frac{d_2}{\sqrt{c_2^2 + d_2^2}}\hat{r_1} + \frac{c_2}{\sqrt{c_2^2 + d_2^2}}(\hat{r_0} - \frac{b_2'}{c_2}))^2 \geqslant K_{\alpha/2}^2 \sigma^2$,
   $\hat{r_1} \geqslant \frac{1}{d_2}\sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma, \quad 0 \leqslant \hat{r_2} < \frac{c_2}{c_2^2 + d_2^2}\hat{r_0} + \frac{d_2}{c_2^2 + d_2^2}\hat{r_1} - \frac{b_2'}{c_2^2 + d_2^2}\}$,
6. $\{c_2\hat{r_0} + d_2\hat{r_1} + \hat{r_2} \geqslant b_2' + \sqrt{1 + c_2^2 + d_2^2} K_{\alpha/2}\sigma$,
   $\hat{r_1} \geqslant \frac{1}{d_2}\sqrt{c_2^2 + d_2^2} K_{\alpha/2}\sigma, \quad \hat{r_2} \geqslant \frac{c_2}{c_2^2 + d_2^2}\hat{r_0} + \frac{d_2}{c_2^2 + d_2^2}\hat{r_1} - \frac{b_2'}{c_2^2 + d_2^2}\}$.

rejection region upper bound $\underline{\hat{\beta}}$

1. $\{\hat{\beta}_0 \leqslant \mu, \quad \hat{\beta}_1 < \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 < 0\},$

2. $\{(\hat{\beta}_0 - \mu - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma)^2 + (1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2})^2 {S_{X_2}}^2 {\hat{\beta}_2}^2$

   $\geqslant (1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2})K_{\alpha/2}^2\sigma^2,$

   $\hat{\beta}_1 < \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$

   $0 \leqslant \hat{\beta}_2 < (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}\},$

3. $\{\hat{\beta}_0 + X_{02}\hat{\beta}_2 \leqslant \mu - (\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}} - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}})K_{\alpha/2}\sigma,$

   $\hat{\beta}_1 < \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 \geqslant (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}\},$

4. $\{\hat{\beta}_0 + X_{01}\hat{\beta}_1 \leqslant \mu - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$

   $\hat{\beta}_1 \geqslant \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 < 0\},$

5. $\{{S_{X_2}}^2{\hat{\beta}_2}^2 + \frac{1}{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}(X_{01}\hat{\beta}_1 + \hat{\beta}_0 - \mu)^2 \geqslant K_{\alpha/2}^2\sigma^2,$

   $\hat{\beta}_1 \geqslant \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, 0 \leqslant \hat{\beta}_2 < (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}\},$

6. $\{\hat{\beta}_0 + X_{01}\hat{\beta}_1 + X_{02}\hat{\beta}_2 \leqslant \mu - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}K_{\alpha/2}\sigma,$

   $\hat{\beta}_1 \geqslant \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 \geqslant (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}\}.$

4.4 when $X_{01} < 0$ and $X_{02} > 0$

rejection region lower bound

1. $\{\hat{\beta}_0 \geqslant \mu, \quad \hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 < 0\},$

2. $\{(\hat{\beta}_0 - \mu + \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma)^2 + (1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2})^2 {S_{X_2}}^2 {\hat{\beta}_2}^2$

   $\geqslant (1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2})K_{\alpha/2}^2\sigma^2,$

   $\hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$

   $0 \leqslant \hat{\beta}_2 < (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}\},$

3. $\{\hat{\beta}_0 + X_{02}\hat{\beta}_2 \geqslant \mu + (\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}} - \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}})K_{\alpha/2}\sigma,$

   $\hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 \geqslant (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}\},$

4. $\{\hat{\beta}_0 + X_{01}\hat{\beta}_1 \geqslant \mu + \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$

   $\hat{\beta}_1 \geqslant -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 < 0\},$

5. $\{{S_{X_2}}^2{\hat{\beta}_2}^2 + \frac{1}{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}(X_{01}\hat{\beta}_1 + \hat{\beta}_0 - \mu)^2 \geqslant K_{\alpha/2}^2\sigma^2,$

   $\hat{\beta}_1 \geqslant -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$

   $0 \leqslant \hat{\beta}_2 < (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}\},$

6. $\{\hat{\beta}_0 + X_{01}\hat{\beta}_1 + X_{02}\hat{\beta}_2 \geqslant \mu + \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}K_{\alpha/2}\sigma,$

   $\hat{\beta}_1 \geqslant -\frac{1}{X_{01}}\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \hat{\beta}_2 \geqslant (\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \mu)\frac{-X_{02}}{{S_{X_2}}^2}\frac{1}{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}\}.$

rejection region upper bound

1. $$\{\hat{\beta}_0 > \mu, \quad \hat{\beta}_1 < 0, \quad \hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma\},$$

2. $$\{(\hat{\beta}_0 - \mu - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma)^2 + (\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n})S_{X_1}^2{\hat{\beta}_1}^2$$
$$\geqslant (\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n})E_{\alpha/2}^2\sigma^2,$$
$$0 \leqslant \hat{\beta}_1 < (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}, \hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma\},$$

3. $$\{\hat{\beta}_0 + X_{01}\hat{\beta}_1 \geqslant \mu + (\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{{X_{01}^*}^2}{{S_{X_1}}^2}+1+\frac{1}{n}} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}})E_{\alpha/2}\sigma,$$
$$\hat{\beta}_1 \geqslant (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}, \hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma\},$$

4. $$\{\hat{\beta}_0 + \hat{\beta}_2 X_{02} \geqslant \mu + \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma,$$
$$\hat{\beta}_1 < 0, \quad \hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma\},$$

5. $$\{(\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)^2\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}} + {S_{X_1}}^2{\hat{\beta}_1}^2 \geqslant E_{\alpha/2}^2\sigma^2,$$
$$0 \leqslant \hat{\beta}_1 < (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}, \hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma\},$$

6. $$\{\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} \geqslant \mu + \sqrt{\frac{{X_{01}^*}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma,$$
$$\hat{\beta}_1 \geqslant (\hat{\beta}_0 - \mu + X_{02}\hat{\beta}_2)\frac{X_{01}}{{S_{X_1}}^2}\frac{1}{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}, \quad \hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+1+\frac{1}{n}}E_{\alpha/2}\sigma\}.$$

Chapter 5

When σ^2 is Unknown

When σ^2 is unknown, recall the hypothesis test (3.1)

$$\begin{aligned} G_{0L} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \leqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0 \\ G_{0U} &: \beta_0 + \beta_1 X_{01} + \beta_2 X_{02} \geqslant \mu \qquad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0 \\ \text{and } G_1 &: \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0 \end{aligned} \tag{5.1}$$

Test G_{0L} against $G_1 - G_{0L}$. In terms of r, the test becomes to $G_{01} : 0 \leqslant r_2 \leqslant b_1 - c_1 r_0 - d_1 r_1, 0 \leqslant r_1$ against $G_{11} : r_1 \geqslant 0, r_2 \geqslant 0$. Then the LRT is

$$\Lambda = (\frac{\sigma^{*2}}{\bar{\sigma}^2})^n/2,$$

where σ^{*2} is the MLE of σ^2 under G_{01} and $\bar{\sigma}^2$ is the MLE of σ^2 under G_{11}. Hence the LRT reject G_{01} for large value of test statistic,

$$\lambda = 1 - \Lambda^2/n = 1 - \frac{vS^2 + ||\hat{r} - r^*||^2}{vS^2 + ||\hat{r} - \bar{r}||^2}.$$

Even we can change its form, Mukerjee and Tu (1995) have shown some difficulties of using this test statistic. Peiris and Bhattacharya (2016) proposed a test statistic T^2 which rejects G_{01} with large values, where $T^2 = \frac{||\bar{r}-r^*||^2}{S^2}$ and I use that for my forgoing discussion. The test reduced to the $\bar{\chi}^2_{01}$ test when σ^2 is known by replacing S^2 with σ^2. As Peiris and Bhattacharya (2016) shown, the least favorable null distribution of LRT is

$$Pr(LRT \leqslant C_\alpha | \hat{r} = L) = \sum_{i=0}^{3} w_i P(F_{i,n-3} \leqslant C^2_\alpha / i)$$

where $F_{i,n-3}$ is the F-distribution with i and n-3 degrees of freedom. If i=0, Let $P(F_{i,n-3} \leqslant C^2_{\alpha/2}/i) = 1$. And the critical values C_α can be computed using the equation

$$\alpha = w_1 P(F_{1,n-3} \leqslant C^2_\alpha) + w_2 P(F_{2,n-3} \leqslant C^2_\alpha/2) + w_3 P(F_{3,n-3} \leqslant C^2_\alpha/3)$$

Then the table of critical values are given by Peiris and Bhattacharya (2016) in appendix.

