Quasi-Static Griffith Fracture Evolution with Boundary Loads

by

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Abstract

In this dissertation, we study the well-posedness of a variational formulation for modeling quasi-static evolution of cracks in elastic materials under boundary loads. Quasi-static evolution of fracture for displacement loads, i.e., Dirichlet boundary conditions, has been studied extensively in the past couple of decades, using models based on global and local minimization. However, boundary loads, i.e., Neumann boundary conditions, had been seen as problematic with the usual variational formulation, due to a straightforward non-existence argument.

Recently, a variational formulation, namely *dual minimization*, was proposed as a method for finding solutions for fracture problem with boundary loads. Adopting this method, we study existence of quasi-static fracture evolutions under time-varying boundary loads.

Global minimizers of the quasi-static Dirichlet problem have always balanced the sum of stored elastic plus crack dissipated surface energies. Nonetheless, even though our formulation for the quasi-static Neumann problem is based on global minimization, we show that evolutions here do not necessarily satisfy this energy balance, and describe how there can be decreases in the energy. Note that decrease in the sum of stored and dissipated energies in time might be expected since the effect of kinetic energy caused by the jumps in the evolution of cracks is not considered in the quasi-static energy equation. We also give estimates on how big energy drops can be.

Also, in a separate problem, we prove that a regularized Ambrosio-Tortorelli type energy functional that models fracture in layered structures with interfaces $\Gamma-$converges to a sharp interface energy, where the surface energy of a crack at the interface is proportional to an *effective* toughness, that in a sense averages the toughness of the interface and the bulk materials, whereas away from the interface, it is proportional to the toughness of the bulk.

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Chapter 1

Introduction

This dissertation is focused on a variational model of fracture evolution in materials, in the presence of boundary loads. The past twenty five years have seen an extensive development of variational models and analysis of fracture, see [9] for a recent overview. Rigorous analysis of these models has introduced new mathematical challenges and led to better understanding of fracture phenomena in materials. Variational models have been pretty successful in predicting paths of fracture in materials and have had an impact on the study of mechanics of fracture in the engineering community – primarily through their connection to phase-field models. In the current study, we propose a variational formulation to find evolution of cracks in materials under boundary loads which had been seen as inconsistent with variational models in the literature. This is the first analysis of quasi-static fracture with boundary loads.

In this chapter, we will set the stage for the results presented in this dissertation. First, in section 1.1, we will provide some background on variational models of fracture and that gives us an opportunity to introduce some of the notation. We will also survey some of the useful references to review some of the developments and challenges in the field. Then in section 1.2, we will give a brief description of the main results and outline the structure of the material presented in the ensuing chapters.

A portion of the background material below is adapted from Casey Richardson's Ph.D. dissertation [17].

1.1 Background

In the context of modeling fracture, the primary goal is to formulate a well-posed model that can *predict crack evolution*, which includes determining both when pre-existing cracks will run and the paths that such cracks take through the material. Although models with such predictive capability are quite recent, the central idea in the theory of brittle fracture was proposed by Griffith in 1920 [12]. He formulated the following criterion for two dimensional crack propagation: a pre-existing crack can run only when the elastic energy that is released by cracking, per unit length of crack, exceeds the toughness of the material. Precisely, he defined the energy release rate (in two dimensions) as:

$$G := \frac{d\mathcal{W}}{dl}$$

where $\mathcal{W}$ is the bulk elastic energy stored in the material and l is the length of the crack. Then, the *Griffith criterion* states that the crack will not run if G is less than the toughness of the material, and can run if G equals the toughness. Implicit in this criterion is a view of fracture as a balance between the energy that is required to create new crack - which Griffith implied is proportional to the length of the crack - and the elastic energy that is released when the material cracks. However, notice that the Griffith criterion only provides a rule for determining when cracks grow; the path of the crack must be known a priori.

This assumption and the restriction to two dimensions were eliminated using methods in the Calculus of Variations. This progress happened in the time of development of Free Discontinuity Problems and the theory of Special Functions of Bounded Variation (SBV), see section 1.1.1 for definitions. Using u to map Ω to its deformed configuration, with the discontinuity set of u identifying the crack set, Ambrosio and Braides [1] proposed to model static fracture by minimizing

$$u \mapsto \mathcal{W}(u) + \mathcal{H}^{N-1}(S_u) \tag{1.1}$$

over $u \in SBV(\Omega)$, $u = g$ (given) on $\partial\Omega$; here $\mathcal{H}^{N-1}$ denote the $(N-1)-$dimensional Hausdorff measure, S_u denotes the (approximate) discontinuity set of u, and

$$\mathcal{W}(u) = \int_\Omega W(\nabla u)\, dx,$$

where W is the elastic energy density and ∇u is the deformation gradient. Notice that the term $\mathcal{H}^{N-1}(S_u)$ models the surface energy of the crack. For an admissible function, one can generally create a competitor with lower elastic energy by using more discontinuities, but at the cost of the surface energy of the additional discontinuity set. Thus (1.1) captures the competition between crack "length" and elastic energy release that is the core feature of the Griffith criterion, and the location of the crack is determined by this energy minimization. The existence of a minimizer for (1.1), given typical assumptions on W, follows from the compactness of the space SBV, see Remark 1.1.2 below.

For crack evolution, in the realm of *quasi-statics* – that is assuming that the rate of change in the problem parameters (Dirichlet boundary conditions, boundary loads, body forces) is small compared to the time it takes the body to reach elastic equilibrium – Francfort and Marigo [11] proposed the following model: first discretize time, at each time-step solve an appropriate static problem (where, since cracks cannot heal, (1.1) is slightly modified to penalize only new discontinuities), and then find the time-continuous evolution by taking the limit as the size of the time-steps goes to zero. The main issue is to show that this limit satisfies the properties of a quasi-static evolution: loosely stated, at each time the crack set and deformation satisfy a minimality property and that the crack evolution satisfies an energy balance, which relates the stored elastic energy plus the dissipated energy to the work done by loading. These properties were proven for the time-continuous limit, first in two dimensions with certain geometric constraints on the crack sets by Dal Maso and Toader [7], and then in the general SBV setting by Francfort and Larsen [10] and for quasi-convex W with a standard $p-$growth condition by Dal Maso, Francfort and Toader [4]. This result was extended to the case of hyperelastic materials in the finite elasticity framework with the non-interpenetration condition by Dal Maso and Lazzaroni [6].

The general postulate for the evolution of crack in the above quasi-static models is *global minimization*, that is at each time, the material wants to minimize the sum of its bulk and surface energies among all competitors. Therefore, for a pair of displacement-crack (u, Γ), an *irreversible quasi-static evolution* of minimum energy configurations is a function $t \mapsto (u(t), \Gamma(t))$ which satisfies the following conditions:

(a) irreversibility: for all $s > t$, $\Gamma(s) \supset \Gamma(t)$;

(b) global stability: for all t the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t, i.e.,

$$\mathcal{W}(u(t)) + \mathcal{H}^{N-1}(\Gamma(t)) \leq \mathcal{W}(v) + \mathcal{H}^{N-1}(\Gamma')$$

for all admissible pairs (v, Γ'), where $\Gamma' \supset \Gamma(t)$, $v = g(t)$ on $\partial\Omega$, and g is the prescribed boundary condition;

(c) energy balance: the increment in stored energy plus the energy spent in crack increase equals the work of external forces, that is for all t_1, t_2 with $t_1 < t_2$,

$$\mathcal{W}(u(t_2))+\mathcal{H}^{N-1}(\Gamma(t_2)) = \mathcal{W}(u(t_1))+\mathcal{H}^{N-1}(\Gamma(t_1))+\int_{t_1}^{t_2}\int_{\partial\Omega} \dot{g}(s)\partial_\nu u(s)\, d\mathcal{H}^{N-1}ds.$$

However, there are some inconsistencies within this formulation and the main culprit is the jumps in time in the cracks. First off, the jumps in the cracks make the quasi-static assumption dubious since it is no longer valid to assume that the material's response time to reach equilibrium is short. Moreover, the energy balance (c) does not account for the effect of kinetic energy caused by these jumps in $t \mapsto \Gamma(t)$ and subsequently in $t \mapsto u(t)$.

Therefore, to mitigate the effect of possibly unnecessary jumps in the evolution of cracks, models based on *local minimization* were studied. Inspired by De Giorgi's minimizing movements approach mentioned in [1], Dal Maso and Toader in [8] adopted a local minimization method (with certain geometric constraints on the cracks) to construct discrete-time evolutions. The continuous-time limit achieved from these approximations satisfies some minimality different from that of evolutions found by global minimization. Moreover, the associated energy is of bounded variation, does not increase, and can have negative jumps in time. Hence, this model is more realistic in that if the cracks have jumps in time, then energy decreases (as a compensation for not including the kinetic energy in the equation).

Later in 2009, Larsen in [14] (see also [15]) proposed a new definition of stability for the evolutions, namely $\varepsilon -$ *stability*, that is somewhere in between local and global minimization and admits evolutions that are stable in approximation, under this new definition, and in fact implies local minimality. Here too, the energy does not increase in time and may present negative jumps, unless $t \mapsto \Gamma(t)$ is continuous.

To summarize, the more realistic models inspired by local minimization techniques propose that we substitute the statements (b) and (c) above with the following:

(b′) local stability: for all t the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t among competitors *"close enough"* to $(u(t), \Gamma(t))$, that is

$$\mathcal{W}(u(t)) + \mathcal{H}^{N-1}(\Gamma(t)) \leq \mathcal{W}(v) + \mathcal{H}^{N-1}(\Gamma')$$

for pairs (v, Γ') *"close enough"* to $(u(t), \Gamma(t))$, with $\Gamma' \supset \Gamma(t)$, $v = g(t)$ on $\partial\Omega$;

(c′) energy does not increase: the increment in stored energy plus the energy spent in crack increase *is less than or equal to* the work of external forces, that is for all t_1, t_2 with $t_1 < t_2$,

$$\mathcal{W}(u(t_2)) + \mathcal{H}^{N-1}(\Gamma(t_2)) \leq \mathcal{W}(u(t_1)) + \mathcal{H}^{N-1}(\Gamma(t_1)) + \int_{t_1}^{t_2} \int_{\partial\Omega} \dot{g}(s) \partial_\nu u(s)\, d\mathcal{H}^{N-1} ds.$$

Remark 1.1.1. As we elaborate in section 3.3, even though our quasi-static variational formulation for the boundary load problem is based on global minimization, we will show that there are evolutions that can decrease the energy.

1.1.1 Mathematical Preliminaries

Central to our formulation and all of the introduction above is the space of *Special Functions of Bounded Variation*, SBV for short. Briefly stated, $SBV(\Omega)$ is the space of functions $u \in BV(\Omega)$ such that the singular part of their distributional derivative, Du, is concentrated on the set where u is (approximately) discontinuous, a countably $(N-1)-$rectifiable subset of $\mathbb{R}^N$. So, $u \in SBV(\Omega)$ if and only if $u \in BV(\Omega)$ and the distributional derivative of u has the following decomposition

$$Du = \nabla u \; d\mathcal{L}^N + [u] \; \nu \; \mathcal{H}^{N-1} \lfloor S_u,$$

where $\nabla u \in L^1(\Omega)$ is the density of the absolutely continuous part, $[u]$ stands for the jump of u along S_u and ν denotes the approximate unit normal to S_u.

