Cracks in Half Plane, Cracks in Discs

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by

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Abstract

This project starts from an eigenvalue problem of Steklov type which models displacement fields occurring during the destabilization of faults in elastic media. We introduce and study in details the functional space V, a generalized solution space for this eigenvalue problem. The original formulation valid for half planes is then extended to problems in disks by conformal mapping.

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Contents

List of Figures

List of Algorithms

1 Introduction

Understanding the dynamics of slow slip events on geological faults can be vitally important in seismology. By slow event, we mean important slip taking place on an intermediate time scale (i.e. minutes to months). This is much longer than seismic time scales (seconds) but much shorter than geological time scales (hundreds of years). Since slow slip events are aseismic (i.e. there is no associated seismic wave), their detection is possible only by means of modern GPS techniques which can resolve "less-than-cm" surface displacements. Two types of phenomena can be related to slow slip events: silent earthquakes and nucleation (or initiation) phases for (ordinary) earthquakes. Either phenomenon can be modeled using the same physics (slip weakening of friction force) in association with the same mathematics which involve eigenvalue analysis [5] [6] [7].

Accounts of silent earthquakes in subduction zones near Japan, New Zealand, Alaska and Mexico were recently reported in the literature. Silent earthquakes are rather large ($6 \leq Magnitude \leq 8$) and produce surface displacements (range about 2-6 cm) that can be picked up by GPS techniques.

The earthquake nucleation (or initiation) phase, which precedes dynamic rupture, was uncovered by detailed seismological observations and recognized in laboratory experiments. Important physical properties of the nucleation phase (characteristic time, critical fault length, etc) are obtained through simple mathematical properties of unstable evolution. Early detection of the nucleation phase from surface displacements has the potential to play a key role in short time prediction of large earthquakes.

Unstable evolution can be modeled using linear stability analysis, which leads to a static eigenvalue problem. We will denote EP this eigenvalue problem throughout the rest of this thesis. In order to provide a mathematical study of eigenvalue problem EP, we introduce the functional space V, defined as the closure of all continuously differentiable functions in $\mathbb{R}^2$ with compact support, with regard to the L^2 norm of the gradient. The space V has the following property: V contains strictly the Sobolev space $H^1(\mathbb{R}^2)$. We provide a rigorous proof this fact in this thesis. We also give a striking example, with proof, of a function in V that tends uniformly to infinity at infinity.

Eigenvalue problem EP can also be defined in the lower half plane with Neumann boundary condition at the surface. It then serves as a model for destabilization of strike slip (or sometimes called antiplane) faults. In order to apply this destabilization model to the case of cracks in elastic rods, we use a conformal mapping from the lower half plane to the unit disk, which we will denote Φ. We then turn our attention to the following related question: what are all the diffeomorphisms Ψ that will map harmonic functions in a open set U of $\mathbb{R}^2$ to harmonic functions in a open set V of $\mathbb{R}^2$? We prove that a necessary and sufficient condition on such mappings Ψ, is that on each connected component of U, Ψ is associated to either a holomorphic function or to the conjugate of a holomorphic function.

Assuming that a crack in the lower half plane has a linear shape, we study its image under the conformal mapping Ψ. The image is a circular arc for which we determine the center, the radius, and angular span. If the depth of the linear crack tends to infinity, we show that its image under Ψ approaches the point $(1, 0)$, and we give asymptotic estimates of corresponding center, radius, and angular span. This of particular importance since the crack inverse problem was entirely solved for linear cracks in half space in [1], under the assumption that these cracks were far enough from the surface. We plan to find an analogous solution to the crack inverse problem in elastic rods in some future work.

2 Problem Statement

We denote by D the lower half plane $D = \{(x_1, x_2)|x_2 < 0\}$ in the non-dimensional coordinate system Ox_1x_2. Its boundary, denoted by

$$\Gamma_{obs} := \{(x_1, x_2)|x_2 = 0\}, \tag{1}$$

is called the "surface observation" boundary. Let Γ be a bounded connected arc, called cut, crack or fault, included in D, which will be assumed as a smooth oriented curve with no double points. Our problem is formulated in a non-dimensional coordinate system, which means that we chose a characteristic length L. A natural choice for L is provided by relating it to the physical length of the fault. In our coordinate system we decide to fix the length of the fault, by imposing $|\Gamma| = 2$. Let $(x_1(v), x_2(v))$, $v \,\epsilon\, [-1, 1]$ be the arc length parametric equations for Γ. We take the unit normal n to be indirectly perpendicular to the tangent vector. We denote $\Omega = \Omega(\Gamma)$ the open set, $\Omega := D/\Gamma$: it has the fault Γ as an internal boundary.

Figure 1 shows a possible choice for Γ , a line segment of length 2 with angle of inclination α and depth d.

2.1 Direct Eigenvalue Problem (EP):

Let us start by defining the direct problem. We consider the following (Steklov type) eigenproblem involving the Laplace operator: find $\Upsilon : \Omega \to \mathbb{R}$ and $\beta \,\epsilon\, \mathbb{R}$ such that

$$div(\nabla\Upsilon) = 0 \ in \ \Omega \tag{2}$$

$$\partial_n\Upsilon = 0 \ on \ \Gamma_{obs} \tag{3}$$

$$[\partial_n\Upsilon] = 0 \ on \ \Gamma_{obs} \tag{4}$$

$$[\partial_n\Upsilon] - \beta[\Upsilon] = 0 \ on \ \Omega \tag{5}$$

where Υ satisfies some decay at infinity, and where we have denoted using [ ] the jump across Γ (i.e. $[w] = w^+ - w^-$, where $w^+(x) = \lim_{t \to 0^+} w(x + tn(x))$, $w^-(x) = \lim_{t \to 0^-} w(x + tn(x)))$ and $\partial_n = \nabla \bullet n$ the corresponding normal derivative, with the unit normal n pointing towards the positive side. Let us remark that the above eigenvalue problem, associated with the wave equation with a special boundary condition (i.e. Robin type with opposite sign), depends only on the position and shape of Γ. All the physical properties (elasticity, friction, loads, etc) of the system are concentrated in the non-dimensional parameter β and its associated eigenvector.

2.2 Physical Significance of the Eigenvalue Problem

Let us now describe the static (or quasi-static) problem associated to this friction law. These processes correspond to "slow slip events," which characterize crustal displacements developing on intermediate time scales (days, month). Compared to geological time scales, these phenomena are sufficiently rapid to have been referred to as "silent earthquakes", because at their time scale the crust is essentially behaving elastically, as for earthquakes. Note that the time scale governing usual earthquakes is of the order of seconds: the process is then fully dynamic. Even if the formulation is quite different in that case, the same approach is valid during the first part of the initiation (or nucleation) phase. The dynamical process is then quite slow and the same eigenvalue analysis is applicable.

Using the method of linear stability analysis, we interpret the function Υ as an approximation to the surface displacements of the system for some relatively long period of time. In order to obtain a reasonable approximation, we use the following constituitive relations. The equilibrium equation for this system yields (2). Since there is a stress-free condition imposed on the surface of the earth, we have that (3) follows. On the interface Γ the shear stress has no jumps and a frictional contact is supposed to act, implying (4).

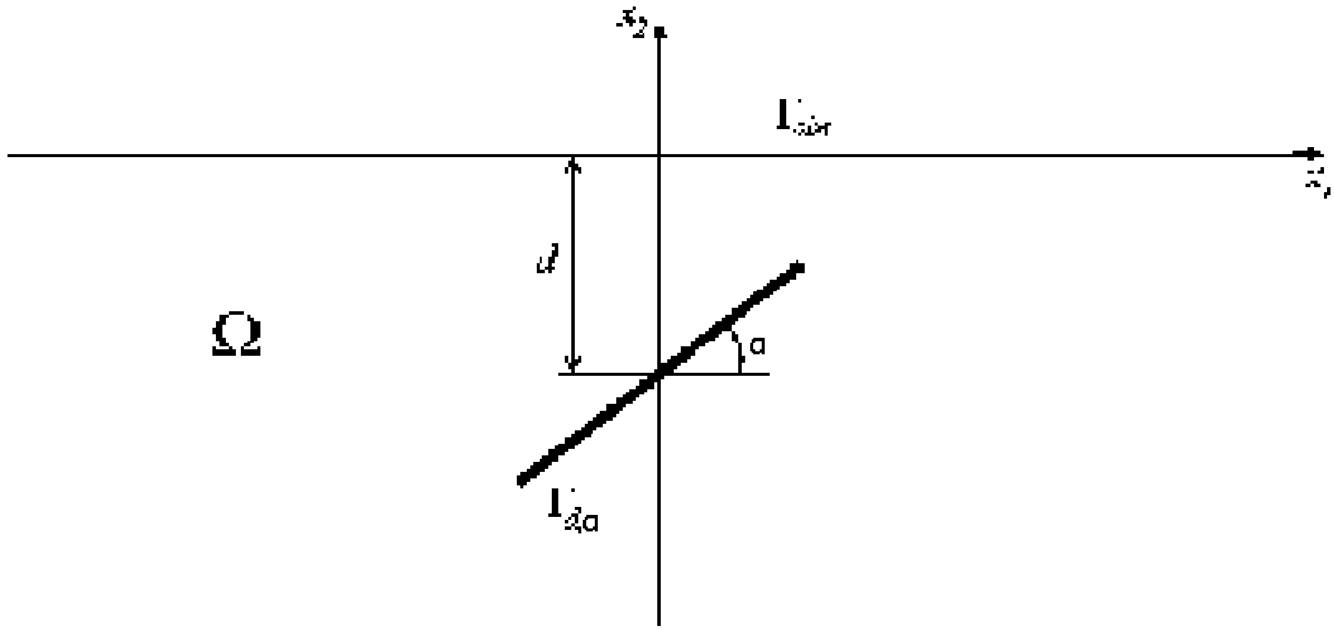

Figure 1: The fault Γ in the lower half plane.

3 The Functional Space V

We have that the solution of our eigenvalue problem to be in the functional space $V(\Omega) = H^1_B(\Omega)$ [1] [2], where $H^1_B(\Omega)$ is the space of functions in $H^1(\Omega)$ with bounded support. We first consider a simpler space, termed V, defined as the closure of $C_0^\infty(\mathbb{R}^2)$ with the norm $\|u\| = (\int_{\mathbb{R}^2} |\nabla u|^2)^{1/2}$. The results which we obtain in V can be generalized to $V(\Omega)$.

We have that $u \,\epsilon\, V$ if $\exists\, u_n \,\epsilon\, C_0^\infty(\mathbb{R}^2)$ such that $\|\nabla u - \nabla u_n\|_{L^2} \to 0$ as $n \to \infty$.

We first look at the various properties of V.

3.1 Equivalence Classes in V

Let u be a constant c. Then we have that setting $u_n = 0$ satisfies the above requirements, so V does contain constants. However, since V is a Banach Space, the $\|u\| = 0$ iff $u = 0$. Therefore we have that constants are equivalent to 0 in V.

Then for any function $u \,\epsilon\, V$ and $c \,\epsilon\, \mathbb{R}$ we have that $u \cong u + c$ in V.

