The Interplay Between Graph Theory and Delta Matroids

by
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Abstract

After Bouchet introduced the concept of delta matroid, it became a novel extension of matroid theory. Throughout the past decades, a strong connection between graph theory and delta matroid theory was developed. With the guidance of the article by Moffatt, we explored the concepts of delta matroids from a graph theorist's persepective.

Contents

Chapter 1

Introduction

In this project report, we give a summary of a journal article written by Iain Moffatt. The main focus of this report is to grasp the essence of some delta matroid theory and see how it is related to graph theory. We will first give a review of elementary graph theory, a definition of delta matroid. From there, we will demonstrate how we can construct delta matroids through various types of operations. Finally, we will explore some topological graph theory, and investigate its connection with delta matroid theory. Most of the concepts discussed in this report have similarities shared by these two seemingly different fields of mathematics.

In the original paper, Moffatt left out a bunch of exercises to the readers, aiming to help them understand the concepts of delta matroid theory without a prior knowledge. Throughout this report, we will give proof and explanation to the important results in delta matroid theory.

Chapter 2

Background

In this chapter, we lay out the necessary background in graph theory and matroid theory for our later discussions.

2.1 Graph Theory

Definition. A *graph* G is an ordered pair of disjoint sets (V, E) such that F is a subset of the set V^2 of unordered pairs of V. The set V is the set of *vertices*, usually denoted by $V(G)$, and E is the set of *edges*, denoted by $E(G)$. An edge $\{u, v\}$ is said to join the vertices u and v and is denoted by uv. The *order* of a graph G, denoted by $|G|$, is equal to $|V(G)|$.

Definition. We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V \subset V', E \subset E'$.

Example. The *empty graph* of order n is a graph G, where $|G| = n, E = \emptyset$.

Definition. The set of vertices adjacent to a vertex $x \in G$, is called the *neighborhood* of x, denoted by $\Gamma(x)$. The *degree* of x is $d(x) = |\Gamma(x)|$. If every vertex of graph G has degree k, we say G is $k - regular$.

Definition. The *adjacency matrix* $A = A(G) = (a_{ij})$ of a graph G is a $0 - 1$ matrix where $a_{ij} = 1$ if and only if $v_i v_j$ is an edge.

Example. The *cubical graph* Q_3 is a 3-regular graph, and C_4 is one of its subgraphs.

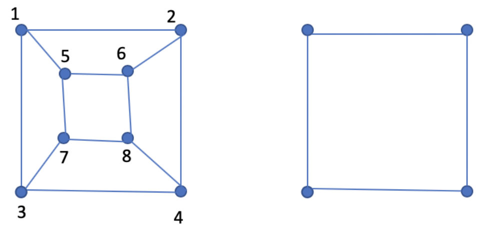

FIGURE 2.1: The graph Q_3, and one of its subgraphs C_4.

Example. The adjacency matrix for Q_3 (of the specific labeling in Figure 2.1) is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Notice that the adjacency of a graph is not unique. Depending on the numbering of vertices, one may end up with different matrices, however, for each adjacency matrix A of G, A is always symmetric.

A well-known problem related to graph theory is the Seven Bridges of Königsberg. The solution to this notable problem introduces one of the most important aspects of a graph G: whether G is *connected* and *Eulerian*. The *connectivity* of a graph G is closely related to real world application problems, for instance, network flow problems. We first give some basic definitions about connectivity of a graph G.

Definition. A *walk* W of a graph G is an alternating sequence of vertices and edges. A *path* P of a graph G is of the form

$$V(P) = \{v_0, v_1, v_2, \cdots, v_l\}, E(P) = \{v_0v_1, v_1v_2, \cdots, v_{l-1}v_l\}.$$

We call P a $v_0 - v_l$ path, and of course, it is also a $v_l - v_0$ path. P has *end vertives* v_0, v_l, and it is of *length* l. From this notation, we conclude that a path is a walk with distinct vertices. A walk is a *trail* if all the edges are distinct, moreover, a trail is called a *circuit* if its end vertices coincide, i.e., a *closed trail*. If a walk $W = v_0 v_1 \cdots v_l$ is such that it has length at least 3, and the vertices $v_i, 0 < i < l$ are all distinct from each other and v_0, then we say W is a *cycle*.

Example. In the labeled graph Q_3, here are some examples of walk, path, trail, circuit.
A walk $W = 1, 15, 5, 57, 7, 75, 5$.
A path $P = 1, 15, 5, 57, 7$.
A trail $T = 1, 12, 2, 26, 6$.
A circuit $C = 1, 15, 5, 57, 7, 73, 3, 31, 1$.

Definition. A graph G is *connected* if for every pair of distinct vertices $\{u, v\}$, a $u - v$ path exists.

Example. Q_3 is a connected graph but the following graph is not.

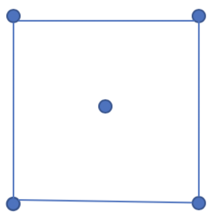

FIGURE 2.2: A disconnected graph. No path from any vertices to the one in the middle.

Now we establish the concept of *Eulerian Graph*, which will be reintroduced later when building connections with delta matroids.

Definition. A circuit in a graph G containing all the edges is said to be an *Euler circuit*, a trail containing all edges is an *Euler trail*. G is *Eulerian* if it has an Euler circuit.

Example. Q_3 does not have an Euler circuit, hence it is not Eulerian, but the complete graph K_5 is Eulerian. As a side note, a *complete graph* is a simple undirected (no direction on edges) graph where each pair of distinct vertices is connected by a unique edge.

This concept of Eulerian graph was introduced by Leonhard Euler in 1736, when he was invited to solve the the puzzle of Seven Bridges of Königsberg. The following theorem characterizes the basic property about Eulerian graphs.

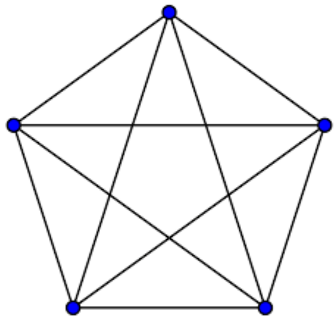

FIGURE 2.3: The complete graph K_5 is Eulerian.

Theorem (Euler's Theorem)[2.] A non-trivial connected graph has an Euler circuit if and only if each vertex has even degree.

This theorem is relatively intuitive to understand, in order for an Euler circuit to exist, whenever an edge 'comes' to a vertex, it must 'leave' the vertex at some point in the circuit, hence they pair up. Up to this point, we have covered most of the fundamental concepts about graph theory that we will use in this report, next we give a definition for delta matroids.

