Graphs and Their Cycle Spectra

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Abstract

We explain the main results and proof techniques used in Martin Merker's 2021 paper "Gaps in the cycle spectrum of 3-connected cubic planar graphs", as well as a family of counterexamples to a conjecture of Merker.

Acknowledgements

Thank you to Brigitte Servatius for advising me these past 4 years and igniting my interest in graph theory. I would also like to thank my friends and family who have always supported me throughout my academic career and otherwise.

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Chapter 1: Introduction

Throughout this paper, I will be referring to the glossary that appears in Chapter 5. The definitions throughout the glossary were adapted from the 5th edition of "Introduction to Graph Theory" by Robin J Wilson [2]. Therefore, the various terms used throughout the paper are primarily used as he used them.

On May 22, 2019, Martin Merker submitted a mathematical paper to be published titled "Gaps in the cycle spectrum of 3-connected cubic planar graphs." [1] The final version of the paper was published in January, 2021 in the Journal of Combinatorial Theory. In Merker's paper, he attempts to prove that "for every natural number k, every sufficiently large 3-connected cubic planar graph has a cycle whose length is in $[k, 2k+9]$." He also shows that "this bound is close to being optimal by constructing, for every even $k \geq 4$, an infinite family of 3-connected cubic planar graphs that contain no cycle whose length is in $[k, 2k+1]$."

The cycle spectrum of a graph G is the set of the lengths of the cycles in G. A gap in the cycle spectrum C(G) of a graph G is an interval of integers not contained in C(G), but G contains a cycle with length greater than b, that is, G has a circumference (27) greater than b. Consider Figure 1.1. It is easy to see that there are only cycles of length 3 and 11 for this graph. Therefore, the cycle spectrum of this graph is {3, 11}, and there is a gap [4, 10].

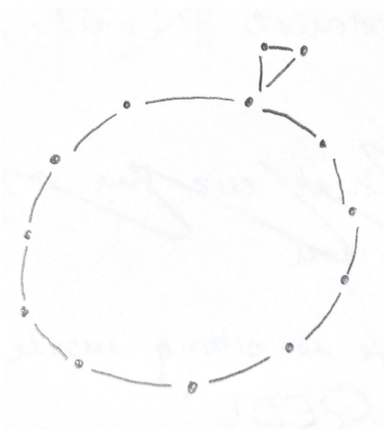

Figure 1.1: A graph whose cycle spectrum is [11, 3].

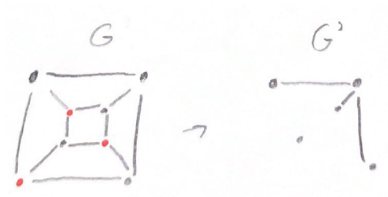

Figure 1.2: By removing the red vertices of G, the graph becomes disconnected, shown in G'.

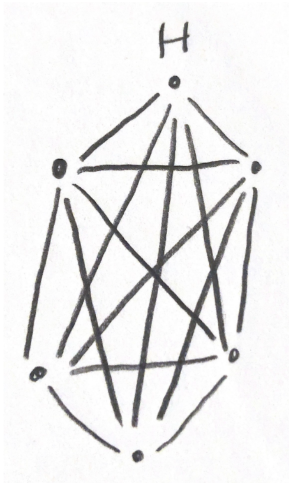

Figure 1.3: The graph H, also known as K_6.

Now, what is a 3-connected cubic planar graph? A graph is 3-connected when you must delete 3 vertices in order to disconnect the graph. Graph G of Figure 1.2 is 3-connected. By removing the vertices marked in red, G becomes G', a disconnected graph. A graph is cubic, if each vertex of the graph has degree 3. The graph G in Figure 1.2 is also cubic. A planar graph can be drawn in the plane with no crossings. The graph G in Figure 1.2 is also planar, as opposed to graph H in Figure 1.3, which is non-planar. Therefore, the cube graph, G in Figure 1.2, is an example of a 3-connected cubic planar graph.

So Merker is trying to prove that every sufficiently large 3-connected cubic planar graph has a cycle spectrum that contains a cycle of length [k, 2k+9]. He also seeks to prove that this bound is close to optimal.

Only a year after the publication of Merker's paper, Carol Zamfirescu submitted his own paper to disprove elements of Merker's paper. Zamfirescu's paper has yet to be published. In the paper, Zamfirescu specifically proves that for $k \geq 6$, there exists an infinite family of 3-connected cubic planar graphs of circumference at least k, whose cycle spectrum contains no elements of [k, 2k+2]. [3]

Chapter 2: Gaps in the Cycle Spectrum of 3-connected Cubic Planar Graphs

In his paper, Merker is trying to prove that "for every natural number k, every sufficiently large 3-connected cubic planar graph has a cycle whose length is in $[k, 2k+9]$. We also show that this bound is close to being optimal by constructing, for every even $k \leq 4$, an infinite family of 3-connected cubic planar graphs that contain no cycle whose length is in $[k, 2k+1]$" [1].

Suppose that $k = 0$. According to Merker, every sufficiently large, 3-connected cubic planar graph must have a cycle whose length is in $[0, 9]$. Suppose that $k = 1$. Then, the interval becomes $[1, 11]$. This continues for all natural numbers.

Merker's conjecture says that a graph must be sufficiently large, but what does that mean? The smallest cycle (24) an applicable graph could have in its cycle spectrum (25) is of length 9, when $k = 0$. This implies that the smallest possible sufficiently large graph has 10 vertices, as the graph must also be cubic (16), planar (48), and 3-connected (40).

Figure 2.1 provides an example of a 3-connected cubic planar graph graph on 10 vertices. Its cycle spectrum is $\{3, 4, 5, 6, 7, 8, 9, 10\}$.

To prove his conjecture, Merker proves a series of theorems. Before moving to the main proof, we will examine the proofs of these theorems

"Proposition 1. Let $a, b \in \mathbb{N}$ with $3 \leq a \leq b$. The interval $[a, b]$ is a gap of some 2-connected cubic planar graph if and only if $a = 3$, $b \leq 4$, or $a = 4$, and $b \geq 5$." [1].

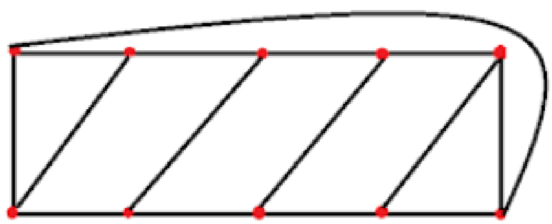

Figure 2.1: A 3-connected cubic planar graph

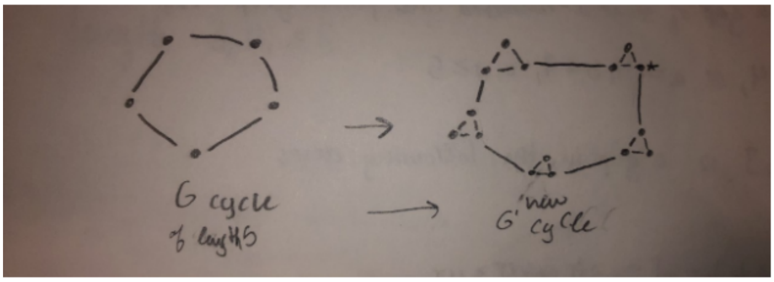

Figure 2.2: C5 with vertices converted to triangles

Proposition 1 implies that
1) If $a = 3$, then $[a, b] = [3, 3]$ or $[3, 4]$
2) If $a = 4$, then $[a, b] = [4, 4], [4, 5], [4, 6], [4, 7], [4, 8]$, or $[4, 9]$.
3) If $a = 5$, then $[a, b] = [5, n]$, where $n \geq 5$

According to Merker, every 2-connected cubic plane graph contains a face (49) of length 3, 4, or 5. Therefore, the maximum girth (26) of a 2-connected cubic planar graph is 5, and the girth must be at least 3. With this knowledge, we know that if there is a gap (28) of any cubic plane 2-connected graph of the form [3, b], b must be either 3 or 4. By definition, $[3, 4]$ is a gap of any 2-connected plane graph of girth 5. Merker claims that if a graph G is cubic, planar, and 2-connected, then replacing each of G's vertices with triangles results in a graph G' such that $C(G') \cap [4, 9] = 0$. Figure 2.2 shows an example of a cycle of length 5 whose vertices have been replaced with triangles. In Figure 2.2, $C(G') = \{3, 10, 11, 12, 13, 14, 15\}$. Thus, $C(G') \cap [4, 9] = 0$. This proves Merker's second claim.

Now assume $a \geq 5$, and let $C = v_1, v_2, v_3, ..., v_{3k}, v_1$ be a cycle on $3k$ vertices where $3k > b$. Let G be the graph consisting of C and k vertices u_1, u_2, ..., u_k such that u_i is joined to v_{3i-2} and v_{3i-1}, for $i \in \{1, ..., k\}$ Figure 2.3 is an example of such a graph, for k=3, a=5, and b=8.

The graph in Figure 2.3 is a Hamiltonian (30), 2-connected cubic planar graph. Figure 2.4 shows one of its Hamilton cycles.

By looking at Figure 2.3, it is easy to see that the only cycles of size less than $3k$ (in this case, $3k = 9$), have size 3 or 4. Thus, the proposition is proven.

"Conjecture 3: If $k \in \mathbb{N}$ with $k \geq 2$ and G is a 3-connected cubic planar graph of circumference at least k, then $C(G) \cap [k, 2k + 2] \neq 0$."

Conjecture 3 pertains to larger graphs, which have the largest possible gaps across all 3-connected cubic planar graphs [1].

"Conjecture 4: There exists $c \in \mathbb{N}$ such that $C(G) \cap [k, 2k + c] \neq 0$ for every $k \in \mathbb{N}$ and 3-connected planar graph G of circumference at least k."

Conjecture 4 takes Merker's primary conjecture and modifies it to include non-cubic graphs. The difference between Conjecture 4 and Merker's primary conjecture is the

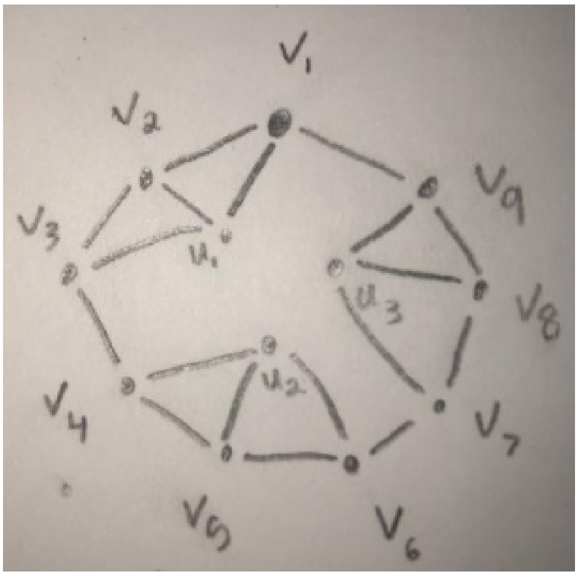

Figure 2.3: Merker's described graph, for k=3, a=5, and b=8

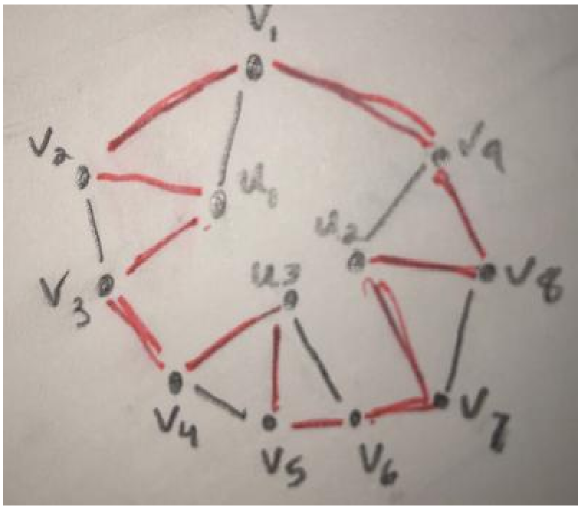

Figure 2.4: One of the Hamilton cycles of the graph in Figure 2.3

interval being intersected with the graph's cycle spectrum. The gap in the cycle spectrum of the graph in Conjecture 4 depends on k.

