Sequential Quadratic Programming-Based Contingency Constrained Optimal Power Flow

by

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Abstract

The focus of this thesis is formulation and development of a mathematical framework for the solution of the contingency constrained optimal power flow (OPF) based on sequential quadratic programming. The contingency constrained optimal power flow minimizes the total cost of a base case operating state as well as the expected cost of recovery from contingencies such as line or generation outages. The sequential quadratic programming (SCP) OPF formulation has been expanded in order to recognize contingency conditions and the problem is solved as a single entity by an efficient interior point method. The new formulation takes into account the system corrective capabilities in response to contingencies introduced through ramp-rate constraints. Contingency constrained OPF is a very challenging problem, because each contingency considered introduces a new problem as large as the base case problem. By proper system reduction and benefits of constraint relaxation (active set) methods, in which transmission constraints are not introduced until they are violated, the size of the system can be reduced significantly Therefore, restricting our attention to the active set constraint set makes this large problem significantly smaller and computationally feasible.

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Contents

1 Introduction

1.1 Background and Motivation

Optimal power flow (OPF) as an optimization method for an energy management system (EMS) control center was developed in the 1960s and 1970's and since then has been an important function as a standard application. The classical OPF formulations were pioneered by Carpentier [1] and Dommel and Tinney [2]. Since then a great deal of research has been done and various optimization techniques have been used in order to find efficient solutions to this non-linear optimization problem.

OPF is a tool used for both the operation and planning of a power system. It can be intuitively explained in the following way. If we are to supply a given demand, and if we have generation units committed (participating in the dispatch), OPF gives an answer as to how much power each unit has to produce (dispatch) as well as how to adjust transformer settings in order to supply demand most economically, while respecting all the constraints imposed on the system.

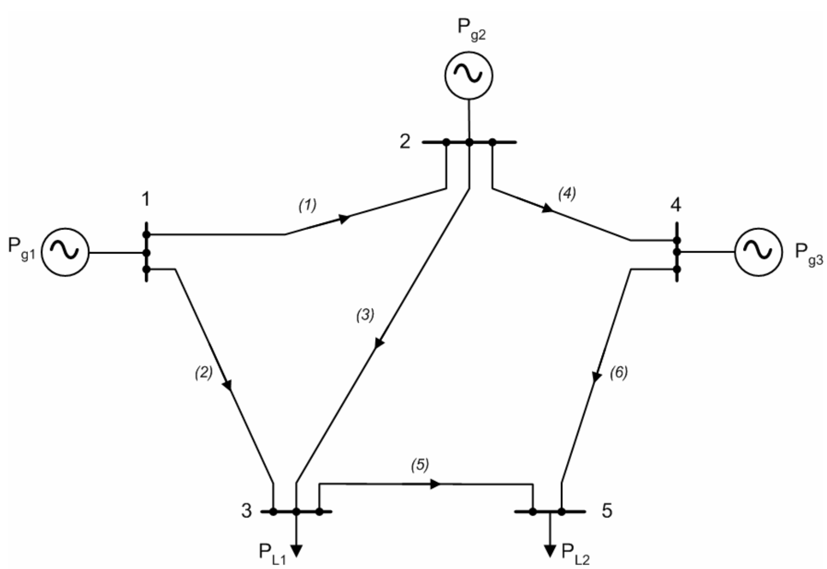


Fig. 1. Five-bus power system

The five bus network Fig. 1. will be considered as an example. Generators P_{g1}, P_{g2} P_{g3} are dispatchable sources of active and reactive power. For now let's assume that control variables are just active power generation P_{g1}, P_{g2} P_{g3}. Buses 3 and 5 are purely load buses P_{L1}, P_{L2}. The OPF problem is to minimize total cost of generation (P_{g1}, P_{g2} and P_{g3}) while satisfying the scheduled load, voltage, line flow and generation limits.

OPF is a computationally intensive tool when analyzing many generation plants, transmission lines and demands. Finally the engineering constraints and economic objectives for system operations are combined by formulating and solving the optimal power flow problem. OPF is used in economic analysis of the power system as well. Also, the OPF problem, besides generation dispatch, computes short-run marginal costs for each resource used in providing power as well as at each bus, which can be helpful in the design of transmission pricing and expansion policies. Marginal cost is considered an important concept in the design of emerging markets. However, it must be recognized that OPF marginal costs are static snapshots of the system conditional on one assumed set of supply and demand. In order to reflect multiple time periods (even for a few hours) OPF marginal prices would be calculated very often.

A contingency is a loss of one or more transmission equipment and/or generation units. Loss of a transmission line is usually due to a storm or automatic relaying action. The occurrence of a contingency is unpredictable; therefore it is of great importance for a system to operate in a such a way that corrective actions can be taken. Contingency analysis is often the most time-consuming function in an Energy Management System (EMS). A transmission system is said to be "secure" if it could continue to provide power that meets demand even if a contingency were to occur.

In the case of a generation outage, the lost generation will be supplied by the remaining generators, according to some specified redistribution pattern. Some plants need significant time to increase power and can not respond promptly to the contingency occurrence. It may be not possible to rely entirely on the economically most attractive plant to increase the output as fast as necessary; thus other, more expensive plants must be used at least temporarily.

The OPF formulation can include constraints that represent operation of the system after the occurrence of contingencies. Contingency constrained OPF tells how to dispatch power capacities and controlled components of the system if serious disturbances were to occur anywhere in the system. This solution of the OPF problem takes into account the security of the system and also allows the OPF to dispatch the system in defensive manner.

The list of all possible contingencies is very long, and considering all of those would be demanding in time as well as computational sense. Therefore it is important to distinguish those contingencies which produce post-contingency violations, and reduce the constraint list to those that results in violation. Most of the cases have no violations and can be disregarded. We have to find a way to select contingencies in such a way that only those that are likely to result in an overload or voltage limit violation will actually be studied in detail. The other cases will go unanalyzed.

Contingency constrained OPF may have many different scenarios, and involve simulation of system flows for each possible major disruption to the system, including an unplanned power outage, or a line outage (caused by lightning strike for example). If some disruptive event would be particularly problematic, special dispatch patterns including load shedding should be considered. In that case load shedding can be incorporated as a control as long as it is given an artificially high cost. Otherwise, the cheapest solution would be to shed as much load as possible.

If K is the total number of contingencies considered, each step of the algorithm requires solution of K+1 quadratic programming problems (one for each contingency and base case). Therefore efficient solution of this very large optimization problem is crucial.

1.2 Contribution of the Thesis

A mathematical framework for the sequential quadratic programming-based contingency constrained OPF is given. A potentially big problem is solved as a single entity using an interior point method and constraint relaxation (active set) method. This formulation takes into account the system corrective capabilities in response to contingencies. A program based on the proposed framework was written and contingency cases that consider line outages have been studied.

1.3 Outline of the Thesis

- Chapter 2 starts with an overview of the OPF. In particular, it defines basic terms associated with the OPF problem like operation objective, constraints, interior point method, and constraint relaxation. A sequential quadratic programming algorithm for base case OPF is reviewed. The chapter concludes with the formulation of a fast decoupled version of sequential quadratic programming.
- Chapter 3 gives a step by step formulation of sequential quadratic programming-based contingency constrained OPF. The chapter concludes with an outline of the algorithm.
- Chapter 4 presents numerical results, from applying the algorithm to the IEEE 14 and 30 bus networks, with concentration on line outages.
- Chapter 5 summarizes the thesis and discusses directions for future work.

2 Sequential Quadratic Programming Based Optimal Power Flow

2.1 Introduction

In this chapter, an overview of the OPF problem formulation will be given and methods used in its solution will be explained. The sequential quadratic programming (SQP) approach to the base case OPF as presented in [5], [6] will be briefly reviewed and a formulation based on decoupled power flow will be derived. The major feature of the SQP formulation is that the algorithm is divided into an outer linearization and an inner optimization loop. The system to be solved in the inner loop is of the size of the active set, which is potentially small.

The main benefit of the fast decoupled formulation is that the Jacobian matrix and some of the terms calculated through the iterative process are constant, greatly reducing the computational effort in factorization.

2.2 Operational objectives

As mentioned at the beginning, the OPF formulation has a single objective function. The most common objective functions are: minimum cost of operation, minimum active power transmission losses, minimum deviation from the specified point, minimum number of controls rescheduled. The most common objective function to be minimized is the cost of operation, which will be our objective function as well. The objective function usually depends on variables with direct cost (power generation) and variables without direct cost (voltage magnitude). Load shedding can be incorporated in the objective as well. It must be incorporated via a very high cost; otherwise the cheapest solution would be to shed as much load as possible.

The minimum cost of generation objective function is a sum of the costs of the generators participating in the dispatch. A critical part of this formulation is modeling the cost curves. The cost of thermal units is derived from the heat-rate curves which are quite often far from convex. Because convexity of the objective function is one of the assumptions for optimization methods employed in the solution of the OPF problem, a first approach is to approximate cost curves as convex polynomials. Other approximations, such as using an arbitrary number of linear segments are acceptable as well.
In our formulation cost curves are approximated by a quadratic polynomial of the form:

$$c_g(p_g) = ap_g^2 + bp_g + c$$

where p_g is in MW (or per unit) output of the generator and a, b, c are constant coefficients.

2.3 Constraints

As we stated, the OPF is a constrained optimization problem. The set of constraints can be divided into equality constraints and inequality constraints. The equality constraint set typically consists of power balance (active and reactive) at each node of the network which result from Kirchhoff's current law.
Another set of constraints are inequality constraints, which are usually limits resulting from network component limitations. A common set of inequality constraints consists of:

- Generator power constraints (P and Q)
- Line power constraints (P)
- Voltage, tap ratios, and phase shifter angle constraints

Generators are rated by maximum apparent power (S_{max}) which they can produce. The combination of P, Q produced by a generator must obey the apparent circle equation $P^2 + Q^2 \le S_{\max}^2$. The maximum active power (P_{max}) produced by generator is limited by the turbine's physical limits, while maximum reactive power (Q_{max}) is often determined so that heating of the rotor is within a prespecfied tolerance. Likewise, a minimum generation level is usually specified. Therefore for each generator in the network is subject to the following constraints:

$$P_k^{\min} \leq P_k \leq P_k^{\max}$$
$$Q_k^{min} \leq Q_k \leq Q_k^{max}$$

Besides generators, transformers provide an additional means of control of the flow of both active and reactive power. There are two types of controllable transformers: tap changers and phase shifters, although some transformers regulate both the magnitude and phase angle. Controllable transformers are those which provide a small adjustment of voltage magnitude, usually in the range ±10% or which shift the phase angle of the line voltages. A type of transformer designed for small adjustments of voltage rather than for changing voltage levels is called a regulating transformer.

2.4 Mathematical formulation of the OPF

Optimal power flow is formulated mathematically as the following constrained nonlinear optimization problem:

$$\begin{array}{ll} minimize & c(x,u) \\ subject\ to & g(x,u) = 0 \\ & f(x,u) \leq 0 \end{array} \tag{1}$$

The objective function is a scalar function. Two types of variables appear in the above optimization problem: x is a set of state variables (voltage magnitudes v and phase angles θ for each node in the network) and u is the set of controllable quantities in the system (generator outputs, adjustable transformers)

$$x = \begin{pmatrix} v \\ \theta \end{pmatrix} \qquad x \in \Re^{2n}$$

where n denotes the number of nodes (buses) in the network.

$$u = \begin{pmatrix} p_g \\ q_g \\ t_b \\ \varphi \end{pmatrix} \qquad\qquad u \in \Re^{n_u}$$

where n_u is the number of control variables: active power (p_g) reactive power (q_g) tap changing transformers (t_b) phase shifting transformers(φ).

The equality constraints *g(x,u)* are power balance equations (active and reactive) for each node in the network, occasionally augmented by a few special equality constraints such as specifying voltage at voltage controlled buses.
Inequalities *f(x,u)* are the limits on the control variables u, and the operating limits on the power system. Limits on the control variables are known as a "hard" limits (i.e., violation of these limits is not allowed) and operating limits are known as "soft" limits (i.e., small violations are tolerable). The set of inequality constraints prevent of dispatching generation that will lead to violating system limits.

In the past three decades, various optimization techniques have been proposed to solve the nonlinear OPF problem expressed in (1). A few implementations have been very successful. Difficulties with various techniques usually either from unacceptable time consumption for a problems involving large power networks. Techniques that are proposed can be categorized as:

- Gradient methods – these were the first approach to solving OPF and showed very slow convergence properties
- Sequential linear programming (SLP) algorithms based on the linearization of the original OPF problem. In an outer linearization loop the objective function and constraints (power flow equation) are linearized. The SLP problem is of the form:

$$\begin{aligned} &\min imize && c_x^T \Delta x + c_u^T \Delta u \\ &subject\ to && G_x \Delta x + G_u \Delta u = -g \\ &&& F_x \Delta x + F_u \Delta u \leq -f \end{aligned}$$

- Sequential quadratic programming (SQP) algorithms use the second order derivatives to improve the convergence rate. At each outer iteration the objective is approximated as a quadratic with a linear constaraint set. This SQP problem is solved iteratively until convergence is attained.

The above techniques vary in speed, cost of computation, and convergence properties.

In order to solve the optimization problem stated in (1), we have to develop necessary conditions for a minimum of the objective function subject to the given constraints. Therefore we will form the Lagrange function. The Lagrange function is formed by adding constraint functions multiplied by an undetermined multiplier vector (Lagrange multiplier λ and π) to the objective function. It is very important that Lagrange multipliers can be viewed as the optimization variables of auxiliary optimization problems, called *dual problems,* which will be helpful in applying interior point methods. The dual problem objective has the same optimal value and has as optimal solutions the Lagrange multipliers of the original problem.

Before defining the Lagrange function, we will convert the inequality constraint to equality constraint by adding a nonnegative slack variable (*s*). With the introduction of a slack variable the OPF is formulated as:

$$\begin{aligned} &\min imize && c(x,u) \\ &subject\ to && g(x,u) = 0 \\ &&& f(x,u) + s = 0 \\ &&& s \geq 0 \end{aligned}$$

The Lagrangian function for this problem is:

$$L = c(x,u) + \lambda^T g(x,u) + \pi^T (f(x,u) + s)$$

The necessary conditions for an extreme value of the objective function results when we take the partial derivative of the Lagrange function with respect to each variable and set those derivatives to zero. Those conditions are known as Karush-Kuhn-Tucker (KKT) conditions. Convergence is attained when the Karush-Kuhn-Tucker necessary conditions for optimality have been satisfied within practical accuracy.

$$\nabla_x L = \nabla_x c(x,u) + G_x^T \lambda + F_x^T \pi = 0$$

$$\nabla_u L = \nabla_u c(x,u) + G_u^T \lambda + F_u^T \pi = 0$$

$$\nabla_\lambda L = g(x,u) = 0$$

$$\nabla_\pi L = f(x,u) + s = 0$$

$$\nabla_s L = \Pi s = 0 \qquad \{\text{complementary slackness condition}\}$$

$$s, \pi \geq 0$$

where: $\Pi = diag(\pi_i)$

$$G_x = \frac{\partial g(x,u)}{\partial x} \qquad G_x \in \Re^{2n \times 2n}$$

$$G_u = \frac{\partial g(x,u)}{\partial u} \qquad G_u \in \Re^{2n \times n_u}$$

$$F_x = \frac{\partial f(x,u)}{\partial x} \qquad F_x \in \Re^{n_c \times 2n}$$

$$F_u = \frac{\partial f(x,u)}{\partial u} \qquad F_u \in \Re^{n_c \times n_u}$$

The structure of the above Jacobian matrices is discussed in Appendix I

The complementary slackness condition means that whenever the constraint $f(x,u) \leq 0$ is slack (meaning that $f(x,u) < 0$ and consequently $s > 0$) the constraint $\pi \geq 0$ must not be slack (meaning that $\pi = 0$) and vice versa.

Although a first idea to solve the above system of KKT conditions might be by direct application of Newton's method, experience shows that the domain of convergence can be quite small in many cases, a condition that ultimately leads to failure to converge. A more reliable and powerful is idea that of a barrier function and an interior point method.

2.5 Interior Point Method

The interior point method was developed by Nerendras Karamarkar in 1984 for linear programming, although many of the component ideas were known earlier. Experience indicates that the interior point method is algorithm of choice when solving large-scale problems, which OPF definitely is. The algorithm used for years for solving linear programming problems is the simplex method, which moves from one vertex of the feasible region to another while constantly attempting to improve the value of the objective function. An interior point method implies that progress towards a solution is made through the interior of the feasible region rather than its vertices. Karamarkar discovered how to trace such a path quickly.

There are three versions of the interior point method algorithm, the primal, the dual and the primal-dual. The primal-dual algorithm has been found to be very robust and is the method we use in this work.

The framework for developing an interior point method consists of three important parts:

- A barrier method for optimization with inequalities
- The Lagrange method for optimization with equalities
- Newton's method for solving the KKT conditions

After the transformation of inequality into equality constraints and introducing slack variables, one expands the cost function with a barrier function. The barrier or penalty function accommodates nonnegativity constraints on slack variables. A barrier function is continuous and grows without bound as any of the slack variables approach 0 from positive values (from the interior of their feasible region). The most common example of barrier function and the form we will use is

$$b(\mu, s) = -\mu \sum_{i=1}^{n} ln\, s_i$$

where μ is a scalar parameter called the barrier parameter. The value of μ is varied as the solution of the OPF progresses.

After introducing the barrier function, we can write the modified OPF formulation:

$$\begin{aligned} &minimize && c(x,u) - \mu \sum_{i=1}^{n} \ln s_i \\ &subject\ to && g(x,u) = 0 \\ & && f(x,u) + s = 0 \end{aligned}$$

The Lagrange function for this problem is:

$$L = c(x,u) - \mu \sum_{i=1}^{n} \ln s_i + \lambda^T g(x,u) + \pi^T (f(x,u) + s)$$

and the KKT conditions are:

$$\begin{aligned} &\nabla_x L = \nabla_x c(x,u) + G_x^T \lambda + F_x^T \pi \\ &\nabla_u L = \nabla_u c(x,u) + G_u^T \lambda + F_u^T \pi \\ &\nabla_\lambda L = g(x,u) \\ &\nabla_\pi L = f(x,u) + s \\ &\nabla_s L = -\mu \frac{1}{s_i} + \pi_i \qquad for\ i = 1 \ldots n \end{aligned}$$

The complementary slackness condition in the primal-dual interior point formulation is replaced by:

$$\Pi s = \mu e$$

where e is a vector of ones of appropriate dimension.

