Homogenization of an Elasto-Plastic Problem

by

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Abstract

This project presents the homogenization analysis for a static contact problem with slip dependent friction between an elastic body and a rigid foundation. The homogenization for the static eigenvalue problem associated to this model is studied. We prove that the eigenvalues are of order ε.

We obtain the limit problem for the contact model.

The analysis is carried out by using the Γ-convergence theory.

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1. Introduction

This project presents the homogenization analysis for a static contact problem with slip dependent friction between an elastic body and a rigid foundation. The homogenization for the static eigenvalue problem associated to this model is studied. We prove that the eigenvalues are of order ε.

We obtain the limit problem for the contact model.

The analysis is carried out by using the Γ-convergence theory.

The material is organized as follows:
1. in the first section a general presentation of the theory of Γ-convergence is provided.
2. the second section presents results concerning the heat transfer problem trough a three dimensional body divided in two parts by a hyperplane with periodically distributed small holes.
3. in the third section, the results presented in section 2 are generalized in order to analyze the elasto-plastic contact problem.
4.The last part is reserved for partial conclusions and future ideas about the spectral problem associated. It needs to be mentioned that the last section in this project is original and will be submitted for publication.

2. Elements of Γ-convergence of functionals

In this section we present the general results from the theory Γ-convergence of functionals,which will be used later on the work.
These results can be found in [10].

Definition 1 *Assume that X satisfies the first axiom of countability. Let (F_h) be a sequence of functions from X into $\overline{\mathbf{R}}$.*

The function $F' = \Gamma - \liminf_{h\longrightarrow\infty} F_h$ is characterized by the following properties:
(a)for every $x \in X$ and for every sequence (x_h) converging to x in X

$$F'(x) \leq \liminf_{h\longrightarrow\infty} F_h(x_h) \tag{2.1}$$

(b) for every $x \in X$ there exists a sequence (x_h) converging to x in X such that

$$F'(x) = \liminf_{h\longrightarrow\infty} F_h(x_h). \tag{2.2}$$

The function $F'' = \Gamma - \limsup_{h\longrightarrow\infty} F_h$ is characterized by the following properties *(c) for every $x \in X$ and for every sequence (x_h) converging to x in X it is*

$$F''(x) \leq \limsup_{h \longrightarrow \infty} F_h(x_h) \tag{2.3}$$

(d)for every $x \in X$ there exists a sequence (x_h) converging to x in X such that

$$F''(x) = \limsup_{h \longrightarrow \infty} F_h(x_h) \tag{2.4}$$

The sequence (F_h) Γ-converges to F if and only if $F' = F''$. Therefore $(F_h)\Gamma-$converges to F if and only if the following conditions are satisfied:
(e) for every $x \in X$ and for every sequence (x_h) converging to x in X it is

$$F(x) \leq \liminf_{h \longrightarrow \infty} F_h(x_h) \tag{2.5}$$

(f) for every $x \in X$ there exists a sequence (x_h) converging to x in X such that

$$F(x) = \lim_{h \longrightarrow \infty} F_h(x_h) \tag{2.6}$$

We give next some properties of Γ-limits.

Proposition 1 *Let $G : X \to R$ be a continuous function. Then*

$$\Gamma - \liminf_{h \longrightarrow \infty} (F_h + G) = \Gamma - \liminf_{h \longrightarrow \infty} F_h + G \tag{2.7}$$

$$\Gamma - \limsup_{h \longrightarrow \infty} (F_h + G) = \Gamma - \limsup_{h \longrightarrow \infty} F_h + G. \tag{2.8}$$

Proof. We will prove the first statement, the second one being similar.
So let $F = \Gamma - \liminf_{h \longrightarrow \infty} F_h$. We will show that

$$\Gamma - \liminf_{h \longrightarrow \infty} (F_h + G) = F + G.$$

To do this let $x \in X$ and $x_h \to x$. Now using the fact that G is a continuous function and the definition of F, we have

$$\liminf_{h \longrightarrow \infty} (F_h(x_h) + G(x_h)) = \liminf_{h \longrightarrow \infty} F_h(x_h) + G(x) \geq F(x) + G(x).$$

Next let $x \in X$. We have that there is a sequence (x_h) convergent to x in X such that

$$\liminf_{h \longrightarrow \infty} F_h(x_h) = F(x)$$

Then again by continuity of G we have $\liminf_{h\longrightarrow\infty}(F_h(x_h)+G(x_h)) = F(x)+G(x)$. Q. E. D

Next we introduce the minimizers and give a few properties related to the convergence of minimizers via Γ-convergence.

Definition 2 *For every function $F : X \to \overline{\mathbf{R}}$ we denote by $M(F)$ the (possibly empty) set of all minimizers of F in X ,i. e. ,*

$$M(F) = \{x \in X : F(x) = \inf_{y\in X} F(y)\}. \tag{2.9}$$

In order to state a complete result,which includes also the case where the functions F_h do not attain their minimum on X, we introduce the notion of $\varepsilon - minimizer$.

Definition 3 *Let $F : X \to \overline{\mathbf{R}}$ be a function and let $\varepsilon > 0$. An ε-minimizer of F in X is a point $x \in X$ such that*

$$F(x) \leq (\inf_{y\in X} F(y) + \varepsilon) \bigvee (-\frac{1}{\varepsilon}). \tag{2.10}$$

The set of all $\varepsilon - minimizers$ of F in X will be denoted by $M_\varepsilon(F)$.

It is clear that if $\inf_{y\in X} F(y) > -\infty$ and ε is small enough,then x is an ε-minimizer of F in X if and only if

$$F(x) \leq \inf_{y\in X} F(y) + \varepsilon \tag{2.11}$$

If $F \geq 0$ this is true for every $\varepsilon > 0$. The term $-1\backslash\varepsilon$ appears in the definition only to deal with the case $\inf_{y\in X} F(y) = -\infty$ in a unified way. For any $F : X \to \mathbf{R}$,it is easy to see that x is a minimizer of F in X if and only if x is an ε-minimizer of F in X for every $\varepsilon > 0$,i. e. ,

$$M(F) = \bigcap_{\varepsilon>0} M_\varepsilon(F) \tag{2.12}$$

Next we give three results about the convergence of minimizers without proofs. For the proofs the reader can consult [10]

Proposition 2 *For every $h \in \mathbf{N}$ let x_h be a minimizer of F_h in X(or more generally an ε_h-minimizer,where (ε_h) is a sequence of positive real numbers converging to 0). If (x_h) converge to x in X,then x is a minimizer of F' and F'' and*

$$F'(x) = \liminf_{h\longrightarrow\infty} F_h(x_h), \qquad F''(x) = \limsup_{h\longrightarrow\infty} F_h(x_h) \tag{2.13}$$

Proposition 3 *Assume that* (F_h) $\Gamma -$ *converges to a function* F *in* X. *For every* $h \in \mathbf{N}$ *be a minimizer of* F_h *in* X*(or more generaly,an* ε_h*-minimizer where* (ε_h) *is a sequence of positive real numbers converging to* 0*). If* x *is a clauster point of* (x_h)*,then* x *is a minimizer of* F *in* X *and*

$$F(x) = \limsup_{h \longrightarrow \infty} F_h(x_h) \tag{2.14}$$

If (x_h) *converges to* x *in* X*,then* x *is a minimizer of* F *in* X *and*

$$F(x) = \lim_{h \rightarrow \infty} F_h(x_h) \tag{2.15}$$

We give bellow a theorem for convergence of minimizers in case of an equi-coercive family of functionals.

Proposition 4 *Suppose that* (F_h) *is equi-coercive and* $\Gamma -$ *converge to a function* F*,with a unique minimum point* x_0 *in* X.
let (x_h) *be a sequence in* X *such that* x_h *is an* $\varepsilon_h -$ *minimizer for* F_h *in* X*for every* $h \in \mathbf{N}$*,where* ε_h *is a sequence of positive real numbers converging to zero.*
Then (x_h) *converge to* x_0 *in* X *and* $F_h(x_h)$ *converges to* $F(x_0)$.

We remember that we are in a very general setting,i. e X is a topological vector space over the real numbers. $F^{'}$ and $F^{''}$ have the same signification as above. We give next a few properties of $\Gamma -$ *limits* wich will be very useful further. The proofs for all these properties are elementary and can be found in [10].
Let F_h be a family of functionals considered like before.

Proposition 5 *If each function* F_h *is convex,then* $F^{''}$ *is convex.*

In general the above statement is not true for $F^{'}$.

Proposition 6 *Suppose that each function* F_h *is even. Then* $F^{'}$ *and* $F^{''}$ *are even.*

The analogue for odd functions does not hold in general.