3.2 Is Either $H^1(\mathbb{R}^2)$ or V Included in the Other?

3.2.1 <u>**Theorem 1:**</u> V is not a Subset of $H^1(\mathbb{R}^2)$

Proof: To show $V \nsubseteq H^1(\mathbb{R}^2)$, we consider functions of the form $f(r,\theta) = \frac{1}{(1+r^2)^\alpha}$.

We first notice that $\int(\int_0^\infty |f(r,\theta|^2 \, r dr) d\theta = 2\pi \int_0^\infty |f(r,\theta)|^2 \, r dr$ since $f(r,\theta)$ is only a function of r.

Then we have that

$$2\pi \int_0^\infty |f(r,\theta)|^2 \, r dr = 2\pi \int_0^\infty \frac{r}{(1+r^2)^{2\alpha}} dr \tag{6}$$

Since r is nonnegative, then $(1+r^2)^{2\alpha \geq} r^{4\alpha}$ so that

$$2\pi \int_0^\infty \frac{r}{(1+r^2)^{2\alpha}} dr \leq 2\pi \int_1^\infty \frac{r}{r^{4\alpha}} dr + c = 2\pi \int_1^\infty \frac{1}{r^{4\alpha-1}} dr + c \tag{7}$$

where c is the value of the first integral from 0 to 1.

By the p-test, this integral in convergent when $4\alpha - 1 > 1$, or $\alpha > \frac{1}{2}$.

Then $f(r,\theta) \,\epsilon\, L^2((\mathbb{R}^2)$ for $\alpha > 1/2$.

Taking the gradient of $f(r,\theta)$ we have that$\nabla f(r,\theta) = (\frac{-2\alpha r}{(1+r^2)^{\alpha+1}}, 0)$.

Then the L^2norm of $\nabla f(r,\theta)$ is given by

$$2\pi \int_0^\infty |\nabla f(r,\theta)|^2 \, r dr = 2\pi \int_0^\infty \frac{4\alpha^2 r^2}{(1+r^2)^{2\alpha+2}} r dr \leq 8\pi\alpha^2 \int_1^\infty \frac{r^3}{r^{4\alpha+4}} dr + c = 8\pi\alpha^2 \int_1^\infty \frac{1}{r^{4\alpha+1}} dr + c \tag{8}$$

where c is the value of the second integral from 0 to 1.

This converges by the p-test when $4\alpha + 1 > 1$, or $\alpha > 0$.

Therefore, $f(r,\theta) \notin H^1(\mathbb{R}^2)$ when $0 < \alpha \leq \frac{1}{2}$, and if $\alpha > \frac{1}{2}$, then $f(r,\theta) \in H^1(\mathbb{R}^2)$.

From now on, let $0 < \alpha \leq \frac{1}{2}$.

Define $p(r)$ as the plateau function which is 1 when $|r| \leq 1$, 0 for $|r| \geq 2$, and decreasing inbetween.

We have that $p(r/n)$ is $C_0^\infty(\mathbb{R}^2)$, so that the product $f(r,\theta)p(r/n) \,\epsilon\, C_0^\infty(\mathbb{R}^2)\ \forall n$.

We now consider the integral $\lim_{n\to\infty} \int_0^\infty |\nabla f(r,\theta) - \nabla(f(r,\theta)p(r/n))|^2 \, r dr$.

By expansion, $\nabla(f(r,\theta)p(r/n)) = (\nabla f(r,\theta))p(r/n) + f(r,\theta)(\nabla p(r/n))$.

On the interval $[0,n]$, this simplifies to $\nabla f(r,\theta)$ since $p(r/n) = 1$ and $\nabla p(r/n) = 0$.

Then our integral on $[0,n]$ is 0.

We then consider the interval $[2n,\infty]$.

Since $p(r/n) = 0$, we are left with computing $\lim_{n\to\infty} \int_{2n}^\infty |\nabla f(r,\theta)|^2 \, r dr$.

From (8) we obtain

$$\lim_{n\to\infty} \int_{2n}^{\infty} |\nabla f(r,\theta)|^2 \, r dr \leq \lim_{n\to\infty} \int_{2n}^{\infty} \frac{4\alpha^2}{r^{4\alpha+1}} dr = \lim_{n\to\infty} \frac{\alpha}{(2n)^{4\alpha}} = 0 \tag{9}$$

Now we are left with the interval $[n,2n]$. We can write this as

$$\lim_{n\to\infty} \int_n^{2n} |\nabla f(r,\theta)(1-p(r/n)) - f(r,\theta)\nabla p(r/n)|^2 \, r dr \tag{10}$$

Defining $\nabla p(r/n)) = \frac{p'(r/n)}{n}$, and using the fact that $p(r/n)$ is inifnitely differentiable,

we can bound $|p'(r/n)|$ in $[n,2n]$ by some $q < \infty$.

Since $1 - p(r/n) \leq 1$, we can rewrite (10) as

$$\leq \lim_{n\to\infty} \int_n^{2n} (|\nabla f(r,\theta)| + \frac{q}{n} |f(r,\theta)|)^2 r dr \tag{11}$$

$$= \lim_{n \to \infty} \int_{n}^{2n} (|\nabla f(r,\theta)|^2 + \frac{2q}{n} |f(r,\theta)| |\nabla f(r,\theta)| + \frac{q^2}{n^2} |f(r,\theta)|^2) r dr \tag{12}$$

We first examine $\lim_{n \to \infty} \int_n^{2n} \frac{q^2}{n^2} |f(r,\theta)|^2 \, r dr$.

From (8) we obtain $\lim_{n \to \infty} \int_n^{2n} \frac{q^2}{n^2} |f(r,\theta)|^2 \, r dr \leq \lim_{n \to \infty} \frac{q^2}{n^2} \int_n^{2n} \frac{1}{r^{4\alpha-1}} dr$.

For $\alpha \geq 1/4$, we have $4\alpha - 1 \geq 0$, so that $\frac{1}{r^{4\alpha-1}} \leq 1$, so

$$\lim_{n \to \infty} \frac{q^2}{n^2} \int_{n}^{2n} \frac{1}{r^{4\alpha-1}} dr \leq \lim_{n \to \infty} \frac{q^2}{n^2} \int_{n}^{2n} 1 dr = \lim_{n \to \infty} \frac{q^2}{n} = 0 \tag{13}$$

For $0 < \alpha < 1/4$, we have that $\frac{1}{r^{4\alpha-1}} = r^{\beta}$, for some $\beta < 1$. Therefore $\frac{1}{r^{4\alpha-1}} \leq (2n)^{\beta} < 2n^{\beta}$. Then

$$\lim_{n \to \infty} \frac{q^2}{n^2} \int_{n}^{2n} \frac{1}{r^{4\alpha-1}} dr < \lim_{n \to \infty} \frac{q^2}{n^2} (2n - n)(2n^{\beta}) = \lim_{n \to \infty} \frac{2q^2}{n^{1-\beta}} = 0 \tag{14}$$

We now examine $\lim_{n \to \infty} \int_n^{2n} |\nabla f(r,\theta)|^2 \, r dr$.

From (79) we obtain

$$\lim_{n \to \infty} \int_{n}^{2n} |\nabla f(r,\theta)|^2 \, r dr \leq \lim_{n \to \infty} \int_{n}^{2n} \frac{4\alpha^2}{r^{4\alpha+1}} dr = \lim_{n \to \infty} \frac{-\alpha}{(2n)^{4\alpha}} + \frac{\alpha}{(n)^{4\alpha}} = 0 \tag{15}$$

We only have $\lim_{n \to \infty} \int_n^{2n} \frac{2q}{n} |f(r,\theta)| \, |\nabla f(r,\theta)| \, r dr$ to examine.

Then

$$\lim_{n \to \infty} \int_{n}^{2n} \frac{2q}{n} |f(r,\theta)| \, |\nabla f(r,\theta)| \, r dr = \lim_{n \to \infty} \int_{n}^{2n} \frac{2q}{n} (\frac{1}{(1+r^2)^{\alpha}})(\frac{2\alpha r^2}{(1+r^2)^{\alpha+1}}) dr \tag{16}$$

$$= \lim_{n \to \infty} \frac{4q\alpha}{n} \int_{n}^{2n} \frac{r^2}{(1+r^2)^{2\alpha+1}} dr \leq \lim_{n \to \infty} \frac{4q\alpha}{n} \int_{\mathrm{n}}^{2n} \frac{1}{r^{4\alpha}} dr \tag{17}$$

Since $\frac{1}{r^{4\alpha}} \leq \frac{1}{n^{4\alpha}}$, (17) becomes

$$\leq \lim_{n \to \infty} \frac{4q\alpha}{n} \int_{\mathrm{n}}^{2n} \frac{1}{n^{4\alpha}} dr \leq \lim_{n \to \infty} \frac{4q\alpha}{n^{4\alpha}} = 0 \tag{18}$$

Therefore defining $f_{\mathrm{n}} = f(r,\theta) p(r/n)$, we see that $f \, \epsilon \, V, f \notin H^1(\mathbb{R}^2)$ for these values of α.

3.2.2 **Theorem 2:** $H^1(\mathbb{R}^2)$ is a Proper Subset of V

We will now show $H^1(\mathbb{R}^2) \subset V$.

Let $u \epsilon H^1(\mathbb{R}^2)$. Then $u \epsilon L^2(\mathbb{R}^2)$ and $\nabla u \epsilon L^2(\mathbb{R}^2)$. Define $u_n = u(r,\theta)p(r/n)$.

Then we wish to show $\|\nabla u - \nabla u_n\|_{L^2} \to 0$ as $n \to \infty$.

We have $\lim\limits_{n \to \infty} \|\nabla u - \nabla u_n\|_{L^2} = \lim\limits_{n \to \infty} \int_0^\infty |\nabla u(r,\theta) - \nabla(u(r,\theta)p(r/n))|^2 \, rdr$.

As before, the integral on $[0,n]$ is clearly 0, and on the interval $[2n,\infty]$ we are left with $\int_{2n}^\infty |\nabla u(r,\theta)|^2 \, rdr$.

Let $\varepsilon > 0$ be chosen arbitrarily small.

Since $\nabla u \epsilon L^2(\mathbb{R}^2)$, we have that $\exists\, n_1$ s.t. $\forall\, n > n_1 \int_{2n}^\infty |\nabla u(r,\theta)|^2 \, rdr < \varepsilon$.

Therefore the integral on $[2n,\infty]$ is bounded by ε .

The integral on $[n,2n]$ reduces to $\lim\limits_{n \to \infty} \int_n^{2n} |\nabla u(r,\theta)(1-p(r/n)) - u(r,\theta)\nabla p(r/n)|^2 \, rdr$.

Applying Minkowski's Inequality we are left with showing that

$\lim\limits_{n \to \infty} \int_n^{2n} |\nabla u(r,\theta)|^2 \, rdr \to 0$ and $\lim\limits_{n \to \infty} \int_n^{2n} \frac{q^2}{n^2} |u(r,\theta)|^2 \, rdr \to 0$.

For our first integral, we have that $\forall\, \varepsilon > 0$ and $\forall\, n > 2n_1$, then $\int_n^{2n} |\nabla u(r,\theta)|^2 \, rdr < \varepsilon$.