In general terms, the next step would be to apply Newton's method to the KKT conditions, in other words to linearize the KKT conditions. Those linearized KKT conditions can be interpreted as KKT conditions of the quadratic Lagrangian function, and that is the origin of the name "sequential quadratic programming" (SQP). At each iteration the linearized KKT conditions are the KKT conditions of a quadratic subproblem. More details about the solution process will be deferred to later chapters. While the concentration will be on the SQP techniques presented in [5], [6].

2.6 Constraint Relaxation Method

In order to make the OPF algorithm efficient another very important method known as a *constraint relaxation* or an *active set method* will be employed. In this technique, we ignore constraints until they are violated. Thus the set of active inequality constraints is identified by the set of indices of the constraints that are satisfied as equations (i.e. $f(x,u)=0$). The set of inequality constraints whose indices lie in the active set are said to be *active* or *binding* while the remainder are *inactive*. The inactive constraints may be ignored. The Lagrange multipliers for inequality constraints become nonzero only when the inequalities become active (binding) or all in the active set.

Generally, only a small percentage of the total transmission constraints become active, greatly reducing the size of the system. Numerical examples presented in [14] show significant reduction in problem size achieved in practice by the active set method. The aim of the algorithm must be to discover which constrains are active. A heruristic such as adding to the active set just the most violated of the newly constraints and discarding the remaining violations has proven to be very efficient. Thus, the algorithm to be explained in chapter 3 will rely on an active set method, one of the key tools in building an efficient algorithm. For the contingency constrained OPF active set is built for each contingency as well as for the base case.

2.7 Full AC case

The OPF formulation considered is:

$$
\begin{array}{ll}
\text{minimize} & c(x,u)-\mu\sum_{i=1}^{n} \ln s_i \\
\text{subject to} & g(x,u)=0 \\
 & f(x,u)+s=0
\end{array}
$$

The Lagrangian function for this problem is:

$$L = c(x,u) - \mu \sum_{i=1}^{n_c} ln\, s_i + \lambda^T g(x,u) + \pi^T (f(x,u) + s)$$

and the KKT conditions are:

$$\nabla_x L = \nabla_x c(x,u) + G_x^T \lambda + F_x^T \pi = 0$$
$$\nabla_u L = \nabla_u c(x,u) + G_u^T \lambda + F_u^T \pi = 0$$
$$\nabla_\lambda L = g(x,u) = 0$$
$$\nabla_\pi L = f(x,u) + s = 0$$
$$\nabla_s L = -\mu \frac{1}{s_i} + \pi_i = 0 \qquad for\ i = 1 \ldots n_c$$

In the general AC case the matrix G_x has following block matrix form:

$$G_x = \begin{pmatrix} G_{Pv} & G_{Pa} \\ G_{Qv} & G_{Qa} \end{pmatrix}$$

Including a power balance equation for the reference bus (subscript r denotes the reference bus), and linearizing the above KKT conditions will produce

$$W_{xx}\Delta x + W_{xu}\Delta u + G_x^T \lambda + G_{rx}^T \lambda_r + F_x^T \pi = b_x \tag{2}$$
$$W_{ux}\Delta x + W_{uu}\Delta u + G_u^T \lambda + G_{ru}^T \lambda_r + F_u^T \pi = b_u \tag{3}$$
$$G_x \Delta x + G_u \Delta u = b_\lambda \tag{4}$$
$$G_{rx} \Delta x + G_{ru} \Delta u = b_r \tag{5}$$
$$F_x \Delta x + F_u \Delta u + s = b_\pi \tag{6}$$
$$\Pi S e = \mu e \tag{7}$$

The right hand side is:

$$b_u = -\nabla_u c(x,u)$$
$$b_x = -\nabla_x c(x,u)$$
$$b_\lambda = -g(x,u)$$

$$b_r = -g_r(x,u)$$

$$b_\pi = -f(x,u)$$

The Hessian matrix (W_{xx}, W_{xu}, W_{ux}, W_{uu}) is a symmetric matrix of second partial derivatives. Each element of the Hessian terms is a linear combination of the second partial derivatives of the power flow equation. The elements of the Hessian matrix represent the coupling between the variables θ, V, transformer turns ratio against each other and are usually very small. This property is exploited in the decoupled formulation of OPF [9] where second order terms are set to zero. The formula for the Hessian elements follows:

$$W_{xx} = \nabla^2_{xx} c(x,u) + \sum_{i=1}^{n}\left(\frac{\partial^2 g_{pi}}{\partial x^2}\lambda_{pi} + \frac{\partial^2 g_{qi}}{\partial x^2}\lambda_{qi}\right) + \sum_{i=1}^{n_c}\frac{\partial^2 f_i}{\partial x^2}\pi_i$$

$$W_{xu} = \nabla^2_{xu} c(x,u) + \sum_{i=1}^{n}\left(\frac{\partial^2 g_{pi}}{\partial x \partial u}\lambda_{pi} + \frac{\partial^2 g_{qi}}{\partial x \partial u}\lambda_{qi}\right) + \sum_{i=1}^{n_c}\frac{\partial^2 f_i}{\partial x \partial u}\pi_i$$

$$W_{ux} = W_{xu}^T$$

$$W_{uu} = \nabla^2_{uu} c(x,u) + \sum_{i=1}^{n}\left(\frac{\partial^2 g_{pi}}{\partial u^2}\lambda_{pi} + \frac{\partial^2 g_{qi}}{\partial u^2}\lambda_{qi}\right) + \sum_{i=1}^{n_c}\frac{\partial^2 f_i}{\partial u^2}\pi_i$$

In the above system the reference bus power balance equation is included through terms G_{rx}, G_{ru}, and λ_r. Recall that the power flow Jacobian is singular; therefore the power balance equality for the reference bus is replaced with an equality constraint forcing the reference bus angle to be zero (see Appendix I).

If we express λ and Δx from equations (2) and (4) we will get

$$\begin{pmatrix}\Delta x \\ \lambda\end{pmatrix} = \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0\end{pmatrix}^{-1}\begin{pmatrix} b_x - W_{xu}\Delta u - G_{rx}^T\lambda_r - F_x^T\pi \\ b_\lambda - G_u \Delta u\end{pmatrix} \tag{8}$$

Substituting equation (8) into the equations (3), (5), (6), (7) yields the following reduced system:

$$\overline{W}_{uu}\Delta u + \overline{G}_{ru}^T\lambda_r + \overline{F}_u^T\pi = \overline{b}_u$$
$$\overline{G}_{ru}\Delta u = \overline{b}_r$$
$$\overline{F}_u\Delta u + s = \overline{b}_\pi$$
$$\Pi S e = \mu e$$

The definitions of the terms in the reduced system and the computational procedure for their calculation can be found in Appendix III. A more detailed derivation can be found in [6].

If we expand the above system of equations about s and π we get:

$$\overline{W}_{uu}\Delta u + \overline{G}_{ru}^T\lambda_r + \overline{F}_u^T\Delta\pi = \overline{b}_u - \overline{F}_u^T\pi$$
$$\overline{G}_{ru}\Delta u = \overline{b}_r$$
$$\overline{F}_u\Delta u + \Delta s = \overline{b}_\pi - Se$$
$$\Pi\Delta s + S\Delta\pi = \mu e - \Pi S e$$

If we eliminate Δs from the last equation

$$\Delta s = \Pi^{-1}(\mu e - \Pi S e - S\Delta\pi)$$

The system in matrix form will be:

$$\begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T & \overline{F}_u^T \\ \overline{G}_{ru} & 0 & 0 \\ \overline{F}_u & 0 & -\Pi^{-1}S \end{pmatrix}\begin{pmatrix} \Delta u \\ \lambda_r \\ \Delta\pi \end{pmatrix} = \begin{pmatrix} \overline{b}_u - \overline{F}_u^T\pi \\ \overline{b}_r \\ \overline{b}_\pi - \mu\Pi^{-1}e \end{pmatrix} \tag{9}$$

In order to compute with this system, the following factorization will be performed

$$U^T D U = \begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T \\ \overline{G}_{ru} & 0 \end{pmatrix}$$

and following variables introduced:

$$F^T = \begin{pmatrix} \overline{F}_u^T \\ 0 \end{pmatrix}$$

$$y = \begin{pmatrix} \Delta u \\ \lambda_r \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \bar{b}_u \\ \bar{b}_r \end{pmatrix}$$

$$\hat{b}_\pi = \bar{b}_\pi - \mu \Pi^{-1} e$$

The system (9) can be rewritten in the following way:

$$\begin{pmatrix} U^T D U & F^T \\ F & -\Pi^{-1} S \end{pmatrix} \begin{pmatrix} y \\ \Delta\pi \end{pmatrix} = \begin{pmatrix} b_1 - F^T \pi \\ \hat{b}_\pi \end{pmatrix}$$

Its solution can be written:

$$y = U^{-1} D^{-1} U^{-T} (b_1 - F^T \pi - F^T \Delta\pi) \tag{10}$$

$$F\, y - \Pi^{-1} S\, \Delta\pi = \hat{b}_\pi \tag{11}$$

after substitution of equation (10) into (11) and some algebra we get:

$$\left(-\Pi^{-1} S - F U^{-1} D^{-1} U^{-T} F^T\right) \Delta\pi = \hat{b}_\pi - F U^{-1} D^{-1} U^{-T} (b_1 - F^T \pi) \tag{12}$$

To calculate $\Delta\pi$ from the above equation, introduce

$$\bar{F} = U^{-T} F^T$$

where $\bar{F}$ is calculated by fast forward substitution

$$U^T \bar{F} = F^T$$

Now equation (12) can be solved by writing it as

$$\left(-\Pi^{-1} S - \bar{F}^T D^{-1} \bar{F}\right) \Delta\pi = \hat{b}_\pi - \bar{F}^T D^{-1} U^{-T} b_1 + \bar{F}^T D^{-1} \bar{F} \pi$$

$$\left(-\Pi^{-1} S - C\right) \Delta\pi = \hat{b}_\pi - \bar{F}^T D^{-1} \bar{b}_1 + C\pi \tag{13}$$

where

$$C = \bar{F}^T D^{-1} \bar{F}$$

$$\overline{b}_1 = U^{-T} b_1$$

In order to calculate $\Delta\pi$ from equation (13), a $U^T DU$ factorization of the symetric matrix $\Pi^{-1}S - C$ must be performed. The latter is potentially a small dense matrix with the size of the active constraint set. When we have calculated $\Delta\pi$, we can obtain Δs

$$\Delta s = \Pi^{-1}(\mu e - \Pi S e - S\Delta\pi)$$

and y can be calculated in three steps (forward/backward substitution and division by diagonal) from the equation (10).

Now that we have Δu and λ_r , therefore Δx and λ can be calculated as well from equation (8). The complete algorithm can be found in [6].

2.8 Fast Decoupled case

What the decoupled power flow does is decomposes the load flow problem into real and reactive subproblems. A fast and reliable load flow calculation based on the decoupling of the active and reactive subproblems may be essential for the computationally intensive contingency constrained OPF. This section will give formulation of the sequential quadratic programming OPF based on fast decoupled power flow.

The formulation of the fast decoupled OPF is the same as for the AC case

$$\begin{aligned} &\textit{minimize} && c(x,u) - \mu \sum_{i=1}^{n} \ln s_i \\ &\textit{subject to} && g(x,u) = 0 \\ & && f(x,u) + s = 0 \end{aligned}$$

where power balance equations $g(x,u)=0$ are defined according to fast decoupled power flow [9] approach:

$$B'\Delta\theta = \Delta P$$

$$B''\Delta V = \Delta Q$$

The matrices B' and B'' are defined in Appendix I

The problem Lagrangian with logarithm barrier function is:

$$L(x,u,\lambda,\pi) = c(x,u) - \mu\sum_{i=1}^{n_c} ln\, s_i + \lambda^T g(x,u) + \pi^T(f(x,u)+s)$$

The KKT conditions from the problem Lagrangian are given by:

$$\nabla_x L = \nabla_x c(x,u) + G_x^T\lambda + F_x^T\pi = 0$$

$$\nabla_u L = \nabla_u c(x,u) + G_u^T\lambda + F_u^T\pi = 0$$

$$\nabla_\lambda L = g(x,u) = 0$$

$$\nabla_\pi L = f(x,u) + s = 0$$

$$\nabla_s L = \Pi s = \mu e$$

Recall that in the general case, the matrix G_x has form:

$$G_x = \begin{pmatrix} G_{Pv} & G_{Pa} \\ G_{Qv} & G_{Qa} \end{pmatrix}$$

while in the decoupled power flow case the matrix G_x can be written in the 2×2 block matrix form

$$G_X = \begin{pmatrix} 0 & G_{Pa} \\ G_{QV} & 0 \end{pmatrix} = \begin{pmatrix} 0 & B' \\ B'' & 0 \end{pmatrix}$$

The linearized KKT conditions with respect to x and u only are:

$$W_{uu}\Delta u + W_{ux}\Delta x + G_u^T\lambda + G_{ru}^T\lambda_r + F_u^T\pi = b_u$$

$$W_{xu}\Delta u + W_{xx}\Delta x + G_x^T\lambda + G_{rx}^T\lambda_r + F_x^T\pi = b_x$$

$$G_u\Delta u + G_x\Delta x = b_\lambda$$

$$G_{rx}\Delta x + G_{ru}\Delta u = b_r$$

$$F_u\Delta u + F_x\Delta x + s = b_\pi$$

$$\Pi S e = \mu e$$

where right hand side is:

$$b_u = -\nabla_u c(x,u)$$

$$b_x = -\nabla_x c(x,u)$$

$$b_\lambda = -g(x,u)$$

$$b_r = -g_r(x,u)$$

$$b_\pi = -f(x,u)$$

The Hessian terms as defined for general case can be further simplified in the decoupled case. The second order derivatives corresponding to both equality and inequality constraints are zero. Therefore:

$$W_{xx} = \nabla^2_{xx} c(x,u)$$

$$W_{xu} = \nabla^2_{xu} c(x,u)$$

$$W_{ux} = W^T_{xu}$$

$$W_{uu} = \nabla^2_{uu} c(x,u)$$

The generator cost function as defined is quadratic and depends just on the control variables u; thus the only nonzero term is $W_{uu} = \nabla^2_{uu} c(x,u)$, and it is constant due to the quadratic cost function.

Our system will have following form:

$$W_{uu}\Delta u + G_u^T \lambda + G_{ru}^T \lambda_r + F_u^T \pi = b_u \quad (14)$$

$$G_x^T \lambda + G_{rx}^T \lambda_r + F_x^T \pi = b_x \quad (15)$$

$$G_u \Delta u + G_x \Delta x = b_\lambda \quad (16)$$

$$G_{ru} \Delta u + G_{rx} \Delta x = b_r \quad (17)$$

$$F_u \Delta u + F_x \Delta x + s = b_\pi \quad (18)$$

$$\Pi S e = \mu e \quad (19)$$

System reduction will be conducted first with the expression for λ and Δx from equations (15) and (16). That process yields:

$$\begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}\begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = \begin{pmatrix} b_\lambda^{(k)} - G_u \Delta u \\ b_x^{(k)} - G_{rx}^T \lambda_r - F_x^T \pi \end{pmatrix}$$

$$\begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \left(\begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} - \begin{pmatrix} G_u \\ 0 \end{pmatrix} \Delta u - \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix} \lambda_r - \begin{pmatrix} 0 \\ F_x^T \end{pmatrix} \pi \right) \tag{20}$$

The matrix $\begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}$ in the fast decoupled model is a constant matrix and is factored only once.

Substitution of (20) into the remaining equations of the system is the next step. Equation (14) can be rewritten as:

$$W_{uu}\Delta u + \begin{pmatrix} 0 & G_u^T \end{pmatrix}\begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} + G_{ru}^T \lambda_r + F_u^T \pi = b_u$$

After substituting equation (20) it has form:

$$W_{uu}\Delta u + \begin{pmatrix} 0 & G_u^T \end{pmatrix}\begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \left(\begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} - \begin{pmatrix} G_u \\ 0 \end{pmatrix} \Delta u - \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix} \lambda_r - \begin{pmatrix} 0 \\ F_x^T \end{pmatrix} \pi \right)$$
$$+ G_{ru}^T \lambda_r + F_u^T \pi = b_u$$

The following variables are introduced

$$\overline{W}_{uu} = W_{uu} - \begin{pmatrix} 0 & G_u^T \end{pmatrix}\begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1}\begin{pmatrix} G_u \\ 0 \end{pmatrix} = W_{uu}$$

$$\overline{F}_u = F_u - \begin{pmatrix} 0 & F_x \end{pmatrix}\begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ G_u \end{pmatrix}$$

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} 0 & G_{rx} \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_u \end{pmatrix}$$

$$\overline{b}_u = b_u - \begin{pmatrix} 0 & G_u^T \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix}$$

Equation (14) can be rewritten

$$\overline{W}_{uu} \Delta u + \overline{G}_{ru}^T \lambda_r + \overline{F}_u^T \pi = \overline{b}_u$$

Next, equation (17) can be rewritten in the matrix form:

$$G_{ru} \Delta u + \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = b_r$$

substituting equation (20) into (17) gives

$$\left(G_{ru} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} G_u \\ 0 \end{pmatrix} \right) \Delta u - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix} \lambda_r -$$
$$- \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F_x^T \end{pmatrix} \pi = b_r - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix}$$

The following variables for the modified equation (17) can be introduced

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} G_u \\ 0 \end{pmatrix}$$

$$\overline{b}_r = b_r - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix}$$

and following two terms are equal to zero

$$\begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix}$$

$$\begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F_x^T \end{pmatrix}$$

With terms defined above equation (17) can be rewritten

$$\overline{G}_{ru} \Delta u = \overline{b}_r$$

Equation (18) in the matrix form can be rewritten in the following way:

$$F_u \Delta u + \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} + s = b_\pi$$

Substituting equation (20) into (18) yields

$$\left(F_u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} G_u \\ 0 \end{pmatrix} \right) \Delta u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix} \lambda_r -$$

$$- \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F_x^T \end{pmatrix} \pi + s = b_\pi - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix}$$