Definition 4 *Let* p *be a real number. We say that function* $F : X \rightarrow \overline{\mathbf{R}}$ *is positively homogenuous of degree* p *if* $F(tx) = t^p F(x)$ *for every* $t > 0$ *and for every* $x \in X$.

Proposition 7 *Suppose that each function* F_h *is positively homogenuous of degree* p. *Then* $F^{'}$ *and* $F^{''}$ *are positively homogenuous of degree* p.

Definition 5 *We say that a function $F : X \to [0;\infty]$ is a (non-negative) quadratic form(with exetended real values) if there is a linear subspace Y of X and a symetric bilinear form $B : Y \times Y \to \mathbf{R}$ such that*

$$F(x) = \begin{cases} B(x,x) & \text{if } x \in Y \\ +\infty & \text{if } x \in X \backslash Y \end{cases}$$

Obviously we can see that every non-negative quadratic form is convex.

Proposition 8 *Suppose that (F_h) $\Gamma-converges$ to a function F,and that each function F_h is a non-negative quadratic form. Then F is a non-negative quadratic form.*

The last proposition we use is so called the local property of $\Gamma - limits$.

Proposition 9 *If two sequences of functions (F_h) and (G_h) coincide on an open subset U of X then their $\Gamma - lowerlimits$ as well as their $\Gamma - upperlimits$ coincide on U.*

3. Neuman's Strainer

In this chapter we present a practical problem in homogenization theory,called Neuman's Strainer. This problem models a heat propagation in a three dimensional body cutted by a hyperplane such that the heat can propagate only through small "holles" (barrieres) periodic distributed on the hyperplane. This model has a profound significance in acoustic also and was first described by Marchenko & Hruslov[Reference],Sanchez-Palencia[reference]. The physical model is the following: Let the domain $\mathcal{D}$ in $\mathbf{R}^3$ and the hyperplane Σ. $\mathcal{D}$ is divided in two subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$ by Σ. The two subdomains are connected trough many small "holles" (bariers) periodic distributed on Σ. Let $\Sigma_d = \partial\mathcal{D}\backslash\Sigma$. On Σ let us define an ε-periodic structure. In each small cell of size ε is centered a disc β_ε^i of radius $r_\varepsilon \leq \frac{\varepsilon}{2}$. Let us denote

$$\beta_\varepsilon = \bigcup_i \beta_\varepsilon^i \qquad \mathcal{D}_\varepsilon = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \beta_\varepsilon. \qquad \Gamma_f^\varepsilon = \Sigma\backslash\beta_\varepsilon. \tag{3.16}$$

Consider now the elliptic problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{on } \mathcal{D}_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial n_1} = \frac{\partial u_\varepsilon}{\partial n_2} = 0 & \text{on } \Gamma_f^\varepsilon \\ u_\varepsilon = 0 & \text{on } \Sigma_d \end{cases} \tag{3.17}$$

The holes (barriers) $\beta_\varepsilon = \cup\beta_\varepsilon^i$ are obtained by taking an r_ε-homotety of a fixed hole $\beta_0 \subset (-\frac{1}{2},+\frac{1}{2})^{N-1}$ which is then translated in order to obtain an ε-periodic configuration in all direction of Σ
The "density of the holes" is described by r_ε.
Γ-convergence approach for this problem is due to Atouch [11] and Damlamian [12].
The heat temperature u_ε,for a given source f is a solution of

$$\min_{u\in H_0^1(\mathcal{D}_1\cup\mathcal{D}_2)} \{F^\varepsilon(u) - \int_{\mathcal{D}} fu\} \tag{3.18}$$

where $F^\varepsilon : H_0^1(\mathcal{D}_1 \cup \mathcal{D}_2) \to \overline{\mathbf{R}}^+$ is equal to

$$F^\varepsilon(u) = \int_{\mathcal{D}_1\cup\mathcal{D}_2} \mid \nabla u \mid^2 dx + I_{K^\varepsilon}(u) \tag{3.19}$$

where

$$K^\varepsilon = \{u \in H^1(\mathcal{D}_1 \cup \mathcal{D}_2)/ \quad [u] = 0 \quad on \quad \beta_\varepsilon\} \tag{3.20}$$

Important here is that

$$u \in H_0^1(\mathcal{D}_\varepsilon) \Leftrightarrow u \in H^1(\mathcal{D}_1 \cup \mathcal{D}_2) \quad [u] = 0 \quad on \quad \beta_\varepsilon$$

We can see obviously that the family of functionals $\{F^\varepsilon; \varepsilon \to 0\}$ is uniformly coercive on $X = H^1(\mathcal{D}_1 \cup \mathcal{D}_2)$.
So we can deduce from Proposition 3 in the first chapter that the limit analysis for this problem is equivalent to the study of Γ-convergence of the sequence of functionals $\{F^\varepsilon : X \longrightarrow \overline{\mathbf{R}}^+ \quad \varepsilon \to 0\}$ in the weak topology of X.
The main result is:

Proposition 10 *Let the capacity of β_0 be*
$cap\beta_0 = inf\{\int_{\mathbf{R}^3} \mid \nabla w(x) \mid^2 dx \quad /w \in H^1(\mathbf{R}^3) \quad w = 1 \quad on \quad \beta_0\}$. *Then we have*

$$\Gamma - \lim_{\varepsilon\to 0} F^\varepsilon(u) = F(u) = \int_{\mathcal{D}_1\cup\mathcal{D}_2} \mid \nabla u \mid^2 dx + C\int_\Sigma [u]^2 d\sigma \tag{3.21}$$

where

$$C = \begin{cases} 0 & if \ \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon^2} = 0 \\ \frac{cap\beta_0}{4} & if \ \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon^2} = 1 \\ +\infty & if \ \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon^2} = +\infty \end{cases} \tag{3.22}$$

Proof. We will use the direct method for Γ-convergence in order to prove the result stated in the theorem.
This is a two step algorithm:
1. First step.
Intuition of Γ-limit using various properties of Γ-convergence see Propositions[4-8] in

the first chapter.
2. Second step.
Rigorous identification of Γ-limit. This is usually done using the Definition 1,for Γ-convergence of Functionals.
Let's proceed with the first step for our problem.
We observe that $F^{\varepsilon} : X \to \overline{\mathbf{R}}^{+}$ can be written in the following form,

$$F^{\varepsilon}(u) = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla u \mid^2 + \int_{\Sigma} a_{\varepsilon}(x)[u]^2 dx \tag{3.23}$$

$$a_{\varepsilon}(x) = \begin{cases} 0 & \text{if } x \in \Gamma_f^{\varepsilon} \\ +\infty & \text{if } x \in \beta_{\varepsilon} \end{cases}$$

Let us now assume (for the moment) that $\Gamma - \lim_{\varepsilon \to 0} F^{\varepsilon} = F$ and examine the form of the limit functional.

$$F(u) = \min\{\lim_{\varepsilon \to 0} \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla u \mid^2 dx; \quad u_{\varepsilon} \rightharpoonup u; \quad [u_{\varepsilon}] = 0 \quad on \quad \beta_{\varepsilon}\} \tag{3.24}$$

Writing $u_{\varepsilon} = u - z_{\varepsilon}$,$z_{\varepsilon} \rightharpoonup 0$ we have

$$F(u) = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla u \mid^2 dx + G([u]) \tag{3.25}$$

where

$$G([u]) = \min\{\lim_{\varepsilon \to 0} \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla z_{\varepsilon} \mid^2 dx; \quad z_{\varepsilon} \rightharpoonup 0 \quad [z_{\varepsilon}] = [u] \quad on \quad \beta_{\varepsilon}\} \tag{3.26}$$

Now because of the Propositions 1,7 and 8 we have that $G([u]) = C \int_{\Sigma} [u]^2 d\sigma$. This last relation and (2. 25) gives us

$$F(u) = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla u \mid^2 dx + C \int_{\Sigma} [u]^2 d\sigma \tag{3.27}$$

Taking now $u = +\frac{1}{2}$ on $\mathcal{D}_1$ and $u = -\frac{1}{2}$ on $\mathcal{D}_2$ we have

$$Cmeas\Sigma = min\{\lim_{\varepsilon \to 0} \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla u_{\varepsilon} \mid^2 dx; \quad u_{\varepsilon} \to u, \quad [u_{\varepsilon}] = 0 \quad on \quad \beta_{\varepsilon}\} \tag{3.28}$$

But because of the odd property of the function u, i.e. $u(x_1, x_2, -x_3) = -u(x_1, x_2, x_3)$, approaching it with functions equal to zero on β_{ε} or by functions with the jump equal to zero on β_{ε} are equivalent problems!! This is the reason way the Neuman Strainer problem admits a similar proof as the problem considered by Cioranescu and Murat (see [13] where the holes have dimension three). The second step in our proof is the identification of the Γ-limit. We will give an outline of the proof. The complete proof can be found in [11] and [12].

