Transfer Theory and Homology

by

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Abstract

In this thesis, we develop three interpretations of the *transfer* homomorphism: via topological spaces, simplicial homology, and group theory. Historically, the first appearance was in group theory, from the permutation action of G on cosets of a subgroup. The constructions in topology and homological algebra can be interpreted this way through results on the fundamental groups and homology groups of covering spaces. The goal of this thesis is to describe enough background in each area to define and interpret the transfer homomorphism and to give some applications of the transfer in finite group theory.

Acknowledgments

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Chapter 1

Topological Motivations

The transfer homomorphism was initially investigated by Burnside and Frobenius around 1900 as part of their investigations of the structure of finite groups. With the development of homology and homotopy in the 1930s, it was realised that the transfer is a special case of a more general homological construction. In this chapter we develop the concepts necessary to understand the motivation for the transfer map.

1.1 Topological Spaces

While we will define and discuss general topological spaces, we will not be terribly interested in details of point-set topology, so we will mostly focus on the case of manifolds. Later, we will need some additional generality when we discuss Eilenberg-MacLane spaces. General background on topology can be found in the textbook of Bredon [1] and Hatcher [5].

We start our discussion from the general concept of a topological space, focusing in particular on manifolds are what we are interested later. Then we will look at examples of some commonly seem topological spaces.

Definition 1.1 (Topological Space)**.** Let X be a nonempty set and τ be a collection of subsets of X satisfying the following:

(i) $X \in \tau$.

(ii) $\emptyset \in \tau$.

(iii) Intersection of finite number of elements from τ is in τ.
That is, if $S_1, ..., S_n \in \tau$, then $\bigcap_{i=1,\dots,n} S_i \in \tau$.

(iv) Union of arbitrary number of elements from τ is in τ.
That is, if for each $\alpha \in I$, $S_\alpha \in \tau$, then $\bigcup_{\alpha \in I} S_\alpha \in \tau$.

The pair (X, τ) is called a *topological space*, and τ is called the *topology* on set X. Elements of τ are called *open sets*. It is sometimes denoted as X for short.

Remark 1.1. Note the topology on a set depends on how the open sets are defined. The *standard topology* on $\mathbb{R}$ is generated by open intervals of the form (a, b) for $a, b \in \mathbb{R}$. That is: an open set in the standard topology is constructed from finite intersections and arbitrary unions of open intervals. There are also non-standard topologies: for example the *discrete topology* on set X is a topology that takes every subset of X as an open set. In the *cofinite topology* a set is open if and only if the complement of the set is finite.

Many important topological spaces are induced from metric spaces. We briefly discuss some metric spaces here for getting some intuition.

Definition 1.2 (Metric Space)**.** A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions: $\forall x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) If $x \neq y$ then $d(x, y) > 0$.

(iii) $d(x, y) = d(y, x)$.

(iv) the Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

A *metric space* is a set X equipped with a metric d.

Example 1.1 ($\mathbb{R}^n$)**.** Recall that $\mathbb{R}^n$ is a metric space when we define the metric to be the distance function:

$$d(x, y) = \sqrt{(x - y) \cdot (x - y)}\,.$$

Definition 1.3 (Open ball)**.** We define the *open ball* of radius r around $x \in \mathbb{R}^n$ to be the set

$$B_r(x) = \{y \in \mathbb{R}^n : d(x, y) < r\}\,.$$

The collection of all open balls in $\mathbb{R}^n$ is a *basis* for the standard topology on $\mathbb{R}^n$. In other words, every open set in $\mathbb{R}^n$ in the standard topology is constructed by taking finite intersections and arbitrary unions of open balls.

More generally, any metric on a set X induces a topology, but not every topological space can be induced from a metric. The next definition contains some standard terminology for discussing topological spaces. For the spaces that we consider, the intuitive notions of interior, boundary, etc. will suffice, but in the interests of being mathematical precise we include formal definitions.

Definition 1.4. Let (X, τ) be a topological space.

(i) Let $S \subseteq X$. A *neighborhood* of S is a subset $V \subseteq X$ that contains an open set U such that:

$$S \subseteq U \subseteq V$$

(ii) Given $a \in X$, a *neighborhood* of a is subset V of X that contains an open set U such that:

$$a \in U \subseteq V$$

(iii) A subset $S \subseteq X$ is called *closed* if its complement S^c is open.

(iv) Let $S \subseteq X$, a point $x \in X$ is in the *closure* of S if every neighbourhood of x has non-empty intersection with S.

(v) Let $S \subseteq X$, a point $x \in X$ is in the *boundary* of S if $x \in \overline{S} \cap \overline{S^c}$.

(vi) Let $S \subseteq X$, a point $x \in X$ is in the *interior* of S if there is an open set V such that $x \in V$ and $V \subseteq S$.

1.2 Maps Between Topological Spaces

In analysis, continuity is defined in the "$\epsilon - \delta$" language, this definition of continuity makes sense for functions between metric spaces. To generalize continuity to arbitrary topological spaces a little more generality is required.

Definition 1.5 (Continuous function between topological spaces)**.** Let X, Y be topological spaces, a function $f : X \to Y$ is *continuous* if for all open set $V \subseteq Y$, $f^{-1}(V) = \{x \in X : f(x) = V\}$ is open in X.

Remark 1.2. When $f : X \to Y$ is continuous functions between metric spaces X, Y, the open set definition of continuity is equivalent to the $\epsilon - \delta$ definition of continuity.

To show the "open set" continuity implies the "$\epsilon - \delta$" continuity, consider continuous function $f : X \to Y$ between metric spaces X, Y. Given $x_0 \in X$ and $\epsilon > 0$, the open ball $B_\epsilon(x_0) = \{y \in Y : d(f(x_0), y) < \epsilon\}$ is open in Y and therefore $f^{-1}(B_\epsilon(x_0))$ is open in X. Since $x_0 \in f^{-1}(B_\epsilon(x_0))$, we choose $\delta = min\{d(x_0, b) : b \in \partial f^{-1}(B_\epsilon(x_0))\}$. Then for $x \in X$, $d(x, x_0) < \delta$ implies $x \in f^{-1}(B_\epsilon(x_0))$ and hence $f(x) \in B_\epsilon(x_0)$, we have $d(f(x), f(x_0)) < \epsilon$.

To show the "$\epsilon - \delta$" continuity implies "open set" continuity. Let $f : X \to Y$ be a given continuous function between metric spaces X, Y. Given open set $V \in Y$, let $x_0 \in f^{-1}(V)$, then $f(x_0) \in V$. Choose $\epsilon = min\{d(f(x_0), b) : b \in \partial V\}$. Then by the $\epsilon - \delta$ continuity of f, $\exists \delta > 0$ such that $\forall x \in X$, $d(x_0, x) < \delta$ implies $d(f(x_0), f(x)) < \epsilon$, which is the same as $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq V$. It follows $B_\delta(x_0) \subseteq f^{-1}(V)$. Since this holds for any $x_0 \in f^{-1}(V)$, $f^{-1}(V)$ is open.

Example 1.2 (Continuous functions)**.** We give some examples of continuous functions between topological spaces.

- A constant function $f : X \to Y$ defined as $x \mapsto c$ for all $x \in X$ with some $c \in Y$ is a continuous function. Take any open set $V \in Y$, $f^{-1}(V)$ is either $\emptyset$ or X, both are open.

- The distance function $f : \mathbb{R}^n \to \mathbb{R}$ to a fixed point $x_0 \in \mathbb{R}^n$ defined as $x \mapsto d(x, x_0)$ is continuous. Take any open set $(a, b) \in \mathbb{R}$, $f^{-1}(a, b)$ is an open set in $\mathbb{R}^n$.

- The indicator function $f_U : \mathbb{R} \to \{0, 1\}$ is defined as $x \mapsto 1$ if $x \in U$ and $x \mapsto 0$ if $x \notin U$ for some $U \subseteq \mathbb{R}$. Let $\mathbb{R}$ be equipped with the standard topology and $\{0, 1\}$ be equipped with the discrete topology. Then f_U is a discontinuous function.

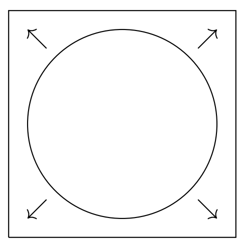
Definition 1.6 (Homeomorphism)**.** Given topological spaces X, Y, a function $f : X \to Y$ is called a *homeomorphism* is:

(i) f is bijective.

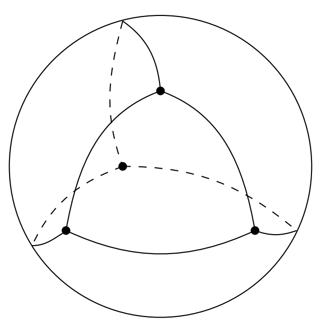
(ii) Both f and f^{-1} are continuous.

If there is a homeomorphism between spaces X and Y we say X and Y are *homoemorphic*, equivalently, if one can be continuously deformed from another.

Example 1.3 ($\mathbb{S}^1$ and square)**.** A circle is homeomorphic to a square since the deformation between them is bijective, both deformation and its inverse are both continuous. Thus $\mathbb{S}^1$ is homeomorphic to a square.



Example 1.4 ($\mathbb{S}^2$ and tetrahedron)**.** A tetrahedron can be continuously deformed to a sphere. Each point on the surface is identified by a one to one correspondence before and after the deformation. This deformation is a homeomorphism since it is bijective and each open set on sphere correspond to an open set on the tetrahedron and thus continuous in both direction. The sphere is therefore homeomorphic to a tetrahedron.



1.3 Topological Manifolds

Definition 1.7 (Topological Manifolds)**.** A topological space X is said to be a n-dimensional *Topological Manifold* if it is:

(i) Hausdorff.

(ii) Second countable.

(iii) Locally Euclidean.

Recall that *Hausdorff* is the property that $\forall x, y \in X$, $\exists U, V$ open such that $x \in U$, $y \in V$ and $U \cap V \neq \emptyset$, *Second countable* means that the basis of X has countable size. Finally locally Euclidean is $\forall x \in X$, $\exists U$ that is open and homeomorphic to the Euclidean n-ball $B_n = \{x \in \mathbb{R}^n : d(x, 0) < 1\}$, or equivalently if U is homeomorphic to $\mathbb{R}^n$. We will say *manifold* to refer *topological manifold* after now.

The motivation for the three conditions are: (i) allows points to be distinguishable on the topological space. Motivation for (ii) is that it ensures the topological space is not too large and exclude pathological examples. Condition (iii) enables us to map the local subsets from a manifold to the Euclidean space where we can easily handle. Note that Hausdorff and second countability are hereditary properties. More discussion on properties of manifolds can be found in Tu [12]. Below are some examples of topological spaces and we skip some of their verifications.

Example 1.5 ($\mathbb{R}^n$)**.** $\mathbb{R}^n$ is homeomorphic to itself by the identity map $f_{id} : x \mapsto x$, $\forall x \in \mathbb{R}$.

Example 1.6 ($\mathbb{S}^n$)**.** A n-sphere (contained in $\mathbb{R}^{n+1}$) has the property that every point $x \in \mathbb{S}^n$ has a neighborhood homeomorphic to $\mathbb{R}^n$ and thus is a manifold.

Example 1.7 ($\mathbb{T}^n$)**.** The n-dimentional torus is also a manifold since all $x \in \mathbb{T}^n$ has neighborhood homeomorphic to $\mathbb{R}^n$.

Definition 1.8 ($P(\mathbb{R}^n)$)**.** Given space $\mathbb{R}^{n+1}$, we define the equivalence relation "$\sim$" on $\mathbb{R}^{n+1} \setminus \{0\}$: $(x_0, ..., x_n) \sim (y_0, ..., y_n)$ if and only if $(x_0, ..., x_n) = \lambda(y_0, ..., y_n)$ for some nonzero $\lambda \in \mathbb{R}$. The projective space is defined to be:

$$\begin{aligned} P(\mathbb{R}^n) &= (\mathbb{R}^{n+1} \setminus \{0\})/\text{``} \sim \text{''} \\ &= \{\text{Lines through 0 in } \mathbb{R}^{n+1}\} \end{aligned}$$

Example 1.8 (Real Projective Plane)**.** Take the vector space $\mathbb{R}^3$, by definition $P(\mathbb{R}^3)$ is given as:

$$\begin{aligned} P(\mathbb{R}^3) &= (\mathbb{R}^3 \setminus \{0\})/\text{``} \sim \text{''} \\ &= \{\text{Lines through 0 in } \mathbb{R}^3\} \end{aligned}$$

To see the projective plane from a geometric perspective, we take the vector space $\mathbb{R}^3$ and consider an plane $z = \alpha$ for $\alpha \neq 0$ in $\mathbb{R}^3$. Lines in $\mathbb{R}^3$ through 0 form elements of $\mathbb{P}^2$. Each line intersects the plane at a unique point except lines on $z = 0$. The projective plane $\mathbb{P}^2$ thus have a one to one correspondence to the points on the plane $z = \alpha$ excepts the lines on $z = 0$. If we consider lines on $z = 0$ as a projective line, the projective plane can be identified by the plane $z = \alpha$ added with the projective line (refered as "line at ∞").

Definition 1.9 (Alternate definition of real projective plane)**.** It is more often we see the projective plane is constructed by a sphere $\mathbb{S}^2$ with antipodal points identified. That is, the projective plane is given as the quotient space of $\mathbb{S}^2$ with the relation "$\sim$": $x \sim -x$. Then $P\mathbb{R}^2 = \mathbb{S}^2/(x \sim -x)$. Then if we take the upper hemisphere of $\mathbb{S}^2$, an element x of $P\mathbb{R}^2$ can be represented by the point of intersection on the upper hemisphere. For every point not on the equator we obtained unique identification. For the points on the equator of $\mathbb{S}^2$, they correspond to the lines on the $z = 0$ plane. We we refer them as "points at infinity".

Although a manifold is locally homeomorphic to the Euclidean space, its global properties are different from Eulidean Spcace. To observe the difference on the global structure, we will need to review charts and continuous transitions of a manifold.

Definition 1.10 (Charts)**.** On a manifold X, a *Chart* (or *Coordinate Chart*) is a pair (U, ϕ) where U is an open set and ϕ is a homeomorphism such that $\phi : U \to V \subseteq \mathbb{R}^n$.

Thus for a point $u \in U$, the chart ϕ assigns u a coordinate in $\mathbb{R}^n$. Precisely, $\phi(u) = (f_1(u), ..., f_n(u))$ where f_i's are functions from U to $\mathbb{R}$. We will look at a simple example of charts defined on a sphere:

Example 1.9 (Charts of Sphere)**.** Let ϕ be the homeomorphism from the northern hemisphere to an open disk in $\mathbb{R}^2$, then ϕ is a chart. Similarly we obtain a chart by taking homeomorphism from southern hemisphere to an open disk in $\mathbb{R}^2$.

Definition 1.11 (Atlas)**.** Given manifold X, a set of charts $\{\phi_i : i \in \mathbb{N}\}$ with domains U_i is called an *atlas* of X if the set covers X: $X \subseteq \bigcup_{i \in I} U_i$.

Example 1.10 (Cover as Atlas)**.** Consider $X = \mathbb{R}$ as a manifold, then set of all open intervals covers $\mathbb{R}$ and thus is the atlas of X.

When there are different charts with overlapping domain, the overlapping part can be mapped to different coordinates in $\mathbb{R}^n$. The transition map allows their coordinates to be identified from one to another.

Definition 1.12 (Transition map)**.** Given manifold X, two charts ϕ, ψ with overlapping domain U, V. The *transition maps* between ϕ and ψ are:

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$

The composite function $\psi \circ \phi^{-1}$ is called the transition map from ϕ to ψ, similarly $\phi \circ \psi^{-1}$ is from ψ to ϕ. We say two charts are *compatible* if transition maps in both direction are C^∞(or "smooth").

These transition maps are useful as their differentiability will define a new class of manifold called the differentiable manifolds. Moreover when they are infinitely differentiable, we call such manifolds *smooth manifolds*.

Remark 1.3. We now use the term *Atlas* to refer smooth atlas which is a collection of compatible charts covers the space.

Definition 1.13 (Smooth Manifold)**.** A *smooth manifold* is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure given by the *Maximal Atlas* $\mathcal{A}$ of M where the charts in $\mathcal{A}$ are pairewise compatible.

Remark 1.4. By *Maximal Atlas* we mean an atlas such that no bigger atlas contains it. A topological space either has a Maximal Atlas or not. This is a result of the property that every atlas is contained in a maximal atlas. Readers who feel unfamiliar with these may consult any standard textbook on smooth manifolds.

1.4 Fundamental Group

In the study of Algebraic Topology, the principal idea is to associate every topological space an algebraic interpretation such that it simplifies the situation. We will soon review some ideas about fundamental groups and covering spaces as they will be useful when study any particular topological space.

The idea of fundamental group is to associate a group to the space we would like to study, the structure of the group captures properties of the space and implies more information about the space. We follow the ideas of Kuga [8] in our exposition of fundamental groups and covering spaces.

Definition 1.14 (Homotopy)**.** Let $f, g : X \to Y$. We say f and g are *homotopic* if one can be continuously deformed to another. More formally, let $I = [0, 1]$. Then f is *homotopic* to g if there exist a family of maps $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. F is called a *homotopy* from f to g and we denote $f \simeq g$ when such F exists.

Example 1.11 (Homotopic path)**.** Recall that a *path* γ on X is a continuous function function $f : I \to X$ where $f(0)$ is the initial point and $f(1)$ is the endpoint. Two paths $\alpha, \beta : I \to X$ are homotopic if $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$ and one can be continuously deformed to another.

The idea of homotopy between paths allows explicit expression of the continuous deformation of paths with fixed endpoints, vice versa such continuous deformation gives a homotopy.

Example 1.12. Consider $f(x) = \sin(x)$ and $g(x) = 0$ on $[0, 2\pi]$. A homotopy $F : [0, 2\pi] \to \mathbb{R}$ can be given as:

$$F(x,t) = (1-t)\sin(x)$$

then $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Definition 1.15 (Loop)**.** The embedding $\gamma : I \to X$ of interval $I = [0, 1]$ to X is called a *loop* when $\gamma(0) = \gamma(1)$. Equivalently, the embedding $\mathbb{S}^1 \to X$ defines a loop too.

Proposition 1.1 (Homotopy as equivalence relation)**.** Homotopy is an *equivalence relation* on the set of maps from X to Y.