The following variables can be defined:

$$\hat{F}_u = F_u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} G_u \\ 0 \end{pmatrix}$$

$$\overline{b}_\pi = b_\pi - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix}$$

and following two terms are zero

$$\begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F_x^T \end{pmatrix}$$

$$\begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix}$$

With terms defined above equation (18) can be rewritten

$$\hat{F}_u \Delta u + s = \overline{b}_\pi$$

In order to prove that our system is symmetric we have to show that $\hat{F}_u^T = \overline{F}_u^T$.
From the definition of term $\overline{F}_u$ follows

$$\overline{F}_u = F_u - \begin{pmatrix} 0 & F_x \end{pmatrix} \begin{pmatrix} G_x^{-1} & 0 \\ 0 & G_x^{-T} \end{pmatrix} \begin{pmatrix} 0 \\ G_u \end{pmatrix}$$

applying block multiplication yields

$$\overline{F}_u = F_u - F_x G_x^{-T} G_u$$

and taking transpose of the above term yields

$$\overline{F}_u^T = F_u^T - G_u^T G_x^{-1} F_x^T$$

The same set of operations can be performed on term $\hat{F}_u$

$$\hat{F}_u = F_u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x^{-1} & 0 \\ 0 & G_x^{-T} \end{pmatrix} \begin{pmatrix} G_u \\ 0 \end{pmatrix}$$

$$\hat{F}_u = F_u - F_x G_x^{-1} G_u$$

$$\hat{F}_u^T = F_u^T - G_u^T G_x^{-T} F_x^T$$

G_x is symmetric matrix therefore follows $G_x^{-1} = G_x^{-T}$ which ultimately leads to the conclusion that $\hat{F}_u^T = \overline{F}_u^T$

Now the reduced quadratic problem will have the same form as in the AC case but with terms $\overline{W}_{uu}$ and $\overline{G}_{ru}$ that will remain constant during iteration process.

$$\overline{W}_{uu} \Delta u + \overline{G}_{ru}^T \lambda_r + \overline{F}_u^T \pi = \overline{b}_u$$

$$\overline{G}_{ru} \Delta u = \overline{b}_r$$

$$\overline{F}_u \Delta u + s = \overline{b}_\pi$$

$$\Pi S e = \mu e$$

Detailed computational procedure for calculation of terms in the reduced system can be found in Appendix III. Expand this system of equations about s and π

$$\overline{W}_{uu}\Delta u + \overline{G}_{ru}^T\lambda_r + \overline{F}_u^T\Delta\pi = \overline{b}_u - \overline{F}_u^T\pi$$

$$\overline{G}_{ru}\Delta u = \overline{b}_r$$

$$\overline{F}_u\Delta u + \Delta s = \overline{b}_\pi - S$$

$$\Pi\Delta s + S\Delta\pi = \mu e - \Pi S e$$

Eliminate Δs from the last equation

$$\Delta s = \Pi^{-1}\left(\mu e - \Pi S e - S\Delta\pi\right)$$

and substitute in the rest of the system to obtain:

$$\begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T & \overline{F}_u^T \\ \overline{G}_{ru} & 0 & 0 \\ \overline{F}_u & 0 & -\Pi^{-1}S \end{pmatrix}\begin{pmatrix} \Delta u \\ \lambda_r \\ \Delta\pi \end{pmatrix} = \begin{pmatrix} \overline{b}_u - \overline{F}_u^T\pi \\ \overline{b}_r \\ \overline{b}_\pi - \mu\Pi^{-1}e \end{pmatrix} \tag{21}$$

The next step is the same as for the full AC case explained before, with the observation that the U^TDU factorization is performed just once because the matrices $\overline{W}_{uu}$ and $\overline{G}_{ru}$ are constant

$$U^TDU = \begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T \\ \overline{G}_{ru} & 0 \end{pmatrix}$$

and following variables introduced:

$$F^T = \begin{pmatrix} \overline{F}_u^T \\ 0 \end{pmatrix}$$

$$y = \begin{pmatrix} \Delta u \\ \lambda_r \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \overline{b}_u \\ \overline{b}_r \end{pmatrix}$$

$$\hat{b}_\pi = \overline{b}_\pi - \mu \Pi^{-1} e$$

Then, system (21) can be rewritten on following way

$$\begin{pmatrix} U^T D U & F^T \\ F & -\Pi^{-1} S \end{pmatrix} \begin{pmatrix} y \\ \Delta\pi \end{pmatrix} = \begin{pmatrix} b_1 - F^T \pi \\ \hat{b}_\pi \end{pmatrix}$$

Following the same steps as in full AC case $\Delta\pi$ is solved from the following equation

$$\left(-\Pi^{-1} S - C\right) \Delta\pi = \hat{b}_\pi - \overline{F}^T D^{-1} \overline{b}_1 + C\pi$$

When we have calculated $\Delta\pi$, we can obtain Δs

$$\Delta s = \Pi^{-1} \left(\mu e - \Pi S e - S \Delta\pi\right)$$

and y.

Finally, Δx and λ can be calculated as well from equation (20):

$$\begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix} \begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = \left(\begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} - \begin{pmatrix} G_u \\ 0 \end{pmatrix} \Delta u - \begin{pmatrix} 0 \\ G_{rx}^T \end{pmatrix} \lambda_r - \begin{pmatrix} 0 \\ F_x^T \end{pmatrix} \pi \right)$$

Partitioning Δx and λ as shown below. Call the right hand side of the equation a vector *q,* and partition it similarly:

$$\Delta x = \begin{pmatrix} \Delta v \\ \Delta\theta \end{pmatrix} \qquad \Delta\lambda = \begin{pmatrix} \lambda_p \\ \lambda_q \end{pmatrix} \qquad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

Because we are dealing with decoupled power flow equations, we need only to find LU factors of the blocks B' and B'' in above block matrix. This simplification is another computational savings.

This computation can be conducted as follows

$$B' = U_1^T D_1 U_1$$

$$B'' = U_2^T D_2 U_2$$

$$\begin{pmatrix} 0 & B' & 0 & 0 \\ B'' & 0 & 0 & 0 \\ 0 & 0 & 0 & B'' \\ 0 & 0 & B' & 0 \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta\theta \\ \lambda_p \\ \lambda_q \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$B'\,\Delta\theta = q_1 \quad \Leftrightarrow \quad U_1^T D_1 U_1\,\Delta\theta = q_1$$

$$B''\,\Delta v = q_2 \quad \Leftrightarrow \quad U_2^T D_2 U_2\,\Delta v = q_2$$

$$B''\,\Delta\lambda_q = q_3 \quad \Leftrightarrow \quad U_2^T D_2 U_2\,\Delta\lambda_q = q_3$$

$$B'\,\Delta\lambda_p = q_4 \quad \Leftrightarrow \quad U_1^T D_1 U_1\,\Delta\lambda_p = q_4$$

Each of the unknowns $\Delta\theta$, Δv, $\Delta\lambda_p$, $\Delta\lambda_q$ is calculated by forward/backward substitution and division by the diagonal.

3 Contingency Constrained OPF via Sequential Quadratic Programming

3.1 Introduction

A contingency is an unpredictable disturbance to the transmission or generation facilities. Contingency constrained OPF recognizes the need to operate the system successfully, i.e. within operating limits when a contingency occurs. It has been recognized that with the basic OPF formulation it may not be possible to keep the system in a normal state after a contingency occurs.

By introducing contingencies into the problem we are introducing uncertainty. Contingency constrained OPF answers how to dispatch power capacities and control components of the system to accommodate serious disturbances anywhere in the system. This optimization problem is a cumbersome computational problem when all possible contingencies are considered.

Contingency constrained OPF can be formulated on two ways:

- so called 'safe' or 'preventive' contingency constrained OPF, which does not allow any rescheduling in response to a contingency
- contingency constrained OPF with corrective rescheduling which allows control actions shortly after the occurrence of a contingency

A first possible approach to contingency constrained OPF is the following framework, in which the base case is simply expanded to include contingency constraints.

$$
\begin{array}{lll}
\text{minimize} & c(x,u) & \\
\text{subject to} & g(x,u)=0 & \\
 & f(x,u)\le 0 & \\
 & g_\omega(x_\omega,u)=0 & \omega=1,\dots,k \\
 & f_\omega(x_\omega,u)\le 0 & \omega=1,\dots,k
\end{array}
$$

where: x is the pre-contingency (base case) state vector, x_ω is the post-contingency state vector for contingency ω and u is the control vector for the base case as well as for each contingency case. Solution of this constrained optimization problem is so-called 'safe' solution, which means that the control vector u is calculated such that the system operates successfully under all contingencies. This formulation, does not take into account the ability to change control settings in the event of a contingency. An approach which allows adjustment of control variables after the occurrence of a contingency is known as a corrective rescheduling method.

The importance of corrective rescheduling will be illustrated in the following simple example presented in [4]. In a simple power system shown on Fig. 2.1. generators 1 and 2 are participating in a dispatch to supply a 200 MW load at bus 2.

Generator	Min. Generation [MW]	Max Generation [MW]	Incremental cost [$/MW]
1	50	200	1
2	0	120	2

Table 2.1. Generator data

Line	Max. line flow [MW]
1	100
2	120

Table 2.2. Line data

Fig. 2.1. Two-bus System

Since generator 1 has the lower incremental cost, a pure economic dispatch yields following solution:

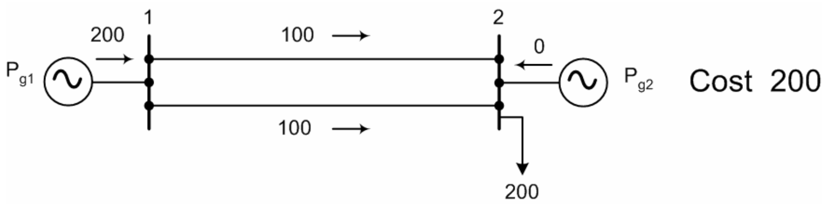


Fig. 2.2. Pure economic dispatch

Now consider a contingency in which line 2 (notice that line 2 has higher capacity) is out. Performing ‘safe’ contingency constrained OPF will result in the dispatch shown in Fig. 2.3. That dispatch guarantees that the system will operate successfully under the base case and the contingency case.

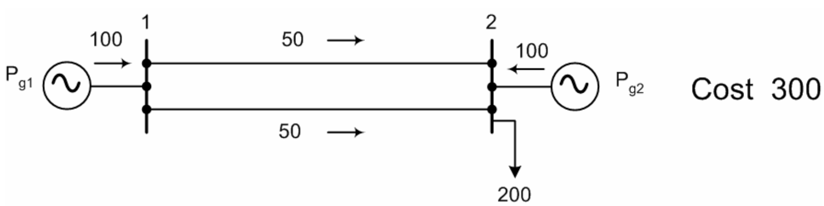


Fig. 2.3. ‘safe’ solution

It can be seen from Fig. 2.3. that an increase in security comes with an increase in cost, because the more expensive generator 2 participates in the dispatch and the less expensive generator 1 has a lower output than if the contingency did not occur.

Finally, consider a corrective rescheduling method. Suppose that the corrective capabilities of generators 1 and 2 are 40 MW and 35 MW respectively, meaning that each can increase its generation by these amounts in response to a contingency occurrence. This expression is known as a generator ramping constraint and can be formulated as the constraints

$$\left| p_{g1} - p_{g1}^{c} \right| \leq 40MW$$

$$\left| p_{g2} - p_{g2}^{c} \right| \leq 35MW$$

This situation will allow the cheaper generator 1 to dispatch at a higher generation value than that obtained by the ‘safe’ solution; the preventive dispatch will produce lower operating cost. This scenario is presented in Fig. 2.4.

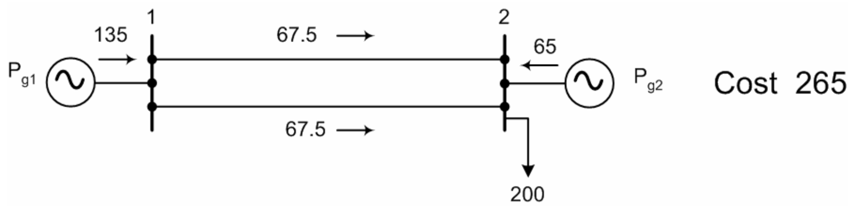


Fig. 2.4. Contingency constrained dispatch with corrective rescheduling

It is easy to demonstrate that contingency constrained dispatch with corrective rescheduling will not lead to overloads in the case of a line outage because generator 1 can be redispatched to a lower value according to its ramp rate constraint. Therefore the same level of security is obtained with a lower production cost (265 instead of 300). From the previous example it can be noticed that the formulation without redispatch is a conservative formulation. That formulation is conservative because it forbids post-contingency changes in control settings.

Contingency constrained OPF with rescheduling is implemented by adding additional set of constraints to the base case problem formulation. Each additional set describes a contingency by a set of power balance equations, inequality constraints, and a new type of constraint known as ramp-rate (or coupling or intertemporal) constraints. These ramp-rate constraints take into account the system’s corrective capabilities after the outage has occurred.

The problem formulation that includes ramp-rate constraints is known as OPF with post-contingency corrective rescheduling. The post contingency state of the power system is that immediately after a contingency occurs in which some line limits, bus limits or other constraints might be violated. In this state some corrective actions must be taken, which place the system in a secure state. This problem was first presented in [4] using Bender’s decomposition, although without modeling the probability of occurrence of the different contingencies.

3.2 Mathematical formulation of the contingency constrained OPF

The contingency constrained OPF problem may be formulated as a single optimization problem which includes a base case and a set of contingency cases coupled with intertemporal constraints (ramp-rate generator limits).
The mathematical formulation of contingency constrained OPF with corrective rescheduling is as follows:

$$\begin{array}{lll}
minimize & c(x,u) + E_{\omega}\{c_{\omega}(u_{\omega})\} & \\
subject\ to & g(x,u) = 0 & \\
 & f(x,u) \leq 0 & \\
 & g_{\omega}(x_{\omega},u_{\omega}) = 0 & \omega = 1,\ldots,k \\
 & f_{\omega}(x_{\omega},u_{\omega}) \leq 0 & \omega = 1,\ldots,k \\
 & h(u,u_{\omega}) \leq 0 &
\end{array} \qquad (1)$$

where:

$g\ (x,u)$ – power balance equations for base case

$f\ (x,u)$ – set of inequality constraints for base case

$g_{\omega}\ (x_{\omega}, u_{\omega})$ – power balance equations for each contingency case

$f_{\omega}\ (x_{\omega}, u_{\omega})$ – set of inequality constraints for contingency case

$h\ (u, u_{\omega})$ – ramp-rate constraints

E_{ω} – denotes expected value

ω – is the set of possible contingencies ω=1…k

The objective function in (1) includes the total cost of operation in the pre-contingency or base case as well as the expected cost of recovery from all contingencies. The additional set of state and control variables (x_{ω},u_{ω}) consists of the state variables under contingency and the control actions (post-contingency control adjustments) taken in response to contingency ω.

As mentioned before, the number of possible contingencies (including multiple outages) can be enormous. Not all contingencies have the same likelihood of occurrence, which leads us to assigning a probability to each contingency considered. Thus, by modeling contingency probabilities we can formulate the optimal power flow as a stochastic programming problem. This formulation is also called the stochastic OPF. We may assign a probability of an outage which in general is not a uniform; e.g., some lines are more prone to outages due to lightning than others.

Because we are dealing with a finite set of events, the expected value is computed by summation

$$E_{\omega}\{c_{\omega}(u_{\omega})\} = \sum_{\omega=1}^{k} p_{\omega} c_{\omega}(u_{\omega})$$

Furthermore, if we assume a linear cost function for each contingency, we have

$$E_{\omega}\{c_{\omega}(u_{\omega})\} = \sum_{\omega=1}^{k} p_{\omega} d_{\omega}^{T} u_{\omega}$$

Now let us introduce ramp-rate constraints. Ramp-rate limits are inequality constraints of the following form:

$$\underline{\Delta} \leq u - u_{\omega} \leq \overline{\Delta} \qquad \omega = 1 \ldots k$$

where $\underline{\Delta}$ and $\overline{\Delta}$ are lower and upper ramp-rate limits on change in generation level. The ramping constant Δ is usually defined as a percentage of generator capacity (i.e. 10% – 15%)

The set of ramp-rate constraints on each contingency can be organized on following way:

$$h_\omega = \begin{pmatrix} u_1 - u_{\omega 1} \le \overline{\Delta}_1 \\ u_2 - u_{\omega 2} \le \overline{\Delta}_2 \\ \vdots \\ u_{ng} - u_{\omega ng} \le \overline{\Delta}_{ng} \\ -u_1 + u_{\omega 1} \le \underline{\Delta}_1 \\ -u_2 + u_{\omega 2} \le \underline{\Delta}_2 \\ \vdots \\ -u_{ng} + u_{\omega ng} \le \underline{\Delta}_{ng} \end{pmatrix} \qquad \text{with dimension } (2n_g \times 1)$$

The contingency constrainted OPF will be solved as a single entity by an interior point method. The general OPF formulation can be transformed by introducing slack variables (s, s_ω, σ) and transforming inequality constraints into equality constrains.

$$\begin{array}{lll} \textit{minimize} & c(x,u) + E_\omega\{c_\omega(u_\omega)\} & \\ \textit{subject to} & g(x,u) = 0 & \\ & f(x,u) + s = 0 & \\ & g_\omega(x_\omega, u_\omega) = 0 & \omega = 1,\ldots,k \\ & f_\omega(x_\omega, u_\omega) + s_\omega = 0 & \omega = 1,\ldots,k \\ & h_\omega(u\ ,u_\omega) + \sigma\ = 0 & \\ & s \ge 0\ ,\ s_\omega \ge 0\ ,\ \sigma\ \ge 0 & \omega = 1,\ldots,k \end{array} \qquad (2)$$

The slack variables must be non-negative; these non-negativity constraints will be imposed by adding a logarithmic barrier function to the objective

$$-\mu\left(\sum_{i=1}^{n_c} ln\, s_i - \sum_{\omega=1}^{k}\ \sum_{i=1}^{n_{c\omega}} ln\, s_{\omega_i} - \sum_{i=1}^{n_r} ln\, \sigma_i\right)$$

The barrier parameter μ is a positive number that is forced to decrease to zero iteratively; n_c, $n_{c\omega}$ are the number of inequality constraints for the base and contingency cases respectively; and n_r is the number of ramp-rate constraints.