"Lemma 5: Let G be a 3-connected cubic plane graph and $k, c \in \mathbb{N}$. If the circumference of G is at least k, then $C(G) \cap [k, 2k+c] \neq 0$ or G has a facial cycle of length greater than $2k + c$."

Merker uses proof by contradiction to prove this lemma, as so:

Suppose $C(G) \cap [k, 2k+c] = 0$, and G has a facial cycle (50) of length less than k. Let C be a cycle of length greater than $2k+c$ with minimal faces. Minimal faces implies that there are no excess faces in the interior of C. Therefore, there are at least 3 facial cycles in the interior of C. For every edge of the cycle C, let C_e denote the facial cycle of the interior of C which is incident with edge e. Now let De denote the symmetric difference (29) of C and C_e. Therefore, $D_e = (C \cup C_e) \setminus (C \cap C_e)$.

Merker mentions that D_e is a union of cycles for every $e \in E(C)$. This is true because C_e is an interior facial cycle of C, which is made up of at least 3 other facial cycles. The other 2 or more facial cycles in C remain intact despite the removal of Ce. Merker states that there exists an edge $f \in E(C)$ such that D_f is itself a cycle. For this to be true, Cf would have to be incident to only one other facial cycle. If C_f was incident to more than 2 cycles, it would create a gap within D_f. This would result in D_f containing multiple cycles. Since every facial cycle has length less than k, $|E(D_f)| \geq |E(C)| - k$. And because $|E(C)| \geq 2k+c$ and $C(G) \cap [k, 2k+c] = 0$, $|E(D_f) > 2k+c$. This is where the contradiction lies. D_f has fewer faces than C in its interior, which contradicts our choice of C. This ends Merker's proof of Lemma 5.

"Lemma 6: Let G be a 3-connected cubic plane graph and $k \in \mathbb{N}$. If $C(G) \cap [k, 2k] = 0$ and G contains a facial cycle C of length at least $2k+1$, then G contains a 2-connected subgraph G' such that

1. no two facial cycles of G' of length less than k intersect,

2. $E(C) \subseteq E(G')$,

3. every edge of G' which is part of a 2-edge-cut is incident with a face longer than $2k$ and a face shorter than k."

We know that every cycle is going to be either short or long because $C(G) \cap [k, 2k] = 0$ and G has a facial cycle of length at least $2k+1$. The cycle is short when its length is less than k, and the cycle is long when its length is greater than $2k+1$. Merker claims that G must contain a 2-connected subgraph (18) G' with the conditions listed above. To prove this, Merker constructs a sequence of 2-connected graphs $G_0, G_1, ...G_n$ by successively "gluing" adjacent short faces together. Merker defines gluing two faces together as deleting their intersected edges and deleting any isolated vertices that occur from these deletions. Figure 2.5 shows an example of gluing.

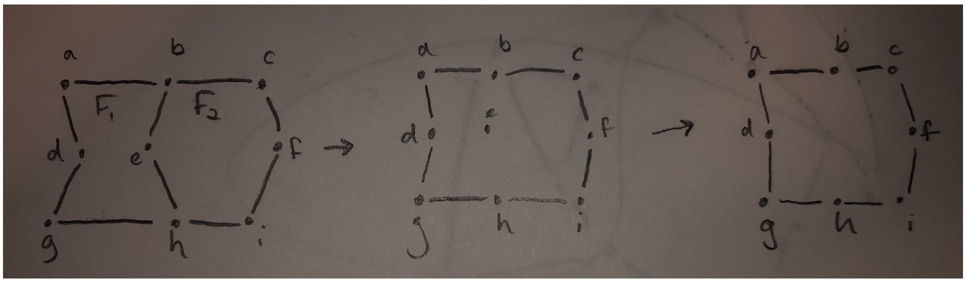

Figure 2.5: The process of gluing 2 cycles

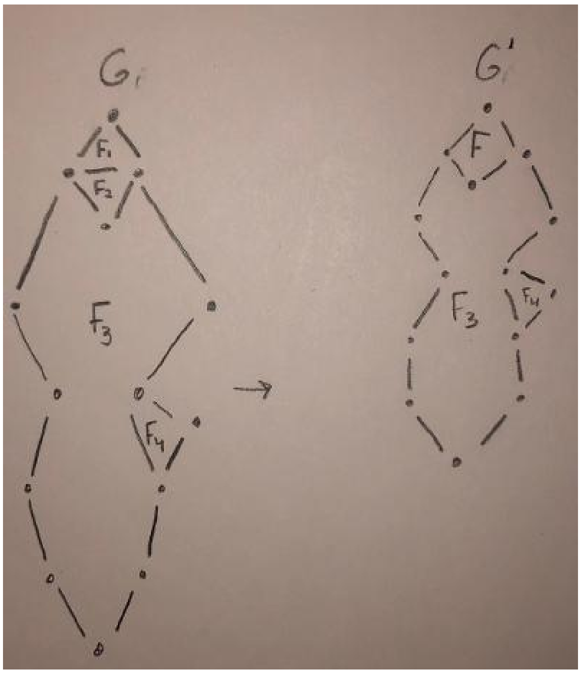

Figure 2.6: The final step in the gluing process

Suppose $G = G_0$. Upon gluing 2 faces together, G_0 becomes G_1, etc. If G_i, for $0 < i < n$, does not contain 2 adjacent short faces, then $G_i = G'$ and the process ends. If G_i has at least 2 adjacent short faces, the gluing continues. Suppose F_1 and F_2 are both short adjacent faces in G_i. By gluing F_1 and F_2 together, you create a new face F. The boundary of F may contain cycles, but they will all be short. Figure 2.6 provides an example of a graph G, where $k = 5$, $2k = 10$, and $|C| = 12$.

In Figure 2.6, only faces F1 and F2 are glued, as they are the only 2 adjacent short faces.

1 is proven by definition of the problem since all of the adjacent short cycles are glued together. 2 is proven as 3 is a long cycle, so it would not be glued or modified in this process. Since Figure 2.6 is only a portion of a graph, this condition is not as easily shown. Therefore, we will examine Merker's reasoning for this condition. We know that G' is 2-connected. From there, Merker states that if e_1 and e_2 make a 2-edge-cut (35),

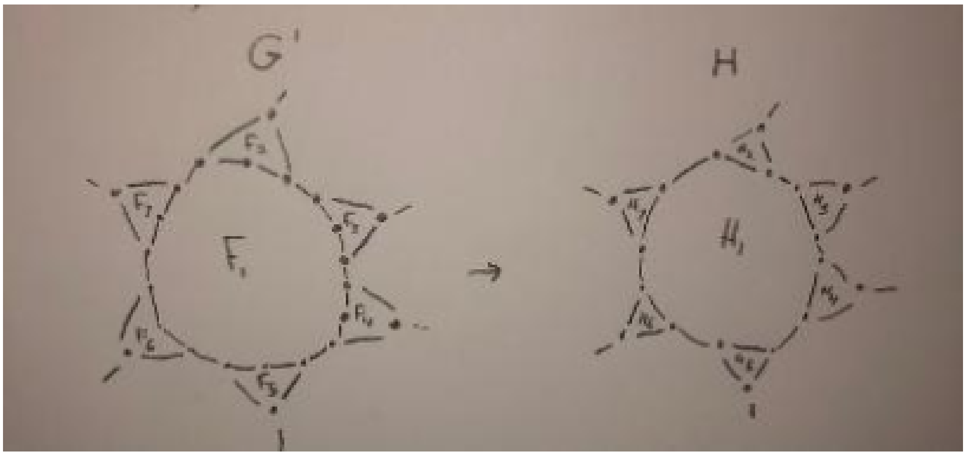

Figure 2.7: An example of suppression from G' to H.

then there are 2 faces F_1 and F_2 that are incident with both e_1 and e_2. At least one of these faces must be long, since there are no 2 adjacent short faces in G'. If both F_1 and F_2 are long, they must also be faces of G, since long facial cycles are not affected by gluing. However, that would mean e_1 and e_2 would disconnect G, which contradicts that G is 3-connected. Therefore, each edge of a 2-edge-cut is incident with a short face and a long face, satisfying 3.

Now, Merker solves his main conjecture: "Theorem 2: If $k \in \mathbb{N}$ and G is a 3-connected cubic planar graph of circumference at least k, then $C(G) \cap [k, 2k+9] \neq 0$. In Proposition 1, Merker generalized the gaps of 2-connected cubic planar graphs. In Theorem 2, he is expanding this to 3-connected cubic planar graphs. Merker proves Theorem 2 using contradiction. Define G as a 3-connected cubic planar graph, with circumference (27) greater than or equal to k. Fix an embedding of G on the plane (47). Suppose $C(G) \cap [k, 2k+9] = 0$. This would imply that all of the cycles of G are either short or long. In this proof, a short cycle has length less than k, and a long cycle has length greater than $2k+9$. It is known that the circumference of G is greater than or equal to k, so there must exist a cycle of G that is long. From Lemma 5, we know that there is a long facial cycle in G.

Now consider Lemma 6. G has a 2-connected subgraph G' that has conditions 1, 2, and 3, as described. Merker now suppresses all vertices of degree 2 of G' to create a new graph H. Since no two short faces are adjacent, the vertices of degree 2 will now become vertices of degree 3 using suppression. An example of this suppression is shown in Figure 2.7.