For other σ^2 unknown cases, the rejection regions are very similar to the corresponding σ^2 known cases with replacing σ with S and obtaining C_α from above equation. I replace the Z_α with $t_{v,\alpha}$ in the boundaries of rejection region and prediction intervals when $\{X_{01} > 0, X_{02} > 0\}$ and $\{X_{01} < 0, X_{02} < 0\}$.

Chapter 6

Prediction Intervals

In this chapter, I summarize all the formulas for the prediction intervals for all the possible sign constraints of X_{01} and X_{02}. When σ^2 is known, the formulas for prediction intervals have similar formats as the formulas when σ^2 is unknown. Hence, I only provide formulas of the prediction intervals when σ^2 is unknown.

6.1 when $X_{01} > 0$ and $X_{02} > 0$

Lower Boundaries,

$$L_P = 1.\ \hat{\beta}_0 - C_{\alpha/2} S \sqrt{1 + \frac{1}{n}} \qquad \text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 < 0,$$

$$2.\ \hat{\beta}_0 - \sqrt{(C_{\alpha/2}^2 S^2 - {S_{X_2}}^2 {\hat{\beta}_2}^2)(1 + \frac{1}{n})}$$

$$\text{if} \quad \hat{\beta}_1 < 0 \quad 0 \leqslant \hat{\beta}_2 < C_{\alpha/2} S \sqrt{\frac{X_{02}^2}{(1 + \frac{1}{n}){S_{X_2}}^4 + X_{02}^2 {S_{X_2}}^2}},$$

$$3.\ \hat{\beta}_0 - \sqrt{(C_{\alpha/2}^2 S^2 - {S_{X_1}}^2 {\hat{\beta}_1}^2)(1 + \frac{1}{n})}$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < C_{\alpha/2} S \sqrt{\frac{X_{01}^2}{(1 + \frac{1}{n}){S_{X_1}}^4 + X_{01}^2 {S_{X_1}}^2}} \quad \hat{\beta}_2 < 0,$$

4. $\hat{\beta}_0 - \sqrt{(C_{\alpha/2}^2 S^2 - {S_{X_1}}^2{\hat{\beta}_1}^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)(1 + \frac{1}{n})}$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < C_{\alpha/2}S\sqrt{\frac{X_{01}^2(1 - \frac{S_{X2}^2{\hat{\beta}_2}^2}{C_{\alpha/2}^2 S^2})}{(1+\frac{1}{n}){S_{X_1}}^4 + X_{01}^2{S_{X_1}}^2}}$$

$$0 \leqslant \hat{\beta}_2 < C_{\alpha/2}S\sqrt{\frac{X_{02}^2(1 - \frac{S_{X1}^2{\hat{\beta}_1}^2}{C_{\alpha/2}^2 S^2})}{(1+\frac{1}{n}){S_{X_2}}^4 + X_{02}^2{S_{X_2}}^2}},$$

5. $\hat{\beta}_0 + \hat{\beta}_2 X_{02} - C_{\alpha/2}S\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}}$

$$\text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 \geqslant C_{\alpha/2}S\sqrt{\frac{X_{02}^2}{(1+\frac{1}{n}){S_{X_2}}^4 + X_{02}^2{S_{X_2}}^2}},$$

6. $\hat{\beta}_0 + \hat{\beta}_1 X_{01} - C_{\alpha/2}S\sqrt{\frac{X_{01}^2}{{S_{X_1}}^2} + 1 + \frac{1}{n}}$

$$\text{if} \quad \hat{\beta}_1 \geqslant C_{\alpha/2}S\sqrt{\frac{X_{01}^2}{(1+\frac{1}{n}){S_{X_1}}^4 + X_{01}^2{S_{X_1}}^2}} \quad \hat{\beta}_2 < 0,$$

7. $\hat{\beta}_0 + \hat{\beta}_2 X_{02} - \sqrt{({C_{\alpha/2}}^2 S^2 - {S_{X_1}}^2{\hat{\beta}_1}^2)(1 + \frac{1}{n} + \frac{X_{02}^2}{{S_{X_2}}^2})}$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < \frac{C_{\alpha/2}S}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{X_{01}^2}(\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n})}}$$

$$\hat{\beta}_2 > C_{\alpha/2}S\sqrt{\frac{X_{02}^2(1 - \frac{S_{X1}^2{\hat{\beta}_1}^2}{C_{\alpha/2}^2 S^2})}{(1+\frac{1}{n}){S_{X_2}}^4 + X_{02}^2{S_{X_2}}^2}},$$

8. $\hat{\beta}_0 + \hat{\beta}_1 X_{01} - \sqrt{({C_{\alpha/2}}^2 S^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)(1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2})}$

$$\text{if} \quad \hat{\beta}_1 > C_{\alpha/2}S\sqrt{\frac{X_{01}^2(1 - \frac{S_{X2}^2{\hat{\beta}_2}^2}{C_{\alpha/2}^2 S^2})}{(1+\frac{1}{n}){S_{X_1}}^4 + X_{01}^2{S_{X_1}}^2}}$$

$$0 \leqslant \hat{\beta}_2 < \frac{C_{\alpha/2}\sigma}{\sqrt{{S_{X_2}}^2 + \frac{{S_{X_2}}^4}{X_{02}^2}(\frac{X_{01}^2}{{S_{X_1}}^2} + 1 + \frac{1}{n})}},$$

$$9.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} - C_{\alpha/2} S \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}$$

$$\text{if} \quad \hat{\beta}_1 > \frac{C_{\alpha/2} S}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{X_{01}^2}\left(\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n}\right)}}$$

$$\hat{\beta}_2 > \frac{C_{\alpha/2} \sigma}{\sqrt{{S_{X_2}}^2 + \frac{{S_{X_2}}^4}{X_{02}^2}\left(\frac{X_{01}^2}{{S_{X_1}}^2} + 1 + \frac{1}{n}\right)}}.$$

Upper Boundaries,

$$U_P = 1.\ \hat{\beta}_0 + \hat{\beta}_2 X_{02} \qquad \text{if} \quad \hat{\beta}_1 < -\frac{1}{X_{01}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}} T_{\alpha/2} S \quad \hat{\beta}_2 \geqslant 0,$$

$$2.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} \qquad \text{if} \quad \hat{\beta}_1 \geqslant 0 \quad \hat{\beta}_2 < -\frac{1}{X_{02}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}} T_{\alpha/2} S,$$

$$3.\ \hat{\beta}_0 \qquad \text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 < min\{0, -\frac{1}{X_{02}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}} T_{\alpha/2} S\},$$

$$4.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} + \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}} T_{\alpha/2} S \quad \text{otherwise.}$$

6.2 when $X_{01} < 0$ and $X_{02} < 0$

Lower Boundaries,

$$L_P = 1.\ \hat{\beta}_0 + \hat{\beta}_2 X_{02} \qquad \text{if} \quad \hat{\beta}_1 < \frac{1}{X_{01}} \sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} T_{\alpha/2} S \quad \hat{\beta}_2 \geqslant 0,$$

$$2.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} \qquad \text{if} \quad \hat{\beta}_1 \geqslant 0 \quad \hat{\beta}_2 < \frac{1}{X_{02}} \sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} T_{\alpha/2} S,$$

$$3.\ \hat{\beta}_0 \qquad \text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 < min\{0, \frac{1}{X_{02}} \sqrt{1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2}}^2}} T_{\alpha/2} S\},$$

$$4.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} - \sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{{X_{02}}^2}{{S_{X_2 68}}^2}} T_{\alpha/2} S \quad \text{otherwise.}$$