We introduce now the test functions $\{w_\varepsilon; \varepsilon \to 0\}$. Let B_ε be the sphere of diameter ε included in each cell of side ε (in $\mathbf{R}^3$!) and take w_ε the capacity potential of β_ε in B_ε:

$$\begin{cases} -\Delta w_\varepsilon = 0 & \text{in } B_\varepsilon \backslash \beta_\varepsilon \\ w_\varepsilon = 1 & \text{in } \beta_\varepsilon \\ w_\varepsilon = 0 & \text{on } \partial B_\varepsilon \end{cases} \tag{3.29}$$

w_ε is then extended by periodicity to the layer of size ε around Σ and then to the hole $\mathbf{R}^3$ by zero. Because of the symmetry of the problem we have $\frac{\partial w_\varepsilon}{\partial n} = 0$ on Γ_f^ε. Let $\overline{w_\varepsilon} = 1 - w_\varepsilon$. It can easily be seen that $\overline{w_\varepsilon} \rightharpoonup 0$ weakly in $H^1(\mathcal{D})$. Indeed we can see $\overline{w_\varepsilon}$ bounded in $H^1(\mathcal{D})$ hence relatively compact in $L^2(\mathcal{D})$.
Let $X_\varepsilon = 1$ on $\cup_i Y_\varepsilon^i \backslash B_\varepsilon^i$ and zero elsewhere. Then we have

$$(\overline{w_\varepsilon} - 1)X_\varepsilon = 0 \quad \text{on } \mathbf{R}^3 \tag{3.30}$$

But $X_\varepsilon \rightharpoonup \theta$ with $0 < \theta < 1$. So every $s - L^2(\mathcal{D})$ limit value of $\overline{w_\varepsilon}$ of the sequence $\{w_\varepsilon; \varepsilon \to 0\}$ from (2. 30) satisfies $(\overline{w} - 1)\theta = 0$ 0n $\mathbf{R}^3$. So $\overline{w_\varepsilon} \to 1$ in $L^2(\mathcal{D})$. So $w_\varepsilon \to 0$ in $L^2(\mathcal{D})$.
We will prove next a very helpful convergence result.

Lemma 1

$$\int_{\mathcal{D}_1 \cup \mathcal{D}_2} |\nabla w_\varepsilon|^2 dx \to \begin{cases} 0 & if\ r_\varepsilon \ll \varepsilon^2 \\ cap\beta_0 & if\ r_\varepsilon = \varepsilon^2 \\ +\infty & if\ r_\varepsilon \gg \varepsilon^2 \end{cases}$$

Proof. We remember that $\overline{w_\varepsilon} = 1 - w_\varepsilon$. For $\overline{w_\varepsilon}$ we have

$$\begin{cases} -\Delta \overline{w_\varepsilon} = 0 & \text{in } B_\varepsilon \backslash \beta_\varepsilon \\ \overline{w_\varepsilon} = 1 & \text{in } \beta_\varepsilon \\ \overline{w_\varepsilon} = 0 & \text{on } \partial B_\varepsilon \end{cases} \tag{3.31}$$

Now a variational formulation of the above problem gives us that $\overline{w_\varepsilon}$ is the solution of the following minimization problem:

$$min\{\int_{Y_\varepsilon} |\nabla w|^2 dx, \quad w = 0 \quad on \quad \beta_\varepsilon, \quad w = 1 \quad on \quad Y_\varepsilon \backslash B_\varepsilon\} \quad where \quad Y_\varepsilon = \varepsilon(-\frac{1}{2}, +\frac{1}{2})^N \tag{3.32}$$

So

$$\int_{\mathcal{D}} |\nabla w_\varepsilon|^2 dx = \int_{\mathcal{D}} |\nabla \overline{w_\varepsilon}|^2 dx \approx \frac{meas\Sigma}{\varepsilon^2} \int_{Y_\varepsilon} |\nabla w_\varepsilon|^2 dx \tag{3.33}$$

because the microscopic cells β_ε^i included in Σ is equal with $\frac{meas\Sigma}{\varepsilon^2}$. Thus

$$\int_{\mathcal{D}} |\nabla w_\varepsilon|^2 dx \approx \frac{meas\Sigma}{\varepsilon^2} min\{\int_{B_\varepsilon} |\nabla w|^2 dx, \quad w = 0 \quad on \quad \beta_\varepsilon, \quad w = 1 \quad on \quad \partial B_\varepsilon\} \tag{3.34}$$

By changing the scale $x = r_\varepsilon y$ we have

$$\int_{\mathcal{D}} \mid \nabla w_\varepsilon \mid^2 dx \approx \frac{meas\Sigma}{\varepsilon^2} min\{\int_{B_\varepsilon / r_\varepsilon} \frac{\mid \nabla w \mid^2}{{r_\varepsilon}^2} {r_\varepsilon}^3 dy / \quad w = 0 \quad on \quad \beta_0, \quad w = 1 \quad on \quad \partial B_\varepsilon / r_\varepsilon\} \tag{3.35}$$

we have that $r_\varepsilon \beta_0 = \beta_\varepsilon$ has been transformed into the initial hole β_0.
Therefore

$$\int_{\mathcal{D}} \mid \nabla w_\varepsilon \mid^2 dx \approx meas\Sigma \frac{r_\varepsilon}{\varepsilon^2} min\{\int_{B_\varepsilon / r_\varepsilon} \mid \nabla w \mid^2 dy / \quad w = 0 \quad on \quad \beta_0, \quad w = 1 \quad on \quad \partial B_\varepsilon / r_\varepsilon\} \tag{3.36}$$

when $\varepsilon \rightarrow 0$ this last minimum converge to $cap_{\overline{\mathbf{R}}^3} \beta_0$. So we obtain

$$\int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla w_\varepsilon \mid^2 dx \approx cap_{\overline{\mathbf{R}}^3} \beta_0 meas(\Sigma) \frac{r_\varepsilon}{\varepsilon^2}. \tag{3.37}$$

We can describe now for every $u \in X$ an approximating sequence v_ε. We will take $u = v \in C^\infty(X)$ first,passing to the limit after that.
Let v_ε to be the solution of the minimization problem:

$$min\{ \ F^\varepsilon(v_\varepsilon) \mid v_\varepsilon \rightarrow v\} \tag{3.38}$$

Now for $v = (v_1, v_2) \in C^\infty(\mathcal{D}_1) \times C^\infty(\mathcal{D}_2)$ we take $v_\varepsilon = (v_1, v_2) - w_\varepsilon(r_1, r_2) = (v_1 - w_\varepsilon r_1, v_2 - w_\varepsilon r_2)$ where w_ε is the solution of (2. 29),and $r_1 \in H^1(\mathcal{D}_1)$ and satisfies $r_1/_\Sigma = \frac{1}{2}[v]$ and $r_2 \in H^1(\mathcal{D}_2)$ and satisfies $r_2/_\Sigma = -\frac{1}{2}[v]$.
Clearly $v_\varepsilon \rightarrow v$ in $w - X$ and $[v_\varepsilon] = [v] - (r_1/_\Sigma - r_2/_\Sigma) = 0$ on β_ε.
Now

$$F^\varepsilon(v_\varepsilon) = \int_{\mathcal{D}_\varepsilon} \mid \nabla v_\varepsilon \mid^2 dx = \sum_{i=1,2} \int_{\mathcal{D}_\varepsilon^i} \mid \nabla v_i - \nabla w_\varepsilon \cdot r_i - w_\varepsilon \nabla r_i \mid^2 dx \sim$$

$$\sim \sum_{i=1,2} \int_{\mathcal{D}_\varepsilon^i} \mid \nabla v_i \mid^2 dx + \sum_{i=1,2} \int_{\mathcal{D}_\varepsilon^i} \mid \nabla w_\varepsilon \mid^2 {r_i}^2 \rightarrow$$

$$\rightarrow \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \mid \nabla v \mid^2 dx + \frac{C}{4} \cdot \int_\Sigma [v]^2 d\sigma$$

Of course we took v smooth enough and now we will use a density argument and a diagonalization procedure in order to prove that there is a sequence $\{u_\varepsilon \rightarrow u\}$ such that $F^\varepsilon(u_\varepsilon) \rightarrow F(u)$. Indeed noticing that F is continuous on X for the norm topology ,given $u \in X$ let

$$v_k \rightarrow u \quad in \quad X, then \quad F(v_k) \rightarrow F(u) \quad when \quad k \rightarrow +\infty.$$

From the preceding argument for every $k \in \mathbf{N}$ there exist an approximating sequence $\{v_{k,\varepsilon} : \varepsilon \rightarrow 0\}$ such that

$$(v_{k,\varepsilon}, F^\varepsilon(v_{k,\varepsilon})) \rightarrow (v_k, F(v_k)) \rightarrow (u, F(u)).$$

We will use next the following diagonalization lemma:

Lemma 2 *Let (M,τ)be a metrizable space and $\{x_{\nu,\mu}/\nu \in \mathbf{N}; \mu \in \mathbf{N}\}$ a double indexed sequence in M such that:*
$x_{\nu,\mu} \to x_\mu$ in τ-topology of M,for $\nu \to +\infty$,and
$x_\mu \to x$ in τ-topology of M for $\mu \to +\infty$.
Then there exists a mapping $\nu \to \mu(\nu)$ increasing to $+\infty$ such that
$x_{\nu,\mu(\nu)} \to x$ in τ-topology of M for $\nu \to +\infty$.