For the second integral, since $u \epsilon L^2(\mathbb{R}^2)$, then $\exists\, n_2$ s.t. $\forall\, n > n_2 \int_n^{2n} \frac{q^2}{n^2} |u(r,\theta)|^2 \, rdr < \varepsilon$.

Then we our left with $\lim\limits_{n \to \infty} \|\nabla u - \nabla u_n\|_{L^2} < 3\varepsilon$, and since ε was arbitrary, we let $\varepsilon \to 0$ and $u \epsilon V$.

3.3 Special Functions in V

From the previous section, we constructed a family of functions that converges to 0 at ∞ in V.

We now construct a function which converges uniformly to infinity at ∞, and is still in V.

Proposition: Show the function $f(r,\theta) = ln(ln(2+r^2))$ is in V but not in H^1.

We have $2\pi \int_0^\infty |f(r,\theta)|^2 \, rdr$ diverges since $f(r,\theta)$ tends to ∞ as r tends to ∞, so $f(r,\theta)$ is not in H^1.

We now define $f_{\text{n}} = (ln(ln(2+r^2)) - ln(ln(n)) - ln2)p(r/n)$.

Then we wish to show that

$$\lim_{n \to \infty} \int_0^\infty |\nabla f(r,\theta)(1-p(r/n)) - (f(r,\theta) - ln(ln(n)) - ln2)\nabla p(r/n)|^2 \, rdr = 0 \tag{19}$$

The integral is 0 on the interval $[0,n]$ for the same reasoning as in (9), and on the interval $[2n,\infty)$ we are left with $\lim\limits_{n \to \infty} \int_{2n}^\infty |\nabla f(r,\theta)|^2 \, rdr$, where $\nabla f(r,\theta) = \frac{2r}{(2+r^2)(ln(2+r^2))}$.

Then we can rewrite our integral above as

$$\lim_{n \to \infty} \int_{2n}^\infty \frac{4r^3}{(2+r^2)^2(ln(2+r^2))^2} dr \tag{20}$$

Since $(2+r^2)^2 \geq r^4$, we have that (20) is $\leq \lim\limits_{n \to \infty} \int_{2n}^\infty \frac{4}{r(ln(2+r^2))^2} dr$.

Then applying $(ln(2+r^2))^2 \geq (ln(r^2))^2 = (2ln(r))^2 = 4(ln(r))^2$,

we have that (20) is $\leq \lim_{n\to\infty} \int_{2n}^{\infty} \frac{1}{r(ln(r))^2} dr = \lim_{n\to\infty} \frac{-1}{ln(r)} |_{2n}^{\infty} = \lim_{n\to\infty} \frac{1}{ln(2n)} = 0$.

Therefore (19) reduces to (12) .

We first consider $\lim_{n\to\infty} \int_n^{2n} |(f(r,\theta) - ln(ln(n)) - ln2)\nabla p(r/n)|^2 \, rdr$.

We apply that $|\nabla p(r/n)| \leq \frac{q}{n}$, $r \leq 2n$.

Then the integral above is $\leq \lim_{n\to\infty} \frac{q^2}{n^2}(2n-n)(2n) \underset{r\epsilon[n,2n]}{\mathrm{Max}} |f(r,\theta) - ln(ln(n)) - ln2|^2$.

To calculate this maxima, we first notice that $ln(ln(2+n^2)) > ln(ln(n^2)) = ln2 + ln(ln(n))$,
so that $f(r,\theta) - ln(ln(n)) - ln2 > 0$ in our interval.

We now remove the absolute values and take the gradient to obtain $\frac{4r(ln(ln(2+r^2)) - ln(ln(n)) - ln2)}{(2+r^2)(ln(2+r^2))}$.
In our interval it follows that this expression is always positive, so the maximum is taken at $r = 2n$,
so that $\underset{r\epsilon[n,2n]}{\mathrm{Max}} |f(r,\theta) - ln(ln(n)) - ln2|^2 = ((ln(ln(2+(2n)^2)) - ln(ln(n)) - ln2)^2$.

Since $ln(ln(2+(2n)^2)) - ln(ln(n)) - ln2 = ln\frac{ln(2+4n^2)}{2ln(n)} < ln\frac{ln(5n^2)}{2ln(n)} = ln\frac{ln5+2ln(n)}{2ln(n)}$, assuming $n > \sqrt{2}$.
Then $\underset{r\epsilon[n,2n]}{\mathrm{Max}} |f(r,\theta) - ln(ln(n)) - ln2|^2 < (ln(1+\frac{ln5}{2ln(n)}))^2$.

Therefore $\lim_{n\to\infty} \frac{q^2}{n^2}(2n-n)(2n) \underset{r\epsilon[n,2n]}{\mathrm{Max}} |f(r,\theta) - ln(ln(n)) - ln2|^2 = 0$.

We now examine $\lim_{n\to\infty} \int_n^{2n} |\nabla f(r,\theta)|^2 \, rdr = \lim_{n\to\infty} \int_n^{2n} \frac{4r^3}{(2+r^2)^2(ln(2+r^2))^2} dr$

From above, this is $\leq \lim_{n\to\infty} \int_n^{2n} \frac{1}{r(ln(r))^2} dr = \lim_{n\to\infty} \frac{-1}{ln(2n)} + \frac{1}{ln(n)} = 0$.

Next, we observe $\lim_{n\to\infty} \int_n^{2n} \frac{2q}{n} |f(r,\theta) - ln(ln(n)) - ln2| \, |\nabla f(r,\theta)| \, rdr$.

Applying that $f(r,\theta) - ln(ln(n)) - ln2 < f(r,\theta)$,

our limit becomes $\lim_{n\to\infty} \int_n^{2n} \frac{2q}{n} \frac{2r^2 ln(ln(2+r^2))}{(2+r^2)(ln(2+r^2))} dr < \lim_{n\to\infty} \int_n^{2n} \frac{2q}{n} \frac{2ln(ln(2+r^2))}{ln(2+r^2)} dr$,

and using $ln(2+r^2) \approx 2ln(r)$, we have $\approx \lim_{n\to\infty} \int_n^{2n} \frac{2q}{n} \frac{ln(2ln(r))}{ln(r)} dr$,

and this simplifies to $\approx \lim_{n\to\infty} \int_n^{2n} \frac{2q}{n} \frac{ln(ln(r))}{ln(r)} dr \leq \lim_{n\to\infty} \frac{2q}{n}(2n-n)\frac{ln(ln(n))}{ln(n)}$.

By L'Hopital's rule, $\lim_{n\to\infty} \frac{ln(ln(n))}{ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{ln(n)}\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{ln(n)} = 0$.

Then (19) tends to 0 as n tends to ∞.
Therefore defining $f_{\mathrm{n}} = (ln(ln(2+r^2)) - ln(ln(n)) - ln2)p(r/n)$, we see that $f \,\epsilon\, V, f \notin H^1$.

4 Mapping the Problem 2.1 into the Unit Disk

Eigenvalue formulation may be used in the study of other crack problems, such as cracks in an elastic cylindrical rod, assuming we can measure the surface displacements as well. However for this problem, we seek a suitable geometry, such as the Unit Disk.

We introduce the following conformal map $\Phi(z) = \frac{z+i}{z-i}$, see Appendix 10.1 for our choice of notations.

4.1 Properties of the Mapping $\Phi(z) = \frac{z+i}{z-i}$

4.1.1 $\Phi(z)$ is a Bijection from Ω to $U/\{(1,0)\}$

Injection: Assume for $z_1, z_2 \in \Omega$ that $\Phi(z_1) = \Phi(z_2)$.

Then

$$\frac{z_1+i}{z_1-i} = \frac{z_2+i}{z_2-i} \Rightarrow (z_1+i)(z_2-i) = (z_1-i)(z_2+i) \Rightarrow z_1z_2 + i(z_2-z_1) = z_1z_2 + i(z_1-z_2) \quad (21)$$

which implies $z_1 - z_2 = z_2 - z_1$, therefore $z_1 = z_2$ and $\Phi(z)$ is an injection.

Surjection: We must show $\Phi : \Omega \to U/\{(1,0)\}$, where U denotes the closed unit disk.

Let $z_0 \in U/\{1\}$, then we claim $z_0 = \frac{z+i}{z-i}$ for some $z \in \Omega$.

Then

$$z_0(z-i) = z+i \Rightarrow z(z_0-1) = i(z_0+1) \Rightarrow z = \frac{i(z_0+1)}{z_0-1} \quad (22)$$

We must now show that $z \in \Omega$. Let $z_0 = x_0 + iy_0$.

Then

$$z = \frac{i(x_0+iy_0+1)}{x_0+iy_0-1} = \frac{-y_0+i(x_0+1)}{x_0-1+iy_0} = \frac{(-y_0+i(x_0+1))(x_0-1-iy_0)}{(x_0-1)^2+y_0^2} \quad (23)$$

For $z = x+iy$ to be in Ω, $y \leq 0$ must follow. Then $(x_0-1)(x_0+1)+y_0^2 \leq 0$ must be true.

Then $x_0^2 + y_0^2 \leq 1$, which is obvious since $z_0 \in U/\{(1,0)\}$.

Therefore $\Phi(z)$ is surjective.

4.1.2 Orientation of the Boundary of the Unit Disk

Consider $z \in \Omega$, where $z = x+iy$.

Then

$$|\Phi(z)| = \left|\frac{z+i}{z-i}\right| = \sqrt{\frac{(x+i(y+1))(x-i(y+1))}{(x+i(y-1))(x-i(y-1))}} = \sqrt{\frac{x^2+(y+1)^2}{x^2+(y-1)^2}} \quad (24)$$

We have that $|\Phi(z)| < 1$ for $y < 0$ and $|\Phi(z)| = 1$ for $y = 0$.

This implies that $\Phi : \mathbb{R} \to \partial U/\{(1,0)\} = \partial\Phi(\Omega)$.

We now consider the function $\Phi(x) = \frac{x+i}{x-i}$ for $x \in \mathbb{R}$.

Then

$$\Phi(x) = \frac{(x+i)(x+i)}{(x+i)(x-i)} = \frac{x^2+2ix-1}{x^2+1} = (\frac{x^2-1}{x^2+1}, \frac{2x}{x^2+1}) = (\Phi_1(x), \Phi_2(x)) \quad (25)$$

We have that Φ_1 is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

Likewise, Φ_2 is decreasing on $(-\infty, -1] \cup [1, \infty)$ and increasing on $[-1, 1]$.

It follows that $\lim_{x \to \pm\infty} \Phi_1(x) = 1$, $\lim_{x \to \pm\infty} \Phi_2(x) = 0$, $\Phi_1(0) = -1 = \Phi_2(-1)$, and $\Phi_2(1) = 1$.

Then by continuity of Φ on $\mathbb{R}$, the fact that $\Phi(x)$ is valued on the boundary, and the previous statements, we conclude that the boundary has a counter-clockwise rotation about the origin.

5 Transformations Preserving Harmonic Functions

Let U and V be open subsets of $\mathbb{R}^2$. Given any harmonic function $u(x_1, x_2)$ in U, and for some bijective C^1 transformation $\Psi : V \to U$, we wish to determine all necessary and sufficient conditions on Ψ such that $u(\Psi(x_1, x_2))$ is harmonic in V. Refer to 10.1 for the homogeneous first order linear case.

