Implied Volatility and Extracted Risk Neutral Density of VIX Options during the Crisis and Relatively Calm Periods

By

Patchara Santawisook

A Thesis submitted to the Faculty of

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfillment of the requirements for the Degree of Master of Science

in

Financial Mathematics

May 2015

Advisor: Stephan Sturm

Department Head: Luca Capogna

Abstract

The 2008 financial crisis provides a valuable opportunity to study empirical data of market volatility during severe financial crisis. In this thesis, we study the implied volatility of VIX options during the crisis (2008) and a relatively calm period (2011). We present a method of calculating the implied volatility of VIX options and fit the implied volatilities using a 4^{th} degree spline interpolation and propose method of extracting risk neutral density from fitted data. We analyze the slope and the level of the fitted implied volatility of VIX options during those periods. The results show that the level of the implied volatility of VIX options is higher and the slope is flatter during the distressed market compared to the relative calm periods.

Acknowledgments

I would like to thank my advisor, Professor Stephan Sturm, who has given me a lot of invaluable advice, provided me guidance and suggestion to make this project possible. I also would like to thank my parents for supporting me throughout my education at WPI.

Table of Contents

Chapter 1 Introduction

Not only is the 2008 financial crisis used as a reference point for the next inevitable crisis, which may happen in the near future, but it also provides an opportunity to study the empirical data of market volatility during severe financial crisis. The VIX index is an index trying to measure the reflection of the investors' view of future expected volatility of the S&P 500 stock index. For instance, when the market declines, the VIX index tends to rise as the investors' fear increases. Alternatively, when the financial market becomes less intense, the investors' fear subsides, and the VIX index tends to go down. We are interested in looking at the implied volatility of VIX options during periods of the 2008 financial crisis, when VIX hit the highest level in history, and during periods of 2011 when S&P downgraded the US market from AAA (excellent) to AA+ (outstanding), which led the VIX to spike since March 2009 (Pepitone, 2011). As an increasing or decreasing in option prices depends on the demand for option contracts, market forces dictate the implied volatility. Therefore, implied volatility can be used as a prediction of the future volatility of the market. In this project, we will study how implied volatility of VIX options behave before and after each incident.

We present the method of how to calculate the implied volatility of VIX options. We fitted the implied volatilities to the 4^{th} degree spline interpolation, and proposed the method of extracting risk neutral density from fitted data based on (Figlewski, 2009). We then analyzed the slope and the level of the implied volatility of VIX options during periods of 2008 and 2011.

We found that the level of the implied volatility of VIX options is higher and the slope is flatter during the distressed market compared to the relative calm periods. We have also attempted to fit the empirical market data to the SVI, a widely used parametric form so that we could use it to approximate the implied volatility at strike that option price may not be available; however, we could not get a good fit due to the convexity, hence a research on finding a better parameterization method should be done so that it allows us to extrapolate the implied volatility of VIX options at extreme strikes where the data in the market may not be available.

Chapter 2 Literature review

In the first part of this chapter, we give an introduction to the European type options, the Black-Scholes model for valuating options and implied volatility. Then we move on to the Heston model, a popular stochastic volatility model used to valuate option price. In the second part we introduce VIX futures and VIX options and show how to estimate the implied volatility of VIX futures out of VIX options.

European options

In this section, we give the brief overview of European options and how to price them using the Black-Scholes model.

A European call option is a contract that gives holders the right, but not obligation to buy a unit of the underlying asset at the predetermined price K (strike price) at the maturity time T. Its payoff at maturity T can be viewed as

$$h(x) = (x-K)^+ = \begin{cases} x-K & \text{if } x > K, \\ 0 & \text{if } x \leq K. \end{cases}$$

Here X_T is the market value of the underlying asset at time T.

A European put option is a contract that gives holders the right, but not obligation to sell a unit of the underlying asset at the predetermined price K (strike) at the maturity time T. Its payoff can be written as

$$h(x) = (K-x)^+ = \begin{cases} K-x & \text{if } x < K, \\ 0 & \text{if } x \geq K. \end{cases}$$

Similar to the call option X_T is the market value of the underlying asset at time T.

An investor who buys a call option profits from increasing of underlying asset while an investor who buys a put option profits from decreasing of underlying asset.

We will give a brief overview of the famous option pricing model, the Black-Scholes model, in the next section.

Black-Scholes model

Assuming a complete market with a constant interest rate r, the price of the European option at time t where $0 < t < T$ can be expressed as conditional expectation under the risk neutral measure,

$$C_t = \mathrm{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^{+}|\mathcal{F}_t\right],$$

$$P_t = \mathrm{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(K - S_T)^{+}|\mathcal{F}_t\right],$$

where K is the strike price of the European options, S_T is the underlying asset traded at time T, and $(\mathcal{F}_t)$ is the filtration generated by the price process (S_t).

In 1973, Black and Scholes introduced a closed-form solution for the price of European options (Black & Scholes, 1973). The following idealized conditions need to be met to apply the Black-Scholes model. Ideally the interest rate is constant, the stock price is lognormal distributed with constant variance rate of the return, the stock pays no dividend, buying or selling stock and option has no transaction costs, and borrowing and lending is not limited at a constant interest rate.

Assuming that above conditions are met, the price of European call option is given by

$$C_{BS}(t, x) = xN(d_1) - Ke^{-r(T-t)}N(d_1),$$

where

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, d_2 = d_1 - \sigma\sqrt{(T-t)},$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2}} dy$$

N(z) is the cumulative normal density function.

Essentially, the European put and call options are linked together by the put-call parity. This relationship arises from a situation when call and put options have the same strike, holding a stock and a put position will result in the same payoff as holding a call position. If the parity is violated, an arbitrage opportunity exists.

The put-call parity for European-style options reads

$$C_{BS}(t,x) - P_{BS}(t,x) = x - Ke^{-r(T-t)}.$$

Using put-call parity relation, the formula for the European put option can be written as:

$$P_{BS}(t,x) = Ke^{-r(T-t)}N(-d_2) - xN(-d_1)\,,$$

where d_1 , d_2 and N are the same as the ones of the European call option.

Implied volatility

Notice that by assuming that the interest rate is constant[1], all other parameters in the Black-Scholes formula can be observed; the implied volatility is the only one that needs to be estimated. The implied volatility can be thought of as a characteristic of the option. For instance, with no change in any other parameters in the Black-Scholes formula, if the price of the option contract increases, the resulting implied volatility obtained from this model will also increase. An increasing or decreasing in option prices depends on the demand for option contracts. Market forces dictate the implied volatility. Therefore, implied volatility can be used as a prediction of the future volatility of the market.

Given the observed option price with strike K, and time to maturity T, the implied volatility (IV) can be estimated by matching that option price with the theoretical Black-Scholes option formula—for instance, for the European call option

$$C_{BS}(t,x;T,K;IV) = C_{obs}.$$

Notice that because of the put-call parity, the implied volatilities of the put and call option with the same strike and time to maturity are the same. This seems to be a good approach to calculate the implied volatility. However the implied volatility obtained from

[1] Note that for this project we assume that the interest rate is constant. We use the 3-month treasury yield rate on the trading rate to calculate the implied volatility. The historical interest rate can be obtained from http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

the market data disagrees with the Black-Scholes' assumption that the volatility is constant. The market data suggest an existence of an implied volatility smile, showing that for a given expiration, strike prices of the same underlying asset have different implied volatilities; options whose strike price differs substantially from the underlying asset have higher implied volatility. Since volatilities are not constant but change stochastically over time, we introduce one of the well-known stochastic volatility models, the Heston model, in the next section.

Heston model and SVI

The Heston model can be used to evaluate the European option. The model states that the stock price S_t and its variance v_t satisfy the following stochastic differential equations (Gatheral, The Volatility Surface: A Practitioner's Guide, 2011):

$$dS_t = rS_t dt + \sqrt{v_t} S_t dZ_1,$$

and

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t} dZ_2,$$

with

$$\langle dZ_1, dZ_2 \rangle = \rho dt,$$

where $\sqrt{v_t}$ is the volatility of S_t, η is the volatility of v_t, and λ is the speed of reversion of v_t to its long-term mean $\bar{v}$. Provided by Lewis in 2000 (Lewis, 2000), the implied variance in Heston model as $T \rightarrow \infty$, can be calculated by its model parameters as:

$$v_T(0) = \frac{4\lambda'\bar{v}'}{\eta^2(1-\rho)^2}\{\sqrt{4\lambda'^2 + \eta^2(1-\rho)^2} - 2\lambda'\}$$

where $\lambda' = \lambda - \frac{\rho\eta}{2}$ and $\lambda'\bar{v}' = \lambda\bar{v}$.

It turns out that with large-time to maturity, the implied volatility of the Heston model is consistent with the stochastic volatility inspired (SVI) introduced by Gatheral (Gatheral, 2011). The SVI is the parameterization of implied volatility smile based on the

Heston model. With the SVI parameterization, we have to only calibrate *a, b,* $\tilde{\rho}$, σ and m, and it can be readily used. The SVI parameterization reads

$$\sigma_{BS}^2(k) = a + b\{\tilde{\rho}(k-m) + \sqrt{(k-m)^2 + \tilde{\sigma}^2}\} \qquad (1)$$

where $k = \log\left(\frac{F}{S}\right)$ *a,b,* $\tilde{\rho}$, σ,and m depend on the expiration.

- a provides the overall level of variance
- b controls the angle between the left and right asymptotes of smile
- $\tilde{\rho}$ controls the orientation of the entire curve
- σ controls how smooth the curve is at the money variance
- m shifts the curve left or right

This SVI is used to fit the implied volatility of empirical market options prices, and provides a smooth volatility smile of those options to a single time to maturity T.

Gatheral and Jacquier gave a proof of this by showing that with the appropriate change of variables, the implied volatility in (1) and (2) are the same (Gatheral & Jacquier, 2010). The following is the brief idea of the proof.

According to Forde, Jacquier & Mijatovic, 2010, the implied variance in the Heston model as $T \to \infty$, can take the following forms:

$$\sigma_\infty^2(x) = 2(2V^*(x) - x + 2(\mathbb{I}_{x\in\left(-\frac{\theta}{2},\frac{\bar{\theta}}{2}\right)} - \mathbb{I}_{x\in\mathbb{R}(-\frac{\theta}{2},\frac{\bar{\theta}}{2})})\sqrt{V^*(x)^2 - xV^*(x)} \quad \text{for all } x \in \mathbb{R}, \qquad (2)$$

where $\bar{\theta} \coloneqq \frac{\kappa\theta}{\kappa - \rho\sigma}$, and the function $V^*: \mathbb{R} \to \mathbb{R}_+$ is defined by

$$V^*(x) \coloneqq p^*(x)x - V(p^*(x)), \qquad \text{for all } x \in \mathbb{R},$$

where

$$V(p) \coloneqq \frac{\kappa\theta}{\sigma^2}(\kappa - \rho\sigma p - d(p), \qquad \text{for all } p \in (p_-, p_+),$$

$$d(p) \coloneqq \sqrt{(\kappa - \rho\sigma p)^2 + \sigma^2 p(1-p)^2}, \qquad \text{for all } p \in (p_-, p_+),$$

$$p^*(x) := \frac{\sigma - 2\kappa\rho + (\kappa\theta p + x\sigma)\eta\left(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2\right)^{-\frac{1}{2}}}{2\sigma\bar{\rho}^2}, \quad \text{for all } x \in \mathbb{R},$$

$$\eta := \sqrt{4\kappa^2 + \sigma^2 - 4\kappa\rho\sigma}, \tag{3}$$

$$p_{\pm} := \frac{-2\kappa\rho + \sigma \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\kappa\rho\sigma}}{2\sigma\bar{\rho}^2}, \text{ and}$$

$$\bar{\rho} := \sqrt{1 - \rho^2}. \tag{4}$$

Notice that $x = \log\left(\frac{F}{S}\right)$, with $T \geq 0$ is the time to maturity.

Gatheral presented that the SVI parameterization for the implied variance takes the form[2] (Gatheral, 2004):

$$\sigma_{SVI}^2(x) = \frac{\omega_1}{2}(1 + \omega_2\rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2}, \text{for all } x \in \mathbb{R}, \tag{5}$$

where x= $\log\left(\frac{F}{S}\right)$, ω_1 and ω_2 are the SVI paramether choices in term of the Heston parameters as:

$$\omega_1 := \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}\left(\sqrt{(2\kappa - \rho\sigma)^2 + \sigma^2(1-\rho^2)} - (2\kappa - \rho\sigma)\right), \text{and } \omega_2 := \frac{\sigma}{\kappa\theta}. \tag{6}$$

Let $\Delta(x) := \sqrt{\sigma^2 x^2 + 2\kappa\theta\rho\sigma x + \kappa^2\theta^2}$, η and $\bar{\rho}$ are defined in (3) and (4) respectively. The SVI implied variance of (5) under the change of variable in (6) can be written as

$$\sigma_{SVI}^2(x) = \frac{2}{\sigma^2\bar{\rho}^2}(\eta - (2\kappa - \rho\sigma))(\kappa\theta + \rho\sigma x + \Delta(x)), \text{for all } x \in \mathbb{R}. \tag{7}$$

By simplifying the expression $\sigma_\infty^2(x)$ in (2) with the assumption that $\kappa - \rho\sigma > 0$ we will also have the implied variance in Heston model as the form

$$\sigma_\infty^2(x) = \frac{2}{\sigma^2\bar{\rho}^2}(\eta - (2\kappa - \rho\sigma))(\kappa\theta + \rho\sigma x + \Delta(x)), \text{for all } x \in \mathbb{R},$$

which concludes that $\sigma_{SVI}^2(x)$=$\sigma_\infty^2(x)$ for all $x \in \mathbb{R}$ (Gatheral & Jacquier, 2010).