Upper Boundaries,

$$U_P = 1.\ \hat{\beta}_0 + C_{\alpha/2}S\sqrt{1+\frac{1}{n}} \qquad \text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 < 0$$

$$2.\ \hat{\beta}_0 + \sqrt{(C^2_{\alpha/2}S^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)(1+\frac{1}{n})}$$

$$\text{if} \quad \hat{\beta}_1 < 0 \quad 0 \leqslant \hat{\beta}_2 < C_{\alpha/2}S\sqrt{\frac{X^2_{02}}{(1+\frac{1}{n}){S_{X_2}}^4 + X^2_{02}{S_{X_2}}^2}}$$

$$3.\ \hat{\beta}_0 + \sqrt{(C^2_{\alpha/2}S^2 - {S_{X_1}}^2{\hat{\beta}_1}^2)(1+\frac{1}{n})}$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < C_{\alpha/2}S\sqrt{\frac{X^2_{01}}{(1+\frac{1}{n}){S_{X_1}}^4 + X^2_{01}{S_{X_1}}^2}} \quad \hat{\beta}_2 < 0$$

$$4.\ \hat{\beta}_0 + \sqrt{(C^2_{\alpha/2}S^2 - {S_{X_1}}^2{\hat{\beta}_1}^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)(1+\frac{1}{n})}$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < C_{\alpha/2}S\sqrt{\frac{X^2_{01}(1-\frac{S^2_{X2}{\hat{\beta}_2}^2}{C^2_{\alpha/2}S^2})}{(1+\frac{1}{n}){S_{X_1}}^4 + X^2_{01}{S_{X_1}}^2}}$$

$$0 \leqslant \hat{\beta}_2 < C_{\alpha/2}S\sqrt{\frac{X^2_{02}(1-\frac{S^2_{X1}{\hat{\beta}_1}^2}{C^2_{\alpha/2}S^2})}{(1+\frac{1}{n}){S_{X_2}}^4 + X^2_{02}{S_{X_2}}^2}}$$

$$5.\ \hat{\beta}_0 + \hat{\beta}_2 X_{02} + C_{\alpha/2}S\sqrt{\frac{X^2_{02}}{{S_{X_2}}^2} + 1 + \frac{1}{n}}$$

$$\text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 \geqslant C_{\alpha/2}S\sqrt{\frac{X^2_{02}}{(1+\frac{1}{n}){S_{X_2}}^4 + X^2_{02}{S_{X_2}}^2}}$$

$$6.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} + C_{\alpha/2}S\sqrt{\frac{X^2_{01}}{{S_{X_1}}^2} + 1 + \frac{1}{n}}$$

$$\text{if} \quad \hat{\beta}_1 \geqslant C_{\alpha/2}S\sqrt{\frac{X^2_{01}}{(1+\frac{1}{n}){S_{X_1}}^4 + X^2_{01}{S_{X_1}}^2}} \quad \hat{\beta}_2 < 0$$

7. $\hat{\beta}_0 + \hat{\beta}_2 X_{02} + \sqrt{({C_{\alpha/2}}^2 S^2 - {S_{X_1}}^2 {\hat{\beta}_1}^2)(1 + \frac{1}{n} + \frac{X_{02}^2}{{S_{X_2}}^2})}$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < \frac{C_{\alpha/2} S}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{X_{01}^2}(\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n})}}$$

$$\hat{\beta}_2 > C_{\alpha/2} S \sqrt{\frac{X_{02}^2 (1 - \frac{S_{X1}^2 {\hat{\beta}_1}^2}{C_{\alpha/2}^2 S^2})}{(1 + \frac{1}{n}){S_{X_2}}^4 + X_{02}^2 {S_{X_2}}^2}}$$

8. $\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \sqrt{({C_{\alpha/2}}^2 S^2 - {S_{X_2}}^2 {\hat{\beta}_2}^2)(1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2})}$

$$\text{if} \quad \hat{\beta}_1 > C_{\alpha/2} S \sqrt{\frac{X_{01}^2 (1 - \frac{S_{X2}^2 {\hat{\beta}_2}^2}{C_{\alpha/2}^2 S^2})}{(1 + \frac{1}{n}){S_{X_1}}^4 + X_{01}^2 {S_{X_1}}^2}}$$

$$0 \leqslant \hat{\beta}_2 < \frac{C_{\alpha/2} \sigma}{\sqrt{{S_{X_2}}^2 + \frac{{S_{X_2}}^4}{X_{02}^2}(\frac{X_{01}^2}{{S_{X_1}}^2} + 1 + \frac{1}{n})}}$$

9. $\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} + C_{\alpha/2} S \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}}$

$$\text{if} \quad \hat{\beta}_1 > \frac{C_{\alpha/2} S}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{X_{01}^2}(\frac{X_{02}^2}{{S_{X_2}}^2} + 1 + \frac{1}{n})}} \quad \hat{\beta}_2 > \frac{C_{\alpha/2} \sigma}{\sqrt{{S_{X_2}}^2 + \frac{{S_{X_2}}^4}{X_{02}^2}(\frac{X_{01}^2}{{S_{X_1}}^2} + 1 + \frac{1}{n})}}$$

6.3 when $X_{01} > 0$ and $X_{02} < 0$

Lower Boundaries,

$$L_p\!=1.\ \hat{\beta}_0 \qquad \text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

$$2.\ \hat{\beta}_0 - \sqrt{(\frac{X_{02}^2}{{S_{X2}}^2}+\frac{1}{n}+1)(E_{\alpha/2}^2S^2-{S_{X1}}^2{\hat{\beta}_1}^2)} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < \frac{X_{02}\hat{\beta}_2 + (\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2}} - \sqrt{1+\frac{1}{n}+\frac{X_{02}^2}{{S_{X2}}^2}})E_{\alpha/2}S}{\frac{{S_{X1}}^2}{X_{01}}(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2})}$$

$$\hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

$$3.\ \hat{\beta}_0 + \hat{\beta}_1X_{01} - (\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{1}{n}+1} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1})E_{\alpha/2}S$$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{X_{02}\hat{\beta}_2 + (\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2}} - \sqrt{1+\frac{1}{n}+\frac{X_{02}^2}{{S_{X2}}^2}})E_{\alpha/2}S}{\frac{{S_{X1}}^2}{X_{01}}(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2})}$$

$$\hat{\beta}_2 < \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

$$4.\ \hat{\beta}_0 + \hat{\beta}_2X_{02} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

$$\text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

$$5.\ \hat{\beta}_0 + \hat{\beta}_2X_{02} - \sqrt{(\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1)(E_{\alpha/2}^2S^2-{S_{X_1}}^2{\hat{\beta}_1}^2)}$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < \frac{E_{\alpha/2}S}{\sqrt{{S_{X_1}}^2+\frac{{S_{X_1}}^4}{X_{01}^2}(\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1)}}$$

$$\hat{\beta}_2 \geqslant \frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2}+\frac{1}{n}+1}E_{\alpha/2}S$$

6. $\hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{1}{n} + 1} E_{\alpha/2} S$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{E_{\alpha/2} S}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{X_{01}^2}(\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1)}} \qquad \hat{\beta}_2 \geqslant \frac{1}{X_{02}} \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1} E_{\alpha/2} S$$

Upper Boundaries,

U_P= 1. $\hat{\beta}_0 \qquad \text{if} \quad \hat{\beta}_1 < \frac{1}{X_{01}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}} K_{\alpha/2} S, \quad \hat{\beta}_2 < 0$