The resulting Lagrangian function for the above problem, with a logarithmic barrier function for the interior point method, is:

$$
\begin{aligned}
L = c(x,u) + \lambda^T g(x,u) + \pi^T (f(x,u) + s) + \\
+\sum_{\omega=1}^{k}\left[\lambda_\omega^T g_\omega(x_\omega,u_\omega) + \pi_\omega^T (f_\omega(x_\omega,u_\omega) + s_\omega) + p_\omega d_\omega^T u_\omega\right] + \\
+\gamma^T (h(u,u_\omega) + \sigma) - \mu\left(\sum_{i=1}^{n_c} ln\, s_i - \sum_{\omega=1}^{k}\sum_{i=1}^{n_{c_\omega}} ln\, s_{\omega_i} - \sum_{i=1}^{n_r} ln\, \sigma_i\right)
\end{aligned}
\tag{3}
$$

The stationary point of the Lagrangian function (3) is the solution of following system of KKT conditions:

$$\nabla_x L = \nabla_x c(x,u) + G_x^T \lambda + F_x^T \pi = 0 \tag{4}$$

$$\nabla_u L = \nabla_u c(x,u) + G_u^T \lambda + F_u^T \pi + H_u^T \gamma = 0 \tag{5}$$

$$\nabla_{x_\omega} L = G_{\omega_{x_\omega}}^T \lambda_\omega + F_{\omega_{x_\omega}}^T \pi_\omega = 0 \tag{6}$$

$$\nabla_{u_\omega} L = G_{\omega_{u_\omega}}^T \lambda_\omega + F_{\omega_{u_\omega}}^T \pi_\omega + H_{u_\omega}^T \gamma + p_\omega d_\omega = 0 \tag{7}$$

$$\nabla_\lambda L = g(x,u) = 0 \tag{8}$$

$$\nabla_\pi L = f(x,u) + s = 0 \tag{9}$$

$$\nabla_{\lambda_\omega} L = g_\omega(x_\omega,u_\omega) = 0 \tag{10}$$

$$\nabla_{\pi\omega} L = f_\omega(x_\omega,u_\omega) + s_\omega = 0 \tag{11}$$

$$\nabla_\gamma L = h(u,u_\omega) + \sigma = 0 \tag{12}$$

$$\nabla_s L = \pi - \mu S^{-1} e = 0 \tag{13}$$

$$\nabla_{s_\omega} L = \pi_\omega - \mu S_\omega^{-1} e = 0 \tag{14}$$

$$\nabla_{\sigma_\omega} L = \gamma - \mu \Sigma^{-1} e = 0 \tag{15}$$

$$s \geq 0,\; s_\omega \geq 0,\; \sigma \geq 0$$

$$for\;\; \omega = 1 \ldots k$$

where: $S = diag(s)$ $S_\omega = diag(s_\omega)$, $\Sigma = diag(\sigma)$ and e is vector of ones of appropriate dimension; i.e. $e = (1 \ldots 1)^T$

Any point that satisfies above (KKT) conditions is said to be a first-order critical point for the problem The last three of these conditions (13), (14), (15) are known as complementary slackness conditions. The KKT conditions represent a system of nonlinear equations.

The first step in a solution process is to apply Newton linearization; therefore, the KKT conditions will be expanded about x, u, x_ω u_ω

$$W_{xu}\Delta u + W_{xx}\Delta x + G_x^T\lambda + F_x^T\pi = -\nabla_x c(x,u) \tag{16}$$

$$W_{uu}\Delta u + W_{ux}\Delta x + G_u^T\lambda + F_u^T\pi + H_u^T\gamma \ = -\nabla_u c(x,u) \tag{17}$$

$$W_{x_\omega u_\omega}\Delta u_\omega + W_{x_\omega x_\omega}\Delta x_\omega + G_{\omega_{x_\omega}}^T\lambda_\omega + F_{\omega_{x_\omega}}^T\pi_\omega = 0 \tag{18}$$

$$W_{u_\omega u_\omega}\Delta u_\omega + W_{u_\omega x_\omega}\Delta x_\omega + G_{\omega_{u_\omega}}^T\lambda_\omega + F_{\omega_{u_\omega}}^T\pi_\omega + H_{u_\omega}^T\gamma \ = -p_\omega d_\omega \tag{19}$$

$$G_u\Delta u + G_x\Delta x = -g(x,u) \tag{20}$$

$$F_u\Delta u + F_x\Delta x + s = -f(x,u) \tag{21}$$

$$G_{\omega_{x_\omega}}\Delta x_\omega + G_{\omega_{u_\omega}}\Delta u_\omega = -g_\omega(x_\omega,u_\omega) \tag{22}$$

$$F_{\omega_{x_\omega}}\Delta x_\omega + F_{\omega_{u_\omega}}\Delta u_\omega + s_\omega = -f_\omega(x_\omega,u_\omega) \tag{23}$$

$$H_u\Delta u + H_{u_\omega}\Delta u_\omega + \sigma \ = -h\,(u,u_\omega) \tag{24}$$

$$\Pi S e = \mu e \tag{25}$$

$$\Pi_\omega S_\omega e = \mu e \tag{26}$$

$$\Gamma\Sigma e = \mu e \tag{27}$$

3.3 Forming a quadratic subproblem

Now we will pose the question: For which optimization problem are these the KKT conditions? First take a look at a Lagrangian that gives the above KKT conditions

$$L = \begin{pmatrix} \nabla c_x^T & \nabla c_u^T & 0 & p_\omega d_\omega^T \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \\ \Delta x_\omega \\ \Delta u_\omega \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Delta x^T & \Delta u^T & \Delta x_\omega^T & \Delta u_\omega^T \end{pmatrix} \begin{pmatrix} W_{xx} & W_{xu} & 0 & 0 \\ W_{ux} & W_{uu} & 0 & 0 \\ 0 & 0 & W_{x_\omega x_\omega} & W_{x_\omega u_\omega} \\ 0 & 0 & W_{u_\omega x_\omega} & W_{u_\omega u_\omega} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \\ \Delta x_\omega \\ \Delta u_\omega \end{pmatrix} +$$

$$+ \lambda^T \left(G_u \Delta u + G_x \Delta x + g(x,u) \right) + \pi^T \left(F_u \Delta u + F_x \Delta x + s + f(x,u) \right) +$$

$$+ \lambda_\omega^T \left(G_{\omega_{x_\omega}} \Delta x_\omega + G_{\omega_{u_\omega}} \Delta u_\omega + g_\omega(x_\omega, u_\omega) \right) + \pi_\omega^T \left(F_{\omega_{x_\omega}} \Delta x_\omega + F_{\omega_{u_\omega}} \Delta u_\omega + s_\omega + f_\omega(x_\omega, u_\omega) \right) +$$

$$+ \gamma^T \left(H_u \Delta u + H_{u_\omega} \Delta u_\omega + \sigma + h(u, u_\omega) \right)$$

$$- \mu \left(\sum_{i=1}^{n_c} \ln s_i - \sum_{\omega=1}^{k} \sum_{i=1}^{n_{c\omega}} \ln s_{\omega_i} - \sum_{i=1}^{n_r} \ln \sigma_i \right)$$

Now we can formulate a quadratic programming subproblem given the above Lagrangian function. The linearized KKT conditions given in (16) – (27) are the KKT conditions for the quadratic programming (QP) problem:

$$\text{minimize}$$

$$\begin{pmatrix} \nabla c_x^T & \nabla c_u^T & 0 & p_\omega d_\omega^T \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \\ \Delta x_\omega \\ \Delta u_\omega \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Delta x^T & \Delta u^T & \Delta x_\omega^T & \Delta u_\omega^T \end{pmatrix} \begin{pmatrix} W_{xx} & W_{xu} & 0 & 0 \\ W_{ux} & W_{uu} & 0 & 0 \\ 0 & 0 & W_{x_\omega x_\omega} & W_{x_\omega u_\omega} \\ 0 & 0 & W_{u_\omega x_\omega} & W_{u_\omega u_\omega} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \\ \Delta x_\omega \\ \Delta u_\omega \end{pmatrix}$$

$$- \mu \left(\sum_{i=1}^{n_c} \ln s_i - \sum_{\omega=1}^{k} \sum_{i=1}^{n_{c\omega}} \ln s_{\omega_i} - \sum_{i=1}^{n_r} \ln \sigma_i \right)$$

such that $$G_u \Delta u + G_x \Delta x = -g(x,u)$$
$$F_u \Delta u + F_x \Delta x \le -f(x,u)$$
$$G_{\omega_{x_\omega}} \Delta x_\omega + G_{\omega_{u_\omega}} \Delta u_\omega = -g_\omega(x_\omega, u_\omega)$$
$$F_{\omega_{x_\omega}} \Delta x_\omega + F_{\omega_{u_\omega}} \Delta u_\omega \le -f_\omega(x_\omega, u_\omega)$$
$$H_u \Delta u + H_{u_\omega} \Delta u_\omega \le -h\,(u, u_\omega)$$

Because the Jacobian matrix is defined as in Appendix I without a power balance equation corresponding to the reference bus, we have to include that power balance equation as well. Therefore, we will add an additional term reference bus in both the base case and the contingency cases.

Base case:

λ_r Lagrangian multiplier for reference bus active power balance

G_{rx} row vector (1×2n) – gradient of active power balance equation at reference bus with respect to state variables

G_{ru} row vector ($1\times n_u$) - gradient of active power balance equation at reference bus with respect to control variables

Contingency case:

$\lambda_{\omega r}$ Lagrangian multiplier for reference bus active power balance

$G_{\omega r_{x_\omega}}$ row vector (1×2n) - gradient of active power balance equation at reference bus with respect to state variables

$G_{\omega r_{u_\omega}}$ row vector ($1\times n_{u\omega}$) - gradient of active power balance equation at reference bus with respect to control variables

$$W_{xu}\Delta u + W_{xx}\Delta x + G_x^T \lambda + G_{rx}^T \lambda_r + F_x^T \pi = b_x \tag{28}$$

$$W_{uu}\Delta u + W_{ux}\Delta x + G_u^T \lambda + G_{ru}^T \lambda_r + F_u^T \pi + H_u^T \gamma = b_u \tag{29}$$

$$W_{x_\omega u_\omega}\Delta u_\omega + W_{x_\omega x_\omega}\Delta x_\omega + G_{\omega_{x_\omega}}^T \lambda_\omega + G_{\omega r_{x_\omega}}^T \lambda_{\omega r} + F_{\omega_{x_\omega}}^T \pi_\omega = b_{x_\omega} \tag{30}$$

$$W_{u_\omega u_\omega}\Delta u_\omega + W_{u_\omega x_\omega}\Delta x_\omega + G_{\omega_{u_\omega}}^T \lambda_\omega + G_{\omega r_{u_\omega}}^T \lambda_{\omega r} + F_{\omega_{u_\omega}}^T \pi_\omega + H_{u_\omega}^T \gamma = b_{u_\omega} \tag{31}$$

$$G_u \Delta u + G_x \Delta x = b_\lambda \tag{32}$$

$$G_{ru}\Delta u + G_{rx}\Delta x = b_r \tag{33}$$

$$F_u\Delta u + F_x\Delta x + s = b_\pi \tag{34}$$

$$G_{\omega_{x_\omega}}\Delta x_\omega + G_{\omega_{u_\omega}}\Delta u_\omega = b_{\lambda_\omega} \tag{35}$$

$$G_{\omega r_{x_\omega}}\Delta x_\omega + G_{\omega r_{u_\omega}}\Delta u_\omega = b_{\omega r} \tag{36}$$

$$F_{\omega_{x_\omega}}\Delta x_\omega + F_{\omega_{u_\omega}}\Delta u_\omega + s_\omega = b_{\pi_\omega} \tag{37}$$

$$H_u\Delta u + H_{u_\omega}\Delta u_\omega + \sigma = b_\gamma \tag{38}$$

$$\Pi S e = \mu e \tag{39}$$

$$\Pi_\omega S_\omega e = \mu e \tag{40}$$

$$\Gamma\Sigma e = \mu e \tag{41}$$

The right hand sides of the above system can be denoted:

$$b_x = -\nabla_x c(x,u)$$

$$b_u = -\nabla_u c(x,u)$$

$$b_{x_\omega} = 0$$

$$b_{u_\omega} = -p_\omega d_\omega$$

$$b_\lambda = -g(x,u)$$

$$b_r = -g_r(x,u)$$

$$b_\pi = -f(x,u)$$

$$b_{\lambda_\omega} = -g_\omega(x_\omega,u_\omega)$$

$$b_{\omega r} = -g_{\omega r}(x_\omega,u_\omega)$$

$$b_{\pi_\omega} = -f_\omega(x_\omega,u_\omega)$$

$$b_\gamma = -h(u,u_\omega)$$

We notice that introducing the reference bus resulted in two more equations, (33) for the base case and (36) for the contingency cases. Equations (28) to (41) form a system of nonlinear equations for the interior point formulation.

The above system will be solved in a way similar to [6] and reviewed in chapter 1. The approach is to eliminate Δx, λ, for the base case and Δx_ω , λ_ω for each contingency case.

Equations (28) and (32) can be rewritten in the matrix form:

$$\begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}\begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = \begin{pmatrix} b_x - W_{xu}\Delta u - G_{rx}^T\lambda_r - F_x^T\pi \\ b_\lambda - G_u \Delta u \end{pmatrix} \tag{42}$$

The corresponding contingency equations (30) and (35) can also be rewritten in the matrix form:

$$\begin{pmatrix} W_{x_\omega x_\omega} & G_{\omega_{x_\omega}}^T \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}\begin{pmatrix} \Delta x_\omega \\ \lambda_\omega \end{pmatrix} = \begin{pmatrix} b_{x_\omega} - W_{x_\omega u_\omega}\Delta u_\omega - G_{\omega r_{x_\omega}}^T \lambda_{\omega r} - F_{\omega_{x_\omega}}^T \pi_\omega \\ b_{\lambda_\omega} - G_{\omega_{u_\omega}} \Delta u_\omega \end{pmatrix} \tag{43}$$

Before we embark on equation solving we should keep in mind that the inverse of the 2×2 block matrix from equations (41) and (42) has the following structure.

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & D \\ E & F \end{pmatrix}$$

With this property many of the terms in the following derivation will be zero. Now substitute equations (42) and (43) in the rest of the system equations.

First equation to be rewritten is equation (29)

$$W_{uu}\Delta u + \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix}\begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} + G_{ru}^T\lambda_r + F_u^T\pi + H_u^T\gamma \ = b_u^{(k)}$$

Substituting equation (42) into above yields

$$W_{uu}\Delta u + \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix}\begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1}\left[\begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} - \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}\Delta u - \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix}\lambda_r - \begin{pmatrix} F_x^T \\ 0 \end{pmatrix}\pi\right] +$$
$$+ G_{ru}^T\lambda_r + F_u^T\pi + H_u^T\gamma \ = b_u$$

$$\left(W_{uu} - \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}\right) \Delta u + \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} +$$
$$+ \left(G_{ru}^T - \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix}\right) \lambda_r +$$
$$+ \left(F_u^T - \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_x^T \\ 0 \end{pmatrix}\right) \pi + H_u^T \gamma \;= b_u$$

In the previous equation, the following variables can be introduced:

$$\overline{W}_{uu} = W_{uu} - \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}$$

$$\overline{b}_u = b_u - \begin{pmatrix} W_{ux} & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix}$$

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}$$

$$\overline{F}_u = F_u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}$$

with these terms, equation (29) can be rewritten as:

$$\overline{W}_{uu} \Delta u + \overline{G}_{ru}^T \lambda_r + \overline{F}_u^T \pi + H_u^T \gamma \;= \overline{b}_u \tag{44}$$

equation (31) in the matrix form:

$$W_{u_\omega u_\omega} \Delta u_\omega + \begin{pmatrix} W_{u_\omega x_\omega} & G_{\omega_{u_\omega}}^T \end{pmatrix} \begin{pmatrix} \Delta x_\omega \\ \lambda_\omega \end{pmatrix} + G_{\omega r_{u_\omega}}^T \lambda_{\omega r} + F_{\omega_{u_\omega}}^T \pi_\omega + H_{u_\omega}^T \gamma \;= b_{u_\omega}$$

Similarly, substituting equation (43) into above yields

$$W_{u_\omega u_\omega}\Delta u_\omega + \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \left[\begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix} - \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix} \Delta u_\omega - \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} \lambda_{\omega r} - \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} \pi_\omega \right] +$$
$$+ G^T_{\omega r_{u_\omega}} \lambda_{\omega r} + F^T_{\omega_{u_\omega}} \pi_\omega + H^T_{u_\omega} \gamma \ = b_{u_\omega}$$

$$\left(W_{u_\omega u_\omega} - \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix} \right) \Delta u_\omega +$$
$$+ \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix} +$$
$$+ \left(G^T_{\omega r_{u_\omega}} - \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} \right) \lambda_{\omega r} +$$
$$+ \left(F^T_{\omega_{u_\omega}} - \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} \right) \pi_\omega + H^T_{u_\omega} \gamma \ = b_{u_\omega}$$

The following matrices are introduced:

$$\overline{W}_{u_\omega u_\omega} = W_{u_\omega u_\omega} - \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix}$$

$$\overline{b}_{u_\omega} = b_{u_\omega} - \begin{pmatrix} W_{u_\omega x_\omega} & G^T_{\omega_{u_\omega}} \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix}$$

$$\overline{G}_{\omega r_{u_\omega}} = G_{\omega r_{u_\omega}} - \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix}$$

$$\overline{F}_{\omega_{u_\omega}} = F_{\omega_{u_\omega}} - \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix}$$

Finally equation (31) can be rewritten as

$$\overline{W}_{u_\omega u_\omega} \Delta u_\omega + \overline{G}^T_{\omega r_{u_\omega}} \lambda_{\omega r} + \overline{F}^T_{\omega_{u_\omega}} \pi_\omega + H^T_{u_\omega} \gamma \ = \overline{b}_{u_\omega} \tag{45}$$

Next equation is (33):

$$G_{ru}\Delta u + \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = b_r$$

Substituting (42) into (33) gives

$$G_{ru}\Delta u + \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \left[\begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} - \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \Delta u - \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix} \lambda_r - \begin{pmatrix} F_x^T \\ 0 \end{pmatrix} \pi \right] = b_r$$

$$\left(G_{ru} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \right) \Delta u + \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} -$$

$$- \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix} \lambda_r - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_x^T \\ 0 \end{pmatrix} \pi = b_r$$

The following matrices are introduced:

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}$$

$$\overline{b}_r = b_r - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix}$$

following matrix inversion structure following two terms are zero

$$\begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_x^T \\ 0 \end{pmatrix}$$

Finally equation (33) can be rewritten as

$$\overline{G}_{ru}\Delta u = \overline{b}_r \tag{46}$$

equation (34) in the matrix form:

$$F_u \Delta u + \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} + s = b_\pi$$

following the substitution of (42) yields

$$F_u \Delta u + \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \left[\begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} - \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \Delta u - \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix} \lambda_r - \begin{pmatrix} F_x^T \\ 0 \end{pmatrix} \pi \right] + s = b_\pi$$

$$\left(F_u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \right) \Delta u + \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} -$$
$$- \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix} \lambda_r - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_x^T \\ 0 \end{pmatrix} \pi + s = b_\pi$$

The following matrices are introduced:

$$\overline{F}_u = F_u - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}$$

$$\overline{b}_\pi = b_\pi - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix}$$