The proof of the lemma can be found in [11].
Using this Lemma we have that there exists an increasing map $\varepsilon \to k(\varepsilon)$ such that

$$(v_{k(\varepsilon),\varepsilon}, F^\varepsilon(v_{k(\varepsilon),\varepsilon})) \to (u, F(u))$$

in strong topology of $L^2 \times \mathbf{R}$ for $\varepsilon \to 0$.
Denoting $u_\varepsilon = v_{k(\varepsilon),\varepsilon}$,from the uniform coercivity of the F^ε,

$$u_\varepsilon \to u \text{ weekly in } X \text{ and} F^\varepsilon(u_\varepsilon) = \Phi(u_\varepsilon) \to F(u)$$

In order to complete the proof of the Γ-convergence result we need to show that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } X \Rightarrow \liminf_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon) \geq F(u)$$

without restriction we can assume $u = (u_1, u_2)$ with u_i regular,and then using a density argument. We can easily see that

$$\int_{\mathcal{D}_i} |\nabla u_\varepsilon|^2 dx \geq 2\int_{\mathcal{D}_i} \nabla u_\varepsilon \nabla v_\varepsilon - \int_{\mathcal{D}_i} |\nabla v_\varepsilon|^2 dx = \int_{\mathcal{D}_i} |\nabla v_\varepsilon|^2 dx + 2\int_{\mathcal{D}_i} \nabla v_\varepsilon(\nabla u_\varepsilon - \nabla v_\varepsilon) dx$$

,for $i = 1, 2$.
From the first part of the proof we have

$$\lim_{\varepsilon\to 0} \int_{\mathcal{D}_1\cup\mathcal{D}_2} |\nabla v_\varepsilon|^2 dx = \int_{\mathcal{D}_1\cup\mathcal{D}_2} |\nabla v|^2 dx + C\int_\Sigma [v]^2 d\sigma = F(v) \qquad (3.39)$$

For any product space $X = Y \times Z$ the scalar product in X is given by
$< u, v > = < u_1, v_1 >_Y + < u_2, v_2 >_Z$ for $u = (u_1, u_2), v = (v_1, v_2)$ in X.
By $|\cdot|$ we will understand the norm in X induced by the above scalar product,or the graph norm.
Now we have

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon) \geq F(v) + 2 \cdot \liminf_{\varepsilon\to 0} I_\varepsilon \qquad (3.40)$$

where

$$I_\varepsilon = \int_{\mathcal{D}_1\cup\mathcal{D}_2} < \nabla v_\varepsilon, \nabla u_\varepsilon > dx - \int_{\mathcal{D}_1\cup\mathcal{D}_2} |\nabla v_\varepsilon|^2 dx.$$

There is a difficulty to estimate the limit in the duality pair $< u_\varepsilon, v_\varepsilon >_{L^2,L^2}$ because we know $v_\varepsilon \rightharpoonup v$ and $u_\varepsilon \rightharpoonup u$ in X. But we have $\nabla v_\varepsilon = (\nabla v_1 - w_\varepsilon \cdot \nabla r_1 - \nabla w_\varepsilon \cdot$

$r_1, \nabla v_2 - w_\varepsilon \cdot \nabla r_2 - \nabla w_\varepsilon \cdot r_2)$.
Thus

$$\int_{\mathcal{D}_1 \cup \mathcal{D}_2} < \nabla v_\varepsilon, \nabla u_\varepsilon > dx =$$

$$\sum_i \int_{\mathcal{D}_i} < \nabla v_i, \nabla u_\varepsilon > dx - \sum_i \int_{\mathcal{D}_i} w_\varepsilon < \nabla r_i, \nabla v_\varepsilon > dx - \sum_i \int_{\mathcal{D}_i} r_i < \nabla w_\varepsilon, \nabla u_\varepsilon > \quad (3.41)$$

Since $w_\varepsilon \to 0$ in $L^2(\mathcal{D})$,$\nabla u_\varepsilon \rightharpoonup \nabla u$ w-$L^2(\mathcal{D}_1) \times L^2(\mathcal{D}_2)$ and v-smooth enough we can say

$$\sum_i \int_{\mathcal{D}_i} < \nabla v_i, \nabla u_\varepsilon > dx \to \int_{\mathcal{D}_1 \cup \mathcal{D}_2} < \nabla v, \nabla u > dx$$

and

$$\sum_i \int_{\mathcal{D}_i} w_\varepsilon < \nabla r_i, \nabla v_\varepsilon > dx \to 0.$$

Thus

$$\liminf_{\varepsilon \to 0} I_\varepsilon = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} < \nabla v, \nabla u - \nabla v > dx - C \int_\Sigma [v]^2 d\sigma + \liminf_{\varepsilon \to 0} \int_{\mathcal{D}_1 \cup \mathcal{D}_2} r < \nabla w_\varepsilon, \nabla u_\varepsilon > dx$$

Using that $r_1 = \frac{1}{2} \cdot [v]$ and $r_2 = -\frac{1}{2}[v]$ and that $\frac{\partial w_\varepsilon}{\partial n} \mid_1 = \frac{\partial w_\varepsilon}{\partial n} \mid_2 = 0$ on Γ_f^ε we have that

$$< -\Delta w_\varepsilon, u_\varepsilon r >_{(X',X)} = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} < \nabla u_\varepsilon, r \cdot \nabla w_\varepsilon > dx + \int_{\mathcal{D}_1 \cup \mathcal{D}_2} u_\varepsilon < \nabla r, \nabla w_\varepsilon > dx + \int_{\beta_\varepsilon} \frac{\partial w_\varepsilon}{\partial n} \cdot u_\varepsilon \cdot [v] d\sigma$$

. Noticing that $u_\varepsilon \to u$ s-$L^2(\mathcal{D}_1) \times L^2(\mathcal{D}_2)$ and $\nabla w_\varepsilon \rightharpoonup 0$ in $L^2(\mathcal{D}_1) \times L^2(\mathcal{D}_2)$

$$\int_{\mathcal{D}_1 \cup \mathcal{D}_2} u_\varepsilon < \nabla r, \nabla w_\varepsilon > dx \to 0$$

and because $\frac{\partial w_\varepsilon}{\partial n} \cdot u_\varepsilon \cdot [v]$ is bounded and $meas(\beta_\varepsilon) \to 0$ when $\varepsilon \to 0$ we get that:

$$\liminf_{\varepsilon \to 0} \int_{\beta_\varepsilon} \frac{\partial w_\varepsilon}{\partial n} \cdot u_\varepsilon \cdot [v] d\sigma = 0.$$

Thus

$$\liminf_{\varepsilon \to 0} I_\varepsilon = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} < \nabla v, \nabla(u - v) > dx - C \int_\Sigma [v]^2 d\sigma + \liminf_{\varepsilon \to 0} J_\varepsilon \quad (3.42)$$

where

$$J_\varepsilon = < -\Delta w_\varepsilon, u_\varepsilon r >_{(X',X)}$$

We remember that we can approximate $u \in X$ with $v \in C^\infty(\overline{\mathcal{D}_1}) \times C^\infty(\overline{\mathcal{D}_2})$. Thus:

$$\lim_{\varepsilon \to 0} J_\varepsilon = C \int_\Sigma uv dx \quad (3.43)$$

The proof of the above result can be done by caring quite similar arguments as in [13], see also [11] and [12]. For the moment let us assume(2. 42) and complete the proof. From(2. 40),(2. 41) and(2. 42),

$$\liminf_{\varepsilon\to 0} F^{\varepsilon}(u_{\varepsilon} \geq F(v) + 2[\int_{\mathcal{D}_1\cup\mathcal{D}_2} < \nabla v, \nabla u - \nabla v > dx - C\int_{\Sigma} v(u-v)dx].$$