Proof. For any $f : X \to Y$, $f \simeq f$ thus the relation is reflexive. For any $f, g : X \to Y$, if the homotopy from f to g is $F : X \times I \to Y$, we let the homotopy from g to f be $G : X \times I \to Y$ by letting $G(x, t) = F(x, 1-t)$. Thus $g \simeq f$ and the relation is symmetric. For any $f, g, h : X \to Y$ with $f \simeq g$ and $g \simeq h$, homotopy from f to h can be obtained by letting $H : X \times I \to Y$ defined piecewise: $H(x, t) = F(x, 2t)$ for $t \in [0, 1/2]$ and $H(x, t) = G(x, 2t - 1)$ for $t \in [1/2, 1]$. $\square$

Definition 1.16 (Homotopy class)**.** By Proposition 1.1 homotopy between X, Y is an equivalence relation, its equivalence classes are called the *homotopy classes.*

Definition 1.17 (Fundamental group)**.** The set of homotopy classes of loops in X based at a point $x_0 \in X$ form a group called the *fundamental group* of X and is denoted as $\pi_1(X, x_0)$. The binary operation $*$ of $\pi_1(X, x_0)$ is defined as the concatenation of (homotopy classes of)loops: if $[f], [g]$ are two homotopy classes, let $[f]\cdot[g]$ be the homotopy class $[h]$ described in proof of Proposition 1.1

Example 1.13 ($\pi(\mathbb{S}^n)$)**.** Recall an *n-sphere* $\mathbb{S}^n$ is the set of points in $\mathbb{R}^{n+1}$ with distance r to a fixed point.

- A 1-*Sphere* $\mathbb{S}^1$ is a circle, the embedding of a loop can be winding the circle any integer number of times, thus $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. More explicitly, let $\gamma_n : I \to \mathbb{S}^1$ defined as $\gamma_n : t \mapsto (\cos(2\pi(nt)), \sin(2\pi(nt)))$ where $n \in \mathbb{Z}$. Each γ_n gives an embedding, the set $\{\gamma_n\}$ has the group structure and is isomorphic to $\mathbb{Z}$.
- A 2-*Sphere* $\mathbb{S}^2$ is a usual "sphere", the embedding of a loop to $\mathbb{S}^2$ is always contractible to a point thus $\pi_1(\mathbb{S}^2) \cong \{1\}$.
- Similarly, all $\mathbb{S}^n$ for $n \geq 2$ have trivial fundamental group.

Example 1.14 ($\pi_1(P\mathbb{R}^2)$)**.** By Definition 1.9 the real projective plane can be constructed by a sphere with the equivalence relation of identifying antipodal points. $P\mathbb{R}^2$ can be identified by the piece of upper hemisphere with the equator(Since one point on the hemisphere is enough to identify the pair). There's only two way to embed a loop to this configuration:

(i) The loop is contained in the interior(upper hemisphere);

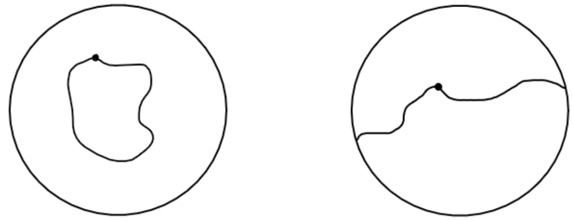


Figure 1.1: Two types of loops in $P\mathbb{R}^2$

(ii) The loop intersects the equator.

In the first case such loop is contractable and the homotopy class of such loop is the set of trival loops which gives the identity element of $\pi_1(P\mathbb{R}^2)$, in the second case the loop is non-contractable and its homotopy class gives an non-identity element of $\pi_1(P\mathbb{R}^2)$. Thus there's only two elements in $\pi_1(P\mathbb{R}^2)$. The only group of order 2 is $\mathbb{Z}_2$, $\pi_1(P\mathbb{R}^2) \cong \mathbb{Z}_2$.

1.5 Covering Spaces

In this section we try to use minimal algebraic topology background and maximal concreteness to cover some of the basic and useful properties we need to study covering spaces. Most of the details of the following discussion can be found in Kuga [8].

Definition 1.18 (Covering Space)**.** Given a topological space X, a space X' is called the *covering space* of X if there is a map $f : X' \to X$ satisfying the following:

(i) f is a continuous surjective function.

(ii) For each point $x \in X$, there exist an open neighbourhood U around x such that $f^{-1}(U)$ is a union of disjoint open sets $\{V_1, V_2, ..., V_n\}$. Each open set V_k maps onto U by f homeomorphically.

Such map f is called a *covering map* from the *covering space* X' to the *base space* X.

Example 1.15 (Covering space of $P\mathbb{R}^2$)**.** Let $X' = \mathbb{S}^2$, then X' is a cover of $P\mathbb{R}^2$ by $f : \mathbb{S}^2 \to P\mathbb{R}$. For every point $x \in P\mathbb{R}^2$, we can find a small open neighbourhood U on $\mathbb{S}^2$ around x such that $f^{-1}(U)$ are the two disjoint open sets that are antipodal to each other.

Definition 1.19 (Sheets)**.** Given open neighbourhood U and the set of copies of $f^{-1}(U)$. The cardinality of the set is said to be the number of *sheets* of the covering space.

Example 1.16 (Circle $\mathbb{S}^1$)**.** Example 1.13 shows the fundamental group of $\mathbb{S}^1$ is the group $\mathbb{Z}$. This illustration of fundamental group can also be interpreted as a covering space of $\mathbb{S}^1$. Precisely, let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be defined as $z \mapsto z^n$ where z is on the unit circle on complex plane. This gives a covering of $\mathbb{S}^1$ by $\mathbb{S}^1$ which has n sheets. Note that covering space is not necessarily unique, another covering map of $\mathbb{S}^1$ can be given as $g : \mathbb{R} \to \mathbb{S}^1$ where $x \mapsto e^{2\pi i x}$ for $x \in \mathbb{R}$.

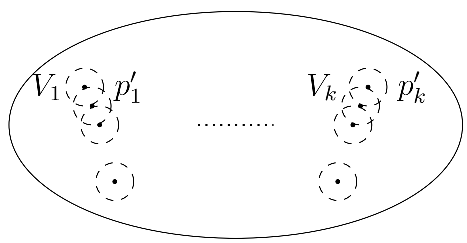
Proposition 1.2 (Concatenation of path)**.** If given $f : X \to X'$, define "$\cdot$" as concatenation of paths, then $f(\alpha \cdot \beta) = f(\alpha)f(\beta)$ where $\alpha, \beta \in X'$.

Proposition 1.3 (Covering preserves homotopy)**.** Given $\alpha \simeq \beta$ in X', then $f(\alpha) \simeq f(\beta)$ in X.

Definition 1.20 (Lifting)**.** Given a covering $f : X' \to X$, a lift of a curve $C \in X$ is a curve $C' \in X'$ such that $f(C') = C$.

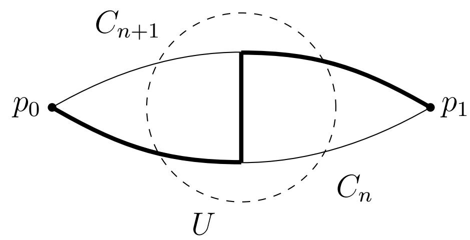
Proposition 1.4 (Lifts of curve uniquely determined by initial point, see Kuga [8] Page 62-64)**.** Give base space X, its cover $f : X' \to X$. Let C be a curve on X. There can be multiple lifts of C exist in X'. When initial point of C' is chosen the lift C' is uniquely determined.

Proof. As stated in the definition of the covering map, if we take a small neighbourhood U of a point p on X, there exists a union of disjoint neighbourhood $V_1, V_2, ..., V_n$ that contains the copies of lifts of p in X'. Suppose for given curve C on X there are multiple lifts $C'_1, C'_2, ..., C'_n$ on X'. Given initial point p on C and there are n copies of lifts of $p : p'_1, p'_2, ..., p'_n$ in $V_1, V_2, ..., V_n$ respectively. Suppose that we begin our lift at $p'_k = (f|_{V_k})^{-1}(p)$ for some k, then we can show the lift of curve C is unique: Starting from the initial point we choose sufficiently small open neighbourhood to be disjoint from other neighbourhoods, the lifting process is unique at each step and eventually we obtain an unique lifted curve started from initial point p'_k. □



Proposition 1.5 (Lifting preserves homotopy, see Kuga [8] Page 66 Preparation Theorem)**.** If given topological space X with covering space X' and covering map $f : X' \to X$, for a pair of homotopic curves C_0, C_1, their lifts C'_0, C'_1 in X' are also homotopic.

Proof. Given $C_0 \simeq C_1$ in X, from definition of homotopy, C_0, C_1 have common initial point p_0 and end point p_1 and they lift to p'_0 and p'_1 in X'. We know C_0 can be continuously deformed to C_1, at each step of the deformation of C_0, we make the deformation sufficiently small so that the different part while deforming C_n to C_{n+1} is contained in a sufficiently small open neighbourhood U. Restricting to an open neighbourhood V_k, the lift $(f|_{V_k})^{-1}(C_n)$ and the lift$(f|_{V_k})^{-1}(C_{n+1})$ coincide in everywhere else except the difference part in the small neighbourhood $U' = (f|_{V_k})^{-1}(U)$. By property of covering map, $f|_{V_k}$ is a homeomorphism between U and U', thus homotopy is preserved by f and lifts of homotopic curves C_n, C_{n+1} must be homotopic in covering space. □



1.6 Covering Space & Fundamental Group

Let X be a manifold, giving a covering $f : X \to X'$ is equivalent to give a subgroup of the fundamental group of X. Our goal is to show that there is a bijection between the fundamental group of the covering space X' and a subgroup of fundamental group of base space X.

Take the base point $o \in X$ and its lift $o' \in X'$, we have two fundamental groups: $\pi_1(X, o)$ for the space X and $\pi_1(X', o')$ for space X'.

The elements in $\pi_1(X, p)$ and $\pi_1(X', p')$ are homotopy classes of loops in X and X' respectively. When the covering map f act on homotopy classes $[\alpha] \in X'$ we denote its image as $f_*([\alpha]) \in X$.

Theorem 1.1 (Bijection between $\pi_1(X')$ and a subgroup of $\pi_1(X)$, see Kuga [8] Theorem 10.1)**.** Let X' be a covering space of X and $f : X' \to X$ be the covering map. The induced map $f_* : \pi_1(X', o') \to \pi_1(X, o)$ is an injection and there is a bijection from $\pi_1(X', o')$ to the image $f_*(\pi_1(X', o'))$ which is a subgroup of $\pi_1(X, o)$.

Proof. To show injection, let $[\gamma] \in Ker(f_*)$ be the set of null homotopic loops in X', $f_*([\gamma]) = [1]$. Suppose there is a different homotopy class of loops $[\beta] \in X'$ such that $f_*([\beta]) = [1]$. By Proposition 1.5, lifts of homotopic loops are homotopic, then $[\beta]$ must be the same as the null homotopy class $[1]$. The kernel is trivial, thus f_* is injective.

To show this is a subgroup, note the property $f(\alpha \cdot \beta) = f(\alpha) f(\beta)$ induces $f_*([\alpha] \cdot [\beta]) = f_*([\alpha]) f_*([\beta])$. Thus f_* is a homomorphism, $f_*(\pi_1(X', o')) \leq \pi_1(X, o)$.

Thus $\pi_1(X', o')$ and $f_*(\pi_1(X', o'))$ has a bijection relationship, and also we have:

$$\pi_1(X', o') \cong f_*(\pi_1(X', o'))$$

$\square$

Remark 1.5. This one-to-one correspondence between a covering of space X and a subgroup of $\pi_1(X, o)$ can be interpreted as "covering space" version of the *Galois Correspondence* of intermediate fields, we will continue investigating this after Theorem 1.2.

Definition 1.21 (Covering transformation)**.** Let $f : X' \to X$ be a covering map. A homeomorphism $t : X' \to X'$ is called a covering transformation (or deck transformation) of $f : X' \to X$ if $f(t(p)) = f(p)$ for all $p \in X'$.

Example 1.17. Recall Example 1.15 shows $\mathbb{S}^2$ is a cover of $P\mathbb{R}^2$. A covering transformation is the antipodal map t given by $x \mapsto -x$ for $x \in \mathbb{S}^2$.

Example 1.18. A typical covering transformation of the covering $g : \mathbb{R} \to \mathbb{S}^1$ in Example 1.16 is translation of integers on $\mathbb{R}$.

Definition 1.22 (Conjugate)**.** If in the covering space X', under the covering transformation map t, for $p_1, p_2 \in X'$ we have $t(p_1) = p_2$, then p_1 and p_2 are called *conjugates* of each other.

Proposition 1.6 (Covering transformations group)**.** Given a covering space X' of X, the set S of all covering transformation form a group under composition and is called the covering transformation group of $f : X' \to X$.

Proof. Identity covering transformation is regarded as the identity element. For any t with $g(p_1) = p_2$, there's an inverse t^{-1} since t is a homeomorphism. If there is t_1, t_2 with $t_1(p_1) = p_2$ and $t_2(p_2) = p_3$, then composition is $t_2(t_1(p_1)) = p_3$ is again a covering transformation. To see associativity, let t_1, t_2, t_3 defined as $t_1(p_1) = p_2$, $t_2(p_2) = p_3$, and $t_3(p_3) = p_4$. Then $(t_3t_2)t_1(p_1) = p_4 = t_3(t_2t_1)(p_1)$. $\square$

Proposition 1.7 (Uniqueness of covering transformation)**.** Let t be a covering transformation, for a pair of conjugates $p_1, p_2 \in X'$ with $t(p_1) = p_2$, the covering transformation t is uniquely determined.

Proof. For the pair $\{p_1, p_2\}$, suppose there's two different covering transformations t, t' both from p_1 to p_2. Then $t(p_1) = p_2 = t'(p_1)$, $t = t'$. $\square$

Proposition 1.8. A coset of $\pi_1(X')$ in $\pi_1(X)$ is the set of loops in X' that lifts to a path from x_0 to x_i for some i where $1 \leq i \leq n$ and n is the number of sheets of X'.

Proof. "$\supseteq$" direction:
Let $\alpha \in \pi_1(X)$ with a lift starts at x_0 ends at x_i. Then αt_i^{-1} is a closed loop based at x_0 thus $\alpha t_i^{-1} \in \pi_1(X')$, hence $\alpha = (\alpha t_i^{-1})t_i \in \pi_1(X')t_i$. Any element in $\pi_1(X)$ that lifts to path from x_0 to x_i is in the coset $\pi_1(X')t_i$.

"$\subseteq$" direction:
Consider coset $\pi_1(X')t_i$, an element in $\pi_1(X')t_i$ is of the form βt_i where $\beta \in \pi_1(X')$ and t_i is a path from x_0 to x_i. Therefore all elements in $\pi_1(X')t_i$ are path from x_0 to x_i. Moreover, any path γ from x_0 to x_i can be written in form of $\beta' t_i$ up to homotopy for some $\beta' \in \pi_1(X')$ because any γ can be obtained by deformation of some $\beta' t_i$. $\square$

Thus there exist bijection between n set of paths who lift to paths end at x_i's and n cosets generated by $\pi_1(X')$ in $\pi_1(X)$. Moreover the t_i are representatives in these cosets and the choice of t_i in each coset does not affect the pre-transfer map because changing t_i to another element in the same coset preserves homotopy in X'.

Definition 1.23 (Galois covering)**.** A covering $f : X' \to X$ is called Galois or normal covering if for every $p \in X$ and every pair of lifts $p'_1, p'_2 \in X'$ of $p \in X$ there is a covering transformation t such that $f(t(p'_1)) = f(p'_2)$.

$$\begin{tikzcd} p_1 \arrow[r, "t"] \arrow[dr, "f"'] & p_2 \arrow[d, "f"] \\ & f(p_1) \end{tikzcd}$$

Theorem 1.2 (Kuga [8] Theorem 11.2)**.** Let $f : X' \to X$ be a Galois covering space of X then:

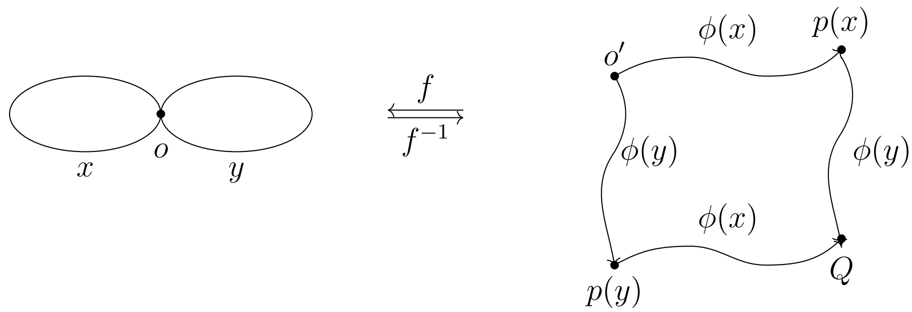
(*i*) $f_*(\pi(X', o')) \lhd \pi_1(X, o)$.

(*ii*) $\pi_1(X, o)/f_*(\pi_1(X', o')) \cong S$

Note: S is the set of all covering transformations correspond to the covering space $f : X' \to X$. $o' \in X'$ is a lift of $o \in X$ by f^{-1}.

Proof. (i). For every element $\alpha \in \pi_1(X, o)$, its lift in X' is not necessarily a loop. Denote the starting point as o' and endpoint as $p(\alpha)$.(Note that o' and p(α) are conjugates to each other). let $\phi(\alpha) \in S$ denote the covering transformation t s.t. $t(o') = p(\alpha)$. We show first ϕ is a homomorphism and then $Ker(\phi) = f_*(\pi_1(X', o'))$.

ϕ is a homomorphism can be illustrated by the following diagram.



Given $x, y \in \pi_1(X, o)$. In one way we apply covering transformation to $p(y) = \phi(x)o'$ gets to the point $Q = \phi(x)\phi(y)o'$, in another way we follow the loop $x \cdot y$ in X, loop x correspond to the curve $o'p(x) \in X'$, loop y correspond to the curve $p(x)Q \in X'$. Thus the point $Q = \phi(x)\phi(y)o' = \phi(x \cdot y)o'$, ϕ is a homomorphism.

Now we show $Ker(\phi) = f_*(\pi_1(X', o'))$.