2.2 Delta Matroids

The definition of delta matroids is closely related to set theory, we assume a basic knowledge about sets and start with symmetric difference of two sets, which is less familiar to some readers.

Definition. Let X, Y be two sets, the *symmetric difference*, also known as the *disjunctive union*, of X and Y, denoted by $X\Delta Y$, is

$$X\Delta Y := (X \cup Y) \setminus (X \cap Y).$$

Example. Let $X = \{a, b, c\}, Y = \{b, c, d\}$, then $X\Delta Y = \{a, d\}$.

Definition. A *set system* is a pair $D = (E, \mathcal{F})$ where E is a set, and $\mathcal{F}$ is a collection of subsets of E. A set system is *proper* if $\mathcal{F}$ is not empty, *trivial* if E is empty.

Definition. A set system $D = (E, \mathcal{F})$ is said to satisfy the *Symmetric Exchange Axiom* if

$$(\forall X, Y \in \mathcal{F})(\forall u \in X\Delta Y)(\exists v \in X\Delta Y)(X\Delta\{u, v\} \in \mathcal{F}).$$

If $D = (E, \mathcal{F})$ satisfies the Symmetric Exchange Axiom, then we say D is a *delta matroid*, and E is the *ground set*, $\mathcal{F}$ is the *feasible set*. In this definition, we allow $u = v$.

One might wonder what exactly is a matroid, and its relation with delta matroid. In fact, matroid theory was introduced much earlier than delta matroid, but since in this report, we are aiming to use graph theory knowledge to study some properties about delta matroids, our focus is on delta matroids. However, there is a relation between matroids and delta matroids, as the names suggest.

Definition. A delta matroid is said to be a *matroid* if all of its feasible sets are of the same size.

When talking about matroids, we usually use the term *bases* in stead of *feasible sets*, therefore, we denote the set system $D = (E, \mathcal{B})$ as a matroid, rather than $\mathcal{F}$ for the collection of feasible sets. The definition given above is equivalent to the following:

Definition. The set system $D = (E, \mathcal{B})$ is a *matroid* if

(i). $\mathcal{B}$ is non-empty.

(ii). For distinct $A, B \in \mathcal{B}$, if $a \in A - B$, then $\exists b \in B - A$ with $(A - a) \cup b \in \mathcal{B}$.

Proposition. The two definitions of matroids are equivalent.

Proof. Suppose B_1, B_2 are distinct members of $\mathcal{B}$, without loss of generality, assume $|B_1| > |B_2|$ and that $|B_1 - B_2|$ is minimalized. Then from the second definition, we can choose $a \in B_1 - B_2$ and $b \in B_2 - B_1$ such that $(B_1 - a) \cup b \in \mathcal{B}$, then $|(B_1 - a) \cup b| = |B_1| > |B_2|$, but $|((B_1 - a) \cup b) - B_2| < |B_1 - B_2|$, contradicting to $|B_1 - B_2|$ being the minimum, hence $|B_1| = |B_2|$. □

Example. We will give one example of delta matroids, a more interesting example will be given in Chapter 4 where we combine graph theory and delta matroid together. Take a look at the adjacency matrix A of Q_3: we label the columns of this matrix 1-8. The ground set E is $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The feasible set $\mathcal{F}$ of E, is defined to be: for each column label, if it labels a basis of the column space of A, then it is in the feasible set. This is also a matroid, since all bases of the column space of a matrix must be of same size. The Symmetric Exchange

Axiom holds because it is analogous to Steinitz Exchange Lemma in linear algebra, which states that for a linearly independent subset of n elements, one may choose to replace some elements in the subset in a way that the resulting subset of n elements still span the entire vector space.

Chapter 3

Manipulations on Delta Matroids

In this chapter, we look into the cardinalities of the sets lying in the feasible sets of delta matroids. From there, we can start constructing delta matroids through some basic operations. These operations: twisting, deleting, and contracting, are fundamental concepts in delta matroid theory.

3.1 Characterizing the Feasible Sets

Definition. A delta matroid is said to be *even* if its feasible sets are either all of odd size, or all of even size. Otherwise, it is said to be *odd*. A delta matroid is said to be *normal* if the empty set is feasible.

Corollary. A matroid is an even delta matroid, since the feasible sets are equicardinal, i.e., the bases.

Definition. The *width* of a delta matroid $D = (E, \mathcal{F})$, we denote it by $w(D)$, is defined to be:

$$w(D) \coloneqq \max_{F \in \mathcal{F}} |F| - \min_{F \in \mathcal{F}} |F|.$$

We also denote $\mathcal{F}_{min}$ to be the collection of all feasible sets in $\mathcal{F}$ of minimum size, similarly, $\mathcal{F}_{max}$ to be the collection of all feasible sets in $\mathcal{F}$ of maximum size, and $\mathcal{F}_{min+k}$ to be the collection of all feasible sets that are exactly k larger than the sets in $\mathcal{F}_{min}$.

From the definition of delta matroids, we have the following interesting result regarding the gap between the collection of feasible sets of different sizes.

Theorem. If a delta matroid has a feasible set X of size k and a larger feasible set, then it has a feasible set Y of size $k+1$ or $k+2$.

This theorem can be verified using $|X \Delta Y|, X, Y \in \mathcal{F}$. From this theorem, one can see that for an even delta matroid, if X is a feasible and there is a larger feasible set, then the maximum gap between feasible sets with different cardinalities is 2. In fact, it is at most 2 for any delta matroids (could be 1 if the delta matroid is odd).

Corollary. Let $D = (E, \mathcal{F})$ be a delta matroid, then $D_{min} \coloneqq (E, \mathcal{F}_{min})$ and $D_{max} \coloneqq (E, \mathcal{F}_{max})$ are matroids, called the lower and upper matroids, respectively.

Proof. We show that D_{min} is a matroid, then proof for D_{max} is analogous.
Let $F_1, F_2 \in \mathcal{F}_{min}$, and let $u \in F_1 - F_2$, then $u \in F_1 \Delta F_2$, since F_1, F_2 are all feasible sets, then Symmetric Exchange Axiom must hold, meaning that $\exists v \in F_1 \Delta F_2$ with $F_1 \Delta \{u, v\} \in \mathcal{F}$. Since $|F_1| = |F_2|$, it must be the case that $v \in F_2 - F_1$, which is the definition of bases of matroid. □

3.2 Operations on Delta Matroids

We can construct many more delta matroids from a given one through operations. In this subsection, we give the three fundamental operations in delta matroid theory and show that delta matroids are closed under such operations. These operations serve as bridges between delta matroid theory and graph theory, where readers can find similar definitions in graph theory. In the later chapter, we will reintroduce these operations and demonstrate how they are applied to the field of graph theory.