Allowing cracks along the boundary of Ω generally involves some notational issues. First introduced in [16], one simple way of incorporating the idea of growth of crack on the boundary of a bounded regular domain Ω with the usual SBV is by defining the space

$$SBV(\overline{\Omega}) := \{u : \overline{\Omega} \to \mathbb{R} \;:\; u|_\Omega \in SBV(\Omega),\ \text{and}\ u|_{\partial\Omega} \in L^1(\partial\Omega; \mathcal{H}^{N-1}\lfloor\partial\Omega)\},$$

with the approximate discontinuity set defined by

$$S_u := S_{u|_\Omega} \cup \{x \in \partial\Omega \;:\; T(u|_\Omega)(x) \neq u(x)\},$$

where T denotes the usual trace operator. This simply means that if the trace of a function in $SBV(\overline{\Omega})$ restricted to Ω does not match its specified boundary value at a point on the boundary, then we have a discontinuity (crack) at that point. Note that the pointwise values above are considered in the sense of precise representatives. Moreover, following the same idea we define a subspace

$$SBV_2(\overline{\Omega}) := \{u \in SBV(\overline{\Omega}) \;:\; \nabla u|_\Omega \in L^2(\Omega)\}.$$

Next, we define a notion of convergence in the space $SBV(\Omega)$ that we will be using throughout the sequel:

Definition 1.1.1 ($SBV-$convergence)**.** We say that a sequence of functions $u_n \in SBV(\Omega)$ converges in the sense of SBV to $u \in SBV(\Omega)$, denoted by $u_n \overset{SBV}{\rightharpoonup} u$, if

$$\begin{aligned}
\nabla u_n &\rightharpoonup \nabla u \text{ in } L^1(\Omega) \\
[u_n]\nu_n \mathcal{H}^{N-1}\lfloor S_{u_n} &\overset{*}{\rightharpoonup} [u]\nu\mathcal{H}^{N-1}\lfloor S_u \text{ as measures,} \\
u_n &\to u \text{ in } L^1(\Omega), \\
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\Omega).
\end{aligned}$$

The following compactness result is due to Ambrosio, which we state it in a way that is easier to apply with our formulation. For the proof of the following and more detail on the SBV theory see [2].

Remark 1.1.2 ($SBV-$compactness). If for a sequence of functions $u_n \in SBV(\Omega)$, the sequences

$$\{\|\nabla u_n\|_{L^2(\Omega)}\}, \quad \{\mathcal{H}^{N-1}(S_{u_n})\}, \quad \{\|u_n\|_{L^\infty(\Omega)}\}$$

are bounded, then there exists a function $u \in SBV(\Omega)$ such that, up to passing to a subsequence, $u_n \overset{SBV}{\rightharpoonup} u$.

1.2 Overview of Dissertation

The main result of this dissertation is the study of well-posedness a variational model of fracture for quasi-static evolution of cracks in materials under boundary loads, which is the subject of Chapter 3. As a prelude to this chapter, we introduce our main tool in the study of existence of solution to a fracture problem with boundary loads in Chapter 2.

In Chapter 2, we first explain why the most natural variational approach to seek existence of solutions to fracture with boundary loads fails. Then, in the next section, we introduce a new variational formulation, *alternate minimization*, devised in [16], that proposes a way of finding solutions to the fracture with boundary loads problem. We define a notion of $failure$ for the material under load and show that if the material does not fail, there exist solutions. We also discuss limitations and small extensions of this method.

In Chapter 3, we first devote a small section on explaining what we mean by *quasi-static* evolutions and a bit discuss the advantages and limitations of this assumption. Then, we move on to seek existence of quasi-static evolutions of cracks with time-varying boundary loads. Employing the alternate minimization approach, we first construct discrete-time solutions. Next, we pass to the limit as the size of the time-step converges zero and finally, extend the limit to the whole time interval to get a continuous-time evolution. In Theorem 3.2.1, we state our main result that the acquired continuous-time evolution satisfies the desired minimality properties of a solution. Finally, we discuss balance of energy for the quasi-static evolution. We show that the associated energy does not increase in time and discuss how there can be decreases in the energy. Moreover, we give estimates on how big energy drops can be.

Finally, in Chapter 4, for a separate problem, we prove a $\Gamma-$convergence result for a phase-field model of fracture for layered structures with interfaces. We show that whenever the fracture toughness of the interface material is less than or equal to the fracture toughness of the bulk material, the surface energy of a crack at the interface is proportional to an *effective* toughness that in a sense averages the two toughnesses; whereas away from the interface it is proportional to the toughness of the bulk material. In our proof, we make the assumption that the interface is made up of a finite union of closed C^1 curves.

Chapter 2

Variational Fracture with Boundary Loads: Alternate Minimization

In this chapter, we introduce a method of finding solutions to a variational model of fracture with boundary loads, namely the *alternate minimization* method. This chapter mainly reproduces the material of [16].

2.1 Introduction

Variational fracture had been seen as incompatible with boundary and body loads due to a straightforward non-existence argument, see [9]. There is an inherent difference in the variational formulation, based on *global minimization*, when there is a specified displacement, i.e., Dirichlet boundary conditions, and when there is a boundary load, i.e., Neumann boundary conditions, or a body load for that matter. For a linear elastic solid, the variational formulation of equilibrium Griffith fracture (based on global minimization) for a given Dirichlet data g and a pre-existing crack set K is to minimize

$$E_D[K](w) := \frac{1}{2}\int_\Omega |\nabla w|^2 + \mathcal{H}^{N-1}(S_w \backslash K) \tag{2.1}$$

over $w \in SBV_2(\overline{\Omega})$ with $w = g$ on $\partial\Omega$. We use a subscript D for the energy E_D to emphasize that only Dirichlet boundary values will be imposed when minimizing this functional. A minimizer w of (2.1), weakly solves the following boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega\backslash(K \cup S_w), \\ w = g & \text{on } \partial\Omega, \\ \partial_\nu w = 0 & \text{on } K \cup S_w, \end{cases} \tag{2.2}$$

and satisfies the Griffith stability criterion, that is for any $\tilde{w} \in SBV_2(\overline{\Omega})$ with $\tilde{w} = g$ on $\partial\Omega$ we have that

$$\frac{1}{2}\int_\Omega |\nabla w|^2 - \frac{1}{2}\int_\Omega |\nabla \tilde{w}|^2 \leq \mathcal{H}^{N-1}\big(S_{\tilde{w}}\backslash(K \cup S_w)\big). \tag{2.3}$$

Now, we decompose the boundary of the domain, $\partial\Omega$, into the disjoint union of measurable subsets $\partial_D\Omega$ and $\partial_N\Omega$, where on $\partial_D\Omega$ we specify a Dirichlet data g and on $\partial_N\Omega$ we specify a Neumann data (boundary load) f. We seek a variational formulation for Griffith fracture with the given boundary data, again based on global minimization. A straightforward answer readily found in the literature, see for example [4], is to minimize the total energy functional

$$w \mapsto \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_{\partial_N\Omega} fw + \mathcal{H}^{N-1}(S_w\backslash K) \tag{2.4}$$

over $w \in SBV_2(\overline{\Omega})$ with $w = g$ on $\partial_D\Omega$. Note that a minimizer of the above (if it exists) weakly satisfies the following boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega\backslash(K \cup S_w), \\ w = g & \text{on } \partial_D\Omega, \\ \partial_\nu w = f & \text{on } \partial_N\Omega, \\ \partial_\nu w = 0 & \text{on } K \cup S_w, \end{cases} \tag{2.5}$$

together with the global stability criterion that for any $\tilde{w} \in SBV_2(\overline{\Omega})$ with $\tilde{w} = g$ on $\partial_D\Omega$ we have that

$$\left(\frac{1}{2}\int_\Omega |\nabla w|^2 - \int_{\partial_N\Omega} fw\right) - \left(\frac{1}{2}\int_\Omega |\nabla \tilde{w}|^2 - \int_{\partial_N\Omega} f\tilde{w}\right) \leq \mathcal{H}^{N-1}\big(S_{\tilde{w}}\backslash(K \cup S_w)\big). \tag{2.6}$$

The problem is that (2.4) cannot be minimized over $SBV_2(\overline{\Omega})$ since for instance, for $f \equiv 1$, we can easily come up with a sequence of functions $w_n \in SBV_2(\overline{\Omega})$ such that $\mathcal{H}^{N-1}(S_{w_n} \cap \partial_N\Omega) > \delta > 0$ for some $\delta > 0$, and $\|w_n\|_{L^\infty(\partial_N\Omega))} \nearrow +\infty$, and therefore, the infimum of the functional (2.4) is $-\infty$ over $SBV_2(\overline{\Omega})$. In other words, the crack set will always disconnect a piece of $\partial_N\Omega$ from the body and send it to infinity thus making the energy arbitrarily low.

Remark 2.1.1. Note that removing the jump set of w from $\partial_N\Omega$ and substituting $\int_{\partial_N\Omega\setminus S_w} fw$ for $\int_{\partial_N\Omega} fw$ in (2.4) will not solve the problem although it rules out the example we gave above. Because the jump set S_w of an admissible function w can branch near $\partial_N\Omega$ and disconnect a piece of the domain enclosed by S_w and $\partial_N\Omega$ from the domain and send it to infinity while $\mathcal{H}^{N-1}(S_w \cap \partial_N\Omega) = 0$ and S_w has no contribution in the integral $\int_{\partial_N\Omega\setminus S_w} fw$.

Looking back at what we are after, we want a variational formulation of Griffith fracture that allows for mixed boundary conditions. So, if we have a function w such that it minimizes E_D subject to $w = g$ on $\partial_D\Omega$ and $w = h$ on $\partial_N\Omega$, for some function h, and happens to satisfy $\partial_\nu w = f$ on $\partial_N\Omega$, then w satisfies the mixed boundary conditions and the crack $K \cup S_w$ satisfies the Griffith stability criterion of (2.6) for all $\tilde{w} \in SBV_2(\overline{\Omega})$ with $\tilde{w} = g$ on $\partial_D\Omega$ and $\tilde{w} = h$ on $\partial_N\Omega$.

In the next section, we introduce a method of finding solutions like the above, namely, the *alternate minimization* method, whereby we solve two variational problems simultaneously. The idea is to keep the Neumann boundary term $\int_{\partial_N\Omega} fw$ and the variation of crack set S_w apart so that they cannot collaborate to make the energy arbitrarily low, as in (2.4). This will lead to a notion of *failure* (defined below) which can potentially predict when a crack can break off a piece of the boundary under loading from the body.

2.2 Alternate Minimization

In this section, we introduce the *alternate minimization* method. In this method, we consider two variational problems in one of which we solve the Neumann problem and in the other we solve the Dirichlet problem for Griffith fracture. When solving the Neumann problem we fix the discontinuity set, and when solving the Dirichlet problem we fix the boundary values, so that we prevent the collaboration of Neumann boundary term and variation in the crack set causing the infimum to be minus infinity. So, we introduce a variational formulation and discuss existence of solutions.

To make the exposition easier, we will consider the case that the Dirichlet boundary condition g is identically zero.

To begin the existence argument, with a given (possibly empty) pre-existing crack set K_0, we first choose a minimizer w_1 of

$$E_N(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_{\partial_N\Omega} fw$$

over $w \in SBV_2(\overline{\Omega})$ with $w = 0$ on $\partial_D\Omega$ and $S_w \subset K_0$. This will be used to supply Dirichlet data on $\partial_N\Omega$ when solving the Dirichlet problem. Note that we use a subscript N for the energy E_N to emphasize that only Neumann problem is solved without allowing variations of discontinuity set. Next, choose a minimizer v_1 of

$$E_D[K_0](w) = \frac{1}{2}\int_\Omega |\nabla w|^2 + \mathcal{H}^{N-1}(S_w \backslash K_0)$$

over $w \in SBV_2(\overline{\Omega})$ with $w = 0$ on $\partial_D\Omega$ and $w = w_1$ on $\partial_N\Omega$.

