Computational Methods for Option Pricing

A Directed Research Project

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Abstract

This paper aims to practice the use of Monte Carlo methods to simulate stock prices in order to price European call options using control variates. American put options are priced using the binomial model separately. Finally, we use the information to form a portfolio position using an Interactive Brokers paper trading account.

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1. Introduction

In this study we use Monte Carlo methods to price options on stocks. Furthermore, we use this information to form our own portfolio positions including both stocks and options.

First of all, we choose some stocks from several economic sectors and find the historical data for these selected asset prices. Using these historical prices, we can estimate the volatility and mean value of the selected stock returns.

Secondly, we use the estimated volatility and mean value as the estimates for the parameters in our pricing model. Based on the estimates for the parameters, we simulate the stock prices in each economic sector together in a multidimensional geometric Brownian motion.

Then according to the simulated stock price paths, we need to price European call options and American put options separately. For the European call options we implement the variance reduction technique: control variates. For the American put option we implement the binomial model.

After pricing options on selected stocks, we compare their prices with the actual trading prices in the market. Then we form our own portfolio positions through buying some stocks and some call or put options on the selected stocks.

Finally, we track the performance of our portfolio in the real market and discuss the major challenges we face in the study.

2. Data

Since this is a short-term prediction, we choose a series of historical data for stock prices in a recent period: the three months between Jan 3, 2011 and Apr 15, 2011. We choose three economic sectors as representatives, which are “airline”, “computer” and “internet” industries.

We also use the 3-month Treasury bill rate as our risk-free rate.

3. Methodology

In this study we primarily practice using Monte Carlo methods, which try to approximate a stock price or an option's value through a series of simulations.

We consider two types of options. One is European call option and the holder can exercise the option only at the expiry. The other is American put option and the holder can exercise the option at any time before the maturity.

In order to simulate the stock price paths, we use geometric Brownian motion, a fundamental model for pricing an asset value. The model assumes that the stochastic process of stock price (St) is a Geometric Brownian motion if log(St) is a Brownian motion.

When pricing European call options we implement the technique of control variates. This variance reduction method aims to construct a new control estimator, which is an unbiased and consistent estimate with minimized variance for our target.

When pricing American put options we implement the binomial model. It shows that at each time point, we need to compare the immediate exercise value with the value associating with holding it and trading it later on. From the expiry date back to today, we need to repeat this comparison at every time point and finally get the put option price.

4. Process

(1) Select Data

Choose three economic sectors: "Airline", "Computer" and "Internet"
Choose several stocks in each sector as Figure 1 and find their historical adjusted closing prices between "Jan 3, 2011" and "Apr 15, 2011":

Airline	AMR	DAL	JBLU	LCC	LUV	UAL
Computer	AAPL	DELL	HPQ	IBM	NCR	
Internet	NTES	SINA	SOHU	YHOO		

Table 1

(2) Estimate Parameters

In each sector, compute the log price returns and then compute the mean vector and the covariance matrix Σ of stock returns. Since these are daily returns, we need to convert the mean vector and covariance matrix in terms of a year based on 252 days per year. We use the 3-month Treasury bill rate 0.1 as our risk-free rate.

(3) Predict Stock Price Movement

In each economic sector, we use a multidimensional geometric Brownian motion model (GBM) to simulate the price paths of the selected stocks.

$$S_i(t_{K+1}) = S_i(t_k) \times \exp((\mu_i - \frac{1}{2}\sigma^2) \times (t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k} \sum_{j=1}^{d} A_{ij}.Z_{k+1,j})$$

$$\underline{Z_k} = [Z_{k,1}, Z_{k,2}, \ldots. Z_{k,d}] \longrightarrow N(\underline{0}, I)$$

(k=0,1…n-1 i=1,2…d)

where A is calculated from the Cholesky factorization of Σ [1]

Take the example of “Internet” sector; we plot the simulated paths for each stock in this sector.