3.2.1 Twisting

Definition. Let $D = (E, \mathcal{F})$ be a delta matroid, $A \subset E$. Let

$$\mathcal{F}' \coloneqq \{X \Delta A : X \in \mathcal{F}\}.$$

Then the *twist* of D by A, denoted $D * A$, is defined as

$$D * A := (D, \mathcal{F}').$$

The *dual* of D, denoted by D^*, is defined as $D^* := D * E$.

Proposition. Delta matroids are closed under twisting: if $D = (E, \mathcal{F})$ is a delta matroid, then $D * A$ is also a delta matroid for each $A \subset E$.

Proof. Let $F_1, F_2 \in \mathcal{F}$, by the definition of twist, we need to verify the Symmetric Exchange Axiom, but $(F_1 \Delta A)\Delta(F_2 \Delta A) = F_1 \Delta F_2$, which is an obvious property for the symmetric difference. Therefore, if F_1, F_2 are feasible, then $F_1 \Delta A, F_2 \Delta A$ are also feasible. □

The twist of a delta matroid is sometimes called a pivoting of a delta matroid, and from its name, some of the properties about a certain delta matroid is preserved under this operation, here are two interesting results.

Proposition. Let $D = (E, \mathcal{F})$ be an even delta matroid, then $D * A$ is also an even delta matroid for each $A \subset E$, i.e., twisting preserves the evenness of a delta matroid.

Proof. Let $F_1, F_2 \in \mathcal{F}$, if D is even, then F_1, F_2 must have all even or odd cardinalities, but in either case, $|F_1 \Delta F_2|$ is even. Let $A \subset E$, then again we have $(F_1 \Delta A)\Delta(F_2 \Delta A) = F_1 \Delta F_2$, which is of even cardinality. □

Proposition. Let $D = (E, \mathcal{F})$ be an even delta matroid, $A, B \subset E$, then $(D * A) * B = D * (A \Delta B)$.

Proof. This result can be proved using associativity of symmetric difference. Let $F \in \mathcal{F}$, then $(F \Delta A)\Delta B = F \Delta(A \Delta B)$. □

3.2.2 Deletion

Before we move on to the second operation, we first need some preliminary concepts.

Definition. Let $D = (E, \mathcal{F})$ be a delta matroid, then an element $e \in E$ is a *loop* if it is not in any feasible set, and a *coloop* if it is in every feasible set.

From this definition, the following proposition is an immediate result.

Proposition. If e is a loop in D, then it is a coloop in D^*.

Proof. Let $D = (E, \mathcal{F})$ be a delta matroid, $e \in E$ loop, then $\forall X \in \mathcal{F}, e \notin X$, then it must be $e \in X \Delta E$. □

Similarly, we have if e is a coloop in D, then it is a loop in D^*.

Definition. Let $D = (E, \mathcal{F})$ be a delta matroid, and $e \in E$, then the definition of D *delete* by e, denoted $D \backslash e$, depends on whether e is a coloop or not.

(i). When e is a coloop,

$$D \backslash e := (E \backslash e, \mathcal{F}'), \mathcal{F}' = \{X \backslash e : X \in \mathcal{F}, e \in X\}.$$

(ii). When e is not a coloop,

$$D \backslash e := (E \backslash e, \mathcal{F}'), \mathcal{F}' = \{X : X \in \mathcal{F}, e \notin X\}.$$

Proposition. Delta matroids are closed under deletion: if $D = (E, \mathcal{F})$ is a delta matroid, then $D \backslash e$ is also a delta matroid.

Proof. Notice that if e is a coloop, by definition, it is in every feasible sets. Let $F_1, F_2 \in \mathcal{F}$, then $e \notin F_1 \Delta F_2$, so removing e from the ground set has no impact on the Symmetric Exchange Axiom. Otherwise, if e is not a coloop, then for all pairs of feasible sets that do not contain e, Symmetric Exchange Axiom holds for them because they are feasible for the original delta matroid $D = (E, \mathcal{F})$. □

3.2.3 Contraction

Definition. Let $D = (E, \mathcal{F})$ be a delta matroid, and $e \in E$, then the definition of D *contract* by e, denoted D/e, depends on whether e is a loop or not.

(i). When e is a loop,

$$D/e := (E \backslash e, \mathcal{F}).$$

(ii). When e is not a loop,

$$D/e := (E\backslash e, \mathcal{F}'), \mathcal{F}' = \{X\backslash e : X \in \mathcal{F}, e \in X\}.$$

Proposition. Delta matroids are closed under contraction: if $D = (E, \mathcal{F})$ is a delta matroid, then D/e, is also a delta matroid.

Proof. Notice that if e is a loop, D and D/e has the same feasible sets since the feasible sets of D has nothing to do with e. Otherwise, if e is not a loop, we restrict to the feasible sets of D that contain e, similar to that of deletion, if all such sets contain e, e cannot be in their symmetric differences, thus the Symmetric Exchange Axiom holds naturally. □

The next identity explains the relationship between twisting, deletion, and contraction.

Proposition. Let $D = (E, \mathcal{F})$ be a delta matroid, $e \in E$, then

$$D/e = (D * e)\backslash e.$$

Proof. This is just an alternate way of writing the special cases for $D/e, D\backslash e$. That is, if $e \in E$ is a loop or a coloop, then we can set $D/e = D\backslash e$. □

With a similar argument, we have $D\backslash e = (D * e)/e$. This is a fundamental result in delta matroid theory, and we will revisit this identity in the next chapter when we introduce the Ribbon Graph Delta Matroids. We now give a simple example of all the concepts introduced in this chapter. Some more complicated examples will be discussed in the next chapter.

Example. Let $D = (E, \mathcal{F})$ be a delta matroid, with $E = \{a, b, c, d, e, f\}$, and

$$\mathcal{F} = \{\{b, d, e\}, \{b, d, f\}, \{c, d, e\}, \{c, d, f\}, \{d, e, f\}, \{a, b, c, d, e\},$$

$$\{a, b, c, d, f\}, \{a, b, d, e, f\}, \{b, c, d, e, f\}\}.$$

This is an even delta matroid, since all of its feasible sets are of odd cardinalities, but it is not normal since the $\emptyset \notin \mathcal{F}$.
The width of this delta matroid is, $w(D) = 5 - 3 = 2$.