Because of the structure of $\begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1}$, the following two identities are true:

$$\begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} G_{rx}^T \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_x^T \\ 0 \end{pmatrix} = 0$$

finally equation (34) can be rewritten as

$$\overline{F}_u \Delta u + s = \overline{b}_\pi \tag{47}$$

The equation (36) is written as:

$$\begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} \Delta x_\omega \\ \lambda_\omega \end{pmatrix} + G_{\omega r_{u_\omega}} \Delta u_\omega = b_{\omega r}$$

substituting (43) into (36) gives

$$\begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \left[\begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix} - \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix} \Delta u_\omega - \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} \lambda_{\omega r} - \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} \pi_\omega \right] +$$
$$+ G_{\omega r_{u_\omega}} \Delta u_\omega = b_{\omega r}$$

$$\left(G_{\omega r_{u_\omega}} + \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix} \right) \Delta u_\omega -$$
$$- \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} \lambda_{\omega r} -$$
$$- \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} \pi_\omega = b_{\omega r} - \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix}$$

The following matrices are introduced:

$$\overline{G}_{\omega r_{u_\omega}} = G_{\omega r_{u_\omega}} + \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix}$$

$$\overline{b}_{\omega r} = b_{\omega r} - \begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix}$$

Furthermore since

$$\begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} G_{\omega r_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} = 0$$

Equation (36) can be rewritten as

$$\overline{G}_{\omega r_{u_\omega}} \Delta u_\omega = \overline{b}_{\omega r} \tag{48}$$

The last equation to be modified is (37)

$$\begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} \Delta x_\omega \\ \lambda_\omega \end{pmatrix} + F_{\omega_{u_\omega}} \Delta u_\omega + s_\omega = b_{\pi_\omega}$$

substituting (43) into (37) yields

$$\begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \left[\begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix} - \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix} \Delta u_\omega - \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} \lambda_{\omega r} - \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} \pi_\omega \right] +$$
$$+ F_{\omega_{u_\omega}} \Delta u_\omega + s_\omega = b_{\pi_\omega}$$

$$\left(F_{\omega_{u_\omega}} - \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix} \right) \Delta u_\omega -$$
$$- \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} \lambda_{\omega r} -$$
$$- \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} \pi_\omega = b_{\pi_\omega} - \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix}$$

The following matrices are introduced:

$$\overline{F}_{\omega_{u_\omega}} = F_{\omega_{u_\omega}} - \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{x_\omega u_\omega} \\ G_{\omega_{u_\omega}} \end{pmatrix}$$

$$\overline{b}_{\pi_\omega} = b_{\pi_\omega} - \begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} \\ b_{\lambda_\omega} \end{pmatrix}$$

Since

$$\begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} G^T_{\omega r_{x_\omega}} \\ 0 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} F_{\omega_{x_\omega}} & 0 \end{pmatrix} \begin{pmatrix} W_{x_\omega x_\omega} & G^T_{\omega_{x_\omega}} \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F^T_{\omega_{x_\omega}} \\ 0 \end{pmatrix} = 0$$

Equation (37) reduces to:

$$\overline{F}_{\omega_{u_\omega}} \Delta u_\omega + s_\omega = \overline{b}_{\pi_\omega} \tag{49}$$

Equations (44) – (49) together with unmodified equations (38) – (41) form reduced system.

3.4 Solving reduced system

Reduced system of equations is:

$$\overline{W}_{uu} \Delta u + \overline{G}^T_{ru} \lambda_r + \overline{F}^T_u \pi + H^T_u \gamma \quad = \overline{b}_u$$

$$\overline{W}_{u_\omega u_\omega} \Delta u_\omega + \overline{G}^T_{\omega r_{u_\omega}} \lambda_{\omega r} + \overline{F}^T_{\omega_{u_\omega}} \pi_\omega + H^T_{u_\omega} \gamma \quad = \overline{b}_{u_\omega}$$

$$\overline{G}_{ru} \Delta u = \overline{b}_r$$

$$\overline{F}_u \Delta u + s = \overline{b}_\pi$$

$$\overline{G}_{\omega r_{u_\omega}} \Delta u_\omega = \overline{b}_{\omega r}$$

$$\overline{F}_{\omega_{u_\omega}} \Delta u_\omega + s_\omega = \overline{b}_{\pi_\omega}$$

$$H_u \Delta u + H_{u_\omega} \Delta u_\omega + \sigma \;\; = b_\gamma$$

$$\Pi S e = \mu e$$

$$\Pi_\omega S_\omega e = \mu e$$

$$\Gamma \Sigma \; e = \mu e$$

where:

$$\Gamma \;\; = diag(\gamma_i)$$

$$\Sigma \;\; = diag(\sigma_i)$$

The first step in solution of the reduced system is to expand system about s, s_ω, π, π_ω and σ which yields:

$$\overline{W}_{uu} \Delta u + \overline{G}_{ru}^T \lambda_r + \overline{F}_u^T \Delta\pi + H_u^T \gamma \;\; = \overline{b}_u - \overline{F}_u^T \pi$$

$$\overline{W}_{u_\omega u_\omega} \Delta u_\omega + \overline{G}_{\omega r_{u_\omega}}^T \lambda_{\omega r} + \overline{F}_{\omega_{u_\omega}}^T \Delta\pi_\omega + H_{u_\omega}^T \gamma \;\; = \overline{b}_{u_\omega} - \overline{F}_{\omega_{u_\omega}}^T \pi_\omega$$

$$\overline{G}_{ru} \Delta u = \overline{b}_r$$

$$\overline{F}_u \Delta u + \Delta s = b_\pi - S e$$

$$\overline{G}_{\omega r_{u_\omega}} \Delta u_\omega = \overline{b}_{\omega r}$$

$$\overline{F}_{\omega_{u_\omega}} \Delta u_\omega + \Delta s_\omega = b_{\pi_\omega} - S_\omega e$$

$$H_u \Delta u + H_{u_\omega} \Delta u_\omega + \Delta\sigma \;\; = b_\gamma - \Sigma e$$

$$\Pi \Delta s + S \Delta\pi = \mu e - \Pi S e$$

$$\Pi_\omega \Delta s_\omega + S_\omega \Delta\pi_\omega = \mu e - \Pi_\omega S_\omega e$$

$$\Gamma \; \Delta\sigma \;\; + \Sigma \;\; \Delta\gamma \;\; = \mu e - \Gamma \; \Sigma \;\; e$$

Variables Δs, Δs_ω, $\Delta\sigma$ can be expressed from the last three equations

$$\Delta s = \Pi^{-1}\left(\mu e - \Pi S e - S \Delta\pi\right)$$

$$\Delta s_\omega = \Pi_\omega^{-1}\left(\mu e - \Pi_\omega S_\omega e - S_\omega \Delta\pi_\omega\right)$$

$$\Delta\sigma = \Gamma^{-1}\left(\mu e - \Gamma\, \Sigma\, e - \Sigma\, \Delta\gamma\right)$$

The next step is to eliminate the slack variables Δs, Δs_ω, $\Delta\sigma$ from the above reduced system

After performing that operation the reduced system will have following matrix form:

$$\begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T & \overline{F}_u^T & 0 & 0 & 0 & H_u^T \\ \overline{G}_{ru} & 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{F}_u & 0 & -\Pi^{-1}S & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{W}_{u_\omega u_\omega} & \overline{G}_{\omega r_{u_\omega}}^T & \overline{F}_{\omega_{u_\omega}}^T & H_{u_\omega}^T \\ 0 & 0 & 0 & \overline{G}_{\omega r_{u_\omega}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{F}_{\omega_{u_\omega}} & 0 & -\Pi_\omega^{-1}S_\omega & 0 \\ H_u & 0 & 0 & H_{u_\omega} & 0 & 0 & -\Gamma^{-1}\Sigma \end{pmatrix} \begin{pmatrix} \Delta u \\ \lambda_r \\ \Delta\pi \\ \Delta u_\omega \\ \lambda_{\omega r} \\ \Delta\pi_\omega \\ \Delta\gamma \end{pmatrix} = \begin{pmatrix} \hat{b}_u \\ \overline{b}_r \\ \hat{b}_\pi \\ \hat{b}_{u_\omega} \\ \overline{b}_{\omega r} \\ \hat{b}_{\pi_\omega} \\ \hat{b}_\gamma \end{pmatrix}$$

where the right hand side vectors are:

$$\hat{b}_u = \overline{b}_u - \overline{F}_u^T \pi$$

$$\hat{b}_{u_\omega} = \overline{b}_{u_\omega} - \overline{F}_{\omega_{u_\omega}}^T \pi_\omega$$

$$\hat{b}_\pi = \overline{b}_\pi - \mu\Pi^{-1}e$$

$$\hat{b}_{\pi_\omega} = \overline{b}_{\pi_\omega} - \mu\Pi_\omega^{-1}e$$

$$\hat{b}_\gamma = b_\gamma - \mu\Gamma^{-1}e$$

The above reduced system is still unacceptably large due to significant number of control variables (u,u_ω). It would be computationally easier to express each of control variables Δu and Δu_ω and further reduce the size of the system. As mentioned before, it is well known that only a small number of the total inequality constraints become active, which makes system significantly smaller. The size of the active constraint set is the size of the Lagrange multiplier vectors π and π_ω respectively, therefore reduced system after eliminating control variables (u, u_ω) will be the size of active sets corresponding to each contingency.

Solving above system is conducted in two stages. First consider base case block

$$\begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T & \overline{F}_u^T \\ \overline{G}_{ru} & 0 & 0 \\ \overline{F}_u & 0 & -\Pi^{-1}S \end{pmatrix} \begin{pmatrix} \Delta u \\ \lambda_r \\ \Delta\pi \end{pmatrix} + \begin{pmatrix} H_u^T \\ 0 \\ 0 \end{pmatrix} \Delta\gamma = \begin{pmatrix} \hat{b}_u \\ \overline{b}_r \\ \hat{b}_\pi \end{pmatrix}$$

$$U^T D U = \begin{pmatrix} \overline{W}_{uu} & \overline{G}_{ru}^T \\ \overline{G}_{ru} & 0 \end{pmatrix}$$

Define

$$F^T = \begin{pmatrix} \overline{F}_u^T \\ 0 \end{pmatrix} \qquad \hat{H}^T = \begin{pmatrix} H_u^T \\ 0 \end{pmatrix}$$

$$y = \begin{pmatrix} \Delta u \\ \lambda_r \end{pmatrix}$$

$$b_0 = \begin{pmatrix} \overline{b}_u \\ \overline{b}_r \end{pmatrix}$$

$$\hat{b}_\pi = \overline{b}_\pi - \mu\Pi^{-1}e$$

$$\begin{pmatrix} U^T D U & F^T \\ F & -\Pi^{-1}S \end{pmatrix} \begin{pmatrix} y \\ \Delta\pi \end{pmatrix} + \begin{pmatrix} \hat{H}^T \\ 0 \end{pmatrix} \Delta\gamma = \begin{pmatrix} b_0 - F^T\pi \\ \hat{b}_\pi \end{pmatrix}$$

Above system is solved on following way:

$$y = U^{-1}D^{-1}U^{-T}(b_0 - F^T\pi - \hat{H}^T\Delta\gamma - F^T\Delta\pi) \tag{50}$$

$$F\,y - \Pi^{-1}S\,\Delta\pi = \hat{b}_\pi \tag{51}$$

after substitution of equation (50) into (51) and some algebra we get:

$$\begin{aligned} &\left(-\Pi^{-1}S - FU^{-1}D^{-1}U^{-T}F^T\right)\Delta\pi - FU^{-1}D^{-1}U^{-T}\hat{H}^T\Delta\gamma = \\ &= \hat{b}_\pi - FU^{-1}D^{-1}U^{-T}\left(b_0 - F^T\pi\right) \end{aligned} \tag{53}$$

to calculate $\Delta\pi$ from the above equation we introduce following:

$$\overline{F} = U^{-T} F^{T} \quad \text{or} \quad \overline{F}^{T} = F U^{-1}$$

$$U^{T} \overline{F} = F^{T}$$

Since F is sparse, $\overline{F}$ calculated by fast forward substitution.

Equation (53) can be calculated as follows

$$\left(-\Pi^{-1} S - \overline{F}^{T} D^{-1} \overline{F}\right) \Delta\pi + \overline{F}^{T} D^{-1} U^{-T} \hat{H}^{T} \Delta\gamma = \hat{b}_{\pi} - \overline{F}^{T} D^{-1} U^{-T} (b_{0} - F^{T} \pi)$$

$$\left(-\Pi^{-1} S - \overline{F}^{T} D^{-1} \overline{F}\right) \Delta\pi + \overline{F}^{T} D^{-1} U^{-T} \hat{H}^{T} \Delta\gamma = \hat{b}_{\pi} - \overline{F}^{T} D^{-1} U^{-T} b_{0} + \overline{F}^{T} D^{-1} U^{-T} F^{T} \pi$$

$$\left(-\Pi^{-1} S - \overline{F}^{T} D^{-1} \overline{F}\right) \Delta\pi + \overline{F}^{T} D^{-1} U^{-T} \hat{H}^{T} \Delta\gamma = \hat{b}_{\pi} - \overline{F}^{T} D^{-1} U^{-T} b_{0} + \overline{F}^{T} D^{-1} \overline{F} \pi$$

$$\left(-\Pi^{-1} S - R\right) \Delta\pi + \overline{F}^{T} D^{-1} \widetilde{H} \Delta\gamma = \hat{b}_{\pi} - \overline{F}^{T} D^{-1} \widetilde{b}_{0} + R\pi \tag{54}$$

where

$$R = \overline{F}^{T} D^{-1} \overline{F}$$
$$\widetilde{H} = U^{-T} \hat{H}^{T}$$
$$\widetilde{b}_{0} = U^{-T} b_{0}$$

Therefore $\widetilde{H}$ is calculated by performing column by column forward substitution and $\widetilde{b}$ is calculated by a single forward substitution

For the general formulation we define following terms

$$C_{0} = -\Pi^{-1} S - R$$
$$V_{0}^{T} = \overline{F}^{T} D^{-1} \widetilde{H}$$
$$r_{0} = \hat{b}_{\pi} - \overline{F}^{T} D^{-1} \widetilde{b}_{0} + R\pi$$

Each contingency block can be analyzed in the same fashion

$$\begin{pmatrix} \overline{W}_{u_\omega u_\omega} & \overline{G}^T_{\omega r_{u_\omega}} & \overline{F}^T_{\omega_{u_\omega}} \\ \overline{G}_{\omega r_{u_\omega}} & 0 & 0 \\ \overline{F}_{\omega_{u_\omega}} & 0 & -\Pi^{-1}_\omega S_\omega \end{pmatrix} \begin{pmatrix} \Delta u_\omega \\ \lambda_{\omega r} \\ \Delta\pi_\omega \end{pmatrix} + \begin{pmatrix} H^T_{u_\omega} \\ 0 \\ 0 \end{pmatrix} \Delta\gamma = \begin{pmatrix} \hat{b}_{u_\omega} \\ \overline{b}_{\omega r} \\ \hat{b}_{\pi_\omega} \end{pmatrix}$$

$$U^T_\omega D_\omega U_\omega = \begin{pmatrix} \overline{W}_{u_\omega u_\omega} & \overline{G}^T_{\omega r_{u_\omega}} \\ \overline{G}^T_{\omega r_{u_\omega}} & 0 \end{pmatrix}$$

Define:

$$F^T_\omega = \begin{pmatrix} \overline{F}^T_{\omega_{u_\omega}} \\ 0 \end{pmatrix} \qquad \hat{H}^T_\omega = \begin{pmatrix} H^T_{u_\omega} \\ 0 \end{pmatrix}$$

$$y_\omega = \begin{pmatrix} \Delta u_\omega \\ \lambda_{\omega r} \end{pmatrix}$$

$$b_\omega = \begin{pmatrix} \overline{b}_{u_\omega} \\ \overline{b}_{\omega r} \end{pmatrix}$$

$$\hat{b}_{\pi_\omega} = \overline{b}_{\pi_\omega} - \mu\Pi^{-1}_\omega e$$

Then

$$\begin{pmatrix} U^T_\omega D_\omega U_\omega & F^T_\omega \\ F_\omega & -\Pi^{-1}_\omega S_\omega \end{pmatrix} \begin{pmatrix} y_\omega \\ \Delta\pi_\omega \end{pmatrix} + \begin{pmatrix} \hat{H}^T_\omega \\ 0 \end{pmatrix} \Delta\gamma = \begin{pmatrix} b_\omega - F^T_\omega \pi \\ \hat{b}_{\pi_\omega} \end{pmatrix}$$

The above system is solved in the following way:

$$y_\omega = U^{-1}_\omega D^{-1}_\omega U^{-T}_\omega (b_\omega - F^T_\omega \pi_\omega - \hat{H}^T_\omega \Delta\gamma - F^T_\omega \Delta\pi_\omega) \tag{55}$$

$$F_\omega y_\omega - \Pi^{-1}_\omega S_\omega \Delta\pi_\omega = \hat{b}_{\pi_\omega} \tag{56}$$

after substitution of equation (55) into (56) and some algebra we get:

$$\begin{aligned} &\left(-\Pi^{-1}_\omega S_\omega - F_\omega U^{-1}_\omega D^{-1}_\omega U^{-T}_\omega F^T_\omega\right)\Delta\pi_\omega - F^T_\omega U^{-1}_\omega D^{-1}_\omega U^{-T}_\omega \hat{H}^T_\omega \Delta\gamma = \\ &= \hat{b}_{\pi_\omega} - F_\omega U^{-1}_\omega D^{-1}_\omega U^{-T}_\omega \left(b_\omega - F^T_\omega \pi\right) \end{aligned} \tag{57}$$

To calculate $\Delta\pi_\omega$ from the above equation we introduce following:

$$\overline{F}_\omega = U_\omega^{-T} F_\omega^T \quad \text{or} \quad \overline{F}_\omega^T = F_\omega U_\omega^{-1}$$