Now letting v tends to u in the norm topology of X and using the continuity of Fwe get

$$\liminf_{\varepsilon\to 0} F^{\varepsilon}(u_{\varepsilon}) \geq F(u)$$

4. Asymptotic analysis of a contact problem

4.1 Problem statement

In this section the elastic contact problem with friction is presented. We consider the three dimensional shearing of a elastic domain $\mathcal{D} \subset \mathbf{R}^3$. If we denote by $u : \mathcal{D} \longrightarrow \mathbf{R}^3$ the displacement field then the elastic constitutive equation and the equilibrium equation read

$$\sigma(u) = \mathcal{A}\varepsilon(u) + \sigma^{\infty}, \quad \mathrm{div}(\mathcal{A}\varepsilon(u)) = 0 \quad \text{in} \quad \mathcal{D}, \tag{4.44}$$

where $\mathcal{A}$ is the fourth order elastic tensor, $\sigma(u)$ is the stress tensor, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is the small strain tensor and $\sigma^{\infty} \in C^0(\bar{\mathcal{D}})$ is the pre-stress. $\mathcal{A}$ is a symmetric and positively defined fourth order tensor, i. e.

$$\mathcal{A}_{ijkl} \in L^{\infty}(\mathcal{D}), \quad \mathcal{A}(x)\varepsilon\cdot\sigma = \mathcal{A}(x)\sigma\cdot\varepsilon, \quad \text{a. e.} \quad x \in \mathcal{D}, \tag{4.45}$$

$$\exists\, a > 0 \text{ such that } \quad \mathcal{A}(x)\varepsilon\cdot\varepsilon \geq a|\varepsilon|^2, \quad \text{a. e.} \quad x \in \mathcal{D}, \tag{4.46}$$

$\forall\, i,j,k,l = \overline{1,3}$ and for all $\sigma, \varepsilon \in \mathbf{R}^{3\times 3}_S$.

The smooth boundary $\Sigma = \partial\mathcal{D}$ is divided into two disjoint parts $\Sigma = \Sigma_d \cup \Gamma_f$, where $\Sigma_d = \partial\bar{\mathcal{D}}$ is the exterior boundary and Γ_f is the interior one (i. e. it's a subset of the interior of $\bar{\mathcal{D}}$) and we be called in the following the fault. For the sake of simplicity on the exterior boundary we shall suppose vanishing displacement conditions

$$u = 0 \quad \text{on} \quad \Sigma_d, \tag{4.47}$$

We suppose that on the fault Γ_f a slip-dependent friction law is modeling the contact and the pre-stress σ^{∞} is such that the fault does not open during the slip:

$$[\sigma(u)n] = 0, \quad [u\cdot n] = 0 \quad \text{on} \quad \Gamma_f, \tag{4.48}$$

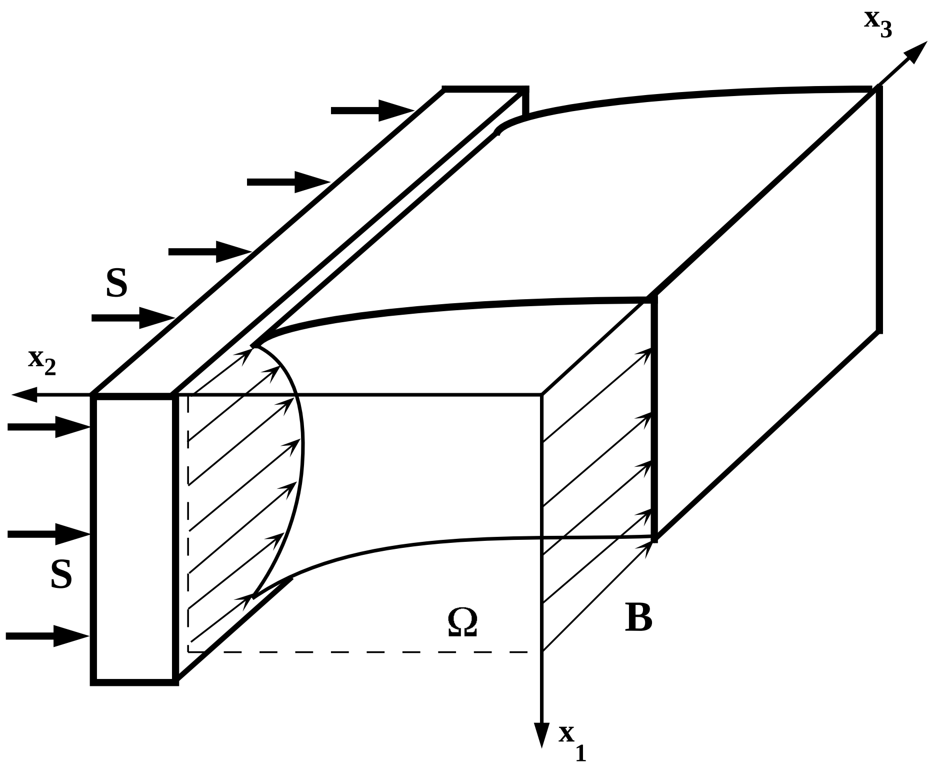

Figure 1: The geometry of the anti-plane problem

$$\sigma_\tau(u) = -\mu(|[u_\tau]|)|\sigma_n(u)|\frac{[u_\tau]}{|[u_\tau]|} \quad \text{if} \quad [u_\tau] \neq 0 \;\text{ on }\; \Gamma_f, \tag{4.49}$$

$$|\sigma_\tau(u)| \leq \mu(0)|\sigma_n(u)| \quad \text{if} \quad [u_\tau] = 0 \;\text{ on }\; \Gamma_f, \tag{4.50}$$

where [ ] denotes the half of the jump across Γ_f, (i. e. $[w] = (w^+ - w^-)/2$), n is the unit normal outwards the positive side of Γ_f, $\sigma_\tau(u) = \sigma(u)n - (\sigma(u)n \cdot n)n$ is the tangential stress, $\sigma_n(u) = \sigma(u)n \cdot n$ is the normal stress, $u_\tau = u - (u \cdot n)n$ is the tangential displacement and $u_n = u \cdot n$ is the normal displacement. Equations (4.49)- (4.50) assert that the tangential (friction) stress is bounded by the normal stress multiplied by the value of the friction coefficient $\mu(0)$. If such a limit is not attained sliding does not occur. Otherwise the friction stress is opposed to the slip $[u_\tau]$ and its absolute value depends on the slip modulus through $\mu(|[u_\tau]|)$. Concerning the regularity of $\mu : \Gamma \times \mathbf{R}^+ \to \mathbf{R}$ we suppose that the friction coefficient is a Lipschitz function, with respect to the slip, and let H be the antiderivative

$$H(x, u) := \int_0^u \mu(x, s)\, ds.$$

We suppose that there exist $L, a, b \geq 0$, and $\gamma \in L^\infty(\Gamma)$ such that

$$|\mu(x, s_1) - \mu(x, s_2)| \leq L|s_1 - s_2|, \quad H(x, s) - \mu(x, 0)s + b\gamma(x)s^2/2 + as^3 \geq 0, \tag{4.51}$$

a. e. $x \in \Gamma_f$, and for all $s, s_1, s_2 \in \mathbf{R}^+$.

A specific friction law with a linear piecewise slip weakening, which is a reasonable approximation of the experimental observations (see [7]), can be written as follows

$$\mu(s) = \begin{cases} \dfrac{\mu_s - \mu_d}{D_c} s + \mu_s & if \ s \leq D_c \\ \mu_d & if \ s \geq D_c \end{cases} \tag{4.52}$$

where $\mu_s > \mu_d$ are the static and, respectively, dynamic friction coefficients and D_c is the critical slip.