Let $A \subset E, A = \{a, b\}$. Then $D * A$ has the same ground set, but with feasible set $\mathcal{F}_1$:

$$\mathcal{F}_1 = \{\{a, d, e\}, \{a, d, f\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, d, e, f\},$$

$$\{c, d, e\}, \{c, d, f\}, \{d, e, f\}, \{a, c, d, e, f\}\}.$$

Notice that the delta matroid $(E, \mathcal{F}_1)$ is again an even delta matroid.
$D\backslash\{b\}$ has ground set $\{a, e, d, e, f\}$, and since b is not a coloop, $D\backslash\{b\}$ has feasible set $\mathcal{F}_2$:

$$\mathcal{F}_2 = \{\{c, d, e\}, \{c, d, f\}, \{d, e, f\}, \{c, d, e\}\}.$$

$D/\{b\}$ has ground set $\{a, e, d, e, f\}$, and since b is not a loop, $D/\{b\}$ has feasible set $\mathcal{F}_3$:

$$\mathcal{F}_3 = \{\{d, e\}, \{d, f\}, \{a, c, d, e\}, \{a, c, d, f\}, \{a, d, e, f\}, \{c, d, e, f\}\}.$$

In fact, we can also compose operations together, then we can get a *minor* of a delta matroid, the result obtained from the original delta matroid through the operations of deletion, contraction, and twisting.

Chapter 4

Topological Graph Theory and Delta Matroids

4.1 Planarity

4.1.1 Planar Graphs

There are certain graphs that can be drawn in a plane without edges crossing, this type of graph is called a *planar* graph. Planar graphs are important in graph theory because they set up the foundation for some famous problems in this field of study, for example, the vertex coloring problem and the Hamiltonian cycle probelm. *Euler's Formula* helps us characterize the planar graphs.

Definition. A *face* of a graph is the connected component if omit the vertices and edges of a plane graph. For each graph, there is one unbounded face, and for the remaining bounded faces, each one is *bounded* by the set of edges in its closure.

Theorem (Euler's Formula). If a connected planar graph G has n vertices, m edges, and f faces, then $n - m + f$=2.

Example. The graph Q_3 is drawn in a way with no edges crossing, so it is planar. To verify the Euler's formula, we have $n = 8$, $m = 12$, and $f = 6$.

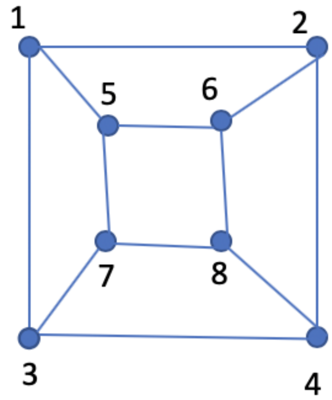

FIGURE 4.1: Q_3.

4.1.2 Embedding Graphs on Surfaces

The complete graph K_5 is not planar. Recall that a minor of a graph is the subgraph obtained from G by a sequence of edge contraction, edge deletion, and vertex deletion. It turns out that any graph that contains K_5 as a minor cannot be drawn in a plane without edges crossing, due to Kuratowski and Wagner. However, we can glue a 'handle' to the plane surface so that the some of the edges can pass through the handle in order to avoid crossing with other edges.

Definition. Let *genus* of a graph G is the an integer n that must be added to the plane so that the embedded graph does not cross. For example, the genus of a planar graph is 0.

A plane with one handle is the same as a torus (or, a 'donut) from a topologist's point of view, and thus we can embed nonplanar graphs in surface so that they do not have edges crossing each other.

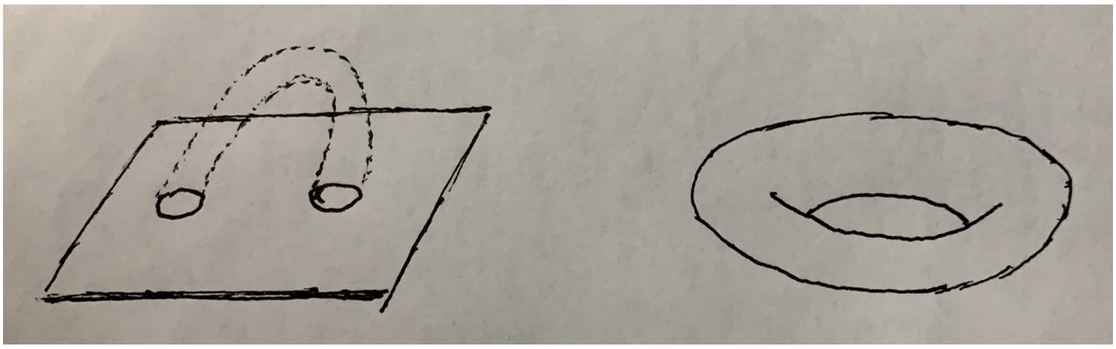

FIGURE 4.2: A Torus.

4.2 Cycle Matroids

In this section, we briefly discuss the concept of *cycle matroids*, they are well-known examples of matroids in a way that they make graph theory and matroid theory compatible with each other. From here, we can move on to the relationship between delta matroids and graphs embedded in surfaces. Before we go further into this concept, a brief review of connected components of a graph G, spanning trees/forests is laid out below. First, we introduce one of the most important concepts in graph theory, *tree*, as it appears very often in network related algorithms. We will follow the definition of a *tree* used by Bollobás'[2].

Definition. Let G be a graph, the following are equivalent:

(i). G a tree.

(ii). G is a minimal connected graph, that is, G connected and if $uv \in E(G)$, then $G - uv$ is disconnected.

(iii). G is maximal acyclic graph, that is, G is acyclic and if u and v are non-adjacent vertices of G, then $G + uv$ contains a cycle.

Definition. A *spanning tree* of a connected graph G is a tree that contains every vertex of G.

Recall that a maximal connected subgraph is a connected component of a graph. We define a *forest* to be a graph, all of whose connected components are trees. The definition of a *spanning forest* is similar to that of a spanning tree. Spanning tree problems are extremely useful in the design of a network, that is, we want to build a network that reaches to every hub node in the target, but at the same time, to minimize the cost of constructing connections between each two nodes. This is called the Minimum (Weight) Spanning Tree Problem (MST) in the area of combinatorics and graph theory. Here are examples to help digest the concepts mentioned above.

Example. A spanning tree of Q_3 is highlighted in red. There are many spanning trees of a given graph. If a graph G is complete with order n, then it has n^{n-2} spanning trees. Furthermore, if we assign weights to each edge in a graph, and we want to find a solution to the MST problem, there could be multiple optimal solutions using greedy algorithms.