$U_\omega^T \overline{F}_\omega = F_\omega^T$ $\overline{F}_\omega$ is calculated by fast forward substitution

now equation (57) can be calculated as follows

$$\left(-\Pi_\omega^{-1} S_\omega - \overline{F}_\omega^T D_\omega^{-1} \overline{F}_\omega\right)\Delta\pi_\omega + \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T} \hat{H}_\omega^T \Delta\gamma = \hat{b}_{\pi_\omega} - \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T}\left(b_\omega - F_\omega^T \pi_\omega\right)$$

$$\left(-\Pi_\omega^{-1} S_\omega - \overline{F}_\omega^T D_\omega^{-1} \overline{F}_\omega\right)\Delta\pi_\omega + \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T} \hat{H}_\omega^T \Delta\gamma = \hat{b}_{\pi_\omega} - \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T} b_\omega + \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T} F_\omega^T \pi_\omega$$

$$\left(-\Pi_\omega^{-1} S_\omega - \overline{F}_\omega^T D_\omega^{-1} \overline{F}_\omega\right)\Delta\pi_\omega + \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T} \hat{H}_\omega^T \Delta\gamma = \hat{b}_{\pi_\omega} - \overline{F}_\omega^T D_\omega^{-1} U_\omega^{-T} b_\omega + \overline{F}_\omega^T D_\omega^{-1} \overline{F}_\omega \pi_\omega$$

$$\left(-\Pi_\omega^{-1} S_\omega - R_\omega\right)\Delta\pi_\omega + \overline{F}_\omega^T D_\omega^{-1} \widetilde{H}_\omega \Delta\gamma = \hat{b}_{\pi_\omega} - \overline{F}_\omega^T D_\omega^{-1} \widetilde{b}_\omega + R_\omega \pi_\omega \tag{58}$$

where

$$R_\omega = \overline{F}_\omega^T D_\omega^{-1} \overline{F}_\omega$$
$$\widetilde{H}_\omega = U_\omega^{-T} \hat{H}_\omega^T$$
$$\widetilde{b}_\omega = U_\omega^{-T} b_\omega$$

As in the base case $\widetilde{H}_\omega$ is calculated by performing column by column forward substitution and $\widetilde{b}_\omega$ is calculated by a single forward substitution

Contingency terms for the general formulation are

$$C_\omega = -\Pi_\omega^{-1} S_\omega - R_\omega \qquad \omega = 1 \ldots k$$
$$V_\omega^T = \overline{F}_\omega^T D_\omega^T \widetilde{H}_\omega \qquad \omega = 1 \ldots k$$
$$r_\omega = \hat{b}_\pi - \overline{F}_\omega^T D_\omega^{-1} \widetilde{b}_\omega + R_\omega \pi_\omega \qquad \omega = 1 \ldots k$$

The last equation of the reduced system is:

$$H_u \Delta u + H_{u_\omega} \Delta u_\omega - \Gamma^{-1}\Sigma \ \Delta\gamma = \hat{b}_\gamma$$

can be rewritten as

$$\hat{H}\ y + \hat{H}_\omega\ y_\omega - \Gamma^{-1}\Sigma \ \Delta\gamma = \hat{b}_\gamma$$

substituting (50) and (55) into above equation yields

$$\begin{aligned} &\hat{H}\, U^{-1} D^{-1} U^{-T} (b_0 - F^T \pi - \hat{H}^T \Delta\gamma - F^T \Delta\pi) + \\ &+ \hat{H}_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} (b_\omega - F_\omega^T \pi_\omega - \hat{H}_\omega^T \Delta\gamma - F_\omega^T \Delta\pi_\omega) - \Gamma^{-1}\Sigma \ \Delta\gamma = \hat{b}_\gamma \end{aligned}$$

$$\begin{aligned} &- \hat{H}\, U^{-1} D^{-1} U^{-T} F^T \Delta\pi - \hat{H}_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} F_\omega^T \Delta\pi_\omega - \\ &- \left(\hat{H}\, U^{-1} D^{-1} U^{-T} \hat{H}^T + \hat{H}_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} \hat{H}_\omega^T + \Gamma^{-1}\Sigma \ \right)\Delta\gamma = \\ &= \hat{b}_\gamma \ - \hat{H}\, U^{-1} D^{-1} U^{-T} b_0 - \hat{H}_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} b_\omega + \hat{H}\, U^{-1} D^{-1} U^{-T} F^T \pi + \hat{H}_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} F_\omega^T \pi_\omega \end{aligned}$$

$$\begin{aligned} &- \widetilde{H}^T\ D^{-1} \overline{F}\ \Delta\pi - \widetilde{H}_\omega^T D_\omega^{-1} \overline{F}_\omega \Delta\pi_\omega - \\ &- \left(\widetilde{H}^T\ D^{-1} \widetilde{H}\ \ + \widetilde{H}_\omega^T\ D_\omega^{-1} \widetilde{H}_\omega + \Gamma^{-1}\Sigma \ \right)\Delta\gamma = \\ &= \hat{b}_\gamma \ - \widetilde{H}^T\ D^{-1} \widetilde{b}_0 - \widetilde{H}_\omega^T\ D_\omega^{-1} \widetilde{b}_\omega + \widetilde{H}^T D^{-1} \overline{F}\ \ \pi + \widetilde{H}_\omega^T\ D_\omega^{-1} \overline{F}_\omega \pi_\omega \end{aligned}$$

$$\begin{aligned} &V_0\, \Delta\pi + V_\omega \Delta\pi_\omega + \left(\widetilde{H}^T\ D^{-1} \widetilde{H}\ \ + \widetilde{H}_\omega^T\ D_\omega^{-1} \widetilde{H}_\omega + \Gamma^{-1}\Sigma \ \right)\Delta\gamma = \\ &= -\hat{b}_\gamma \ + \widetilde{H}^T\ D^{-1} \widetilde{b}_0 + \widetilde{H}_\omega^T\ D_\omega^{-1} \widetilde{b}_\omega - V_0\, \pi - V_\omega \pi_\omega \end{aligned}$$

For the general formulation it will be useful to define following terms

$$M = \widetilde{H}^T\ D^{-1} \widetilde{H} + \sum_{\omega=1}^{k} \widetilde{H}_\omega^T D_\omega^{-1} \widetilde{H}_\omega + \Gamma^{-1}\Sigma$$

$$r_\gamma = -\hat{b}_\gamma \ + \widetilde{H}^T\ D^{-1} \widetilde{b}_0 + \sum_{\omega=1}^{k} \widetilde{H}_\omega^T D_\omega^{-1} \widetilde{b}_\omega - V_0\, \pi - \sum_{\omega=1}^{k} V_\omega \pi_\omega$$

if we denote base case as a case with index zero ($\omega = 0$) above terms can be written in more compact form

$$M = \sum_{\omega=0}^{k} \widetilde{H}_{\omega}^{T} D_{\omega}^{-1} \widetilde{H}_{\omega} + \Gamma^{-1}\Sigma$$

$$r_{\gamma} = -\hat{b}_{\gamma} + \sum_{\omega=0}^{k} \widetilde{H}_{\omega}^{T} D_{\omega}^{-1} \widetilde{b}_{\omega} - \sum_{\omega=0}^{k} V_{\omega}\pi_{\omega}$$

therefore last equation can be rewritten as:

$$V_0\,\Delta\pi + V_{\omega}\Delta\pi_{\omega} + M\Delta\gamma = r_{\gamma} \qquad \omega = 1\ldots k$$

3.5 General contingency constrained OPF formulation

The general formulation with block matrices defined above can be written:

$$\begin{pmatrix} C_0 & & & & & V_0^T \\ & C_1 & & & & V_1^T \\ & & C_2 & & & V_2^T \\ & & & \ddots & & \vdots \\ & & & & C_k & V_k^T \\ V_0 & V_1 & V_2 & \cdots & V_k & M \end{pmatrix} \begin{pmatrix} \Delta\pi_0 \\ \Delta\pi_1 \\ \Delta\pi_2 \\ \vdots \\ \Delta\pi_k \\ \Delta\gamma \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_k \\ r_{\gamma} \end{pmatrix}$$

A procedure for the solution of the above bordered diagonal system suggested in [7] is the following

Since the first K+1 equations have form

$$C_{\omega}\Delta\pi_{\omega} + V_{\omega}^{T}\Delta\gamma = r_{\omega}$$

we can express $\Delta\pi_{\omega}$ as

$$\Delta\pi_{\omega} = C_{\omega}^{-1}\left(r_{\omega} - V_{\omega}^{T}\Delta\gamma\right) \tag{59}$$

The last equation can be written:

$$\sum_{\omega=0}^{k} V_{\omega}\Delta\pi_{\omega} + M\Delta\gamma = r_{\gamma}$$

substituting equation (59) into above equation, results in

$$\sum_{\omega=0}^{k} V_\omega C_\omega^{-1}\left(r_\omega - V_\omega^T \Delta\gamma\right) + M\Delta\gamma = r_\gamma$$

$$\sum_{\omega=0}^{k} V_\omega C_\omega^{-1} r_\omega - \sum_{\omega=0}^{k} V_\omega C_\omega^{-1} V_\omega^T \Delta\gamma + M\Delta\gamma = r_\gamma$$

$$\left(M - \sum_{\omega=0}^{k} V_\omega C_\omega^{-1} V_\omega^T\right)\Delta\gamma = r_\gamma - \sum_{\omega=0}^{k} V_\omega C_\omega^{-1} r_\omega \tag{60}$$

To calculate $\Delta\gamma$ from the above equation we have to introduce following operations: first factor each diagonal block by:

$$C_\omega = U_\omega^T D_\omega U_\omega$$

therefore

$$V_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} V_\omega^T = K_\omega^T D_\omega^{-1} K_\omega$$

where $K_\omega = U_\omega^{-T} V_\omega^T$ is calculated by column by column back substitution and term $\bar{r}_\omega = U_\omega^{-T} r_\omega$ is calculated by single back substitution

$$V_\omega U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} r_\omega = K_\omega^T D_\omega^{-1} \bar{r}_\omega$$

therefore equation (60) can be rewritten:

$$\left(M - \sum_{\omega=0}^{k} K_\omega^T D_\omega^{-1} K_\omega\right)\Delta\gamma = r_\gamma - \sum_{\omega=0}^{k} K_\omega^T D_\omega^{-1} \bar{r}_\omega$$

$\Delta\gamma$ can now be easily solved and then the $\Delta\pi_i$ is solved from (59)

3.6 Algorithm Description

The outer loop algorithm can be outlined as:

Initialize x, u, x_ω, u_ω

while (KKT conditions > ε)

Calculate c, f, g, G, W for base and each contingency

Do the inner loop algorithm

Calculate KKT conditions

end

The inner loop algorithm can be outlined as:

Initialize s, π, s_ω, π_ω for enforced violations

Build $\overline{F}_\omega, R_\omega, \widetilde{H}_\omega, \widetilde{b}_\omega, C_\omega, V_\omega, r_\omega \qquad \omega = 0 \ldots k$

Build M, r_γ

$C_\omega = U_\omega^T D_\omega U_\omega \quad \omega = 0 \ldots k$

$K_\omega = U_\omega^{-T} V_\omega^T \quad \omega = 0 \ldots k$

Initialize μ

while $\mu > \varepsilon$

Calculate $\Delta\gamma$ from

$$\left(M - \sum_{\omega=0}^{k} K_\omega^T D_\omega^{-1} K_\omega \right) \Delta\gamma = r_\gamma - \sum_{\omega=0}^{k} K_\omega^T D_\omega^{-1} \overline{r}_\omega$$

Calculate $\Delta\pi$, $\Delta\pi_\omega$ from

$$\left(-\Pi^{-1} S - R \right) \Delta\pi = \hat{b}_\pi - \overline{F}^T D^{-1} \overline{b}_0 + R\pi - \overline{F}^T D^{-1} \widetilde{H} \Delta\gamma$$

$$\left(-\Pi_\omega^{-1} S_\omega - R_\omega \right) \Delta\pi_\omega = \hat{b}_{\pi_\omega} - \overline{F}_\omega^T D_\omega^{-1} \widetilde{b}_\omega + R_\omega \pi_\omega - \overline{F}_\omega^T D_\omega^{-1} \widetilde{H}_\omega \Delta\gamma$$

Calculate Δs Δs_ω and σ from

$$\Delta s = \Pi^{-1}\left(\mu e - \Pi S e - S\Delta\pi\right)$$

$$\Delta s_\omega = \Pi_\omega^{-1}\left(\mu e - \Pi_\omega S_\omega e - S_\omega \Delta\pi_\omega\right)$$

$$\Delta\sigma = \Gamma^{-1}\left(\mu e - \Gamma\ \Sigma\ e - \Sigma\ \Delta\gamma\ \right)$$

Update π, s and σ

Update μ $\quad \mu^{k+1} = \dfrac{\pi^T s}{\left(n + n_g\right)^2}$

Update following terms

$$C_\omega = -\Pi_\omega^{-1} S_\omega - R_\omega \qquad \omega = 0 \ldots k$$

$$\hat{b}_{\pi_\omega} = \overline{b}_{\pi_\omega} - \mu \Pi_\omega^{-1} e \qquad \omega = 0 \ldots k$$

$$r_\omega = \hat{b}_\pi - \overline{F}_\omega^T D_\omega^{-1} \widetilde{b}_\omega + R_\omega \pi_\omega \qquad \omega = 0 \ldots k$$

$$M = \sum_{\omega=0}^{k} \widetilde{H}_\omega^T D_\omega^{-1} \widetilde{H}_\omega + \Gamma^{-1}\Sigma$$

$$r_\gamma = -\hat{b}_\gamma + \sum_{\omega=0}^{k} \widetilde{H}_\omega^T D_\omega^{-1} \widetilde{b}_\omega - \sum_{\omega=0}^{k} V_\omega \pi_\omega$$

end

Calculate Δu , λ_r , Δu_ω , $\lambda_{\omega r}$ from

$$\begin{pmatrix} \Delta u \\ \lambda_r \end{pmatrix} = U^{-1} D^{-1} U^{-T} (b_0 - F^T \pi - \hat{H}^T \Delta\gamma - F^T \Delta\pi)$$

$$\begin{pmatrix} \Delta u_\omega \\ \lambda_{\omega r} \end{pmatrix} = U_\omega^{-1} D_\omega^{-1} U_\omega^{-T} (b_\omega - F_\omega^T \pi_\omega - \hat{H}_\omega^T \Delta\gamma - F_\omega^T \Delta\pi_\omega)$$

Calculate Δx and Δx_ω from

$$\begin{pmatrix} \Delta x \\ \lambda \end{pmatrix} = \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x - W_{xu}\Delta u - G_{rx}^T \lambda_r - F_x^T \pi \\ b_\lambda - G_u \Delta u \end{pmatrix}$$

$$\begin{pmatrix} \Delta x_\omega \\ \lambda_\omega \end{pmatrix} = \begin{pmatrix} W_{x_\omega x_\omega} & G_{\omega_{x_\omega}}^T \\ G_{\omega_{x_\omega}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{x_\omega} - W_{x_\omega u_\omega} \Delta u_\omega - G_{\omega r_{x_\omega}}^T \lambda_{\omega r} - F_{\omega_{x_\omega}}^T \pi_\omega \\ b_{\lambda_\omega} - G_{\omega_{u_\omega}} \Delta u_\omega \end{pmatrix}$$

Check for new violations

while new violations≠0

Initialize s, π, s_ω, π_ω

Build $\overline{F}_\omega$ $\omega = 0 \ldots k$ for the new violations

Build columns $\overline{F}, R, \widetilde{H}, \widetilde{b}_0, C_0, V_0, r_0$

Build $\overline{F}_\omega, R_\omega, \widetilde{H}_\omega, \widetilde{b}_\omega, C_\omega, V_\omega, r_\omega$ $\omega = 0 \ldots k$

Build M, r_γ

$C_\omega = U_\omega^T D_\omega U_\omega$ $\omega = 0 \ldots k$

$K_\omega = U_\omega^{-T} V_\omega^T$ $\omega = 0 \ldots k$

Initialize μ
while $\mu > \varepsilon$

Calculate $\Delta\gamma$ from

Calculate $\Delta\pi$, $\Delta\pi_\omega$ from

Calculate Δs Δs_ω and σ from

Update π , s and σ

Update μ

Update following terms

$C_\omega = -\Pi_\omega^{-1} S_\omega - R_\omega$ $\omega = 0 \ldots k$

$\hat{b}_{\pi_\omega} = \overline{b}_{\pi_\omega} - \mu \Pi_\omega^{-1} e$ $\omega = 0 \ldots k$

$r_\omega = \hat{b}_\pi - \overline{F}_\omega^T D_\omega^{-1} \widetilde{b}_\omega + R_\omega \pi_\omega$ $\omega = 0 \ldots k$

$$M = \sum_{\omega=0}^{k} \widetilde{H}_\omega^T D_\omega^{-1} \widetilde{H}_\omega + \Gamma^{-1}\Sigma$$

$$r_\gamma = -\hat{b}_\gamma + \sum_{\omega=0}^{k} \widetilde{H}_\omega^T D_\omega^{-1} \widetilde{b}_\omega - \sum_{\omega=0}^{k} V_\omega \pi_\omega$$

end

Calculate Δx and Δx_ω from

Check for new violations

end while new violations

4 Simulation results

The algorithm was tested on two cases the IEEE14 and IEEE 30 bus networks. The ramp-rate constraints coefficient, Δ, is defined as 10% of the generating capacity of each generator. In performing the contingency constrained algorithm, only line outages have been considered. The results of the algorithm are presented by comparing the total cost of a base case solution with the total cost of the contingency case solution for the various of contingency cases considered. The total cost of each contingency case is normalized by the cost of the base case.

4.1 IEEE 14 bus network case

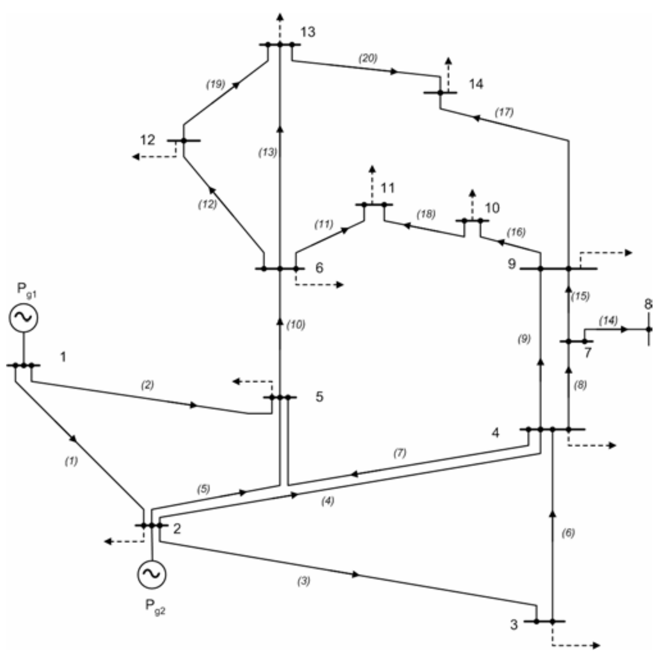


Fig. 4.1. IEEE 14 bus network

	IEEE 14 bus network		
Case	**Number of contingencies n_c**	**CC OPF cost [p.u]**	**Cost increase [%]**
1	5	1.0458	4.58
2	7	1.0634	6.34
3	12	1.0892	8.92

Table 4.1. Cost comparison for IEEE 30 bus. “CC OPF” is contingency constrained OPF.