Merker states that there is a canonical bijection (44) between the faces of G' and H. This is true in Figure 2.7, as H maintains the same number of faces as G'. Merker now defines the following:

1. F(G') denotes the faces of G'

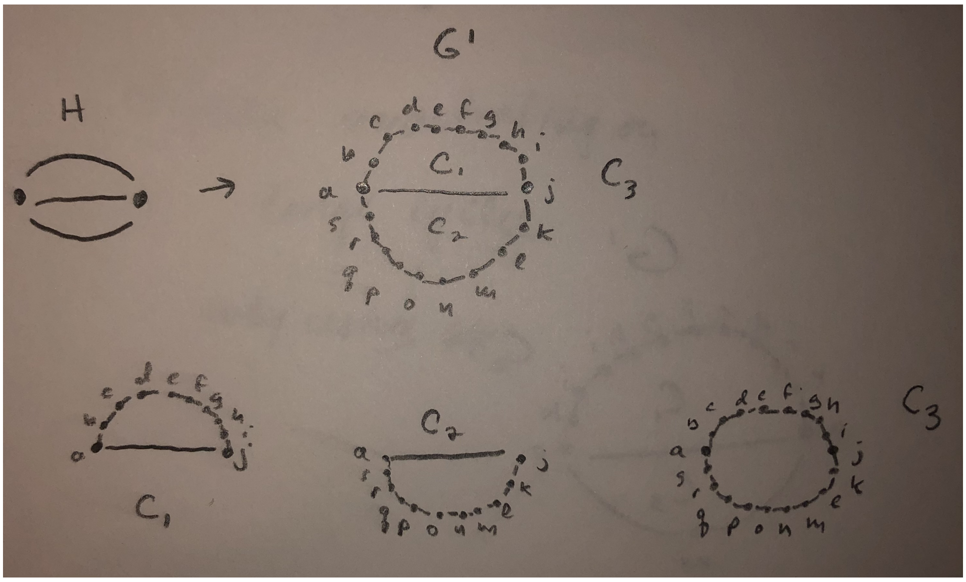

Figure 2.8: This figure shows H as a triple edge, the resulting G', and the cycles of G'.

2. l(F) denotes the length of the face F of G'
3. lh(F) denotes the length of the face that corresponds to F in H
4. X = {F in F(G'): l(F) < k}
5. Y = {F in F(G'): l(F) > 2k + 9}
6. $x = |X|$
7. $y = |Y|$
8. $n = |V(H)|$

By Euler's formula, $x + y = (n/2) + 2$, as H is a 2-connected cubic planar graph. Merker now proves that every facial cycle of H must have length at least 3. To do so, he uses contradiction.

Suppose H is a triple edge, also called a multiple edge of size 3 (8). If H is a triple edge, then G' will have 3 facial cycles (C_1, C_2, and C_3). If you consider the infinite face (49), this is true, as shown in Figure 2.8.

In general, 1 implies that at least 2 of the facial cycles of G' are long. Assume then that C_1 and C_2 are long. Merker states that by C, $|E(C_1 \cap C_2)| \leq 1$. This means that the number of edges in the pairwise intersection (45) of the two long facial cycles is 1. This holds true in Figure 2.8.3 states specifically that every edge of G' which is part of a 2-edge-cut is incident with a face longer than 2k and a face shorter than 2k. If the cycle C_3 is also long, then by C their pairwise intersections would be 1 and G' would be a triple edge. This contradicts 2.

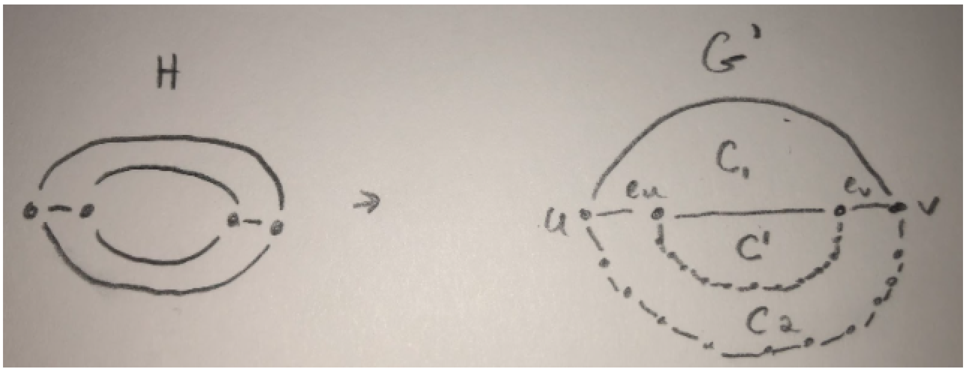

Figure 2.9: This figure shows H with one multiple edge and the resulting G'.

This implies that the facial cycle C_3 must be short. However, this results in another contradiction: $2k+9 < |E(C_1)| = |E(C_1 \cap C_2)| + |E(C_1 \cap C_3)| < 1+k$. $2k+9 < |E(C_1)|$ is true, since C_1 is long. $|E(C_1)| = |E(C_1 \cap C2)| + |E(C_1 \cap C_3)|$ is true, since C1 is pairwise incident with C_2 and C_3. We know that $|E(C_1 \cap C_2)| + |E(C_1 \cap C_3)| < 1+k$ is true because $|E(C_1 \cap C_2) \leq 1|$ as defined, and $|E(C_1 \cap C_3)| < k$, because C_3 is short. Therefore, the contradiction within the inequality is $2k+9 \nleq k+1$.

Suppose H has a facial cycle of length 2, implying that it must have a multiple edge of size 2. Let C' be the corresponding facial cycle in G' and let C_1 and C_2 denote the 2 facial cycles in G' that intersect with C'. Let u, c denote the 2 vertices of degree of degree 3 in F' that are incident with C'. Let e_u, e_v denote the edges incident with u and v that are not incident with C'. Figure 2.9 provides an example.

The edges e_u, $e_v \subseteq E(C_1 \cap C_2)$. e_u and e_v also form a 2-edge cut. By 3, we know that one of the cycles must be short and the other must be long. We will assign C_1 as short and C_2 as long. By 1, C' must also be long. From this information, we form the following inequality: $2k+9 < |E(C')| = |E(C' \cap C_1)| + |E(C' \cap C_2)|$. This implies that $|E(C' \cap C_2)| > k+9$, which contradicts 3. Therefore, every facial cycle of H must have length at least 3.

We now know that the minimum length of a facial cycle in H is 3. By 1, the faces in X are pairwise non-adjacent (46), thus $n = |V(H)| \geq \sum_{F \in X} lh(F) \geq 3x$. This implies that the vertices of H are greater than or equal to the sum of the lengths of the short faces of H, which is greater than or equal to 3x. This implies that $\sum_{F \in Y} lh(F) = \sum_{F \in F(G')} lh(F) - \sum_{F \in X} lh(F) = 2|E(H)| - \sum_{F \in X} lh(F) \leq 3n - 3x$. This means that the sum of the lengths of the long faces of H is equal to the sum of the length of the faces of H minus the sum of the lengths of the short faces of H is equal to 2*(the total edges of H) - the sum of the lengths of the short faces of H $\leq$ 3*(vertices of H) - 3(the number of short faces of G'). In G', each edge incident with a face in Y is also an edge in H or it is incident with a face in X, so by the previous inequality, $\sum_{F \in Y} l(F) \leq \sum_{F \in Y} lh(F) + \sum_{F \in X} l(F) \leq 3(n-x) + (k-1)x = 3n + (k-14)x$.

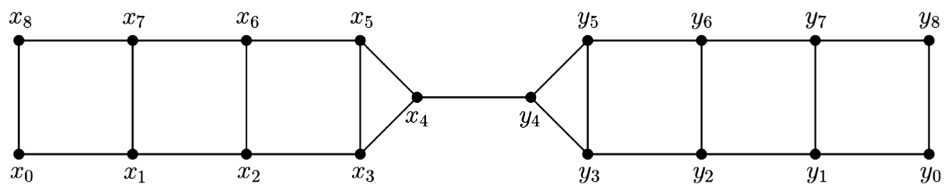

Figure 2.10: Merker's depiction of H_4.

Each face in Y has length at least $2k + 9$ in G', so $\sum_{F \in Y} l(F) \geq (2k + 10)y = (2k + 10)(\frac{n}{2} + 2 - x) > (k + 5)n - (2n + 10)x$. Combining these 2 inequalities, you get $(k+5)n - (2k+10)x < 3n + (k-4)x$, which simplifies to $(k+2)n < 3(k+2)x$, implying that $n < 3x$. This contradicts one of the previous inequalities, ending the proof. Thus, If $k \in \mathbb{N}$ and G is a 3-connected cubic planar graph of circumference at least k, then $C(G) \cap [k, 2k + 9] \neq 0$

Beyond proving the main conjecture, Merker proves that for $k \in \mathbb{N}$, with $k \geq 2$, there exists a 2-connected cubic planar graph G of circumference at least $2k$ such that $C(G) \cap [2k, 4k + 1] = 0$. He proves this to show that $[k, 2k + 9]$ is close to the optimal bound for the main conjecture.

Now, Merker wants to look specifically at graphs with large cycle graphs, by setting $k \geq 2$. He now claims that for every $k \geq 2$, the interval $[2k, 4k+1]$ is a gap of a 3-connected cubic planar graph and thus, [k, 2k+9] is close to optimal. In his main conjecture, Merker is proving that there is not a gap on [k, 2k+9], so by proving this new conjecture, he is saying that there must be a cycle that has a length in [k, 2k). Merker proves it as so:

Theorem: For $k \in \mathbb{N}$ with $k \geq 2$ there exists a 3-connected cubic planar graph G of circumference at least $2k$ such that $C(G) \cap [2k, 4k + 1] = 0$.

Before looking at Merker's proof, we will examine this new theorem. If the circumference of the graph G is at least 2k, we know that it has to be greater than $4k + 1$ since there is no cycle of G in the interval [2k, 4k+1].

For $k \in \mathbb{N}$, let H_k be the graph consisting of two disjoint cycles $C_1 = x_0, x_1, ..., x_{2k}, x_0$ and $C_2 = y_0, y_1, ..., y_{2k}, y_0$, with chords $x_i x_{2k-i}$ and $y_i y_{2k-i}$, for $i = \{1, ..., k-1\}$, as well as an edge joining C_1 and C_2. Merker provides an example, H_4, shown in Figure 2.10. I have also created my own example, H_5, shown in Figure 2.11

Now, let D be a cubic graph and have $e \in E(D)$. D(e, H_k) is the graph obtained from D by replacing e with a copy of H_k. Figure 2.12 provides an example of this edge replacement, using the cube graph as D. Suppose M is a matching (52) in D. Figure 2.13 shows 2 examples of matchings of the cube graph, M_1 and M_2. M_1 is a perfect matching while M_2 is not. Now we define D(M, H_k) as the graph we obtain by successively replacing the edges in M by H_k. Figure 2.14 shows D(M_1, H_4) and D(M_2, H_4). It can also be noted that by using any perfect matching M (53) of D and H_1, you get D in which all of its vertices are replaced with triangles, as shown in Figure 2.15.