"$\subseteq$" direction:

$$\begin{aligned} Ker(\phi) &= \{\alpha \in \pi_1(X, o) : \text{lift } \alpha' \in X' \text{ gives } \phi(\alpha) \text{ connects } o' \text{ to } p(\alpha) = o'\} \\ &\subseteq f_*(\pi_1(X', o')) \end{aligned}$$

"$\supseteq$" direction:

$$\begin{aligned} f_*(\pi_1(X', o')) &= \{\alpha = f_*(\alpha') \in \pi_1(X, o) : \alpha' \in X',\ \phi(\alpha) \text{ connects } o' \text{ to } p(\alpha) = o'\} \\ &\subseteq Ker(\phi) \end{aligned}$$

Thus, $f_*(\pi_1(X', o')) = ker(\phi)$ implies $f_*(\pi_1(X', o')) \lhd \pi_1(X, o)$ and we have shown (i).

To show (ii), we show ϕ is surjective. Take $\alpha \in \pi_1(X, o)$, all lifts of α is a set of conjugates $\{\alpha'_1, \alpha'_2, ..., \alpha'_n\}$ where each pairs of conjugates correspond to a covering transformation and such process consumes all covering transformations. Thus ϕ is surjective. By the fundamental homomorphism theorem,

$$\pi_1(X, o)/f_*(\pi_1(X', o')) \cong Im(\phi) = S$$

□

Remark 1.6 (Analogy of Galois Correspondence)**.** This one-one correspondence between covering spaces of X and subgroups of $\pi_1(X, o)$ is analogous to the *Galois Correspondence* in Galois Theory about field extensions. If we define $Aut(X'/X)$ to be which is the set of deck transformations $t : X' \to X'$. When the cover is Galois, $Aut(X'/X)$ reach the largest cardinality it can be and in this case it is analogous to the Galois extension.

Theorem 1.3 (Number of Sheets in cover is the index of $f_*(\pi_1(X', o'))$)**.** Given Galois covering $f : X' \to X$, the number of sheets of f equals to the index of $f_*(\pi_1(X', o'))$ in $\pi_1(X, o)$.

Proof. To show this we build a bijection between a set of conjugates in X' and the set of cosets of $f_*(\pi_1(X', o'))$ in $\pi_1(X, o)$. For $g \in \pi_1(X, o)$, consider the map $\phi : Hg \to p(g)$ which sends the coset Hg to the endpoint of lift of $h \cdot g$. Here all elements of H lifts to loops base at a fixed point o'.

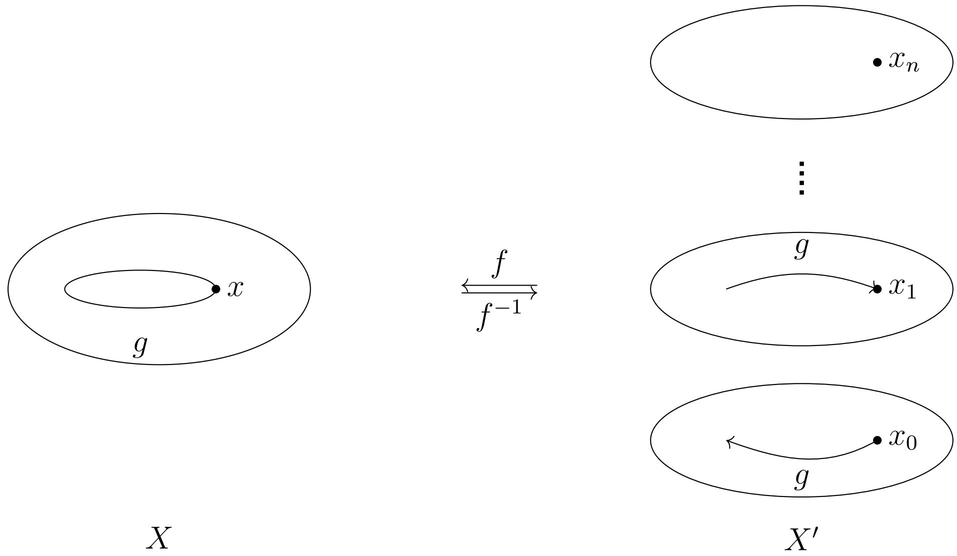
To see injectivity, consider $\phi(Hg_1) = \phi(Hg_2)$, then g_1, g_2 must have same endpoint in X', also they have same initial point o'. Since homotopy curves lift to homotopy curves, $g_1 = g_2$ and $Hg_1 = Hg_2$.

To see surjectivity, consider the set $\{f^{-1}(o)\}$ of all conjugates of o', since o' can be connected to any element in $\{f^{-1}(o)\}$ by a lift of some $g \in \pi_1(X, o)$. Thus ϕ is surjective.

We have now shown this is a bijection and the number of sheets of Galois covering f is exactly the index of $f_*(\pi_1(X', o'))$. □

1.7 Pre-Transfer Map

Given manifold X and its covering space X' with covering map $f : X \to X'$. From $\pi_1(X')$ to $\pi_1(X)$, there is the natural projection from $\pi_1(X')$ to $\pi_1(X)$ induced by f by sending loops in X' to loops in X up to homotopy. On the opposite direction, when lift loops from X to X', there are ways to take elements from $\pi_1(X)$ to $\pi_1(X')$ but there exist no canonical map. In fact some loops in $\pi_1(X)$ might not even lift to closed loop in $\pi_1(X')$. We construct the following way to map elements from $\pi_1(X)$ to $\pi_1(X')$:



Consider the base point x in X' which has n lifts $\{x_0, ..., x_{n-1}\}$. A loop $g \in \pi_1(X)$ with basepoint x lifts to a paths connecting a pair (x_i, x_j) of conjugates determined by initial point. We abuse notation here and denote that g connects (x_i, x_j) pairs. With out loss of generality let initial point to be x_0, g takes x_0 to some x_i. Meanwhile since the covering space is connected there exist a path from x_0 to x_i, denote as t_i. The image of t_i in X begins and ends at x_0, so this is an element of the fundamental group of X. Furthermore, no two of the t_i are homotopic, by the definition of a cover: disjoint open neighbourhoods seperate the x_i's which means that a path between them cannot be contracted homeomorphically to the identity. Thus we obtain a set $T = \{t_0, ..., t_{n-1}\}$ of path that act on $\{x_0, ..., x_{n-1}\}$ as deck transformations: an element of the fundamental group of X may be lifted to any of the x_i and so determines a permutation of the sheets of the cover. (It takes a little more work to verify that these maps are automorphisms of the covering space, but this is true).

Concatenation $t_i g$ is viewed as concatenation of path in X'. We let the dot operation defined as follow: $t_i \cdot g$ is defined to be the unique t_j which maps to the endpoint of the concatenation $t_i g$. This means that $t_i g (t_i \cdot g)^{-1}$ maps x_0 to x_0 and so is an element of the fundamental group of X'.

We can now construct the following map from $\pi_1(X)$ to $\pi_1(X')$:

$$V(g) = \prod_{t_i \in T} t_i g (t_i \cdot g)^{-1}$$

Then $V(g)$ can be interpreted as a sum of loops with basepoint x_0 and we call it the *pre-transfer* map. Consider elements in $\pi_1(X)$, they either lift to loops in X' and belongs to $\pi_1(X')$ or lift to path connecting two fibers of the basepoint. While lifting, fixing the initial point, by Proposition 1.4 the lift is uniquely determined.

Since we have not explicitly imposed an ordering on the elements of T, the map $V(g)$ is well defined only when $\pi_1(X', x')$ is abelian. As a function between abelian groups

$$V : \mathrm{Ab}\pi_1(X, x) \to \mathrm{Ab}\pi_1(X', x_0)$$

V is known as the *Transfer*, and turns out to be a homomorphism. The terms in the product may be reordered as desired, and this is necessary to establish the homomorphism property:

$$\begin{aligned}
V(g_1g_2) &= \prod_{t_i \in T} t_i g_1 g_2 (t_i \cdot (g_1 g_2))^{-1} \\
&= \prod_{t_i \in T} t_i g_1 (t_i \cdot g_1)^{-1} (t_i \cdot g_1) g_2 ((t_i \cdot g_1) \cdot g_2)^{-1} \\
&= \prod_{t_i \in T} t_i g_1 (t_i \cdot g_1)^{-1} \prod_{t_i \in T} (t_i \cdot g_1) g_2 ((t_i \cdot g_1) \cdot g_2)^{-1} \\
&= \prod_{t_i \in T} t_i g_1 (t_i \cdot g_1)^{-1} \prod_{t_j \in T} t_j g_2 (t_j \cdot g_2)^{-1}
\end{aligned}$$

The last equality is only reordering the index of T of the last product. As just discussed earlier the order of t_i won't affect the image when $\pi_1(X')$ is abelian. Thus $V(g_1g_2) = V(g_1)V(g_2)$. As we desired, the tansfer map is a homomorphism when $\pi_1(X')$ is abelian.

To conclude: the fundamental groups of a space and its cover have geometric interpretations, and there is a natural covering map which can be used to show that up to isomorphism

$$\pi_1(X', x_0) \leq \pi_1(X, x)\,.$$

In this section, we have constructed a homomorphism *in the other direction* at the cost of moving to the abelianised groups.

$$V : \mathrm{Ab}\pi_1(X, x) \to \mathrm{Ab}\pi_1(X', x_0)$$

This map is called the *transfer homomorphism*, and in general is neither injective nor surjective. Unfortunately, the abelianisation of the fundamental group does not yet have obvious connections to the geometric structure of the underlying space. In the next chapter we will show that this group is the first in a sequence of *homology* groups, which are of central interest in algebraic topology.

In contrast to the higher homotopy groups, which are mysterious even for the simplest topological spaces (e.g. being unknown even for sufficiently large n-spheres), homology groups can be computed using linear algebra and finite combinatorial approximations of topological spaces. This is the topic of the next chapter.

Chapter 2

Simplicial Homology

In this chapter we develop the beginnings of homology theory. Recall that an *invariant* of a topological space is an object associated to the space which is invariant under homeomorphisms. For example, the dimension of a manifold is an invariant. Thus spaces with the same invariant may be homeomorphic or not, but spaces with distinct invariants are never homeomorphic. The fundamental group is an example of an invariant. In fact, taken together, the set of homotopy groups is a complete invariant for a space: the spaces are homeomorphic if and only if all their homotopy groups are isomorphic. Unfortunately, the homotopy groups are not effectively computable. In this chapter, we define the homology groups of a space, which are effectively computable given a *simplicial resolution* (or *triangulation*) of a topological space. We will begin by developing the machinery of simplicial complexes and chain complexes. Further discussions about simplicial homology can be found in Chapter 2 of Hatcher [5].

2.1 Simplicial Chain Complex

There are many homology theories, but simplicial homology is the most concrete. The idea is to replace a nice topological space with a covering by finitely many contractible open sets. We require also that intersections are simply connected. Then much of the structure of the topological space is captured by a finite combinatorial structure, which is a simplicial chain complex. We define these structures in this section.

Definition 2.1 (Standard n-simplex)**.** The standard n-simplex is defined as follow:

$$\Delta^n = (e_0, ..., e_n) = \{t_0 e_0 + ... + t_n e_n : t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1\}$$

Definition 2.2 (Simplex)**.** Generalized idea of standard n-simplex is called the *n-simplex*, an n-simplex is the image of the standard n-simplex under some homeomorphism.

Definition 2.3 (Triangulation)**.** Given a n-dimensional manifold X, a *triangulation* of X is a decomposition of X into n-simplices satisfying the following conditions:

(i) Interior of the simplexes are disjoint.

(ii) Union of all simplexes is the whole topological space X.

(iii) Intersection of n-simplexes are lower dimensional simplexes.

Note: when applying triangulation to a space, we assume homeomorphism is allowed. Next example is an illustration.

Example 2.1 (Triangulation of Sphere)**.** We can divide the surface of the sphere into 4 pieces and each piece is homeomorphic to a triangle. This makes the surface of the sphere homeomorphic to a tetrahedron as shown in Example 1.4

In fact, it is possible to triangularise any compact manifold with finitely many simplices, though known algorithms often result in impractically large numbers of simplices. Finding triangulations with small numbers of simplices even for well-known topological surfaces is an ongoing project. In the other direction, it is possible to define a topological space by giving a triangulation, which is defined by finite combinatorial data: one needs only to specify the intersections of each pair of simplices. Deciding the topological properties of such spaces from the triangulation is a well-studied problem.

Now we move toward the definition of the homology groups.

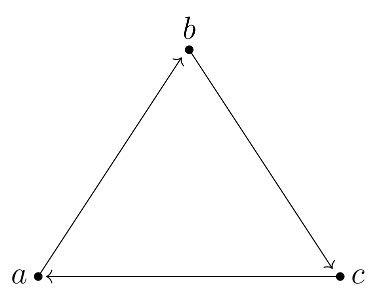
Definition 2.4 (Boundary Map)**.** Let X be a simplicial complex. For $n \geq 1$, the boundary map of X is a map $\partial_n : C_n(X) \to C_{n-1}(X)$ defined as:

$$\partial_n[v_0, ..., v_n] = \sum_{i=0}^{n}(-1)^i[v_0, ..., \hat{v}_i, ...v_n]$$

where $\hat{v}_i$ denotes the term v_i is omitted.

Consider a topological space X, up to homeomorphism, X can be thought as the union of many n-simplex for $n = 0, 1, 2, ...$ etc. We are interested to see if some n-simplex is ought to be the boundary of some $(n+1)$-simplex, for example, given a 2-simplex (triangle), then the boundary of it is made of 1-simplex (lines).

Example 2.2 (Boundary of a Triangle)**.** Suppose given a single triangle with vertices labeled a, b, c. This is a single 2-simplex. The edges are 1 simplxes given by $[a, b]$, $[b, c]$, $[c, a]$.



If we take the boundary twice on the triangle we obtain 0.

$$\partial[a,b,c] = [b,c] - [a,c] + [a,b]$$

$$\begin{aligned}\partial\partial[a,b,c] &= \partial([b,c] - [a,c] + [a,b]) \\ &= (c-b) - (c-a) + (b-a) \\ &= 0\end{aligned}$$

We extend the concept of a single simplical complex to a chain of simplcial complexes now.

Definition 2.5 ($C_n(X)$)**.** Given a simplicial complex X, let $C_n(X)$ be the free abelian group generated by the basis of all n-simplexes in X. An element in $C_n(X)$ is called a n-chain.

Definition 2.6 (Simplicial Chain Complex)**.** Given a simplicial complex X, we associate it a sequence of free abelian groups connected by boundary maps. Each $C_n(X)$ is defined as in Def. 2.5 and let $C_{-1} = 0$.

$$... \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} ...... \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

where each ∂ are boundary maps. For short, we write C_n to refer $C_n(X)$.

Proposition 2.1. For all $n \geq 0$, $\partial_{n-1} \circ \partial_n = 0$.

Proof.

$$\partial_n(x_0, ..., x_n) = \sum_{k=0}^{n} (-1)^k (x_0, ..., \hat{x_k}, ..., x_n)$$

Each term in the summation is of the form $(-1)^i(x_0, ..., \hat{x_i}, ..., x_n)$, apply ∂_{n-1} to it:

$$\begin{aligned}\partial_{n-1}(x_0, ..., \hat{x_i}, ..., x_n) = &\sum_{k=0,k<i}^{i-1} (-1)^k (x_0, ..., \hat{x_k}, ..., \hat{x_i}, ..., x_n) \\ &+ \sum_{k=i+1}^{n} (-1)^{k-1} (x_0, ..., \hat{x_i}, ..., \hat{x_k}, ..., x_n)\end{aligned}$$

Notice that since the $\hat{x_i}$ term disappeared, thus exponent of (-1) after i-th entry alters.
In $\partial_{n-1} \circ \partial_n(x_0, ..., x_n)$, the term $(x_0, ..., \hat{x_i}, ..., \hat{x_j}, ...x_n)$ occurs twice, one in $\partial_{n-1}(x_0, ..., \hat{x_i}, ..., x_j, ...x_n)$, one in $\partial_{n-1}(x_0, ..., x_i, ..., \hat{x_j}, ...x_n)$, with different signs. Thus the composition has $\partial_{n-1} \circ \partial_n = 0$. □

Definition 2.7 (Exact Chain Complex)**.** A simplicial chain complex is said to be *exact* if for all $i \geq 0$, $Ker(\partial_i) = Im(\partial_{i+1})$.

Note: Not all chain complexes are exact. For all chain complexes, $Im(\partial_{n+1}) \subseteq Ker(\partial_n)$, a chain becomes exact only when equality holds.

Definition 2.8 (Homology Group)**.** Given topological space X, we can construct a simplicial chain complex as given in above definition. The homology group at each C_i is defined as:

$$H_i = \frac{Ker\partial_i}{Im\partial_{i+1}}$$

A chain complex is exact if all of its homology groups are 0 (trivial group).

Definition 2.9 (Boundary, Cycle, Homology class)**.** The elements of $Im(\partial_{n+1})$ are *boundaries.* Each one is the boundary of some $(n+1)$-chain element in $C_{n+1}X$. For this reason, an n-cell attached to the simplicial complex along the boundary retracts to a point.

The elements in $ker(\partial_n)$ are *cycles.* Every boundary is a cycle, as is every linear combination of boundaries. But their may be other non-obvious cycles in the space.

The *homology classes* of a topological space are the elements of the quotient group $H_n(X) = ker(\partial_n)/Im(\partial_{n+1})$. These are cosets of the boundaries. We say two cycles f, g are homologous if they represent the same coset, that is, the difference of f and g is a boundary. We denote it as $f \sim g$.

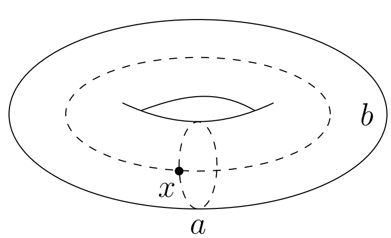
It is interesting to note that two topological spaces can have isomorphic homology groups in every dimension without being homeomorphic – homology is **not** a complete invariant of topological spaces. On the other hand, it is routine to calculate homology once a triangulation is given.

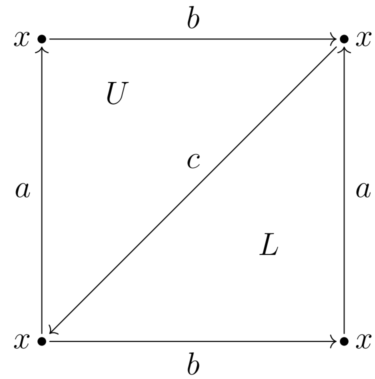
The *Mayer-Vietoris* sequence in homology is a tool which can be used to compute homology of a space X from open subsets X_1, X_2 satisfying $X_1 \cup X_2 = X$, provided the homology of X_1, X_2 and $X_1 \cap X_2$ are known. This generalises easily to open covers by finitely many open sets, and can be used recursively to compute homology. In particular: the main obstruction to computing homology groups is the computation of a triangulation.