Now, by equating the (1) with (5), with the parameter choice in (6), we then have the followings correspondence between the SVI parameters in (1) and the Heston parameters:

[2] Note that this is a different, but equivalent parameterization of (1).

$a = \frac{\omega_1}{2}(1 - \rho^2),$

$b = \frac{\omega_1 \omega_2}{2T},$

$\tilde{\rho} = \rho,$

$m = -\frac{\rho T}{\omega_2},$

$\tilde{\sigma} = \frac{\sqrt{1-\rho^2} T}{\omega_2}.$

This result allows us to extrapolate the implied volatility of options at extreme strikes where the data in the market may not be available by using the equations below.

As $k \to \infty$, the SVI parameterization of the right asymptote becomes:

$$\sigma^2_{BS,R}(k) = a + b(1 + \rho)(k - m).$$

Similarly, as $k \to -\infty$, the SVI parameterization of the left asymptote becomes:

$$\sigma^2_{BS,R}(k) = a - b(1 - \rho)(k - m).$$

Hence, for the left asymptote, the smile has slope $-b\ (1 - \rho)$ and intercept $a + bm\ (1 - \rho)$.

Variance swap and VIX

The CBOE Volatility Index (VIX Index) was first presented in 1993 as the measurement of the market's expectation of 30-day implied volatility based on the S&P 100 Index (OEX Index). In 2003, the CBOE and Goldman Sachs introduced the new method to determine the VIX, which is now based on the S&P 500 index (SPX) to track the expected volatility based on the SPX put and call options over range of strikes. It was not until March 24, 2004 that the CBOE launched the volatility as a tradable asset product, VIX futures, as a trading vehicle that allows investors to trade on VIX. Investors who long VIX futures positions profit from increasing volatility. In February 2006, CBOE launched another volatility product, VIX options. VIX options are European style options, which generally are exercised only on the expiration date. An investor who longs a call VIX option profits from increasing volatility while longs a put VIX option profits from decreasing volatility.

The construction of VIX is based on a method of pricing variance swap; therefore, knowing how to evaluate the price of variance swap would be crucial.

Assuming that σ_t is the volatility of the underlying (the S&P index for this project), for the time period $[t, T]$ where $t < T$, a variance swap is an agreement to exchange $\frac{1}{T-t}\int_t^T \sigma_u^2 du$ for some $E_t\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$ at time t. $\frac{1}{T-t}\int_t^T \sigma_u^2 du$ is the floating leg that equals to the quadratic variation of $\log S_t$, and $E_t\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$ is called the fixed leg or variance swap with an assumption that the contract has no entry cost at initial time t.

Let S_t and σ_t satisfy the following stochastic differential equation

$\frac{dS_t}{S_t} = rdt + \sigma_t dZ_t$ under given risk neutral measure. Using Ito's formula, under the risk neutral measure, we have

$$d\log S_t = \left(r - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dZ_t. \qquad (8)$$

From (8), we can get

$(d\log S_t)^2 = \sigma_t^2 dt$, and $\sigma_t^2 dt = 2\left(\frac{dS_t}{S_t} - d\log S_t\right)$.

So, $E_t\left[\int_t^T \sigma_u^2 du\right] \quad = 2E_t\left[\int_t^T (\frac{dS_u}{S_u} - d\log S_u\right]$

$$= 2E_t\left[\int_t^T \frac{1}{S_u} dS_u - \int_t^T d\log S_u\right]$$

$$= 2r(T-t) - 2E_t\left[\log\frac{S_T}{S_t}\right]. \tag{9}$$

It can be showed that the log contract can be replicated by the prices of European call and put options on the underlying asset by:

$$e^{-r(T-t)}E_t\left[\log\frac{S_T}{S_t}\right] = r(T-t)e^{-r(T-t)} - \int_0^{F_{t,T}} \frac{1}{K^2}(K-S_T)^+ \, dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2}(S_T-K)^+ \, dK. \tag{10}$$

Substituting (10) to (9), we get

$$E_t\left[\int_t^T \sigma_u^2 du\right] = 2e^{r(T-t)}\int_0^{F_{t,T}} \frac{1}{K^2}(K-S_T)^+ \, dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2}(S_T-K)^+ \, dK.$$

At time t, a variance swap $E_t\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right]$ is

$$\frac{2e^{r(T-t)}}{T-t}\int_0^{F_{t,T}} \frac{1}{K^2}(K-S_T)^+ \, dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2}(S_T-K)^+ \, dK.$$

Use $\tau = T - t = 30$ days, and take square root of the variance swap to yield the VIX formula at time t:

$$\mathrm{VIX}_t = \sqrt{\frac{2e^{r\tau}}{\tau}\int_0^{F_{t,t+\tau}} \frac{1}{K^2}\mathrm{P}(t,K,t+\tau)\, dK + \int_{F_{t,t+\tau}}^{\infty} \frac{1}{K^2}\mathrm{C}(t,K,t+\tau)\, dK.}$$

Finally, we can approximate VIX^2 value at initial time 0 in the discrete time as

$$\mathrm{VIX}^2 = \frac{2}{T}\sum_{K_i \le F_T} \frac{\Delta K_i}{K_i^2} e^{rT}\mathrm{Put}[K_i] + \frac{2}{T}\sum_{K_i > F_T} \frac{\Delta K_i}{K_i^2} e^{rT}\mathrm{Call}[K_i].$$

where

$F_T = e^{rT}$ is the forward price on the stock expired at T and r is the risk-free interest.

As discussed in (Institutional White Papers), VIX is a volatility index composed of options, with the price of each option reflecting the market's expectation of future volatility. The equation used to calculate the VIX is based on the method of pricing the variance swap on S&P500 index. The only difference is the correction term, $-\frac{1}{T}\left[\frac{F}{K_0}-1\right]^2$, which is added into the value of variance swap to improve the accuracy of an approximation. It is the correction for the discrepancy between K_i and the forward price F_T when $K_i \neq F_T$. The VIX calculation is described below

$$\mathrm{VIX}^2 = \frac{2}{T}\sum_{K_i \leq F_T}\frac{\Delta K_i}{K_i^2}e^{rT}\mathrm{Put}[K_i] + \frac{2}{T}\sum_{K_i \geq F_T}\frac{\Delta K_i}{K_i^2}e^{rT}\mathrm{Call}[K_i] - \frac{1}{T}\left[\frac{F}{K_0}-1\right]^2,$$

where F is the forward index level desired from index options prices and K_0 is the first strike below the forward index level F.

Implied volatility of VIX options

In this section, we discuss how to obtain the implied volatility of VIX options from the market data. Recall from the implied volatility section that given the observed call and put prices from the market data $\tilde{C}_t$ and $\tilde{P}_t$, the implied volatility σ_{IV}^C and σ_{IV}^P, respectively, can be calculated using Black-Scholes model

$$\tilde{C}_t = C_t(S_t, T, K, r, \sigma_{IV}^C),$$

$$\tilde{P}_t = P_t(S_t, T, K, r, \sigma_{IV}^P),$$

where S_t is an underlying asset, T is time to maturity, K is the strike price, and r is constant risk free rate.

VIX options are options on VIX futures. By treating VIX futures as underlying, the implied volatility $\sigma_{IV}^{C,VIX-F}$ and $\sigma_{IV}^{P,VIX-F}$ of VIX futures is given by

$$\tilde{C}_t^{VIX} = C_t(F_t, T, K, r, \sigma_{IV}^{C,VIX-F}),$$

$$\tilde{P}_t^{VIX} = P_t(F_t, T, K, r, \sigma_{IV}^{P,VIX-F}),$$

where F_t, the future price at time t, is equal to $\mathrm{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]$ under the risk neutral measure, and $(\mathcal{F}_t)$ is the filtration generated by the process (S_t). Treating the VIX itself as the underlying (S_t), the option prices on VIX futures under the risk neutral measure read

$$C_t^{VIX} = \mathrm{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(F_T(T+t_0, r) - K)^+|\mathcal{F}_t\right],$$

$$P_t^{VIX} = \mathrm{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\big(K - F_T(T+t_0, r)\big)^+ |\mathcal{F}_t\right],$$

where t_0 is the forward time of the futures, which is 30 days in practice, and $F_T(T+t_0, r) = e^{rt_0}S_T$ is the future price at time T. Simplify the equations above, we get

$$C_t^{VIX} = e^{rt_0}C_t(S_t, T, e^{-rt_0}K, r, \sigma),$$
$$P_t^{VIX} = e^{rt_0}P_t(S_t, T, e^{-rt_0}K, r, \sigma).$$

Thus, we can extract the implied volatility from the market data of VIX options as

$$\tilde{C}_t^{VIX} = e^{rt_0}C_t(S_t, T, e^{-rt_0}K, r, \sigma_{IV}^{C,VIX}),$$
$$\tilde{P}_t^{VIX} = e^{rt_0}P_t(S_t, T, e^{-rt_0}K, r, \sigma_{IV}^{P,VIX}).$$

Chapter 3 The data and methodology

The data

This study focuses on the market conditions around October 2008 and September 2011 events. We take the historical VIX options prices data of six months before and after each event for total of thirteen months. For each monthly options contract, we collected put and call options prices of each available strike price that expired 71 days later. The data were obtained from the Market Data Express (MDX). In year 2008 before the financial crisis, the strike prices were available from $10 to $50, and after the crisis the strike prices were available up to $70 in that year. In 2011, the strike prices from $10 to $80 for put and call options were available for all months, and some months were available up to $100. The spreads bid-ask were relatively wide compared to the bid or ask price: the maximum are $1.6 and $1, the minimum are $0.05 and $0.05, and the average are $0.36 and $0.23 for the 2008 and 2011 events respectively. Figlewski suggests to use either the transaction prices or the midpoints of quoted bid-ask spreads as the market's options prices for S&P500 options (Figlewski, 2009). However, options transactions take place irregularly and only few strikes are frequently traded. For this project, the midpoints of bid-ask spreads were used as the market's option prices as the best available data for VIX options, even if trades might not occur.

The table below describes the layout of the VIX data:

Column Label	Data Type	Description
TRADE_DT	yyyymmdd	The date the trades occurred on
UNDLY	varchar(1-6)	The underlying stock, index or financial instrument
CLS	varchar(1-3)	The option trading class symbol
EXPR_DT	yyyymmdd	The date the option expires
STRK_PRC	decimal(8.3)	The explicit dollar.cent strike price of the option.
PC	char(1)	'C' or 'P'
OIT	integer(8)	The 'open-interest' in this series

VOL	integer(8)	The number of option 'contracts' traded in this series on this day.
HIGH	decimal(7.2)	The highest trade price in this series on this day (null if no trades occurred).
LOW	decimal(7.2)	The lowest trade price in this series on this day (null if no trades occurred).
OPEN	decimal(7.2)	The trade price on the first trade in this series on this day (null if no trades occurred).
LAST	decimal(7.2)	The trade price on the last trade in this series on this day (null if no trades occurred).
L_BID	decimal(7.2)	The bid price on the last quote in this series on this day.
L_ASK	decimal(7.2)	The ask price on the last quote in this series on this day.
UNDL_PRC	decimal(7.2)	The closing price on the associated underlying instrument on this day.
S_TYPE	char(10)	Leap, non-Leap, Weekly
P_TYPE	char(15)	Equity, Index, Interest Rate

From the table above, TRADE_DT, UNDLY, EXPR_DT, STRK_PRC, PC, L_BID, L_ASK were used to calculate implied volatility of VIX options. Since PC was stored as the char(1), it is customary to convert this char(1) to 0 or 1, where 1 represents C (call option) and 0 represents P (put option). Note that the data stored under UNDL_PRC are the VIX index not VIX futures. Since the VIX options are options on VIX futures, we will need VIX futures as the underlying asset for option pricing. The closing price of VIX futures corresponding to the VIX options prices on each trading day were attained from the VIX Futures Historical Market Data on the CBOE website. The implied volatility of VIX options is calculated based on the theoretical Black-Scholes pricing formula using MATLAB software as in Appendix A.

Data cleaning

Due to inconsistencies of the market data, the data must be treated to remove noises. We followed the data cleaning procedure introduced in Figlewski with some modifications (Figlewski, 2009). For each monthly option contract, do the following:

(i) Calculate midpoints of bid-ask spreads of put and call options as the market's put and call option prices.

(ii) Calculate implied volatilities of midpoints put and call options.

(iii) Filter out all options whose bid quotes are less than 0.25 or have no implied volatility value

(iv) Blend put and call implied volatilities in the region around at the money level in order to smooth out the jump at the transition point.

Let S_0 be the current underlying VIX future value, we blend the put and call implied volatilities two strike levels below and above S_0. For instance, if the VIX future is currently at 22.75 and the available strikes of the VIX options are

..., 18, 19, 20, 21, 22, 23, 24, 25 ...

We will choose to blend the implied volatilities of put and call options whose strike are 21, 22, 23 and 24.

Let X_{min} and X_{max} be the minimum and maximum strikes that are in the range of strikes that we want to blend, the blended implied volatility of each applicable strike value X can be calculated as:

$$IV_{blend}(\mathrm{X}) = \mathrm{w}\, IV_{put}(\mathrm{X}) + (1-\mathrm{w})\, IV_{call}(\mathrm{X})$$

$$\text{where} \qquad w = \frac{X_{max}-X}{X_{max}-X_{min}}.$$

(v) Combine implied volatilities by taking put implied volatilities for strikes up to strike immediately before X_{min}, blended implied volatilities for strikes from X_{min} to X_{max}, call implied volatilities for strikes greater than X_{max}.