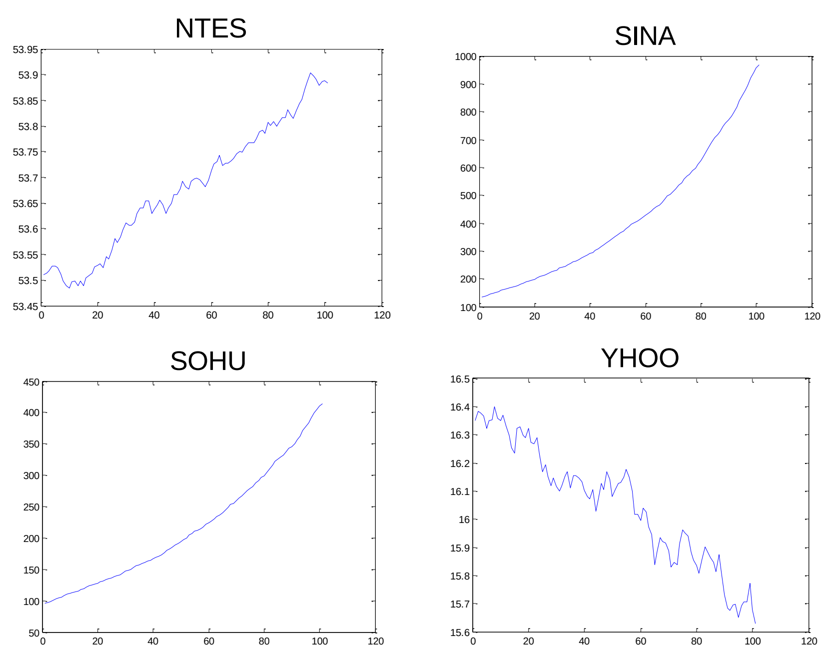

Table 2

[1] Monte Carlo Methods in Financial Engineering, Paul Glasserman, August 7, 2003

As we have seen above, the price of YHOO tends to decrease in the future and the other three stocks all experienced an upward trend. In addition, SINA and SOHU both have a stable increasing trend.

Since the payoff for call option is max (S_T-K,0) and the payoff for put option is max (K-S_T,0), when the underlying price increases, we can buy a call option to have positive payoff at expiry and when the underlying price decreases, we can buy a put option based on it to have positive payoff at expiry.

Thus in the industry "Internet", in order to hedge the decreasing trend of YHOO stock price, we will buy a put option based on YHOO stock. Also, for the stocks SINA and SOHU, we will buy two call options separately based on them. We apply the same analysis process to the other two economic sectors. We can form the corresponding call or put options based on those selected stocks. We will form our initial portfolio including 15 stocks and 10 options as the follows:

"Airline"	AMR	DAL	JBLU	LCC	LUV	UAL
Option		Call		Call		Call
"Computer"	AAPL	DELL	HPQ	IBM	NCR	
Option	Call	Put	Put	Put		
"Internet"	NTES	SINA	SOHU	YHOO		
Option		Call	Call	Put		

Table 3

Next, we need to determine which option to buy. If we fix the same maturity for these options, we need to choose the specific option of specific strike price by comparing our computed option price with the market option price.

(4) Price the European Call Option

In this study in order to form the option positions on Apr 19, 2011, we use the stock prices at Apr 18, 2011 as the initial stock prices. We also calculate the time to maturity T in terms of years. In our portfolio all options will mature on May 20, 2011 and over the period from Apr 19 to May 20, the number of trading days is 23. Thus, the time to maturity T is set to be 23/252=0.09127.

First we need to simulate the price paths of the underlying stocks in each sector. Since an option is priced under the martingale measure, which is also known as the risk-neutral measure, we should simulate our price paths of underlying stocks under martingale measure instead of any other measure. In another words, in the simulation of the stock price paths, all stock returns have the same drift—the risk-free interest rate (r) instead of the estimated mu in our prediction of stock prices.

$$S_i(t_{K+1}) = S_i(t_k) \times \exp((r - \frac{1}{2}\sigma^2) \times (t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k} \sum_{j=1}^{d} A_{ij}.Z_{k+1,j})$$

$$\underline{Z_k} = [Z_{k,1}, Z_{k,2}, \ldots. Z_{k,d}] \longrightarrow N(\underline{0}, I)$$

(k=0,1…n-1   i=1,2…d)

where A is calculated from the Cholesky factorization of Σ [2]

Then, using the variance reduction technique of control variates, we can price the European call options.