We repeat this process recursively to find sequences $\{w_j\}$, $\{v_j\}$ and $\{K_j\}$ such that for $j \geq 1$,

$$K_j := K_{j-1} \cup S_{v_j},$$

w_j minimizes

$$E_N(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_{\partial_N\Omega} fw$$

over $w \in SBV_2(\overline{\Omega})$ with $w = 0$ on $\partial_D\Omega$ and $S_w \subset K_{j-1}$, and v_j minimizes

$$E_D[K_{j-1}](w) = \frac{1}{2}\int_\Omega |\nabla w|^2 + \mathcal{H}^{N-1}(S_w \backslash K_{j-1})$$

over $w \in SBV_2(\overline{\Omega})$ with $w = 0$ on $\partial_D\Omega$ and $w = w_j$ on $\partial_N\Omega$. Note that for all $j \in \mathbb{N}$,

$$K_j \supset K_{j-1}.$$

We also let

$$K_\infty := \cup_{j\in\mathbb{N}} K_j. \tag{2.7}$$

The hope is that the sequence $\{w_j\}$ weakly converges to a w_∞ that minimizes both E_N and $E_D[K_\infty]$ over the appropriate classes of competitors and satisfies the desired boundary conditions. In the following remark, we discuss what could go wrong in the above minimization process.

Remark 2.2.1. Note that there is a potential issue in minimizing the Neumann energy E_N above, since it can happen that at some stage j, the crack K_{j-1} disconnects a piece of $\partial_N\Omega$ from the domain and similar to the non-existence argument earlier the infimum of E_N is $-\infty$, and there is no minimizer. Moreover, this failure can occur in the limiting case, as j tends to infinity. If the sequence $\{\|w_j\|_{L^\infty(\partial_N\Omega)}\}$ is not bounded, then as $j \to \infty$, the displacement $w_j|_{\partial_N\Omega}$ becomes arbitrarily large. A similar failure can happen if K_∞ (the limiting crack set - defined in (2.7)) breaks off a piece of $\partial_N\Omega$.

In fact, this lack of solution is due to the Neumann problem and not caused by the variational formulation. There is necessarily a possibility of material failure when there are loads and fracture. Therefore, we must allow for the possibility that the material fails under the boundary load. This is all encapsulated in the definition below.

Remark 2.2.2. At this point, a natural question or even objection may be, how is this variational formulation different from minimizing (2.4) that we discussed in the previous section in terms of existence of solutions? The key difference is that in the latter, non-existence is guaranteed due to the interaction between the Neumann boundary term and variation of crack set. Here, lack of solution is only a possibility and a result of material failure under the loading. So, this approach can potentially predict whether a body Ω with an initial crack K_0 under a boundary load f breaks, or not.

Definition 2.2.1 (Non-failure)**.** We say the material *does not fail* under the boundary load f if the following hold:

(i) Each w_j exists, and $\sup_{j\in\mathbb{N}} \|w_j\|_\infty < +\infty$,

(ii) $\mathrm{Cap}(K_\infty \cap \mathcal{N}_\varepsilon(\partial_N\Omega)) \to 0$ as $\varepsilon \to 0$,

where $\|w_j\|_\infty := \max\{\|w_j\|_{L^\infty(\Omega)}, \|w_j\|_{L^\infty(\partial\Omega)}\}$, and $\mathcal{N}_\varepsilon(\partial_N\Omega) := \{x \in \overline{\Omega} : \mathrm{dist}(x, \partial_N\Omega) < \varepsilon\}$ is the $\varepsilon-$neighborhood of the Neumann part of the boundary.

Remark 2.2.3. Note that condition (ii) of the above definition allows for some interaction between the crack set K_∞ and $\partial_N\Omega$. For example, in the two dimensional setting, K_∞ is allowed to intersect with $\partial_N\Omega$ normally at finitely many points (without branching, of course).

Now, we are in a position to state our static existence result in view of Definition 2.2.1:

Theorem 2.2.1. *If the material does not fail under the boundary load $f \in L^\infty(\partial_N\Omega)$, then $\mathcal{H}^{N-1}(K_\infty) < +\infty$, and there exists $w_\infty \in SBV_2(\overline{\Omega})$ such that, up to a subsequence,*

$$w_j \rightharpoonup w_\infty \text{ in } SBV(\Omega)$$

as $j \to \infty$, w_∞ minimizes $E_D[K_\infty]$ over $\{w \in SBV_2(\overline{\Omega}) \ : \ w = w_\infty \text{ on } \partial\Omega\}$, and it minimizes E_N over $\{w \in SBV_2(\overline{\Omega}) \ : \ S_w \subset K_\infty, w = 0 \text{ on } \partial_D\Omega\}$. Moreover,

$$Tw_j \to Tw_\infty \text{ in } L^2(\partial_N\Omega), \tag{2.8}$$

and

$$\frac{1}{2}\int_\Omega |\nabla w_j|^2 \nearrow \frac{1}{2}\int_\Omega |\nabla w_\infty|^2 \tag{2.9}$$

as $j \to \infty$.

The proof will be done in multiple lemmas as follows. In the first lemma, we present two useful properties of the sequences $\{w_j\}$ and $\{K_j\}$, in the second lemma we show existence of w_∞ and it minimizing $E_D[K_\infty]$, and finally in the third lemma, we prove its minimality for E_N, convergence of traces and Dirichlet energies.

From here on, we will assume that Definition 2.2.1 holds.

Lemma 2.2.1. *For all $j \geq 2$,*

1.

$$\frac{1}{2}\int_\Omega |\nabla w_j|^2 - \int_{\partial_N\Omega} f w_j + \mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \leq \frac{1}{2}\int_\Omega |\nabla w_{j-1}|^2 - \int_{\partial_N\Omega} f w_{j-1}, \tag{2.10}$$

2. monotonicity of the Dirichlet energy of the sequence of minimizers $\{w_j\}$:

$$\frac{1}{2}\int_\Omega |\nabla w_{j-1}|^2 \leq \frac{1}{2}\int_\Omega |\nabla w_j|^2. \tag{2.11}$$

Proof. Fix $j \geq 2$. Note that since $S_{v_{j-1}} \subset K_{j-1}$ and $v_{j-1} = 0$ on $\partial_D\Omega$, from minimality of w_j we have that

$$\frac{1}{2}\int_\Omega |\nabla w_j|^2 - \int_{\partial_N\Omega} f w_j \leq \frac{1}{2}\int_\Omega |\nabla v_{j-1}|^2 - \int_{\partial_N\Omega} f v_{j-1}. \tag{2.12}$$

Moreover, from minimality of v_{j-1}, since $w_{j-1} = v_{j-1}$ on $\partial_N\Omega$, we get that

$$\frac{1}{2}\int_\Omega |\nabla v_{j-1}|^2 + \mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \leq \frac{1}{2}\int_\Omega |\nabla w_{j-1}|^2, \tag{2.13}$$

or equivalently

$$\frac{1}{2}\int_\Omega |\nabla v_{j-1}|^2 - \int_{\partial_N\Omega} f v_{j-1} + \mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \leq \frac{1}{2}\int_\Omega |\nabla w_{j-1}|^2 - \int_{\partial_N\Omega} f w_{j-1}. \tag{2.14}$$

So, putting (2.12) and (2.14) together, we get (2.10).

Next, notice that w_j is an admissible variation for its minimality for E_N, and thus we have that

$$\int_\Omega \nabla w_j \cdot \nabla w_j = \int_{\partial_N\Omega} f w_j. \tag{2.15}$$

Therefore, (2.10) in view of the above yields

$$\frac{1}{2}\int_\Omega |\nabla w_{j-1}|^2 + \mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \leq \frac{1}{2}\int_\Omega |\nabla w_j|^2,$$

which implies (2.11) since $\mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \geq 0$.

□

Lemma 2.2.2. *$\mathcal{H}^{N-1}(K_\infty) < +\infty$, and there exists $w_\infty \in SBV_2(\overline{\Omega})$ such that, up to a subsequence,*

$$w_j \rightharpoonup w_\infty \text{ in } SBV(\Omega)$$

as $j \to \infty$, and w_∞ minimizes $E_D[K_\infty]$ over $\{w \in SBV_2(\overline{\Omega}) \ : \ w = w_\infty \text{ on } \partial\Omega\}$.

Proof. Note that from the condition (i) of Definition 2.2.1, $\{\|w_j\|_{L^\infty(\partial_N\Omega)}\}$ is bounded, and hence, (2.15) implies that the sequence of Dirichlet energies $\{\frac{1}{2}\|\nabla w_j\|_{L^2(\Omega)}\}$ is bounded. Moreover, by (2.11) it is monotonic, so it converges and let C_L be its limit. Consequently, the sequence of Neumann energies converges:

$$E_N(w_j) = \frac{1}{2}\int_\Omega |\nabla w_j|^2 - \int_{\partial_N\Omega} f w_j = -\frac{1}{2}\int_\Omega |\nabla w_j|^2 \to -C_L,$$

as $j \to \infty$.

Next, we look at $\mathcal{H}^{N-1}(K_j)$. Rewriting (2.10), we have that

$$\mathcal{H}^{N-1}(K_{j-1} \backslash K_{j-2}) + E_N(w_j) \leq E_N(w_{j-1}).$$

Summing up the above over $j = 2, 3, \dots, i$ gives

$$\mathcal{H}^{N-1}(K_{i-1} \backslash K_0) + E_N(w_i) \leq E_N(w_1).$$

Note that $\{E_N(w_i)\}$ is bounded, and therefore, in view of the above so is $\{\mathcal{H}^{N-1}(K_i)\}$. Then, letting $i \to \infty$, since $\{K_j\}$ is an increasing sequence, $\mathcal{H}^{N-1}(K_{i-1} \backslash K_0) \to \mathcal{H}^{N-1}(K_\infty \backslash K_0)$, and the above becomes

$$\mathcal{H}^{N-1}(K_\infty \backslash K_0) - C_L \leq E_N(w_1),$$

so that $\mathcal{H}^{N-1}(K_\infty) < +\infty$.

Now, since the sequences $\{\|w_j\|_\infty\}$, $\{\|\nabla w_j\|_{L^2(\Omega)}\}$ and $\{\mathcal{H}^{N-1}(K_j)\}$ are bounded, by SBV compactness (see Remark 1.1.2), there exists a $w_\infty \in SBV_2(\overline{\Omega})$ such that $\{w_j\}$, up to passing to a subsequence, $SBV-$converges to it, where

$$w_\infty|_{\partial_D\Omega} := 0 \text{ and } w_\infty|_{\partial_N\Omega} := T w_\infty.$$

Moreover, $S_{w_\infty} \subset K_\infty$, where the proof is similar to that of (3.23) in [10].

Next, we establish that the energies of v_j's also converge. Note that $\mathcal{H}^{N-1}(K_{j-1} \backslash K_{j-2}) \to 0$ as $j \to \infty$, so (2.13) implies that

$$\limsup_{j\to\infty} \frac{1}{2}\int_\Omega |\nabla v_j|^2 \leq C_L.$$

On the other hand, since $w_{j-1} = v_{j-1}$ on $\partial_N\Omega$, (2.12) implies that

$$\liminf_{j\to\infty} \frac{1}{2}\int_\Omega |\nabla v_j|^2 \geq C_L,$$

and therefore, in view of above we get that $\frac{1}{2}\int_\Omega |\nabla v_j|^2 \to C_L$.