2. $\hat{\beta}_0 - \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}} K_{\alpha/2} S + \sqrt{(1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2})(K_{\alpha/2}^2 S^2 - {S_{X_2}}^2 {\hat{\beta}_2}^2)}$

$$\text{if} \quad \hat{\beta}_1 < \frac{1}{X_{01}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}} K_{\alpha/2} S$$

$$0 \leqslant \hat{\beta}_2 < \frac{(\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2}} - \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X1}}^2}}) K_{\alpha/2} S + X_{01} \hat{\beta}_1}{\frac{{S_{X2}}^2}{X_{02}}(1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2})}$$

3. $\hat{\beta}_0 + X_{02} \hat{\beta}_2 + (\sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2} + \frac{X_{02}^2}{{S_{X_2}}^2}} - \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}}) K_{\alpha/2} S$

$$\text{if} \quad \hat{\beta}_1 < \frac{1}{X_{01}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}} K_{\alpha/2} S$$

$$\hat{\beta}_2 \geqslant \frac{(\sqrt{\frac{1}{n} + \frac{X_{01}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2}} - \sqrt{\frac{1}{n} + \frac{X_{01}^2}{{S_{X1}}^2}}) K_{\alpha/2} S + X_{01} \hat{\beta}_1}{\frac{{S_{X2}}^2}{X_{02}}(\frac{1}{n} + \frac{X_{01}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2})}$$

4. $\hat{\beta}_0 + X_{01} \hat{\beta}_1 + \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}} K_{\alpha/2} S$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{1}{X_{01}} \sqrt{1 + \frac{1}{n} + \frac{X_{01}^2}{{S_{X_1}}^2}} K_{\alpha/2} S, \quad \hat{\beta}_2 < 0$$

$$5.\ \hat{\beta}_0 + X_{01}\hat{\beta}_1 + \sqrt{(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2})(K_{\alpha/2}^2 S^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)}$$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}S,$$

$$0 \leqslant \hat{\beta}_2 < \frac{1}{\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}}\frac{-X_{02}}{{S_{X_2}}^2}K_{\alpha/2}S$$

$$6.\ \hat{\beta}_0 + X_{01}\hat{\beta}_1 + X_{02}\hat{\beta}_2 + \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}K_{\alpha/2}S$$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}S,$$

$$\hat{\beta}_2 \geqslant \frac{1}{\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}}\frac{-X_{02}}{{S_{X_2}}^2}K_{\alpha/2}S$$

6.4 when $X_{01} < 0$ and $X_{02} > 0$

Lower Boundaries,

$$L_P = 1.\ \hat{\beta}_0 \qquad \text{if} \quad \hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 < 0$$

$$2.\ \hat{\beta}_0 + \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma - \sqrt{(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2})(K_{\alpha/2}^2\sigma^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)}$$

$$\text{if} \quad \hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma$$

$$0 \leqslant \hat{\beta}_2 < \frac{(\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}} - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2}})K_{\alpha/2}\sigma + X_{01}\hat{\beta}_1}{\frac{{S_{X2}}^2}{X_{02}}(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2})}$$

3. $$\hat{\beta}_0 + X_{02}\hat{\beta}_2 - \left(\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}} - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}\right)K_{\alpha/2}\sigma$$

$$\text{if}\quad \hat{\beta}_1 < -\frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma$$

$$\hat{\beta}_2 \geqslant \frac{(\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}} - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2}})K_{\alpha/2}\sigma + X_{01}\hat{\beta}_1}{\frac{{S_{X2}}^2}{X_{02}}(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X1}}^2}+\frac{X_{02}^2}{{S_{X2}}^2})}$$

4. $$\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma$$

$$\text{if}\quad \hat{\beta}_1 \geqslant -\frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma, \quad \hat{\beta}_2 < 0$$

5. $$\hat{\beta}_0 + X_{01}\hat{\beta}_1 - \sqrt{(1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2})(K_{\alpha/2}^2\sigma^2 - {S_{X_2}}^2{\hat{\beta}_2}^2)}$$

$$\text{if}\quad \hat{\beta}_1 \geqslant -\frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$$

$$0 \leqslant \hat{\beta}_2 < \frac{1}{\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}}\frac{X_{02}}{{S_{X_2}}^2}K_{\alpha/2}\sigma$$

6. $$\hat{\beta}_0 + X_{01}\hat{\beta}_1 + X_{02}\hat{\beta}_2 - \sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}K_{\alpha/2}\sigma$$

$$\text{if}\quad \hat{\beta}_1 \geqslant -\frac{1}{X_{01}}\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}}K_{\alpha/2}\sigma,$$

$$\hat{\beta}_2 \geqslant \frac{1}{\sqrt{1+\frac{1}{n}+\frac{X_{01}^2}{{S_{X_1}}^2}+\frac{X_{02}^2}{{S_{X_2}}^2}}}\frac{X_{02}}{{S_{X_2}}^2}K_{\alpha/2}\sigma$$

Upper Boundaries,

$$U_P = 1.\ \hat{\beta}_0 \qquad \text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 < -\frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n}}E_{\alpha/2}\sigma$$

$$2.\ \hat{\beta}_0 + \sqrt{(\frac{X_{02}^2}{{S_{X2}}^2} + \frac{1}{n} + 1)(E_{\alpha/2}^2\sigma^2 - {S_{X1}}^2{\hat{\beta}_1}^2)} + \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < \frac{X_{02}\hat{\beta}_2 + (\sqrt{1 + \frac{1}{n} + \frac{X_{02}^2}{{S_{X2}}^2}} - \sqrt{1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2}})E_{\alpha/2}\sigma}{\frac{{S_{X1}}^2}{X_{01}}(1 + \frac{1}{n} + \frac{{X_{01}^*}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2})}$$

$$\hat{\beta}_2 < -\frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$3.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} + (\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{1}{n} + 1} - \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1})E_{\alpha/2}\sigma$$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{X_{02}\hat{\beta}_2 + (\sqrt{1 + \frac{1}{n} + \frac{X_{02}^2}{{S_{X2}}^2}} - \sqrt{1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2}})E_{\alpha/2}\sigma}{\frac{{S_{X1}}^2}{X_{01}}(1 + \frac{1}{n} + \frac{{X_{01}}^2}{{S_{X1}}^2} + \frac{X_{02}^2}{{S_{X2}}^2})}$$

$$\hat{\beta}_2 < -\frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$4.\ \hat{\beta}_0 + \hat{\beta}_2 X_{02} + \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$\text{if} \quad \hat{\beta}_1 < 0 \quad \hat{\beta}_2 \geqslant -\frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$5.\ \hat{\beta}_0 + \hat{\beta}_2 X_{02} + \sqrt{(\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1)(E_{\alpha/2}^2\sigma^2 - {S_{X_1}}^2{\hat{\beta}_1}^2)}$$

$$\text{if} \quad 0 \leqslant \hat{\beta}_1 < \frac{E_{\alpha/2}\sigma}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{{X_{01}^*}^2}(\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1)}}$$

$$\hat{\beta}_2 \geqslant -\frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$6.\ \hat{\beta}_0 + \hat{\beta}_1 X_{01} + \hat{\beta}_2 X_{02} + \sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{{X_{01}}^2}{{S_{X_1}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

$$\text{if} \quad \hat{\beta}_1 \geqslant \frac{E_{\alpha/2}\sigma}{\sqrt{{S_{X_1}}^2 + \frac{{S_{X_1}}^4}{{X_{01}}^2}(\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1)}}, \hat{\beta}_2 \geqslant -\frac{1}{X_{02}}\sqrt{\frac{X_{02}^2}{{S_{X_2}}^2} + \frac{1}{n} + 1}E_{\alpha/2}\sigma$$

Chapter 7

Example

The data set of the Study on the Efficacy of Nosocomial Infection Control(SENIC Project) consists of a random sample of 113 hospitals selected from the original 338 hospitals surveyed. The varibales of interest are the length of patient stay in a hospital in days(Y) as a function of patient's age in years(X_1') and infection risk(X_2'). Then we normalized X_1' and X_2' to satisfy those model assumptions. First multiply the data matrix $X = (X_1, X_2)$ with the negative one half power of the variance-covariance matrix $R_2^{-1/2}$ to get transformed data matrix. Then the column means were substracted from each column to centralize the data. Finally the new data sets satisfy the model assumption, which are $\sum X_{1i} = 0$, $\sum X_{2i} = 0$ and $\sum X_{1i}X_{2i} = 0$. The following table gives the 95% prediction intervals for a new observation Y. We compare the length of prediction intervals to decide the efficiency of the prediction intervals.