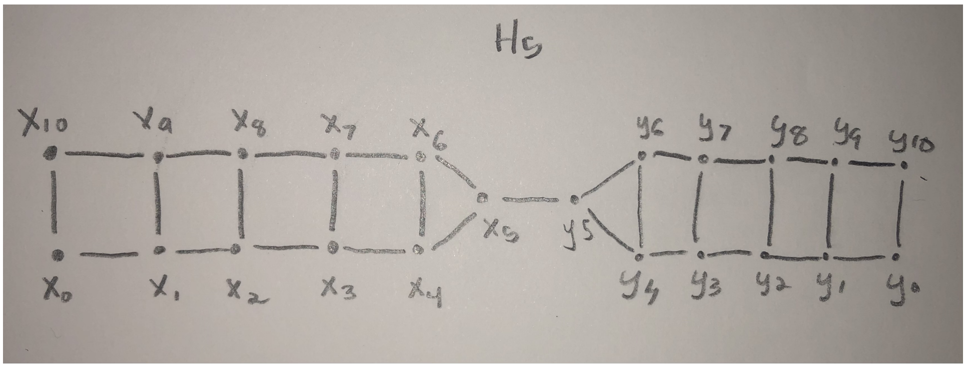

Figure 2.11: H_5

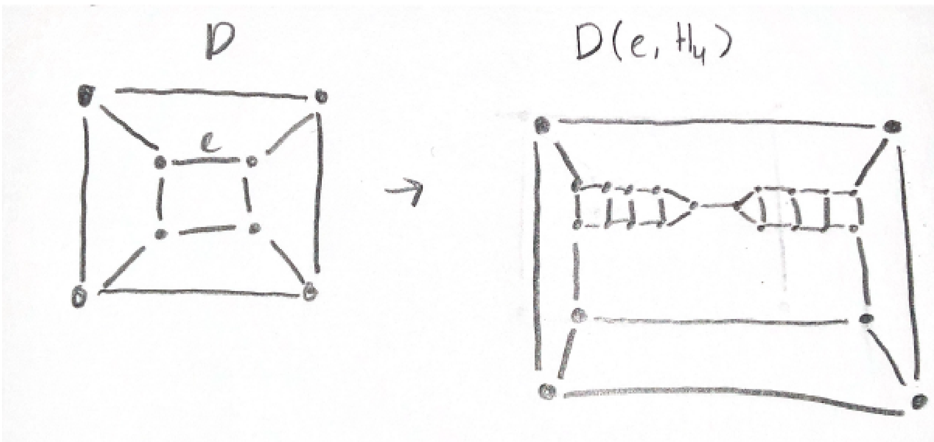

Figure 2.12: The edge e of the cube graph D being replaced by H_4.

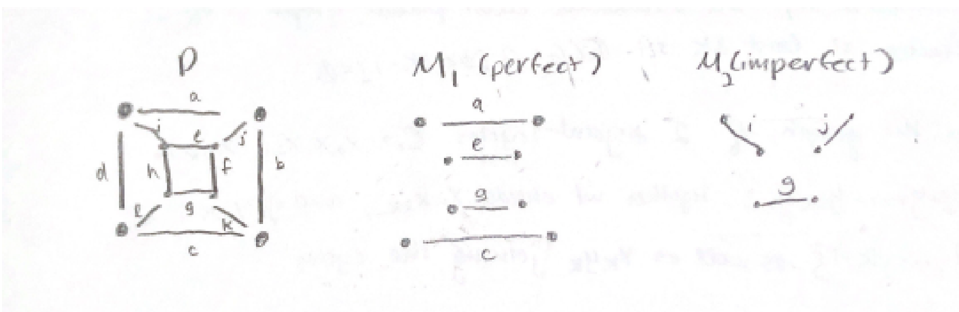

Figure 2.13: The cube graph D and 2 of its possible matchings, M_1 and M_2.

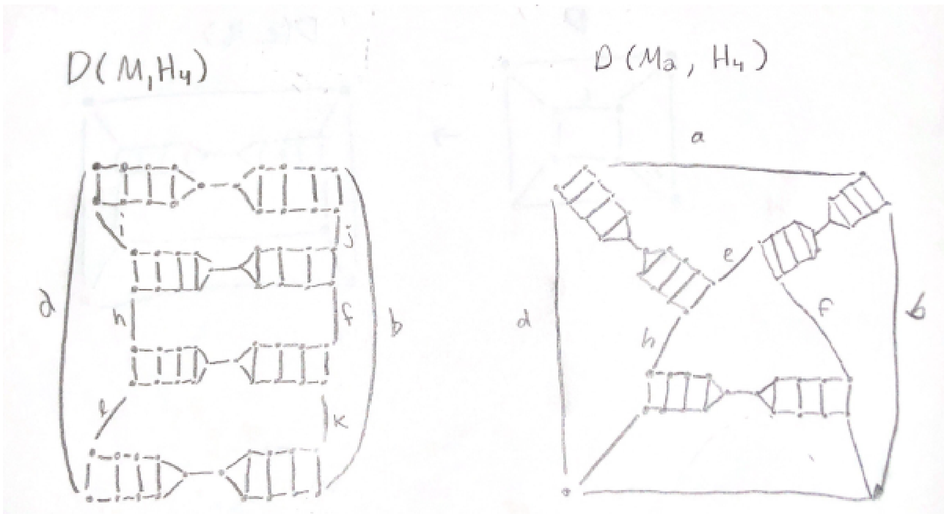

Figure 2.14: D(M_1, H_4) and D(M_2, H_4)

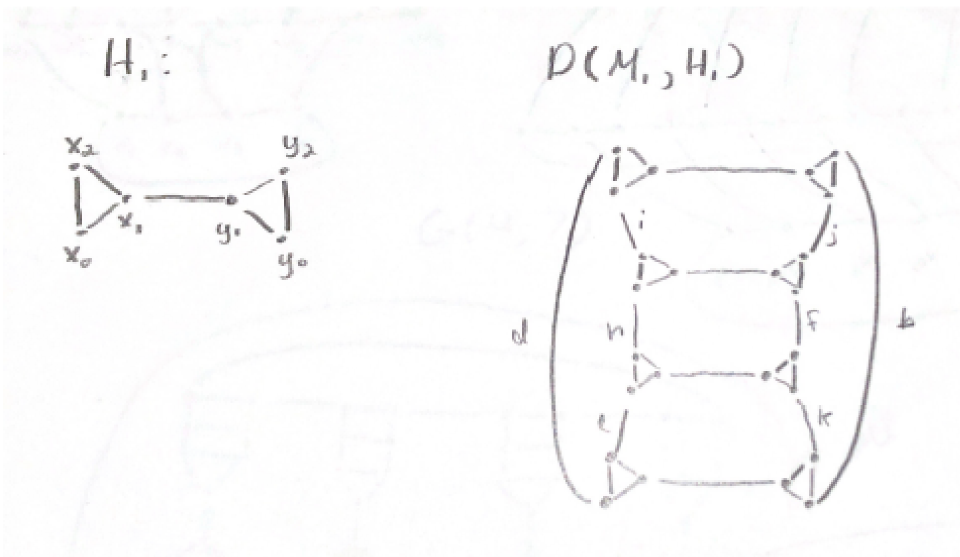

Figure 2.15: H_1 and D(M_1, H_1)

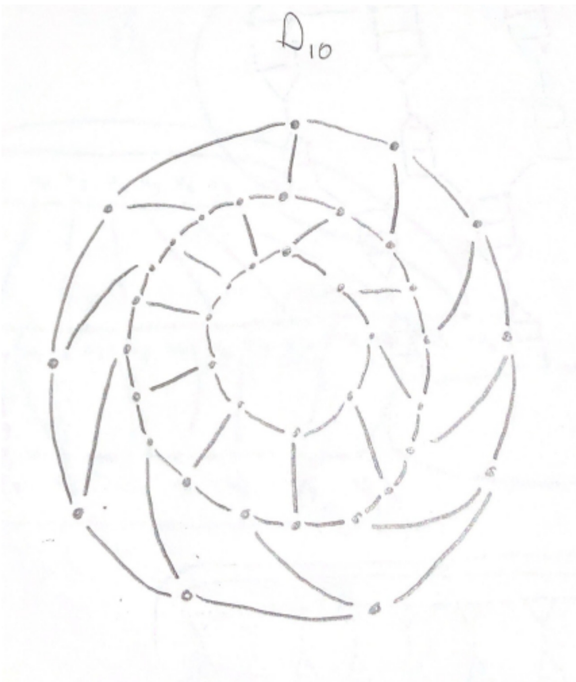

Figure 2.16: D_{10}

For $n \in \mathbb{N}$, $n \geq 4k + 2$, let D_n be a graph consisting of three cycles $C_1 = u_1, ..., u_n, u_1$, $C_2 = v_1, ... v_n, v_1$, and $C_3 = w_1, ..., w_{2n}$, together with edges $v_i w_{2i+1}$ and $u_i w_{2i}$ for all $i \in \{1, .., n\}$. Figure 2.16 shows D_{10} and Figure 2.17 shows D_4.

Now, let M denote the perfect matching in D consisting of the edges in the form $v_i w_{2i-1}$ and $u_i w_{2i}$ for $i \in \{1, ..., n\}$. Figure 2.19 shows the perfect matching, as described, for D_{10}. Now define G(n,k) = D_n(M, H_{k-1}). Merker uses the example G(4, 3), shown in Figure 2.18. I have also created an example, G(10, 2), shown in Figure 2.20. It is easy to see that both G(4, 3) and G(10, 2) are 3-connected cubic planar graphs, as are all graphs in this format.

G(n,k) is a 3-connected cubic planar graph, and since n $\geq$ 24k +2, the circumference of G(n,k) is greater than 2k, but is C(G(n, k)) $\cap$ [2k, 2k+1] = 0?

Since $n \geq 4k + 2$, the circumference of $G(n, k)$ is greater than $2k$, now we must show that $C(G(n, k)) \cap [2k, 4k + 1] = 0$. Let C be a cycle of $G(n, k)$ and suppose the length of the cycle is between 2k and 4k + 1. As a reminder, $G(n, k) = D_n(M, H_{k-1})$. We know that the circumference of $H_{k-1} = 2k - 1$, so since $2k - 1 < 2k$, C isn't in one of the copies of H_{k-1}. Thus, C' corresponds to a cycle in D, if we compress each of the H_{k-1}'s into vertices. For an example, Figure 2.20 would be compressed into Figure 2.15. However, it is important to note that the length of C is greater than C'. The shortest cycle of D not containing any edge of M has length n. $2n > n$, so C' must contain at least one edge in M. However, there are no cycles that contain only one edge of M. So, C' must contain two edges in M. Since edges are used from M, we must remember that we compressed

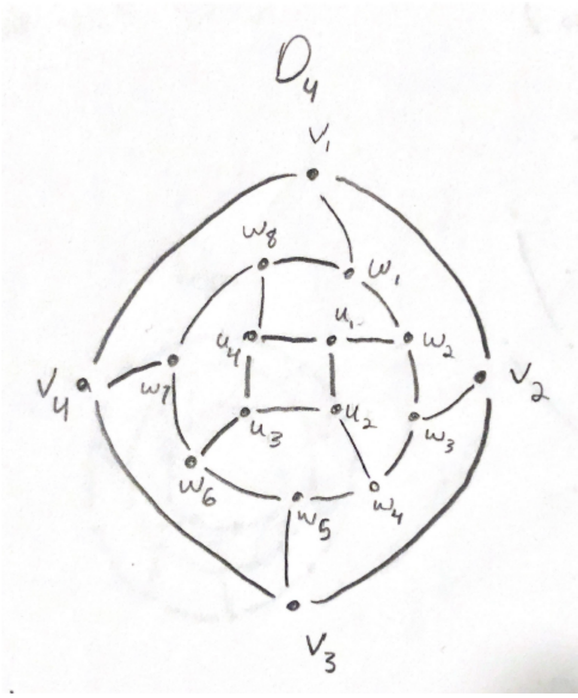

Figure 2.17: D_4

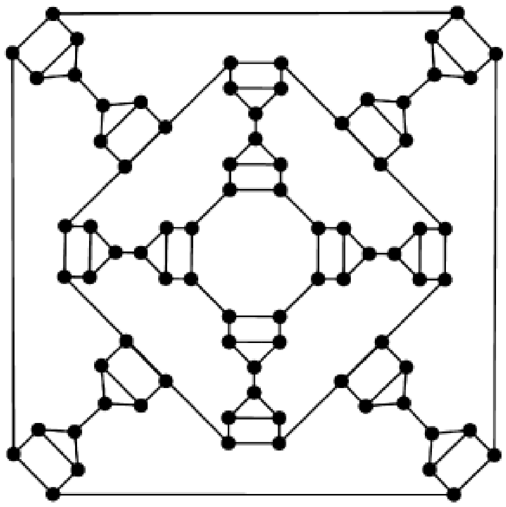

Figure 2.18: G(4,3) [1]