Spline interpolation and SVI

Spline interpolation is a popular interpolation tool that can be used to fill in intermediate values between available data points. This interpolation technique can be used to generate a dense set of data points in order to smooth out the curve of the usable VIX options' implied volatilities. As suggested by Figlewski, interpolating the implied volatilities with a 4^{th} order spline or higher produces a reasonably smooth curve (Figlewski, 2009). In this project, we use the 4^{th} degree spline interpolation with 1 knot to attain the curve of the VIX implied volatilities by applying the least-squares spline approximation MATLAB *spap2* function to the remaining implied volatilities after being cleaned.

As mentioned earlier, the implied volatility of the Heston model converges to SVI as time to maturity grows large. An SVI fit works well with the implied volatilities of S&P 500 options. We propose to fit the SVI to the IVs of VIX options using MATLAB *lsqnonlin* and *mySVIvalue* functions to solve the non-linear data-fitting problems in MATLAB *svi* function. The fitted results from 4^{th} degree spline interpolation and the SVI to the VIX options implied volatilities are compared, and the results can be view in Chapter 4.

Risk neutral density of VIX

Risk neutral density has been a topic of interest to investors and researchers, as it contain informations about the market's expectations. Several works has been done to extract and interpret the risk neutral density including Figleskwi, where he proposes step by step method to extract the market's risk neutral density out of the S&P 500 options (Figlewski, 2009). In this project, we followed the method introduced by Figleskwi to extract the risk neutral density out of VIX options with the following modifications.

Let C, S, K, r and T be call price of option, underlying asset at time 0, the strike price, riskless interest rate, and time to maturity, respectively. Denote f(x) as the risk neutral probability density function (RND) and F(x) is the corresponding risk neutral distribution function. Under the risk neutrality, the price of the call option at time 0 is the expected value of the payoff value discounting back at a risk free rate:

$$C = e^{-rT}E[(S_T - K)^+]$$

with the risk neutral density (RND) of the stock price, we have

$$C = \int_K^\infty e^{-rT}(X - K)f(X)dX.$$

Taking the derivative of C with respect to K,

$$\frac{\partial C}{\partial K} = -e^{-rT}\int_K^\infty f(X)dX = -e^{-rT}[1 - F(K)],$$

where F is the risk neutral distribution.

Therefore, the risk neutral distribution reads,

$$F(K) = e^{rT}\frac{\partial C}{\partial K} + 1.$$

In practice, if we are given the sequence of the options price, C_i with strike prices K_i, for $i = 1, \dots, n$, an approximation of F(K) centered on K_i can be approximated as,

$$F(K_i) \approx e^{rT}\left[\frac{C_{i+1} - C_{i-1}}{K_{i+1} - K_{i-1}}\right] + 1.$$

The RND can be calculated by taking the derivative of the risk neutral distribution with respect to K.

$$f(K) = \frac{\partial F(K)}{\partial K} = e^{rT}\frac{\partial^2 C}{\partial K^2}.$$

In practice, the RND, $f(K_n)$, is approximated as

$$f(K_n) = e^{rT}\frac{C_{n+1} - 2C_n + C_{n-1}}{(\Delta K)^2}. \quad (*)$$

This procedure is for extracting the RND out of the empirical call option prices. The RND can also be obtained from the empirical put option prices using a similar procedure. We will estimate the RND of VIX options using the call prices for this project. The method of obtaining the RND is illustrated.

(i) Retrieve the combined implied volatilities of call and put options after the data has been treated from the data cleaning procedure.

(ii) Use MATLAB *spap2* function to fit the 4^{th} degree spline interpolation to the implied volatilities in (i) to get the smooth curve of implied volatilities out of the usable data.

(iii) Retrieve the dense implied volatilities corresponding to the strikes from the fitted spline by MATLAB *fnval* function.

(iv) Convert the implied volatility associated with each strike in (iii) to the VIX call option prices using MATLAB *myBS* function.

(v) Estimate the RND from the call options resulting from (iv).

Notice that the RNDs we are getting from the available options in (iv) do not extend to the left tail and the right tail of the RND curve. Figlewski suggests using the GEV method to get the tails of the RND. However, we propose a simple way to get the tails of the RND by:

(vi) Let K_1 be the lowest strike available in (iii), setting the implied volatilities of the strike prices that are less than K_1 to be equal to the implied volatility of K_1 and proceed by the same method as (iv) and (v) to get the call prices and RND associated with each strike, we get the RND of the left tail.

(vii) Let K_n be the largest strike available in (iii), setting the implied volatilities of the strikes prices that are greater than K_n to be equal to the implied volatility of K_n and proceed by the same steps as (iv) and (v) to get the call prices and RND associated with each strike greater than K_n to get the RND of the right tail.

(viii) Combine the RND in (v), (vi), and (vii) to get the whole RND level.

Note the MATLAB scripts/functions can be found in Appendix A

Chapter 4 Results

April 2008 to April 2009

This section illustrates the results from fitting the implied volatilities to the SVI and the 4th degree spline interpolation of the April 2008 to April 2009 VIX options. We then extracted risk neutral density from implied volatility curve produced by the spline interpolation. The results show that during the normal market condition, the level of implied volatility of VIX options is lower compared to that of a distressed market, as the expectation of risk level is lower; market participants do not expect a high magnitude of movement of the underlying asset (VIX futures). As a result, the range actively traded strikes is narrower during this time than during time of crisis.

During the distressed market condition, we found that the level of implied volatility of options is higher compared to a relatively calm period. The increased level of implied volatility is caused by investors' uncertainty about the direction and magnitude of the underlying asset movement. This is why the range of actively traded strikes is wider.

Figures 1 and 2 below show trends of the implied volatility and the extracted RNDs of 71 days to maturity of the April 2008 to May 2008 VIX options, during the normal market condition.

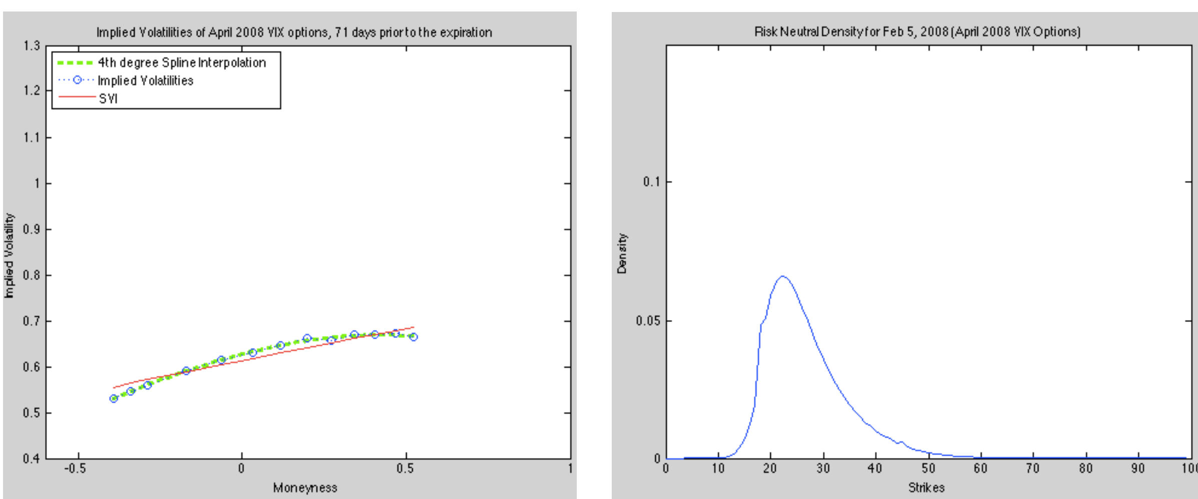


Figure 1 Left: The implied volatilities of April 2008 VIX options traded on Feb 5, 2008 (71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

For the implied volatility plot in Figure 1, at-the-money (ATM) implied volatility is 0.0064. A slope of ATM implied volatility is 0.0064. The underlying asset level of the options is 26.56. This is at a time of pre-crisis when the market is in the normal condition. From the RND plot on the right, we can see that the range of strikes is around +-10 from the underlying asset level. Note that the average closing price for VIX from January 1, 1990 to August 11, 2014 is 20.02 (Rhoads, 2014).

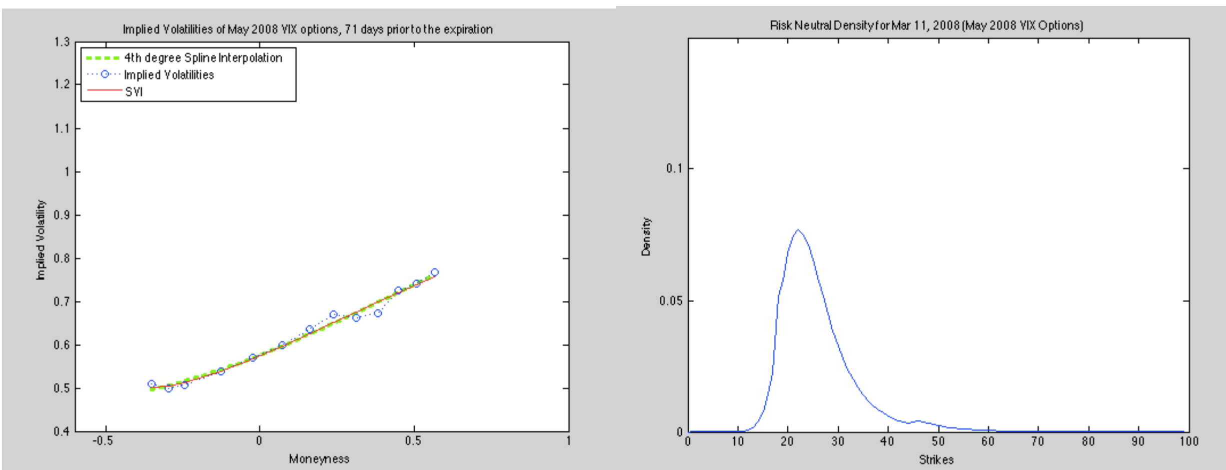


Figure 2 Left: The implied volatilities of May 2008 VIX options traded on March 11, 2008 (71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 2 shows the implied volatility and the RND plot of May 2008 VIX futures traded on March 11, 2008 when the VIX futures closed at 25.49. In the figure on the right, most strike prices are traded at around +- 10 from the VIX futures closing level. At this date, the slope of ATM implied volatility is 0.0105, steeper than the previous month. The level of ATM implied volatility is 0.5782.

Figures 16 to 20 in Appendix B show the plots of implied volatility and the extracted RNDs of 71 days to maturity of the June 2008 to October 2008 VIX options, They show similar trends as Figure 1 and 2 as they are in the same market condition.

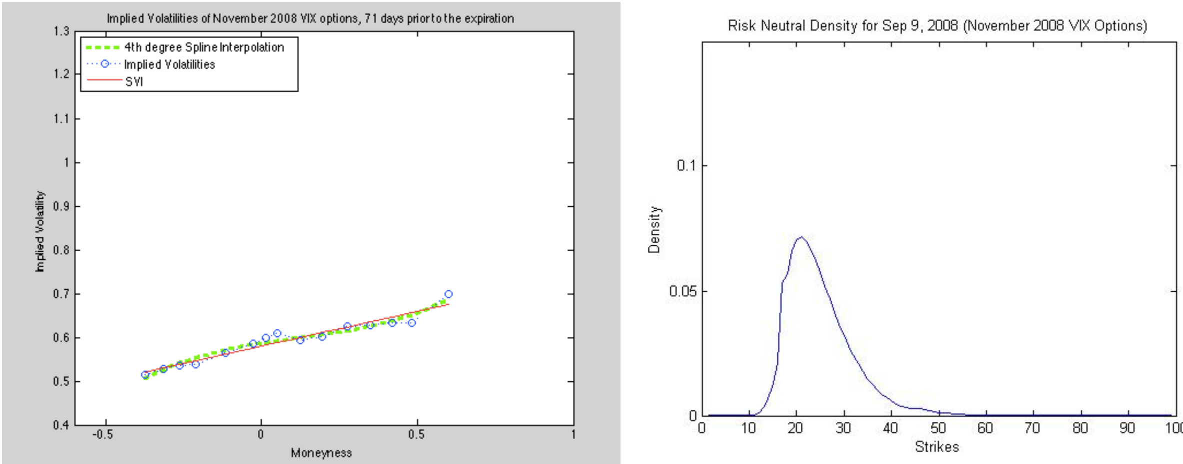


Figure 3 Left: The implied volatilities of November 2008 VIX options traded on September 9, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 3 on the left shows the implied volatilities of November 2008 VIX options traded on September 9, 2008. This is the time when the U.S. stock market was about to crash. The level of ATM implied volatility is slightly less than the one of October 2008 VIX options as show in Figure 20 in Appendix B. The curve of the implied volatility of this month option data becomes flatter as the ATM implied volatility slope is 0.0058, lower than the ATM implied volatility slope of October 2008 VIX options, which is 0.014. The RND curve is slight wider and its height is lightly shorter than the October 2008 data. The VIX futures closing price is 24.60 on this day.