To determine these conditions we consider various harmonic functions.

5.1 Three Simple Examples of Harmonic Functions

Harmonic Polynomials: All holomorphic functions are expressable by power series, and since the real and imaginary parts of $P(x+iy)$ form a power series, we have that $ReP(x+iy)$ and $ImP(x+iy)$ are harmonic for any polynomial P.

Radial Functions: We have that

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f(r)}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f(r)}{\partial \theta^2} = 0 \Rightarrow \frac{\partial}{\partial r}(r\frac{\partial f(r)}{\partial r}) = 0 \Rightarrow r\frac{\partial f(r)}{\partial r} = a \Rightarrow f(r) = aln(r) + b \tag{26}$$

where $a, b \in \mathbb{R}$.

Angular functions: From (21), we see

$$\frac{1}{r^2}\frac{\partial^2 f(\theta)}{\partial \theta^2} = 0 \Rightarrow f(\theta) = a\theta + b \tag{27}$$

where $a, b \in \mathbb{R}$.

Exponential Functions:

$$e^{(b_1 x_1 + b_2 x_2)} \tag{28}$$

where $b_1, b_2 \in \mathbb{C}$, $b_1^2 = -b_2^2$.

5.2 Conditions to Preserve Harmonicity

Given that $\Delta u = 0$ in U, find $\Delta \widetilde{u}$ in V, where $\Psi(x_1, x_2) = (\Psi_1(x_1, x_2), \Psi_2(x_1, x_2))$, $y_1 = \Psi_1(x_1, x_2)$, $y_2 = \Psi_2(x_1, x_2)$, and $\widetilde{u}(x_1, x_2) = u(\Psi(x_1, x_2))$.

Since $\widetilde{u}(y_1, y_2) = u(x_1, x_2)$, we have that

$$\Delta_{\mathrm{x}} u(x_1, x_2) = \Delta_{\mathrm{x}} \widetilde{u}(y_1, y_2) = 0 \tag{29}$$

To reach this conclusion using an alternate method, see 10.3.

We first calculate all partial derivatives for $\widetilde{u}$.

$$\frac{\partial \widetilde{u}}{\partial x_1} = \frac{\partial \widetilde{u}}{\partial y_1}\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \widetilde{u}}{\partial y_2}\frac{\partial \Psi_2}{\partial x_1} \tag{30}$$

$$\frac{\partial \widetilde{u}}{\partial x_2} = \frac{\partial \widetilde{u}}{\partial y_1}\frac{\partial \Psi_1}{\partial x_2} + \frac{\partial \widetilde{u}}{\partial y_2}\frac{\partial \Psi_2}{\partial x_2} \tag{31}$$

$$\frac{\partial^2 \widetilde{u}}{\partial x_1^2} = \frac{\partial^2 \widetilde{u}}{\partial y_1^2}(\frac{\partial \Psi_1}{\partial x_1})^2 + 2\frac{\partial^2 \widetilde{u}}{\partial y_1 \partial y_2}\frac{\partial \Psi_1}{\partial x_1}\frac{\partial \Psi_2}{\partial x_1} + \frac{\partial^2 \widetilde{u}}{\partial y_2^2}(\frac{\partial \Psi_2}{\partial x_1})^2 + \frac{\partial \widetilde{u}}{\partial y_1}\frac{\partial^2 \Psi_1}{\partial x_1^2} + \frac{\partial \widetilde{u}}{\partial y_2}\frac{\partial^2 \Psi_2}{\partial x_1^2} \tag{32}$$

$$\frac{\partial^2 \widetilde{u}}{\partial x_2^2} = \frac{\partial^2 \widetilde{u}}{\partial y_1^2}(\frac{\partial \Psi_1}{\partial x_2})^2 + 2\frac{\partial^2 \widetilde{u}}{\partial y_1 \partial y_2}\frac{\partial \Psi_1}{\partial x_2}\frac{\partial \Psi_2}{\partial x_2} + \frac{\partial^2 \widetilde{u}}{\partial y_2^2}(\frac{\partial \Psi_2}{\partial x_2})^2 + \frac{\partial \widetilde{u}}{\partial y_1}\frac{\partial^2 \Psi_1}{\partial x_2^2} + \frac{\partial \widetilde{u}}{\partial y_2}\frac{\partial^2 \Psi_2}{\partial x_2^2} \tag{33}$$

Now we have from (62) that

$$\Delta_{\mathrm{x}}\widetilde{u}(y_1,y_2)=\frac{\partial^2\widetilde{u}}{\partial x_1^2}+\frac{\partial^2\widetilde{u}}{\partial x_2^2}= \tag{34}$$

$$\frac{\partial^2\widetilde{u}}{\partial y_1^2}((\frac{\partial\Psi_1}{\partial x_1})^2+(\frac{\partial\Psi_1}{\partial x_2})^2)+2\frac{\partial^2\widetilde{u}}{\partial y_1\partial y_2}(\frac{\partial\Psi_1}{\partial x_1}\frac{\partial\Psi_2}{\partial x_1}+\frac{\partial\Psi_1}{\partial x_2}\frac{\partial\Psi_2}{\partial x_2})+\frac{\partial^2\widetilde{u}}{\partial y_2^2}((\frac{\partial\Psi_2}{\partial x_1})^2+(\frac{\partial\Psi_2}{\partial x_2})^2) \tag{35}$$

$$+\frac{\partial\widetilde{u}}{\partial y_1}(\frac{\partial^2\Psi_1}{\partial x_1^2}+\frac{\partial^2\Psi_1}{\partial x_2^2})+\frac{\partial\widetilde{u}}{\partial y_2}(\frac{\partial^2\Psi_2}{\partial x_1^2}+\frac{\partial^2\Psi_2}{\partial x_2^2})=0 \tag{36}$$

Or equivalently,

$$\frac{\partial^2\widetilde{u}}{\partial y_1^2}\left|\nabla_{\mathrm{x}}\Psi_1\right|^2+2\frac{\partial^2\widetilde{u}}{\partial y_1\partial y_2}(\nabla_{\mathrm{x}}\Psi_1\bullet\nabla_{\mathrm{x}}\Psi_2)+\frac{\partial^2\widetilde{u}}{\partial y_2^2}\left|\nabla_{\mathrm{x}}\Psi_2\right|^2+\frac{\partial\widetilde{u}}{\partial y_1}\Delta_{\mathrm{x}}\Psi_1+\frac{\partial\widetilde{u}}{\partial y_2}\Delta_{\mathrm{x}}\Psi_2=0 \tag{37}$$

Find necessary and sufficient conditions on Ψ_1,Ψ_2 so that $\Delta_{\mathrm{y}}\widetilde{u}(y_1,y_2)=0$.
Let $u(x_1,x_2)=x_1$, then all partial derivative terms are 0 except $\frac{\partial\widetilde{u}}{\partial y_1}$, and we see that this implies $\Delta_{\mathrm{x}}\Psi_1=0$. Similarly, if we let $u(x_1,x_2)=x_2$, we obtain $\Delta_{\mathrm{x}}\Psi_2=0$.
We now let $u(x_1,x_2)=e^{(x_1+ix_2)}$. Then $\frac{\partial^2\widetilde{u}}{\partial y_1^2}=e^{(y_1+iy_2)}$, $\frac{\partial^2\widetilde{u}}{\partial y_1\partial y_2}=ie^{(y_1+iy_2)}$, $\frac{\partial^2\widetilde{u}}{\partial y_2^2}=-e^{(y_1+iy_2)}$.
Dividing by $e^{(y_1+iy_2)}$, applying that $\Delta_{\mathrm{x}}\Psi_1=\Delta_{\mathrm{x}}\Psi_2=0$, and setting the real and imaginary parts equal to 0, we obtain that $\nabla_{\mathrm{x}}\Psi_1\bullet\nabla_{\mathrm{x}}\Psi_2=0$ and $\left|\nabla_{\mathrm{x}}\Psi_1\right|^2-\left|\nabla_{\mathrm{x}}\Psi_2\right|^2=0$.
Therefore we have that Ψ_1,Ψ_2 must satisfy

$$\Delta_{\mathrm{x}}\Psi_1=\Delta_{\mathrm{x}}\Psi_2=0 \tag{38}$$

$$\nabla_{\mathrm{x}}\Psi_1\bullet\nabla_{\mathrm{x}}\Psi_2=0 \tag{39}$$

$$|\nabla_{\mathrm{x}}\Psi_1|=|\nabla_{\mathrm{x}}\Psi_2| \tag{40}$$

Assuming these conditions hold on $\Psi(x_1+ix_2)=\Psi_1(x_1,x_2)+i\Psi_2(x_1,x_2)$, we show that on each connected component V_j of V , Ψ or its conjugate is holomorphic.
We have that the gradients are orthogonal, and are of equivalent norm,
so $\frac{\partial\Psi_1}{\partial x_1}=\frac{\partial\Psi_2}{\partial x_2}$, $\frac{\partial\Psi_1}{\partial x_2}=-\frac{\partial\Psi_2}{\partial x_1}$ or $\frac{\partial\Psi_1}{\partial x_1}=\frac{\partial\Psi_2}{\partial x_2}$, $\frac{\partial\Psi_1}{\partial x_2}=\frac{\partial\Psi_2}{\partial x_1}$.
Denote $A_j=\{(x_1,x_2)\,\epsilon\,V\ :\ \frac{\partial\Psi_1}{\partial x_1}=\frac{\partial\Psi_2}{\partial x_2}\}$. A_j is closed by definition of continuous functions.
However, the complement of A_j in V_j ($\{(x_1,x_2)\,\epsilon\,V_j\ :\ \frac{\partial\Psi_1}{\partial x_1}=-\frac{\partial\Psi_2}{\partial x_2}\}$) is closed by the same reasoning.
As V_j is connected, and A_j is both open and closed, we have that $A_j=V_j$ or $A_j=\emptyset$.
We proved the following theorem, where, as previously, for ease of notations, we give the two functions $(x_1,x_2)\rightarrow(\Psi_1(x_1,x_2),\Psi_2(x_1,x_2)),x_1+ix_2\rightarrow\Psi_1(x_1,x_2)+i\Psi_2(x_1,x_2)$ the same name Ψ.

5.2.1 Theorem 3

Let U and V be open subsets of $\mathbb{R}^2$, and $\Psi : V \to U$ some bijective C^1 transformation.
The following properties are equivalent:

(i) For any harmonic function u defined on U, $u \circ \Psi$ is harmonic on V.

(ii) On each connected component of V, Ψ or its conjugate $\overline{\Psi}$ is holomorphic.

6 How is Eigenvalue Problem (EP) Transformed Under $\Phi(z) = \frac{z+i}{z-i}$?

We have from the previous section necessary and sufficient conditions to preserve harmonic functions. We now determine the normal derivative after transformation, and simplify using the previous conditions.

6.1 Preserving Normal Derivatives

We have that $\partial_n = n \bullet \nabla_{\mathrm{x}}$for $u(x_1, x_2)$. Find a relation between $\partial_n u$ and $\partial_{\widetilde{n}} \widetilde{u}$, where $\partial_{\widetilde{n}} = \widetilde{n} \bullet \nabla_{\mathrm{y}}$.