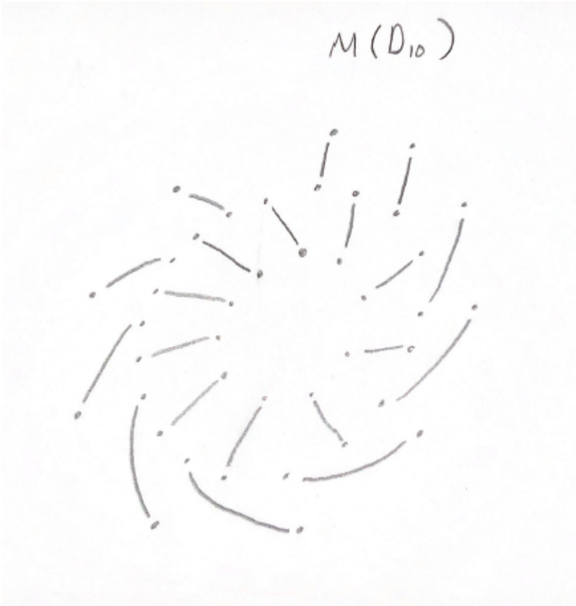

Figure 2.19: A perfect matching of D_{10}.

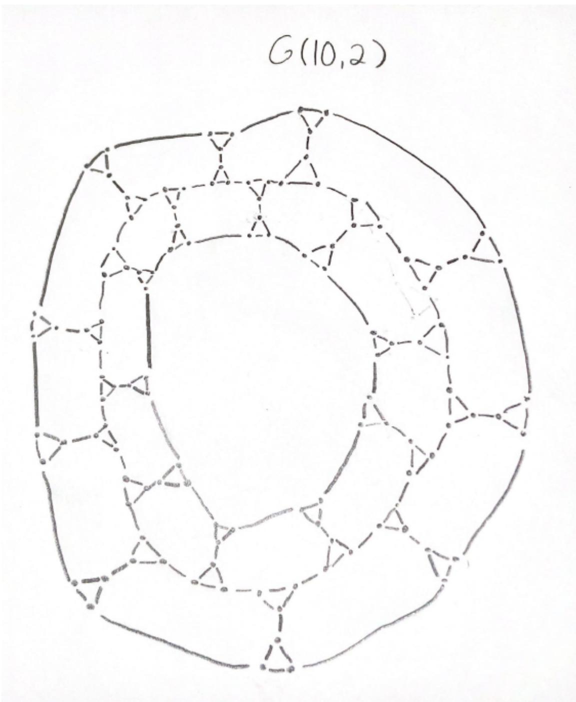

Figure 2.20: G(10,2)

the H_{k-1}'s into vertices. The edges specifically correspond to paths between the vertices X_0 and Y_0, which have length at least $2k-1$. For a specific example, refer to the path from x_0 to y_0 in Figure 2.11. C' contains two consecutive edges that aren't in M. These two edges correspond to a path (22) of length at least 3 in C. Since the girth of D is 5, $|E(C)| \geq |E(C')| + 2(2k-2) + 1 \geq 5 + 4k - 3 = 4k + 2$. This contradicts our choice of C. Therefore, it is proven that $C(G) \cap [2k, 4k+1] = 0$ for $k \geq 2$.

Chapter 3: Counterexamples to a conjecture of Merker on 3-connected cubic planar graphs with a large cycle spectrum gap

In this chapter, we will be discussing Zamfirescu's paper "Counterexamples to a conjecture of Merker on 3-connected cubic planar graphs with a large cycle spectrum gap."

Merker claims that for any non-negative integer $k \geq 2$, where G is a 3-connected cubic planar graph, $C(G) \cap [k, 2k+2] \neq 0$. Zamfirescu claims that this does not hold for a certain infinite family of graphs where $k \geq 6$. Before explaining this infinity family of graphs, he proves that Merker's conjecture is valid for $2 \leq k \leq 5$ [3]).

By Euler's formula (51), every cubic plane graph contains a face of length 3, 4, or 5. We have proven this in Chapter 2. Zamfirescu states that this is enough to show that the conjecture holds for k = {2, 3}. If $k = 2$ for a graph G_1, then there must be a cycle C with length l, where $l \in [2, 6]$. This is true, as there is a facial cycle (50) of length 3, 4, or 5 for the graph G_1. If $k = 3$ for a graph G_2, then there must be a cycle C with length l, where $l \in [3, 8]$. This is true, as there is a facial cycle of length 3, 4, or 5 for the graph G_2.

This logic does not apply when $k = 4$ or $k = 5$. This is because the graph could have a facial cycle of size 3. Zamfirescu proves that $k = 5$ holds with Merker's conjecture through induction.

Suppose the conjecture is untrue for $k = 5$. Then, there exists a 3-connected cubic plane graph G of circumference (27) at least 5, where $C(G) \cap [5, 12] = 0$. If there exists a cycle of length 3 or 4 in G, it must be the boundary of a face. Any 2 facial cycles of size 3 or 4 must be disjoint, since $5 \notin C(G)$ and $6 \notin C(G)$. An example of why they must be disjoint is shown in Figure 3.1.

Now, Zamfirescu contracts every cycle of length 3 and 4 of G to a vertex to obtain G'.

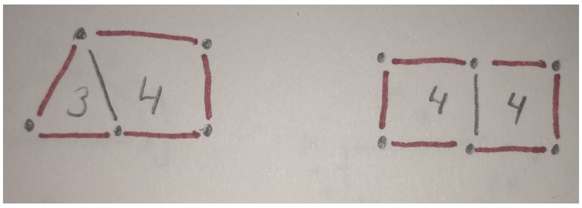

Figure 3.1: Examples of facial cycles of length 3 and 4 connected.

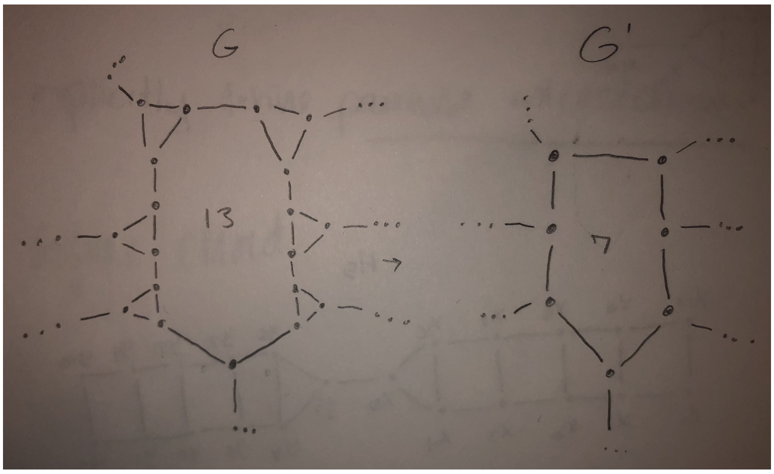

Figure 3.2: The transition from G to G'

An example of this process is shown in Figure 3.2.

Aside from the short cycles, G only has cycles of length greater than 13. So G' is a planar, 3-connected graph with no cycle of length less than 7. This is a contradiction, as there is now a cycle whose length is in [5, 12].

Zamrfirescu does not explain his argument for $k = 4$, but expresses that it is similar to his argument for $k = 5$. I will prove it now.

Suppose the conjecture is untrue for $k = 4$. Then, there exists a 3-connected cubic plane graph of G of circumference at least 4, with $C(G) \cap [4, 10] = 0]$. If there exists a cycle of length 3, then it must be the boundary of a face. Any 2 facial sizes of size 3 must be disjoint, since $5 \notin C(G)$. An example of why they must be disjoint is shown in Figure 3.3.

Contract every cycle of length 3 of G to a vertex to obtain G'. An example of this process is shown in Figure 3.4.

Aside from the short cycles, G only has cycles of length 11 or greater. So G' is a planar, 3-connected graph with no cycles of length less than 6. This is a contradiction, as there is now a cycle whose length is in $[4, 10]$.

While Zamfirescu agrees that Merker's conjecture holds for $k \in [2, 5]$, he now proves that the conjecture does not hold for an infinite family of graphs where $k \geq 6$, for k is even.

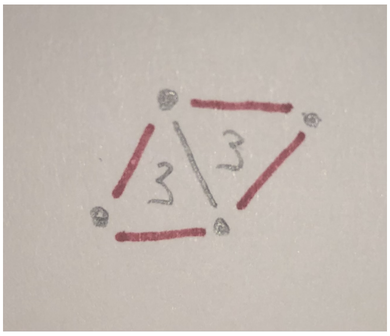

Figure 3.3: Two facial cycles of length 3 connected, creating a 4 cycle.

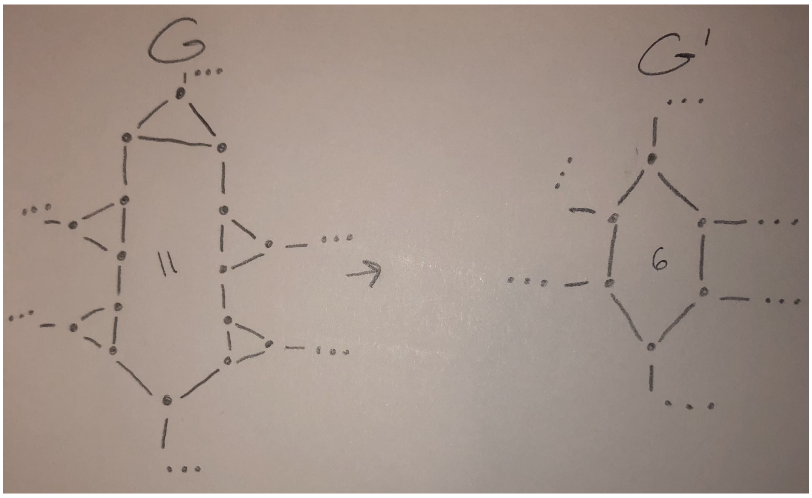

Figure 3.4: The transition from G to G'.