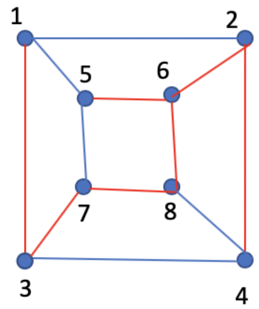

FIGURE 4.3: One spanning tree of Q_3.

We can prove by induction that a spanning tree of a connected graph G of order n has order $n-1$, and a spanning forest of a graph G of order n with k connected components has order $n-k$. Now we can give the definition of a *cycle matroid*.

Definition. Let $G = (V, E)$ be a graph (need not to be connected). Let

$$\mathcal{B} := \{F \subset E(G) : F \text{ is the edge set of a spanning forest of G}\}$$

Then $C(G) := (E, \mathcal{B})$ is the *cycle matroid* of G.

We observe that if two spanning forests (i.e., two members of $\mathcal{B}$) differ in more than one edge, then the Symmetric Exchange Axiom is easily satisfied. If two spanning forests differ in one edge, this is the case where allow $u = v$ in the definition of the axiom. Therefore, for a graph G, $C(G)$ is a delta matroid, but all spanning forests are of the same order, thus each member in the bases has the same size, which make $C(G)$ a matroid.

Example. Let G be the graph shown below, with labeled edges, we can define a cycle matroid of G. The cycle matroid $C(G)$ has ground set $E = \{1, 2, 3, 4, 5, 6\}$,

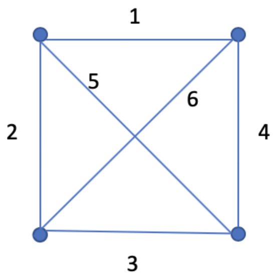

FIGURE 4.4: G with labelled edges

and feasible set

$$\mathcal{F} = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{2,4,5\},\{2,4,6\},\{2,3,4\}\}.$$

4.3 Ribbon Graphs

In this section, we introduce ribbon graphs, which are graphs in surfaces. We explore how this type of graphs tie up with the concept of delta matroids. First, we give the definition of a *ribbon graph*.

Definition. A ribbon graph $\mathbb{G} = (V, E)$ is a surface with boundary represented as the union of two sets of discs, a set V of *vertives* and a set E of *edges* such that

(i). The vertices and edges intersect in disjoint line segments.

(ii). Each such line segment lies on the boundary of precisely one vertex and precisely one edge.

(iii). Every edge contains exactly two such line segments.

This definition is relatively abstract, since this is in the field of topological graph theory, it might be easier to visualize the edges of a ribbon graph as rectangles with two opposite sides 'glued' to the vertices, just like the formation of Möbius band. Below is an example of how we form a ribbon graph from a graph drawing on a torus.

Example. Here is an example of drawing a ribbon graph from an embedded graph on a torus.

4.3.1 Operations on Edges of Ribbon Graphs

We have introduced some operations on delta matroids, now is time to see how they can be applied to ribbon graphs as well. We shall first discuss deletion of an edge in ribbon graphs, as it is the easiest to illustrate.

Definition. Let $\mathbb{G} = (V, E)$ be a ribbon graph, $v \in V, e \in E$, then $\mathbb{G}$ *delete* e, denoted by $\mathbb{G}\backslash e$ is the ribbon graph obtained from $\mathbb{G}$ by removing the edge e, $\mathbb{G}\backslash v$ is the ribbon graph obtained from $\mathbb{G}$ by removing the vertex v and all of its incident edges.

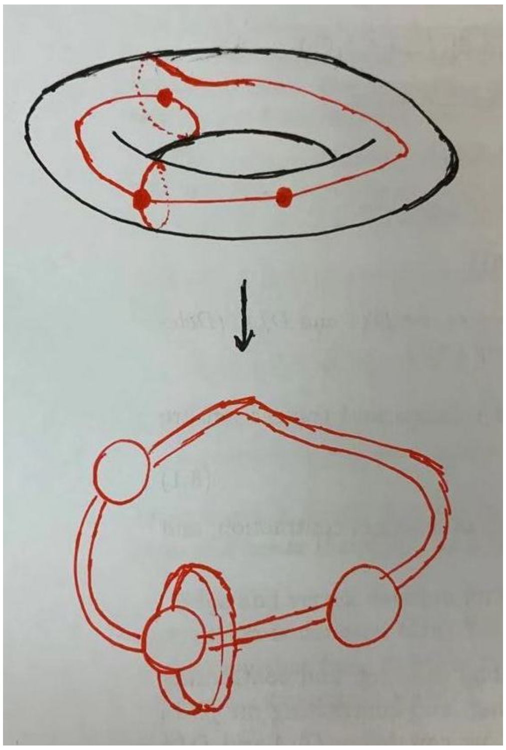

FIGURE 4.5: A Ribbon Graph.

This is a more or less straightforward definition for deletion, it can be applied to graphs in general. Before we move on to contraction, it is necessary that we talk about the *arrow presentation* of ribbon graphs first, as it gives a simplified way to draw ribbon graph. We summarize the procedure of drawing an arrow presentation in the following steps:

(i). Replace each vertex with a 'circle'.

(ii). For each 'rectangle' attached to the vertex, we define a 'flow direction' on the side that is glued to the vertex.

(iii). Follow this direction through the entire rectangle, mark the directions on the circle.

(iv). Repeat for all 'rectangles'.

Example. Here is an example of how to draw the arrow presentation of a ribbon graph.

With the help of arrow presentation of ribbon graphs, we can define *contraction* of a ribbon graph.