7.1 When σ^2 is known

7.1.1 When $X_{01} > 0$ and $X_{02} > 0$

We choose the normalized data $(x_{01}, x_{01}) = (0.9, 0.6)$. Then obtain $c_1 = \frac{S_{X_2}}{x_{02}}\sqrt{1 + 1/n}$ and $d_1 = \frac{S_{X_2}x_{01}}{S_{X_1}x_{02}}$. Then $w_1 = |tan^{-1}(c_1)|$ and $w_2 = tan^{-1}(\frac{c_1}{d_1})$. Then I find $i = 12w_1/\pi$ and $j = 12w_2/\pi$. And I get the approximated range of i and j which are from three to five. So we can decide which part of critical values should be used. Then find critical value from Table A.1 by linear interpolation. Peiris and Bhattacharya (2016). In the formula of prediction upper bound, $Z_{\alpha/2} = 1.96$ whihc the 95 percent quantile of normal distribution. We consider the sample variance $S^2 = \sum(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2/(n - 3)$ as the known population variance σ^2. Then compare the length of restricted prediction interval with unrestricted one.

7.1.2 When $X_{01} < 0$ and $X_{02} < 0$

I choose the normalized data $(x_{01}, x_{01}) = (-3.0, -0.2)$. Then follow the similar precedures to obtain the restricted prediction interval.

7.1.3 When $X_{01} > 0$ and $X_{02} < 0$

I choose the normalized data $(x_{01}, x_{01}) = (2.0, -0.2)$. Then obtain $c_1 = \frac{S_{X_2}}{x_{02}}\sqrt{1 + 1/n}$ and $d_1 = \frac{S_{X_2}x_{01}}{S_{X_1}x_{02}}$. Then $\theta_1 = cos^{-1}(\frac{|d_2|}{\sqrt{1+c_2^2+d_2^2}})$ and $\theta_2 = cos^{-1}(\frac{1}{\sqrt{1+c_2^2+d_2^2}})$. Then I find $i_1 = 12w_1/\pi$ and $i_2 = 12w_2/\pi$. And I get the approximated range of i_1 and i_2 which are from three to five. So I can decide which part of critical values should be used. Let $j = 6$, then the chi-bar-square distribution where $X_{01} > 0$ and $X_{02} < 0$ is same as chi-bar-square distribution where $X_{01} > 0$ and $X_{02} > 0$. So I can use critival values from Table A.1. Then find critical value from the last row in Table

A.1 by linear interpolation. (Peiris and Bhattacharya, (2016)). I consider the sample variance $S^2 = \sum(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2/(n-3)$ as the known population variance σ^2.

Then compare the length of restricted prediction interval with unrestricted one.

7.1.4 When $X_{01} < 0$ and $X_{02} > 0$

I choose the normalized data $(x_{01}, x_{01}) = (-2.0, -1.9)$ and $(x_{01}, x_{01}) = (-0.5, -1.9)$. Then follow the similar precedures to obtain the restricted prediction interval.

When σ is known		
(X_{01}, X_{02})	Restricted	Unrestricted
(2.0 -0.2)	(6.921838, 9.445020)	(6.979653, 13.352616)
(-2.0, 1.9)	(11.535312, 14.158488)	(7.628682, 14.098118)
(-0.5, 1.9)	(11.534532 14.728804)	(8.220635, 14.588695)
(0.9, 0.6)	(7.042015, 13.731856)	(7.437521, 13.731856)
(-3.0, -0.2)	(5.108080, 12.314902)	(5.108080, 11.615757)

I find that lengths of prediction intervals strictly depended on the values of (x_{01}, x_{02}). So prediction intervals for the mean response should be calculated using both restricted and non-restricted formulas to find the most efficient result.

7.2 When σ^2 is unknown

7.2.1 When $X_{01} > 0$ and $X_{02} > 0$

We choose the normalized data $(x_{01}, x_{01}) = (0.9, 0.6)$. Then obtain $c_1 = \frac{S_{X_2}}{x_{02}}\sqrt{1 + 1/n}$ and $d_1 = \frac{S_{X_2} x_{01}}{S_{X_1} x_{02}}$. Then $w_1 = |tan^{-1}(c_1)|$ and $w_2 = tan^{-1}(\frac{c_1}{d_1})$. Then I find $i = 12w_1/\pi$ and $j = 12w_2/\pi$. And I get the approximated range of i and j which are from three

to five. So we can decide which part of critical values should be used. Then find critical value from Table A.2 by linear interpolation. (Peiris and Bhattacharya, (2016)). In the formula of prediction upper bound, $t_{\alpha/2,110}$ whihc the 95 percent quantile of t-distribution with 110 degrees of freedom. We consider the sample variance $S^2 = \sum(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2/(n-3)$ to replace σ^2.
Then compare the length of restricted prediction interval with unrestricted one.

7.2.2 When $X_{01} < 0$ and $X_{02} < 0$

I choose the normalized data $(x_{01}, x_{01}) = (-3.0, -0.2)$. Then follow the similar precedures to obtain the restricted prediction interval.

7.2.3 When $X_{01} > 0$ and $X_{02} < 0$

I choose the normalized data $(x_{01}, x_{01}) = (2.0, -0.2)$. Then obtain $c_1 = \frac{S_{X_2}}{x_{02}}\sqrt{1+1/n}$ and $d_1 = \frac{S_{X_2}x_{01}}{S_{X_1}x_{02}}$. Then $\theta_1 = cos^{-1}(\frac{|d_2|}{\sqrt{1+c_2^2+d_2^2}})$ and $\theta_2 = cos^{-1}(\frac{1}{\sqrt{1+c_2^2+d_2^2}})$. Then I find $i_1 = 12w_1/\pi$ and $i_2 = 12w_2/\pi$. Let $j = 6$, I use F-distribution to replace chi-square distribuion in least favorable null distribution. then the "F-bar distribution" where $X_{01} > 0$ and $X_{02} < 0$ is same as "F-bar distribution" where $X_{01} > 0$ and $X_{02} > 0$. So I can use critival values from Table A.2. Then find critical value from the last column in Table A.2 by linear interpolation. (Peiris and Bhattacharya (2016)). And the sample variance $S^2 = \sum(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2/(n-3)$
Then compare the length of restricted prediction interval with unrestricted one.

7.2.4 When $X_{01} < 0$ and $X_{02} > 0$

I choose the normalized data $(x_{01}, x_{01}) = (-2.0, -1.9)$ and $(x_{01}, x_{01}) = (-0.5, -1.9)$. Then follow the similar precedures to obtain the restricted prediction interval.

When σ is unknown		
(X_{01}, X_{02})	Restricted	Unrestricted
(2.0 -0.2)	(6.885129 9.445026)	(6.944209 13.388060)
(-2.0, 1.9)	(11.534747 14.195788)	(7.592701, 14.134098)
(-0.5, 1.9)	(11.533960 14.767218)	(8.185218, 14.624112)
(0.9, 0.6)	(6.998580, 13.766863)	(7.402514, 13.766863)
(-3.0, -0.2)	(5.071886, 12.366717)	(5.071886, 11.651951)

We find that lengths of prediction intervals strictly depended on the values of (x_{01}, x_{02}). In some cases, our new prediction intervals work better that original ones. So prediction intervals for the mean response can be calculated using both restricted and non-restricted formulas to find the most efficient result.