Now we proceed to prove minimality of w_∞ for $E_D[K_\infty]$. Suppose there exists $\phi \in SBV_2(\overline{\Omega})$ with $\phi = 0$ on $\partial_D\Omega$ such that the variation $w_\infty + \phi$ has less energy, i.e.

$$\frac{1}{2}\int_\Omega |\nabla(w_\infty + \phi)|^2 + \mathcal{H}^{N-1}(S_\phi \backslash K_\infty) < \frac{1}{2}\int_\Omega |\nabla w_\infty|^2.$$

This is equivalent to the energy difference being negative:

$$\int_\Omega \nabla w_\infty \cdot \nabla \phi + \frac{1}{2}\int_\Omega |\nabla \phi|^2 + \mathcal{H}^{N-1}(S_\phi \backslash K_\infty) =: \eta < 0.$$

But, since $\nabla w_j \rightharpoonup \nabla w_\infty$ in $L^2(\Omega)$ and $\mathcal{H}^{N-1}(K_\infty \backslash K_j) \to 0$, the left hand side of the above is equal to the limit of

$$\int_\Omega \nabla w_j \cdot \nabla \phi + \frac{1}{2}\int_\Omega |\nabla \phi|^2 + \mathcal{H}^{N-1}(S_\phi \backslash K_j).$$

Hence, since $\frac{1}{2}\int_\Omega |\nabla w_j|^2 - \frac{1}{2}\int_\Omega |\nabla v_j|^2 \to 0$, for large enough j we have that

$$\frac{1}{2}\int_\Omega |\nabla(w_j + \phi)|^2 + \mathcal{H}^{N-1}(S_\phi \backslash K_j) < \frac{1}{2}\int_\Omega |\nabla w_j|^2 + \frac{\eta}{2} < \frac{1}{2}\int_\Omega |\nabla v_j|^2,$$

or equivalently

$$\frac{1}{2}\int_\Omega |\nabla(w_j + \phi)|^2 + \mathcal{H}^{N-1}(S_\phi \backslash K_{j-1}) < \frac{1}{2}\int_\Omega |\nabla v_j|^2 + \mathcal{H}^{N-1}(S_{v_j} \backslash K_{j-1}),$$

which contradicts the minimality of v_j for $E_D[K_{j-1}]$ and concludes the proof. □

Lemma 2.2.3. *w_∞ minimizes E_N over $\{w \in SBV_2(\overline{\Omega}) \ : \ S_w \subset K_\infty, w = 0$ on $\partial_D\Omega\}$. Moreover,*

$$Tw_j \to Tw_\infty \text{ in } L^2(\partial_N\Omega), \tag{2.16}$$

and

$$\frac{1}{2}\int_\Omega |\nabla w_j|^2 \nearrow \frac{1}{2}\int_\Omega |\nabla w_\infty|^2 \tag{2.17}$$

as $j \to \infty$.

Proof. To see why w_∞ minimizes E_N let $\psi \in SBV_2(\overline{\Omega}) \cap L^\infty(\Omega)$ with $\psi = 0$ on $\partial_D\Omega$ and $S_\psi \subset K_\infty$. In view of condition (ii) of Definition (2.2.1), we can choose a family of functions $\phi_\varepsilon \in C^1(\mathbb{R}^N)$ such that $\phi_\varepsilon = 0$ on a neighborhood of $K_\infty \cap \mathcal{N}_\varepsilon(\partial_N\Omega)$, $1 - \phi_\varepsilon \in C^1_c(\mathbb{R}^N)$, $0 \le \phi_\varepsilon \le 1$, and $\nabla\phi_\varepsilon \to 0$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Notice that this implies $\phi_\varepsilon \to 1$ in $L^2(\Omega)$ as well as $\mathcal{H}^{N-1}-$a.e. on $\partial_N\Omega$, since subsets of $\partial_N\Omega$ with positive $\mathcal{H}^{N-1}-$measure have positive capacity.

Then $\phi_\varepsilon\psi \in SBV_2(\overline{\Omega})$ with $S_{\phi_\varepsilon\psi} \subset K_\infty$. Note also that $\phi_\varepsilon\psi \in H^1(\mathcal{N}_\varepsilon(\partial_N\Omega)^\circ)$, where the superscript $(.)^\circ$ denotes the interior. So we can choose $\psi_\varepsilon \in H^1(\Omega)$ such that $\psi_\varepsilon = \phi_\varepsilon\psi$ on $\partial\Omega$.

Next, from the minimality of w_j for E_N, the fact that ψ_ε is an admissible variation of w_j for E_N, and the convergence of w_j to w_∞, we have

$$\int_{\partial_N\Omega} f\psi_\varepsilon = \lim_{j\to\infty}\int_\Omega \nabla w_j \cdot \nabla\psi_\varepsilon = \int_\Omega \nabla w_\infty \cdot \nabla\psi_\varepsilon.$$

But, since $\psi_\varepsilon - \phi_\varepsilon\psi \in SBV_2(\overline{\Omega})$ with $\psi_\varepsilon - \phi_\varepsilon\psi = 0$ on $\partial\Omega$ and $S_{\psi_\varepsilon - \phi_\varepsilon\psi} \subset K_\infty$, it is an admissible variation for minimality of w_∞ for E_D, so,

$$\int_\Omega \nabla w_\infty \cdot \nabla(\psi_\varepsilon - \phi_\varepsilon\psi) = 0,$$

which in view of the above and the definition of ψ_ε gives

$$\int_\Omega \nabla w_\infty \cdot \nabla(\phi_\varepsilon\psi) = \int_\Omega \nabla w_\infty \cdot \nabla\psi_\varepsilon = \int_{\partial_N\Omega} f\psi_\varepsilon = \int_{\partial_N\Omega} f\phi_\varepsilon\psi.$$

Taking the limit as $\varepsilon \to 0$ in the above, since $\nabla(\phi_\varepsilon\psi) = \psi\nabla\phi_\varepsilon + \phi_\varepsilon\nabla\psi \rightharpoonup \nabla\psi$ in $L^2(\Omega)$, and $\phi_\varepsilon\psi \to \psi$ in $L^1(\partial_N\Omega)$ (since ϕ_ε converges to 1 $\mathcal{H}^{N-1}-$a.e. on $\partial_N\Omega$ and applying the dominated convergence theorem), proves minimality of w_∞ for E_N with the assumption that $\psi \in L^\infty(\Omega)$. The general result follows by approximating in L^∞.

To show the convergence of traces, we note that for all $\varepsilon > 0$,

$$\phi_\varepsilon w_j \to \phi_\varepsilon w_\infty \text{ in } L^2(\Omega), \text{ and } \nabla(\phi_\varepsilon w_j) \rightharpoonup \nabla(\phi_\varepsilon w_\infty) \text{ in } L^2(\Omega), \tag{2.18}$$

as $j \to \infty$, where the first convergence follows from $0 \le \phi_\varepsilon \le 1$ and $w_j \to w_\infty$ in $L^2(\Omega)$, and the second convergence follows from $0 \le \phi_\varepsilon \le 1$, $\nabla w_j \rightharpoonup \nabla w_\infty$ in $L^2(\Omega)$, together with $\nabla\phi_\varepsilon \in L^2(\Omega)$ and $w_j \overset{*}{\rightharpoonup} w_\infty$ in $L^\infty(\Omega)$, which all come from the definition of ϕ_ε and $SBV-$convergence of w_j to w_∞. Then, since $S_{w_j} \subset K_j \subset K_\infty$, $\phi_\varepsilon w_j$ belong to $H^1(\mathcal{N}_\varepsilon(\partial_N\Omega)^\circ)$, and by (2.18) so does $\phi_\varepsilon w_\infty$, hence, (2.18) in view of Remark 2.2.4

below implies that

$$T(\phi_\varepsilon w_j) \to T(\phi_\varepsilon w_\infty) \text{ in } L^2(\partial_N\Omega) \tag{2.19}$$

as $j \to \infty$, for all $\varepsilon > 0$. Next, since $T(\phi_\varepsilon w_j) = \phi_\varepsilon T(w_j)$ and $T(\phi_\varepsilon w_\infty) = \phi_\varepsilon T(w_\infty)$ (because $\phi_\varepsilon \in C^1(\mathbb{R}^N)$), $\phi_\varepsilon \to 1$ $\mathcal{H}^{N-1}-$a.e. on $\partial_N\Omega$ and $Tw_j, Tw_\infty \in L^\infty(\partial_N\Omega)$, by the bounded convergence theorem we get that $T(\phi_\varepsilon w_j) \to Tw_j$ and $T(\phi_\varepsilon w_\infty) \to Tw_\infty$ in $L^2(\partial_N\Omega)$, as $\varepsilon \to 0$. In fact, note that since $\{\|w_j\|_{L^\infty(\partial_N\Omega)}\}$ is bounded, the limit

$$\int_{\partial_N\Omega} |T(\phi_\varepsilon w_j) - Tw_j|^2 = \int_{\partial_N\Omega} |\phi_\varepsilon Tw_j - Tw_j|^2 \le \sup_{j\in\mathbb{N}} \|w_j\|^2_{L^\infty(\partial_N\Omega)} \int_{\partial_N\Omega} |\phi_\varepsilon - 1|^2 \to 0 \text{ as } \varepsilon \to 0,$$

is uniform in j. Therefore, given $\eta > 0$, we can choose $\varepsilon > 0$ small enough so that

$$\|T(\phi_\varepsilon w_\infty) - Tw_\infty\|_{L^2(\partial_N\Omega)} < \eta \text{ and } \|T(\phi_\varepsilon w_j) - Tw_j\|_{L^2(\partial_N\Omega)} < \eta \quad \forall j \in \mathbb{N}.$$

Hence, by the triangle inequality and the above

$$\|Tw_j - Tw_\infty\|_{L^2(\partial_N\Omega)} \le 2\eta + \|T(\phi_\varepsilon w_j) - T(\phi_\varepsilon w_\infty)\|_{L^2(\partial_N\Omega)},$$

which upon letting $j \to \infty$ and $\eta \to 0$, in view of (2.19), gives that

$$Tw_j \to Tw_\infty \text{ in } L^2(\partial_N\Omega), \text{ as } j \to \infty. \tag{2.20}$$

Finally, invoking the facts that w_j's and w_∞ minimize E_N, together with (2.20), yield

$$\int_\Omega |\nabla w_j|^2 = \int_{\partial_N\Omega} f w_j \to \int_{\partial_N\Omega} f w_\infty = \int_\Omega |\nabla w_\infty|^2.$$

□

In the following remark, we show convergence of traces of H^1-functions under weak H^1-convergence:

Remark 2.2.4. For $u_n, u \in H^1(\Omega)$ with Ω a bounded Lipschitz domain, if $u_n \to u$ in $L^2(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$, then

$$Tu_n \to Tu \quad \text{in} \quad L^2(\partial\Omega).$$

This is a consequence of the following estimate from Theorem 1.5.1.10 in [13], that for Ω as above, there exists a constant C depending only on Ω such that for all $u \in H^1(\Omega)$

and all $\epsilon \in (0,1)$,

$$\int_{\partial\Omega} |Tu|^2 \leq C\left(\sqrt{\epsilon}\int_\Omega |\nabla u|^2 + \frac{1}{\sqrt{\epsilon}}\int_\Omega |u|^2\right).$$

Notice that by the weak convergence of gradients, $\{\|\nabla u_n\|_{L^2(\Omega)}\}$ is bounded, say by C_1, and by the lower semicontinuity of the weak limit so is $\|\nabla u\|_{L^2(\Omega)}$, and also $\int_\Omega |u_n - u|^2 \to 0$, hence,

$$\limsup_{n\to\infty}\int_{\partial\Omega} |T(u_n - u)|^2 \leq 4CC_1^2\sqrt{\epsilon},$$

which gives the result by letting $\epsilon \to 0$.

We conclude this section and chapter with a few remarks.

Remark 2.2.5. A natural question, since we show existence only when there is not failure, is whether failure is common, or even certain. We note that, as we mentioned in the introduction, every solution w to a pure Dirichlet problem is also a solution to a mixed problem, as we can designate part of the boundary that is away from the crack as $\partial_N\Omega$, and set $f := \partial_\nu w$. Then, w is a solution to variational fracture with boundary load f on $\partial_N\Omega$. This shows that the formulation here is not vacuous. Furthermore, studying conditions on Ω and f guaranteeing existence (or non failure) seems to be an interesting direction to explore.