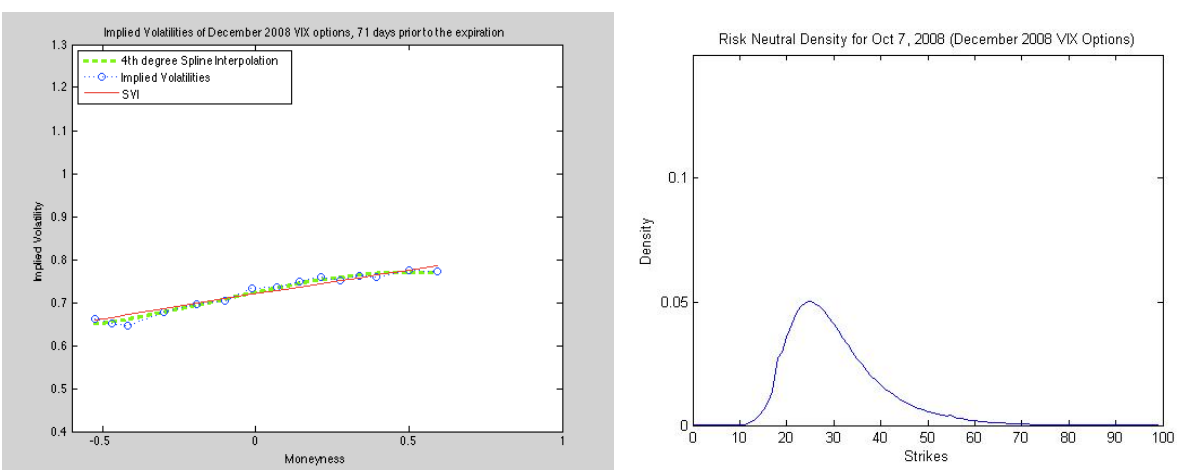


Figure 4 Left: The implied volatilities of December 2008 VIX options traded on October 7, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 4 shows the implied volatilty plot of December 2008 VIX options traded during the time of crisis on October 7, 2008. We clearly see that the level of implied volatilty moves up as the ATM implied volatilty is 0.7244. We also see that the curve of the implied volatility is flat. This accompanies the wider range of traded strikes and shorter height of the RND.

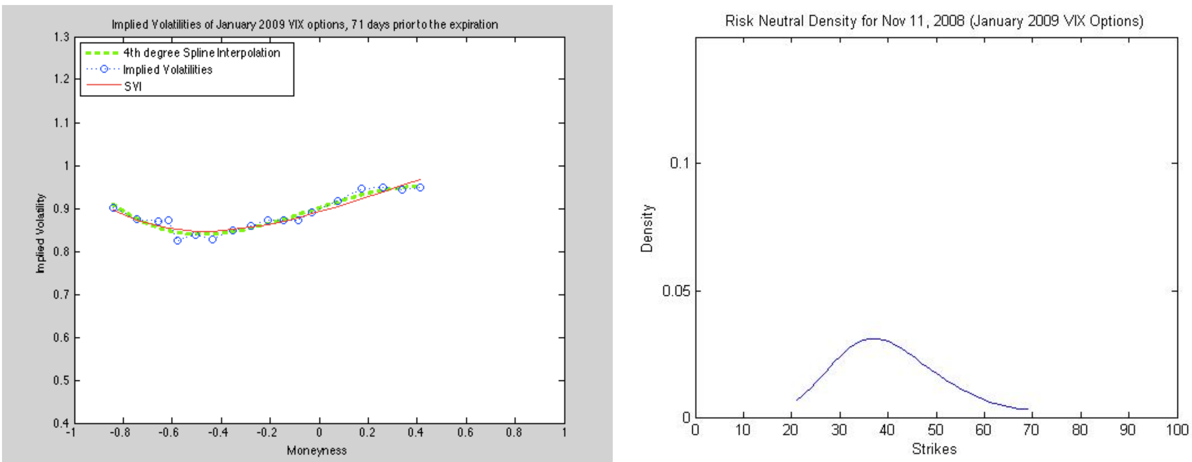


Figure 5 Left: The implied volatilities of January 2009 VIX options traded on November 11, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph

The VIX options implied volatilities of January 2009 traded on November 11, 2008 are higher than the previous month as the ATM implied volatility is 0.8971 with a slope of 0.0046. Noticeably, the flat implied volatility curve is accompanied by the wide RNDs curve with low height. The VIX futures closed at 46.26 on this day, which is considerably higher than the average closing price for the VIX from January 1, 1990 to August 11, 2014, 20.02.

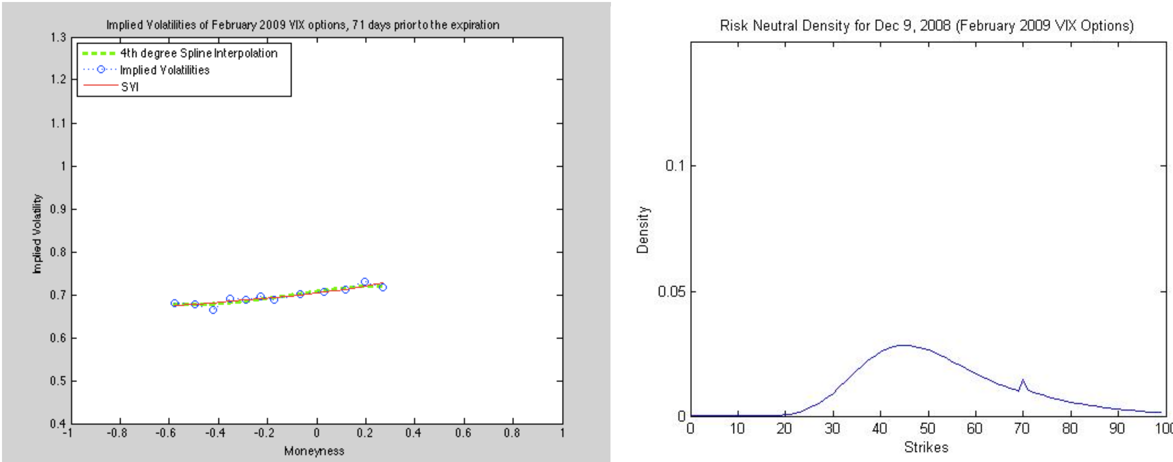


Figure 6 Left: The implied volatilities of February 2009 VIX options traded on December 9, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

The closing level of VIX futures of February 2009 VIX options traded on December 9, 2008 is 53.38. This level is significantly higher than the average VIX closing price. The level of ATM implied volatility is 0.7075, which is lower than the previous months but it is significantly higher than the implied volatility level of VIX options during the normal market condition. The slope of the ATM implied volatility is 0.0016, which is smaller than any other implied volatility slopes we analyzed. As we can see, the RND curve has wide tails with a short height.

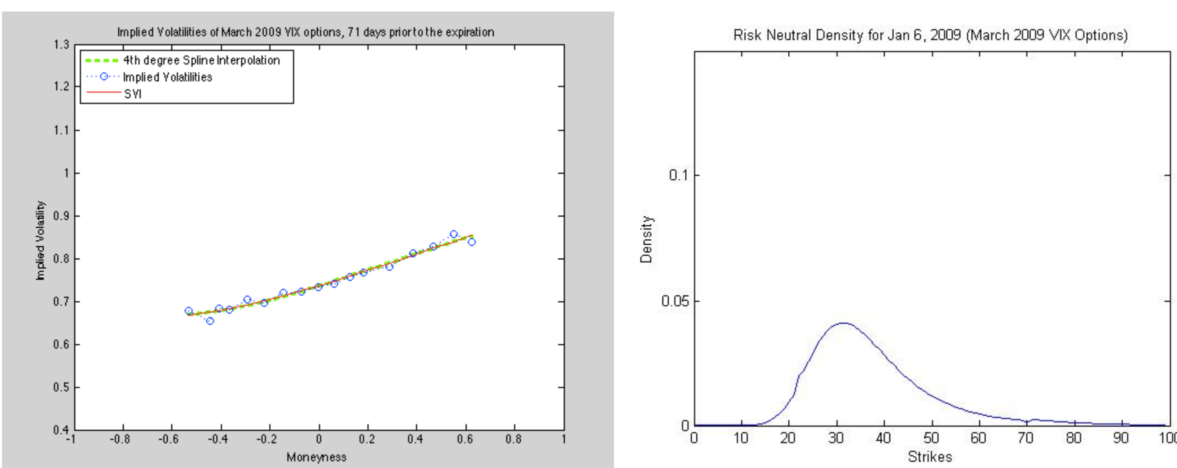


Figure 7 Left: The implied volatilities of March 2009 VIX options traded on January 6, 2009, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 7 on the left above shows the implied volatility curve of March 2009 VIX options traded on January 6, 2009. The closing price of VIX futures on this day is 37.50, which is significantly lower than the previous month. The ATM implied volatility is 0.7335 with a slope of 0.0047. We see that the height of the RND is higher than the previous month and the tails are narrower.

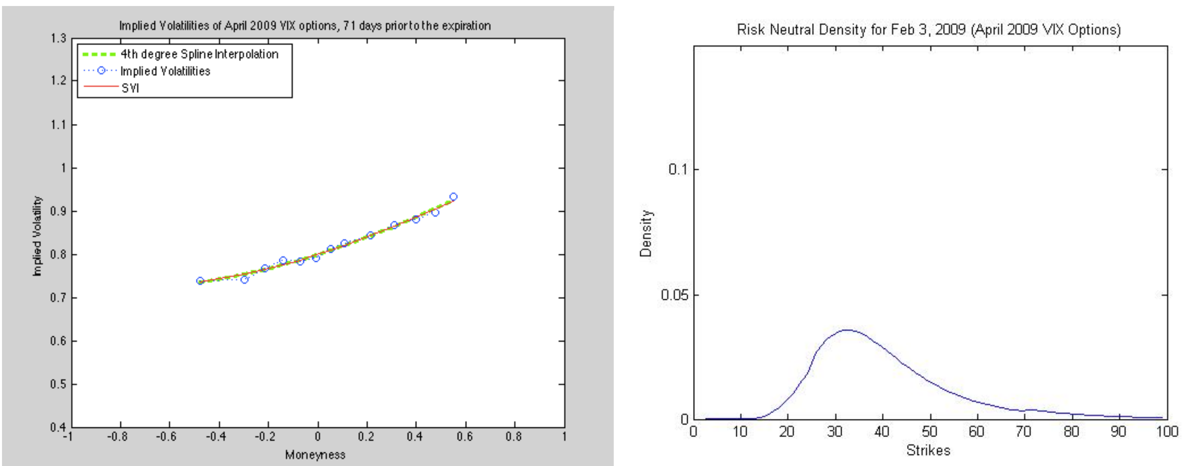


Figure 8 Left: The implied volatilities of April 2009 VIX options traded on February 3, 2009, 71 days prior to the expiration. Right: Right: The RND extracted from the IVs of the left graph.

Figure 8 demonstrates the implied volatilty curve of April 2009 VIX options traded on Febuary 3, 2009. At this time there is still an uncertainity about the future of the market. The ATM implied volatility is still high at 0.7997 with a slope at this point of 0.0043.

Comments on the implied volatilities and the RNDs of April 2008 to April 2009 VIX options

VIX option expiration	ATM volatility	Slope	Underlying asset level
16-Apr-08	0.6664	0.0064	26.56
21-May-08	0.5782	0.0105	25.49
18-Jun-08	0.5471	0.0193	23.95
16-Jul-08	0.5665	0.0142	21.20
20-Aug-08	0.6664	0.0064	23.88
17-Sep-08	0.6388	0.0105	23.46
22-Oct-08	0.6062	0.0141	23.00
19-Nov-08	0.5886	0.0058	24.60
17-Dec-08	0.7244	0.0046	30.31
21-Jan-09	0.8971	0.0053	46.26
18-Feb-09	0.7075	0.0016	53.38
18-Mar-09	0.7335	0.0047	37.50
15-Apr-09	0.7997	0.0043	40.30

Table 1: The ATM slopes and volatilities of VIX options from April 2008 to April 2009, 71 days before the expirations

Note that we are looking at the implied volatilities of 71 days prior to when the VIX options expire for each month. During the time before the crisis, one can see from the table that the range of level of ATM volatility is 0.5471 to 0.6664 for April 2008 to November 2008. The levels of ATM volatility from December 2008 to April 2009 are all above 0.7000. We can see a significant increase on the level of volatilities. One other thing to notice is the slope of the implied volatilities at ATM. Starting from November 2008, the slopes of implied volatilities have decreased compared to the months before November. The implied volatilities after November have flatter curves compared to the months before. These flatter curves are accompanied with the shorter height and longer tail RNDs.

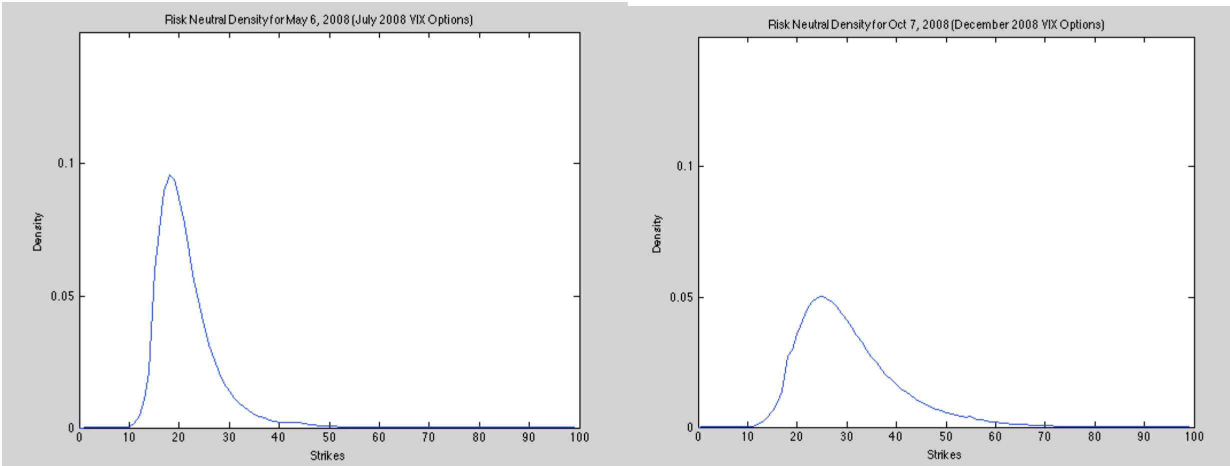


Figure 9 Left: the RND of July 2008 VIX options traded on May 6, 2008. Right: the RND of December 2008 VIX options traded on October 7, 2008.