In our study we need to price a European call option with discounted payoff Y. Assume Y is a function of the price path {St: 0<t<T}.
First we form n independent price path replications: $S_1, S_2, \ldots, S_n$ over [0,T] of S.
Then, we form the control variate estimator:

$$Yi(b) = Yi - b \times [Si(T) - E(S(T))]$$

$$\overline{Y}(b) = \frac{1}{n} \times \sum [Yi - b \times [Si(T) - E(S(T))]] \quad i = 1, 2, \ldots n$$

$$E(S(T)) = e^{rT} \times S(0)$$

Thus, our target is to solve:

$$\overline{Y}(b) = \frac{1}{n} \times \sum [Yi - b \times [Si(T) - e^{rT} \times S(0)]] \quad i = 1, 2, \ldots n$$

Where, Y is the discounted payoff:

$$Yi = e^{-rT} \times \max(Si(T) - K, 0) \quad i = 1, 2, \ldots n$$

Also, the optimal b is given by:

$$b* = \frac{\text{cov}(S(T), Y)}{\text{var}(S(T))}$$

Therefore, $\overline{Y}(b)$ is the price of the European call option.

Using the theory above, we can price those call options in Table 2. The results are as follows:

[2] Monte Carlo Methods in Financial Engineering, Paul Glasserman, August 7, 2003

Industry	Airline			Computer	Internet	
Underlying	**DAL**	**LCC**	**UAL**	**AAPL**	**SINA**	**SOHU**
Option type	call	call	call	call	Call	Call
Initial Price	9.14	8.33	21.22	331.85	135.13	96.07
Strike price	8	8	19	325	135	90
Time to maturity(year)	0.09127	0.09127	0.09127	0.09127	0.09127	0.09127
Computed price	1.473	0.83768	3.273	7.2244	6.437	12.1521
Previous close price	1.3035	0.61872	3.27	19.0906	15.169	11.2314
Bid price	1.120	0.52	3.02	16.45	14.52	12.10
Ask Price	1.50	1.02	3.68	17.20	15.43	13.4
Trade price	1.35	0.83	3.4	16.95	14.6	13.2

Table 4

From the above results, we decide to buy options by comparing their computed prices and previous closing prices in the market. If the computed price is higher than the market price, we buy it, because the lower market price means that the market underestimates the option. On the contrary, if the computed price is lower than the market price, we do not buy it, since the higher market price means that the market overestimates the option. Also, these computed prices should be between the bid-ask prices in order to be traded successfully.

Thus, by the above analysis, we would drop the AAPL call and SINA call, since their computed prices are too much lower than market prices to be compared. As a result, we would form our call option positions as below:

Industry	Airline			Internet
Underlying	**DAL**	**LCC**	**UAL**	**SOHU**
Option type	call	call	call	Call
Initial Price	9.14	8.33	21.22	96.07
Strike price	8	8	19	90
Time to maturity(year)	0.09127	0.09127	0.09127	0.09127
Computed price	1.473	0.83768	3.273	12.1521
Previous close price	1.3035	0.61872	3.27	11.2314
Bid price	1.120	0.52	3.02	12.10
Ask Price	1.50	1.02	3.68	13.4
Trade price	1.35	0.83	3.4	13.2

Table 5

(5) Price the American Put Option

Before pricing put options, we need to use the binomial tree model to simulate the price path of the underlying stock. Here we assume that the log return of the stock would move from one time point to the next by either an upward factor u with probability p or a downward factor d with probability 1-p.

Calculate the up factor and down factor, assume ΔX is the movement of log returns:[3]

$$\Delta X = \sqrt{\sigma^2 \Delta t + (r - \frac{1}{2}\sigma^2)^2 \times \Delta t^2}$$
$$\Delta Xu = \Delta X$$
$$\Delta Xd = -\Delta X$$

Calculate the probabilities:[4]

$$Pu = \frac{1}{2} + \frac{(r - \sigma^2/2) \times \Delta t}{2\Delta X}$$
$$Pd = 1 - Pu$$

Since we suppose the movement is about the log return, we need to exponentiate to simulate the stock price path:

$$S(i, j) = S(0) \times \exp[(j - i) \times \Delta Xu + (i - 1) \times \Delta Xd]$$

where j is the time step, i is the # of state variable at that time

We will simulate the selected stock price as in the following example of YHOO stock:

16.35	16.495	16.642	16.79	16.939	17.089	17.241	17.394	17.549	17.705
0	16.206	16.35	16.495	16.642	16.79	16.939	17.089	17.241	17.394
0	0	16.063	16.206	16.35	16.495	16.642	16.79	16.939	17.089
0	0	0	15.922	16.063	16.206	16.35	16.495	16.642	16.79
0	0	0	0	15.782	15.922	16.063	16.206	16.35	16.495
0	0	0	0	0	15.643	15.782	15.922	16.063	16.206
0	0	0	0	0	0	15.505	15.643	15.782	15.922
0	0	0	0	0	0	0	15.368	15.505	15.643
0	0	0	0	0	0	0	0	15.233	15.368
0	0	0	0	0	0	0	0	0	15.099