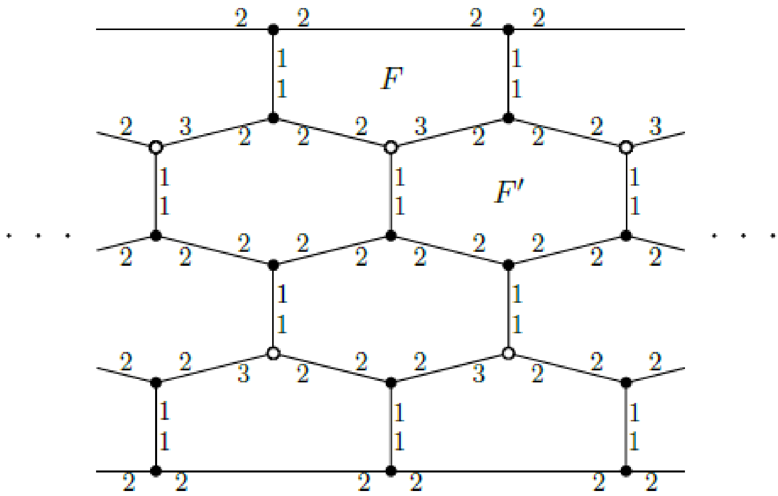

Figure 3.5: Zamfirescu's graph H [3]

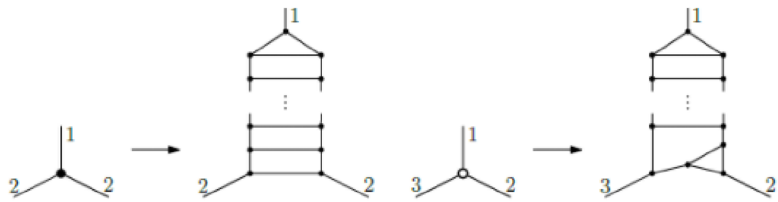

Figure 3.6: Zamfirescu's graph for showing the operations described as A and B [3].

"Theorem: For any integer $k \geq 6$, there exists an infinite family of 3-connected cubic planar graphs of circumference at least k, whose cycle spectrum contains no element of $[k, 2k+2]$."

Zamfirescu constructed the graph H, shown in Figure 3.5. The graph's "left-most and right-most parts should be identified in the obvious way, where the boundary cycles of the two faces incident only with pentagons (top and bottom of Figure 3.5) may have any length of at least $2r+8$ (this yields the advertised infinite family). The vertices of H are either black or white, as shown" [3]. The complete version of this graph can be very large, so has been omitted from the paper.

Now, consider the operations A and B, defined by Zamfirescu in Figure 3.6, where A is the first transformation and B is the second. Zamfirescu refers to all of the horizontal lines in the transformation as rungs, denoted r. In each operation, a cubic vertex of H is replaced with the plane graph A_r and B_r. Refer to Figures 3.7 and 3.8 for examples of the individual operations (showing A_4 and B_5 respectively).

To use the operations A and B on H, replace each black vertex with a copy of A_{r+2} and each white vertex with a copy of B_r, respecting the orientations in the Figure 3.5 by the numbers 1, 2, and 3. Figure 3.9 shows an example of this, with $r = 4$.

The graph remains 3-connected and cubic. The circumference A_{r+2} and B_r is $2r+5$.

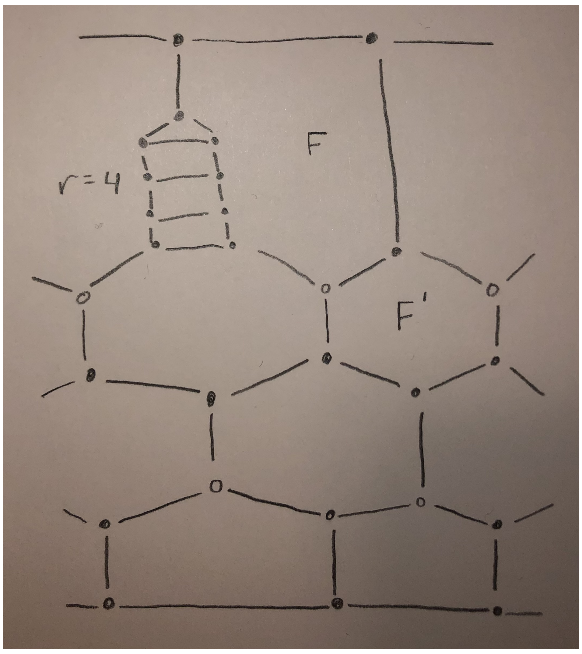

Figure 3.7: The operation A, for r = 4.

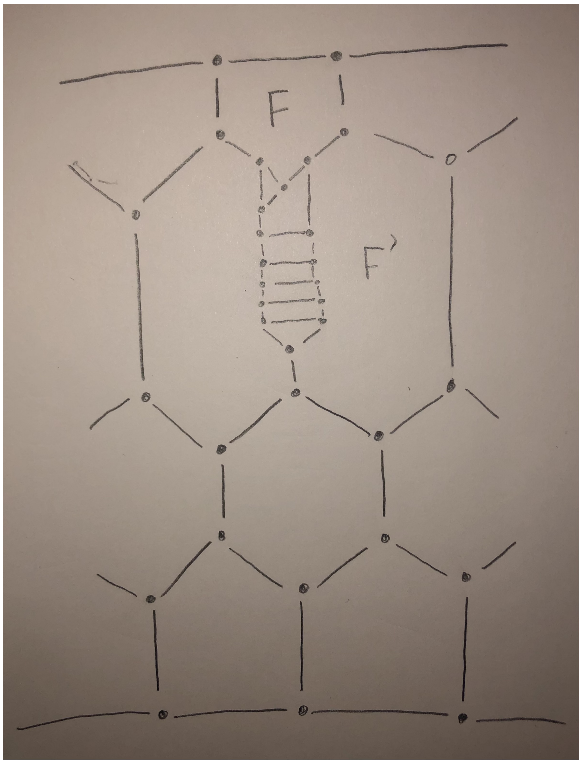

Figure 3.8: The operation B, for r = 5.

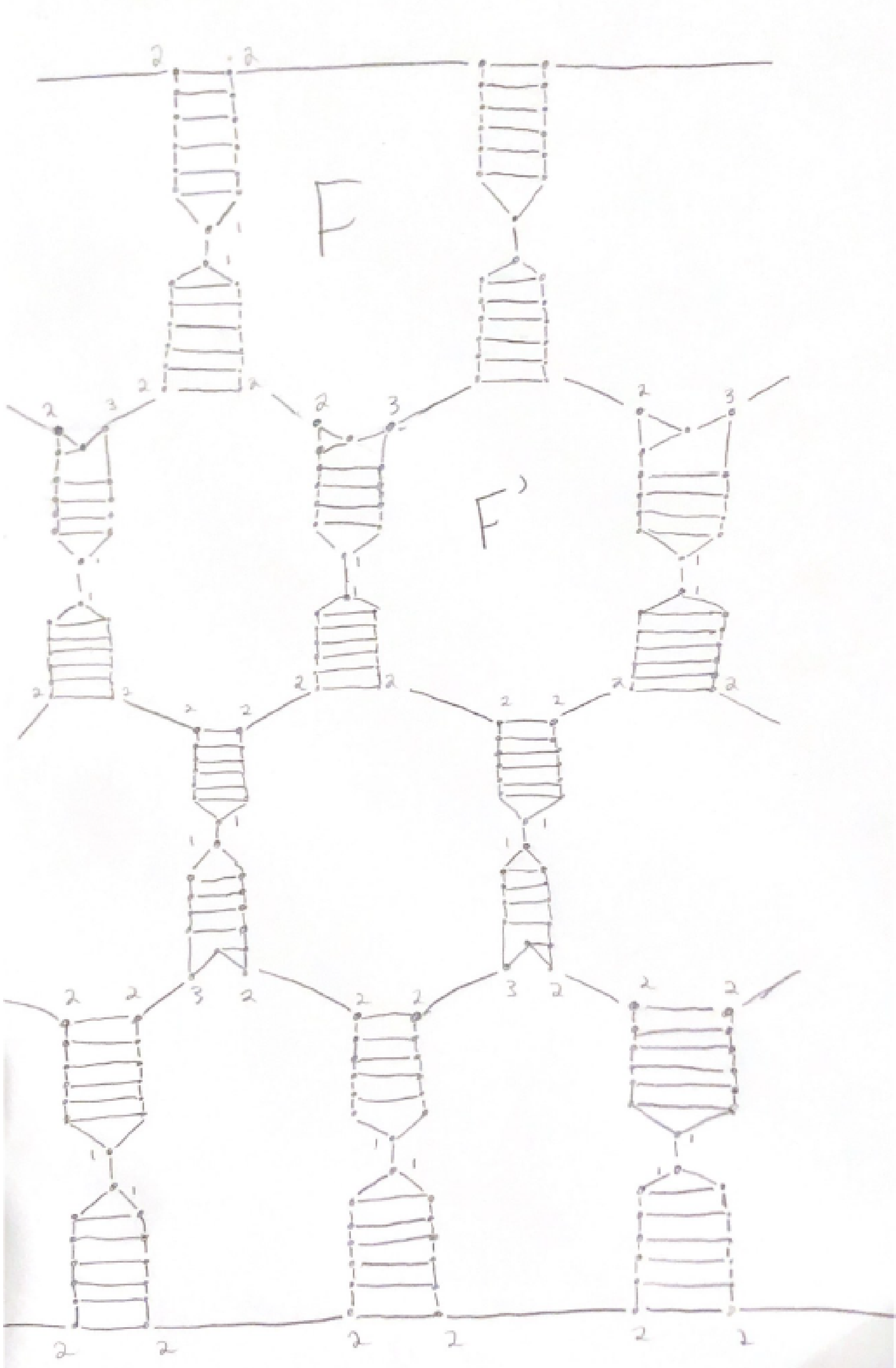

Figure 3.9: A graph in which all of the operations A and B are completed. The graph is incomplete.

This works with the example of $r = 4$ in Figure 3.9. The circumference of A_6 is 13, and the circumference of B_4 is 13. By construction, any cycle in G of length greater than $2r + 5$ has length at least $4r + 15$, which is the length of the cycle bounding the face F and also of the cycle bounding the face F'. The example in Figure 3.9 holds for this condition. The facial cycles of F and F' both have length 31, which is $4r + 15$ for $r = 4$. Thus, for every $l \in 2r + 6, ..., 4r + 14$, the graph G contains no cycle of length l. Now, set $k = 2r + 6$. Merker's conjecture states that for a cubic, planar, 3-connected graph and $K \geq$, $C(G) \cap [k, 2k + 2] \neq 0$. However, for $k = 2r + 6$, there is no cycle in the range $[2r + 6,\ 4r + 12]$. Therefore, this is a valid counterexample to Merker's conjecture.

Zamfirescu also discusses Figure 2.19 from Merker's paper. He discusses how Merker does not satisfy the own rules of his problem. In his proof, Merker uses the graph G(4, 3). However, the chosen $k = 4$ and $n = 3$ do not satisfy the condition $n \geq 4k + 2$.