Figure 9 displays the comparison of two RND curves from two different market conditions. The left graph shows RND curve during normal market condition, and the right graph shows the RND curve at the beginning of severe market condition. We clearly see that the RND curve of the left graph is thinner and its height is taller compared to the right one; this indicates that during the normal market condition smaller range of strike prices are traded. While the shorter height and longer tails RND curve on the right can be translated as people fear of an uncertain market condition so that the wider range of strike prices are traded.

March 2011 to March 2012

This section illustrates the results from fitting the implied volatilities to the SVI and the 4th degree spline interpolation, and extracting risk neutral density out of the implied volatility produced by the spline interpolation of the March 2011 to March 2012 VIX options. This period of time is relatively calm compared to the year of 2008. However there is a relatively high spike in September's of VIX index due to the US market's downgrade. We found that the trends of the ATM level of implied volatility and the RNDs are similar to the one of 2008. When the market is distressed, the levels of implied volatility move higher and the height of the RNDs move lower accompanying the wider RNDs tails.

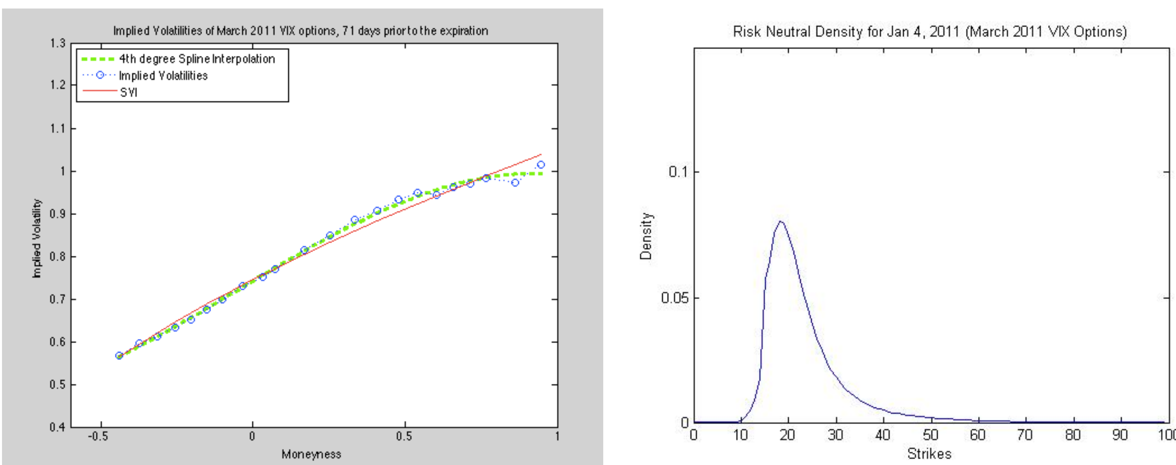


Figure 10 Left: The implied volatilities of March 2011 VIX options traded on January 4, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

For the implied volatility plot in Figure 10, the ATM implied volatility is 0.0.7448. A slope of ATM implied volatility is 0.0182. The underlying asset level of the options is 23.25. The market is relatively calm at this point. From the RND plot on the right, we can see that the range of strikes is around +-10 from the underlying asset level.

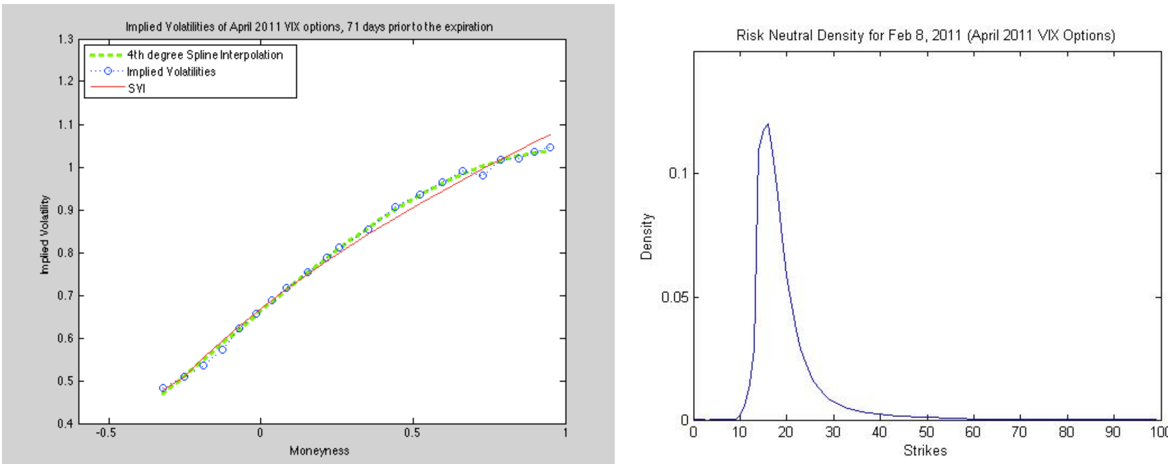


Figure 11 Left: The implied volatilities of April 2011 VIX options traded on February 8, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 11 shows the implied volatility and the RND plot of April 2011 VIX futures traded on February 8, 2011 when the VIX futures closed at 19.30. Notice that as the underlying asset moves lower, the RND height is taller and the RND tails are shorter. At this date, the slope of the ATM implied volatility is 0.0313. The level of the ATM implied volatility is 0.6629.

Figures 21 to 25 in Appendix B show the plots of implied volatility and the extracted RNDs of 71 days to maturity of the May 2011 to September 2011 VIX options, They show similar trends as Figure 10 and 11 as they are in the same market condition, and the underlying assets are between 19.30 to 22.45, which are closed to the average VIX index we mentioned earlier.

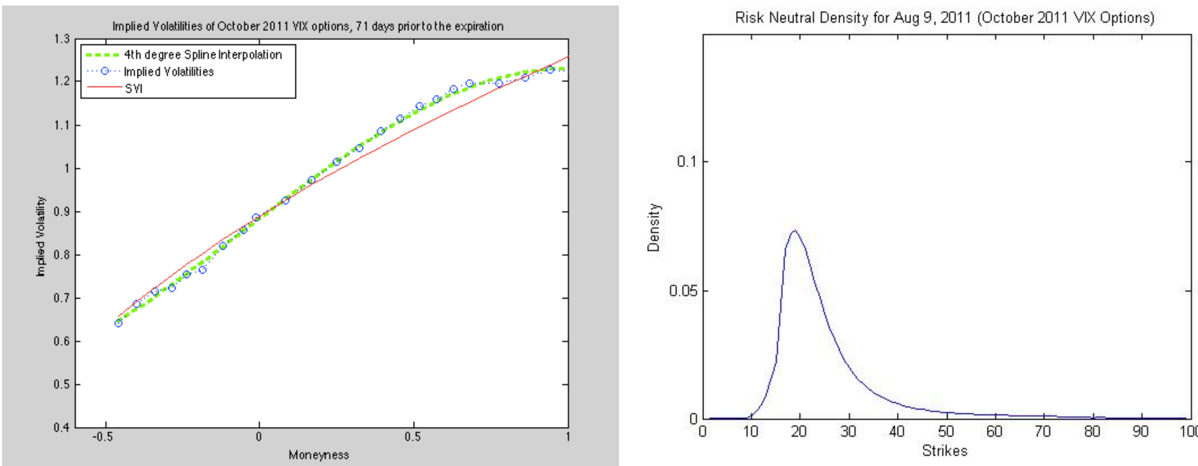


Figure 12 Left: The implied volatilities of October 2011 VIX options traded on August 9, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 12 on the left shows the implied volatilities of October 2011 VIX options traded on August 9, 2011. This is the time when the VIX was about to spike in 2011. The level of the ATM implied volatility is higher than the one of September 2011 VIX options as show in Figure 25 in Appendix B. The curve of the implied volatility of this month's option data becomes flatter as the ATM implied volatility slope is 0.0215, lower than the ATM implied volatility slope of September 2011 VIX options, which is 0.0300. The RND curve is slightly wider and its height is slightly shorter than the September 2011 data. The VIX futures closing price is 25.30 on this day.

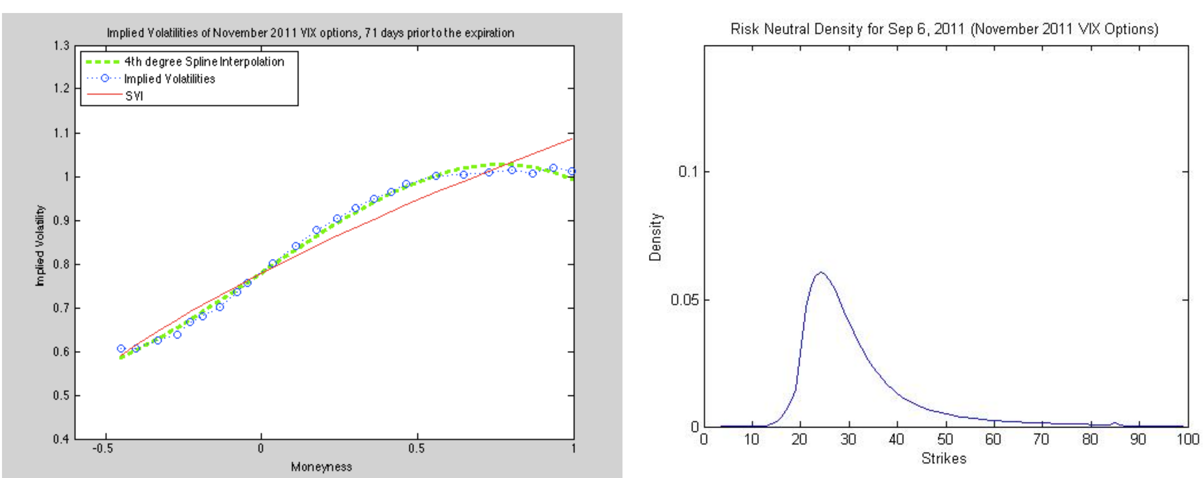


Figure 13 Left: The implied volatilities of November 2011 VIX options traded on September 6, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 13 demonstrates the implied volatilty curve of November 2011 VIX options traded on September 6, 2011. The ATM implied volatility is still high at 0.7997 with a slope at this point of 0.0043. Notice that as the trading day gets closer to the day that VIX spikes, the RND tails are wider, and its height is shorter.

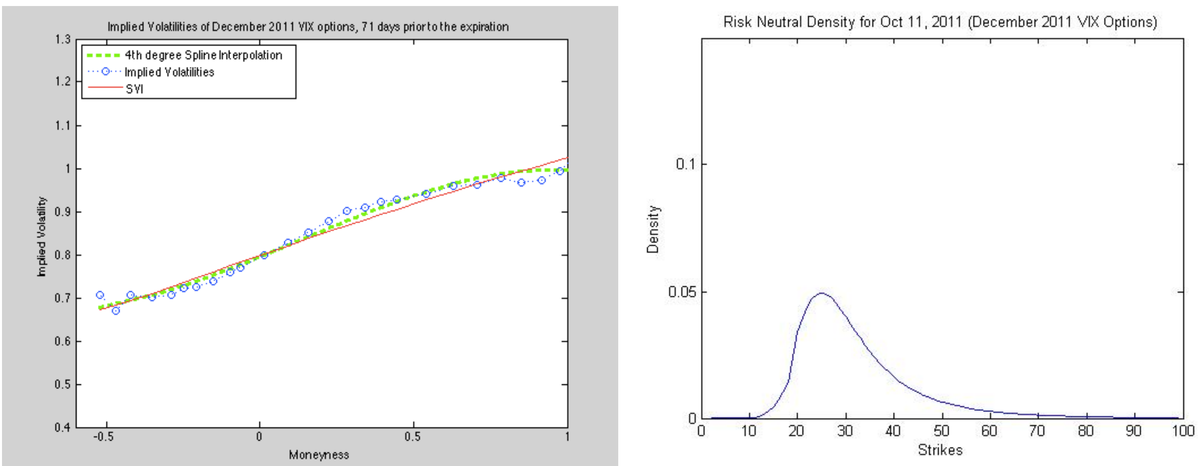


Figure 14 Left: The implied volatilities of December 2011 VIX options traded on October 11, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

Figure 14 shows the implied volatilty plot of December 2011 VIX options traded during the time of high VIX futures level October 11, 2011. We clearly see that the level of implied volatilty moves up as the ATM implied volatilty is 0.8042. We also see that the curve of the implied volatility is flat. This accompanies the wider range of traded strikes and shorter height of the RND as we observed before.