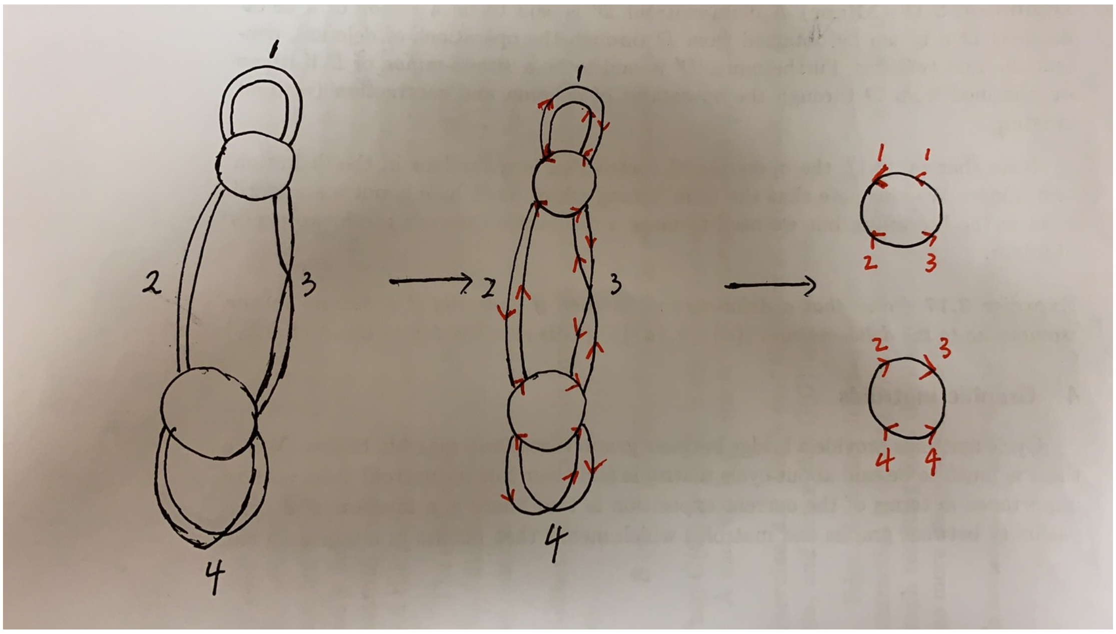

FIGURE 4.6: Arrow Representation of Ribbon Graph.

Definition. Let $\mathbb{G} = (V, E)$ be a ribbon graph, $e \in E$, then $\mathbb{G}$ *contract* e, denoted by $\mathbb{G}/e$ is the ribbon graph obtained by the following procedure:

(i). Use arrow presentation to describe $\mathbb{G}$.

(ii). Delete the e labelled arrows and the curves they originally lie on, add arc connecting the tips and tails of the arrows.

Example. Using the same example as the one in arrow presentation, we demonstrate how to contract edge 3 in the ribbon graph.

4.3.2 Partial Duals and Connection to Group Theory

Recall that in the section where we talked about faces of a graph G, in this section, we construct the *dual graph* of G as well as a *partial dual*. Moffatt provided a connection between operations on ribbon graphs and group theory. Later on we will use this result again to further explore its connection with Eulerian delta matroid.

The *dual* of a ribbon graph, $\mathbb{G}^*$, follows the same construction as that of a planar graph, we will skip the details and only give an example here. What's more interesting is the construction of *partial dual* with respect to e, $\mathbb{G}^e$. Again, we

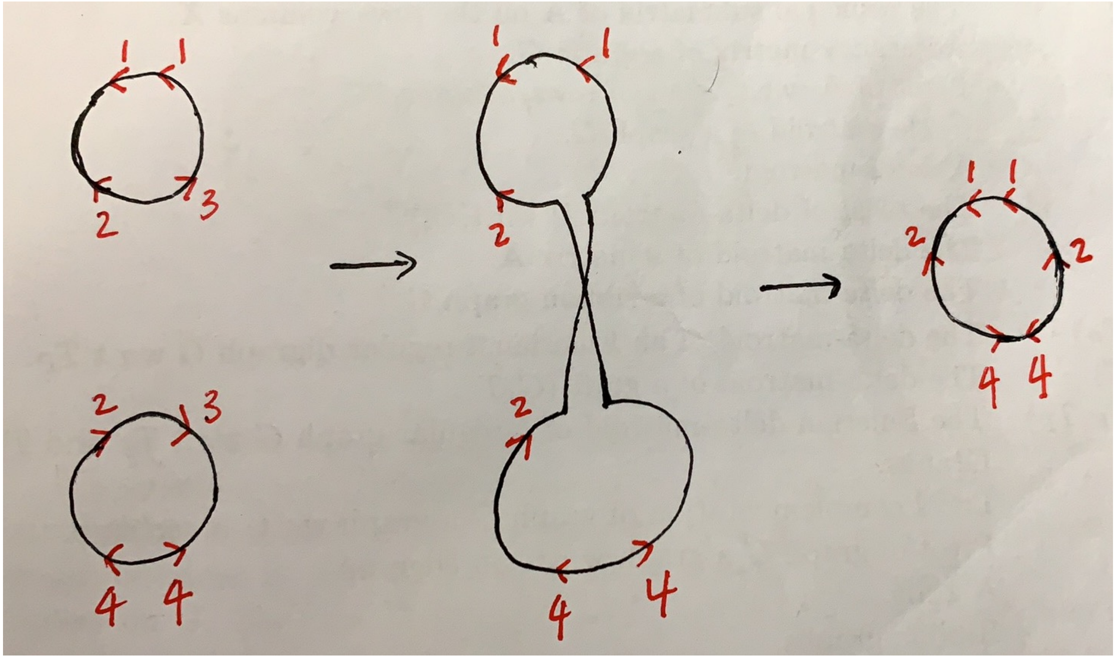

FIGURE 4.7: $\mathbb{G}/3$.

need the arrow representation of ribbon graphs and apply the 'splicing' procedure used by Moffatt. The illustration is shown below.

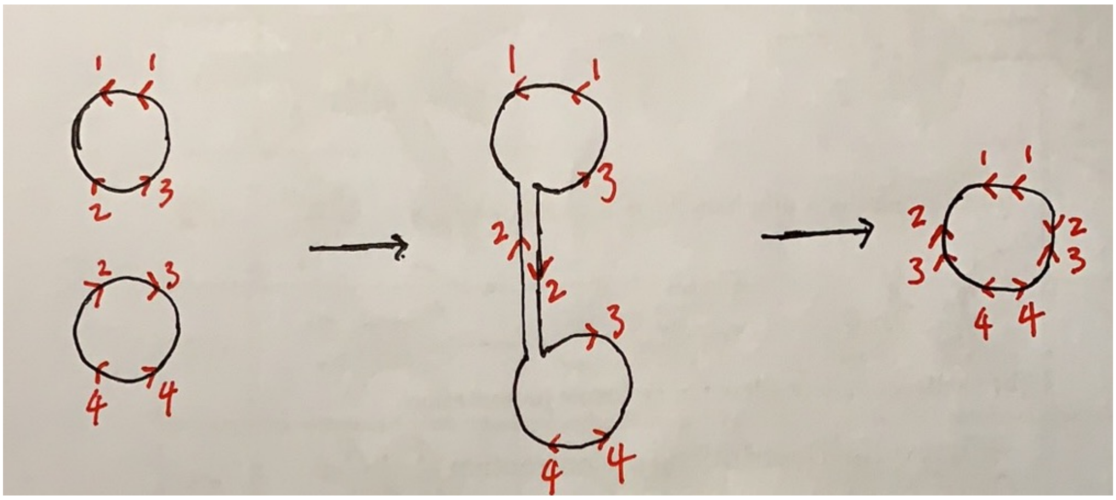

FIGURE 4.8: The partial dual $\mathbb{G}^2$.