Chapter 4:   Future Research

The most apparent next step to take in this problem would be to confirm that the bound [k, 2k+9] (for $k \geq 2$) is optimal. Zamfirescu discovered an infinite set of counterexamples for every even integer $k \geq 6$. Who is to say that there isn't a case where the odd integers above 6 provide a counterexample, or even integers less than 6. Finding such a set, or proving that there never could be one, should be the next course of action. It would also be worthwhile to find further examples of graphs to confirm Merker's conjecture.

Chapter 5: Glossary

Definition 1 ***Vertex (node):*** *The fundamental component of graphs. For a graph to exist, it must contain at least one vertex. In a diagram,a vertex is often represented as a point and labeled with a letter. In Figure 5.1, graph G contains the vertices A, B, C, D, F, and G.*

Definition 2 ***Edge:*** *A connection that exists between two vertices. In a diagram, an edge is represented as a straight line. They are often denoted as two letters, which are the vertices that are joined by the edge. In Figure 5.1, G has edges AB, AB, AD, DC, AC, BC, FG, and BD.*

Definition 3 ***Adjacent:*** *Two vertices are adjacent if there is an edge between them. In Figure 5.1, C and B are adjacent vertices, while H and K are not.*

Definition 4 ***Incident:*** *Two edges are incident if there exists a vertex between them. In Figure 5.1, AB and DC are incident edges, while KH and LI are not.*

Definition 5 ***Degree of a vertex:*** *The amount of edges joined to a vertex. In Figure 5.1, the degree of J is 4, the degree of F is 1, and the degree of A is 4.*

Definition 6 ***Average Degree:*** *The degree of each vertex of a graph divided by the total number of vertices.*

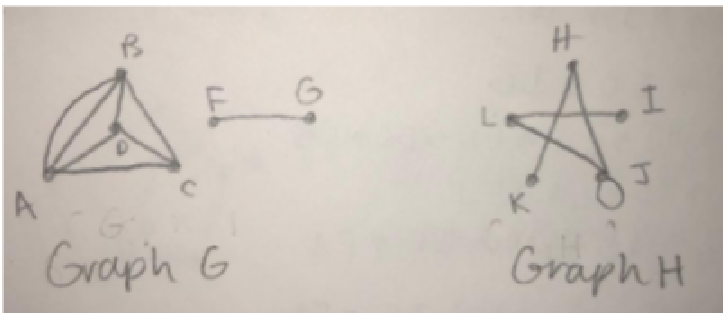

Figure 5.1: Graph G and H

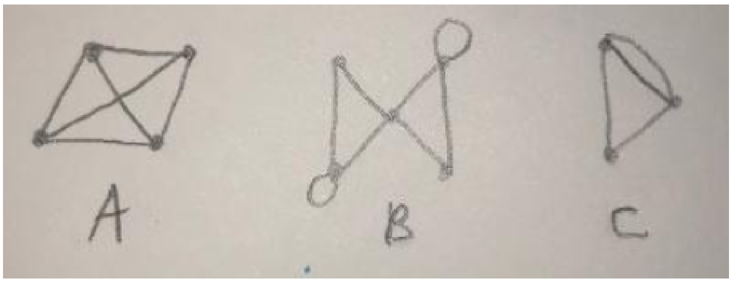

Figure 5.2: Graphs A, B, and C

Definition 7 ***Handshaking Lemma:*** *For all graphs G, the sum of the degrees of all of the vertices in V(G) is an even number. The sum of the degrees is also exactly twice the amount of edges in E(G). From the handshaking lemma, we also find that there is always an even number of vertices with odd degrees in a graph.*

Definition 8 ***Multiple edges:*** *When two vertices have more than one edge going between them. In Figure 5.1, there is a multiple edge in graph G between A and B*

Definition 9 ***Loop:*** *An edge joining a vertex to itself. In a diagram, a loop is often represented as a circle attached to its vertex. In Figure 5.1, there is a loop in graph H at vertex J.*

Definition 10 ***Vertex Set:*** *The set of vertices contained in a graph G, denoted as V(G). In Figure 5.1, V(G) = {A, B, C, D, F, G} and V(H)= {H, I, J, K, L}.*

Definition 11 ***Edge Set:*** *The set of edges contained in a graph G, denoted as E(G). In Figure 5.1, E(G) = {AB, AB, AD, DC, AC, BC, FG, BD} and E(H) = {KH, HJ, JJ, JL, JI}.*

Definition 12 ***Graph:*** *A graph G is composed of a vertex set and an edge set. The edge set can be empty, while the vertex set must have at least one element. Graphs can also contain multiple loops and edges. There are many unique properties of graphs, such as connectivity, planarity, colorability, etc. Each of these properties exists to explain the meaning behind graphs and the patterns that form between them.*

Definition 13 ***Simple graph:*** *A simple graph is particularly defined as a graph with no loops or multiple edges. In some instances, simple graphs are just called graphs. In Figure 5.2, A is an example of a simple graph, as it has no loops or multiple edges. However, both B and C are not simple graphs, as B has loops and C has multiple edges.*

Definition 14 ***Empty Graph:*** *A graph G in which E(G) is empty and V(G) has at least one element. This means that the graph contains no edges: only vertices.*

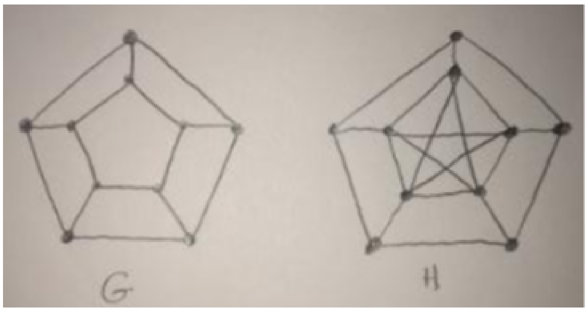

Figure 5.3: Graphs G and H

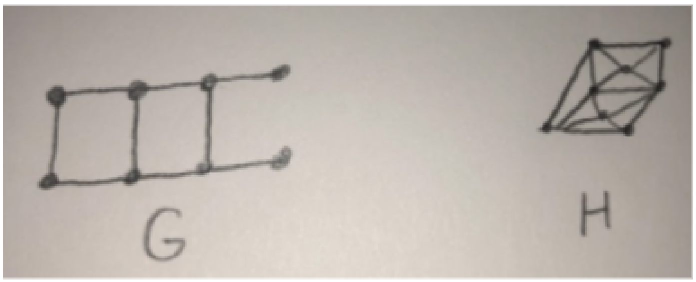

Figure 5.4: Graphs G and H

Definition 15 ***Regular graph:*** *A graph where each vertex has the same degree. In Figure 5.3, G is an example of a regular graph, while H is not. The degree of each vertex in G is 3, while H has vertices of degree 3 and 4.*

Definition 16 ***Cubic graph:*** *A regular graph of degree 3. In Figure fig53, G is an example of a cubic graph.*

Definition 17 ***Subcubic graphs:*** *A graph in which each vertex has a degree of 3 or less. In Figure 5.4, G is a subcubic graph since its vertices have degrees 1, 2, and 3. H is not subcubic, as it has vertices of degrees 3, 4, and 5.*

Definition 18 ***Subgraph:*** *A subgraph of a graph G is a graph whose vertices belong to V(G) and edges belong to E(G). To find a subgraph of any graph G, one can simply delete*

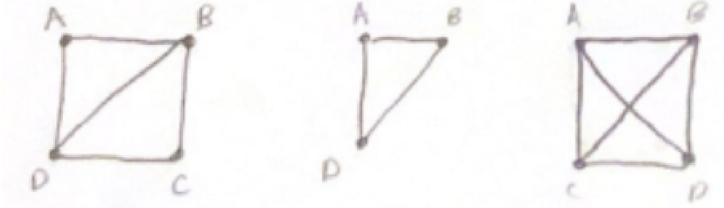

Figure 5.5: 3 graphs

some of its vertices and/or edges. In Figure 5.5, the second graph is a subgraph of the first, while the third is not a subgraph of the first.

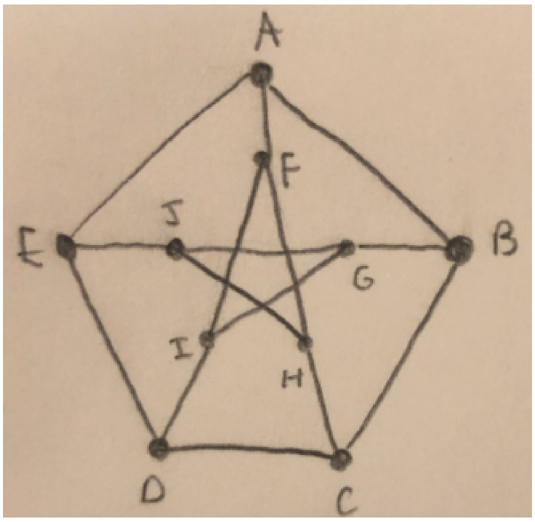

Figure 5.6: The Petersen graph

Definition 19 ***Walk:*** *A sequence of edges in which any two consecutive edges are incident or identical. In Figure 5.6, one example of a walk is AB → BG → GI→ IF → AF → AB → BC. A counterexample would be AB → IG → BC, as these edges are not incidental or identical.*

Definition 20 ***Length:*** *The number of edges in a walk. In Figure 5.6, the length of the walk AB → BG → GI is 3.*

Definition 21 ***Trail:*** *A walk in which all edges are distinct. In Figure 5.6, an example of a trail would be BC → DC → ID → IG. A counterexample would be DC → HC → HJ → EJ → ED → DC.*

Definition 22 ***Path:*** *A walk in which no vertex is crossed more than once. In Figure 5.6, an example of a path would be AB → BC → CD → ED → EJ. A counterexample would be FH → HJ → JG → GI → IF → FA, as the vertex F is crossed more than once.*

Definition 23 ***Closed:*** *A path or trail that ends on the same vertex it began on is considered closed. In Figure 5.6, AB → BC → CD → DI → IF → FA is a closed trail, since it ends on the same vertex it began on and does not repeat any edges.*

Definition 24 ***Cycle:*** *A closed path with at least one edge. In Figure 5.6, AB → BG → GJ → JH → HF → FA is an example of a cycle.*

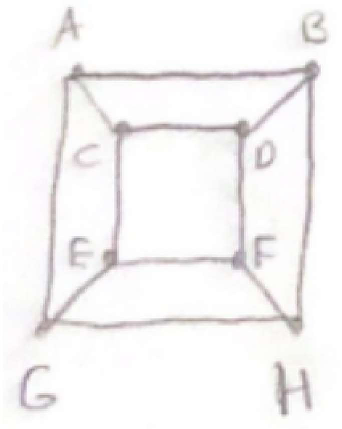

Figure 5.7: The cube graph

Definition 25 ***Cycle spectrum:*** *The set of cycle lengths in a graph G, denoted C(G). In Figure 5.6, the cycle spectrum of the graph is C(G) = 5, 6, 8, 9.*

Definition 26 ***Girth:*** *The length of a graph G's shortest cycle. The girth of the graph in Figure 5.6 is 5.*

Definition 27 ***Circumference*** *The length of a graph G's longest cycle. In Figure 5.6, the circumference is 9.*

Definition 28 ***Spectrum Gap:*** *An interval [a, b] is a gap of a graph G if G has a circumference greater than b and the cycle spectrum of G contains no elements from the interval [a, b].*