We have that

$$\nabla_{\mathrm{x}} u(x_1, x_2) = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}) = (\frac{\partial \widetilde{u}}{\partial y_1}\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \widetilde{u}}{\partial y_2}\frac{\partial \Psi_2}{\partial x_1}, \frac{\partial \widetilde{u}}{\partial y_1}\frac{\partial \Psi_1}{\partial x_2} + \frac{\partial \widetilde{u}}{\partial y_2}\frac{\partial \Psi_2}{\partial x_2}) \tag{41}$$

This last expression can be written as $J^T \nabla_{\mathrm{y}} \widetilde{u}(y_1, y_2)$, where J^T is the transpose of the Jacobian Matrix.

The unit outward normal n can be expressed as $n = \frac{\nabla \mathrm{x} F}{\|\nabla_{\mathrm{x}} F\|}$, where $F(x_1, x_2) = 0$ on U.

By the derivation above, we have that this becomes $\widetilde{n} = \frac{J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2)}{\|J^T \nabla_{\mathrm{y}} \widetilde{u}(y_1, y_2)\|}$.

Therefore we see that

$$\partial_n u = \frac{J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2)}{\left\| J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \right\|} \bullet J^T \nabla_{\mathrm{y}} \widetilde{u}(y_1, y_2) \tag{42}$$

We now assume that Ψ_1, Ψ_2 satisfy all sufficient conditions from 5.2.

These conditions express that J is the multiple of an isometry, more precisely $JJ^T = |\nabla_{\mathrm{x}} \Psi_1|^2 I_2$.

We first calculate $J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \bullet J^T \nabla_{\mathrm{y}} \widetilde{u}(y_1, y_2)$, which can be written as $\nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \bullet JJ^T \nabla_{\mathrm{y}} \widetilde{u}(y_1, y_2)$.

Therefore $J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \bullet J^T \nabla_{\mathrm{y}} \widetilde{u}(y_1, y_2) = ((\nabla_{\mathrm{y}} \widetilde{u}) \bullet (\nabla_{\mathrm{y}} \widetilde{F})) \|\nabla_{\mathrm{x}} \Psi_1\|$.

Also, we have that

$$\left\| J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \right\|^2 = J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \bullet J^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) = \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) \bullet JJ^T \nabla_{\mathrm{y}} \widetilde{F}(y_1, y_2) = \|\nabla_{\mathrm{x}} \Psi_1\|^2 \left\| \nabla_{\mathrm{y}} \widetilde{F} \right\|^2 \tag{43}$$

Then $\partial_n u = \frac{((\nabla_{\mathrm{y}} \widetilde{u}) \bullet (\nabla_{\mathrm{y}} \widetilde{F})) \|\nabla_{\mathrm{x}} \Psi_1\|}{\left\| \nabla_{\mathrm{y}} \widetilde{F} \right\| \|\nabla_{\mathrm{x}} \Psi_1\|} = \frac{(\nabla_{\mathrm{y}} \widetilde{u}) \bullet (\nabla_{\mathrm{y}} \widetilde{F})}{\left\| \nabla_{\mathrm{y}} \widetilde{F} \right\|} = \partial_{\widetilde{n}} \widetilde{u}$.

6.2 Problem Statement in the Unit Disk

We notice from above that letting $\Phi(z) = \frac{z+i}{z-i} = \Psi : \Omega \to U/\{(1,0)\}$, then since Φ is a conformal map, it satisfies all of the necessary and sufficient conditions. To see this, we have that Φ is holomorphic, so that Φ satisfies the Cauchy-Riemann equations, and it is trivial to see that these equations satisfy our conditions. We define $\widetilde{\Upsilon} = \Upsilon(\Phi)$, and since Φ preserves normal derivatives, we have $\partial_n \Upsilon = \partial_{\widetilde{n}} \widetilde{\Upsilon}$. We also define $\widetilde{\Gamma_{obs}}$ as the set $\Phi(\Gamma_{obs})$.

Then we have that 2.1 reduces to the following:

$$div(\nabla \widetilde{\Upsilon}) = 0 \ in \ U/\{(1,0)\} \tag{44}$$

$$\partial_{\widetilde{n}} \widetilde{\Upsilon} = 0 \ on \ \widetilde{\Gamma_{obs}} \tag{45}$$

$$[\partial_{\widetilde{n}} \widetilde{\Upsilon}] = 0 \ on \ \widetilde{\Gamma_{obs}} \tag{46}$$

$$[\partial_{\widetilde{n}} \widetilde{\Upsilon}] - \beta[\widetilde{\Upsilon}] = 0 \ on \ U/\{(1,0)\} \tag{47}$$

We now look at various properties of the crack Γ after transformation by Φ.

7 Mapping Line Segments into the Unit Disk

We now determine the set $\Phi(\Gamma)$.

7.1 Explicit Equations for Images of Segments Under $\Phi(z) = \frac{z+i}{z-i}$.

Let $x(t) = (cos\alpha)t + a$ and $y(t) = (sin\alpha)t + b$ be our line segment Γ,

for $0 \leq t \leq p$, where p must be determined.

Since Γ has length 2, we have that $\int_0^p \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = 2$.

Since $\frac{dx}{dt} = cos\alpha$ and $\frac{dy}{dt} = sin\alpha$, we have that

$$\int\limits_0^p \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \int\limits_0^p \sqrt{(cos\alpha)^2 + (sin\alpha)^2} dt = \int\limits_0^p 1 dt = p = 2 \tag{48}$$

We will now calculate $\Phi(z(t))$, where $z(t) = x(t) + iy(t)$.

Then it follows from (89) that

$$\Phi(z(t)) = 1 + \frac{2i}{cos\alpha t + a + (sin\alpha t + b - 1)i} \tag{49}$$

for $0 \leq t \leq 2$.

An example of such a line segment can be seen in Figure 1.

7.2 Φ Maps Line Segments into Circular Arcs

It is a classical result from complex analysis that our conformal map Φ transforms lines into circles, and therefore line segments to circular arcs by compactness and connectedness under a continuous transformation. However, we approach the problem assuming no knowledge of complex analysis, in order to obtain analytic expressions for the curvature, angular span, radius, etc.

We first begin with a classical result from calculus.

7.2.1 Circles and Constant Curvature

We wish to show that if a curve defined by the parametric equations $f(t) = (f_1(t), f_2(t))$ has a constant curvature $k > 0$, then the curve defined by $f(t)$ is a circle.

We have that $\frac{v \times a}{|v|^3} = k$, where $|v| = 1$ by arc length parametrization.

Then

$$|v \times a| = k = \begin{vmatrix} i & j & k \\ \frac{df_1}{dt} & \frac{df_2}{dt} & 0 \\ \frac{d^2 f_1}{dt^2} & \frac{d^2 f_2}{dt^2} & 0 \end{vmatrix} = \frac{df_1}{dt}\frac{d^2 f_2}{dt^2} - \frac{df_2}{dt}\frac{d^2 f_1}{dt^2} \tag{50}$$

Since v has a constant magnitude, we have that

$$(\frac{df_1}{dt})^2 + (\frac{df_2}{dt})^2 = 1. \tag{51}$$

Differentiating both sides of (51) w.r.t. t we see that

$$2\frac{df_1}{dt}\frac{d^2 f_1}{dt^2} + 2\frac{df_2}{dt}\frac{d^2 f_2}{dt^2} = 0 \Rightarrow -\frac{df_1}{dt}\frac{d^2 f_1}{dt^2} = \frac{df_2}{dt}\frac{d^2 f_2}{dt^2} \tag{52}$$

To utilize this expression, we multiply (50) by $\frac{df_1}{dt}$ to obtain

$$(\frac{df_1}{dt})^2\frac{d^2f_2}{dt^2} - \frac{df_2}{dt}\frac{df_1}{dt}\frac{d^2f_1}{dt^2} = k\frac{df_1}{dt} \tag{53}$$

By substituting (52) into (53) we have

$$(\frac{df_1}{dt})^2\frac{d^2f_2}{dt^2} + (\frac{df_2}{dt})^2\frac{d^2f_2}{dt^2} = k\frac{df_1}{dt} \Rightarrow ((\frac{df_1}{dt})^2 + (\frac{df_2}{dt})^2)\frac{d^2f_2}{dt^2} = \frac{d^2f_2}{dt^2} = k\frac{df_1}{dt} \tag{54}$$

We now obtain the simple differential equation

$$\frac{d^2f_2}{dt^2} = k\frac{df_1}{dt}. \tag{55}$$

Integrating both sides of (55) w.r.t t gives

$$\frac{df_2}{dt} = kf_1(t) + b \tag{56}$$

Repeating this process, except now multiplying (50) by $\frac{df_2}{dt}$, we obtain

$$\frac{d^2f_1}{dt^2} = -k\frac{df_2}{dt} \Rightarrow \frac{df_1}{dt} = -kf_2(t) + c \tag{57}$$

Substitution from (56) yields

$$\frac{d^2f_1}{dt^2} = -k(kf_1(t) + b) = -k^2f_1(t) - kb \tag{58}$$

The homogeneous solution to (58) is clearly seen to be $h(t) = Asin(kt) + Bcos(kt)$, and the particular solution is $p(t) = \frac{-b}{k}$.
Then

$$f_1(t) = h(t) + p(t) = Asin(kt) + Bcos(kt) - \frac{b}{k} \tag{59}$$

From (57) and (59) it follows that

$$f_2(t) = -Acos(kt) + Bsin(kt) + \frac{c}{k} \tag{60}$$

By (51) we have that

$$(kAcos(kt) - kBsin(kt))^2 + (kAsin(kt) + kBcos(kt))^2 = 1 \Rightarrow A^2 + B^2 = \frac{1}{k^2} \tag{61}$$

so that $B = \pm\sqrt{\frac{1}{k^2} - A^2}$.
Then we have that

$$(f_1(t) + \frac{b}{k})^2 + (f_2(t) - \frac{c}{k})^2 = \frac{1}{k^2} \tag{62}$$

$$f_1(t) = Asin(kt) \pm \sqrt{\frac{1}{k^2} - A^2}cos(kt) - \frac{b}{k} \tag{63}$$

$$f_2(t) = -Acos(kt) \pm \sqrt{\frac{1}{k^2} - A^2}sin(kt) + \frac{c}{k} \tag{64}$$

7.2.2 The Curve Defined by $\Phi(z(t))$ has Constant Curvature

From the previous result, and all we need to prove is that the map of Γ under Φ has constant curvature. Let $x(t) = (cos\alpha)t + a$ and $y(t) = (sin\alpha)t + b$ for $0 \le t \le 2$. Then this is a line segment of length 2 in Ω. We first calculate the magnitude of $\frac{d\Phi}{dt}$. To do this we calculate $\frac{d\Phi_1}{dt}$ and $\frac{d\Phi_2}{dt}$.
To understand where the following equations are obtained, we refer to 10.1.
From (91) we have that

$$\frac{d\Phi_1}{dt} = \frac{2\frac{dy}{dt}}{x^2+y^2-2y+1} - \frac{2(y-1)(2x\frac{dx}{dt}+2y\frac{dy}{dt}-2\frac{dy}{dt})}{(x^2+y^2-2y+1)^2} \tag{65}$$

.