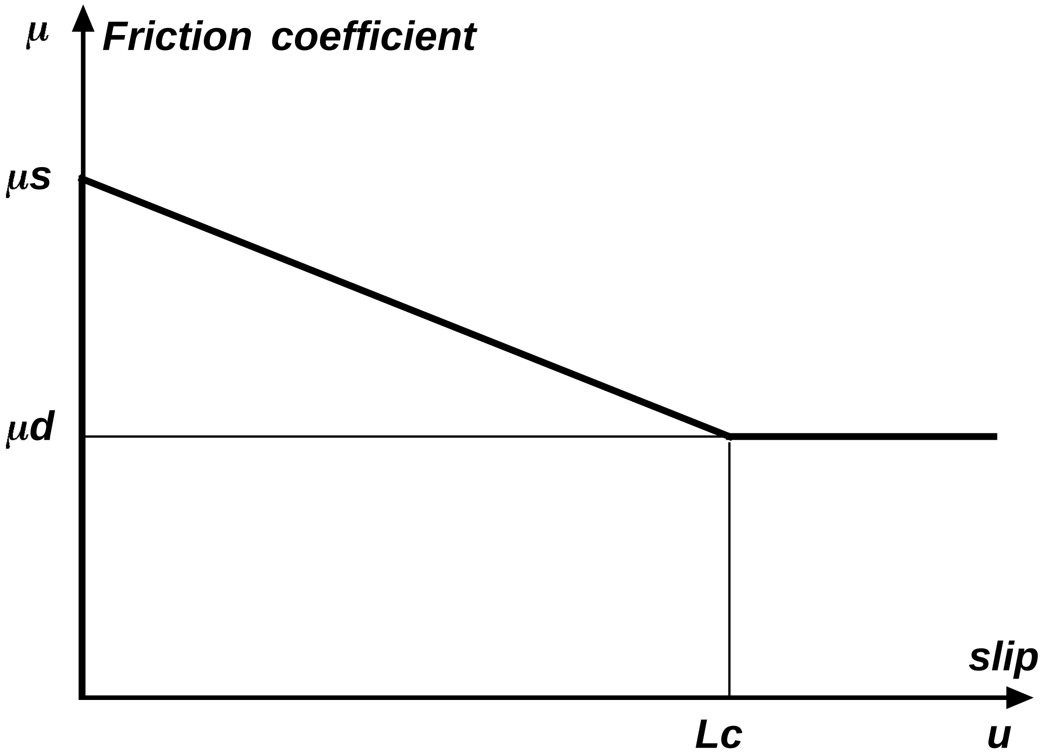

Figure 2: The friction law where μ_s and μ_d are the static and respectively the dynamic friction coefficients and L_c is the critical slip

We shall suppose in the following that Γ_f is a subset of $\Pi = \{x_3 = 0\}$, and $\mathcal{D}$ is symmetric with respect to it. As in [4] the following symmetries will be considered : $u_1(x_1, x_2, -x_3) = -u_1(x_1, y, x_3)$, $u_2(x_1, x_2, -x_3) = -u_2(x_1, x_2, x_3)$, $u_3(x_1, x_2, -x_3) = u_3(x_1, x_2, x_3)$, $\sigma_{13}(x_1, x_2, -x_3) = \sigma_{13}(x_1, x_2, x_3)$, $\sigma_{23}(x_1, x_2, -x_3) = \sigma_{23}(x_1, x_2, x_3)$ and $\sigma_{33}(x_1, x_2, -x_3) + \sigma_{33}(x_1, x_2, x_3) =$
$\sigma_{33}^{\infty}(x_1, x_2, -x_3) + \sigma_{33}^{\infty}(x_1, x_2, x_3)$.

The condition of continuity of the stress vector (4.48) on the fault plane Γ_f gives

$$\sigma_{13}(x,0^-) = \sigma_{13}(x,0^+) = \sigma_{13}(x,0), \quad \sigma_{23}(x,0^-) = \sigma_{23}(x,0^-)(t,x,0^+) = \sigma_{23}(x,0^-)(x,0), \tag{4.53}$$

where $x = (x_1, x_2)$ and $(x, 0)$ belongs to Σ_0 the intersection of $\bar{\mathcal{D}}$ with the plane Π. The normal stress $\sigma_n = \sigma_{33}$ does not present any variation during the slip

$$\sigma_{33}(x,0^+) = \sigma_{33}(x,0^-) = \sigma_{33}^{\infty}(x,0) \tag{4.54}$$

will be denoted by $S(x,0) := -\sigma_{33}^{\infty}(x,0)$ and we suppose that $S \in L^{\infty}(\Gamma_f)$ and $S \geq 0$. The tangential displacement is vanishing outside Γ_f

$$u_\tau(x,0^+) = u_\tau(x,0^-) = 0, \quad \text{for all } (x,0) \in \Sigma_0 \setminus \Gamma_f, \tag{4.55}$$

and the jump on Γ_f is the given by

$$[u_\tau(x,0)] = u_\tau(x,0^+) = -u_\tau(x,0^-), \quad \text{for all } (x,0) \in \Gamma_f. \tag{4.56}$$

Let denote by $\Omega := \mathcal{D} \cap \{x_3 > 0\}$ and by $\Gamma_d := (\Sigma_d \cap \{x_3 > 0\}) \cup \Sigma_0 \setminus \Gamma_f$ which implies that $\partial\Omega = \Gamma_d \cup \Gamma_f$. From the above symmetry properties we can restrict ourselves to find the displacement field u on Ω, the upper half of the domain $\mathcal{D}$.

Let us denote by V the closed subspace of $[H^1(\Omega)]^3$ given by

$$V := \{v \in [H^1(\Omega)]^3 / v = 0 \quad \text{on} \quad \Gamma_d\}. \tag{4.57}$$

From (6. 1), (6. 2) and the Korn's inequality one can easily deduce that the following inner product

$$< u, v >_V := \int_\Omega \mathcal{A}\varepsilon(u) \cdot \varepsilon(v), \quad \forall u, v \in V, \tag{4.58}$$

generates a norm, denoted by $\| \ \|_V$, which is equivalent with the natural norm on $[H^1(\Omega)]^3$.

Since the normal stress is prescribed on Γ_f (see 4.54) we have, as in [2], the following variational formulation of the mechanical problem (4.44), (4.47)-(4.50)

$$u \in V, \ < u, u - v >_V + j(u,u) - j(u,v) \leq f(u - v), \quad \forall v \in V, \tag{4.59}$$

where $j : V \times V \longrightarrow \mathbf{R}_+$ and $f : V \longrightarrow \mathbf{R}$ are given by

$$j(u,v) = \int_{\Gamma_f} S\mu(|u_\tau|)|v_\tau|, \qquad f(v) = -\int_{\Gamma_f} \sigma_\tau^{\infty} \cdot v_\tau, \quad \forall u, v \in V. \tag{4.60}$$

Let us introduce the energy function $\mathcal{W} : V \longrightarrow \mathbf{R}$ given by

$$\mathcal{W}(v) = \frac{1}{2}\|v\|_V^2 + \int_{\Gamma_f} SH(|v_\tau|) - f(v), \quad \forall v \in V, \tag{4.61}$$

and let us recall from [2] the following result :

Theorem 1 *If $u \in V$ is a local minimum for $\mathcal{W}$, then u is a solution of (4.59). Moreover there exists at least a global minimum for $\mathcal{W}$, i. e. there exists $u \in V$ such that*

$$\mathcal{W}(u) \leq \mathcal{W}(v), \qquad \forall v \in V. \tag{4.62}$$

We now define the equivalent (or macroscopic) and the perturbed (or microscopic) problems. Let $\Gamma_f^0 \subset \Sigma_0$ be the large scale (or equivalent) fault with a characteristic length L. In order to define the local problem let us define on $\Gamma_f^0 \subset \Sigma_0$ a ε- periodic structure:In each small cell of size ε is centered a disc β_ε^i of radius $r_\varepsilon \leq \frac{\varepsilon}{2}$. The holes β_ε^i are obtained taking an r_ε-homothetic of a fixed hole strongly included in $Y = (-\frac{1}{2}, \frac{1}{2})^{\mathbf{N}-1}$ which is translated in order to obtain ε- periodic configuration in all directions of $\Gamma_f^0 \subset \Sigma_0$. As before we define $\Gamma_d^\varepsilon := \partial\Omega \setminus \Gamma_f^\varepsilon$, and

$$V_\varepsilon := \{v \in [H^1(\Omega)]^3 / v = 0 \quad \text{on} \quad \Gamma_d^\varepsilon\}. \tag{4.63}$$

$$\mathcal{W}_\varepsilon(v) = \frac{1}{2}\|v\|_{V_\varepsilon}^2 + \int_{\Gamma_f^\varepsilon} SH(|v_\tau|) + \int_{\Gamma_f^\varepsilon} \sigma_\tau^\infty \cdot v_\tau, \quad \forall v \in [H^1(\Omega)]^3. \tag{4.64}$$