Definition 29 ***Symmetric Difference:*** *The symmetric difference of cycles is the set of edges that are in the cycles C and D but not in their intersection. For example, in Figure 5.6, consider the cycles* $C = A \to F \to H \to I \to E \to A$ *and* $D = B \to G \to I \to F \to A \to B$*. The symmetric difference between them would be* $\{AF\}$*.*

Definition 30 ***Hamiltonian graph:*** *A graph G in which there exists a cycle passing exactly once through each vertex of G. The cycle is specifically referred to as the Hamiltonian cycle of G. Figure 5.7 is Hamiltonian; its Hamiltonian cycle is AB → BD → CD → CE → EF → FH → HG → GA.*

Definition 31 ***Connected graph:*** *A graph where each pair of vertices has a path between them. Both G and H in Figure 5.8 are connected graphs since each of their vertices have paths between them. F is not connected since there is no path between Q and S (among other vertex pairs).*

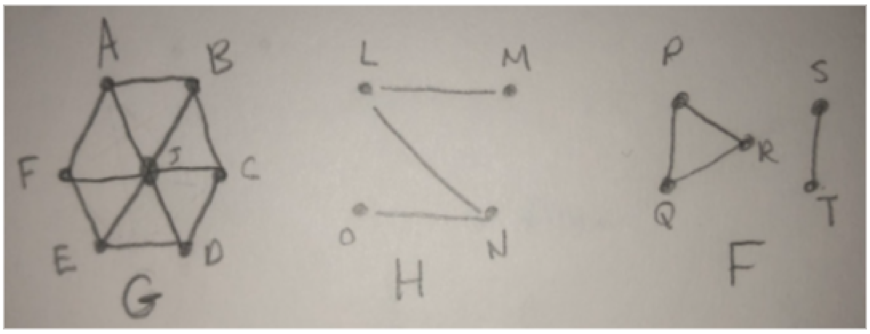

Figure 5.8: Graphs G, H, and F

Definition 32 ***Disconnected graph:*** *A graph in which a path does not exist between each pair of its vertices. In Figure 5.8, F is a disconnected graph, as there is no path between P and S. G and H are both connected.*

Definition 33 ***Disconnecting set:*** *A set of edges whose removal disconnects a connected graph G. if the graph is already disconnected, the disconnecting set increases the number of components in the graph. An example of a disconnecting set in graph G of Figure 5.8 is JF, JC, JB, JA, JB, FA, BC.*

Definition 34 ***Cutset:*** *A disconnecting set such that no subset of the cutset is a disconnecting set. In Figure 5.8, one cutset of G is {FA, AJ, JB, BC}. This separates the edge AB from the rest of the graph.*

Definition 35 ***Edge connectivity:*** *The minimum amount of edges you need to delete in order to disconnect G, denoted as $\lambda(G)$. It is also the size of the smallest cutset. The edge connectivity of graph G in Figure 5.8 is 4, while the edge connectivity of graph H is 1.*

Definition 36 ***K-edge connected:*** *A graph is k-edge connected if k is less than or equal to its edge connectivity. In Figure 5.8, G is 4-edge connected and H is 1-edge connected.*

Definition 37 ***Separating set:*** *A set of vertices whose deletion disconnects G. In Figure 5.8, {J, F, C, E} is an example of a separating set of G.*

Definition 38 ***Connectivity:*** *The size of the smallest separating set in G, denoted $\kappa(G)$. In Figure 5.8, the connectivity of G is 3, as the smallest separating set is {F, J, C}.*

Definition 39 ***K-connected:*** *G is k-connected if k is less than or equal to its connectivity . In Figure 5.8, graph G is 3-connected.*

Definition 40 ***3-connected graph:*** *A graph G in which the size of the separating set is 3 (you must eliminate 3 vertices in order to disconnect G). In Figure 5.8, G is an example of a 3-connected graph, while H and F are not 3-connected.*

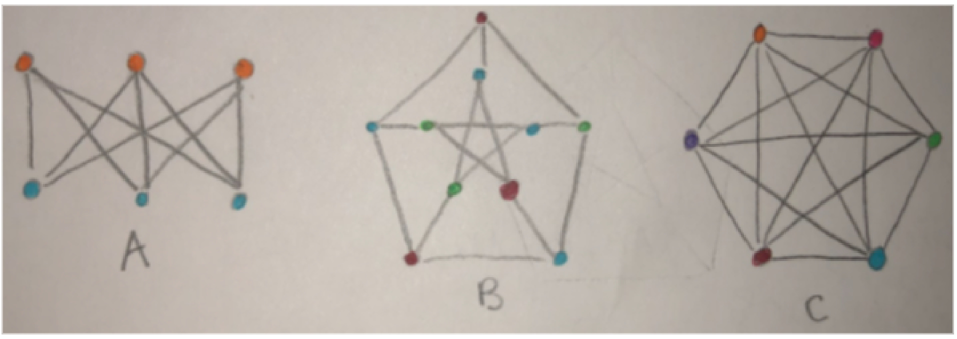

Figure 5.9: Graphs A, B, and C with colored vertices

Definition 41 ***K-colorable:*** *For a graph G without any loops, we can assign one of k colours to each vertex of G so that adjacent vertices have different colors. in Figure 5.9, A is 2-colorable, B is 3-colorable, and C is 6-colorable, as shown.*

Definition 42 ***Chromatic number (k-chromatic):*** *If a graph G is k-colorable, but not (k-1)-colorable, then its chromatic number is k (AKA, the chromatic number is the minimum amount of colors that can be used with the vertices of the graph). In Figure 5.9, the chromatic numbers of A, B, and C are 2, 3, and 6*

Definition 43 ***Bipartite graph:*** *A 2-colorable graph. In practical use, its vertices are often split into two groups: one colored black and the other colored white. The black vertices are only adjacent to white vertices and vice versa. In Figure 5.9, A is bipartite while B is not.*

Definition 44 ***Bijection:*** *A bijection between two sets implies that each element of one set is paired with only one element from the other.*

Definition 45 ***Pairwise Intersection:*** *Pairwise intersection, in relation to cycles, is the intersection of each of the pairs of cycles in a graph. For example, in Figure 5.8 graph G, consider the pair $A \to B \to J \to B$ and $B \to C \to J \to B$. Their intersection is BJ. This could be done with each pairs of cycles in G. While in some cases this could be referring to the vertices of the cycles, we will assume it means edges for the scope of this paper.*

Definition 46 ***Pairwise Non-adjacent:*** *Pairwise non-adjacent, in relation to faces, implies that any pair of faces are non-adjacent by an edge. For example, in Figure 5.11* *the pairs A and C, and B and D, are pairwise non-adjacent. In some cases, this could be looked at with vertices, however this changes the scope of the problem. For example, in Figure 5.11, that would cause the pairs previously mentioned to be adjacent.*

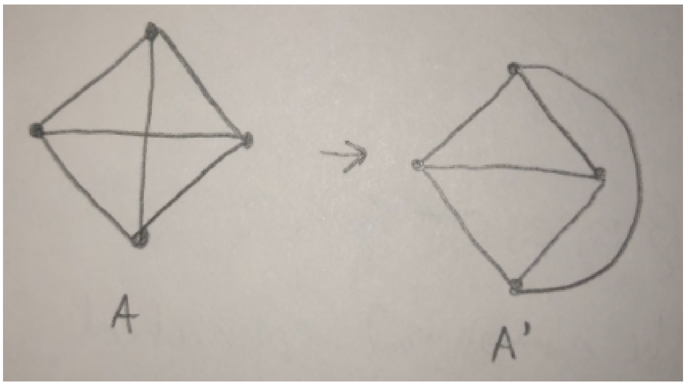

Figure 5.10: Graphs A and A'

Definition 47 ***Plane Graph:*** *A graph that has no crossings as drawn in the plane (no two edges intersect geometrically except at a vertex to which both edges are incident). In Figure 5.10, A' is a plane graph, while A is not.*

Definition 48 ***Planar graph:*** *A graph that can be drawn in the plane with no crossings. In Figure 5.10, both A and A' are planar graphs, since both can be drawn in the plane without crossings. They are the same graph, except A is drawn with crossings while A' is not.*

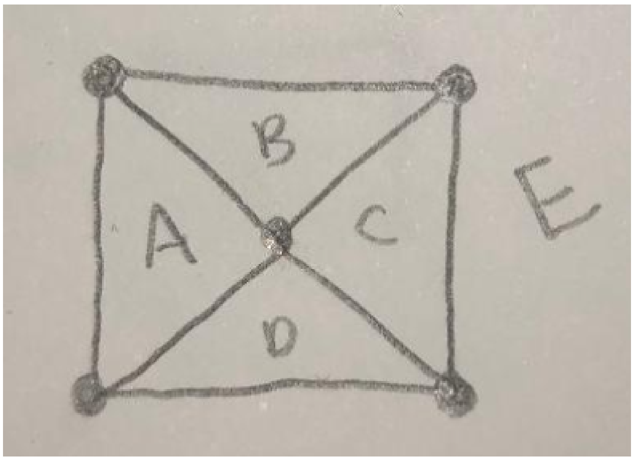

Figure 5.11: A graph with labelled faces

Definition 49 ***Faces:*** *Regions within a graph that are separated by edges. In Figure 5.11, there are 5 faces: A, B, C, D, and E. E is considered the infinite face, as it is made up of the plane surrounding the graph.*

Definition 50 ***Facial Cycle:*** *A cycle that creates a border around a face. For example, in Figure 5.8 graph G, the cycle AB, BJ, JA is a facial cycle.*

Definition 51 ***Euler's Formula:*** *Let G be a plane drawing of a connected planar graph, and let n,m, and f denote respectively the number of vertices, edges, and faces of G. Then, n - m + f = 2. In Figure 5.11, the formula holds true. There are 5 vertices, 8 edges, and 5 faces. 5 - 8 + 5 = 2 is correct.*

Definition 52 ***Matching:*** *Let G(V, E) be a graph. A matching of G is a set of edges in G that are not adjacent. In 5.8, specifically the graph G, the edges AB, FJ, and ED make up a matching of G.*

Definition 53 ***Perfect Matching:*** *Let G(V, E) be a graph. A perfect matching of G is a set of edges in G that are not adjacent and are incident with all the vertices in G. For example, in 5.8, The edges LM and ON make up a perfect matching of G, since they are not adjacent yet contain each of the vertices of H.*

Bibliography

[1] Martin Merker. "Gaps in the cycle spectrum of 3-connected cubic planar graphs". In: *J. Combin. Theory Ser. B* 146 (2021), pp. 68–75. ISSN: 0095-8956. DOI: 10.1016/j.jctb.2020.08.002. URL: https://doi.org/10.1016/j.jctb.2020.08.002.

[2] Robin J. Wilson. *Introduction to Graph Theory*. New York: Prentice Hall/Pearson, 2010. ISBN: 027372889X 9780273728894.

[3] Carol T. Zamfirescu. *Counterexamples to a conjecture of Merker on 3-connected cubic planar graphs with a large cycle spectrum gap*. 2020. arXiv: 2009.00423 [math.CO].