Remark 2.2.6. Note that since $S_{w_\infty} \subset K_\infty$, if we replace K_∞ with S_{w_∞} in the previous lemmas, there is no effect on the energy of w_∞, but there is an increase in the energy of competitors, or there is a reduction in the class of competitors, so then w_∞ minimizes $E_D[S_{w_\infty}]$ over $\{w \in SBV_2(\overline{\Omega}) \ : \ w = w_\infty \text{ on } \partial\Omega\}$, and it minimizes E_N over $\{w \in SBV_2(\overline{\Omega}) \ : \ S_w \subset S_{w_\infty}, w = 0 \text{ on } \partial_D\Omega\}$.

Remark 2.2.7. We can now claim that w_∞ actually does minimize the total energy functional

$$w \mapsto \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_{\partial_N\Omega} fw + \mathcal{H}^{N-1}(S_w \backslash K_\infty),$$

if the class of competitors is restricted to

$$\{w \in SBV_2(\overline{\Omega}) \ : \ w = w_\infty \text{ on } \partial\Omega\} \cup \{w \in SBV_2(\overline{\Omega}) \ : \ S_w \subset K_\infty, w = 0 \text{ on } \partial_D\Omega\}.$$

That is, competitors are not allowed to simultaneously vary both their boundary data on $\partial_N\Omega$ and the crack set. But this is consistent with Griffith's idea that cracks compete with elastic energy, not boundary loads.

Remark 2.2.8. Another natural variational approach seems to be minimizing

$$\frac{1}{2}\int_\Omega |\nabla w|^2 + \mathcal{H}^{N-1}(S_w)$$

over the class

$$\{w \;:\; w \text{ minimizes } E_N(v) \text{ over } v \in SBV_2(\overline{\Omega}), v = 0 \text{ on } \partial_D\Omega, S_v \subset S_w\},$$

since the jump set is fixed when minimizing the Neumann energy. This would be incorrect, however, since as S_w grows, the class of minimizers for E_N grows, so E_N decreases, which means the elastic energy *increases*, since $E_N(v) = -\frac{1}{2}\int_\Omega |\nabla v|^2$. The solution to this minimization problem will therefore necessarily be $S_w = \emptyset$.

Remark 2.2.9. Note that a variational formulation similar to the above will not work in the case of a body load F as well. Minimizing $\frac{1}{2}\int_\Omega |\nabla w|^2 + \mathcal{H}^{N-1}(S_w)$ over the class

$$\{w \;:\; -\Delta w = F \text{ in } \Omega\backslash S_w, w = 0 \text{ on } \partial\Omega\},$$

or in its weak variational form

$$\left\{w \;:\; w \text{ minimizes } v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega Fv \text{ over } v \in SBV_2(\overline{\Omega}), v = 0 \text{ on } \partial\Omega, S_v \subset S_w\right\},$$

will result in $S_w = \emptyset$, with the same exact reasoning as in the previous remark.

Chapter 3

Quasi-Static Fracture Evolution with Boundary Loads

In this chapter, we apply the alternate minimization method introduced in the previous chapter to seek existence of solutions to a *quasi-static* fracture evolution problem with time-varying boundary loads.

We first briefly explain what we mean by a quasi-static evolution. Next, we show the construction of discrete-time evolutions. Then, we get a continuous-time evolution by first passing to the limit of the discrete-time evolutions as the size of time-step approaches zero, and then by extending the result to the whole time interval. We show that this continuous-time limit has the desired minimality properties. Finally, we discuss balance of energy for evolutions and show that there exist evolutions that can decrease the total energy in time and not satisfy the classical quasi-static energy balance known to hold in the case of Dirichlet boundary conditions.

3.1 Quasi-Static Evolution

Quasi-static models are based on the assumption that whatever is driving the motion, e.g., loading, varies slowly in time compared to the elastic wave speed of the material. In other words, the rate of change in the problem parameters (Dirichlet boundary conditions, boundary loads, body forces) is small compared to the time it takes the body to reach elastic equilibrium [5, 17].

More precisely, for a given varying load $f(t)$ (which in our case is a boundary load) on a time interval $[0, T]$, one can consider the rescaled problem corresponding to $f_\varepsilon(t) := f(\varepsilon t)$ on the dilated time interval $[0, T/\varepsilon]$. If the corresponding physical solution (presumably to the dynamic problem) is $u_\varepsilon(t)$, one needs to rescale again in order to take the limit as $\varepsilon \searrow 0$, since the limit of $f_\varepsilon(t)$ is constant in time. Therefore, it is natural to define $u^\varepsilon(t) := u_\varepsilon(t/\varepsilon)$ for $t \in [0, T]$. Setting $u(t)$ to be the limit as $\varepsilon \searrow 0$, it is reasonable to suppose (assuming some damping in the dynamics and that $t \mapsto f(t)$ is not wild) that $u(t)$ is in elastic equilibrium at every t, corresponding to the load $f(t)$ [5].

The first *quasi-static* variational model of fracture based on minimization of the sum of elastic and surface energies was first introduced in 1998 [11]. The idea explained above underlies all quasi-static models of fracture developed ever since, with the only debate being over whether the overall state, made up of both the displacement and crack set, should be a global minimizer of the total energy, a local minimizer, or something in between [14].

The main problem with the quasi-static fracture models concerns jumps in time of the crack set, for which the quasi-static assumption—that while the crack grows the material is always in elastic equilibrium—is dubious. The point is that if in the $\varepsilon \searrow 0$ limit the crack jumps, there is no reason to think that $u_\varepsilon(t)$ varies slowly, even though $f_\varepsilon(t)$ does. This issue also gives rise to a modeling challenge for quasi-static models, that of energy balance. We will expand on this topic more in section 3.3 of this chapter, but all proposed quasi-static models of fracture based on global minimization with non-zero Dirichlet boundary conditions balance the sum of elastic and surface energies with the external work done on the body oblivious to the effect of kinetic energy caused by the jumps in cracks. This energy balance was questioned in models inspired by local minimization techniques, see section 1.1 for more detail.

Even though quasi-static models are the first natural step towards modeling evolution of cracks, due to their simplicity and evasion from having to deal with the wave equation, it is generally agreed that dynamic models need to be considered, and then quasi-static limits can be analyzed. This would help clarify whether cracks jump as soon as the material is not a global minimizer, as proposed in [11], or if jumps only occur to ensure the material is a local minimizer, or if jumps occur based on a condition somewhere in between global and local minimality, as in [14].

3.2 Existence

In this section, we first construct discrete-time evolutions. Then, take the limit as the size of time-step goes to zero and extend the solution to the whole time interval. Our main result is stated in Theorem 3.2.1.

3.2.1 Constructing Discrete-Time Evolutions

For simplicity, we work with the normalized time interval $[0, 1]$. Let I_∞ be a countable dense subset of $[0, 1]$, and for each $n \in \mathbb{N}$, let

$$I_n := \{0 = t_0^n < t_1^n < \cdots < t_n^n = 1\} \subset I_\infty$$

be such that $\{I_n\}$ forms an increasing sequence of nested sets whose union is I_∞, i.e.,

$$\forall n \in \mathbb{N} \quad I_n \subset I_{n+1}, \quad I_\infty = \cup_{n\in\mathbb{N}} I_n.$$

We set $D_n := \sup_{k\in\{1,\ldots,n\}}(t_k^n - t_{k-1}^n)$, and note that $D_n \searrow 0$ as $n \to \infty$.

Remark 3.2.1. Note that throughout the dissertation, we will not specify the dependence of functions on the spatial variable. For example,

$$f(t) := f(t,.).$$

For each $n \in \mathbb{N}$ and $t \in [0,1]$ we define

$$f_n(t) := f_k^n := f(t_k^n) \quad \text{for} \quad t \in [t_k^n, t_{k+1}^n).$$

We assume that the boundary load $f \in W^{1,1}\big([0,1]; L^\infty(\partial_N\Omega)\big)$, so for each $t \in I_\infty$, $f(t) = \lim_{n\to\infty} f_n(t)$, strongly in $L^\infty(\partial_N\Omega)$, and in fact, since $\{I_n\}$ is an increasing nested sequence, $f(t) = f_n(t)$, for large enough n.

Now, for every time step of every partition of the time interval, we perform the alternate minimization described in the previous chapter. So, for fixed $n \in \mathbb{N}$ and fixed $k \in \{0, 1, \ldots, n\}$, at time t_k^n, we get that there exist sequences $\{w_j\}$, $\{v_j\}$ and $\{K_j\}$ with $K_0 = \Gamma_{k-1}^n$, such that w_j's minimize

$$E_N[f_k^n](w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_{\partial_N\Omega} f_k^n w$$

over $\{w \in SBV_2(\overline{\Omega}) \ : \ S_w \subset K_{j-1}, \ w = 0 \text{ on } \partial_D\Omega\}$ and v_j's minimize

$$E_D[K_{j-1}](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_v \backslash K_{j-1})$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = w_j \text{ on } \partial\Omega\}$ with $K_j = K_{j-1} \cup S_{v_j}$. Notice that for the initial time $k = 0$, $K_0 = \Gamma_{-1}^n = \Gamma_0$, where Γ_0 is a possibly empty pre-existing crack set.

Then, by Theorem 2.2.1 in section 2.2, if the material does not fail under the load f_k^n, up to passing to a subsequence if necessary, we let

$$u_k^n := SBV - \lim_{j\to\infty} w_j, \quad \Gamma_k^n := \bigcup_{j=0}^{\infty} K_j, \tag{3.1}$$

and it follows again from Theorem 2.2.1 that u_k^n is a minimizer of $E_N[f_k^n]$ over $\{v \in SBV_2(\overline{\Omega}) \ : \ S_v \subset \Gamma_k^n, \ v = 0 \text{ on } \partial_D\Omega\}$, and a minimizer of $E_D[\Gamma_k^n]$ over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = u_k^n \text{ on } \partial\Omega\}$. Also, following the same exact approach in proving convergence of traces in (2.20), in view of (3.1) we have that

$$w_j \to u_k^n \ \text{ in } \ L^2(\partial_N\Omega). \tag{3.2}$$

Moreover, since u_k^n minimizes $E_N[f_k^n]$, it follows that

$$\int_\Omega \nabla u_k^n \cdot \nabla \xi = \int_{\partial_N \Omega} f_k^n \xi \qquad \forall \xi \in SBV_2(\overline{\Omega}) \quad \text{with} \quad S_\xi \subset \Gamma_k^n, \quad \text{and} \quad \xi = 0 \quad \text{on} \quad \partial_D \Omega. \tag{3.3}$$

Also, (2.10) and (2.17), respectively, imply that

$$\frac{1}{2}\int_\Omega |\nabla w_j|^2 - \int_{\partial_N\Omega} f_k^n w_j + \mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \le \frac{1}{2}\int_\Omega |\nabla w_{j-1}|^2 - \int_{\partial_N\Omega} f_k^n w_{j-1}, \tag{3.4}$$

and

$$\frac{1}{2}\int_\Omega |\nabla w_j|^2 \nearrow \frac{1}{2}\int_\Omega |\nabla u_k^n|^2. \tag{3.5}$$

We then define for $t \in [t_k^n, t_{k+1}^n)$ and $k = 0, 1, \ldots, n$,

$$u_n(t) := u_k^n \quad \text{and} \quad \Gamma_n(t) := \Gamma_k^n. \tag{3.6}$$

Notice that with this notation, for every $t \in I_n$ (in fact, for every $t \in I_\infty$), $u_n(t)$ minimizes

$$E_N[f_n](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N\Omega} f_n(t) v \tag{3.7}$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ S_v \subset \Gamma_n(t),\ v = 0 \text{ on } \partial_D\Omega\}$ and minimizes

$$E_D[\Gamma_n](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_v\backslash\Gamma_n(t)) \tag{3.8}$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = u_n(t) \text{ on } \partial\Omega\}$. Moreover,

$$\mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_n\\ \tau\le t}} S_{u_n(\tau)}\backslash\Gamma_n(t)\Big) = 0. \tag{3.9}$$

In fact, with the above definition for u_n and Γ_n, I_n in (3.9) can be replaced by I_∞. Also, (3.3) becomes

$$\int_\Omega \nabla u_n(t)\cdot\nabla\xi = \int_{\partial_N\Omega} f_n(t)\xi \quad \forall \xi \in SBV_2(\overline{\Omega}) \ \text{ with } \ S_\xi \subset \Gamma_n(t), \ \text{ and } \ \xi = 0 \ \text{ on } \ \partial_D\Omega. \tag{3.10}$$

Remark 3.2.2. From now on, we use phrases E_N-*minimality* and E_D-*minimality* of a pair (u_n, Γ_n) to refer to the facts that it minimizes (3.7) and (3.8), respectively.