Our aim is to study, the asymptotic behavior, when $\varepsilon \to 0$, of the solutions u_ε of the following minimum problem

$$u_\varepsilon \in V_\varepsilon \quad \mathcal{W}_\varepsilon(u_\varepsilon) \leq \mathcal{W}_\varepsilon(v), \qquad \forall v \in V_\varepsilon. \tag{4.65}$$

‘Let

$$V_1 = \{v \in [H^1(\Omega)]^3 / v = 0 \quad on \quad (\Sigma_d \cap \{x_3 > 0\}) \cup (\Sigma_0 \backslash \Gamma_f^0)\} \tag{4.66}$$

From Korn's inequality we know that the norm generated by (3. 57) on V_1 will be equivalent with the usual norm on $[H^1(\Omega)]^3$ That is
$\exists C_1, C_2 \geq 0$ s. t $C_1 || \ v \ ||_{H^1}^2 \leq || \ v \ ||_{V_1}^2 \leq C_2 || \ v \ ||_{H^1}^2$
Thus for all $v \in V_1$ there exists $\lambda_v \in \mathbf{R}^+$ such that $\lambda_v = \frac{||v||_{V_1}^2}{||v||_{H^1}^2}$.
Also by Poincare Inequality we can say that
$\exists C_1, C_2 \geq 0$ s. t $C_1 || \ \nabla v \ ||_{L^2}^2 \leq || \ v \ ||_{H^1}^2 \leq C_2 || \ \nabla v \ ||_{L^2}^2 \quad \forall v \in V_1$.
So we can say that

$$\exists \lambda_v \in \mathbf{R}^+ s.t \lambda_v = \frac{|| \ v \ ||_{V_1}^2}{|| \ \nabla v \ ||_{L^2}^2} \tag{4.67}$$

Let's rewrite the functional $\mathcal{W}_\varepsilon$.

$$\mathcal{W}_\varepsilon(v) = \frac{1}{2} \ || \ v \ ||_{V^1}^2 + \int_{\Gamma_f^0} S \cdot H(| \ v_\tau \ |) + \int_{\Gamma_f^0} \sigma_\tau^\infty \cdot v_\tau + I_{K^\varepsilon}(v) \quad \forall v \varepsilon V_1 \tag{4.68}$$

where $I_{K^\varepsilon}(u) = \{u \in [H^1(\Omega_1)]^3 \mid u = 0 u \in \varepsilon\bar{\beta}\}$.
Using Prop 10, and the symmetry of the set $\mathcal{D}$ with respect to Σ_0 we have

4.2 Homogenization Result

One the main results of this work is presented in this section. The limit homogenized problem is obtained,using the direct method of Γ-convergence.

Lemma 3 *For* $F_\varepsilon(v) = \frac{1}{2} \parallel v \parallel_{V_1}^2 + I_{K^\varepsilon}(v)$ *we have*

$$\Gamma - \lim_{\varepsilon \to 0} F_\varepsilon(v) = F(v) = \frac{1}{2} \parallel v \parallel_{V_1}^2 + \lambda_v C \int_{\Gamma_f^0} v^2$$

The proof follows immediately for Prop 10, symmetry, and () and re placing Σ_0 with Γ_f^0.
Let us call

$$L_1(v) = \int_{\Gamma_f^0} S \cdot H(\mid v_\tau \mid).$$

We will need the following Lemma:

Lemma 4 *Let* $\Omega \subset \mathbf{R}^d$ *be as above and let* $\alpha \in [2, \frac{2(d-1)}{d-2}]$ *if* $d \geq 3$ *and* $\alpha \geq 2$ *if* $d -> 2$. *Then, for* $\beta -> \frac{d(\alpha-2)+2}{2\alpha}$ *if* $d \geq 3$ *or if* $d -> 2$ *and* $\alpha -> 2$, *and for all* $\beta \in]\frac{\alpha-1}{\alpha}, 1[$ *if* $d -> 2$ *and* $\alpha > 2$, *there exists a constant* $C -> C(\beta)$ *such that:*

$$||v||_{L^\alpha(\Gamma)} \leq C||v||_{L^2(\Omega)}^{1-\beta} ||v||_{H^1(\Omega)}^{\beta}, \quad \forall v \in H^1(\Omega). \tag{4.69}$$

The proof of this Lemma can be found in [1].

In the above inequality we can take $d = 3, \beta_0 \in (1/2, 1)$ and $\alpha = 2$
So $\parallel \gamma(v) \parallel_{L^2(\Gamma_f^0)} \leq C \cdot \parallel v \parallel_{L^2(\Omega)}^{1-\beta} \cdot \parallel v \parallel_{H^1(\Omega)}^{\beta}$.
Let $u_n \to uw - H^1$. Thus $u_n \to us\text{-}L^2$, by Sobolev imbedding. Thus

$$\parallel \gamma(v) \parallel_{L^2(\Gamma_f^0)} \leq C \parallel v \parallel_{L^2(\Omega_1)}^{1-\beta} \parallel v \parallel_{H^1}^{\beta} \leq C \cdot \parallel v \parallel_{L^2(\Omega)}^{1-\beta}$$

So

$$\parallel \gamma_\tau(u_n - u) \parallel_{L^2(\Gamma_f^0)} \leq \parallel \gamma(u_n) - \gamma(u) \parallel_{L^2(\Gamma_f^0)} \leq C \cdot \parallel u_n - u \parallel_{L^2(\Omega)}^{1-\beta} \to 0.$$

whereby C we understand a general positive constant.
Then we get

$$\gamma_\tau(u_n) \xrightarrow{L^2} \gamma_\tau(u).$$

But $H(x,u) = \int_0^u \mu(x,s)ds$ for $u\varepsilon\mathbf{R}^+$. Thus

$$\int_{\Gamma_f^0} | H(x, | \gamma_\tau(u_n) |) - H(x, | \gamma_\tau(u) |) | = \int_{\Gamma_f^0} | \int_{|\gamma_\tau(u)|}^{|\gamma_\tau(u_n)|} \mu(x,s)ds | \leq$$

$$\leq \int_{\Gamma_f^0} || \gamma_\tau(u_n) | - | \gamma_\tau(u) || \cdot \mu_0$$

$$\leq \int_{\Gamma_f^0} | \gamma_\tau(u_n) - \gamma_\tau(u) | \cdot \mu_0 \leq \int_{\Gamma_f^0} | \gamma_\tau(u_n) - \gamma_\tau(u) |^2 \cdot \mu_0 \to 0.$$

So we proved that

$$\int_{\Gamma_f^0} S \cdot H(| \gamma_\tau(u_n) |) \to \int_{\Gamma_f^0} S \cdot H(| \gamma_\tau(u) |) \quad \forall \ u_n \rightharpoonup u \ \ w - H^1.$$

in a similar manner we have that
$L_2(v) = \int_{\Gamma_f^0} \sigma_\tau^\infty \cdot v_\tau$ it will be a continuous functional, in weak topology of V_1: using there property ( ) from Chapter 1 we have that

$$\Gamma - \lim_{\varepsilon \to 0} W_\varepsilon(v) = W(v) = \frac{1}{2} \parallel v \parallel_{V_1}^2 + \int_{\Gamma_f^0} S \cdot H(| v_\tau) |) + \int_{\Gamma_f^0} \sigma_\tau^\infty \cdot v_\tau + \lambda_v \cdot C \int_{\Gamma_f^0} v^2$$

where

$$\lambda_v = \frac{\parallel v \parallel_{V_1}^2}{\parallel \nabla v \parallel_{L^2}^2} \quad \forall v.$$

4.3 Associated Spectral Problem

In this section we study the spectral problem,associated with the initial contact problem.
As we have seen in [2] the spectral problem is important in the analysis of the stability of solution for the contact problem. We give a boundedness result concerning eigenvalues for the ε problem.