4.2 IEEE 30 bus network case

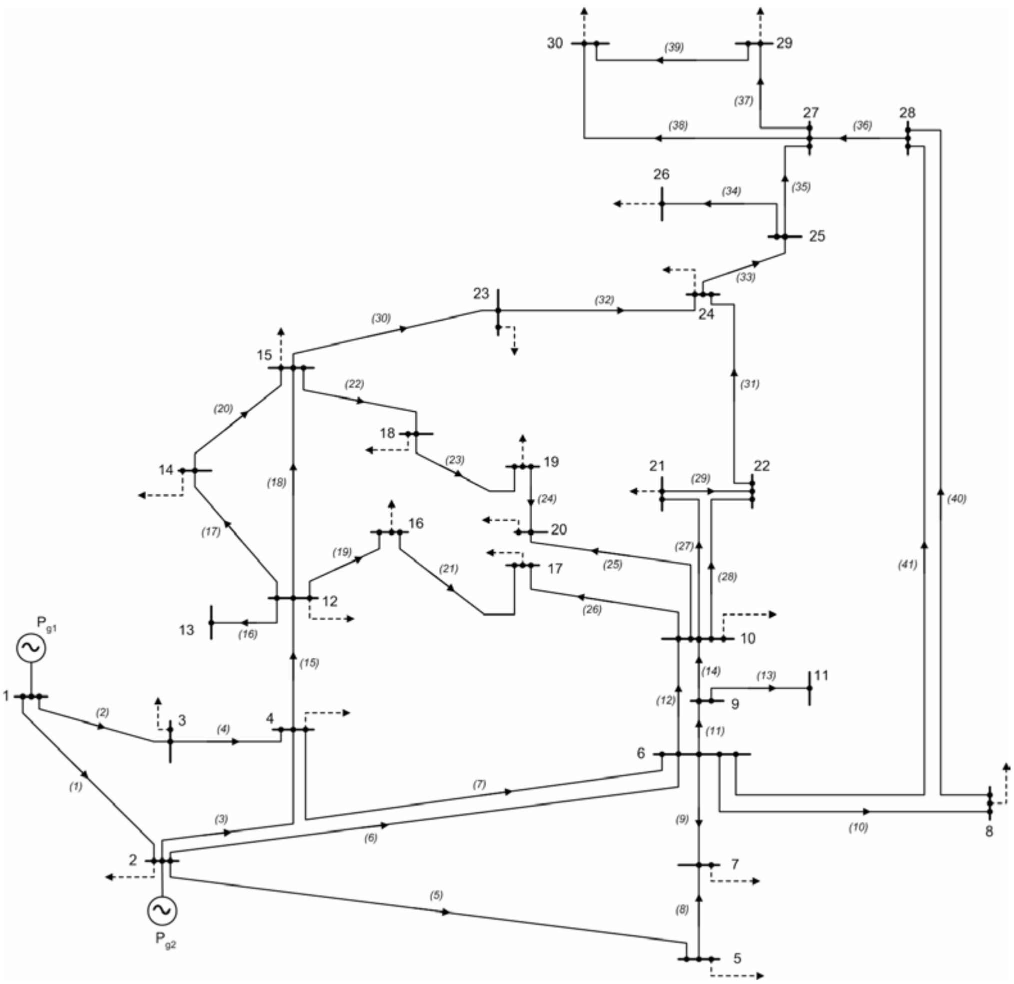


Fig. 4.2. IEEE 30 bus network

	IEEE 30 bus network		
Case	**Number of contingencies n_c**	**CC OPF cost [p.u]**	**Cost increase [%]**
1	10	1.0521	5.21
2	14	1.0773	7.73
3	21	1.1196	11.96

Table 4.2. Cost comparison for IEEE 30 bus

5 Conclusion and Future Work

5.1 Conclusion

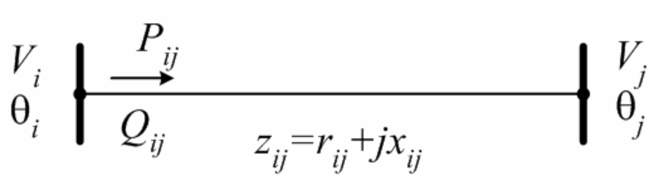
Contingency constrained OPF is a very challenging and computationally demanding optimization problem. The number of contingency cases considered can be very large. Each contingency considered introduces a new problem as large as the base case. Therefore, efficient solution of CC OPF is crucial. This work presents a new formulation based on sequential quadratic programming. The algorithm is based on an interior point method and constraint relaxation or active set method. Restricting our attention to the active constraint set makes this large problem significantly smaller and computationally feasible.

5.2 Future work

There are several directions in which the research presented here can be extended.

- Include load shedding as a control variable
- From chapter 3 it can be seen that the decomposition technique applied in the development of CC OPF produces promising framework for solving large power system cases. In order to apply the proposed algorithm to practical size networks (118 bus, 300 bus test cases and larger) we need to improve computational efficiency by employing sparse matrix techniques.
- Develop a fast-decoupled implementation of the algorithm
- Monte Carlo simulation with importance sampling combined with CC OPF in large networks shows promise to be a good technique in analyzing multiple contingencies

Appendix I: Power Balance, Jacobian and Hessian Equations



For power flow equations:

Active power flow

$$P_{ij} = g_{ij}V_i^2 - g_{ij}V_iV_j\cos\theta - b_{ij}V_iV_j\sin\theta$$

Reactive power flow

$$Q_{ij} = -b_{ij}V_i^2 + b_{ij}V_iV_j\cos\theta - g_{ij}V_iV_j\sin\theta$$

If we denote $\theta=\theta_1-\theta_2$ (angle difference)

$P_1=f(V_1,V_2,\theta)$

Power flow Jacobian and Hessian have following form:

$$\nabla^2 P_1(V_1,V_2,\theta) = \begin{pmatrix} \frac{\partial^2 P_1}{\partial V_1^2} & \frac{\partial^2 P_1}{\partial V_1 \partial V_2} & \frac{\partial^2 P_1}{\partial V_1 \partial\theta} \\ \frac{\partial^2 P_1}{\partial V_2 \partial V_1} & \frac{\partial^2 P_1}{\partial V_2^2} & \frac{\partial^2 P_1}{\partial V_2 \partial\theta} \\ \frac{\partial^2 P_1}{\partial\theta \partial V_1} & \frac{\partial^2 P_1}{\partial\theta \partial V_2} & \frac{\partial P_1}{\partial\theta^2} \end{pmatrix}$$

where:

$$\frac{\partial P_{ij}}{\partial V_i} = 2g_{ij}V_i - g_{ij}V_j\,cos\,\theta - b_{ij}V_j\,sin\,\theta$$

$$\frac{\partial P_{ij}}{\partial V_j} = -g_{ij} V_i \cos\theta - b_{ij} V_i \sin\theta$$

$$\frac{\partial P_{ij}}{\partial \theta} = g_{ij} V_i V_j \sin\theta - b_{ij} V_i V_i \cos\theta$$

In can be shown that element $\frac{\partial^2 P_1}{\partial V_2^2}$ is zero, so that Hessian matrix has following form:

$$\nabla^2 P_1(V_1, V_2, \theta) = \begin{pmatrix} \frac{\partial^2 P_1}{\partial V_1^2} & \frac{\partial^2 P_1}{\partial V_1 \partial V_2} & \frac{\partial^2 P_1}{\partial V_1 \partial \theta} \\ \frac{\partial^2 P_1}{\partial V_2 \partial V_1} & 0 & \frac{\partial^2 P_1}{\partial V_2 \partial \theta} \\ \frac{\partial^2 P_1}{\partial \theta \partial V_1} & \frac{\partial^2 P_1}{\partial \theta \partial V_2} & \frac{\partial P_1}{\partial \theta^2} \end{pmatrix}$$

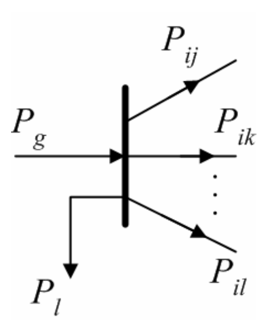
The same set of equations can be written for the reactive power flow $Q_1=f(V_1,V_2,\theta)$

Similar equations can be written for the to end of the line $P_2=f(V_1,V_2,\theta)$

$$\nabla^2 P_2(V_1, V_2, \theta) = \begin{pmatrix} 0 & \frac{\partial^2 P_2}{\partial V_1 \partial V_2} & \frac{\partial^2 P_2}{\partial V_1 \partial \theta} \\ \frac{\partial^2 P_2}{\partial V_2 \partial V_1} & \frac{\partial^2 P_2}{\partial V_2^2} & \frac{\partial^2 P_2}{\partial V_2 \partial \theta} \\ \frac{\partial^2 P_2}{\partial \theta \partial V_1} & \frac{\partial^2 P_2}{\partial \theta \partial V_2} & \frac{\partial P_2}{\partial \theta^2} \end{pmatrix}$$

The same set of equations can be written for the reactive power flow $Q_2=f(V_1,V_2,\theta)$

Power balance equation



$$\sum_j P_{ij} + P_l - P_g = 0$$

Let's denote above power balance equation for node i as $g_i(x,u)=0$ and form vector of dimension $2n$ whose elements are power balance equations for active and reactive power at each node in the network.

Therefore power balance Jacobian can be defined as a matrix of first partial derivatives of power balance equation with respect of state variables. Mathematicaly this is:

$$G_x = \frac{\partial g(x,u)}{\partial x} \qquad G_x \in \Re^{2n \times 2n}$$

The number of rows is $2n$ because we have active and reactive power balance at each node, and the number of columns is $2n$ because we have $2n$ state variables (voltage magnitude and phase angle at each bus)

$$G_X = \begin{pmatrix} G_{PV} & G_{Pa} \\ G_{QV} & G_{Qa} \end{pmatrix}$$

Block matrices G_{Pv} G_{Pa} G_{Qv} G_{Qa} have dimension $n \times n$ (and have bus admittance sparsity structure).

$$G_{Pv} = \begin{pmatrix} \sum_i \frac{dP_i}{dV_1} & \frac{dP_{12}}{dV_2} & \cdots & \frac{dP_{1j}}{dV_j} & \cdots & \frac{dP_{1n}}{dV_n} \\ \frac{dP_{21}}{dV_1} & \sum_i \frac{dP_i}{dV_2} & \cdots & \frac{dP_{2j}}{dV_j} & \cdots & \frac{dP_{2n}}{dV_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{dP_{j1}}{dV_1} & \frac{dP_{j2}}{dV_2} & \cdots & \sum_i \frac{dP_i}{dV_j} & \cdots & \frac{dP_{jn}}{dV_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{dP_{n1}}{dV_1} & \frac{dP_{n2}}{dV_2} & \cdots & \frac{dP_{nj}}{dV_j} & \cdots & \sum_i \frac{dP_i}{dV_n} \end{pmatrix}$$

Because of the property of power network, most of the partial derivatives in the above matrix would be zero.

$$G_{Pa} = \begin{pmatrix} \sum_i \frac{dP_i}{d\theta_1} & \frac{dP_{12}}{d\theta_2} & \cdots & \frac{dP_{1j}}{d\theta_j} & \cdots & \frac{dP_{1n}}{d\theta_n} \\ \frac{dP_{21}}{d\theta_1} & \sum_i \frac{dP_i}{d\theta_2} & \cdots & \frac{dP_{2j}}{d\theta_j} & \cdots & \frac{dP_{2n}}{d\theta_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{dP_{j1}}{d\theta_1} & \frac{dP_{j2}}{d\theta_2} & \cdots & \sum_i \frac{dP_i}{d\theta_j} & \cdots & \frac{dP_{jn}}{d\theta_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{dP_{n1}}{d\theta_1} & \frac{dP_{n2}}{d\theta_2} & \cdots & \frac{dP_{nj}}{d\theta_j} & \cdots & \sum_i \frac{dP_i}{d\theta_n} \end{pmatrix}$$

The above matrices are singular, therefore reference bus must be introduced. Every network has a reference bus (bus with phase angle equal to zero). For reference bus instead of equality constraint for active power balance, we impose constraint which says reference angle equal to zero, therefore if we chose bus one to be reference bus (common case although any bus can be reference), first row in matrix G_{Pv} instead a form

$$\left(\sum_i \frac{dP_1}{dV_1} \quad \frac{dP_{12}}{dV_2} \quad \cdots \quad \frac{dP_{1j}}{dV_j} \quad \cdots \quad \frac{dP_{jn}}{dV_n} \right)$$

has a following form:

$$\left(0 \quad 0 \quad \cdots \quad 0 \quad \cdots \quad 0 \right)$$

Also first row in matrix G_{Pa} instead a form

$$\left(\sum_i \frac{dP_1}{d\theta_1} \quad \frac{dP_{12}}{d\theta_2} \quad \cdots \quad \frac{dP_{1j}}{d\theta_j} \quad \cdots \quad \frac{dP_{jn}}{d\theta_n} \right)$$

has a following form (where 1 stays for a reference bus)

$$\left(1 \quad 0 \quad \cdots \quad 0 \quad \cdots \quad 0 \right)$$

Power balance Jacobian with respect of control variables (u) can be defined as follows:

$$G_u = \frac{\partial g(x,u)}{\partial u} \qquad G_u \in \Re^{2n \times n_u}$$

and written in following block matrix structure:

$$G_u = \begin{pmatrix} G_{Pu} \\ G_{Qu} \end{pmatrix}$$

where

$$G_{Pu} = \frac{\partial g_P(x,u)}{\partial u} \qquad G_{Pu} \in \Re^{n \times n_u}$$

$$G_{Qu} = \frac{\partial g_Q(x,u)}{\partial u} \qquad G_{Qu} \in \Re^{n \times n_u}$$

Block matrices G_{Pu} and G_{Qu} are build on following way:

$$G_{pu} = \begin{cases} 1 & (i,j) \quad \text{i- generator bus; j – position of Pg in } n_u \text{ vector} \\ \dfrac{\partial P_{lj}}{\partial t_b} & (l,k) \quad \text{l – transformer line from bus; k – } t_b \text{ control in } n_u \text{ vector} \\ \dfrac{\partial P_{jl}}{\partial t_b} & (j,k) \quad \text{j – transformer line to bus; k – } t_b \text{ control in } n_u \text{ vector} \end{cases}$$

$$G_{qu} = \begin{cases} 1 & (i,j) \quad \text{i- generator bus; j– position of Qg in } n_u \text{ vector} \\ \dfrac{\partial Q_{lj}}{\partial t_b} & (l,k) \quad \text{l – transformer line from bus; k – } t_b \text{ control in } n_u \text{ vector} \\ \dfrac{\partial Q_{jl}}{\partial t_b} & (j,k) \quad \text{j – transformer line to bus; k – } t_b \text{ control in } n_u \text{ vector} \end{cases}$$

Appendix II: Fast Decoupled Power Flow

Applying Newton's method to the power flow equation results in the most robust power flow algorithm. Drawback to its use is the fact that the terms in the Jacobian matrix must be recalculated each iteration. Linearized power balance equations can be stated as follows:

$$\Delta P_i = \sum_{k=1}^{N} \frac{\partial P_i}{\partial \theta_k} \Delta\theta_k + \sum_{k=1}^{N} \frac{\partial P_i}{\partial V_k} \Delta V_k \tag{1}$$

$$\Delta Q_i = \sum_{k=1}^{N} \frac{\partial Q_i}{\partial \theta_k} \Delta\theta_k + \sum_{k=1}^{N} \frac{\partial Q_i}{\partial V_k} \Delta V_k \tag{2}$$

or in the matrix form

$$\begin{pmatrix} \Delta P \\ \Delta Q \end{pmatrix} = \begin{pmatrix} H & N \\ J & L \end{pmatrix} \begin{pmatrix} \Delta\theta \\ \Delta V \end{pmatrix}$$

Fast decoupled formulation [9] is obtained by neglecting the coupling submatrices N and J according to the following assumptions:

- real power is little influenced by changes in voltage magnitude $\frac{\partial P}{\partial V}$
- insensitivity of reactive power to changes in phase angle $\frac{\partial Q}{\partial \theta}$

Recall power flow equations for both active and reactive power:

$$P_{ik} = g_{ik} V_i^2 - g_{ik} V_i V_k \cos\theta - b_{ik} V_i V_k \sin\theta$$

$$Q_{ik} = -b_{ik} V_i^2 + b_{ik} V_i V_k \cos\theta - g_{ik} V_i V_k \sin\theta$$

where $\theta=\theta_i-\theta_k$ is a angle difference

Partial derivatives of the power balance equation are as follows:

$$\frac{\partial P_{ik}}{\partial \theta} = V_i V_k \left(g_{ik} \sin\theta - b_{ik} \cos\theta \right) \tag{3}$$

$$\frac{\partial Q_{ik}}{\partial V_k} = -V_i\left(g_{ik}\, sin\,\theta - b_{ik}\, cos\,\theta\right) \tag{4}$$

In practical power system following assumptions are almost always valid:

$$cos\,\theta \approx 1 \qquad \text{and} \qquad g_{ik}\, sin\,\theta << b_{ik}$$

Therefore following equations are a good approximation of (3) and (4)

$$\frac{\partial P_{ik}}{\partial \theta} = -V_i V_k b_{ik} \tag{5}$$

$$\frac{\partial Q_{ik}}{\partial V_j / V_k} = -V_i V_k b_{ik} \tag{6}$$

The power flow adjustment according to (1) and (2) can be written

$$\Delta P_i = \frac{\partial P_i}{\partial \theta_k} \Delta\theta_k$$

$$\Delta Q_i = \frac{\partial Q_i}{\partial V_k / V_k} \frac{\Delta V_k}{V_k}$$

After substituting (5) and (6) in the above equations

$$\Delta P_i = -V_i V_k b_{ik} \Delta\theta_k \tag{7}$$

$$\Delta Q_i = -V_i V_k b_{ik} \frac{\Delta V_k}{V_k} \tag{8}$$

Following simplification will be made:

- Equations (7) and (8) will be divided by V_i
- We will assume $V_k \cong 1$

Therefore equations (7) and (8) will have following form

$$\frac{\Delta P_i}{V_i} = -b_{ik} \Delta\theta_k \tag{9}$$

$$\frac{\Delta Q_i}{V_i} = -b_{ik}\Delta V_k \tag{10}$$

Equations (9) and (10) can be generalized in a following matrix form

$$\begin{pmatrix} \frac{\Delta P_1}{V_1} \\ \frac{\Delta P_2}{V_2} \\ \vdots \\ \frac{\Delta P_n}{V_n} \end{pmatrix} = \begin{pmatrix} -b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & -b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & -b_{nn} \end{pmatrix} \begin{pmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \vdots \\ \Delta\theta_n \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} \frac{\Delta Q_1}{V_1} \\ \frac{\Delta Q_2}{V_2} \\ \vdots \\ \frac{\Delta Q_n}{V_n} \end{pmatrix} = \begin{pmatrix} -b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & -b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & -b_{nn} \end{pmatrix} \begin{pmatrix} \Delta V_1 \\ \Delta V_2 \\ \vdots \\ \Delta V_n \end{pmatrix} \tag{12}$$

Matrix equation (11) can be simplified with a assumption $r_{ik} << x_{ik}$ Finally equations (11) and (12) can be written as:

$$\frac{\Delta P}{V} = B'\Delta\theta$$

$$\frac{\Delta Q}{V} = B''\Delta V_k$$

Terms in the B' matrix are:

$$B'_{ik} = -\frac{1}{x_{ik}}$$ assuming a branch from i to k (zero otherwise)

$$B'_{ii} = \sum_{k=1}^{N} \frac{1}{x_{ik}}$$

Terms in the B'' matrix are:

$$B''_{ik} = -B_{ik} = -\frac{x_{ik}}{r_{ik}^2 + x_{ik}^2}$$

$$B''_{ii} = \sum_{k=1}^{N} -B_{ik}$$

Appendix III: Matrix Calculation Details

This appendix will present efficient way to calculate variables which appear in both base and contingency case as well as terms which appear in fast-decouple formulation.