Figures 25 to 27 in Appendix B show the plots of implied volatility and the extracted RNDs of 71 days to maturity of the January 2012 and February 2012 VIX options. They show similar trends as Figure 14 as they are in the same market condition, and their underlying assets are high.

Comments to the fitted SVI and 4th degree spline interpolations to the implied volatilities of March 2011 to March 2012 VIX options

Figures 10 to 14 show that the SVI fits VIX implied volatilities do not work well since all of the implied volatilities' curves are concave down. The 4th degree spline interpolation with 1-knot fits the implied volatilities of VIX options much better than the SVI fit. Hence we continue choosing the implied volatilities produced by the spline interpolation instead of the implied volatilities produced from the SVI to estimate the risk neutral density of March 2011 to March 2012 VIX options.

VIX option expiration	ATM volatility	Slope	Underlying asset level
16-Mar-11	0.7448	0.0182	23.25
20-Apr-11	0.6629	0.0313	19.30
18-May-11	0.7329	0.0228	22.45
15-Jun-11	0.7102	0.0327	21.10
20-Jul-11	0.7612	0.0337	19.70
17-Aug-11	0.7229	0.0312	20.30
21-Sep-11	0.7738	0.0300	21.30
19-Oct-11	0.8840	0.0215	25.30
16-Nov-11	0.7860	0.0151	31.40
21-Dec-11	0.8042	0.0088	32.00
18-Jan-12	0.8611	0.0109	29.95
15-Feb-12	0.8011	0.0120	30.20
21-Mar-12	0.7711	0.0196	25.05

Table 2: The ATM slopes and volatilities of VIX options from March 2011 to March 2012, 71 days before the expirations

Note again that we are looking at the implied volatilities of 71 days prior to when the VIX options expire for each month. During the period of March 2011 to March 2012, VIX reach its highest point around September 2011. During these time periods, we see a similar trend of implied volatilities as when VIX spiked in 2008. From table 2, the level of ATM implied volatilities have increased since October 2011 accompanied with the decrease in slopes and flatter curves of implied volatilities.

Discussion

Figure 1 to 15 show that the SVI fits VIX implied volatilities well if they are concave up or straight lines. However, SVI does not produce a good fit to implied volatilities if they are neither of them. As introduced previously, the SVI parameterization reads, $\sigma_{BS}^2(k) = a + b\{\tilde{\rho}(k-m) + \sqrt{(k-m)^2 + \tilde{\sigma}^2}\}$, and the right asymptote of the SVI parameterization as $k \to \infty$ reads $\sigma_{BS,R}^2(k) = a + b(1+\rho)(k-m)$. Notice that the term adding to a in the SVI equation is always positive and convex with respect to k. Hence since the SVI is convex, we cannot use it to fit with the implied volatilities that do not have the same convex shape.

The 4[th] degree spline interpolation with 1-knot fits the implied volatilities of VIX options much better than the SVI fit. It provides a smooth curve through the available implied volatilities of VIX options without any restriction to the shape of the curve. This difference allows us to choose the implied volatilities produced by the spline interpolation instead of the implied volatilities produced from the SVI to estimate the risk neutral density.

Chapter 5 Conclusion and recommendations

Conclusion

The 4th degree spline interpolation fit well to the implied volatility of VIX options. With careful observation, we found that as the trading days get closer to the time of the crisis, the level of the implied volatility of VIX options is higher, and the curve of the implied volatility get flatter. If the trading days were in the time during the distressed market, the implied volatility will remain in the high level as the level of the underlying assets are still high. The extracted RND height is taller with the narrower tail during the relatively calm market condition, and the RND height will be shorter with wider tails during the distressed market condition. The SVI does not fit VIX implied volatility well due to the concavity.

Recommendation and future study

(i) The SVI parameterization poorly fits VIX implied volatilities as it does not allow concavity. Future research should be done on finding a parametric method that fits the actual VIX implied volatilities better.

(ii) Due to the inability to adequately account for obtaining the implied volatility at strikes that option prices are not available, we set the implied volatility of those strikes equal to the nearest available implied volatility. With this method we can get the option prices of those strikes to extract the RND tails. However, we do not get the smooth RND tails, a better method should be found to generate the smooth tails of the RND of VIX options.

(iii) For future study, we should analyze more frequently over time to expiration for the same month option to see the how the implied volatilities behave.

References

Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*, 637-654.

Buhler, H. (2008). *Volatility Markets.* Consistent modeling, hedging and practical implimentation of Variance Swap Market Models. VDM Verlag Dr. Mller.

Figlewski, S. (2009, February). Estimating the Implied Risk Neutral Density for the U.S. Market Portfolio. *Volatility and Time Series Econometrics: Essay in Honor of Robert F. Engle, NYU Working Paper*(No. FIN-08-004). Retrieved from http://ssrn.com/abstract=1354492

Forde, M., Jacquier, A., & Mijatovic, A. (2010, October 27). Asymptotic formulae for implied volatility in the Heston model. *Proceedings of the Royal Society, 466*(2124), 3593-3620. Retrieved from http://rspa.royalsocietypublishing.org/content/466/2124/3593

Gairat, A. (n.d.). *Variance swaps.* Retrieved 12 10, 2014, from IVolatility.com: http://www.ivolatility.com/doc/VarianceSwaps.pdf

Gatheral, J. (2004, May 26). Global Derivatives & Risk Management 2004. *A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives*. Retrieved from http://bfi.cl/papers/

Gatheral, J. (2011). *The Volatility Surface: A Practitioner's Guide.* Hoboken, New Jersey, United States of America: John Wiley & Sons.

Gatheral, J., & Jacquier, A. (2010). Convergence of Heston to SVI. *Quantitative Finance*, 1129-1132.

Institutional White Papers. (n.d.). Retrieved 12 10, 2014, from CBOE: https://www.cboe.com/Institutional/RCInstitutional.aspx

Lewis, A. L. (2000). *Option Valuation Under Stochastic Volatility: With Mathematica Code.* Finance Press.

Martin, I. (2011, March). Simple Variance Swaps. *NBER Working Paper Series*. Cambridge, Massachusetts, United States of America. Retrieved from http://www.nber.org/papers/w16884

Pepitone, J. (2011, August 8). *Fear index soars to 2-year high*. Retrieved 11 10, 2014, from CNN Money: http://money.cnn.com/2011/08/08/markets/vix_fear_index/

Rhoads, R. (2014, August 12). *CBOE Options Hub*. Retrieved April 27, 2015, from A Historical Perspective on Current VIX Levels: http://www.cboeoptionshub.com/2014/08/12/historical-perspective-current-vix-levels-2/

Appendix A: MATLAB scripts

```
%% Matlab script to calculate implied volatility of VIX options
%% input data/accessing data
mydata = days13mo; %% input data as matrix
putcall = mydata(:,6);
mylist = [2]; % to get the begin line of each month data
for i=2:size(mydata(:,6))-1,
    if putcall(i+1) ~= putcall(i)
        mylist=[mylist i+1];
    end
end
mylist = [mylist length(mydata(:,6))+1];
[m,n] = size(mylist);
% the following is to store the data into the array
% ie. List_k is the list of strikes, where list_k{i} is the vector of strikes
of month i
for i=1:n-1
    list_k{i}=mydata(mylist(i):mylist(i+1)-1,5); % list of strikes of options
for each month i
    list_bid{i}=mydata(mylist(i):mylist(i+1)-1,13); % list of bid price of
options for each month i
    list_ask{i}=mydata(mylist(i):mylist(i+1)-1,14); % list of ask of options
for each month i
    list_mean{i}=(list_bid{i}+list_ask{i})/2;
    % list_s{i}=mydata(mylist(i):mylist(i+1)-1,15); % underlying asset, each
list has only one value
end

% 3 mo. daily treasury yield curve rates on the trading date
%r = [0.14 0.15 0.11 0.07 0.03 0.05 0.03 0.03 0.02 0.02 0.01 0.01 0.02];
r = [2.19 2.19 1.48 1.48 1.41 1.41 1.63 1.63 2.02 2.02 1.86 1.86 1.87 1.87
1.66 1.66 0.82 0.82 (0.29+0.18)/2 (0.29+0.18)/2 0.03 0.03 0.14 0.14 0.32 0.32
]/100;
S = [26.56 26.56 25.49 25.49 23.95 23.95 21.2 21.2 23.88 23.88 23.46 23.46 23
23 24.6 24.6 30.31 30.31 46.26 46.26 53.38 53.38 37.5 37.5 40.3 40.3];
% reference
% http://www.treasury.gov/resource-center/data-chart-center/interest-
rates/Pages/TextView.aspx?data=yieldYear&year=2008

T = 71/365; % 71 day to maturity
t0=30/365;

%% BS for Bid price of call and put options
% The following this the way that the data are stored using array.
% i=1, the data of call options of the first month data
% i=2, the data of put options of the first month data
% i=3, the data of call options of the second month data
% i=4, the data of put options of the second month data
% and so on..
for i=1:n-1,
    if mod(i,2) ~= 0
        for k=1:length(list_k{i})

ivbid{i}(k)=volafun(S(i),list_k{i}(k),T,r(i),t0,'call',list_bid{i}(k));
```

```
ivask{i}(k)=volafun(S(i),list_k{i}(k),T,r(i),t0,'call',list_ask{i}(k));

ivmean{i}(k)=volafun(S(i),list_k{i}(k),T,r(i),t0,'call',list_mean{i}(k));
            K{i}(k) = log(list_k{i}(k)/(S(i)*exp(r(i)*T)));
        end
    elseif  mod(i,2) == 0

        for j=1:length(list_k{i})

ivbid{i}(j)=volafun(S(i),list_k{i}(j),T,r(i),t0,'put',list_bid{i}(j));

ivask{i}(j)=volafun(S(i),list_k{i}(j),T,r(i),t0,'put',list_ask{i}(j));

ivmean{i}(j)=volafun(S(i),list_k{i}(j),T,r(i),t0,'put',list_mean{i}(j));
           K{i}(j) = log(list_k{i}(j)/(S(i)*exp(r(i)*T)));
        end
    end
end
```

```
% function to calculate the implied volatility of options using bisection
% method
function [sig] = volafun(S,K,T,r,t0,type,market)
   sig = 0.5; % the initial value
   sig_max = 20; % max implied volatilty 20 = 2000%
   sig_min = 0.00001; % min implied volatilty
   count = 0;
   err = myBS(S,K,T,t0,r,sig,type)-market;
   while (abs(err) > 0.00001) && (count < 3000)
       if err <0
           sig_min = sig;
           sig = (sig_max + sig)/2;
       else
           sig_max = sig;
           sig = (sig_min+sig)/2;
       end
       err = myBS(S,K,T,t0,r,sig,type)-market;
       count = count+1;
   end
   if count == 3000
       sig = NaN;
   end
end
```

```
% function to price VIX options based on Black-Scholes option pricing
% treating the VIX futures as the underlying

function [price] = myBS(S,K,T,t0,r,sig,type)
    d1 = (log(S/(K*exp(-r*t0)*exp(-r*T))) + (r + sig^2/2)*T) / (sig*sqrt(T));
    d2 = d1 - sig*sqrt(T);
  if strcmp(type,'call')
    price = exp(r*t0)*S*normcdf(d1,0,1) - K*exp(-r*T)*normcdf(d2,0,1);
  else if strcmp(type,'put')
    price = K*exp(-r*T+r*t0)*normcdf(-d2,0,1) - S*exp(-r*t0)*normcdf(-
d1,0,1);
      end
  end
end



%% SVI fitting step
function [F] = svi(M,LM)
xdata = M(:,1);
ydata = M(:,2);
logmoneyness = LM;
helper = @(x) ydata - transpose(sqrt(max(mySVIvalue(x,logmoneyness),0)));

startingValues=[0.1;0.5;0.1;-0.5;0.005]; %% initial value
lowerBounds = [-1,0,-2,-2,-10^3];
upperBounds = [1,10^3,10^3,2,10^3];
options = optimset ( 'MaxIter' , 50 );
aa = lsqnonlin(helper,startingValues,lowerBounds,upperBounds,options);
sviValue = sqrt(max(mySVIvalue(aa,logmoneyness),0));
plot(xdata,ydata,'o')
hold on
plot(xdata, sviValue)
F = aa;



% function for the SVI parameterization
function y = mySVIvalue(x,xdata)
    y = x(1)+x(2)*(x(5)*(xdata-x(3))+sqrt((xdata-x(3)).^2+x(4).^2));
end
```

```
% function to produce the RND curve
function [Y] = rnd(M, s, r)

T=71/365;
t0=30/365;
x = M(:,1);
y = M(:,2);

%Least-squares approximation by splines
figure
spl2 = spap2(1, 4, x, y);  % 1 knot, 4th order
fnplt(spl2,':g',2);
hold on
plot(x,y,':bo');
hold off
title('Figure: Implied Vola from all calls and puts minimum bid price 0.025')
xlabel('VIX') % x-axis label
ylabel('Implied Volatility') % y-axis label


%% compute the dense set of implied volatility out of Least-squares
approximation by splines
% extract dense set of IV from the fitted spline curve
m = length(ceil(x(1)):1:floor(x(length(x))));
for ii=1:m,
    IVinterpolated(ii) = fnval(spap2(1, 4, x, y),ii+x(1)-1);
end

KK = ceil(x(1)):1:floor(x(length(x)));
% the implied volatility corresponding to the above strikes
IVnew = IVinterpolated;

% and then convert them back into the options prices
for jj = 1:size(KK,2),
    newPrices(jj) = myBS(s,KK(jj),T,t0,r,IVnew(jj),'call');
end

% to obtain the RND from the new set of option Prices
for kk = 2:size(KK,2)-1,
    RND(kk) = exp(r*T)*[newPrices(kk+1)-newPrices(kk-1)]/(KK(kk+1)-KK(kk-
1))+1;
end

% calculate the density of the newPrices
for kk = 2:size(KK,2)-1,
    densityCall(kk) = exp(r*T)*[newPrices(kk+1)-2*newPrices(kk)+newPrices(kk-
1)]/(KK(kk+1)-KK(kk))^2;
end
figure
plot(KK(2:size(KK,2)-1),densityCall(2:m-1))

%% Adding Tails
KK1 = 0:1:ceil(x(1))-1;
for ii = 1:size(KK1,2),
    newPricesleft(ii) = myBS(s,KK1(ii),T,t0,r,IVnew(1),'call');
```

```
end
KK3 = floor(x(length(x)))+1:1:100;
for ii = 1:size(KK3,2),
    newPricesright(ii) = myBS(s,KK3(ii),T,t0,r,IVnew(m),'call');
end

allPrices1 =
[transpose(newPricesleft);transpose(newPrices);transpose(newPricesright)];
allPrices =transpose(allPrices1);

allK=0:1:100;

Z=[transpose(allK) allPrices1];
for ss = 2:size(allK,2)-1,
    AdensityCall(ss) = exp(r*T)*[Z(ss+1,2)-2*Z(ss,2)+Z(ss-1,2)]/(Z(ss+1,1)-
Z(ss,1))^2;
end

Y=[transpose(allK(:,1:100)),transpose(AdensityCall)];
% deleting row that has negative RND
deleterow1 = false(size(Y, 1), 1);
% loop over all lines
for n = 1:size(Y, 1)
  if Y(n, 2) < 0
    deleterow1(n) = true;
  end
end
% delete rows
Y(deleterow1,:) = [];

%for the left tail, if a strike below x(1)+1 has higher RND value, remove it
deleterow2 = false(size(Y, 1), 1);
temp = densityCall(2);
for aa=1:size(Y(1:x(1)+1,2));
    if Y(aa,2)> temp
         deleterow2(aa) = 1;
    end
end
Y(deleterow2,:) = [];

plot(Y(:,1),Y(:,2))
axis([0,100,0,0.15])
title('Risk Neutral Density for __enter trading date_ (_enter the month of_
VIX Options)')
xlabel('Strikes') % x-axis label
ylabel('Density') % y-axis label
```