For each fixed ε, let now L_p be the space $[L^p(\Gamma_f^\varepsilon)]$

$$L_p = \{z \in [L^p(\Gamma_F^\varepsilon)]^N \mid z(x) \cdot n(x) = 0 \text{ a. e. } \ x\varepsilon\Gamma_f^0\} \ \ 1 \leq p \leq \infty.$$

For $p < 2(N-1)(N-2)$ let us denote $\cdot$ by $\gamma_\tau : \quad V_1^\varepsilon \to L_p$ the compact operator which associates to all $v \in V_1^\varepsilon$ the tangential component of its trace on Γ_f^ε.

$$\gamma_\tau(v) = v - (v \cdot n)n \ \ \text{along}\Gamma_f^\varepsilon \quad \forall v \in V_1^\varepsilon.$$

Let $V_2^\varepsilon = \ker\gamma_\tau$ be given by: $V_2^\varepsilon = \{v \in V_1^\varepsilon \mid \gamma_\tau(v) = 0\}$ and let V_3^ε be the subspace of V_1^ε which is orthogonal to V_2^ε i. e.

$$V_3^\varepsilon = V_2^\varepsilon = \{v \in V_1^\varepsilon \mid < v, w >_V = 0 \ \ \forall w \in V_2^\varepsilon\}.$$

By similar arguments as in [2] we can show that for the static spectral problem, stated in [2] as: find $v: \Omega \to \mathbf{R}^N, v \neq 0$ and $b \in \mathbf{R}$ such that

$$\begin{aligned} \sigma(v) &= \mathcal{A}\varepsilon(v), \mathrm{div}\sigma(v) = 0 \text{ in } \Omega \\ v &= 0 \text{ or } \Gamma_d \\ \sigma(v)\mu \cdot n &= 0, \ \ \sigma_\tau(v) = bv_\tau \text{ on} \Gamma_f^\varepsilon. \end{aligned}$$

We can find that, for each ε fixed we have

Lemma 5 *There exists an increasing and positive sequence $(b_n^\varepsilon)_{n\geq 1}$ of eigenvalues for the above spectral problem and $b_n^\varepsilon \overset{n\to\infty}{\to} \infty$. To each b_n it corresponds a finite dimensional subspace of eigenfunctions $W_n \subset V_2^\varepsilon$. Moreover we have*

$$b_1^\varepsilon \parallel \gamma_\tau(v) \parallel_{L^2}^2 \leq \parallel v \parallel_{V^\varepsilon}^2 \forall v \in V_1^\varepsilon.$$

For fixed $\varepsilon > 0$, this can be carried out in a similar way, as in [2].

Now let us prove that

Theorem 2 *Let $\{b_n^\varepsilon\}_{n=1}^\infty$ be the sequence of eigenvalues for the static spectral problem stated above. Then for each $n \geq 1$, there is a constant $C_n > 0$, independent with respect to ε, such that*

$$0 < b_n^\varepsilon \leq C_n \varepsilon \ \ \forall \varepsilon > 0.$$

Proof. We will use the minimax principle for eigenvalues due to Rayleigh-Ritz,

$$b_n^\varepsilon = \min_{\substack{S_n \subset V_1^\varepsilon \\ \dim_{S_n = n}}} \max_{\substack{v \in S_n \\ v \neq 0}} R_\varepsilon(v)$$

where $R_\varepsilon(v) = \frac{a^\varepsilon(v,v)}{(v,v)_{\Gamma_f^\varepsilon}}$, and $a^\varepsilon(u,v) = \langle u, v\rangle_{V_1^\varepsilon}, \forall u, v \varepsilon V_1^\varepsilon$ and $(u,v)_{\Gamma_f^\varepsilon} = \int_{\Gamma_f^\varepsilon} \gamma_\tau(u) \cdot \gamma_\tau(v) d\sigma$.

We will use in what's following the next lemma.

Lemma 6 *Let S_n be the space generated by the first n eigenvalues of the following problem*

$$(***) \begin{cases} \delta(v) = \lambda v & in \ \ \Omega \\ v \varepsilon H_0^1. \end{cases}$$

then

$$dim(S_n \mid U) = n \ \ \forall U \subset \Omega.$$

The proof of this result can be done in a similar manner as in Vanninathan (see [9]). So we have that $\dim(S_n \mid \Omega) = n$. We can chose S_n to be the space generated by the first n eigenvalues of the problem $(***)$.

Thus,

$$b_n^\varepsilon \leq \min_{\substack{S_n \subset V_1^\varepsilon \\ \dim_{S_n = n}}} \max_{\substack{v \in S_n \\ v \neq 0}} R_\varepsilon(v) \leq C\lambda_n \cdot \max_{v \in S_n} \frac{\int_\Omega v^2}{\int_{\Gamma_f^\varepsilon} \gamma_\tau^2(v)}$$

where λ_n is the n^th eigenvalue for problem $(***) and we used (3.67)$.

We will prove that

$$\max_{v \in S_n} \frac{\int_\Omega v^2}{\int_{\Gamma_f^\varepsilon} \gamma_\tau^2(v)} \leq C_n \varepsilon.$$

By contradiction if it is false then one can have a subsequence still denoted by ε and $v_\varepsilon \in S_n$ such that

$$\| v_\varepsilon \|_{L^2(\Omega)} = 1 \text{ and}$$

$$\varepsilon \int_{\Gamma_f^\varepsilon} v_\varepsilon^2 \to 0.$$

Because S_n has finite dimension, the sequence $\{v_\varepsilon\}$ it will be bounded in V_1.

Similarly as in [VA] one can obtain that $\int_{\Omega \setminus \Gamma_f^\varepsilon} \to 0 \Leftrightarrow \int_\Omega v_\varepsilon^2 \to 0$ which contradicts the above relation,that the norm of v_ε is 1.

5. Conclusions and Future Work

So we can make the following conclusions: 1.We obtain the homogenized problem in the space V_1,and we can observe that in the homogenized problem the friction coefficient will increase with a positive quantity,namely $\lambda v \cdot cap(\frac{\beta_0}{4})$ when the holes dimensions is $r_\varepsilon = \varepsilon^2$. Otherwise when this is not the case the friction coeficient may be ∞ if $r_\varepsilon \ll \varepsilon^2$ or 0 if $r_\varepsilon \gg \varepsilon^2$. 2.In order to be able to disccuss the stability of solutions for the homogenized solution we need to analyse the spectral problem associated. We have seen that the eigenvalues of the spectral problem do not converge when $\varepsilon \to 0$ to the eigenvalue of an homogenized problem,because they are of order ε. But we expect to extend the results of [9] and to obtain a similar homogenized result,i.e, That the sequence $\frac{b_n^\varepsilon}{\varepsilon}$ converge to the eigenvalue λ of a limit problem,and there exists an extension operatorP_ε for the associated eigenvectors,such that at least on a subsequence we will have $P_\varepsilon u_\varepsilon \to u \quad in \quad H_0^1 \quad weakly$, where u is the eigenvector of the limit problem associated to λ. Also for better approximation of the ε solution we will prove the existence of corrector for both problems,contact problem and the spectral problem.

References

[1] Ioan R. Ionescu, Cristian Dascalu and Michel Campillo, *Slip-weakening friction on a periodic syatem of faults: Spectral analysis*, Z. angew. Math. Phys., 53 (2002) 980-995.

[2] Ioan R. Ionescu, Jean-Claude Paumier, *On the contact problem with slip displacement dependent friction in elastostatics*, Int. J. Engng. Sci., 34, 4 (1996) 471-491.

[3] M. Campillo, P. Favreau, I.R. Ionescu and C. Voisin, *On the effective friction law of an heterogeneous fault, J. Geophys. Res.*, vol. 106, B8 (2001) 16307-16322.

[4] P. Favreau, M. Campillo and I. R. Ionescu, *Initiation of Instability under Slip Dependent Friction in Three Dimension*, Journal of Geophysical Research, 107 (B7), (2002).

[5] Ionescu I.R., *Viscosity solutions for dynamic problems with slip-rate dependent friction*, Quart. Appl. Math. vol. LX, No. 3 (2002) 461-476.

[6] G. Nguetseng and E. Sanchez-Palencia, *Stress cocentration for defects distributed near a surface*, in Local Effects in the Analysis of Structures, P. Ladevèze ed., Elsevier, Amsterdam, (1985).

[7] M. Ohnaka, Y. Kuwahara and K. Yamamoto, *Constitutive relations between dynamic physical parameters near a tip of the propagation slip during stick-slip shear failure*, Tectonophysics, vol. 144 (1987)109-125.

[8] H. Perfettini, M. Campillo and I. R. Ionescu, *Rescaling of the weakening rate*, Geophysical Journal Letters, 2003.

[9] M. Vanninathan, *Sur quelques problemes d'homogeneisation dans les equations aux derivees partielles*, These d'Etat, Univ. Pierre et Marie Curie, (1979).

[10] G. Dal Maso, *An Introduction to* Γ*-Convergence*, Birkhauser, Boston (1993).

[11] H. Attouch, *Variational Convergence for Functions and Operators*, Pitman, Boston, (1984).

[12] A. Damlamian, *Le probleme de la passoire de Neumann*, Rend. Sem. Mat. Univ. Pol. Torino, 43, 3 (1985) 427-450.

[13] D. Cioranescu and F. Murat, *Un terme etrange venu d'ailleurs*, Nonlinear Partial Differential Equations and their Applications, College de France Seminar, vol 2-3, Pitman, Boston, (1982) 98-138, 154-178