Finally, since a topological space can be triangulated in many different ways, it is not clear that homology groups are an invariant of a manifold rather than a triangulation, but in fact this is the case. We refer the reader to Brown [2] for a proof of this fact.

2.2 Some Computed Examples

Example 2.3 ($H_n(\mathbb{T}^2)$)**.** Given a torus, we are can represent it and its triangulation as follow:





The simplicial chain associated with torus is given as:

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$C_i = 0$ for $i \geq 3$ since in this case no simplexes exist for $i \geq 3$ on a torus. We can compute the homology at each C_i:

- $C_0 = \langle x \rangle$ where x is a single vertex.
- $C_1 = \langle a, b, c \rangle$ where a, b, c are the 1-simplexes in the triangularization.
- $C_2 = \langle U, L \rangle$ where U, L denotes the upper and lower triangles in the triangulation.

Note: $\langle ... \rangle$ denotes the spanning set.

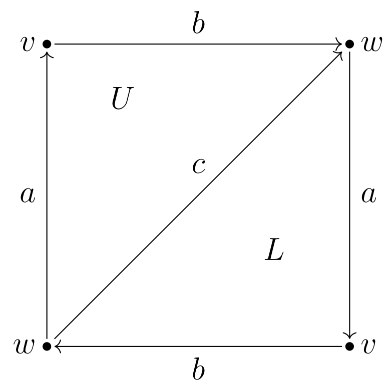
$$H_0 = \frac{Ker(\partial_0)}{Im(\partial_1)} = \frac{\langle x \rangle}{\{0\}} \cong \mathbb{Z}$$

$$H_1 = \frac{Ker(\partial_1)}{Im(\partial_2)} = \frac{\langle a, b, c \rangle}{\langle a + b + c \rangle} \cong \frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$H_2 = \frac{Ker(\partial_2)}{Im(\partial_3)} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

Since $Ker(\partial_2)$ is generated by single element $U - L$, $\langle U - L \rangle \cong \mathbb{Z}$ For higher dimension, all $H_i = 0$ with $i \geq 3$.

Example 2.4 ($H_n(P\mathbb{R}^2)$)**.** As discussed in Example 1.14. The real projective plan can be identified by the upper hemisphere plus the equator. Thus it can be viewed as a closed disk where the boundary is the "line at ∞". In order to triangularize it, consider continuously deform the disk to a square. Thus a triangularization can be done as following:



The simplicial chain complex associated with $P\mathbb{R}^2$ is given as:

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Similarly, $C_i = 0$ for $i \geq 3$ since in this case no simplexes exist for $i \geq 3$ in $P\mathbb{R}^2$. At each C_i, we have:

- $C_0 = \langle v, w \rangle$ where v, w are points on the "equator".
- $C_1 = \langle a, b, c \rangle$ where a, b, c are all 1-simplexes in the triangularization.
- $C_2 = \langle U, L \rangle$ where U, L denotes the upper and lower triangles in the triangulation.

$$H_0 = \frac{Ker(\partial_0)}{Im(\partial_1)} = \frac{\langle v, w \rangle}{\langle v - w \rangle} \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \cong \mathbb{Z}$$

$$H_1 = \frac{Ker(\partial_1)}{Im(\partial_2)} = \frac{\langle a + b, c \rangle}{\langle a + b - c, a + b + c \rangle} = \frac{\langle a + b, c \rangle}{\langle a + b - c, 2c \rangle} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z} \oplus 2\mathbb{Z}} \cong \mathbb{Z}_2$$

$$H_2 = \frac{Ker(\partial_2)}{Im(\partial_3)} = \frac{0}{0} \cong 0$$

$H_i = 0$ for $i \geq 3$.

2.3 Singular Homology

After above discussion and examples about simplicial homology, we observe that the simplicial homology can get difficult to compute when topological space gets complicated. It is natural to introduce the "singular homology" as a generalization of simplicial homology.

The singular n-simplex is the image of the standard n simplex under a (not necessarily bijective) continuous map.

Definition 2.10 (Singular n-simplex)**.** Given a topological space X, a singular n- simplex in X is a continuous map $\sigma : \Delta^n \to X$.

For singular homology, we define the n-chains to be linear combinations of singular n-simplexes. This space is typically infinite dimensional, but typically there are only finitely many distinct singular n-simplexes up to homotopy. The chain complex is defined as before, as are the homology groups. Singular homology is more flexible than simplicial homology: it allows for calculation of homology groups in a wider range of contexts. This is somewhat analogous to the difference between Riemann integration and Lebesgue integration: where both are defined they always agree, but Lebesgue (at the cost of a slightly more difficult definition) allows for computations in some additional cases.

2.4 $n = 1$ Case of Hurewicz Theorem

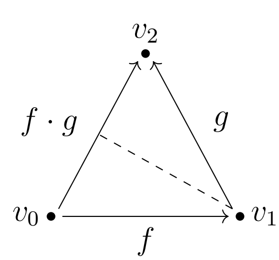
In this section we prove a special case of a famous theorem of Hurewicz, which relates the first homology group to the fundamental group of a topological space. Hatcher [5] gives a topological verification of $Ab(\pi_1(X, x_0)) \cong H_1(X)$ using singular homology in §2.A, we give full details now.

Theorem 2.1. For a path connected space X, $Ab(\pi_1(X, x_0)) \cong H_1(X)$.

Proof. By regarding loops in $\pi_1(X, x_0)$ as 1-dimensional singular cycle, we obtain a homomorphism $h : \pi_1(X, x_0) \to H_1(X)$ where h is surjective and the commutator subgroup of $\pi_1(X, x_0)$ is the kernel of h. Then this implies there is an isomorphism between $Ab(\pi_1(X, x_0))$ and $H_1(X)$. To verify this claim, we first show the following properties holds:

(i) If $f \in \pi_1(X, x_0)$ is a constant path, then $f \sim 0$ in $H_1(X)$.
f is viewed as a trivial loop (a constant path) here. Thus it is also a cycle. In $H(X)$, $h(f) = 0$. Explicitly, f is the boundary of the constant 2-simplex $\sigma = (v_0, v_1, v_2)$ where $v_0 = v_1 = v_2$ and $\partial(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1) = f - f + f = f$ since each of the (v_i, v_j) is a constant path again.

(ii) $f \cdot g \sim f + g$. Let $\sigma = (v_0, v_1, v_2)$ be a 2-simplex, if f and g are two paths given as edge (v_0, v_1) and (v_1, v_2) respectively, we denote the concatenation of the two paths f and g as $f \cdot g$. $f \cdot g$ is also the composition of projections of f and g onto the edge (v_0, v_2). Taking boundary on σ, $\partial(\sigma) = g - f \cdot g + f$. Since the difference between $f + g$ and $f \cdot g$ is a boundary, we have $f \cdot g \sim f + g$.



(*iii*) If $f \simeq g$ in $\pi_1(X, x_0)$, then $f \sim g$ in $H_1(X)$.
Consider a homotopy $F : I \times I \to X$ from f to g. Split the $I \times I$ into two 2-simplces σ_1 and σ_2. Taking boundary of $\sigma_1 - \sigma_2$, the diagonal cancels, (v_0, v_2) and (v_1, v_3) are constant path thus homologous to 0 by (i). We have $\partial(\sigma_1 - \sigma_2) = f + b - g - a$. Since F is a homotopy defined on $I \times I$, a, b are constant paths. By (i), a, b both are boundaries of constant 2-simplices. Thus the difference of $f - g$ must be a boundary, and therefore $f \sim g$.

$$\begin{tikzcd}
v_2 \arrow[r, "g"] & v_3 \\
v_0 \arrow[u, "a"] \arrow[r, "f"'] \arrow[ur, "\sigma_2", "\sigma_1"'] & v_1 \arrow[u, "b"']
\end{tikzcd}$$

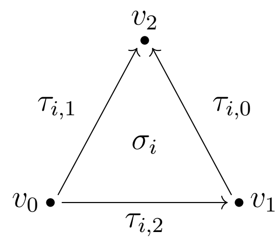
(*iv*) $\overline{f} \sim -f$. $\overline{f}$ denotes the inverse path of f. This is a result from and (ii) and (i): $f + \overline{f} \sim f \cdot \overline{f} \sim 0$, thus $\overline{f} \sim -f$.

(iii) shows h is a well-defined map. (ii) shows that if given $f \cdot g$ in $\pi_1(X, x_0)$, $h(f \cdot g) = h(f + g) = h(f) + h(g)$. Thus h is a homomorphism.

To see h is a surjection, consider a 1-cycle $\sum_i n_i \sigma_i$ representing a given element of $H_1(X)$. We can relabel the σ_i's so that the coefficient is all 1: write $n\sigma_i$ as $\sigma_i + ... + \sigma_i$ and then relabel as $\sigma_i + ... + \sigma_{i+n}$ for multiple of the same simplex, for any $n_i = -1$, write $1\sigma_i'$ to replace $-1\sigma_i$ where σ_i' is the same simplex with opposite orientation. Thus the cycle can be written as $\sum_i \sigma_i$. Now suppose there is some σ_i is not a loop, then the fact that $\partial(\sum_i \sigma_i) = 0$ will imply there must be some σ_j such that $\sigma_1 \cdot \sigma_j$ is defined. Use (iii) and denote $\sigma_k = \sigma_i \cdot \sigma_j$ and use σ_k to replace $\sigma_i \cdot \sigma_j$ in the summation. repeat this process and update the summation, then we will end up with the summation $\sum_i \sigma_i$ being a sum of loops where each σ_i is a loop. Given X is path connected, for each loop σ_i, we connect x_0 to the basepoint of σ_i and denote this path as γ, then $\gamma \cdot \sigma_i \cdot \overline{\gamma}$ is the element in $\pi_1(X, x_0)$ that maps to the element in $H_1(X)$ represented by $\sum_i \sigma_i$. Explicitly, $h(\gamma \cdot \sigma_i \cdot \overline{\gamma}) \sim h(\sigma_i)$ where $h(\sigma_i)$ denotes the image 1-cycle in $H_1(X)$.

To see the commutator subgroup of $\pi_1(X, x_0)$ is contained in $ker(h)$, notice that $H_1(X)$ is an abelian group under addition. Thus any $fgf^{-1}g^{-1}$ becomes 1 in $H_1(X)$ thus $[\pi_1(X, x_0), \pi_1(X, x_0)] \subseteq ker(h)$.

To obtain the opposite direction, we show that every element in $ker(h)$ is a commutator in $\pi_1(X, x_0)$. Take $f \in Ker(h)$, then in $H_1(X)$, f is a 1-cycle which is a boundary of some 2-chain $\sum_i n_i \sigma_i$. Similarly we can rewrite $\sum_i n_i \sigma_i$ such that $n_i = \pm 1$ in the way we did previously. Each σ_i correspond to a 2-simplex. If we apply boundary, $\partial(\sigma_i) = \tau_{i,0} - \tau_{i,1} + \tau_{i,2}$ as shown in the following figure:



Then $f = \partial(\sum_i n_i\sigma_1) = \sum_i n_i\partial(\sigma_i) = \sum_{i,j}(-1)^j n_i\tau_{i,j}$. Since f is a singular 1-cycle in $H_1(X)$, $\sum_{i,j}(-1)^j n_i\tau_{i,j}$ must have everything inside of the cycle canceled: by choosing proper orientation some edges $\tau_{i,j}$ will appear twice with opposite sign thus will be canceled. The remaining edges $\sum_{i,j}(-1)^j n_i\tau_{i,j}$ form f. Since f is a cycle thus is null homotopic, we can now consider a homotopy σ that deforms every vertex in this 2-chain $\sum_i n_i\sigma_i$ to the point x_0. Abusing the notation, we denote the newly deformed singular 2-chain again as $\sum_i n_i\sigma_i$, this chain has every edge being a loop based at x_0. Since in $\sum_{i,j}(-1)^j n_i\tau_{i,j}$ we had all edges in the interior canceled, the homology class $[\sum_{i,j}(-1)^j n_i\tau_{i,j}]$ can be written as $\sum_{i,j}(-1)^j n_i[\tau_{i,j}]$. Moreover, $\sum_{i,j}(-1)^j n_i[\tau_{i,j}] = \sum_i n_i[\partial(\sigma_i)]$. Since now in the new chain $\sum_i n_i\sigma_i$ every edge is a loop at x_0, in $[f] = \sum_i n_i[\partial(\sigma_i)]$, all $[\partial(\sigma_i)] = [\tau_{i,0}] - [\tau_{i,1}] + [\tau_{i,2}] = 0$, this implies $[\tau_{i,2}] = -([\tau_{i,0} - [\tau_{i,1}])$. In $\pi_1(X, x_0)$, this is expressed as $[\tau_{i,2}] = ([\tau_{i,0}] \cdot [\tau_{i,1}]^{-1})^{-1}$. Thus $[\partial(\sigma_i)]$ is in the form of an element followed by its inverse, $[\partial(\sigma_i)]$ is in the commutator subgroup for all i. As a result f is in the commutator subgroup. $\square$

The higher homology groups are **not** the abelianisations of the higher homotopy groups, though the Hurewicz theorem generalises to show that this relation holds in the smallest dimension where either group is non-zero.

2.5 Eilenberg-Maclane Space

Recall that topological spaces X and Y have the same homotopy type if and only if $\pi_i(X, x) = \pi_i(Y, y)$ for some fixed points x and y. This does not imply that X and Y are homeomorphic, since for example any contractible space has all homotopy groups trivial. (A homeomorphism requires a bijection between spaces.) But spaces with the same homotopy type have the same singular and simplicial homology and cohomology groups.

Eilenberg-Maclane spaces have a unique non-trivial homotopy group. For a group G and positive integer n, an Eilenberg-Maclane space of type $K(G, n)$ has $\pi_n(X)$ isomorphic to G, and all other homotopy groups trivial. Note that the homology groups of an Eilenberg-Maclane space may be non-trivial in infinitely many dimensions.

In applications, the most useful Eilenberg-Maclane spaces are of type $K(G, 1)$, that is, spaces with only $\pi_1(X)$ being nontrivial and $\pi_n(X) = 0$ for all $n > 1$. Such spaces are also called *aspherical spaces* and their properties are completely determined by the fundamental group. For example, the unit circle $\mathbb{S}^1$ is an Eilenberg Maclane space of type $K(\mathbb{Z}, 1)$. More generally, Eilenberg-Maclane spaces exist for any finitely presented group.

According to Hatcher [5] §1.B, to construct a $K(G,1)$ space for arbitrary given group G, the following steps suffice:

(*i*) Construct a simplicial complex EG whose n-simplices are $(n+1)$ tuples: $[g_0, ..., g_n]$ where $g_i \in G$ for $1 \le i \le n$. The boundary of an n-simplex is given by the formula

$$\delta([g_0, ..., g_n]) = \prod_{i=1}^{n} (-1)^i [g_0, ..., \hat{g}_i, \ldots, g_n] \,.$$

It can be verified that δ is a boundary map. (Though we do not require it, the homology of G can be computed from this description.)

(*ii*) Consider homotopy $h(t) : EG \to EG$ with $t \in [0,1]$ where h slides every point $x \in [g_0, ..., g_n]$ along the line in $[e, g_0, ..., g_n]$ from x to vertex e where e is the identity in G. Thus h is a homotopy between EG and the point e. So EG has the homotopy type of a point, and all homotopy groups are trivial.

(*iii*) There is action of G on EG: $[g_0, ..., g_n] \mapsto [gg_0, ..., gg_n]$, which is proper and discontinuous. Consider the natural quotient map $EG \to EG/G$, which sends each point to its G-orbit. This gives a covering of $BG = EG/G$ by EG. By the results of Section 1.6, $\pi_1(BG) = G$. It can be shown that all higher homotopy groups of BG remain trivial on passing to the quotient, and so BG is a $K(G,1)$ space.

With the Eilenberg Maclane space of $K(G,1)$, at least the first homotopy group of any random G can be defined as the first homotopy group of its corresponded space BG. It is a well-known fact that homology is determined by homotopy type, so that we can refer to the homology of a $K(G,1)$ space as the homology of G without ambiguity.

2.6 Transfer Map in Simplicial Homology

Let X' be a covering space of X. In Chapter 1 we proved that there is a natural injective homomorphism $\pi_1(X', x) \to \pi_1(X, x)$. At the end of that Chapter, we observed that there is a mysterious "pre-transfer map" (which is not a homomorphism!) in the direction $\pi_1(X, x) \to \pi_1(X', x)$. We observed that the obstruction to being a homomorphism is essentially just non-commutativity of $\pi_1(X', x)$.

By Theorem 2.1, the first homology group is the abelianisation of the fundamental group. In this section we show that the pre-transfer on fundamental groups descends to a true homomorphism between homology groups. Note that by the Eilenberg-Maclane construction, every finite group has an associated space $K(G,1)$. By general theorems about covering spaces, there exists a cover with $\pi_1(X', x) \cong H$ for any subgroup H of G. As a result, we will be able to define a homomorphism

$$v : \mathrm{Ab}(G) \to \mathrm{Ab}(H)$$

for any subgroup H of a finite group G. We will state the remaining results in this section in sufficient generality to apply both to covers of topological spaces and to subgroups of finite groups.

We begin with an insight into the group of deck transformations of a covering space.

Proposition 2.2. Given topological space X and a finite covering $f : X' \to X$, there exists a canonical bijection between cosets of $\pi_1(X', x')$ in $\pi_1(X, x)$ and conjugates of x' over x.

Proof. Let s be a deck-transformation. Since $s(x)$ is conjugate to x, the image of a path from x to $s(x)$ in X' is an element of $\pi_1(X)$. Conversely, any two paths between x and $s(x)$ differ by an element of $\pi_1(X', x')$ and so the deck transformations sending x to $s(x)$ all correspond to elements of this coset of $\pi_1(X', x')$ in $\pi_1(X, x)$. $\square$

Now we introduce some notation for our treatment of the transfer homomorphism (unifying notation for finite groups and covering spaces).

(i) Let G be the group of deck transformations, which is isomorphic to $\pi_1(X)$.

(ii) Let H be the group $\pi_1(X')$ which is isomorphic to a subgroup of $\pi_1(X)$.

(iii) Let T be a set of representatives from cosets of H in G, which is a transversal of the group of deck transformations by Proposition 1.8.