AC Case

Let's group terms which we have to calculate into two groups.

$$\overline{W}_{uu} = W_{uu} - \begin{pmatrix} W_{xu}^T & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \tag{1}$$

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \tag{2}$$

$$\overline{F}_{u} = F_{u} - \begin{pmatrix} F_{x} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix} \tag{3}$$

Next set of equations:

$$\overline{b}_{u} = b_{u} - \begin{pmatrix} W_{xu}^T & G_u^T \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} \tag{4}$$

$$\overline{b}_{r} = b_{r} - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} \tag{5}$$

$$\overline{b}_{\pi} = b_{\pi} - \begin{pmatrix} F_{x} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix} \tag{6}$$

if we perform LU factorization of the following block matrix

$$U^T D U = \begin{pmatrix} W_{xx} & G_x^T \\ G_x & 0 \end{pmatrix} \qquad \text{U has dimension } (4n \times 4n)$$

and define following variables

$$K = \begin{pmatrix} W_{xu} \\ G_u \end{pmatrix}$$

$$b = \begin{pmatrix} b_x \\ b_\lambda \end{pmatrix}$$

$$g^T = \begin{pmatrix} G_{rx} & 0 \end{pmatrix}$$ of dimension ($1 \times 4n$)

$$F = \begin{pmatrix} F_x & 0 \end{pmatrix}$$ of dimension ($n_c \times 4n$)

equations (1) – (6) can be rewritten

$$\overline{W_{uu}} = W_{uu} - K^T U^{-1} D^{-1} U^{-T} K$$

$$\overline{G}_{ru} = G_{ru} - g^T U^{-1} D^{-1} U^{-T} K$$

$$\overline{F}_u = F_u - F U^{-1} D^{-1} U^{-T} K$$

$$\overline{b}_u = b_u - K^T U^{-1} D^{-1} U^{-T} b$$

$$\overline{b}_r = b_r - g^T U^{-1} D^{-1} U^{-T} b$$

$$\overline{b}_\pi = b_\pi - F U^{-1} D^{-1} U^{-T} b$$

let's simplify calculation by introducing following variables:

$$M = U^{-T} K$$

$$x^T = g^T U^{-1}$$

$$R^T = F U^{-1}$$

$$y = U^{-T} b$$

Matrices *M* and *R* can be calculated by performing column by column forward substitution and vector *x* and *y* by forward substitution through following equations:

$$U^T M = K$$

$$U^T x = g$$

$$U^T R = F^T$$

$$U^T y = b$$

Therefore equations (1) - (6) can be finally calculated by:

$$\overline{W}_{uu} = W_{uu} - M^T D^{-1} M$$

$$\overline{G}_{ru} = G_{ru} - x^T D^{-1} M$$

$$\overline{F}_u = F_u - R^T D^{-1} M$$

$$\overline{b}_u = b_u - M^T D^{-1} y$$

$$\overline{b}_r = b_r - x^T D^{-1} y$$

$$\overline{b}_\pi = b_\pi - R^T D^{-1} y$$

Fast-decouple case

The following matrices are considered in the fast-decoupled case:

$$\overline{F}_u = F_u - \begin{pmatrix} 0 & F_x \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_u \end{pmatrix} \tag{7}$$

$$\overline{G}_u = G_u - \begin{pmatrix} 0 & G_{rx} \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_u \end{pmatrix} \tag{8}$$

$$\overline{b}_u = b_u - \begin{pmatrix} 0 & G_u^T \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} \tag{9}$$

$$\overline{b}_r = b_r - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} \tag{10}$$

$$\hat{b}_\pi = b_\pi - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} \tag{11}$$

Matrix F_x can be written in the following block matrix form

$$F_x = \begin{pmatrix} F_v & F_a \end{pmatrix}$$

where

$$F_v = \frac{\partial f(x,u)}{\partial v} \qquad F_v \in \Re^{n_c \times n}$$

$$F_a = \frac{\partial f(x,u)}{\partial \theta} \qquad F_a \in \Re^{n_c \times n}$$

Block matrix $G_u = \begin{pmatrix} G_{pu} \\ G_{qu} \end{pmatrix}$ is defined in Appendix I

Reference bus Jacobian can be written as

$$G_{rx} = \begin{pmatrix} G_{rv} & G_{ra} \end{pmatrix}$$

where

$$G_{rv} = \frac{\partial g_r(x,u)}{\partial v} \qquad G_{rv} \in \Re^{1 \times n}$$

$$G_{ra} = \frac{\partial g_r(x,u)}{\partial \theta} \qquad G_{ra} \in \Re^{1 \times n}$$

Matrix G_x is of the form

$$G_x = \begin{pmatrix} 0 & B' \\ B'' & 0 \end{pmatrix}$$

B′ and B″are symmetric which means $B'^{-T} = B'^{-1}$ and $B''^{-T} = B''^{-1}$

Recall also that B' and B'' are factored according to the following two expressions just once for the entire iterative process.

$$B' = U_1^T D_1 U_1$$

$$B'' = U_2^T D_2 U_2$$

Considering equation (7)

$$\overline{F}_u = F_u - \begin{pmatrix} 0 & F_x \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_u \end{pmatrix} = F_u - F_x G_x^{-T} G_u$$

$$\overline{F}_u = F_u - \begin{pmatrix} F_v & F_a \end{pmatrix} \begin{pmatrix} 0 & B''^{-T} \\ B'^{-T} & 0 \end{pmatrix} \begin{pmatrix} G_{pu} \\ G_{qu} \end{pmatrix} = F_u - F_a B'^{-T} G_{pu} - F_v B''^{-T} G_{qu}$$

$$\overline{F}_u = F_u - F_a B'^{-1} G_{pu} - F_v B''^{-1} G_{qu}$$

$$\overline{F}_u = F_u - F_a \left(U_1^T D_1 U_1\right)^{-1} G_{pu} - F_v \left(U_2^T D_2 U_2\right)^{-1} G_{qu}$$

$$\overline{F}_u = F_u - F_a U_1^{-1} D_1^{-1} U_1^{-T} G_{pu} - F_v U_2^{-1} D_2^{-1} U_2^{-T} G_{qu}$$

Calculation of the above expression is facilitating by calculating F_1, F_2, G_1, G_2 via column by column forward substitution in a following way

$$F_a U_1^{-1} = F_1^T \qquad \Leftrightarrow \qquad U_1^T F_1 = F_a^T$$

$$F_v U_2^{-1} = F_2^T \qquad \Leftrightarrow \qquad U_2^T F_2 = F_v^T$$

$$U_1^{-T} G_{pu} = G_1 \qquad \Leftrightarrow \qquad U_1^T G_1 = G_{pu}$$

$$U_2^{-T} G_{qu} = G_2 \qquad \Leftrightarrow \qquad U_2^T G_2 = G_{qu}$$

Therefore $\overline{F}_u$ is calculated by

$$\overline{F}_u = F_u - F_1^T D_1^{-1} G_1 - F_2^T D_2^{-1} G_2$$

Next is equation (8)

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} 0 & G_{rx} \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_u \end{pmatrix} = G_{ru} - G_{rx} G_x^{-T} G_u$$

$$\overline{G}_{ru} = G_{ru} - \begin{pmatrix} G_{rv} & G_{ra} \end{pmatrix} \begin{pmatrix} 0 & B''^{-T} \\ B'^{-T} & 0 \end{pmatrix} \begin{pmatrix} G_{pu} \\ G_{qu} \end{pmatrix} = G_{ru} - G_{ra} B'^{-T} G_{pu} - G_{rv} B''^{-T} G_{qu}$$

$$\overline{G}_{ru} = G_{ru} - G_{ra}B'^{-1}G_{pu} - G_{rv}B''^{-1}G_{qu}$$

$$\overline{G}_{ru} = G_{ru} - G_{ra}\left(U_1^T D_1 U_1\right)^{-1} G_{pu} - G_{rv}\left(U_2^T D_2 U_2\right)^{-1} G_{qu}$$

$$\overline{G}_{ru} = G_{ru} - G_{ra}U_1^{-1} D_1^{-1} U_1^{-T} G_{pu} - G_{rv} U_2^{-1} D_2^{-1} U_2^{-T} G_{qu}$$

In order to calculate $\overline{G}_{ru}$ vectors g_1 and g_2 have to be obtained via forward substitution

$$G_{ra} U_1^{-1} = g_1^T \qquad \Leftrightarrow \qquad U_1^T g_1 = G_{ra}^T$$

$$G_{rv} U_2^{-1} = g_2^T \qquad \Leftrightarrow \qquad U_2^T g_2 = G_{rv}^T$$

Finally equation (8) can be calculated by

$$\overline{G}_{ru} = G_{ru} - g_1^T D_1^{-1} G_1 - g_2^T D_{21}^{-1} G_2$$

Equation (9) can be simplified as

$$\overline{b}_u = b_u - \begin{pmatrix} 0 & G_u^T \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} = b_u - G_u^T G_x^{-T} b_x$$

$$\overline{b}_u = b_u - \begin{pmatrix} G_{pu}^T & G_{qu}^T \end{pmatrix} \begin{pmatrix} 0 & B''^{-T} \\ B'^{-T} & 0 \end{pmatrix} \begin{pmatrix} b_v \\ b_a \end{pmatrix} = b_u - G_{qu}^T B'^{-T} b_v - G_{pu}^T B''^{-T} b_a$$

$$\overline{b}_u = b_u - G_{qu}^T B'^{-1} b_v - G_{pu}^T B''^{-1} b_a$$

$$\overline{b}_u = b_u - G_{qu}^T \left(U_1^T D_1 U_1\right)^{-1} b_v - G_{pu}^T \left(U_2^T D_2 U_2\right)^{-1} b_a$$

$$\overline{b}_u = b_u - G_{qu}^T U_1^{-1} D_1^{-1} U_1^{-T} b_v - G_{pu}^T U_2^{-1} D_2^{-1} U_2^{-T} b_a$$

Matrices G_3 and G_4 are calculated via column by column forward substitution and vectors b_1 and b_2 via forward substitution

$$G_{qu}^T U_1^{-1} = G_3^T \qquad \Leftrightarrow \qquad U_1^T G_3 = G_{qu}$$

$$G_{pu}^T U_2^{-1} = G_4^T \qquad \Leftrightarrow \qquad U_2^T G_4 = G_{pu}$$

$$U_1^{-T} b_v = b_1 \qquad \Leftrightarrow \qquad U_1^T b_1 = b_v$$

$$U_2^{-T} b_a = b_2 \qquad \Leftrightarrow \qquad U_2^T b_2 = b_a$$

Equation (9) is calculated on following way

$$\overline{b_u} = b_u - G_3^T D_1^{-1} b_1 - G_4^T D_2^{-1} b_2$$

Equation (10)

$$\overline{b_r} = b_r - \begin{pmatrix} G_{rx} & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} = b_r - G_{rx} G_x^{-1} b_\lambda$$

$$\overline{b_r} = b_r - \begin{pmatrix} G_{rv} & G_{ra} \end{pmatrix} \begin{pmatrix} 0 & B'^{-1} \\ B''^{-1} & 0 \end{pmatrix} \begin{pmatrix} b_{\lambda p} \\ b_{\lambda q} \end{pmatrix} = b_r - G_{ra} B''^{-1} b_{\lambda p} - G_{rv} B'^{-1} b_{\lambda q}$$

$$\overline{b_r} = b_r - G_{ra} \left(U_2^T D_2 U_2 \right)^{-1} b_{\lambda p} - G_{rv} \left(U_1^T D_1 U_1 \right)^{-1} b_{\lambda q}$$

$$\overline{b_r} = b_r - G_{ra} U_2^{-1} D_2^{-1} U_2^{-T} b_{\lambda p} - G_{rv} U_1^{-1} D_1^{-1} U_1^{-T} b_{\lambda q}$$

vectors g_3, g_4, b_3, b_4 are obtained performing forward substitution

$$G_{ra} U_2^{-1} = g_3^T \qquad \Leftrightarrow \qquad U_2^T g_3 = G_{ra}^T$$

$$G_{rv} U_1^{-1} = g_4^T \qquad \Leftrightarrow \qquad U_1^T g_4 = G_{ra}^T$$

$$U_2^{-T} b_{\lambda p} = b_3 \qquad \Leftrightarrow \qquad U_2^T b_3 = b_{\lambda p}$$

$$U_1^{-T} b_{\lambda q} = b_4 \qquad \Leftrightarrow \qquad U_1^T b_4 = b_{\lambda q}$$

Thus, variable $\overline{b_r}$ is calculated

$$\overline{b_r} = b_r - g_3^T D_2^{-1} b_3 - g_4^T D_1^{-1} b_4$$

Finally equation (11)

$$\overline{b_\pi} = b_\pi - \begin{pmatrix} F_x & 0 \end{pmatrix} \begin{pmatrix} G_x & 0 \\ 0 & G_x^T \end{pmatrix}^{-1} \begin{pmatrix} b_\lambda \\ b_x \end{pmatrix} = b_\pi - F_x G_x^{-1} b_\lambda$$

$$\overline{b}_\pi = b_\pi - \begin{pmatrix} F_v & F_a \end{pmatrix} \begin{pmatrix} 0 & B'^{-1} \\ B''^{-1} & 0 \end{pmatrix} \begin{pmatrix} b_{\lambda p} \\ b_{\lambda q} \end{pmatrix} = b_\pi - F_a B''^{-1} b_{\lambda p} - F_v B'^{-1} b_{\lambda q}$$

$$\overline{b}_\pi = b_\pi - F_a \left(U_2^T D_2 U_2\right)^{-1} b_{\lambda p} - F_v \left(U_1^T D_1 U_1\right)^{-1} b_{\lambda q}$$

$$\overline{b}_r = b_r - G_{ra} U_2^{-1} D_2^{-1} U_2^{-T} b_{\lambda p} - G_{rv} U_1^{-1} D_1^{-1} U_1^{-T} b_{\lambda q}$$

$$\overline{b}_\pi = b_\pi - F_a U_2^{-1} D_2^{-1} U_2^{-T} b_{\lambda p} - F_v U_1^{-1} D_1^{-1} U_1^{-T} b_{\lambda q}$$

Following two matrices F_3 and F_4 are obtained via column by column forward substitution

$$F_a U_2^{-1} = F_3^T \qquad \Leftrightarrow \qquad U_2^T F_3 = F_a^T$$

$$F_v U_1^{-1} = F_4^T \qquad \Leftrightarrow \qquad U_1^T F_4 = F_v^T$$

Finally

$$\overline{b}_\pi = b_\pi - F_3^T D_2^{-1} b_3 - F_4^T D_1^{-1} b_4$$

Appendix IV: Implementation of the Active Set Method

Solution if feasible if there is no new violated constraints. Now our concern is how to facilitate computation if new violated constraint is found? When new violated constraint is found, we have to build additional columns of matrix F as well as to update matrices where matrix F appears (i.e., $\overline{F}$).

Now we will show how this can be done with not too much extra work:

First we have to calculate $F_{x\,new}$ and $F_{u\,new}$ (row vector) for each new violated constraint.

$$F_{x\,new} = \frac{\partial f_{new}(x,u)}{\partial x} \qquad \text{dimension } (n_v \times 2n)$$

$$F_{u\,new} = \frac{\partial f_{new}(x,u)}{\partial u} \qquad \text{dimension } (n_v \times n_u)$$

where n_v is the number of new violated constraints

Matrix F appears in the following terms:

$$\overline{F} = U^{-T} F^{T} \quad \text{or} \quad \overline{F}^{T} = F U^{-1}$$

New $\overline{F}$ matrix will be denoted by $\overline{F}_{new}$ and will include new violated constraints

$$\overline{F}_{new} = U^{-T} F_{new}^{T}$$

and

$$F_{new}^{T} = \begin{pmatrix} \overline{F}_{u}^{T} & \overline{F}_{u\,new}^{T} \\ 0 & 0 \end{pmatrix}$$

Next term to be calculated is:

$$\overline{F}_{u\,new} = F_{u\,new} - \begin{pmatrix} F_{x\,new} & 0 \end{pmatrix} \begin{pmatrix} W_{xx} & \overline{G}_{x}^{T} \\ \overline{G}_{x} & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{xu} \\ \overline{G}_{u} \end{pmatrix}$$

remember from Appendix II

$$\overline{F}_{u\,new} = F_{u\,new} - \begin{pmatrix} F_{x\,new} & 0 \end{pmatrix} U^{-1} D^{-1} U^{-T} K$$

$$F = \begin{pmatrix} F_{x\,new} & 0 \end{pmatrix} \qquad \text{of dimension } (n_v \times 4n)$$

calculating

$$R^T = F U^{-1}$$

and if we recall

$$M = U^{-T} K$$

equation

$$\overline{F}_{u\,new} = F_{u\,new} - F U^{-1} D^{-1} U^{-T} K$$

$$\overline{F}_{u\,new} = F_{u\,new} - R^T D^{-1} M$$

Finally,

$$\overline{F}_{new} = U^{-T} \begin{pmatrix} \overline{F}_u^T & \overline{F}_{u\,new}^T \\ 0 & 0 \end{pmatrix}$$

$\overline{F}_{new}$ can be written in the following block matrix form

$$\overline{F}_{new} = \begin{pmatrix} \overline{F} & \overline{F}' \end{pmatrix}$$

we just need to perform forward substitution (fast forward substitution) on additional columns corresponding to new violated constraints and calculate $\overline{F}'$

$$U^T \overline{F}' = \begin{pmatrix} \overline{F}_{u\,new}^T \\ 0 \end{pmatrix} \qquad \overline{F}' \text{ has dimension } (4n \times n_v)$$

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