Appendix B: Implied volatility curves and RNDs figures

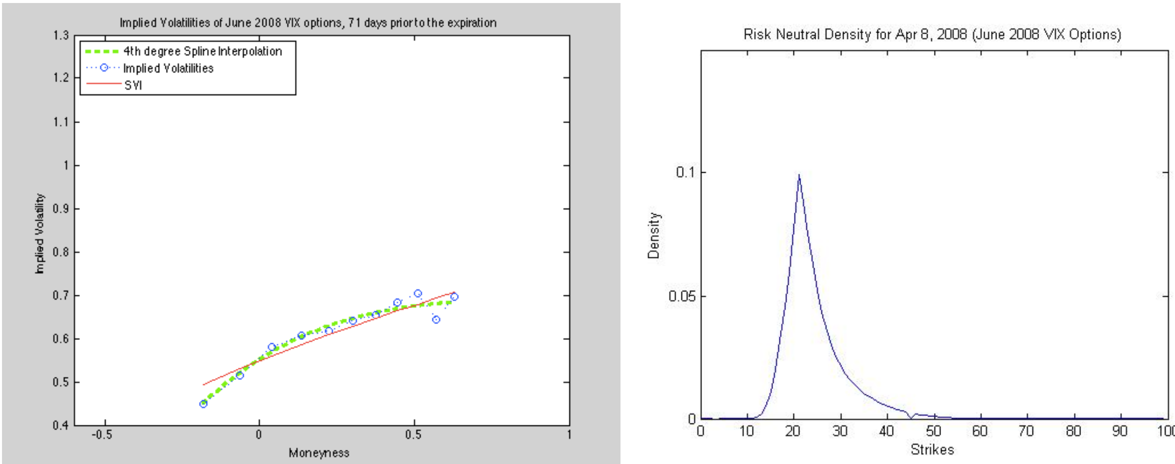


Figure 15 Left: The implied volatilities of June 2008 VIX options traded on April 8, 2008 (71 days prior to the expiration. Right: Right: The RND extracted from the IVs of the left graph.

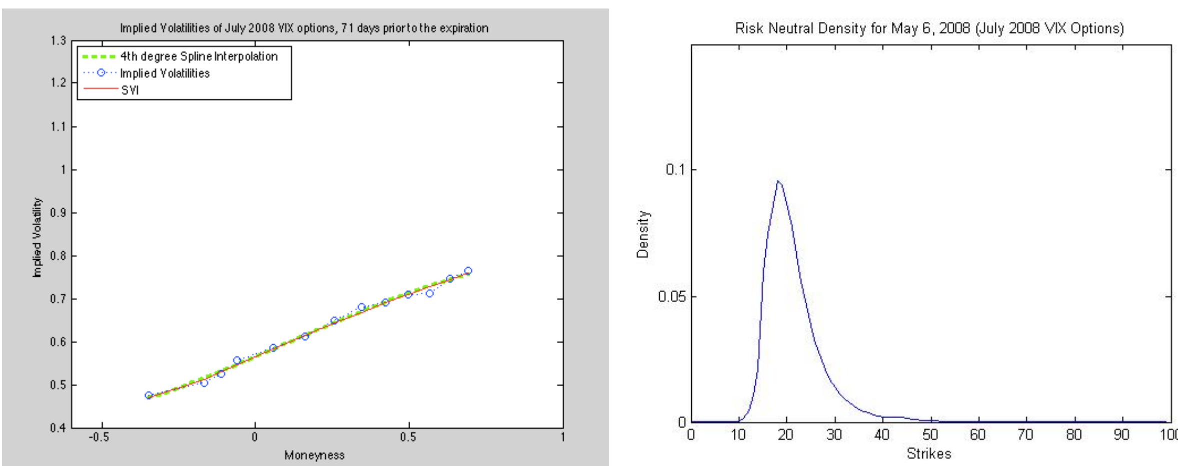


Figure 16 Left: The implied volatilities of July 2008 VIX options traded on May 6, 2008 (71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

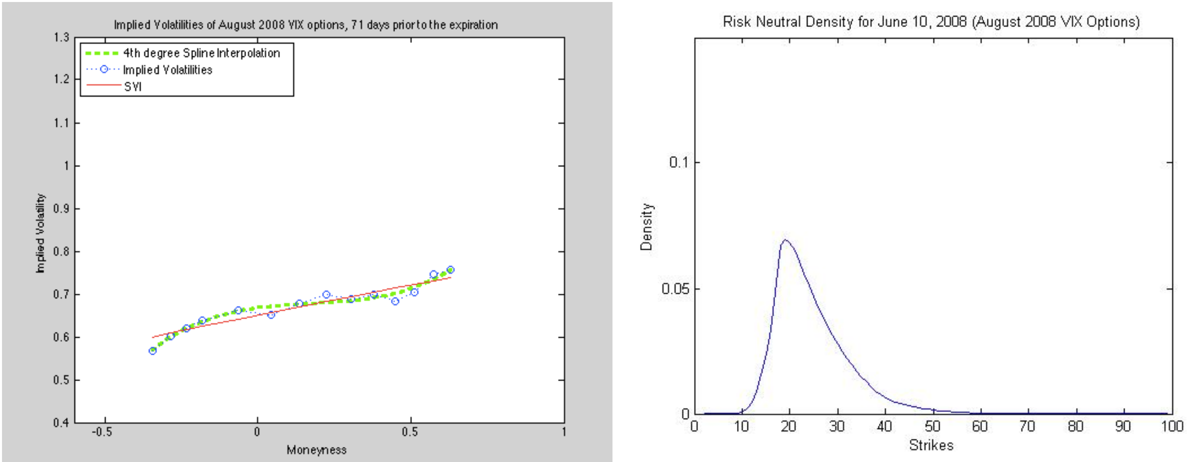


Figure 17 Left: The implied volatilities of August 2008 VIX options traded on June 10, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

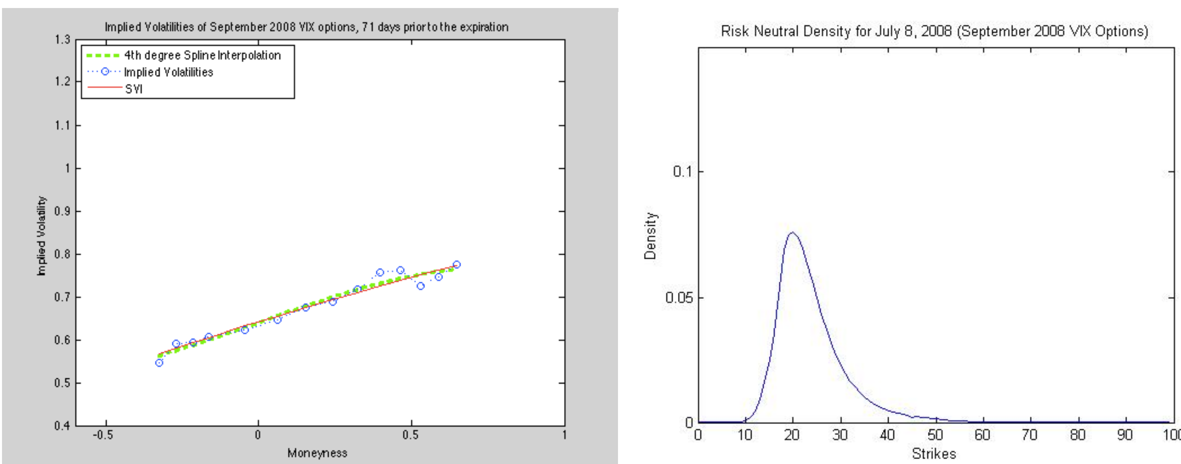


Figure 18 Left: The implied volatilities of September 2008 VIX options traded on July 8, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

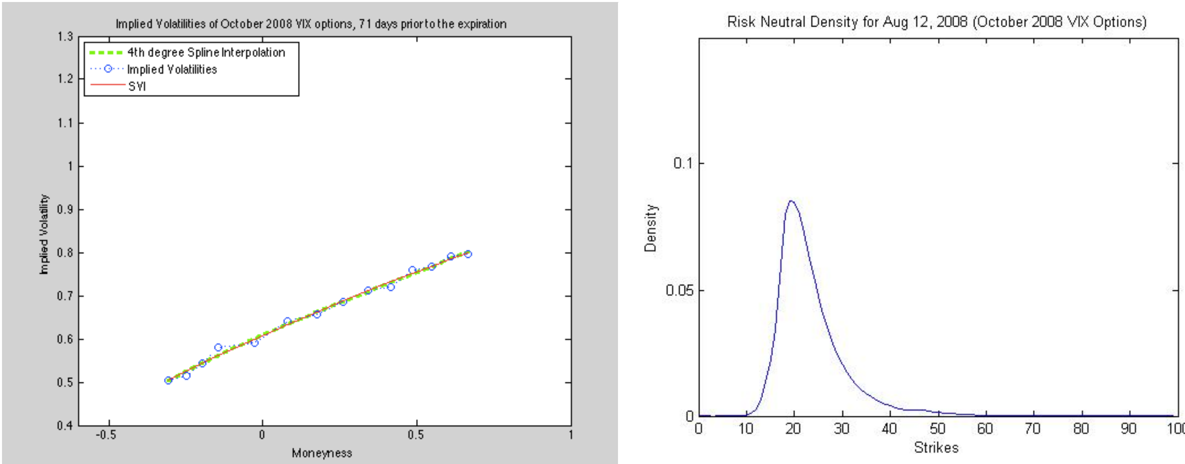


Figure 19 Left: The implied volatilities of October 2008 VIX options traded on August 12, 2008, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

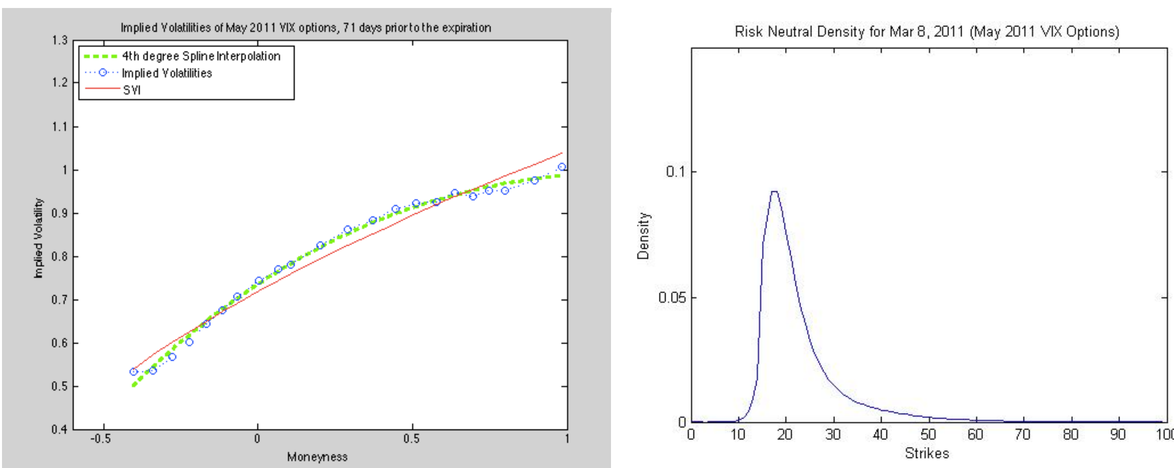
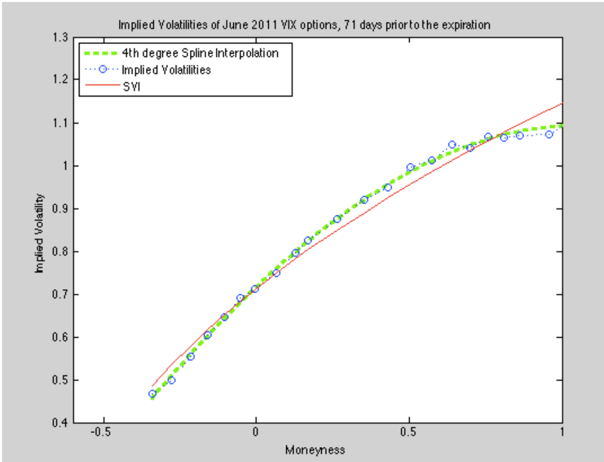


Figure 20 Left: The implied volatilities of May 2011 VIX options traded on March 8, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.