Now we can write the transfer map as follows.

1. Given an element of $\pi_1(X, x)$, there exists a lift to X' which is uniquely determined by the starting point (which must be a conjugate of x'. Consider a lift which begins at a point y. Let t_y be the unique element in T which maps x' to y, and write $t_{y \cdot s}$ for the element which maps from x' to the image of s. Then $t_y s t_{y \cdot s}^{-1}$ is a path in X' beginning and ending at x'. So it is an element of $\pi_1(X', x')$. The transfer is the image of the composition of all such lifts in $\mathrm{Ab}(\pi_1(X', x'))$.

2. Given $g \in G$, there exist unique elements of the transversal T of H such that $g = t_i h$ and $g t_j^{-1} \in H$. The transfer in $\mathrm{Ab}(H)$ is given by

$$v(g) = \prod_{t \in T} tg(t \cdot g)^{-1} \mod H'$$

   Again, each term in the product is an element of H, but there is no natural ordering on the terms, so that the homomorphism is defined only in the abelianisation of H.

Note that a homomorphism $v : G \to Ab(H)$ necessarily has $G' \leq \ker(V)$. By the First Isomorphism Theorem, we refer without ambiguity to v as a homomorphism out of G or out of $Ab(G)$. Note that this result no longer requires any topological notions, the proof works at the level of group theory. It remains only to show that v really is a homomorphism. We leave the proof to Chapter 3 (See Theorem 3.1). In the next chapter, we will also investigate applications of the transfer in the theory of finite groups.

Chapter 3

Transfer Theory in Finite Groups

In Chapter 1, we introduced topological spaces, fundamental groups and covering spaces. We saw in Theorem 1.1 that there is a bijection between subgroups of the fundamental group of a manifold and covers of that manifold. We concluded the chapter with by constructing a natural evaluation map from the fundamental group of a cover into the fundamental group of a space, and an *unnatural* pre-transfer map in the other direction. This map is **not** a homomorphism of fundamental groups.

In Chapter 2, we constructed simplicial complexes and their homology groups. Homology is a coarser invariant than homotopy, and is easier to compute than the homotopy groups (of which the fundamental group is the first). Nevertheless, homology is an invariant of manifolds. Theorem 2.1 shows that the first homology group is the abelianisation of the fundamental group. It turns out that the image of the pre-transfer map in the first homology group is a true homomorphism of groups. More generally there exist transfer homomorphisms between the homology groups of a cover and the homology of the underlying space.

In this chapter, we extend these geometric concepts to finite groups, as follows. An important theorem in algebra, due to MacLane and Eilenberg is that for any group G there exists a topological space which has G as its fundamental group, with all higher homotopy groups vanishing. This is called the Eilenberg-MacLane space for G. We will not be concerned with the construction of these spaces (which typically are not sufficiently geometric to be visualisable). But algebraic techniques establish that these spaces are unique up to homotopy.

It is a deep and surprising fact that, while the homotopy theory of the Eilenberg-MacLane space is not interesting, since the first homotopy group is as intended and all higher homotopy groups vanish, the homology groups turn out to be interesting and to have deep connections to finite groups seemingly unconnected to geometric or topological considerations. Since the Eilenberg-MacLane space is unique, we define the homology of a group to be the homology of the Eilenberg-MacLane space. Transfer maps can be defined at this level.

Historically, the transfer homomorphism pre-dates the development of homology, and can be defined without reference to homology. Nevertheless, understanding the homological motivation for the transfer provides insight into the types of applications it has.

3.1 Definition and Properties of Transfer Map

In this section, we provide a group theoretic verification of the basic properties of the transfer homomorphism. Throughout this section, G is a group, H is a subgroup of G of finite index, and T is a transversal of the cosets of H in G. The following definition captures the action of G on the cosets of H, which are in bijection with the elements of T. In this chapter, theorems and proofs about the transfer homomorphism and related results follow from Isaacs [7].

Definition 3.1 (Transversal)**.** Let $H \subseteq G$, and let T be a transversal of the cosets of H. For $t \in T$ and $g \in G$, define $t \cdot g$ to be the element in T that labels the coset Htg.

Note that the action of G on T is identical to the action of G on the cosets of H as a permutation group.

Before we proceed further it's helpful to review group action and give a concrete example. In most textbook of group theory, group action is defined in following language:

Definition 3.2 (Group action)**.** Let G be a group and S be a set, a left *group action* $\phi : G \times S \to S$ is a map if the following properites are satisfied:

(i) $\forall x \in S$, $1 \cdot x = x$

(ii) $\forall x \in S$, $\forall g, h \in G$, $g \cdot (h \cdot x) = (gh) \cdot x$

Right group action can be defined in similar way.

This is equivalent to the say the group action is a homomorphism from G to $Sym(S)$. That is, the group action is a homomorphism $\phi : G \to Sym(S)$ such that for a fixed $g \in G$: $g \mapsto \sigma_g$ where $\sigma_g \in Sym(S)$. And then for every $s \in S$, $\sigma_g : s \mapsto g \cdot s$.

To see the this is a homomorphsim, take $s \in S$,

$$\begin{aligned} \phi(gh)(s) &= \sigma_{gh}(s) \\ &= (gh) \cdot s \\ &= g \cdot (h \cdot s) \\ &= \sigma_g(\sigma_h s) \\ &= \phi(g)\phi(h)(s) \end{aligned}$$

Example 3.1. Consider D_4, the symmetry group on a square. Take $G = D_4$ and $S = \{1, 2, 3, 4\}$. We can view D_4 as a group acting the set of vertices of a square. r gives permutation (1234): rotate every vertex by 90 degree; f gives permutation (24): fix vertices $1, 3$ and flip $2, 4$.

Lemma 3.1. Suppose $H \subseteq G$ and T is a right transversal. For $t \in T$ and $x, y \in G$, following properties hold:

(i) $t \cdot 1 = t$.

(ii) $(t \cdot x) \cdot y = t \cdot (xy)$.

(iii) $tx(t \cdot x)^{-1} \in H$

Proof. (i) $t \cdot 1$ returns the element in T belongs to coset $Ht1 = Ht$ which is t.

(ii) "$\cdot$" can be thought as group action of G on the cosets of H in G: $Ht_i \cdot g = Ht_i g$. Then t_i's are the labels of the cosets. (ii) follows as an axiom of group action.

(iii) Let $t \in T$ and $x \in G$, then $tx \in Htx = Ht \cdot x$, this implies there's some $h \in H$ such that $tx = h(t \cdot x)$, then $tx(t \cdot x)^{-1} = h \in H$.

□

Definition 3.3 (Transfer Map)**.** Let $H \subseteq G$ be a subgroup with finite index, suppose $M \triangleleft H$ and H/M be abelian, The transfer from G to H/M is defined as a map $v : G \to H/M$ giving by the following:

$$V(g) = \prod_{t \in T} tg(t \cdot g)^{-1}$$

$$v(g) = V(g) \mod M$$

where T is the transversal of G.

With this definition, it is not entirely obvious that the transfer is in fact a homomorphism. Before we verify this, we give an example of the computation of the transfer.

Example 3.2 ($v : D_4 \to \langle r \rangle$)**.** Consider the transfer map from D_4 to its subgroup $H = \langle r \rangle$. Since H is abelian, take $M = H'$ and $H/H' = H$. There's two cosets by Lagrange's theorem: $\{1, r, r^2, r^3\}$ and $\{f, r^2 f, fr^3, fr\}$. Choose $T = \{1, f\}$ as a transversal. Then the transfer map sends the two generators $\{r, f\}$ of D_4 to the their image as below:

$$v(r) = \prod_{t \in T} tr(t \cdot r)^{-1} = [1r(1 \cdot r)^{-1}][fr(f \cdot r)^{-1}] = (r1^{-1})(frf^{-1}) = 1$$
$$v(f) = \prod_{t \in T} tf(t \cdot f)^{-1} = [f(1 \cdot f)^{-1}][f^2(f \cdot f)^{-1}] = (ff^{-1})(f^2 1^{-1}) = 1$$

We skip the remaining elements in D_4 and it can be verified that the transfer map $v : D_4 \to H$ is trivial for all $g \in D_4$. In general this is not true for any group G and its subgroup H. We give another example now, where the transfer is non-trivial.

Now consider the transfer from D_4 to another subgroup $K = \langle f \rangle$. Choose $S = \{1, r\}$ as a transversal. Then,

$$v(r) = \prod_{t \in S} tr(t \cdot r)^{-1} = [1r(1 \cdot r)^{-1}][rr(r \cdot r)^{-1}] = (rr^{-1})(r^2 1) = r^2$$
$$v(f) = \prod_{t \in S} tf(t \cdot f)^{-1} = [1f(1 \cdot f)^{-1}][rf(r \cdot f)^{-1}] = (f1^{-1})(fr^3 r^{-1}) = r^2$$

Next, we establish that the transfer is a homomorphism of groups. The next proof is identical to that at the end of Chapter 2 for simplicial homology, but we include it again so that this chapter on finite groups is self-contained. Then for groups like D_4 it suffice to know what the image of the whole group is with $v(r)$ and $v(f)$ since these are the generates of D_4.

Theorem 3.1 (Transfer is a Homomorphism)**.** Let G be a group, and $H \leq G$ with finite index with transversal T. Let $M \leq H$ be such that H/M is abelian. Then the map $v : G \to H/M$ given by

$$V(g) = \prod_{t \in T} tg(t \cdot g)^{-1}$$
$$v(g) = V(g) \mod M$$

is a homomorphism from G into H/M.

Proof. Let T be a transversal of H, for $t \in T$ and $x, y \in G$, we have $t \cdot (xy) = (t \cdot x) \cdot y$ by Lemma 3.1. Thus we have:

$$t(xy)(t \cdot (xy))^{-1} = (tx(t \cdot x)^{-1})((t \cdot x)y((t \cdot x) \cdot y)^{-1})$$

Both factors on right hand side above are in H by lemma 3.1, thus take transfer on both sides we get:

$$\begin{aligned} \prod_{t \in T} t(xy)(t \cdot (xy))^{-1} &\equiv \prod_{t \in T} tx(t \cdot x)^{-1} \prod_{t \in T} (t \cdot x)y((t \cdot x)y)^{-1} \\ &\equiv \prod_{t \in T} tx(t \cdot x)^{-1} \prod_{t' \in T} t'y(t' \cdot y))^{-1} \\ &\equiv \prod_{t \in T} tx(t \cdot x)^{-1} \prod_{t \in T} ty(t \cdot y))^{-1} \mod M \end{aligned}$$

We have the first congruence since we are modulo M thus terms commute. We have the second congruence since $t \cdot x$ run through all elements of T and t' is relabeling $t \cdot x$. Thus $v(xy) = v(x)v(y)$ and transfer is a homomorphism. □

The transfer map does not depend on choice of left or right transversal, and it does not depend on the choice of transversal since the action of G is on the cosets and t as a label does not affect the image. Next we provide an explicit proof to show the invariance of transversal.

Theorem 3.2 (Invariance of Transversal)**.** Let S and T be two different transversals of $H \subseteq G$, let $M \triangleleft H$ with H/M abelian and H of finite index. Then for $g \in G$:

$$\prod_{t \in T} tg(t \cdot g)^{-1} \equiv \prod_{s \in S} sg(s \cdot g)^{-1} \mod M$$

Proof. Since S and T are two transversals of $H \subseteq G$, for each $t \in T$, there is a unique $s_t \in S$ that lies in the same coset Ht that contains t, we can write $s_t = h_t t$ for some $h_t \in H$. Thus we have:

$$S = \{s_t : t \in T\} = \{h_t t : \ t \in T\}$$

Let $s \in S$ and $g \in G$, then $s = s_t$ for some t and $s \cdot g$ returns the unique element of S lies in $Hsg = Htg = H(t \cdot g)$, thus $s \cdot g = s_{t \cdot g} = h_{t \cdot g}(t \cdot g)$ and we have:

$$sg(s \cdot g)^{-1} = h_t tg(h_{t \cdot g}(t \cdot g))^{-1} = h_t(tg(t \cdot g)^{-1}))h_{t \cdot g}^{-1}$$

$$\begin{aligned}
\prod_{s \in S} sg(s \cdot g)^{-1} &\equiv \prod_{t \in T} h_t(tg(t \cdot g)^{-1})h_{t \cdot g}^{-1} \\
&\equiv \prod_{t \in T} h_t \prod_{t \in T} tg(t \cdot g)^{-1}(\prod_{t \in T} h_{t \cdot g})^{-1} \\
&\equiv \prod_{t \in T} tg(t \cdot g)^{-1} \mod M
\end{aligned}$$

We have the second congruence since it's modulo M and H/M is abelian. The third congruence is obtianed by cancelling the first and last products since $t \cdot g$ runs through all elements of T. Thus we conclude the transfer is independent from choice of transversal. □

As we have illustrated in Chapters 1 and 2, these results can be interpreted in terms of covering space theory and in terms of homology, though here they are formulated entirely in terms of groups. In the next section, we recall some of the theory of finite groups, and explore a few applications of the transfer in finite group theory.

3.2 Sylow Theorems and Topics in Elementary Group Theory

The Sylow theorems provide powerful structural results about subgroups of prime power order in a finite group. Since groups of prime power order have numerous special properties (e.g. non-trivial centers, many normal subgroups) it will be productive to consider the transfer from G into a subgroup of prime power order. Historically, these methods were pioneered by Frobenius and Burnside in the study of finite simple groups.

Lemma 3.2. Let p be a prime and m be an integer not divisible by p, then then number of ways of selecting p^k elements from a set of size $p^k m$ is:

$$\binom{p^k m}{p^k} \equiv m \mod p$$

Proof. We skip the proof here. Details can be found in Lemma 7.5 from Shahriari [11]. □

Names and proof techniques of the Sylow Theorems varies in different textbooks and the Sylow Theory is rich in its materials and applications. More discussions can be found in Chapter 5 of Hall [4] and Chapter 11 of Humphreys [6]. Here we mainly follow the above two reference with minimum prerequisite properties to show the three main theorems.

Theorem 3.3 (Sylow's First Theorem)**.** Let G be a finite group with order $p^k m$ where p^k is a prime power and $p \nmid m$, then G has subgroup of order p^k.

Proof. Let $|G| = p^k m$ and $p \nmid m$. Let $\mathcal{S} = \{S \subseteq G : |S| = p^k\}$, that is, the set of all subsets of G that contains exactly p^k elements. $|\mathcal{S}|$ is the number of ways to choose p^k elements from a set containing $p^k m$ elements. By Lemma 3.2, $|\mathcal{S}| \equiv m \mod p$. Now let G act on $\mathcal{S}$ with action "$\cdot$" : $g \cdot s = gS$ where gS is a left coset of S in G. This defines a group action and produces orbits that partition $\mathcal{S}$. Choose representatives from these orbits and denote as $\{S_1, ...S_r\}$. Then $\mathcal{S} = Orb(S_1) \cup ... \cup Orb(S_r)$ implies $|\mathcal{S}| = |Orb(S_1)| + ... + |Orb(S_r)|$. If every orbit have size divisible by p then p divides $|\mathcal{S}|$. This contradicts $|\mathcal{S}| \equiv m \mod p$. Thus at least one orbit has size not divisible by p. Denote this orbit by S. By the Orbit-Stabilizer Theorem, $p^k \mid |G|$ and $p \nmid |Orb(S)|$ implies $p^k \mid |Stab(S)|$. Thus $p^k \leq |Stab(S)|$. For fixed $s \in S$, if $g \in Stab(S)$ then $g \cdot S = S$ and hence $gs \in S$. Thus the coset $Stab(S)s \subseteq S$. It follows $|Stab(S)| = |Stab(S)s| \leq |S| = p^k$. We must have $|Stab(S)| = p^k$ and $Stab(S) \leq G$ is a subgroup of order p^k. $\square$

Theorem 3.4 (Sylow's Second Theorem)**.** If P is a Sylow p-subgroup of finite G with $|P| = p^k$ and $Q \leq G$ is a p-group($|Q| = p^i$, $i \leq k$). Then for some $g \in G$,

$$Q \subseteq gPg^{-1}$$

Proof. Consider $\mathcal{S} = \{Pg : g \in G\}$, that is, the set of all cosets of P in G. Then $|\mathcal{S}| = |G : P|$ is not divisible by p since P is a Sylow-p subgroup of G. G acts on $\mathcal{S}$ by right multiplication thus P acts on $\mathcal{S}$ too. Then $\mathcal{S}$ can be partitioned into P-orbits. Choose representatives from these orbits and denote as $\{S_1, ..., S_r\}$. Then $\mathcal{S} = Orb(S_1) \cup ... \cup Orb(S_r)$ implies $|\mathcal{S}| = |Orb(S_1)| + ... + |Orb(S_r)|$. If all $|Orb(S_i)|$ for $1 \leq i \leq n$ divides p then $|\mathcal{S}|$ divides p which is a contradiction. Thus there exist at least one orbit S_i with $|S_i|$ not divisible by p. By the Orbit Stabilizer Theorem, $|Orb(S_i)| = |P|/|Stab(S_i)|$ implies $|Orb(S_i)|$ divides $|P| = p^k$. The only possibility is $|Orb(S_i)| = 1$. Thus S_i is a set of single coset Pg for some $g \in G$ and all elements in P fixes Pg, that is, $\forall p \in P$, $Pgp = Pg$. It follows $gp \in Pg$ and $p \in g^{-1}Pg$ for all $p \in P$. Thus $P \subseteq g^{-1}Pg$. $\square$

Corollary 3.1. If P, Q are Sylow p-subgroups of G, then $Q = g^{-1}Pg$ for some $g \in G$.

Proof. By Theorem 3.4, let $P = Q$ then we have $Q \subseteq g^{-1}Pg$ for some $g \in G$. Since P, Q both are Sylow p-subgroups, $|Q| = |P| = |g^{-1}Pg|$ thus must have $P = Q$. $\square$

Theorem 3.5 (Sylow's Third Theorem)**.** Given G of order $p^k m$, where $p \nmid m$, the number n of Sylow p-subgroups is congruent to 1 modulo p.

Proof. Consider the same set $\mathcal{S}$ from proof of Theorem 3.3 with the same group action. As we saw in there, if an orbit in $\mathcal{S}$ containing S have length not divisible by p, then for all $s \in S$, $Stab(S)s \subseteq S$ and $|Stab(S)s| = |S|$. Thus $Stab(S)s = S$, it follows $Stab(S) = Ss^{-1}$ is a Sylow p-subgroup of G. Conjugate this group by s, $s^{-1}(Ss^{-1})s = s^{-1}S$ is again a Sylow p-subgroup. Notice $s^{-1}S \in Orb(S)$, since $|Stab(S)| = |S| = p^k$, by the Orbit Stabilizer Theorem, $|Orb(S)| = m$. Thus we conclude if an orbit has length not divisible by p then it contains a Sylow p-subgroup and has length m.