In the next section, we study the limiting case as $n \to \infty$ for the sequence of discrete-time evolutions $\{(u_n, \Gamma_n)\}$.

3.2.2 Solution on I_∞

In this section, we define a notion of *failure*, inspired by Definition 2.2.1 in Chapter 2, for the quasi-static problem. This together with the minimality properties of the discrete-time evolutions give us certain bounds and subsequently we can find a limit as $n \to \infty$ (which is equivalent to letting the time-step go to zero) and get an evolution defined on I_∞ that satisfy the desired minimality properties, as follows:

Proposition 3.2.1. *If the material does not fail (defined below) under the boundary load $f \in W^{1,1}\big([0,1]; L^\infty(\partial_N\Omega)\big)$, then there exists a pair of displacement-crack $(u_\infty, \Gamma_\infty)$ that satisfies:*

- *For all $t \in I_\infty$, $\mathcal{H}^{N-1}(\Gamma_\infty(t)) < +\infty$, and for all $t_1, t_2 \in I_\infty$ with $t_1 < t_2$, $\Gamma_\infty(t_1) \subset \Gamma_\infty(t_2)$;*
- *for all $t \in I_\infty$, $u_\infty(t)$ minimizes*

$$E_N[f](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N\Omega} f(t)v$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ S_v \subset \Gamma_\infty(t), \ v = 0 \text{ on } \partial_D\Omega\}$ and minimizes

$$E_D[\Gamma_\infty](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_v \backslash \Gamma_\infty(t)),$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = u_\infty(t) \text{ on } \partial\Omega\}$.

The proof of the above follows from the properties of the limits that we find below and Lemma 3.2.1.

Notice that the same failure as explained in the static case (see Remark 2.2.1) can happen for the sequence of discrete-time evolutions $\{(u_n, \Gamma_n)\}$. Specifically, $\{u_n\}$ might blow up on $\partial_N\Omega$ as $n \to \infty$, that is as we refine the time-step. Similarly, the interaction of Γ_n and $\partial_N\Omega$ could worsen as $n \to \infty$. This would mean that the approximation is deteriorating as we refine the time partitioning and can be interpreted as material failure under the quasi-static load f. So, the following definition updates the non-failure definition for the quasi-static evolution:

Definition 3.2.1 (Quasi-static Non-failure)**.** We say the material under the boundary load f does not fail if the following hold:

(i) $\sup_{n\in\mathbb{N}} \sup_{t\in I_\infty} \|u_n(t)\|_\infty < +\infty$,

(ii) $\mathrm{Cap}(\Gamma_n(1)\cap\mathcal{N}_\varepsilon(\partial_N\Omega)) \to 0$ as $\varepsilon\to 0$, uniformly in n.

As a result, if the material does not fail, we get the following uniform bounds:

$$\sup_{n\in\mathbb{N}} \sup_{t\in I_\infty} \|\nabla u_n(t)\|_{L^2(\Omega)} < +\infty, \tag{3.11}$$

$$\sup_{n\in\mathbb{N}} \sup_{t\in I_\infty} \mathcal{H}^{N-1}(\Gamma_n(t)) < +\infty. \tag{3.12}$$

Let us briefly explain why. Note that (3.10) with $\xi = u_n(t)$ implies that for every $t\in I_\infty$,

$$\int_\Omega |\nabla u_n(t)|^2 = \int_{\partial_N\Omega} f_n(t)u_n(t), \tag{3.13}$$

which together with the bound from Definition 3.2.1-(i) and regularity of f gives (3.11). For the bound (3.12), sum the inequality (3.36) (which is derived using only minimality of u_n) over $k=1,\dots,p$:

$$\frac{1}{2}\int_\Omega |\nabla u_p^n|^2 - \int_{\partial_N\Omega} f_p^n u_p^n + \mathcal{H}^{N-1}(\Gamma_p^n\setminus\Gamma_0^n) \le \frac{1}{2}\int_\Omega |\nabla u_0^n|^2 - \int_{\partial_N\Omega} f_0^n u_0^n - \sum_{k=1}^{p}\int_{\partial_N\Omega}(f_k^n - f_{k-1}^n)u_{k-1}^n.$$

Now, for any $n\in\mathbb{N}$ and any $t\in I_\infty$, there exists $p\in\mathbb{N}$ (depending on n) such that $t\in[t_p^n, t_{p+1}^n)$. Using the definitions (3.6) and that $f\in W^{1,1}\big([0,1];L^\infty(\partial_N\Omega)\big)$ in the above, gives

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_n(t)|^2 - \int_{\partial_N\Omega} f_n(t)u_n(t) + \mathcal{H}^{N-1}(\Gamma_n(t)\setminus\Gamma_n(0))\\ \le \frac{1}{2}\int_\Omega |\nabla u_n(0)|^2 - \int_{\partial_N\Omega} f_n(0)u_n(0) - \int_0^t\int_{\partial_N\Omega}\dot f(s)u_n(s)\, d\mathcal{H}^{N-1}ds,\end{aligned}$$

or equivalently in view of (3.13) and monotonicity of $t\mapsto\Gamma_n(t)$,

$$\mathcal{H}^{N-1}(\Gamma_n(t)) \le \frac{1}{2}\int_\Omega |\nabla u_n(t)|^2 - \frac{1}{2}\int_\Omega |\nabla u_n(0)|^2 + \mathcal{H}^{N-1}(\Gamma_n(0)) - \int_0^t\int_{\partial_N\Omega}\dot f(s)u_n(s)\, d\mathcal{H}^{N-1}ds.$$

The first two terms we know are bounded by (3.11) and the third term is independent of n, since $f_n(0)\equiv f(0)$ for all $n\in\mathbb{N}$, and so is bounded by $\mathcal{H}^{N-1}(\Gamma_0)+\mathcal{H}^{N-1}(S_{u_1(0)})$. The last term is bounded by regularity of f and condition (i) of Definition 3.2.1, and so we have (3.12).

Now, from (3.11), (3.12) in view of (3.9), and condition (i), we are in a position to apply the $SBV-$ compactness theorem (see section 1.1.1) along with a diagonalization process to the class $\{u_n(t)|_\Omega \ : \ n \in \mathbb{N}, \ t \in I_\infty\}$ to obtain a subsequence (not relabeled) and $\{u_\infty(t) \in SBV_2(\Omega) \ : \ t \in I_\infty\}$ such that, for all $t \in I_\infty$,

$$\nabla u_n(t) \rightharpoonup \nabla u_\infty(t) \text{ in } L^2(\Omega) \tag{3.14}$$

$$[u_n(t)]\nu_n(t)\mathcal{H}^{N-1}\lfloor S_{u_n(t)|_\Omega} \overset{*}{\rightharpoonup} [u_\infty(t)]\nu_\infty(t)\mathcal{H}^{N-1}\lfloor S_{u_\infty(t)} \text{ as measures,} \tag{3.15}$$

$$u_n(t) \to u_\infty(t) \text{ in } L^2(\Omega), \tag{3.16}$$

$$u_n(t) \overset{*}{\rightharpoonup} u_\infty(t) \text{ in } L^\infty(\Omega), \tag{3.17}$$

as $n \to \infty$, where $[.]$ and ν respectively denote the jump along the jump set and the approximate normal to the jump set of the function in SBV. For this type of convergence we use the terminology $SBV-$convergence throughout the sequel. Note that as a result of the above convergence,

$$\sup_{t\in I_\infty} \|\nabla u_\infty(t)\|_{L^2(\Omega)} < +\infty, \tag{3.18}$$

$$\sup_{t\in I_\infty} \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S_{u_\infty(\tau)}\Big) < +\infty, \tag{3.19}$$

$$\sup_{t\in I_\infty} \|u_\infty(t)\|_{L^\infty(\Omega)} < +\infty. \tag{3.20}$$

The bounds (3.18) and (3.20) are direct consequences of lower semicontinuity from weak and weak* convergences (3.14) and (3.17), in view of the bound (3.11) and condition (i), respectively. The bound (3.19) is also a consequence of lower semicontinuity together with Lemma 3.1 in [10] and the bound (3.12) in view of (3.9). Next, for $t \in I_\infty$, we extend $u_\infty(t)$ to $\overline{\Omega}$ by:

$$u_\infty(t)|_{\partial_D\Omega} := 0, \text{ and } u_\infty(t)|_{\partial_N\Omega} := T(u_\infty(t)).$$

Moreover, notice that if for $t \in I_\infty$ we define

$$\Gamma_\infty(t) := \bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S_{u_\infty(\tau)}, \tag{3.21}$$

and in view of the bound (3.19), we have that for all $t \in I_\infty$,

$$\mathcal{H}^{N-1}(\Gamma_\infty(t)) < +\infty. \tag{3.22}$$

Also, condition (ii) of Definition 2 implies,

$$\mathrm{Cap}(\Gamma_\infty(1) \cap \mathcal{N}_\varepsilon(\partial_N \Omega)) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.23}$$

To see why, note that according to condition (ii) of Definition 3.2.1 above we can choose a family of functions $\phi_\varepsilon \in C^1(\mathbb{R}^N)$ such that $\phi_\varepsilon = 0$ on a neighborhood of $\cup_{n\in\mathbb{N}}\Gamma_n(1) \cap \mathcal{N}_\varepsilon(\partial_N\Omega)$, $1 - \phi_\varepsilon \in C^1_c(\mathbb{R}^N)$, $0 \le \phi_\varepsilon \le 1$, and $\nabla\phi_\varepsilon \to 0$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Notice that this implies $\phi_\varepsilon \to 1$ in $L^2(\Omega)$ as well as $\mathcal{H}^{N-1}-$a.e. on $\partial_N\Omega$, since subsets of $\partial_N\Omega$ with positive $\mathcal{H}^{N-1}-$measure have positive capacity.