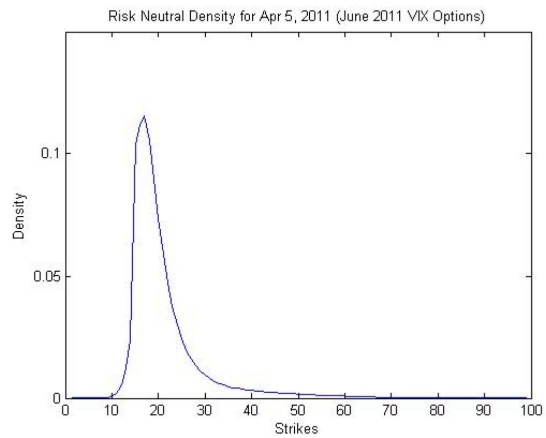
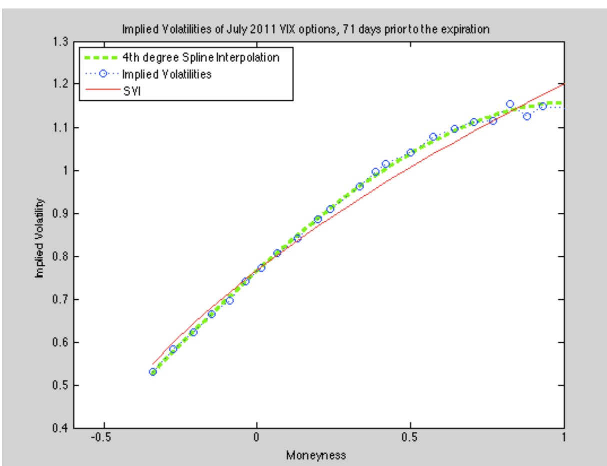


Figure 21 Left: The implied volatilities of June 2011 VIX options traded on April 5, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.



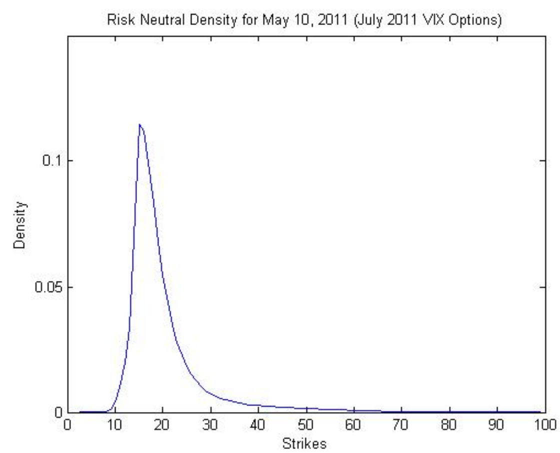


Figure 22 Left: The implied volatilities of July 2011 VIX options traded on May 10, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

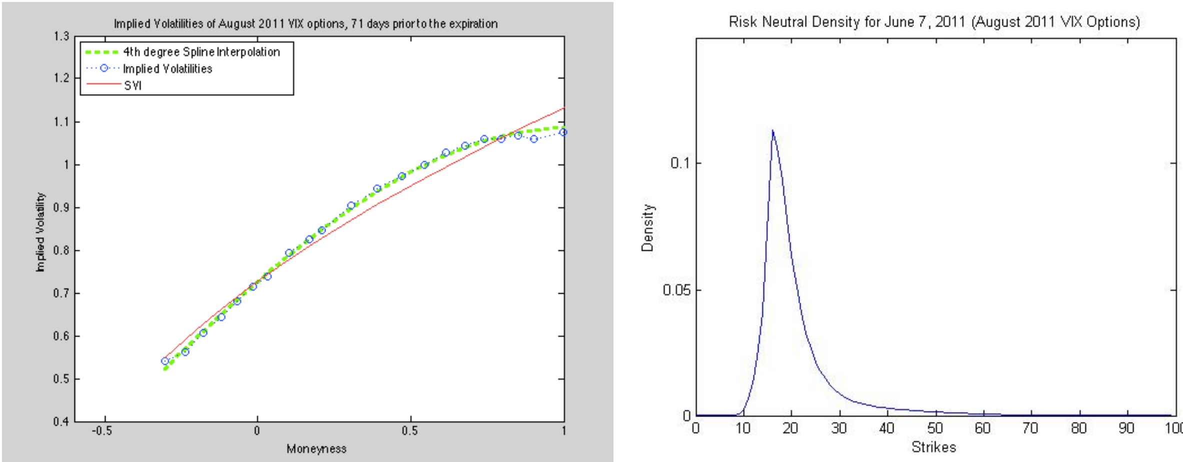


Figure 23 Left: The implied volatilities of August 2011 VIX options traded on June 7, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

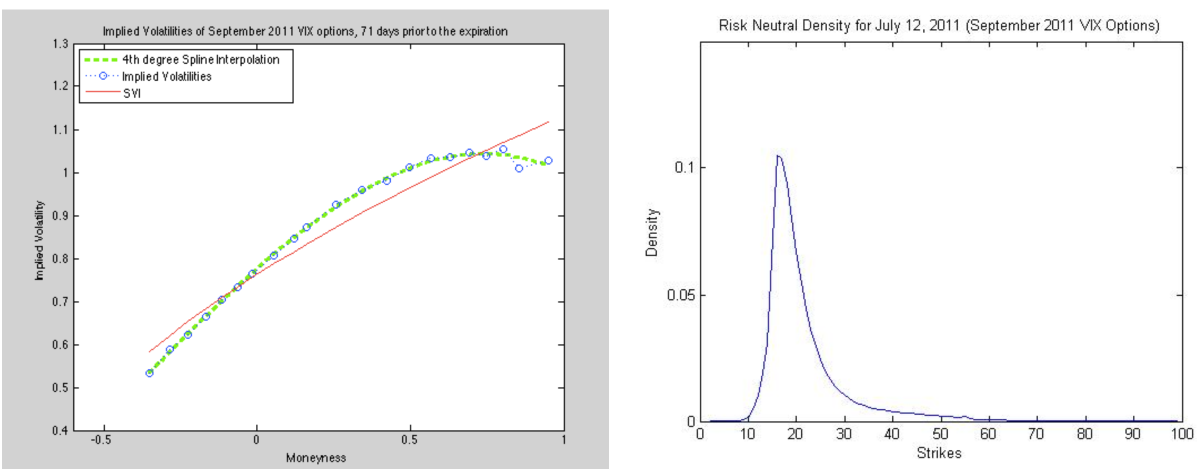


Figure 24 Left: The implied volatilities of September 2011 VIX options traded on July 12, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

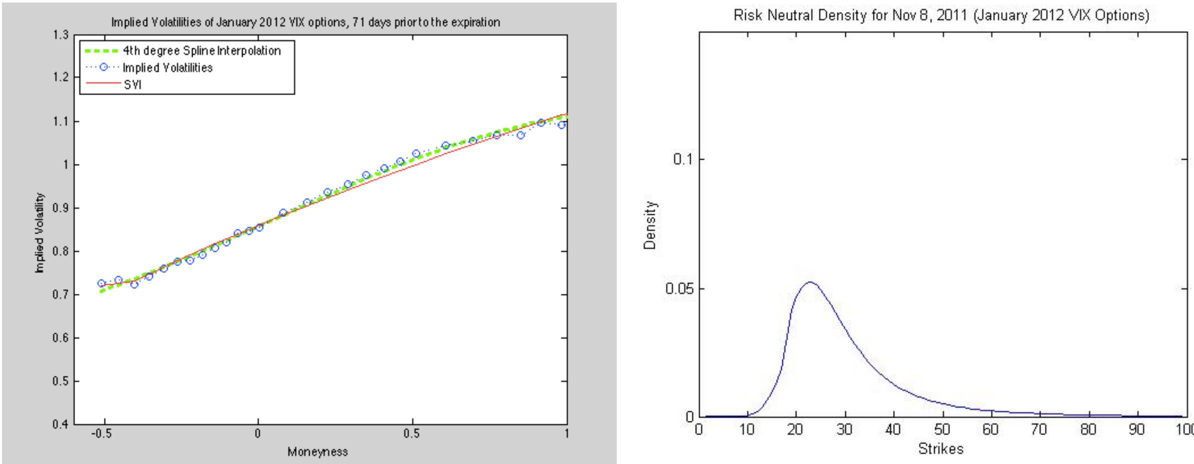


Figure 25 Left: The implied volatilities of January 2012 VIX options traded on November 8, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.

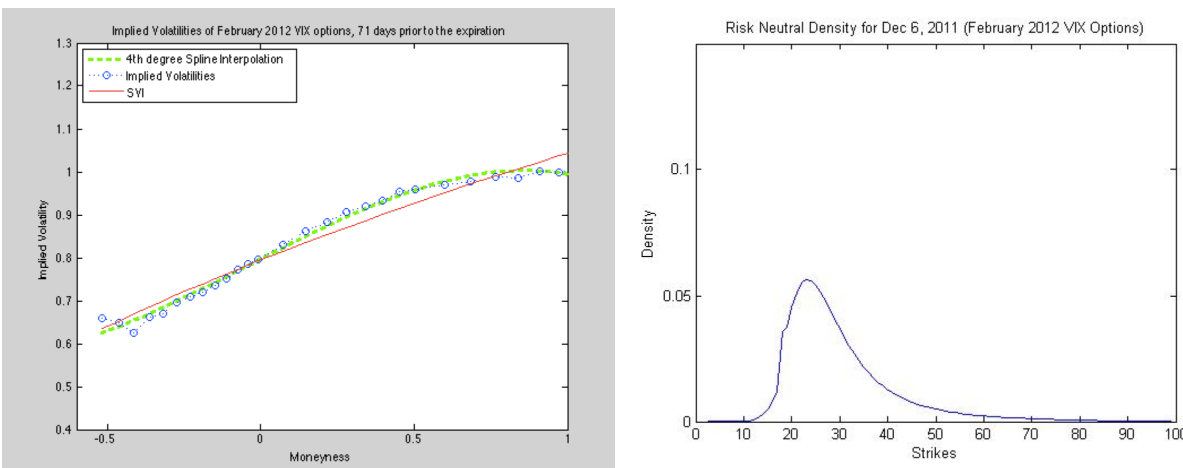
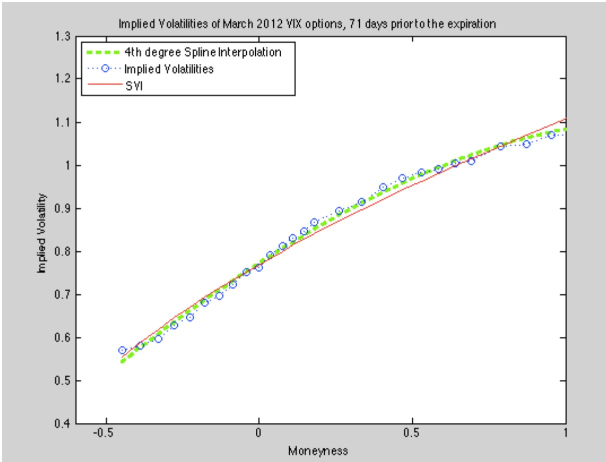


Figure 26 Left: The implied volatilities of February 2012 VIX options traded on December 6, 2011, 71 days prior to the expiration. Right: The RND extracted from the IVs of the left graph.



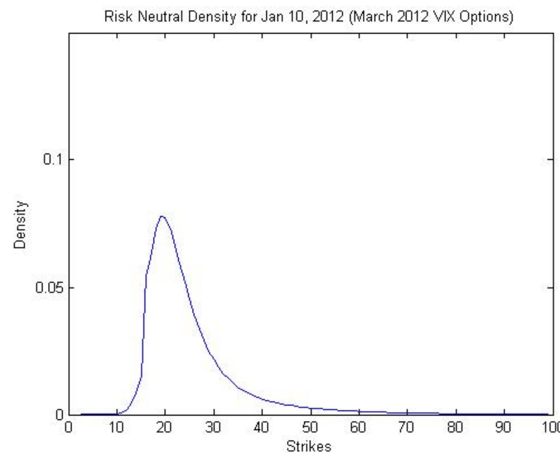


Figure 27 Left: the implied volatilities of March 2012 VIX options traded on January 10, 2012, 71 days prior to the expiration. Right: the RND extracted from the IVs of the left graph.