On the other way around, if an orbit in $\mathcal{S}$ contains a Sylow p-subgroup P, then if $g \in Stab(P)$, there is $gP = P$ and this implies $g \in P$. Thus $Stab(P) \subseteq P$ and hence $Stab(P) = P$. $Orb(P)$ has length m by the Orbit Stabilizer Theorem. Therefore every orbit that contains a Sylow p-subgroup have length m which is not divisible by p.

Now we have every orbit that contains a Sylow p-subgroup has length m, and every orbit that is not divisible by p contains a Sylow p-subgroup and has length m. Next we show every orbit contains at most one Sylow p-subgroup. Suppose P_1, P_2 are Sylow p-subgroups in the same orbit, then $P_1 = gP_2$ for some $g \in G$. This implies $1 \in P_2 \cap gP_2$, that is, the two cosets have non-empty intersection. This is impossible, thus $P_1 = P_2$.

Let n_p denote the number of Sylow p-subgroups in G, then $|\mathcal{S}| \equiv mn_p \mod p$. Apply Lemma 3.2, $|\mathcal{S}| \equiv mn_p \equiv m \mod p$, thus $n_p \equiv 1 \mod p$. □

Application of the Sylow Theorem is useful in classifying simple groups as the theorem tells information about Sylow subgroups. In fact, many application is about using the Sylow Theorem to show existence of proper nontrivial normal subgroup and show the group is not simple, or a group of given order is solvable or not. We will look at some quick results about simple groups.

Corollary 3.2. If a group G with finite order has only one proper nontrivial subgroup of a given order, then this subgroup is normal and G is not simple.

Proof. Observe that $g^{-1}Xg$ is a subgroup of G if and only if X is a subgroup of G. □

Corollary 3.3. n_p divides $|G|/p^k$.

Proof. Take any Sylow p-subgroup P, since every two Sylow p-subgroups are conjugate, by the Orbit Stabilizer Theorem number of conjugates of P equals to the index of $N_G(P)$. Thus $n_p = |G : N_G(P)| = |G|/|N_G(P)|$. Since $P \subseteq N_G(P)$, $N_G(P)$ has order contains p^k as a factor, $|G|/|N_G(P)|$ has no factor of p thus n_p divides $|G|/p^k$. □

Proposition 3.1. A group of order 28 has a normal subgroup of order 7.

Proof. If $|G| = 28$, then the number of Sylow 7-subgroups divides 4 and congruent to 1 modulo 7, thus the number must be 1, this Sylow 7-subgroup is a proper non-trivial normal subgroup and therefore G is not simple. □

Proposition 3.2. If p, q are primes with $q < p$, then any group of order pq has a single subgroup of order p and this subgroup is normal in G.

Proof. The number of Sylow p-subgroups is $n_p \equiv 1 \mod p$, and divides the order of G. The only such integer is 1, by hypothesis. $\square$

More generally, unless $q \equiv 1 \mod p$ the same conclusion holds for q and every group of order pq is cyclic. In the case that $q \equiv 1 \mod p$ there is an additional group, which may be described in terms of the semi-direct product.

Proposition 3.3. Let $\mathbb{Z}_n$ be a cyclic group of order n, written additively. The automorphism group of $\mathbb{Z}_n$ is the multiplicative group $\mathbb{Z}_n^*$.

Proof. Since C_n is generated by 1, for any automorphism α, we have $\alpha(m) = m \cdot \alpha(1)$, so the automorphism is entirely known once the image of 1 is known. Clearly $\alpha(1)$ generates the whole group if and only if $\alpha(1)$ is coprime to n and the result follows. $\square$

Next, we review two ways of factoring a group G into two smaller subgroups, equivalently, we can construct larger group with small groups by taking direct product or semidirect product.

Definition 3.4 (Complement)**.** Given H as a subgroup of G, a subgroup K of G is called a *complement* of H in G if $G = HK$ and $H \cap K = 1$.

Proposition 3.4. Given G and $H, K \leq G$, if $H \cap K = 1$, the set $HK = \{hk : h \in H, k \in K\}$ can be uniquely expressed as product of form hk. Moreover, if both H, K are normal in G, $HK \cong H \times K$.

Proof. The first statement is obvious, to see the second statement we consider $\psi : HK \to H \times K$ defined as $hk \mapsto (h, k)$. Before showing this is a homomorphism, notice $H \lhd G$ implies $k^{-1}hk \in H$ thus $h^{-1}k^{-1}hk \in H$, similarly we have $h^{-1}k^{-1}hk \in K$. Since by hypothesis $H \cap K = 1$ we have $h^{-1}k^{-1}hk = 1$ which means $hk = kh$.

Now let $h_1, h_2 \in H$, $k_1, k_2 \in K$, then

$$\begin{aligned} \psi(h_1k_1h_2k_2) &= \psi(h_1h_2k_1k_2) \\ &= (h_1h_2, k_1k_2) \\ &= (h_1, k_1)(h_2, k_2) \\ &= \psi(h_1k_1)\psi(h_2k_2) \end{aligned}$$

Thus ψ is a homomorphism and it is a bijection since HK can be written uniquely with elements of the form hk. We have $HK \cong H \times K$. $\square$

Definition 3.5 (Direct and semidirect product)**.** Let G be a group and H, K are subgroup of G satisfying the following conditions:

(i) $H \lhd G$ and $K \lhd G$.

(ii) $H \cap K = 1$.

(iii) $G = HK$.

Then G is a *direct product* of H and K.

Now we look at a generalization of the direct product by relaxing the condition of both groups being normal but keep the other conditions:

(i) $H \triangleleft G$.

(ii) $H \cap K = 1$.

(iii) $G = HK$.

What we obtain here is called a *semidirect product*.

In the case of semidirect product, for given $H, K \leq G$, the semidirect product is not necessarily unique (counterexamples can be easily found), H and K are not enough give information of the group G. The extra information is given by action of K on H by conjugation: for $h \in H, k \in K$, $k \cdot h = khk^{-1}$. Now for two elements h_1k_1, h_2k_2 in the semidirect product, the group operation can be well defined:

$$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1k_1h_2k_1^{-1}k_1k_2 \\ &= (h_1k_1h_2k_1^{-1})(k_1k_2) \\ &= hk\end{aligned}$$

where $h = h_1k_1h_2k_1^{-1}$ and $k = k_1k_2$. Now k is a well defined element from K, h involves elements from both K and H, but with the group action above we have $h = h_1(k_1 \cdot h_2)$. Recall Definition 3.2, the action of K on H gives a homommorphism $\phi : K \to Aut(H)$. Thus we write $H \rtimes_\phi K$ to denote the semidirect product of H and K with homomomrphism ϕ from K to $Aut(H)$.

One application of the semidirect product is that it let us classify groups of certain order, for example group of order pq where p, q are distinct primes. Although this method can write group of certain order as factor of two subgroups, it does not imply any group G can be written as semidirect product, for example any group that is simple, or the quaternion group Q_8 where no proper subgroup has a complement.

Example 3.3. We classify the groups of order $2p$ for odd prime p. By the Sylow theorems, a group of order $2p$ has subgroups of orders 2 and p. By Lagrange's theorem these subgroups intersect only in the identity element, and by the counting portion of the Sylow theorems, the group of order p is normal. Hence a group of order $2p$ is a semidirect product with normal subgroup of order p and complement of order 2. Since a semi-direct product is determined by a homomorphism from the complement H into $Aut(N)$ and the automorphism group of C_p is cyclic of order $p-1$, there are just two choices. If the action is trivial, we obtain a cyclic group while if the action is by inversion we obtain a dihedral group.

Generalising the previous example, we can classify all groups of order pq where p and q are distinct primes.

Example 3.4. Suppose G is a group with order pq where $p < q$ are two primes. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. Then apply the Sylow Theorem, $G \cong Q \rtimes_\phi P$ for some $\phi : P \to Aut(Q)$. If $p \nmid |Aut(Q)|$ the only homomomrphism ϕ is the trivial map which implies G is a direct product of two groups of order p and q, thus G is a cyclic group of order pq. Otherwise, there is a nonabelian group of order pq where C_p maps onto the unique cyclic subgroup of order p in $Aut(Q)$. As an example, consider the following element of order 3 in $Aut(\mathbb{Z}_7)$:

$$\sigma = (1, 2, 4)(3, 6, 5)$$

which realises multiplication by 2. This is an automorphism of $\mathbb{Z}_7$, and the group generated by $x \mapsto x + 1$ and σ is a nonabelian group of order 21. As a permutation group, it is generated by the permutations

$$(1, 2, 3, 4, 5, 6, 7), \quad (1, 2, 4)(3, 6, 5)\,.$$

More generally, a nonabelian group of order pq can be constructed for any divisor p of $q - 1$ by taking an element of order p from the multiplicative group the field of order q, together with a generator for the additive group.

3.3 Computations With The Transfer

Often there is not much useful information we can tell from directly applying the transfer map on a group. There is more information can be given from properties of the transfer map. The Transfer Evaluation Lemma helps on evaluation of transfer map and tells more than just evaluation.

Lemma 3.3 (Transfer Evaluation)**.** Let G be a group and $H \leq G$ with finite index, let T be a transversal of H. For a fixed $g \in G$, there exist a subset $T_0 \subseteq T$ and positive integers n_t for $t \in T_0$ with the following properties:

(i) $\sum n_t = |G : H|$

(ii) $tg^{n_t}t^{-1} \in H$ for all $t \in T_0$

(iii) $V(g) \equiv \prod_{t \in T_0} tg^{n_t}t^{-1}$

(iv) If $o(g) < +\infty$, then n_t divides $o(g)$ for every $t \in T_0$.

Proof. *(i)* The cyclic group $\langle g \rangle \subseteq G$ acts on T by the dot "$\cdot$" action, T thus can be divided into $\langle g \rangle$-orbits. Let T_0 be a set of representatives of these orbits and let n_t denote size of the orbit containing t, thus the sum of the sizes of orbits is $\sum n_t = |T| = |G : H|$.

(ii) Take $t \in T$, then the permutation induced by g on the $\langle g \rangle$-orbit containing t is an n_t cycle, elements of the orbit are: $t, t \cdot g, t \cdot g^2, ..., t \cdot g^{n_t - 1}$, thus $t \cdot g^{n_t} = t$ and this implies $Ht = Htg^{n_t}$. Thus $tg^{n_t} \in Htg^{n_t} = Ht$ and hence $tg^{n_t}t^{-1} \in H$.

(*iii*) Since $\forall s \in T$, $sg(s \cdot g) \in H$ by Lemma 3.1, and the product of all such elements with s chosen in some order gives $V(g)$. Now let $s = t \cdot g^i$, then

$$\begin{aligned} sg(s \cdot g)^{-1} &= (t \cdot g^i)g((t \cdot g^i) \cdot g)^{-1} \\ &= (t \cdot g^i)g(t \cdot g^{i+1})^{-1} \\ &\in H \end{aligned}$$

Contribution to $V(g)$ from elements of T that lie in the orbit containing t is:

$$\prod_{i=0}^{n_t-1} (t \cdot g^i)g(t \cdot g^{i+1})^{-1} = tg^{n_t}t^{-1}$$

Then it follows:

$$V(g) = \prod_{t \in T_0} tg^{n_t}t^{-1} \mod M$$

(*iv*) Since $o(g) = k < +\infty$, then $|\langle g \rangle| = k$, the size of every $\langle g \rangle$-orbit on T divides k.

□

For a transfer homomorphism $G \to H$ the Evaluation Lemma is best understood in terms of the coset action of G on H. Often knowing the orbit structure of G on cosets of H is sufficient.

Example 3.5. Let p, q be primes with $p \mid q - 1$. Let G be a group of order pq. Let H be a Sylow p-subgroup of G with generator h and consider the transfer homomorphism $V : G \to H$. By the Sylow theorems, h acts on the cosets of H fixing exactly one point, and moving all others in orbits of length p. So by the Transfer evaluation lemma, all orbits except that of length 1 are trivial. So $V(h) = h$ for any $h \in H$ and the transfer is nontrivial.

We give a concrete example of the use of the Transfer Evaluation Lemma to classify groups of order 30.

Proposition 3.5. Suppose that G is a group of order 30. Then G is isomorphic to one of the following: C_{30}, D_{15}, $D_5 \times C_3$ or $D_3 \times C_5$.

Proof. By the Sylow theorems, G must have a subgroup H of order 2. We consider the transfer $V : G \to H$. Let h be the non-trivial element of H and apply Lemma 3.3.

$$V(h) = \prod_{i=1}^{r} t_i h^{n_i} t_i^{-1}$$

where each term $n_i \in \{1, 2\}$ and $t_i h^{n_i} t_i^{-1} \in H$. Since $\sum_{i=1}^{r} n_i = 15$ an odd number of the n_i must be equal to 1. Furthermore, $t_i h t_i^{-1} = h$ for any term which does not vanish. We conclude that $V(h) = h$ and so the transfer is not identically 0.

By the First Isomorphism Theorem, G has a normal subgroup of order 15. Since $3 \equiv 2 \mod 5$ a group of order 15 must be cyclic. So G is a semi-direct product of a cyclic group of order 15 and a complement of order 2.

Semi-direct products are classified by homomorphisms $H \to Aut(N)$. In this case, $Aut(C_{15}) = Aut(C_3) \times Aut(C_5) \cong C_2 \times C_4$. There are three subgroups of order 2 in $Aut(C_{15})$, inverting an element of order 3, an element of order 5 or both. These give all groups of order 30. □

Let us now give a more general application of the Transfer homomorphism.

Theorem 3.6. Let G be a group and suppose G has center $Z(G)$ of index n. Then the transfer map from G to Z is the map $g \mapsto g^n$. This gives a homomorphism from G into $Z(G)$.

Proof. Choose a transversal T for the cosets of $Z(G)$ in G. Since $Z(G)$ is abelian, there is no need to quotient by commutators. By the Transfer Evaluation Lemma,

$$v(g) = \pi(G) = \prod_{t \in T_0} t g^{n_t} t^{-1}$$

By the Transfer Evaluation Lemma again, for $g \in G$, $tg^{n_t}t^{-1} \in Z(G)$ for all $t \in T$. Since $tg^{n_t}t^{-1} \in Z(G)$, $tg^{n_t}t^{-1} = t^{-1}(tg^{n_t}t^{-1})^{-1}t = g^{n_t}$. Thus

$$v(g) = \pi(G) = \prod_{t \in T_0} g^{n_t} = g^{\sum n_t} = g^n$$

□

As an application of the above result, we recover a well-known result of Schur's. If G is a torsion group with center of finite index, then the commutator subgroup of G is finite. Let $V : G \to Z(G)$ be the transfer homomorphism.

Since $G' \leq \ker(V)$, and $Z(G)$ has finite index in G then $G' \cap Z(G)$ has finite index in G'. Since G' is generated by commutators of a transversal of $Z(G)$, it is finitely generated. And since $G' \cap Z(G)$ has finite index in G' it too is finitely generated. So $G' \cap Z(G)$ is a finitely generated torsion abelian group: so it is finite. But then G' has a finite subgroup of finite index, and so is finite.

Finally, observe that $g(xg^{-1}x^{-1})$ as xgx^{-1} varies over the conjugates of G. These elements are distinct, and are commutators. So every conjugacy class in a group satisfying these hypotheses is finite.

Remark 3.1. Arguments such as the above motivated Burnside to ask whether a finitely generated group in which all elements have order divisible by p is necessarily finite. This was a famous problem in group theory which continues to attract researchers. This lead to the construction of the so called *Tarski monsters*: infinite simple groups in which every element has prime order p (for some prime $p > 10^7 5$). Worse, every non-trivial subgroup is cyclic of order p, so they behave nothing like finite p-groups.

3.4 The Focal Subgroup and Transfer into Sylow Subgroups

The existence of Sylow subgroups is established by the Sylow theorems. One of the original motivations for the study of transfer in group theory was to study simple groups. The transfer can be used for this purpose.

Theorem 3.7. *Let G be a finite group and suppose G has abelian Sylow-p subgroup, then $p \nmid |Z(G) \cap G'|$.*

Proof. Let P be a Sylow p-subgroup of G, and T a transversal of P in G. Suppose that $z \in Z(G) \cap P$ and consider the transfer homomorphism $v : G \to P$.

By hypothesis, $Ptz = Pzt = Pt$. So $t \cdot z = t$ for all $t \in T$ and

$$v(z) = \prod_{t \in T} z = z^{|G:P|}.$$

Since P is Sylow, $|G : P|$ is coprime to p, so $v(z) \neq 1$. But $G' \subseteq \ker(v)$ so $z \notin G'$. In particular, no element of order p is contained in $Z(G) \cap G'$, so the triple intersection $P \cap G' \cap Z(G)$ is trivial. □

Remark 3.2. A group G is a *perfect central extension* of an abelian group Z by a group H if G satisfies the following conditions:

(*i*) $G = G'$, i.e. G is perfect.

(*ii*) G contains a central subgroup isomorphic to Z.

(*iii*) The quotient G/Z is isomorphic to H.

An important example of a perfect central extension is $SL_n(k)$ which is an extension of the central group of scalar matrices isomorphic to a subgroup of k^* by the group $\mathrm{PSL}_n(k)$.

For a fixed perfect group H, Theorem 3.7 gives conditions on the Sylow subgroups of a perfect central of H. For example, if p divides the order of the central subgroup in a perfect central extension, then $|H|$ must be divisible by p^3. In particular, these methods can be used to show that $GL_n(q)$ is not perfect (without direct reference to the determinant).

To go deeper into applications of the transfer we develop the *focal subgroup*, which was introduced by Burnside and developed into a powerful tool in finite group theory by D. G. Higman.

Definition 3.6 (Focal Subgroup)**.** Let $H \subseteq G$, the focal subgroup of H in G is defined as follows:

$$Foc_G(H) = \langle x^{-1}y : x, y \in H, x, y \text{ conjugates in } G \rangle$$

It is immediate from the definition of the focal subgroup that

$$H' \leq Foc_G(H) \leq G' .$$

In particular, $Foc_G(H) \leq \ker(V)$ where $V : G \to H$ is the transfer homomorphism. Of particular interest in the theory of finite simple groups is the distinct between elements of H' are of the form xy^{-1} for H-conjugate elements $x, y \in H$ and the Focal subgroup, in which x, y need only be conjugate in G. This *fusion* of conjugacy classes turns out to have deep implications for the structure of G as a whole.