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Appendix A

R Codes in Example

This is the R code for σ^2 known case. For the comparison, we assum σ^2 equal S^2.

```
library(”expm”)#this is for sigma known case
#import data
SENIC <- read.table(”~/Google Drive/Thesis/SENIC.txt”, header=FALSE,col.names =c(”ID”,”stay”,”Age”,”Risk”,”culturing_ratio”,”X-ray_ratio”,”beds”,”affiliation”,”region”,”daily_census”,”nurses”,”facilities_services”))
Y=SENIC[,2]
X1=SENIC[,3]
X2=SENIC[,4]
#assumption
n=length(X1)
#normalized
X=cbind(X1,X2)
R=matrix(c(var(X1),cov(X1,X2),cov(X2,X1),var(X2)),nrow=2,ncol=2)
newX=t(solve(sqrtm(R))%*%t(X))
newx1=newX[,1]-mean(newX[,1])
```

```
newx2=newX[,2]-mean(newX[,2])
sum(newx1*newx2)#assumption no.3
x1=newx1;x2=newx2
olse=lm(Y~x1+x2)
summary(olse)
S=sqrt(sum((olse$residuals)^2)/(n-3))  #assume sigma=S
Sx1=sqrt(sum(x1^2));Sx2=sqrt(sum(x2^2))
b0=as.numeric(olse$coefficients[1])#9.648319
b1=as.numeric(olse$coefficients[2])#0.08068539   >0
b2=as.numeric(olse$coefficients[3])#0.7601274    >0
#new data
x01=c(-3,-2,-0.5,0.4,0.9,2);x02=c(-3,-1.5,-0.2,0.6,1.9,3)#better be in
    the range
Tab=matrix(data=NA,nrow=length(x01)*length(x02),ncol=6)
#Ca where alpha=0.025
#check the range of i,j. It is a part of critical values table,
#sigma known, use normal-distribution for boundary
CV=matrix(c(2.411,2.357,2.290,2.208,2.357,2.300,2.229,2.142,
              2.290,2.229,2.155,2.063,2.2080,2.142,2.063,1.968),nrow=4,
                ncol=4)#sigma known Table A.1
EK=c(2.361,2.316,2.266,2.2080,2.142,2.063,1.968)# let j=6 Table A.1
for(a in 1:length(x01))
{
  for(b in 1:length(x02))
  {
    c=Sx2/x02[b]*sqrt(1+1/n);d=Sx2*x01[a]/(Sx1*x02[b])
    Tab[length(x01)*(a-1)+b,1]=x01[a]
    Tab[length(x01)*(a-1)+b,2]=x02[b]
    #unrestricted intervals(sigma known)
    I=rep(1,length(x1))
    NEWX=cbind(I,x1,x2)
```

```
    bhat=as.numeric(olse$coefficients)
    Xh=c(1,x01[a],x02[b])
    Yhat=t(Xh)%*%bhat    #estimator
    a2=(sum((olse$residuals)^2)/(n-3))   #sigma^2
    z.quantiles <- qnorm(0.975)   # normal quantile
    a2yhat=a2*(1+t(Xh)%*%solve(t(NEWX)%*%NEWX)%*%Xh)   #sigma^2{pred}
    Tab[length(x01)*(a-1)+b,3]=Yhat-z.quantiles*sqrt(a2yhat)
    Tab[length(x01)*(a-1)+b,4]=Yhat+z.quantiles*sqrt(a2yhat)
    if(x01[a]>0&x02[b]>0)
    {
      w1=abs(atan(c));w2=abs(atan(c/d))
      i=w1*12/pi;j=w2*12/pi
      #cv linear interpolation
      cv1=CV[floor(j)-2,floor(i)-2];cv2=CV[floor(j)-2,floor(i)-1];cv3=CV[floor(j)-1,floor(i)-2];cv4=CV[floor(j)-1,floor(i)-1]
      cv=cv4+(cv3-cv4)*(i-floor(i))+(cv2+(cv1-cv2)*(i-floor(i))-cv4+(cv3-cv4)*(i-floor(i)))*(j-floor(j))
      #lower bound
      if(b1<0&b2<0){
        Tab[length(x01)*(a-1)+b,5]=b0-cv*S*sqrt(1+1/n)
      }else if(b1<0&b2>0&b2-cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2)<0)){
        Tab[length(x01)*(a-1)+b,5]=b0-sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n))
      }else if(b1>0&b1-cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2<0){
        Tab[length(x01)*(a-1)+b,5]=b0-sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n))
      }else if(b1>0&b2>0&b1-sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2-sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))<0){
```

```
      Tab[length(x01)*(a-1)+b,5]=b0-sqrt((cv^2*S^2-Sx1^2*x01[a]^2-Sx2^2*x02[b]^2)*(1+1/n))
    }else if(b1<0&b2-cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-cv*S*sqrt(x02[b]^2/Sx2^2+1+1/n)
    }else if(b2<0&b1-cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-cv*S*sqrt(x01[a]^2/Sx1^2+1+1/n)
    }else if(b1>0&b1-cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))<0&b2-sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n+x02[b]^2/Sx2^2))
    }else if(b2>0&b2-cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))<0&b1-sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n+x01[a]^2/Sx1^2))
    }else if(b1-cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))>0&b2-cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-cv*S*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)
    }
    #upper bound
    if(b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S<0&b2>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]
```

```
      }else if(b2+1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S<0&b1>0){
        Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]
      }else if(b2+1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S<0&b1<0){
        Tab[length(x01)*(a-1)+b,6]=b0
      }else{
        Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S
      }
    }else if(x01[a]>0&x02[b]<0){
      w1=acos(abs(d)/sqrt(1+c^2+d^2));w2=acos(1/sqrt(1+c^2+d^2))
      i=w1*12/pi;j=w2*12/pi
      #cv linear interpolation
      ecv=EK[floor(i)+2]+(EK[floor(i)+1]-EK[floor(i)+2])*(i-floor(i))
      kcv=EK[floor(j)+2]+(EK[floor(j)+1]-EK[floor(j)+2])*(j-floor(j))
      #lower bound
      if(b1<0&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S<0){
        Tab[length(x01)*(a-1)+b,5]=b0
      }else if(b1>0&b1-(x02[b]*b2+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x01[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&b2-1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
        Tab[length(x01)*(a-1)+b,5]=b0-sqrt((x02[b]^2/Sx2^2+1+1/n)*(ecv^2*S^2-Sx1^2*b1^2))-sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S
      }else if(b1-(x02[b]*b2+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x02[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&b2-1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
        Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S
```

```
    }else if(b1<0&&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S
    }else if(b1>0&&b1-(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-sqrt((1+1/n+x02[b]^2/Sx2^2)*(ecv^2*S^2-Sx1^2*b1^2))
    }else if(b1-(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*ecv*S
    }
    #upper bound
    if(b2<0&&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0
    }else if(b2>0&&b2-(x01[a]*b1+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&&b1-1/x01[a]*sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((x01[a]^2/Sx1^2+1+1/n)*(kcv^2*S^2-Sx2^2*b2^2))-sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S
    }else if(b2-(x01[a]*b1+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx2^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&&b1-1/x01[a]*sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S
    }else if(b2<0&&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S
```

```
    }else if(b2>0&b2+(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+sqrt((1+1/n+x01[a]^2/Sx1^2)*(kcv^2*S^2-Sx2^2*b2^2))
    }else if(b2+(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*kcv*S
    }
  }else if(x01[a]<0&x02[b]<0){
    w1=abs(atan(c));w2=abs(atan(c/d))
    i=w1*12/pi;j=w2*12/pi
    #cv linear interpolation
    cv1=CV[floor(j)-2,floor(i)-2];cv2=CV[floor(j)-2,floor(i)-1];cv3=CV[floor(j)-1,floor(i)-2];cv4=CV[floor(j)-1,floor(i)-1]
    cv=cv4+(cv3-cv4)*(i-floor(i))+(cv2+(cv1-cv2)*(i-floor(i))-cv4+(cv3-cv4)*(i-floor(i)))*(j-floor(j))
    #upper bound
    if(b1<0&b2<0){
      Tab[length(x01)*(a-1)+b,6]=b0-cv*S*sqrt(1+1/n)
    }else if(b1<0&b2>0&b2+cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2)<0)){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n))
    }else if(b1>0&b1+cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n))
    }else if(b1>0&b2>0&b1+sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2+sqrt(x02[b]^2*abs(cv^2*S
```

```
      ^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((cv^2*S^2-Sx1^2*x01[a]^2-Sx2^2*x02[b]^2)*(1+1/n))
    }else if(b1<0&b2+cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+cv*S*sqrt(x02[b]^2/Sx2^2+1+1/n)
    }else if(b2<0&b1+cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+cv*S*sqrt(x01[a]^2/Sx1^2+1+1/n)
    }else if(b1>0&b1+cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))<0&b2+sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n+x02[b]^2/Sx2^2))
    }else if(b2>0&b2+cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))<0&b1+sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n+x01[a]^2/Sx1^2))
    }else if(b1+cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))>0&b2+cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+cv*S*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)
    }
    #lower bound
    if(b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S<0&b2>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]
```

```
    }else if(b2-1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S<0&b1>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]
    }else if(b2-1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S<0&b1<0){
      Tab[length(x01)*(a-1)+b,5]=b0
    }else{
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qnorm(0.975)*S
    }
  }else if(x01[a]<0&x02[b]>0){
    w1=acos(abs(d)/sqrt(1+c^2+d^2));w2=acos(1/sqrt(1+c^2+d^2))
    i=w1*12/pi;j=w2*12/pi
    #cv linear interpolation
    ecv=EK[floor(i)+2]+(EK[floor(i)+1]-EK[floor(i)+2])*(i-floor(i))
    kcv=EK[floor(j)+2]+(EK[floor(j)+1]-EK[floor(j)+2])*(j-floor(j))
    #upper bound
    if(b1<0&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0
    }else if(b1>0&b1-(x02[b]*b2-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x01[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&b2+1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((x02[b]^2/Sx2^2+1+1/n)*(ecv^2*S^2-Sx1^2*b1^2))+sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S
    }else if(b1-(x02[b]*b2-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x01[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&b2+1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S
```

```
    }else if(b1<0&&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+sqrt(1+1/n+x02[b]^2/Sx2
          ^2)*ecv*S
    }else if(b1>0&&b1+(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+
        x02[b]^2/Sx2^2)<0&&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*
        S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+sqrt((1+1/n+x02[b]^2/
          Sx2^2)*(ecv^2*S^2-Sx1^2*b1^2))
    }else if(b1+(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b
        ]^2/Sx2^2)>0&&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+sqrt(1+1/n+
          x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*ecv*S
    }
    #lower bound
    if(b2<0&&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,5]=b0
    }else if(b2>0&&b2-(x01[a]*b1-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/
        Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/
        n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&&b1+1/x01[a]*sqrt(x01[a]^2
        /Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,5]=b0-sqrt((x01[a]^2/Sx1^2+1+1/n)*(kcv
          ^2*S^2-Sx2^2*b2^2))+sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S
    }else if(b2-(x01[a]*b1-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)
        -sqrt(1+1/n+x01[a]^2/Sx2^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[
        a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&&b1+1/x01[a]*sqrt(x01[a]^2/Sx1
        ^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-(sqrt(1+1/n+x01[a]^2/
          Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S
    }else if(b2<0&&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-sqrt(1+1/n+x01[a]^2/Sx1
          ^2)*kcv*S
```

```
      }else if(b2>0&b2-(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
        Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-sqrt((1+1/n+x01[a]^2/Sx1^2)*(kcv^2*S^2-Sx2^2*b2^2))
      }else if(b2-(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
        Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*kcv*S
      }
    }
  }
}
Tab
```