Example.

It turns out that we can repeatedly construct the partial dual one edge at a time, in this way we can define a the partial dual of $\mathbb{G}$ with respect to a set of edges $A \subset E$. The following identity is useful:

$$\text{Let } A, B \subset E, \text{then } (\mathbb{G}^A)^B = \mathbb{G}^{A\Delta B}.$$

To see why this is true, first we know that the dual of a dual gives the original graph, hence, when applying partial duals to $\mathbb{G}$ with respect to a sequence of set of edges, we only consider those that are in the symmetric difference, since edges in the intersection of A, B will have no impact on the ribbon graph when perform partial dual twice. Furthermore, If we compare all the examples given so far, a familiar identity appears again:

Proposition. Let $\mathbb{G} = (V, E)$ be a ribbon graph, $e \in E$, then

$$\mathbb{G}/e = \mathbb{G}^e \backslash e.$$

One of the most appealing extension of ribbon graph theory is its relation to group theory, Moffatt left the following claim to the reader and we shall give a proof for this.

Proposition. Consider set S of pairs $(\mathbb{G}, e)$, where $(\mathbb{G}$ is a ribbon graph and e is an edge. Given the following two operations

$$\delta : (\mathbb{G}, e) \to (\mathbb{G}^e, e), \tau : (\mathbb{G}, e) \to (\mathbb{G}^{\tau(e)}, e)$$

where $\mathbb{G}^{\tau(e)}$ is obtained from $\mathbb{G}$ by adding a 'half-twist' to the edge e, i.e., reverse the direction of exactly one $e-$labelled arrow in an arrow presentation of $\mathbb{G}$. Two ribbon graphs are *twisted duals* if one can be obtained from the other by a sequence of operations δ, τ. Then these two operations induce an action of the symmetric group $\langle \delta, \tau | \delta^2, \tau^2, (\tau\delta)^3 \rangle$.

Proof. First, we need to verify that $\delta^2, \tau^2, (\tau\delta)^3$ indeed give the identity of the group. The following figures demonstrate these operations.

To see its connection with the symmetric group S_3, notice that the alternating group A_3 is a normal subgroup of S_3, so A_3 is cyclic, which means it is also abelian. Therefore, the factor group S_3/A_3 is a cyclic group of order 2, thus also abelian. Using this fact, one can see that all the factor groups of S_3 are abelian, if we can show that $\tau\delta \neq \delta\tau$, then we can conclude that δ, τ do not have any other operation in the group since they are not abelian and do not belong to any of the factor groups of S_3. There are many examples, the one illustrated in the figure above is one of them. □

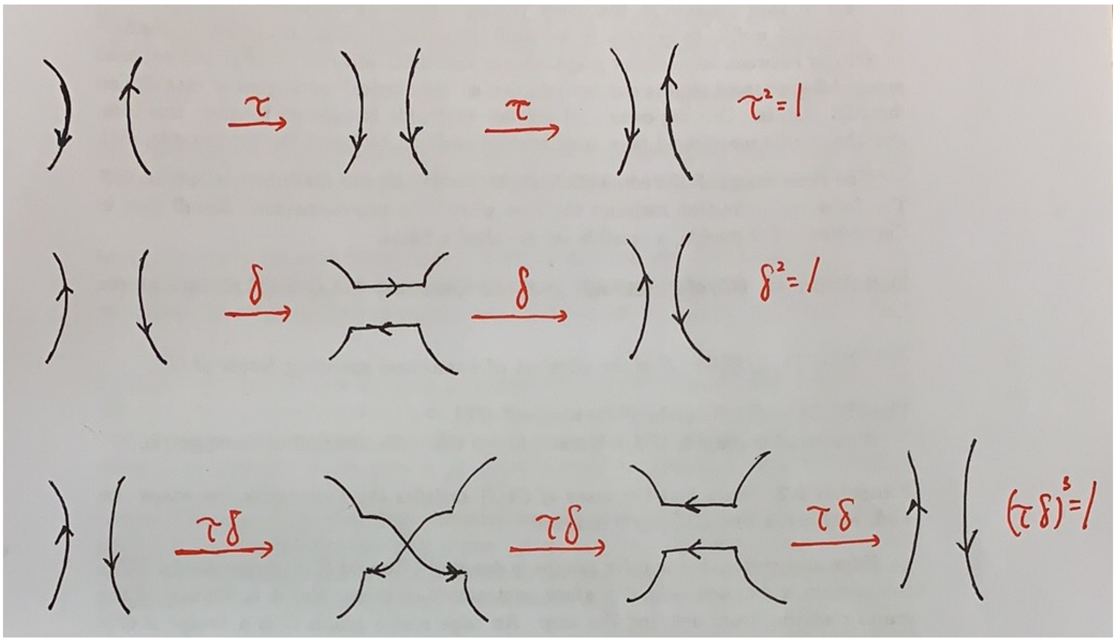

FIGURE 4.9: Group Operations τ, δ

4.4 Ribbon-Graphic Delta Matroids

In the previous section of cycle matroids, we see that members of the feasible sets consist of spanning trees. For ribbon-graphic delta matroids, similar things happen: the feasible set contains spanning quasi-trees, so first, we give a definition of a *quasi-tree*. The reason why we don't use spanning trees for ribbon graph is that they fail to characterize the topological behavior of the ribbon graph, to be more specific, a 'twist' of a 'rectangular band' in the ribbon graph will have the same spanning tree as a 'rectangular band' without a 'twist'.

Definition. A *quasi-tree* is a ribbon graph with only one boundary component, if a ribbon subgraph $\mathbb{H}$ is a quasi-tree and has the same vertex set as $\mathbb{G}$, then we say $\mathbb{H}$ is a *spanning quasi-tree.*

The definition of a ribbon-graphic delta matroid is analogous to that of cycle matroid.

Definition. Let $\mathbb{G} = (V, E)$ be a ribbon graph, and let

$$\mathcal{F} := \{F \subset E : F \text{ is the edge set of a spanning quasi-tree of } \mathbb{G}\}.$$

This is the delta matroid of $\mathbb{G}$, denoted by $D(\mathbb{G}) = (E, \mathcal{F})$.