Since $0 \le \phi_\varepsilon \le 1$, $\nabla\phi_\varepsilon \in L^2(\Omega)$, and for all $t \in I_\infty$, $u_n(t)$ $SBV-$converges to $u_\infty(t)$, we get that for all $\varepsilon > 0$,

$$\phi_\varepsilon u_n(t) \to \phi_\varepsilon u_\infty(t) \text{ in } L^2(\Omega), \text{ and } \nabla(\phi_\varepsilon u_n(t)) \rightharpoonup \nabla(\phi_\varepsilon u_\infty(t)) \text{ in } L^2(\Omega),$$

as $n \to \infty$, which together with the fact that $\phi_\varepsilon u_n(t) \in H^1(\mathcal{N}_\varepsilon(\partial_N\Omega)^\circ)$ imply that $\phi_\varepsilon u_\infty(t) \in H^1(\mathcal{N}_\varepsilon(\partial_N\Omega)^\circ)$. Hence, for all $t \in I_\infty$, $\mathrm{Cap}(S_{u_\infty(t)} \setminus \mathrm{supp}(1-\phi_\varepsilon) \cap \mathcal{N}_\varepsilon(\partial_N\Omega)) = 0$, or equivalently,

$$\mathrm{Cap}(\Gamma_\infty(1) \setminus \mathrm{supp}(1-\phi_\varepsilon) \cap \mathcal{N}_\varepsilon(\partial_N\Omega)) = 0.$$

Then, (3.23) follows provided that $\mathrm{Cap}(\Gamma_\infty(1) \cap \mathrm{supp}(1-\phi_\varepsilon) \cap \mathcal{N}_\varepsilon(\partial_N\Omega)) \to 0$ as $\varepsilon \to 0$. Note that otherwise, in view of condition (ii) of Definition 2, $\mathrm{Cap}(\Gamma_\infty(1) \setminus \Gamma_n(1) \cap \mathrm{supp}(1-\phi_\varepsilon) \cap \mathcal{N}_\varepsilon(\partial_N\Omega)) \nrightarrow 0$, which would imply $\nabla\phi_\varepsilon \nrightarrow 0$ in $L^2(\mathbb{R}^N)$, contradicting the definition of ϕ_ε.

Furthermore, following the same exact approach as in the proof of (2.20) with $\{\phi_\varepsilon\}$ chosen as in the above, we get that for all $t \in I_\infty$,

$$u_n(t) \to u_\infty(t) \text{ in } L^2(\partial_N\Omega). \tag{3.24}$$

We now proceed by investigating the minimality properties of u_∞, which are essentially consequences of minimality of u_n's, convergences (3.14-3.17), and the method of Jump Transfer devised in [10].

Lemma 3.2.1. *For $t \in I_\infty$, $u_\infty(t)$ minimizes*

$$E_N[f](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N\Omega} f(t)v \tag{3.25}$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ S_v \subset \Gamma_\infty(t),\ v = 0 \text{ on } \partial_D\Omega\}$ *and minimizes*

$$E_D[\Gamma_\infty](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_v \backslash \Gamma_\infty(t)), \tag{3.26}$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = u_\infty(t) \text{ on } \partial\Omega\}$. *Further,* $\nabla u_n(t) \to \nabla u_\infty(t)$ *strongly in* $L^2(\Omega)$ *as* $n \to \infty$.

Proof. We start first by proving the E_D-minimality. By definition of $\Gamma_\infty(t)$ in (3.21), for any $\eta > 0$, there exist $0 \le t_1 < \cdots < t_p < t_{p+1} = t$ in I_∞, such that

$$\mathcal{H}^{N-1}\Big(\bigcup_{k=1}^{p+1} S_{u_\infty(t_k)}\Big) \ge \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty \\ \tau \le t}} S_{u_\infty(\tau)}\Big) - \eta. \tag{3.27}$$

Moreover, note that from (3.14) and (3.16) we have that for each $k = 1, \dots, p$,

$$\nabla u_n(t_k) \rightharpoonup \nabla u_\infty(t_k) \text{ in } L^2(\Omega), \tag{3.28}$$

$$u_n(t_k) \ \to \ u_\infty(t_k) \text{ in } L^2(\Omega), \tag{3.29}$$

as $n \to \infty$.

Next, take φ to be an arbitrary element in $\{v \in SBV_2(\overline{\Omega}) \ : \ v = 0 \text{ on } \partial\Omega\}$ with $\mathcal{H}^{N-1}(S_\varphi) < +\infty$. Thanks to Theorem 2.8 in [10], based on (3.28) and (3.29), there exists a sequence $\{\varphi_n\} \subset SBV_2(\Omega)$ with $\varphi_n = 0$ on $\partial\Omega$ such that

$$\nabla\varphi_n \to \nabla\varphi \text{ in } L^2(\Omega), \tag{3.30}$$

$$\limsup_{n\to\infty} \mathcal{H}^{N-1}\Big(S_{\varphi_n} \backslash \bigcup_{k=1}^{p} S_{u_n(t_k)}\Big) \le \mathcal{H}^{N-1}\Big(S_\varphi \backslash \bigcup_{k=1}^{p} S_{u_\infty(t_k)}\Big). \tag{3.31}$$

Now, from the minimization in (3.8), in view of (3.9), we get

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla u_n(t)|^2 &\le \frac{1}{2}\int_\Omega |\nabla(u_n(t)+\varphi_n)|^2 + \mathcal{H}^{N-1}\Big(S_{u_n(t)+\varphi_n} \backslash \Gamma_n(t)\Big) \\
&\le \frac{1}{2}\int_\Omega |\nabla(u_n(t)+\varphi_n)|^2 + \mathcal{H}^{N-1}\Big(S_{u_n(t)+\varphi_n} \backslash \bigcup_{k=1}^{p+1} S_{u_n(t_k)}\Big) \\
&= \frac{1}{2}\int_\Omega |\nabla u_n(t)|^2 + \int_\Omega \nabla u_n(t)\cdot\nabla\varphi_n + \frac{1}{2}\int_\Omega |\nabla\varphi_n|^2 + \mathcal{H}^{N-1}\Big(S_{u_n(t)+\varphi_n} \backslash \bigcup_{k=1}^{p+1} S_{u_n(t_k)}\Big),
\end{aligned}$$

or equivalently

$$0 \leq \int_\Omega \nabla u_n(t) \cdot \nabla \varphi_n + \frac{1}{2} \int_\Omega |\nabla \varphi_n|^2 + \mathcal{H}^{N-1}\Big(S_{\varphi_n} \setminus \bigcup_{k=1}^{p} S_{u_n(t_k)}\Big).$$

However, in view of (3.14), (3.30), and (3.31), we may take the limsup as $n \to \infty$ of the right hand side of the above and, upon adding $\frac{1}{2}\|\nabla u_\infty(t)\|^2_{L^2(\Omega)}$, obtain

$$\frac{1}{2} \int_\Omega |\nabla u_\infty(t)|^2 \leq \frac{1}{2} \int_\Omega |\nabla (u_\infty(t) + \varphi)|^2 + \mathcal{H}^{N-1}\Big(S_{u_\infty(t)+\varphi} \setminus \bigcup_{k=1}^{p+1} S_{u_\infty(t_k)}\Big),$$

which concludes the proof of minimality of $u_\infty(t)$ for $E_D[\Gamma_\infty]$ in view of (3.27), taking into account that $\eta > 0$ was arbitrary and $\mathcal{H}^{N-1}(S_\varphi) < +\infty$.

To prove the E_N-minimality, let $\psi \in SBV_2(\overline{\Omega}) \cap L^\infty(\Omega)$, $S_\psi \subset \Gamma_\infty(t)$, $\psi = 0$ on $\partial_D\Omega$ be given. In view of (3.23), we can choose $\phi_\varepsilon \in C^1(\mathbb{R}^N)$ such that $\phi_\varepsilon = 0$ on a neighborhood of $\Gamma_\infty(1) \cap \mathcal{N}_\varepsilon(\partial_N\Omega)$, $1 - \phi_\varepsilon \in C^1_c(\mathbb{R}^N)$, $0 \leq \phi_\varepsilon \leq 1$, and $\nabla\phi_\varepsilon \to 0$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Notice that this implies $\phi_\varepsilon \to 1$ in $L^2(\Omega)$ as well as $\mathcal{H}^{N-1}-$a.e. on $\partial_N\Omega$, since subsets of $\partial_N\Omega$ with positive $\mathcal{H}^{N-1}-$measure have positive capacity.

Note that $\phi_\varepsilon\psi \in SBV_2(\overline{\Omega})$ with $S_{\phi_\varepsilon\psi} \subset \Gamma_\infty(t)$, so $\phi_\varepsilon\psi \in H^1(\mathcal{N}_\varepsilon(\partial_N\Omega)^\circ)$ and we can choose $\psi_\varepsilon \in H^1(\Omega)$ such that $\psi_\varepsilon = \phi_\varepsilon\psi$ on $\partial\Omega$.

Next, from the E_N-minimality of $u_n(t)$ for E_N, the fact that ψ_ε is an admissible variation of $u_n(t)$ for E_N, and the convergence of $u_n(t)$ to $u_\infty(t)$ and $f_n(t)$ to $f(t)$, we have

$$\int_{\partial_N\Omega} f(t)\psi_\varepsilon = \lim_{n\to\infty} \int_{\partial_N\Omega} f_n(t)\psi_\varepsilon = \lim_{n\to\infty} \int_\Omega \nabla u_n(t) \cdot \nabla\psi_\varepsilon = \int_\Omega \nabla u_\infty(t) \cdot \nabla\psi_\varepsilon.$$

But, since $\psi_\varepsilon - \phi_\varepsilon\psi \in SBV_2(\overline{\Omega})$ with $\psi_\varepsilon - \phi_\varepsilon\psi = 0$ on $\partial\Omega$ and $S_{\psi_\varepsilon - \phi_\varepsilon\psi} \subset \Gamma_\infty(t)$, it is an admissible variation for the E_D-minimality of $u_\infty(t)$, so,

$$\int_\Omega \nabla u_\infty(t) \cdot \nabla(\psi_\varepsilon - \phi_\varepsilon\psi) = 0,$$

which in view of the above and the definition of ψ_ε gives

$$\int_\Omega \nabla u_\infty(t) \cdot \nabla(\phi_\varepsilon\psi) = \int_\Omega \nabla u_\infty(t) \cdot \nabla\psi_\varepsilon = \int_{\partial_N\Omega} f\psi_\varepsilon = \int_{\partial_N\Omega} f\phi_\varepsilon\psi.$$

Taking the limit as $\varepsilon \to 0$ in the above, since $\nabla(\phi_\varepsilon \psi) = \psi \nabla \phi_\varepsilon + \phi_\varepsilon \nabla \psi \rightharpoonup \nabla \psi$ in $L^2(\Omega)$, and $\phi_\varepsilon \psi \to \psi$ in $L^1(\partial_N \Omega)$ (since ϕ_ε converges to 1 $\mathcal{H}^{N-1}-$a.e. on $\partial_N \Omega$ and applying the dominated convergence theorem), proves the E_N-minimality of $u_\infty(t)$ with the assumption that $\psi \in L^\infty(\Omega)$. The general result follows by approximating in L^∞.

Finally, from (3.13), strong convergence of $f_n(t)$ to $f(t)$, convergence (3.24) and E_N-minimality of u_∞ we get that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n(t)|^2 = \lim_{n\to\infty} \int_{\partial_N \Omega} f_n(t) u_n(t) = \int_{\partial_N \Omega} f(t) u_\infty(t) = \int_\Omega |\nabla u_\infty(t)|^2.$$

Therefore, appealing to the weak convergence in (3.14), we get the strong convergence $\nabla u_n(t) \to \nabla u_\infty(t)$ in $L^2(\Omega)$.