GeneralExamplePI(known).R

This is the R code for σ^2 unknown case.

```
library("expm")#this is for sigma unknown case
#import data
SENIC <- read.table("~/Google Drive/Thesis/SENIC.txt", header=FALSE,col.names =c("ID","stay","Age","Risk","culturing_ratio","X-ray_ratio","beds","affiliation","region","daily_census","nurses","facilities_services"))
Y=SENIC[,2]
X1=SENIC[,3]
X2=SENIC[,4]
#assumption
n=length(X1)
#normalized
X=cbind(X1,X2)
R=matrix(c(var(X1),cov(X1,X2),cov(X2,X1),var(X2)),nrow=2,ncol=2)
```

```
newX=t(solve(sqrtm(R))%*%t(X))
newx1=newX[,1]-mean(newX[,1])
newx2=newX[,2]-mean(newX[,2])
sum(newx1*newx2)#assumption no.3
x1=newx1;x2=newx2
olse=lm(Y~x1+x2)
summary(olse)
S=sqrt(sum((olse$residuals)^2)/(n-3))
Sx1=sqrt(sum(x1^2));Sx2=sqrt(sum(x2^2))
b0=as.numeric(olse$coefficients[1])#9.648319
b1=as.numeric(olse$coefficients[2])#0.08068539    >0
b2=as.numeric(olse$coefficients[3])#0.7601274     >0
#new data
x01=c(-3,-2,-0.5,0.4,0.9,2);x02=c(-3,-1.5,-0.2,0.6,1.9,3)#better be in the range
Tab=matrix(data=NA,nrow=length(x01)*length(x02),ncol=6)
#Ca where alpha=0.025
#check the range of i,j. It is a part of critical values table,
#sigma unknown, use t-distribution for boundary when x
CV=matrix(c(2.444,2.388,2.320,2.236,2.388,2.329,2.257,
            2.167,2.320,2.257,2.181,2.087,2.236,2.167,2.087,1.990),nrow=4,ncol=4)#sigma unknown
EK=c(2.393,2.347,2.295,2.236,2.167,2.087,1.990)#v=110 let j=6 Table A.2
for(a in 1:length(x01))
{
 for(b in 1:length(x02))
  {
    c=Sx2/x02[b]*sqrt(1+1/n);d=Sx2*x01[a]/(Sx1*x02[b])
    Tab[length(x01)*(a-1)+b,1]=x01[a]
    Tab[length(x01)*(a-1)+b,2]=x02[b]
    #unrestricted intervals
```

```
  newdata = data.frame(x1=x01[a],x2=x02[b])
  predict(olse, newdata, interval="predict")
  Tab[length(x01)*(a-1)+b,3]=predict(olse, newdata, interval="predict")[2]
  Tab[length(x01)*(a-1)+b,4]=predict(olse, newdata, interval="predict")[3]
  if(x01[a]>0&x02[b]>0)
  {
    w1=abs(atan(c));w2=abs(atan(c/d))
    i=w1*12/pi;j=w2*12/pi
    #cv linear interpolation
    cv1=CV[floor(j)-2,floor(i)-2];cv2=CV[floor(j)-2,floor(i)-1];cv3=CV[floor(j)-1,floor(i)-2];cv4=CV[floor(j)-1,floor(i)-1]
    cv=cv4+(cv3-cv4)*(i-floor(i))+(cv2+(cv1-cv2)*(i-floor(i))-cv4+(cv3-cv4)*(i-floor(i)))*(j-floor(j))
    #lower bound
    if(b1<0&b2<0){
      Tab[length(x01)*(a-1)+b,5]=b0-cv*S*sqrt(1+1/n)
    }else if(b1<0&b2>0&b2-cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2)<0)){
      Tab[length(x01)*(a-1)+b,5]=b0-sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n))
    }else if(b1>0&b1-cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2<0){
      Tab[length(x01)*(a-1)+b,5]=b0-sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n))
    }else if(b1>0&b2>0&b1-sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2-sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))<0){
      Tab[length(x01)*(a-1)+b,5]=b0-sqrt((cv^2*S^2-Sx1^2*x01[a]^2-Sx2^2*x02[b]^2)*(1+1/n))
```

```
    }else if(b1<0&b2-cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-cv*S*sqrt(x02[b]^2/Sx2^2+1+1/n)
    }else if(b2<0&b1-cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-cv*S*sqrt(x01[a]^2/Sx1^2+1+1/n)
    }else if(b1>0&b1-cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))<0&b2-sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n+x02[b]^2/Sx2^2))
    }else if(b2>0&b2-cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))<0&b1-sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n+x01[a]^2/Sx1^2))
    }else if(b1-cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))>0&b2-cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-cv*S*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)
    }
    #upper bound
    if(b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S<0&b2>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]
    }else if(b2+1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S<0&b1>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]
```

```
      }else if(b2+1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S<0&b1<0){
        Tab[length(x01)*(a-1)+b,6]=b0
      }else{
        Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S
      }
    }else if(x01[a]>0&x02[b]<0){
      w1=acos(abs(d)/sqrt(1+c^2+d^2));w2=acos(1/sqrt(1+c^2+d^2))
      i=w1*12/pi;j=w2*12/pi
      #cv linear interpolation
      ecv=EK[floor(i)+2]+(EK[floor(i)+1]-EK[floor(i)+2])*(i-floor(i))
      kcv=EK[floor(j)+2]+(EK[floor(j)+1]-EK[floor(j)+2])*(j-floor(j))
      #lower bound
      if(b1<0&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S<0){
        Tab[length(x01)*(a-1)+b,5]=b0
      }else if(b1>0&b1-(x02[b]*b2+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x01[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&b2-1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
        Tab[length(x01)*(a-1)+b,5]=b0-sqrt((x02[b]^2/Sx2^2+1+1/n)*(ecv^2*S^2-Sx1^2*b1^2))-sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S
      }else if(b1-(x02[b]*b2+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x02[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&b2-1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
        Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S
      }else if(b1<0&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
        Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S
```

```
    }else if(b1>0&b1-(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-sqrt((1+1/n+x02[b]^2/Sx2^2)*(ecv^2*S^2-Sx1^2*b1^2))
    }else if(b1-(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&b2-1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*ecv*S
    }
    #upper bound
    if(b2<0&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0
    }else if(b2>0&b2-(x01[a]*b1+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&b1-1/x01[a]*sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((x01[a]^2/Sx1^2+1+1/n)*(kcv^2*S^2-Sx2^2*b2^2))-sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S