From (92) we have that

$$\frac{d\Phi_2}{dt} = \frac{2\frac{dx}{dt}}{x^2+y^2-2y+1} - \frac{2x(2x\frac{dx}{dt}+2y\frac{dy}{dt}-2\frac{dy}{dt})}{(x^2+y^2-2y+1)^2} \tag{66}$$

We now calculate

$$\left|\frac{d\Phi}{dt}\right|^2 = (\frac{d\Phi_1}{dt})^2 + (\frac{d\Phi_2}{dt})^2 \tag{67}$$

$$(\frac{d\Phi_1}{dt})^2 = \frac{4}{(x^2+y^2-2y+1)^2}((\frac{dy}{dt})^2 - \frac{2(y-1)\frac{dy}{dt}(2x\frac{dx}{dt}+2(y-1)\frac{dy}{dt})}{x^2+y^2-2y+1} + \frac{(y-1)^2(2x\frac{dx}{dt}+2(y-1)\frac{dy}{dt})^2}{(x^2+y^2-2y+1)^2} \tag{68}$$

$$(\frac{d\Phi_2}{dt})^2 = \frac{4}{(x^2+y^2-2y+1)^2}((\frac{dx}{dt})^2 - \frac{2x\frac{dx}{dt}(2x\frac{dx}{dt}+2(y-1)\frac{dy}{dt})}{x^2+y^2-2y+1} + \frac{x^2(2x\frac{dx}{dt}+2(y-1)\frac{dy}{dt})^2}{(x^2+y^2-2y+1)^2}) \tag{69}$$

Since $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 1$, we have from (67) that

$$\left|\frac{d\Phi}{dt}\right|^2 = \frac{4}{(x^2+y^2-2y+1)^2}(1 + \frac{-(2x\frac{dx}{dt}+2\frac{dy}{dt}(y-1))(2x\frac{dx}{dt}+2(y-1)\frac{dy}{dt}) + (2x\frac{dx}{dt}+2(y-1)\frac{dy}{dt})^2}{x^2+y^2-2y+1}) \tag{70}$$

Upon simplification we obtain

$$\left|\frac{d\Phi}{dt}\right|^2 = \frac{4}{(x^2+y^2-2y+1)^2} = \frac{4}{(t^2+2(acos\alpha+(b-1)sin\alpha)t+a^2+b^2-2b+1)^2} \tag{71}$$

$$\left|\frac{d\Phi}{dt}\right| = \frac{2}{t^2+2(acos\alpha+(b-1)sin\alpha)t+a^2+b^2-2b+1} \tag{72}$$

To calculate the curvature k we apply $\frac{\left|\frac{d\Phi}{dt}\times\frac{d^2\Phi}{dt^2}\right|}{\left|\frac{d\Phi}{dt}\right|^3} = k$.
We've already calculated $\left|\frac{d\Phi}{dt}\right|$, so we now calculate

$$\left|\frac{d\Phi}{dt}\times\frac{d^2\Phi}{dt^2}\right| = \begin{vmatrix} i & j & k \\ \frac{d\Phi_1}{dt} & \frac{d\Phi_2}{dt} & 0 \\ \frac{d^2\Phi_1}{dt^2} & \frac{d^2\Phi_2}{dt^2} & 0 \end{vmatrix} = \frac{d\Phi_1}{dt}\frac{d^2\Phi_2}{dt^2} - \frac{d\Phi_2}{dt}\frac{d^2\Phi_1}{dt^2} \tag{73}$$

Since this calculation is quite messy, I employed the assistance of Maxima and obtained

$$\left|\frac{d\Phi}{dt} \times \frac{d^2\Phi}{dt^2}\right| = \frac{-8(asin\alpha - bcos\alpha + cos\alpha)}{(t^2 + 2(sin\alpha(b-1) + acos\alpha)t + a^2 + b^2 - 2b + 1)^3} \tag{74}$$

Then

$$k = \frac{\left|\frac{d\Phi}{dt} \times \frac{d^2\Phi}{dt^2}\right|}{\left|\frac{d\Phi}{dt}\right|^3} = \frac{\frac{-8(asin\alpha - bcos\alpha + cos\alpha)}{(t^2+2(sin\alpha(b-1)+acos\alpha)t+a^2+b^2-2b+1)^3}}{(\frac{2}{t^2+2(acos\alpha+(b-1)sin\alpha)t+a^2+b^2-2b+1})^3} = (b-1)cos\alpha - asin\alpha \tag{75}$$

Notice that $k = 0$ when $(b-1)cos\alpha = asin\alpha$. This condition signifies that the line supporting the line segment Γ passes through the point (0,1), the singluar point of Φ.
We have that the radius of the circle supporting the circular arc $\Phi(\Gamma)$ is given by

$$r = \frac{1}{|k|} = \frac{1}{|(1-b)cos\alpha + asin\alpha|} \tag{76}$$

To see that $\Phi(z)$ indeed maps line segments to circular arcs, see Figure 2.

7.2.3 Arclength and Angular Span

We utilize the fact that $s = r\theta$ to find an analytic equation for s and θ.
From (72) we have that

$$s = \int_0^2 \left|\frac{d\Phi}{dt}\right| dt = \int_0^2 \left|\frac{2}{t^2 + 2(acos\alpha + (b-1)sin\alpha)t + a^2 + b^2 - 2b + 1}\right| dt \tag{77}$$

Using quadratic formula on the bottom we see that

$$t = \frac{-2(acos\alpha + (b-1)sin\alpha) \pm 2\sqrt{(acos\alpha + (b-1)sin\alpha)^2 - (a^2 + b^2 - 2b + 1)}}{2} \tag{78}$$

Since

$$(acos\alpha + (b-1)sin\alpha)^2 = a^2cos^2\alpha + (b-1)^2sin^2\alpha + 2a(b-1)sin(2\alpha) \tag{79}$$

it follows that the discrminant is

$$-a^2sin^2\alpha - (b-1)^2cos^2\alpha + 2a(b-1)sin(2\alpha) = -(asin\alpha - (b-1)cos\alpha)^2 \tag{80}$$

Then applying (78) with (80) we have that

$$t = -(acos\alpha + (b-1)sin\alpha) \pm i(asin\alpha - (b-1)cos\alpha). \tag{81}$$

It follows that these roots are never real since that would imply $asin\alpha = (b-1)cos\alpha$, which is impossible.
Now we have that

$$s = \int_0^2 \frac{2}{t^2 + 2(acos\alpha + (b-1)sin\alpha)t + a^2 + b^2 - 2b + 1} dt \tag{82}$$

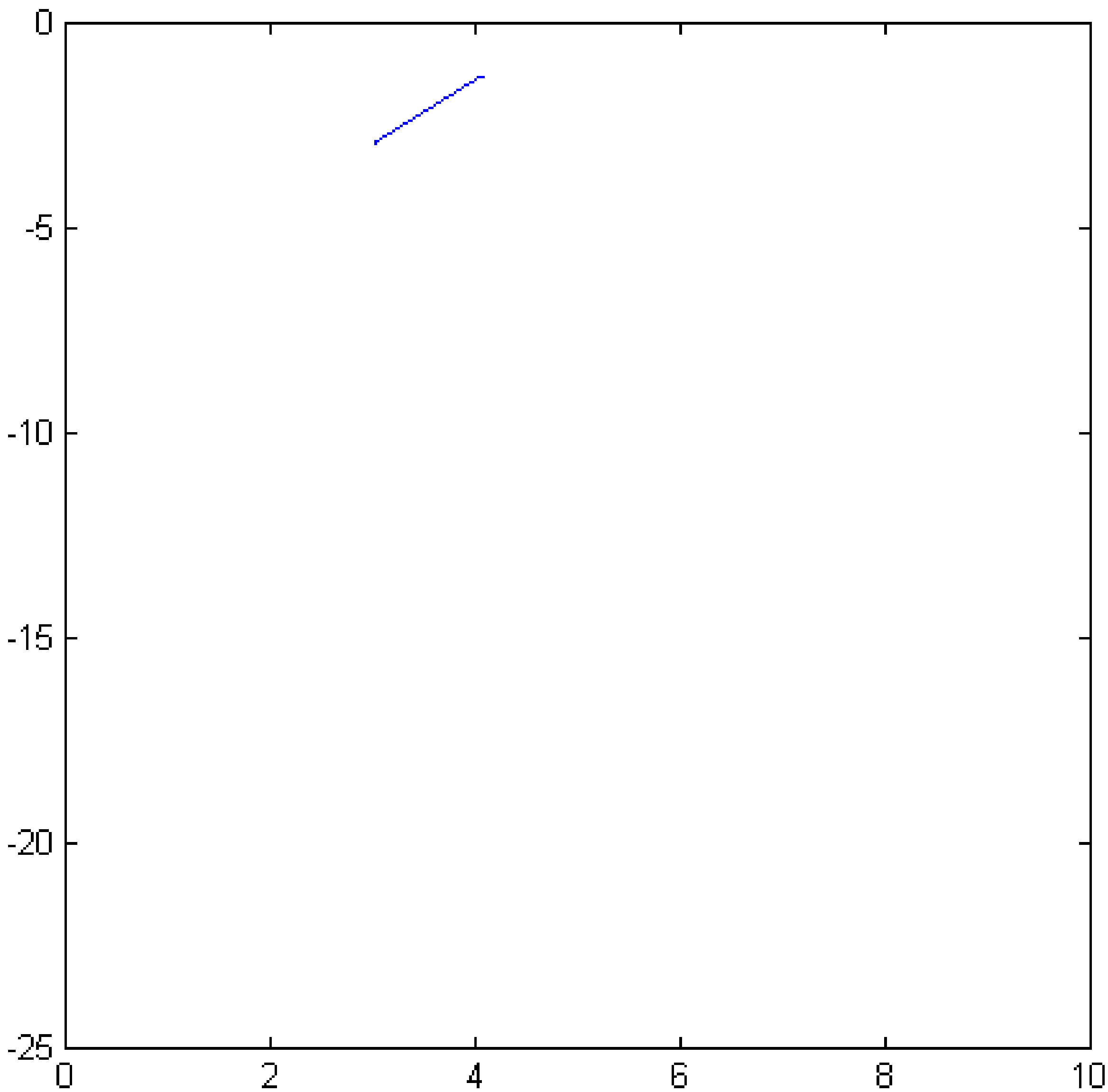

Figure 2: The line segment generated by paramgraph(1,3,-3,2), see algorithm 1, in the lower half plane.