Table 6

[3] Implementing Models in Quantitative Finance: Methods and Cases, Gianluca Fusai and Andrea Roncoroni, Dec, 2007
[4] Implementing Models in Quantitative Finance: Methods and Cases, Gianluca Fusai and Andrea Roncoroni, Dec, 2007

Then we need to price the put option. American put options allow the holder to be able to exercise the option at any point in time. In another words, at every time point before expiry, the holder would face a problem of either exercising early or holding the option and selecting an optimal strategy later. Thus the put option price would be the maximum of the early exercise payoff and the value associating with holding it. From the expiry date back to today we need to repeat this comparison at every time point and finally get the put option price.

$$P(i,T) = \max(K - S(i,T), 0)$$

$$P(i, j) = \max\{\max(K - S(i, j)), e^{-r\Delta t}[Pu \times P(i, j+1) + Pd \times P(i+1, j+1)]\}$$ [5]

Using the theory above we price the put options summarized in Figure 2. The results are as follows:

	Computer			Internet
	DELL	**HPQ**	**IBM**	**YHOO**
Option type	put	put	put	put
Initial Price	14.71	39.75	165.94	16.35
Strike price	15	41	170	17
Time to maturity	0.09127	0.09127	0.09127	0.09127
Computed price	0.7426	2.0434	5.7668	1.36077
Bid price	0.620	1.76	6.80	1.11
Ask Price	0.90	2.20	7.25	1.78
Trade price	0.73	1.88	7.1	1.2

Table 7

We use the same analysis for choosing call options. Thus we will not buy IBM, since its computed value is less than market price, leading to an overestimation in the market.

We would form our put option positions as below:

	Computer		Internet
	DELL	**HPQ**	**YHOO**
Option type	put	put	put
Initial Price	14.71	39.75	16.35
Strike price	15	41	17
Time to maturity	0.09127	0.09127	0.09127
Computed price	0.7426	2.0434	1.36077
Bid price	0.620	1.76	1.11
Ask Price	0.90	2.20	1.78
Trade price	0.73	1.88	1.2

Table 8

[5] Implementing Models in Quantitative Finance: Methods and Cases, Gianluca Fusai and Andrea Roncoroni, Dec, 2007

5. Performance

Through above calculation and analysis we form our portfolio position including 15 stocks, 4 call options, and 3 put options as the following table summarizes:

	Stock	Call Option	Computed price	Bid-Ask Price	Trade price	Put Option	Computed price	Bid-Ask Price	Trade price
Airline	**AMR**	**DAL**	1.473	1.120-1.50	1.35				
	DAL	**LCC**	0.83768	0.52-1.02	0.83				
	JBLU	**UAL**	3.273	3.02-3.68	3.4				
	LCC								
	LUV								
	UAL								
Computer	**AAPL**					**DELL**	0.7426	0.620-0.90	0.73
	DELL					**HPQ**	2.0434	1.76-2.20	1.88
	HPQ								
	IBM								
	NCR								
Internet	**NTES**	**SOHU**	12.1521	12.10-13.4	13.2	**YHOO**	1.36077	1.11-1.78	1.2
	SINA								
	SOHU								
	YHOO								

Table 9

After forming our portfolio positions, we can observe its performance in the market. During this period the stock prices are experienced a stable movement. However, the call and put option prices change a lot, resulting in extremely large profit or loss. As a result, during the last week our portfolio has an accrued a small loss.

6. Conclusion

Comparing our calculated option prices with the market trade price, we can see that the difference between call options is relatively larger than that between American put options. This means that the error in pricing call options is relatively larger than that in pricing American put options. This might be the result of our estimate for the volatility of the historical stock returns. Under the risk-neutral measure, the option prices are primarily determined by the volatility instead of the drift. Thus the estimate for volatility is important for our pricing. In further study we might use other methods to estimate the volatility of the historical stock returns instead of computing the covariance matrix directly. For example, we might use an industry factor model to relate the selected stock returns in an economic sector to the corresponding industry index.

7. References

1. Paul Glasserman, 2003, *Monte Carlo Methods in Financial Engineering*, Springer, New York City, 596 P.
2. Gianluca Fusai and Andrea Roncoroni, 2008, *Implementing Models in Quantitative Finance: Methods and Cases*, Springer, New York City, 631 P.
3. The stock price data used in the project was obtained through Yahoo Finance, http://biz.yahoo.com/r/ (May 2, 2011)