The next theorem characterises the Focal subgroup of a Sylow p-subgroup precisely, and is the first important application of the transfer in finite group theory. It is due to Burnside.

Theorem 3.8 (Focal Subgoup)**.** Let G be a finite group, and $H \subseteq G$ be a Sylow p-subgroup of G. Take the transfer map $v : G \to H/H'$, then

$$Foc_G(H) = H \cap G' = H \cap ker(v)$$

Proof. First, we notice that

$$Foc_G(H) \subseteq H \cap G' \subseteq H \cap Ker(v) .$$

The first inclusion follows from the elements of $Foc_G(H)$ being commutators in G. Since the image of v is abelian, $G' \leq \ker(v)$. We need only to show that $H \cap ker(v) \subseteq Foc_G(H)$.

Equivalently, if $h \in \ker(v)$ then h is a product of commutators in G. Write $P_0 = P \cap G'$. By the Transfer Evaluation Lemma,

$$v(h) = \prod_{t \in T'} t h^{n_t} t^{-1} \text{mod } P_0$$

for some subset T' of a transversal of P in G. Since P/P' is abelian, we may introduce an additional factor

$$v(h) = \prod_{t \in T'} h^{n_t} \left(h^{-n_t} t h^{n_t} t^{-1} \right) \text{mod } P' .$$

Since the second term is a commutator, we find that

$$v(h) = \prod_{t \in T'} h^{n_t} = h^{|G:P|} \mod P_0 .$$

Now, since P is a Sylow p-subgroup, $|G : P|$ is coprime to p, so that $V : P \to P/P_0$ is surjective. Hence $P \cap \ker(v) \leq P_0$, as required. □

Remark 3.3. Groups of prime power order have the property that $P' < P$, that is the abelianisation is always non-trivial. As a result, the Focal subgroup theorem can interpreted as stating that fusion always occurs in the Sylow p-subgroups of a simple group. This observation has led to much work on so-called *fusion systems* of finite groups.

We now have a precise description of the kernel of the transfer into a Sylow p-subgroup. In particular, the transfer is non-trivial if and only if $P_0 \neq P$. The next result shows that knowledge of $N_G(P)$ is sufficient to determine the focal subgroup of an abelian Sylow p-subgroup.

Remark 3.4. Recall that the *centraliser* of a subgroup $H \leq G$ is the subgroup $C_G(H) = \{g \in G : [g, H] = 1\}$. The *normaliser* of H is the group $N_G(H) = \{g \in G : gHg^{-1} = H\}$. The centraliser is normal in the normaliser, and H is normal in both. The quotient $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.

Lemma 3.4 (Burnside's lemma)**.** Let P be an abelian Sylow p-subgroup of a finite group G. If $x, y \in P$ are conjugate in G then there exists $n \in N_G(P)$ such that $n^{-1}xn = y$.

Proof. By hypothesis, there exists $g \in G$ such that $y = g^{-1}xg$. Since $x \in P$, it follows that $y \in P^g$ where P^g is a conjugate of P. Since the Sylow p-subgroups of G are abelian, $P^g \in C_G(y)$. But by hypothesis, $C_G(P) \in C_G(y)$, also.

By order considerations, P and P^g are Sylow p-subgroups of $C_G(y)$ and so are conjugate by some element $n \in C_G(y)$. So $P^n = P^g$. So $gc^{-1} \in N_G(P)$ and $x^{gc^{-1}} = y$ as required. ☐

Recall that a *normal p-complement* of G is a normal subgroup N such that G/N is isomorphic to a Sylow p-subgroup of G.

Theorem 3.9 (Burnside's Normal p-Complement theorem)**.** Let G be a finite group and let $P \in Syl_p(G)$, suppose $P \subseteq Z(N_G(P))$, then G has a normal p-complement.

Proof. Take P to be abelian, we have $P \subseteq C_G(P)$, by the previous Burnside's Lemma, any two elements $x, y \in P$ are G-comjugate are also conjugate in $N_G(P)$, but P is in the center of $N_G(P)$, x, y must be the same element. Therefore no two distinct elements of P can conjugate in G.
By the Focal subgroup Theorem, we have $Foc_G(P) = 1 = P \cap ker(v)$ with transfer map $v : G \to P$
Now $ker(v) \lhd G$, then $Ker(v)P$ has order $\frac{|ker(v)||P|}{|ker(v) \cap P|} = |ker(v)||P|$, since P is a sylow p-subgroup we know p does not divide $|ker(v)|$, thus $|G : ker(v)| = |v(g)|$ is a power of p, we have now shown that $ker(v)$ is a normal p-complement. ☐

As an easy corollary of the p-complement theorem we have the following results.

Proposition 3.6. Let p be the smallest prime which divides $|G|$. If the Sylow p-subgroup of G is cyclic then G has a normal p-complement.

Proof. Write P for a Sylow p-subgroup of G. Observe that $|Aut(C_{p^n})| = p^{n-1}(p-1)$. So all prime divisors of this order are $\leq p$. So by hypothesis, $N_G(P) = C_G(P)$ and no fusion occurs in P, so G has a normal p-complement. ☐

We conclude with an application to finite simple groups.

Theorem 3.10. Let P be an abelian 2-group such that $Aut(P)$ is a 2-group. Then P is not the Sylow 2-subgroup of a finite simple group.

Proof. We will show that any group having P as a Sylow 2-subgroup has a normal 2-complement.

First, the Focal Subgroup Theorem states that the image of the transfer homomorphism $v : G \to P$ is isomorphic to $P/P \cap [G, G]$. In particular the image is trivial if and only if $P \subseteq [G, G]$. But Lemma 3.4 shows that fusion in P is controlled by the normaliser $N_G(P)$. Since P is abelian and Sylow, we know that $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P)$ of odd order.

Finally, we use that $Aut(P)$ has no subgroups of odd order to see that $N_G(P)$ has trivial action on P. Hence $v(G) \cong P$ and G has a normal 2-complement. In particular, G cannot be simple. □

The next result gives some examples of 2-groups satisfying the hypotheses of the theorem.

Proposition 3.7. Let P be a product of cyclic 2-groups of distinct orders. Then $Aut(P)$ is a 2-group.

Proof. Multiplication by an odd integer is an automorphism of the cyclic group of order 2^k. Clearly there are 2^{k-1} such automorphisms, so $Aut(C_{2^k})$ is of order 2^{k-1}.

If P is a direct product of non-isomorphic groups, then each factor is characteristic and automorphisms map these direct factors to themselves. So $Aut(P)$ is a direct product of the automorphism groups of the individual factors. □

Note that the condition on the distinct factors cannot be relaxed. The Sylow 2-subgroup of the simple group A_5 is isomorphic to $C_2 \times C_2$. The methods developed in this section require that a group of order 3 acts to permute the three subgroups C_2 of the Sylow 2-subgroup, ensuring that the transfer into this subgroup is trivial.

Transfer arguments become more complicated when the Sylow 2-subgroups of G are nonabelian, though much can still be said. We refer the reader to Chapter 10 of Robinson's monograph for further applications of the transfer [10]. Many of the deeper applications of transfer theory for finite simple groups appears in the literature as research on fusion systems.

Chapter 4

Further Interpretations and Applications of the Transfer

In this thesis we have explored the topological motivations for the transfer homomorphism and its applications in finite group theory. In this section we sketch some further homological and representation theoretic ways of understanding the transfer homomorphism.

4.1 G-modules, Homology and Cohomology

Dual to homology theories are cohomology theories. These may be interpreted in a few different ways, depending on context. For our purposes the most convenient is to develop an analogy to the dual of a vector space.

Definition 4.1. A G-module is an abelian group M together with an *action* of G on M which satisfies

$$g(m_1 + m_2) = gm_1 + gm_2\,,$$

for all m_1 and m_2.

Since an abelian group is a $\mathbb{Z}$-module, a G module is precisely the same thing as a $\mathbb{Z}G$ module. Note that a G-module structure on a vector space V is precisely the same thing as a representation of G on V. The study of G-modules should be considered a generalisation of representation theory.

Next, we introduce a chain complex for a finite group G. Define $\mathbb{Z}G^n$ to be the free module with basis $\{[g_1, \ldots, g_n] : g_i \in G\}$ with boundary map

$$\delta[g_1, \ldots, g_n] = \prod_{i=1}^{n} (-1)^i [g_1, \ldots, \hat{g}_i \ldots, g_n]\,.$$

As noted in chapter 2, this chain complex is exact, and so has trivial homology. Taking the G-coinvariant submodule at each point in the complex gives a *derived chain complex*, as follows

$$\ldots \to M_n \to M_{n-1} \to M_{n-2} \to \ldots$$

where a basis for M_n is given by the orbits of G on $\mathbb{Z}G^n$ and the boundary maps are chosen so that implicit diagram commutes.

The homology groups of G are then defined to be the quotients $Im(\delta_n)/Ker(\delta_{n-1})$. To construct the cohomology groups, one may consider the modules of homomorphisms from $\mathbb{Z}G^n$ into a fixed module M. Observe now that in the derived resolution we obtain homomorphisms

$$\mathbb{Z}G^n \to \mathbb{Z}G^{n-1} \to M$$

which suggests that there is an induced *coboundary* map from $\mathbb{Z}G^n \to M$ which factors through $\mathbb{Z}G^{n-1}$. As a result, we obtain a chain complex

$$\ldots \to \mathrm{Hom}(\mathbb{Z}G^n, M) \to \mathrm{Hom}(\mathbb{Z}G^{n+1}, M) \to \ldots$$

where as before the cohomology groups are defined as quotients $Im(\delta_n)/Ker(\delta_{n+1})$. We do not delve into the details of cohomology of finite groups, which is a large and well-developed subject. Details may be found in Section 17.2 of Dummit and Foote [3]. Two observations are important:

1. Under fairly general conditions, the first homology and first cohomology groups are isomorphic, but not in a canonical way. (This is exactly analogous to the existence of a non-canonical isomorphism between a finite dimensional vector space and its dual).

2. Cohomology is a dual theory to homology in the sense that all arrows are reversed. Just as there is a natural evaluation map $Ab(H) \to Ab(G)$ and an unnatural transfer map $Ab(G) \to Ab(H)$, there is a natural map $Hom(G, M) \to Hom(H, M)$ and an unnatural transfer map $Hom(H, M) \to Hom(G, M)$. These are often encountered in representation theory: they are respectively the restriction and induction maps. These satisfy many relations, most famously Frobenius reciprocity.

Throughout this thesis, we chose to emphasise topological and group theoretic methods over those of homological algebra. In fact, it is entirely possible to develop the transfer homomorphism in a homological framework (though this can appear unmotivated without the topological notions). This approach to homology is developed fully by Dummit and Foote in Chapter 17.

4.2 Induced Representations

Finally, we can describe Marshall Hall's representation with theoretic treatment of the transfer homomorphism. To do so, we will need some material on induced representations.

Given an arbitrary subgroup H of G with finite index n, Frobenius constructed a method to find the representation of G from representation of H. We suppose χ is a representation of H. To extend χ to a representation of G, we consider the cosets

$\{Ht_1, ..., Ht_n\}$ where $T = t_1, ..., t_n$ is the set of transversal of H in G, we can write G as: $G = Ht_1 \cup ... \cup Ht_n$. For every $g \in G$, defined $\dot{\chi}(g)$ as:

$$\dot{\chi}(g) = \begin{pmatrix} \chi(t_1 g t_1^{-1}) & \chi(t_1 g t_2^{-1}) & ... & \chi(t_1 g t_n^{-1}) \\ ... & ... & & ... \\ \chi(t_n g t_1^{-1}) & \chi(t_n g t_2^{-1}) & ... & \chi(t_n g t_n^{-1}) \end{pmatrix}$$

Where if $t_i g_t k^{-1} \notin H$ for some i, j, denote $\chi(t_i g_t k^{-1}) = 0$ in the (i, j) entry of $\dot{\chi}(g)$. It terms out that $\dot{\chi}$ is indeed a representation of G, that is, $\forall g, h \in G$, $\dot{\chi}(g)\dot{\chi}(h) = \dot{\chi}(gh)$.

There's two cases arise when considering matrix multiplication of $\dot{\chi}(g)$ and $\dot{\chi}(h)$ and in both cases the equality holds. We omit the proof here where readers can find details in §3.1 of Ledermann [9]. The character $\dot{\chi}$ is called *induced representation* of χ.

With above construction, we can consider the special case that representation ρ of G is induced from a one-dimensional representation of a subgroup H.

Definition 4.2 (Monomial Representation)**.** Given a group G, a representation ρ of G is said to be monomial if there is a subgroup H of G with one-dimensional representation ρ' such that ρ is induced by ρ'.

We may interpret a monomial representation as an induced representation as follows. The proof is omitted since it follows immediately from a careful application of the relevant definitions.

Theorem 4.1. The following are equivalent, for a representation ρ of a group G.

- (i) There exists a G-invariant system of one-dimensional subspaces spanning V, with the stabiliser of a single subspace denoted by H.
- (ii) ρ is a transitive monomial representation.
- (iii) ρ is induced from a one-dimensional representation of H.

Recall that an element of $\mathrm{Hom}(G, \mathbb{C})$ is a one-dimensional character of G. Suppose that ρ is a monomial representation of G. Then

$$g \mapsto \det(\rho(g))$$

is a such a homomorphism by the multiplicative property of determinants. The *natural* restriction map

$$\mathrm{Hom}(G, \mathbb{C}) \to \mathrm{Hom}(H, \mathbb{C})$$

is obtained from restricting a character of G to one of H. There is a more subtle map:

$$\chi \in \mathrm{Hom}(H, \mathbb{C}) \to \det \cdot \chi_H^G \in \mathrm{Hom}(G, \mathbb{C})$$

obtained from the composition of the determinant map with the induced representation. This is precisely the *cohomological* version of the transfer map.

We can proceed even a step further. Recall that we have shown that the first homology group of G is canonically isomorphic to $Ab(G)$, while the first cohomology group of G with coefficients in $\mathbb{C}$ is $\mathrm{Hom}(G, \mathbb{C})$, the group of linear characters of $Ab(G)$. By the well-known duality of abelian groups, the first homology and first cohomology groups are isomorphic (but not canonically isomorphic). It follows that the natural evaluation map for homology can be interpreted as a *corestriction* map and that the transfer in homology is *coinduction*. This interpretation leads to further insights into the relations between the transfer in homology and cohomology which we do not pursue further.

We conclude with the observation that Hall's definition of the transfer is non-canonically isomorphic to the usual definition, since he defines the cohomological version of the transfer rather than the homological version. One is non-trivial if and only if the other is, so for the purposes of group theory (where typically one wants to verify that certain groups are not perfect) Hall's version has the advantage of allowing the immediate application of methods from character theory and representation theory.

Definition 4.3 (Monomial permutation)**.** Consider a set $S = \{u_1, ..., u_n\}$ that can be multiplied from the left by elements of a group H with the following rules: $\forall u_i \in S$, $h_i, h_j \in H$, $1u_i = u_i$ and $h_i(h_j u_k) = (h_i h_j)u_k$. Then a monomial permutation M is defined to be a mapping: $s_i \mapsto h_{ij}u_j$ where $i = 1, ..., n$ and $j = j(i)$.

Proposition 4.1 (Monomial permutations form a group)**.** Define the product of two mapping as following: if $M_1 : u_i \mapsto h_{ij}u_j$, $M_2 : u_j \mapsto h_{jk}u_k$, let $M_1M_2 : u_i \mapsto h_{ij}h_{jk}u_k$. Then all such mappings form a group.

Proof. Each mapping $M : u_i \mapsto h_{ij}u_j$ can be written as a matrix of dimension $n \times n$ with h_{ij} on the ij-th entry for every pair of i, j and 0 in the remaining entries. Then it is the same as the usual matrix algebra but with matrices of entries from H. Identity is just the identity matrix, associativity follows from matrix algebra property. Composition of two mapping is multiplication of two matrices and thus closed under composition. For a given mapping, for each h_{ij} in its matrix, we construct a new matrix assigns h_{ij}^{-1} in the ji-th entry and zero everywhere else, then this gives the inverse element. □

Proposition 4.2. If we let M denote the group of all such mappings on a set S, then mappings of the form $u_i \mapsto h_{ii}u_j$ form a normal subgroup D under multiplication.

Proof. Similar to above, write elements of D in matrix form it is easily seen they form a subgroup and any elements of the form mdm^{-1} where $m \in M$ and $d \in D$ is again a diagonal matrix thus D is normal. □

The quotient group M/D is a set of cosets of form mD where $m : u_i \mapsto h_{ij}u_j$ is a monomial permutation on set S. Then M/D can be viewed as the symmetric group of permutations of elements $u_1, ..., u_n$. In general, for any subgroup G of M, for $g \in G$: $u_i \mapsto h_{ij}u_j$, we can define map $g^* : u_i \mapsto u_j$ induced by g. Then these induced elements g^* preserves structure of G, $\chi : g \mapsto g^*$ is a homomorphism and the induced group G^* from G is a permutation group on set S.

Definition 4.4 (Transitive monomial permutation group)**.** We say a monomial permutation group $G \leq M$ is transitive if the induced permutation group G^* is transitive.

Theorem 4.2. Let G be a group with subgroup K of finite index n and $G = K + Kx_2 + ... + Kx_n$. If $K \to H$ is a homomorphism of K on to H, then there is a transitive monomial representation of G with multiplier H: $\forall g \in G$, let $x_i g = k_{ij} x_j$ where $i = 1, ..., n$, $j = j(i)$ and $k_{ij} \in K$. Let the homomorphism from K to H be $k_{ij} \mapsto h_{ij}$. Then $\pi(g) : s_i \mapsto h_{ij} s_j$ is a transitive monomial representation with multiplier H.