Example. Here is a ribbon graph with labeled edges, notice that a spanning quasi-tree must contain the same vertex set as $\mathbb{G}$, to save space, we give all the spanning quasi-trees in terms of the edge labeling, therefore each member of the feasible set is the label of edges used to construct a spanning quasi-tree. The set of spanning

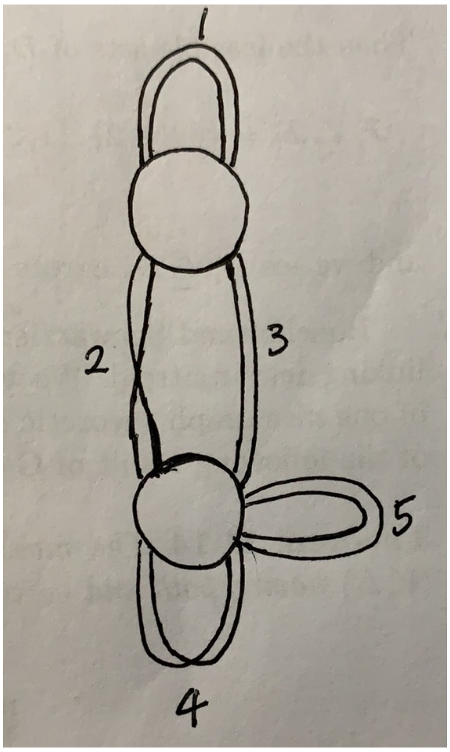

FIGURE 4.10: $\mathbb{G}$.

quasi-trees, i.e., the feasible set is

$$\mathcal{F} = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{2, 5\}, \{3, 5\}, \{1, 2, 3\}, \{1, 2, 4\},$$

$$\{1, 3, 4\}, \{1, 2, 5\}, \{1, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 4\}, \{1, 2, 4, 5\}, \{1, 2, 35\},$$

$$\{1, 2, 3, 4\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 34, 5\}\}.$$

We briefly mentioned that using spanning trees of $\mathbb{G}$ fails to capture the unique character of ribbon graphs. We denote the spanning trees of $\mathbb{G}$ as $C(\mathbb{G})$, notice that among all members of the feasible set of $D(\mathbb{G}) = (E, \mathcal{F})$, those with minimum sizes are actually the bases for $C(\mathbb{G})$, this is because $D(\mathbb{G}_{min})$ is the most 'economical' way of selecting edges in a spanning quasi-tree, as long as the edges that connect all vertices make the subgraph of $\mathbb{G}$ one boundary component. Hence, if we have a connected ribbon graph with n vertices, then to find a spanning quasi-tree with minimum number of edges, it is equivalent to find a spanning tree, which is of order $n-1$.

4.5 Eulerian Delta Matroid

Bouchet introduced this delta matroid[4], and our interest is in the set of 4−regular graphs and their Eulerian circuits. The graphs need not to be simple, that is, we can have multiple edges connecting a pair of vertices, as long as the entire graph is 4−regular. We first consider the *bitransition* of a half-edge (to avoid the case of a loop, where an edge has two endvertices that coincide). Three types of a bitransition can happenat a given vertex, shown below. A *transition system* is

(a) A vertex v (b) Bitransitions

FIGURE 4.11: Bitransitions of a vertex v in a 4-regular graph [5].

a collection of bitransitions, one at each vertex. The transition system forms an Eulerian circuit of graph G, as it gives directions of how one edge passes through a given vertex. For each of the 3 bitransitions at a vertex, we specify one to be *forbidden*, the other two to be *allowed*, and one of the allowed bitransitions to be *preferred*. Bouchet denoted T_F to be the system with all forbidden bitransitions, T_P to be the system with all preferred bitransitions. Then

$$D(G, T_F, T_P) \coloneqq (V, \mathcal{F}) \text{ is a delta matroid.}$$

$\mathcal{F}$ is the collection of subsets U of V such that there exists an allowed transition system of G with preferred bitransitions at vertices in U. The proof for $D(G, T_F, T_P)$ has an one-to-one correspondence with ribbon-graphic delta matroid is somewhat intriguing, we will give a summary of the sketch of it. We first construct the *medial* graph by placing a vertex at each edge of G and drawing edges that follow the boundaries of a faces. Then we aim to solve the 2-coloring problem of the medial graph, which corresponds to whether the faces of original graph G cross each other or not. Moffatt has laid out six possibilities of coloring of the medial graph, which correspond to an embedded cycle family of graphs. As a conclusion, the Eulerian delta matroid is an extension of the ribbon-graphic delta matroid, and ribbon-graphic delta matroids serve as a translation of the words in delta matroid theory into ribbon graph theory. We end this chapter with an example of an Eulerian delta matroid.

Example. The graph shown here is a 4-regular graph with labelled vertices and edges, as well as the three bitransitions at vertex 1. We define the forbidden

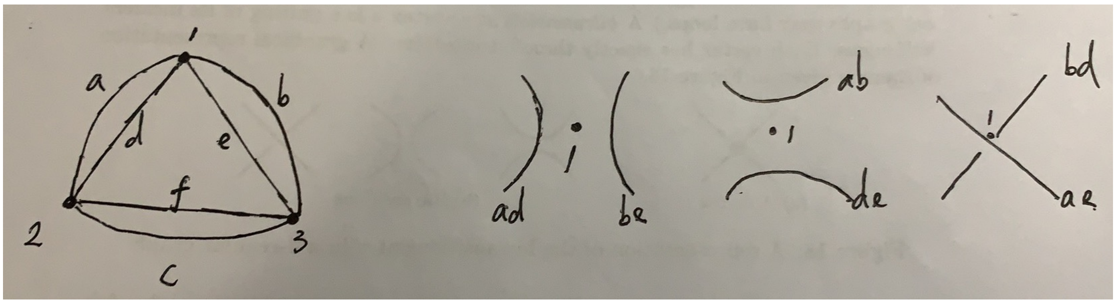

FIGURE 4.12: $\mathbb{G}$.

bitransition at vertex 1 to be ab, de, at vertex 2 to be af, cd, and at vertex 3 to be be, cf. We prefer the bitransition at vertex 1 to be ad, be, at vertex 2 to be ac, df, and at vertex 3 to be bc, ef. Then we have the following Eulerian circuits:

- $adcbef$, which uses preferred bitransitions at 1 and 3.
- $acedfb$, which uses preferred bitransitions at 2.
- $becadf$, which uses preferred bitransitions at 1 and 2. Hence the delta matroid has the feasible set $\{\{1,3\},\{2\},\{1,2\}\}$.

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