    }else if(b2-(x01[a]*b1+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx2^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&b1-1/x01[a]*sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S
    }else if(b2<0&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S
    }else if(b2>0&b2+(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
```

```
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+sqrt((1+1/n+x01[a]^2/Sx1^2)*(kcv^2*S^2-Sx2^2*b2^2))
    }else if(b2+(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*kcv*S
    }
  }else if(x01[a]<0&x02[b]<0){
    w1=abs(atan(c));w2=abs(atan(c/d))
    i=w1*12/pi;j=w2*12/pi
    #cv linear interpolation
    cv1=CV[floor(j)-2,floor(i)-2];cv2=CV[floor(j)-2,floor(i)-1];cv3=CV[floor(j)-1,floor(i)-2];cv4=CV[floor(j)-1,floor(i)-1]
    cv=cv4+(cv3-cv4)*(i-floor(i))+(cv2+(cv1-cv2)*(i-floor(i))-cv4+(cv3-cv4)*(i-floor(i)))*(j-floor(j))
    #upper bound
    if(b1<0&b2<0){
      Tab[length(x01)*(a-1)+b,6]=b0-cv*S*sqrt(1+1/n)
    }else if(b1<0&b2>0&b2+cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2)<0)){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n))
    }else if(b1>0&b1+cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n))
    }else if(b1>0&b2>0&b1+sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))<0&b2+sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((cv^2*S^2-Sx1^2*x01[a]^2-Sx2^2*x02[b]^2)*(1+1/n))
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    }else if(b1<0&b2+cv*S*sqrt(x02[b]^2/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+cv*S*sqrt(x02[b]^2/Sx2^2+1+1/n)
    }else if(b2<0&b1+cv*S*sqrt(x01[a]^2/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+cv*S*sqrt(x01[a]^2/Sx1^2+1+1/n)
    }else if(b1>0&b1+cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))<0&b2+sqrt(x02[b]^2*abs(cv^2*S^2-Sx1^2*b1^2)/((1+1/n)*Sx2^4+x02[b]^2*Sx2^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+sqrt((cv^2*S^2-Sx1^2*b1^2)*(1+1/n+x02[b]^2/Sx2^2))
    }else if(b2>0&b2+cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))<0&b1+sqrt(x01[a]^2*abs(cv^2*S^2-Sx2^2*b2^2)/((1+1/n)*Sx1^4+x01[a]^2*Sx1^2))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+sqrt((cv^2*S^2-Sx2^2*b2^2)*(1+1/n+x01[a]^2/Sx1^2))
    }else if(b1+cv*S/sqrt(Sx1^2+Sx1^4/x01[a]^2*(x02[b]^2/Sx2^2+1+1/n))>0&b2+cv*S/sqrt(Sx2^2+Sx2^4/x02[b]^2*(x01[a]^2/Sx1^2+1+1/n))>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+cv*S*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)
    }
    #lower bound
    if(b1-1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S<0&b2>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]
    }else if(b2-1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S<0&b1>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]
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    }else if(b2-1/x02[b]*sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S<0&b1<0){
      Tab[length(x01)*(a-1)+b,5]=b0
    }else{
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*qt(0.975,n-3)*S
    }
  }else if(x01[a]<0&x02[b]>0){
    w1=acos(abs(d)/sqrt(1+c^2+d^2));w2=acos(1/sqrt(1+c^2+d^2))
    i=w1*12/pi;j=w2*12/pi
    #cv linear interpolation
    ecv=EK[floor(i)+2]+(EK[floor(i)+1]-EK[floor(i)+2])*(i-floor(i))
    kcv=EK[floor(j)+2]+(EK[floor(j)+1]-EK[floor(j)+2])*(j-floor(j))
    #upper bound
    if(b1<0&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0
    }else if(b1>0&b1-(x02[b]*b2-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x01[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&b2+1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+sqrt((x02[b]^2/Sx2^2+1+1/n)*(ecv^2*S^2-Sx1^2*b1^2))+sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S
    }else if(b1-(x02[b]*b2-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S)/(Sx1^2/x01[a]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&b2+1/x02[b]*sqrt(x02[b]^2/Sx2^2+1+1/n)*ecv*S<0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x02[b]^2/Sx2^2))*ecv*S
    }else if(b1<0&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S
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    }else if(b1>0&b1+(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b2*x02[b]+sqrt((1+1/n+x02[b]^2/Sx2^2)*(ecv^2*S^2-Sx1^2*b1^2))
    }else if(b1+(x01[a]/Sx1^2*ecv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&b2+1/x02[b]*sqrt(1+1/n+x02[b]^2/Sx2^2)*ecv*S>0){
      Tab[length(x01)*(a-1)+b,6]=b0+b1*x01[a]+b2*x02[b]+sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*ecv*S
    }
    #lower bound
    if(b2<0&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,5]=b0
    }else if(b2>0&b2-(x01[a]*b1-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))<0&b1+1/x01[a]*sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,5]=b0-sqrt((x01[a]^2/Sx1^2+1+1/n)*(kcv^2*S^2-Sx2^2*b2^2))+sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S
    }else if(b2-(x01[a]*b1-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx2^2))*kcv*S)/(Sx2^2/x02[b]*(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2))>0&b1+1/x01[a]*sqrt(x01[a]^2/Sx1^2+1+1/n)*kcv*S<0){
      Tab[length(x01)*(a-1)+b,5]=b0+b2*x02[b]-(sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)-sqrt(1+1/n+x01[a]^2/Sx1^2))*kcv*S
    }else if(b2<0&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
      Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S
    }else if(b2>0&b2-(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)<0&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
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        Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]-sqrt((1+1/n+x01[a]^2/Sx1^2)*(kcv^2*S^2-Sx2^2*b2^2))
      }else if(b2-(x02[b]/Sx2^2*kcv*S)/sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)>0&b1+1/x01[a]*sqrt(1+1/n+x01[a]^2/Sx1^2)*kcv*S>0){
        Tab[length(x01)*(a-1)+b,5]=b0+b1*x01[a]+b2*x02[b]-sqrt(1+1/n+x01[a]^2/Sx1^2+x02[b]^2/Sx2^2)*kcv*S
      }
    }
  }
}
Tab
```

GeneralExamplePI.R