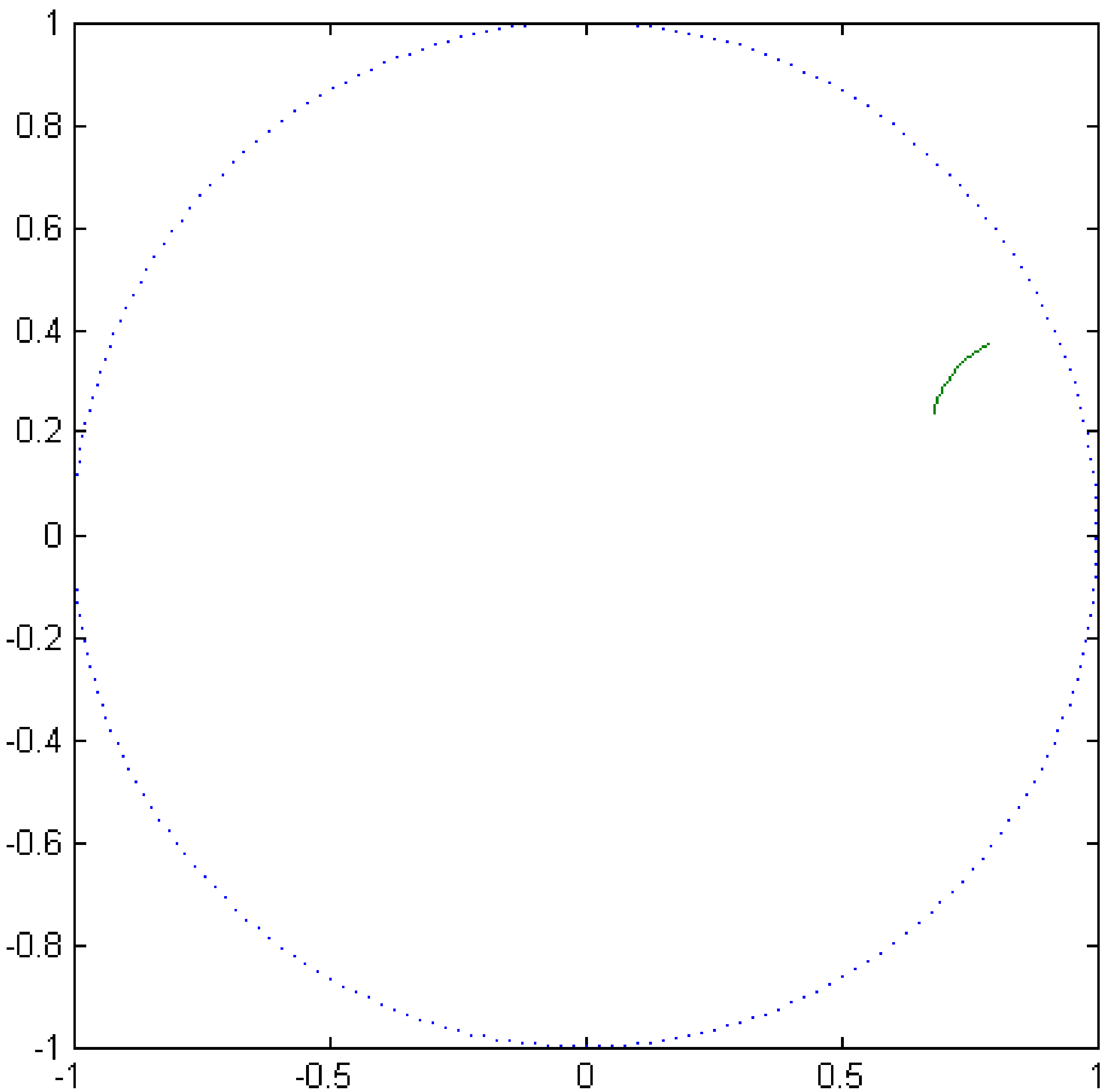

Figure 3: This is the image of the previous segment under Φ.

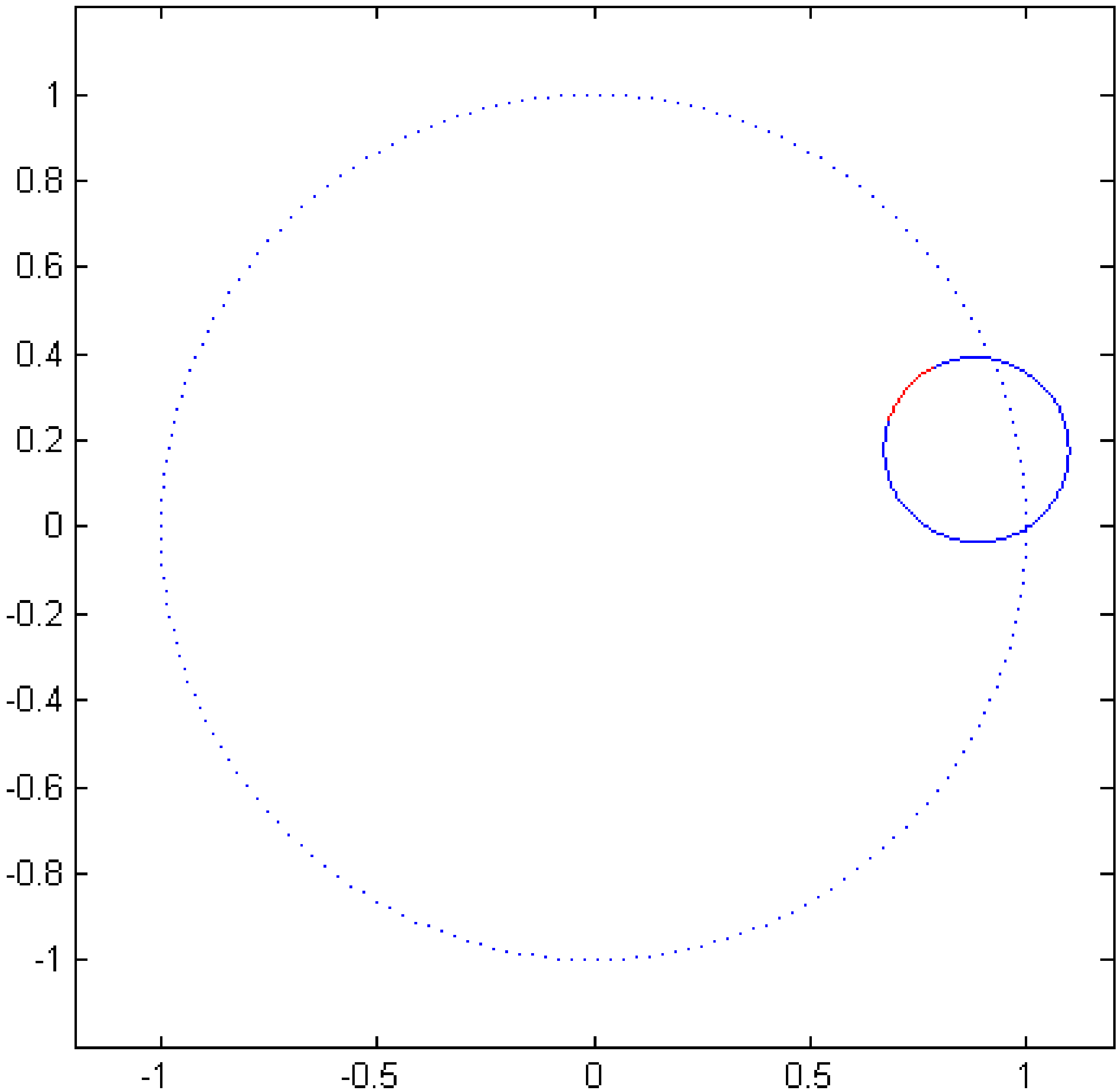

Figure 4: The circular extension of the arc in the previous image through $(1, 0)$

Since $t = 0$ yields $a^2 + (b-1)^2 > 0$, we apply the following integral formula

$$\int \frac{dx}{x^2 + cx + d} = \frac{2}{\sqrt{4d - c^2}} tan^{-1}(\frac{2x + c}{\sqrt{4d - c^2}}) \tag{83}$$

noting that $\sqrt{4b - a^2} = |asin\alpha - (b-1)cos\alpha|$, so that

$$s = \frac{2}{|asin\alpha - (b-1)cos\alpha|}(tan^{-1}(\frac{4 + 2(acos\alpha + (b-1)sin\alpha)}{|asin\alpha - (b-1)cos\alpha|}) - tan^{-1}(\frac{2(acos\alpha + (b-1)sin\alpha)}{|asin\alpha - (b-1)cos\alpha|})) \tag{84}$$

Then

$$\theta = 2(tan^{-1}(\frac{4 + 2(acos\alpha + (b-1)sin\alpha)}{|asin\alpha - (b-1)cos\alpha|}) - tan^{-1}(\frac{2(acos\alpha + (b-1)sin\alpha)}{|asin\alpha - (b-1)cos\alpha|})) \tag{85}$$

7.2.4 Analysis of our Symmetric Solution $\phi_{\mathbf{d}}$ as d Approaches ∞ .

Every ray in Ω^- is mapped to a circle by $\Phi(x, y)$, so the point at infinity is mapped to $(1, 0)$.
Then every circle in U passes through $(1, 0)$ (see Figure 4). Since d is the depth of Γ, we have that d corresponds to the parameter b of the arc. Then letting d tend to ∞ implies that b must tend to $-\infty$.
From (76) we have that $r = \frac{1}{|(1-b)cos\alpha + asin\alpha|}$, as b approaches $-\infty$, r tends to 0 provided $cos\alpha \neq 0$.
If $cos\alpha = 0$, then $sin\alpha = \pm 1$, and $r = \frac{1}{|a|}$. If $a = 0$, we have that $k = 0$.
Therefore if $cos\alpha \neq 0$, r approaches 0, and Γ is mapped to the point $(1, 0)$.
In the case that Γ is a vertical segment, then $\Phi(\Gamma)$ is invariant under vertical displacement.
The importance of d after mapping Γ by Φ is equivalent to the distance between $\Phi(\Gamma)$ and the point $(1, 0)$, assuming Γ is not vertical. In the vertical case, the value of d does not matter.
We first consider if $\Phi(\Gamma)$ crosses the y axis.
We define θ_1 as the angle between the x axis and the line segment connecting $(1, 0)$ and the endpoint of $\Phi(\Gamma)$ above the x axis, while θ_2 is the angle below x axis.
From (92) we have $\Phi_2(x, y) = 0$ implies $x = 0$.
Since $x(t) = cos\alpha t + a$, then $t_1 = \frac{-a}{cos\alpha}$ is the time when $\Phi_2(x, y) = 0$.
Let $0 < t_1 < 2$, we first look at the distance from $(1, 0)$ to $\Phi(\Gamma)$, for certain values of t.
At $t = 0$, we have $(\Phi_1, \Phi_2) = (1 + \frac{2(b-1)}{a^2+b^2-2b+1}, \frac{2a}{a^2+b^2-2b+1})$.
At $t = t_1$, we have $(\Phi_1, \Phi_2) = (1 + \frac{2(-atan\alpha+b-1)}{(-atan\alpha+b)^2-2(-atan\alpha+b)+1}, 0)$.
At $t = 2$, we have $(\Phi_1, \Phi_2) = (1 + \frac{2(2sin\alpha+b-1)}{(2cos\alpha+a)^2+(2sin\alpha+b)^2-2(2sin\alpha+b)+1}, \frac{2(2cos\alpha+a)}{(2cos\alpha+a)^2+(2sin\alpha+b)^2-2(2sin\alpha+b)+1})$.
We now apply Law of Cosines to calculate θ_1, θ_2.
Using Maxima, we obtain that