Proof. Given subgroup $H \leq G$ and $G = K + Kx_2 + ... + Kx_n$. Let $g_1, g_2 \in G$ and suppose $x_i g_1 = k_{ij} x_j$ and $x_j g_2 = k_{ks} x_s$, then $x_i(g_1 g_2) = k_{ij} k_{js} x_s$. Applying the homomorphism $k_{ij} \mapsto h_{ij}$, $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$. Thus π is a representation of G. By applying $\pi(g)$ to al $g \in G$, we generate a permutation group that permutes the cosets of H that is transitive. □

4.3 Transfer map

Suppose we have a monomial representation of a group G with multiplier H defined as $\pi(g) : u_i \mapsto h_{ij} u_j$, $i = 1, ..., n, j = j(i)$. From π, we induce the following mapping:

$$g \mapsto \prod_{i=1}^{n} h_{ij} \bmod H'$$

Then this is a homomorphism from G to H/H'. To see it is a homomorphism, let $g_1, g_2 \in G$, then $g_1 \mapsto \prod_{i=1}^{n} h_{i_1 j_1} \bmod H'$, $g_2 \mapsto \prod_{i=1}^{n} h_{i_2 j_2} \bmod H'$, $g_1 g_2 \mapsto \prod_{i=1}^{n} h_{i_1 j_1} h_{i_2 j_2} \bmod H'$. We define this map more formally as below.

Definition 4.5. If we let $K \leq G$, and let $\phi(kx_j) = x_j$ where $k \in K$,

$$V_{G \to K}(g) = \prod_{i=1}^{n} x_i g \phi(x_i g)^{-1} \bmod K'$$

$V_{G \to K}$ is a map from G to K/K' called the *transfer map* from G to K.

Thus, for a given $g \in G$, its image is:

$$\begin{aligned} V_{G \to K}(g) &= \prod_{i=1}^{n} x_i g \phi(x_i g)^{-1} \bmod K' \\ &= \prod_{i=1}^{n} (k_{ij} x_j) \phi(k_{ij} x_j)^{-1} \bmod K' \\ &= \prod_{i=1}^{n} k_{ij} x_j (x_j)^{-1} \bmod K' \\ &= \prod_{i=1}^{n} k_{ij} \bmod K' \end{aligned}$$

Theorem 4.3. *(i)* The mapping $g \mapsto V_{G\to K}(g)$ is a homomorphism from G to K/K'.

(ii) The transfer map $V_{G\to K}(g)$ is independent of choices of representatives x_i.

(iii) If $T \leq K \leq G$, then $V_{G\to T}(g) = V_{K\to T}(V_{G\to K}(g))$.

Proof. To see (i), we let $g_1, g_2 \in G$, suppose $x_i g_1 = k_{ij} x_j$ and $x_j g_2 = k_{js} x_s$ for $i = 1, .., n$ and $j = j(i)$. Then,

$$V_{G\to K}(g_1) = \prod_{i=1}^{n} k_{ij} \bmod K'$$

$$V_{G\to K}(g_2) = \prod_{j=1}^{n} k_{js} \bmod K'$$

$$V_{G\to K}(g_1 g_2) = \prod_{i=1}^{n} k_{ij} k_{js} \bmod K'$$

Thus $V_{G\to K}(g_1 g_2) = V_{G\to K}(g_1) V_{G\to K}(g_2)$ and $V_{G\to K}$ is a homomorphism.

To see (ii), suppose instead of $\{x_i\}$, we use a different set of transversals $\{x_i'\}$ to represent the cosets. On each coset, since x_i and $x_{i'}$ both represent the same coset, $Kx_i = Kx_i'$, then $k_i x_i = k_{i'} x_{i'}$ for some k_i and $k_{i'}$ and $x_{i'} = k_{i'}^{-1} k_i x_i$.If g act on x_i as $x_i g = k_{ij} x_j$, then $x_{i'} g = k_{i'}^{-1} k_i x_i g = k_{i'}^{-1} k_i k_{ij} x_j = k_{i'}^{-1} k_i k_{ij} k_{j'}^{-1} k_j x_j$ Taking the transfer map with transversals $\{x_{i'}\}$ it becomes:

$$\begin{aligned} V_{G\to K}(g) &= \prod_{i=1}^{n} (k_{i'}^{-1} k_i k_{ij} k_{j'}^{-1} k_j) \bmod K' \\ &= \prod_{i=1}^{n} k_{i'}^{-1} k_i \prod_{i=1}^{n} k_{ij} \prod_{i=1}^{n} k_{j'}^{-1} k_j \bmod K' \\ &= \prod_{i=1}^{n} k_{ij} \bmod K' \end{aligned}$$

To see (iii), suppose $G = K + Kx_2 + ... + Kx_n,\ K = T + Ty_2 + ... + Ty_m$. Then, expressing G in terms of T:

$$\begin{aligned} G =& T + Ty_2 + ... + Ty_m \\ &+ ... \\ &+ Tx_i + Ty_2 x_i + ... + Ty_m x_i \\ &+ ... \\ &+ Tx_n + Ty_2 x_n + ... + Ty_m x_n \end{aligned}$$

Thus $\{x_p y_q\}$ with $p = 1, ..., n$ and $1 = 1, ..., m$ is a new transversal related to subgroup T. For $g \in G$, if $x_i g = k_{ij} x_j$ and $y_r k_{ij} = t_{ijrs} y_s$, then $y_r x_i g = t_{ijrs} y_s x_j$.

$$V_{G\to T}(g) = \prod_{i,r} t_{ijrs} \bmod T'$$

Note also on the set of transversals $\{x_i\}$ where $i = 1, .., n$ related to subgroup K,

$$V_{G\to K}(g) = \prod_i k_{ij} \bmod K'$$

And on the set of transversals $\{y_i\}$ where $i = 1, .., m$ related to subgroup K,

$$V_{K\to T}(k_{ij}) = \prod_r t_{ijrs} \bmod T'$$

Then,

$$\begin{aligned}
V_{G\to T}(g) &= \prod_{i,r} t_{ijrs} \bmod T' \\
&= \prod_i \prod_r t_{ijrs} \bmod T' \\
&= \prod_i (V_{K\to T}(k_{ij})) \bmod T' \\
&= V_{K\to T} \prod_i k_{ij} \bmod T' \\
&= V_{k\to T}(V_{G\to K})
\end{aligned}$$

□

Appendix A

Homology Group Derived From Free Resolution

In this chapter we aim to follow the construction in Brown [2] on deriving the group homology. We will first review some terminologies and construct homology of a group, at the end we show that $Ab(G) \cong H_1(G)$ as we shown in Chapter 2 but in a group theoretic approach. Reader who are interested in further topics of cohomology may find related materials in first three chapter of Brown.

A.1 Review on Modules and Group Rings

Definition A.1 (Left Module)**.** Given a ring R, a *left R-module* is an additive abelian group M where the R action on M is given as following:
For $m, m_1, m_2 \in M$ and $r, r_1, r_2 \in R$,

(i) $r(m_1 + m_2) = rm_1 + rm_2$.

(ii) $(r_1 + r_2)m = r_1 m + r_2 m$.

(iii) $(r_1 r_2)m = r_1(r_2 m)$.

(iv) $1m = m$ when R has an multiplicative identity. This condition is not required for M to be a R-module.

In a similar way we can define a right R-module. If R is commutative then any left R-module M will satisfy the properties for right R-module as well and thus is also a right R-module, we say it is a *R-module*. A R-module M can be also viewed as a "vector space" with the "vectors" from M, and "scalars" from R.

Example A.1. Below are some examples that naturally comes with a module structure:

- A vector space: regard the field of vector space as the ring R and the vectors as elements in the additive abelian group M.

- A ring R: let $M = (R, +)$ be the additive abelian group over the ring R, then M is an R module.
- An abelian group G: let $R = \mathbb{Z}$ and $rm := m + ... + m$ represent the r copies of m for any $r \in \mathbb{Z}$ and $m \in G$.

Definition A.2 (Free Module)**.** Let R be a ring with identity element, and for $n \in \mathbb{N}$, let:

$$R^n = \{(x_1, x_2, ..., x_n) : x_i \in R\}$$

R^n is a R-module. It is closed under entrywise addition. To turn it into a R-module, define the R action:

$$r(x_1, x_2, ..., x_n) = (rx_1, rx_2, ..., rx_n)$$

R^n is called a *free R-module*, in general any R-module isomorphic to the above R^n is a free R-module.

Definition A.3 (Submodule)**.** If M is an R-module, a subset N of M is called a *submodule* if:

- N is a subgroup closed under addition: for all $n_1, n_2 \in N$, $n_1 + n_2 \in N$.
- N is closed under ring action: for all $r \in R$ and $n \in N$, $rn \in N$.

Definition A.4 (Quotient Module)**.** Let M be an R-module and N be a submodule of M, then the *quotient module M/N* is the set of all cosets:

$$M/N = \{m + N : m \in M\}$$

and R act on M/N as: $r(m + N) = rm + N$ for $r \in R$.

Definition A.5 (Module Homomorphism)**.** If M_1, M_2 are R-modules, a map ϕ from M_1 to M_2 is an *R-module homomorphism* if it satisfy the following: for all $m, m_1, m_2 \in M$, and $r \in R$,

- $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$
- $\phi(rm) = r(\phi(m))$

We now explore more on a special kind of module: group ring. It has both ring and free module structure. Later we will use it to define free resolution and from there we develop group homology.

Example A.2 (Group Ring)**.** Given a group G, a ring R, a *group ring RG* has elements of the form: $x = \sum_{g \in G} \alpha_g g$ with $g \in G$ and $\alpha_g \in R$ as coefficient. A Group ring RG is a ring and also a RG module. Thus, Let $M = RG$ be the additive abelian group, then $(RG, +)$ admits RG action as:

$$(\sum \alpha_g g) \cdot (\sum \alpha_h h)$$

where we consider $\sum \alpha_g g$ as the ring element and $\sum \alpha_h h$ as the module element.

Definition A.6 (Augmentation Map)**.** Let $\mathbb{Z}G$ be the group ring, the *augmentation map* is defined as: for $\sum \alpha_g g \in \mathbb{Z}G$,

$$\epsilon(\sum \alpha_g g) = \sum \alpha_g$$

The kernel of augmentation map ϵ is called the *augmentation ideal*.

Example A.3. Take $\mathbb{Z}G$ as a ring and cyclic group G of order n. We obtain a module over $\mathbb{Z}$. Define the module homomorphism $T : \mathbb{Z}G \to \mathbb{Z}G$ by $T(g) = g - 1$ for basis element $g \in \mathbb{Z}G$. Here the group ring $\mathbb{Z}G$ is both a $\mathbb{Z}G$-module and a $\mathbb{Z}$-module, we express it in $\mathbb{Z}$-basis with basis elements from G. For any basis element $g_i \in G$, T act as following:

$$T(g^i) = g^i - g^{i-1}$$

then, for $\sum \alpha_g g \in \mathbb{Z}G$,

$$T(\sum \alpha_g g) = \sum \alpha_g T(g)$$

Example A.4. Similar to above example, we can define a module homomorphism $N : \mathbb{Z}G \to \mathbb{Z}G$ by $N(1) = 1 + g + g^2 + ... + g^{n-1}$ acting on the $\mathbb{Z}$ baiss as:

$$N(g^i) = g^i + g^{i+1} + ... + g^{n-1+i}$$

then, for $\sum \alpha_g g \in \mathbb{Z}G$,

$$N(\sum \alpha_g g) = \sum \alpha_g g + g^2 ... + g^{n-1}$$

Example A.5 (F_n module)**.** Recall the free module defined earlier, given a group G, we can construct $\mathbb{Z}$ modules F_n by elements of the form $(g_0, g_1, ...g_n)$ with each $g_i \in G$. This is a $\mathbb{Z}$ module if we allow addition to be pointwise and $\mathbb{Z}$ act as $(g_0, g_1, ..., g_n)m = (g_0 m, g_1 m, ..., g_n m)$ for $m \in \mathbb{Z}$.

Example A.6. Let M be the F_n module defined above, we can take the quotient of M by dividing the submodule generated by all elements of the form $m - mg$. Denote $N = \langle m - mg \rangle$ to be the submodule. Each element in M/N is of the form $(g_0, ..., g_n) + N$ for some $(g_0, ..., g_n) \in M$. It can be verified that M/N is a quotient module where G acts trivially.
We first show M/N is a quotient module, then show G action on M/N is trivial.
Let $m = (g_0, ..., g_n) \in M = F_n$, by definition,

$$\begin{aligned} M/N &= \{m + N : m \in M\} \\ &= \{(g_0, ..., g_n) + N : (g_0, ..., g_n) \in M\} \end{aligned}$$

M/N is closed under group addition since:

$$\begin{aligned} ((g_0, ..., g_n) + N) + ((h_0, ..., h_n) + N) &= (g_0, ..., g_n) + (h_0, ..., h_n) + N \\ &= (g_0 + h_0, ..., g_n + h_n) + N \end{aligned}$$

$$= (k_0, ...k_n) + N$$

For some element $(k_0, ..., k_n)$ in the module F_n.

Let $r = \sum \alpha_g g \in \mathbb{Z}G$ denote the ring element, M/N is closed under ring action since

$$\begin{aligned} r((g_0, ..., g_n) + N) &= \sum \alpha_g g((g_0, ..., g_n) + N) \\ &= \sum \alpha_g ((g_0 g, ..., g_n g) + N) \\ &= \sum \alpha_g (g_0 g, ..., g_n g) + N \\ &= m' + N \end{aligned}$$

Where $M' = \sum \alpha_g (g_0 g, ..., g_n g)$ is an element in module $M = F_n$.
We have shown M/N is a submodule and we now want to show M/N under G action is invariant.

Since $m - mg \in N$, $N = m - mg + N$, then

$$\begin{aligned} (g_0, ..., g_n) + N &= (g_0, ..., g_n) + (m - mg) + N \\ &= (g_0, ..., g_n) + (g_0, ..., g_n)g^{-1} - (g_0, ..., g_n) + N \\ &= (g_0, ..., g_n)g^{-1} + N \end{aligned}$$

Now if g acts on element $(g_0, ..., g_n) + N$,

$$\begin{aligned} ((g_0, ..., g_n) + N)g &= ((g_0, ..., g_n)g^{-1} + N)g \\ &= (g_0, ..., g_n) + N \end{aligned}$$

A.2 Homology Group of Cyclic Group C_n

Given a group G, we can derive the homology of G by first build a free resolution of G over the group ring $\mathbb{Z}G$, and quotient each module in the resolution to make it G-invariant, this derived chain is called the *co-invariant* chain. The homology of group G will be defined on the co-invariant chain.

Definition A.7 (Chain Complex)**.** We have introduced the simplicial chain complex in Chapter 2, we now generalize the idea from linear combinations of simplexes to modules. In the module case, a *chain complex* is a sequence of R-modules

$$... \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} ... \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where each C_i is a R-module and d_i is a boundary operator(analogous to the simplicial homology). The composition of two adjacent boundary operator satisfy: for all $n \geq 0$, $\partial_n \circ \partial_{n+1} = 0$ for all n.

Definition A.8 (Free Resolution)**.** A *free resolution* of of a $\mathbb{Z}G$-module M is an exact chain complex

$$... \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} ...... \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where each C_i is a free $\mathbb{Z}G$ module.

Example A.7 (Homology of Cyclic Group)**.** Here we show an example of how to compute the homology of a cyclic group G. First we can construct a free resolution of G over group rings $\mathbb{Z}G$.

$$... \xrightarrow{T} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{T} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{0} 0$$

Where the maps are defined as follow:

- 0: $\mathbb{Z} \to 0$, let $z \in \mathbb{Z}$, $0(z) = 0$.
- ϵ: $\mathbb{Z}G \to \mathbb{Z}$, $\epsilon(\sum \alpha_g g) = \sum \alpha_g$.
- T: $\mathbb{Z}G \to \mathbb{Z}G$, let $1 \in \mathbb{Z}G$, $T(1) = g - 1$.
- $N : \mathbb{Z}G \to \mathbb{Z}G$, $N = 1 + g + g^2 + ... + g^{n-1}$.

We show this chain is exact:

The first composition $0 \circ \epsilon = 0$ is trivial.

$$\begin{aligned}
\epsilon \circ T(\sum \alpha_g g) &= \epsilon[(a_0 g + ... + a_{n-1}g^n) - (a_0 + ... + a_{n-1}g^{n-1})] \\
&= (a_0 + ... + a_{n-1}) - (a_0 + ... + a_{n-1}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
T \circ N(\sum \alpha_g g) &= T[(a_0 + ... + a_{n-1}g^{n-1}) + ... + (a_0 g^{n-1} + ... + a_{n-1}g^{-2})] \\
&= [(a_0 g + ... + a_{n-1}) + ... + (a_0 + ... + a_{n-1}g^{-1})] \\
&- [(a_0 + ... + a_{n-1}g^{n-1}) + ... + (a_0 g^{n-1} + ... + a_{n-1}g^{-2})] \\
&= 0
\end{aligned}$$

By repeating the maps N and $t - 1$ we get a periodic resolution. Now quotient the modules and derive the co-invariant modules. From earlier discussion,the co-invariant modules are quotients by $\langle m - mg \rangle$, the induced co-invariant chain needs to make the diagram commute. In this example with cyclic G, the induced complex chain has maps n and 0 alternates where $n : \mathbb{Z} \to \mathbb{Z}$ is a given as: for $z \in \mathbb{Z}$, $n(z) = nz$.

$$\begin{tikzcd}
... \arrow[r, "T"] & \mathbb{Z}G \arrow[r, "N"] \arrow[d, "T"'] & \mathbb{Z}G \arrow[r, "T"] \arrow[d, "T"'] & \mathbb{Z}G \arrow[r, "T"] \arrow[d, "T"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "\epsilon"] & 0 \\
... \arrow[r, "n"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "0"] & 0
\end{tikzcd}$$

Now we can compute the homology group of the induced chain,

$$H_0(G) = \frac{ker(0)}{Im(n)} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

$$H_1(G) = \frac{ker(0)}{Im(n)} = \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \mathbb{Z}_n$$

$$H_2(G) = \frac{ker(n)}{Im(0)} = \frac{\{0\}}{\{0\}} \cong 0$$

$$\dots\dots$$

Where $\{0\}$ denotes the set of zero and 0 denotes trivial group.

As the n and 0 alternates, we obtain the pattern of the n-th homology group:

- If $i = 0$, $H_0 \cong \mathbb{Z}$.
- If i is odd, $H_i \cong \mathbb{Z}_n$
- If i is even, $H_i \cong 0$.

Up to here we have developed homology group of cyclic group G from free resolution. To define homology group for arbitrary G, a more standard method used is to construct the *standard resolution* of F_n's connected by the usual boundary maps. With *bar notation*, one can verify that a map from $H_1(G)$ to $H_1(H)$ for $H \leq G$ can be regarded as the transfer map from $Ab(G)$ to $Ab(H)$. Readers with further interest may work through this in exercise 2 of §3.9 in Brown [2].

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