$$cos(\theta_1) = \frac{(b-1)\frac{(-atan\alpha+b-1)}{|-atan\alpha+b-1|}\frac{|a^2tan^2\alpha-2a(b+1)tan\alpha+b^2-2b+1|}{a^2tan^2\alpha-2a(b+1)tan\alpha+b^2-2b+1)}}{\sqrt{a^2 + b^2 - 2b + 1}} \tag{86}$$

so that $cos(\theta_1) = \frac{1-b}{\sqrt{a^2+b^2-2b+1}}$ for $b \ll 0$, which tends to 1 as b tends to $-\infty$, so that θ_1 tends to 0.
Maxima also gives that

$$cos(\theta_2) = \frac{(2sin\alpha + b - 1)\frac{(-atan\alpha+b-1)}{|-atan\alpha+b-1|}\frac{|a^2tan^2\alpha-2a(b+1)tan\alpha+b^2-2b+1|}{a^2tan^2\alpha-2a(b+1)tan\alpha+b^2-2b+1)}}{\sqrt{4(b-1)sin\alpha + 4acos\alpha + a^2 + b^2 - 2b + 5}} \tag{87}$$

so that $cos(\theta_2) = \frac{-(2sin\alpha+b-1)}{\sqrt{4(b-1)sin\alpha+4acos\alpha+a^2+b^2-2b+5}}$ for $b \ll 0$, which tends to 1 as b tends to $-\infty$, so θ_2 tends to 0. If $t_1 \notin (0,2)$, $\Phi(\Gamma)$ does not cross the y-axis, so that the arc is contained on one side.
We define θ as the angle formed by the two endpoints of $\Phi(\Gamma)$ and the point $(1,0)$.
We use the Law of Cosines again to calculate the value of $cos(\theta)$, which is given by

$$cos(\theta) = \frac{(2(b-1)sin\alpha + 2acos\alpha + b^2 - 2b + a^2 + 1)}{\sqrt{a^2+b^2-2b+1}\sqrt{4(b-1)sin\alpha + 4acos\alpha + a^2 + b^2 - 2b + 5}} \tag{88}$$

Letting b tend to $-\infty$, we see that again $cos(\theta)$ tends to 1, so that θ tends to 0.

8 Plans for Future Work

Ionescu and Volkov studied the inverse problem attached to eigenvalue problem 2.1. More precisely, they sought methods for reconstructing the fault or cut Γ from readings of Υ on the surface $x_2 = 0$ of the lower half plane. Their proposed reconstruction method involved the derivation of asymptotic formulas for Υ and related quantities as the depth of the fault Γ becomes large. Since the dominant part of the asymptotic formula involved relatively simple closed form terms, a reconstruction algorithm was designed based on the computation of moments of Υ. In light of this work for the half plane geometry, we will focus in our future work on three questions pertaining to the case of a disk:

- What does one obtain by mapping the reconstruction algorithm obtained by Ionescu and Volkov to the unit disk?

- If the particular conformal mapping Φ is used, a uniform grid of points on the line $x_2 = 0$ is transformed into a grid of points on the boundary of the unit disk U that accumulates near $(1, 0)$. What does that imply for reconstructing cracks in disks? Is it advantageous in some situations to compose Φ with a suitable rotation?

- Can we obtain a useful asymptotic formula in the case of disks solely based on the assumption that the crack is small with regard to the size of the unit disk?

9 References

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10 Appendix

10.1 Different Ways of Expressing the Mapping Φ

Recall how Φ is defined, $\Phi : \mathbb{C}/\{i\} \to \mathbb{C}$. For ease of notation, we also denote Φ by the mapping $\Phi : \mathbb{R}^2/(0,1) \to \mathbb{R}^2$ defined by $(x,y) \mapsto (Re\Phi(x+iy), Im\Phi(x+iy))$.

$$\Phi(z) = \frac{z+i}{z-i} = 1 + \frac{2i}{z-i} = 1 + \frac{2i(\overline{z}+i)}{(z-i)(\overline{z}+i)} = 1 + \frac{2i(\overline{z}+i)}{|z|^2 + iz - i\overline{z} + 1} \tag{89}$$

For $z = x + iy$, we see that

$$\Phi(x,y) = 1 + \frac{2i(x - i(y-1))}{x^2 + y^2 + i(x+iy) - i(x-iy) + 1} = 1 + \frac{2(y-1) + 2ix}{x^2 + y^2 - 2y + 1} \tag{90}$$

We therefore have that $\Phi(x,y) = \Phi_1(x,y) + \Phi_2(x,y)i$, where

$$\Phi_1(x,y) = 1 + \frac{2(y-1)}{x^2 + y^2 - 2y + 1} \tag{91}$$

$$\Phi_2(x,y) = \frac{2x}{x^2 + y^2 - 2y + 1}. \tag{92}$$

From (22) we have that

$$\Phi^{-1}(w) = \frac{i(w+1)}{w-1} = i + \frac{2i}{w-1} = i + \frac{2i(\overline{w}-1)}{(w-1)(\overline{w}-1)} = i + \frac{2i(\overline{w}-1)}{|w|^2 - \overline{w} - w + 1} \tag{93}$$

When $w = x + iy$, we have that

$$\Phi^{-1}(x,y) = i + \frac{2i(x - iy - 1)}{x^2 + y^2 - 2x + 1} = i + \frac{2y + 2i(x-1)}{x^2 + y^2 - 2x + 1} \tag{94}$$

Then $\Phi^{-1}(x,y) = \Phi_1^{-1}(x,y) + \Phi_2^{-1}(x,y)i$, where

$$\Phi_1^{-1}(x,y) = \frac{2y}{x^2 + y^2 - 2x + 1} \tag{95}$$

$$\Phi_2^{-1}(x,y) = 1 + \frac{2(x-1)}{x^2 + y^2 - 2x + 1} \tag{96}$$

10.2 Composition of Harmonic and Holomorphic Functions

In Section 6.2, we use the fact that Φ satisfies the Cauchy-Riemann equations, and then trivially satisfies all conditions to be a harmonic preserving transformation.

However, we can use the following result to also make the argument that Φ preserves harmonicity.

For a harmonic function $f : \Omega \to \mathbb{R}$ and a holomorphic function $g : U/\{1\} \to \Omega$, show that $\Delta(f \circ g) = 0$ in Ω.

We first utilize the fact that $f(z)$ can be extended locally to a holomorphic function denoted by $u(z)$ such that $u(z) = f(z) + ih(z)\ \forall\, z \in \Omega$. Since composition of holomorphic functions is holomorphic, we have that $u(g(z)) = f(g(z)) + ih(g(z))$ is holomorphic. Since the real and imaginary part of a holomorphic function are harmonic, we have $f(g(z))$ is harmonic, and therefore $\Delta(f \circ g) = 0$ in Ω.

10.3 Transforming Homogeneous First Order Linear PDE's

Let u satisfy in $U \subset \mathbb{R}^2$ the PDE

$$a_1(x_1,x_2)\frac{\partial}{\partial x_1}u(x_1,x_2) + a_2(x_1,x_2)\frac{\partial}{\partial x_2}u(x_1,x_2) + b(x_1,x_2)u(x_1,x_2) = 0 \tag{97}$$

Let $\widetilde{u}(v_1,v_2) = u(\Psi^{-1}(v_1,v_2))$, where $\Psi : U \to V \subset \mathbb{R}^2$, $\Psi \in C^1(U), \Psi^{-1} \in C^1(V)$, and Ψ is a bijection. Find the PDE for $\widetilde{u}$.

We want to find $T : U \to V$, such that $\Psi^{-1}(T(x_1,x_2))$ =(x_1,x_2). Let $T = \Psi$.

Then

$$\widetilde{u}(\Psi(x_1,x_2)) = u(x_1,x_2) \tag{98}$$

Since Ψ is valued on V, Ψ is a vector in $\mathbb{R}^2$, we can assume the form

$$\Psi(x_1,x_2) = (\Psi_1(x_1,x_2), \Psi_2(x_1,x_2)) = (y_1,y_2) \tag{99}$$

Resubstituting these two results above yields

$$\widetilde{a}_1(\frac{\partial\widetilde{u}}{\partial y_1}\frac{\partial\Psi_1}{\partial x_1} + \frac{\partial\widetilde{u}}{\partial y_2}\frac{\partial\Psi_2}{\partial x_1}) + \widetilde{a}_2(\frac{\partial\widetilde{u}}{\partial y_1}\frac{\partial\Psi_1}{\partial x_2} + \frac{\partial\widetilde{u}}{\partial y_2}\frac{\partial\Psi_2}{\partial x_2}) + \widetilde{b}\widetilde{u} = 0 \tag{100}$$

as our new PDE, where $\widetilde{u}, \widetilde{a}_1, \widetilde{a}_2, \widetilde{b}$ are functions of (y_1,y_2), and Ψ_1, Ψ_2 are functions of (x_1,x_2).

Algorithm 1 Paramgraph.m

```
function paramgraph(a,h,k,L)
if nargin ~= 4,
error('Need four arguments');
end
t=0:1/1000:L;
x = cos(a) .* t + h;
y = sin(a) .* t + k;
theta=0:0.01:2*pi;
function changeGraph_Callback(source,eventdata)
switch(get(source,'String'));
case 'map to unit circle'
q1=1+(2.*(y-1))./(x.^2+y.^2-2.*y+1);
q2=(2.*x)./(x.^2+y.^2-2.*y+1);
plot(cos(theta),sin(theta),':',q1,q2);
set(source,'String','extend to circle');
case 'extend to circle'
q1=1+(2.*(y-1))./(x.^2+y.^2-2.*y+1);
q2=(2.*x)./(x.^2+y.^2-2.*y+1);
tn=-1000:1/1000:0;
tp=L:1/1000:1000;
cxn = cos(a) .* tn + h;
cyn = sin(a) .* tn + k;
cxp = cos(a) .* tp + h;
cyp = sin(a) .* tp + k;
q1n=1+(2.*(cyn-1))./(cxn.^2+cyn.^2-2.*cyn+1);
q2n=(2.*cxn)./(cxn.^2+cyn.^2-2.*cyn+1);
q1p=1+(2.*(cyp-1))./(cxp.^2+cyp.^2-2.*cyp+1);
q2p=(2.*cxp)./(cxp.^2+cyp.^2-2.*cyp+1);
plot(cos(theta),sin(theta),':',q1,q2,'r',q1n,q2n,'b',q1p,q2p,'b');
axis([-1.2 1.2 -1.2 1.2]);
set(source,'String','go back to line segment plot');
case 'go back to line segment plot' plot(0:10:10,-24:24:0,'w',x,y);
set(source,'String','map to unit circle');
end
end
figure('Visible','on','Position',[360,500,470,510]);
haxes = axes('Units','Pixels','Position',[50,70,400,400]);
plot(haxes,0:10:10,-24:24:0,'w',x,y);
hbutton = uicontrol('Style','pushbutton','Position',[150,5,150,25],'Callback',@changeGraph_Callback,'String','map
to unit circle');
align(hbutton,'Center','None');
end
```