□

Next, we see how, as a result of the E_N and E_D minimality of discrete time solutions, the energy does not increase. If we define the total energy functional by

$$\mathcal{E}_n(t) := \mathcal{E}[u_n, \Gamma_n, f_n](t) := \frac{1}{2} \int_\Omega |\nabla u_n(t)|^2 - \int_{\partial_N \Omega} f_n(t) u_n(t) + \mathcal{H}^{N-1}(\Gamma_n(t)), \quad (3.32)$$

we have

Lemma 3.2.2. *Given $n \in \mathbb{N}$, for t_{k-1}, t_k in I_n,*

$$\mathcal{E}_n(t_k) \leq \mathcal{E}_n(t_{k-1}) - \int_{\partial_N \Omega} \big(f_n(t_k) - f_n(t_{k-1})\big) u_n(t_{k-1}). \quad (3.33)$$

Proof. The starting point for proving (3.33) is going back to the definition of $u_n(t_k)$ and $\Gamma_n(t_k)$ given at the beginning of this section. So, as in (3.1), let $\{w_j\}$ be a minimizer of $E_N[f_k^n]$ over $\{w \in SBV_2(\overline{\Omega}) \ : \ S_w \subset K_{j-1}, \ w = 0 \text{ on } \partial_D \Omega\}$ where $K_j = K_{j-1} \cup S_{v_j}$ and v_j is a minimizer of $E_D[K_{j-1}]$ over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = w_j \text{ on } \partial\Omega\}$. Notice that summing (3.4) over $2 \leq j \leq i$ yields

$$\frac{1}{2} \int_\Omega |\nabla w_i|^2 - \int_{\partial_N \Omega} f_k^n w_i + \mathcal{H}^{N-1}(K_{i-1} \backslash K_0) \leq \frac{1}{2} \int_\Omega |\nabla w_1|^2 - \int_{\partial_N \Omega} f_k^n w_1. \quad (3.34)$$

On the other hand, since $S_{u_{k-1}^n} \subset \Gamma_{k-1}^n = K_0$, testing the E_N-minimality of w_1 with u_{k-1}^n gives

$$\frac{1}{2} \int_\Omega |\nabla w_1|^2 - \int_{\partial_N \Omega} f_k^n w_1 \leq \frac{1}{2} \int_\Omega |\nabla u_{k-1}^n|^2 - \int_{\partial_N \Omega} f_k^n u_{k-1}^n. \quad (3.35)$$

Note that if we let $i \to \infty$, in view of monotonicity in (3.5), $L^2(\partial_N \Omega)-$convergence of w_i to u_k^n in (3.2), and monotonicity of K_i, we get

$$\frac{1}{2}\int_\Omega |\nabla w_i|^2 \nearrow \frac{1}{2}\int_\Omega |\nabla u_k^n|^2,$$
$$\int_{\partial_N\Omega} f_k^n w_i \to \int_{\partial_N\Omega} f_k^n u_k^n,$$
$$\mathcal{H}^{N-1}(K_{i-1}\backslash K_0) \nearrow \mathcal{H}^{N-1}(\Gamma_k^n \backslash \Gamma_{k-1}^n).$$

Therefore, (3.34) in view of the above and (3.35) yields

$$\frac{1}{2}\int_\Omega |\nabla u_k^n|^2 - \int_{\partial_N\Omega} f_k^n u_k^n + \mathcal{H}^{N-1}(\Gamma_k^n\backslash\Gamma_{k-1}^n) \le \frac{1}{2}\int_\Omega |\nabla u_{k-1}^n|^2 - \int_{\partial_N\Omega} f_k^n u_{k-1}^n, \quad (3.36)$$

which upon adding and subtracting $\int_{\partial_N\Omega} f_{k-1}^n u_{k-1}^n$ to its right hand side and appealing to definition (3.32) gives (3.33).

□

Notice that the reverse direction of the inequality above in Lemma 3.2.2 does not necessarily hold as we explain below in section 3.3. However, as we state in the following proposition, adding a certain amount to the total energy at the later time produces the opposite inequality in Lemma 3.2.2 and gives an upper bound on the amount of drop in energy between two consecutive time steps.

Proposition 3.2.2. *Given $n \in \mathbb{N}$, for t_{k-1}, t_k in I_n,*

$$\mathcal{E}_n(t_k) + \Delta_k^n \ge \mathcal{E}_n(t_{k-1}) - \int_{\partial_N\Omega} \big(f_n(t_k) - f_n(t_{k-1})\big) u_n(t_{k-1}), \quad (3.37)$$

where

$$\begin{aligned}\Delta_k^n := \mathcal{H}^{N-1}(\Gamma_n(t_k)\backslash\Gamma_n(t_{k-1})) &+ \int_{\partial_N\Omega} f_n(t_k)\big(u_n(t_k) - u_n(t_{k-1})\big) \\ &- \int_{\partial_N\Omega} \big(f_n(t_k) - f_n(t_{k-1})\big) u_n(t_{k-1}) \ge 0. \end{aligned} \quad (3.38)$$

Proof. Let us first show that $\Delta_k^n \ge 0$. We start as in the proof of Lemma 3.2.2. Let $\{w_j\}$ and $\{K_j\}$ be such that (3.1) holds. Notice that it follows from the E_N-minimality of w_j that $\int_\Omega |\nabla w_j|^2 = \int_{\partial_N\Omega} f_k^n w_j$, and therefore, (3.4) implies

$$2\mathcal{H}^{N-1}(K_{j-1}\backslash K_{j-2}) \leq \int_{\partial_N\Omega} f_k^n w_j - \int_{\partial_N\Omega} f_k^n w_{j-1},$$

which after summing over $2 \leq j \leq i$ and letting $i \to \infty$ gives,

$$2\mathcal{H}^{N-1}(\Gamma_k^n\backslash\Gamma_{k-1}^n) \leq \int_{\partial_N\Omega} f_k^n u_k^n - \int_{\partial_N\Omega} f_k^n w_1. \tag{3.39}$$

Moreover, testing u_{k-1}^n against E_N-minimality of w_1 yields

$$\frac{1}{2}\int_\Omega |\nabla w_1|^2 - \int_{\partial_N\Omega} f_k^n w_1 \leq \frac{1}{2}\int_\Omega |\nabla u_{k-1}^n|^2 - \int_{\partial_N\Omega} f_k^n u_{k-1}^n,$$

which in view of the facts that $\int_\Omega |\nabla w_1|^2 = \int_{\partial_N\Omega} f_k^n w_1$ and $\int_\Omega |\nabla u_{k-1}^n|^2 = \int_{\partial_N\Omega} f_{k-1}^n u_{k-1}^n$ becomes

$$-\int_{\partial_N\Omega} f_k^n w_1 \leq \int_{\partial_N\Omega} f_{k-1}^n u_{k-1}^n - 2\int_{\partial_N\Omega} f_k^n u_{k-1}^n.$$

Hence, (3.39) in view of the above becomes

$$2\mathcal{H}^{N-1}(\Gamma_k^n\backslash\Gamma_{k-1}^n) + \int_{\partial_N\Omega} (f_k^n - f_{k-1}^n)u_{k-1}^n \leq \int_{\partial_N\Omega} f_k^n (u_k^n - u_{k-1}^n). \tag{3.40}$$

Therefore, using the definition of Δ_k^n in (3.38) together with the above, we get

$$\Delta_k^n \geq 3\mathcal{H}^{N-1}(\Gamma_k^n\backslash\Gamma_{k-1}^n) \geq 0.$$

Now, let

$$\zeta := \arg\min\left\{\int_\Omega |\nabla v|^2 \ : \ v \in SBV_2(\overline{\Omega}),\ v = u_{k-1}^n \text{ on } \partial\Omega,\ S_v \subset \Gamma_k^n\right\}.$$

Note that since $u_{k-1}^n - \zeta = 0$ on $\partial\Omega$, the above implies

$$\int_\Omega \nabla\zeta \cdot \nabla(u_{k-1}^n - \zeta) = 0,$$

or equivalently

$$\int_\Omega \nabla\zeta \cdot \nabla u_{k-1}^n = \int_\Omega |\nabla\zeta|^2.$$

Also, from the E_D-minimality of u_{k-1}^n we get

$$\frac{1}{2}\int_\Omega |\nabla u_{k-1}^n|^2 \leq \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + \mathcal{H}^{N-1}(\Gamma_k^n \backslash \Gamma_{k-1}^n).$$

Hence, in view of the two expressions above we have,

$$\begin{aligned}\int_\Omega |\nabla u_{k-1}^n - \nabla\zeta|^2 &= \int_\Omega |\nabla u_{k-1}^n|^2 - 2\int_\Omega \nabla u_{k-1}^n \cdot \nabla\zeta + \int_\Omega |\nabla\zeta|^2 \\ &= \int_\Omega |\nabla u_{k-1}^n|^2 - \int_\Omega |\nabla\zeta|^2 \leq 2\mathcal{H}^{N-1}(\Gamma_k^n \backslash \Gamma_{k-1}^n). \end{aligned} \tag{3.41}$$

Furthermore, since

$$\int_\Omega |\nabla u_k^n - \nabla\zeta|^2 = \int_\Omega |\nabla u_k^n|^2 - 2\int_\Omega \nabla u_k^n \cdot \nabla\zeta + \int_\Omega |\nabla\zeta|^2,$$

the facts that $\int_\Omega \nabla u_k^n \cdot \nabla\zeta = \int_{\partial_N\Omega} f_k^n \zeta = \int_{\partial_N\Omega} f_k^n u_{k-1}^n$, and $\int_\Omega |\nabla\zeta|^2 \leq \int_\Omega |\nabla u_{k-1}^n|^2$ (which follows from the definition of ζ), imply

$$\int_\Omega |\nabla u_k^n - \nabla\zeta|^2 \leq \int_\Omega |\nabla u_k^n|^2 - 2\int_{\partial_N\Omega} f_k^n u_{k-1}^n + \int_\Omega |\nabla u_{k-1}^n|^2.$$

Also, $\int_\Omega |\nabla u_k^n|^2 = \int_{\partial_N\Omega} f_k^n u_k^n$ and $\int_\Omega |\nabla u_{k-1}^n|^2 = \int_{\partial_N\Omega} f_{k-1}^n u_{k-1}^n$, therefore,

$$\int_\Omega |\nabla u_k^n - \nabla\zeta|^2 \leq \int_{\partial_N\Omega} f_k^n(u_k^n - u_{k-1}^n) - \int_{\partial_N\Omega} (f_k^n - f_{k-1}^n)u_{k-1}^n. \tag{3.42}$$

Thus, Cauchy's inequality, (3.41) and (3.42) imply

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_k^n - \nabla u_{k-1}^n|^2 &\leq \int_\Omega |\nabla u_{k-1}^n - \nabla\zeta|^2 + \int_\Omega |\nabla u_k^n - \nabla\zeta|^2 \\ &\leq 2\mathcal{H}^{N-1}(\Gamma_k^n \backslash \Gamma_{k-1}^n) + \int_{\partial_N\Omega} f_k^n(u_k^n - u_{k-1}^n) - \int_{\partial_N\Omega} (f_k^n - f_{k-1}^n)u_{k-1}^n. \end{aligned}$$

Finally, by adding and subtracting ∇u_k^n in the Dirichlet integral of u_{k-1}^n below, using the above, and that $\int_\Omega \nabla u_k^n \cdot (\nabla u_k^n - \nabla u_{k-1}^n) = \int_{\partial_N\Omega} f_k^n(u_k^n - u_{k-1}^n)$ we get

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla u^n_{k-1}|^2 - \int_{\partial_N\Omega} f^n_{k-1}u^n_{k-1} &= \frac{1}{2}\int_\Omega |\nabla u^n_{k-1} - \nabla u^n_k + \nabla u^n_k|^2 - \int_{\partial_N\Omega} f^n_{k-1}u^n_{k-1} \\
&= \frac{1}{2}\int_\Omega |\nabla u^n_k|^2 - \int_\Omega \nabla u^n_k \cdot (\nabla u^n_k - \nabla u^n_{k-1}) + \frac{1}{2}\int_\Omega |\nabla u^n_k - \nabla u^n_{k-1}|^2 \\
&\quad - \int_{\partial_N\Omega} f^n_{k-1}u^n_{k-1} \\
&\le \frac{1}{2}\int_\Omega |\nabla u^n_k|^2 - \int_{\partial_N\Omega} f^n_k(u^n_k - u^n_{k-1}) + 2\mathcal{H}^{N-1}(\Gamma^n_k \backslash \Gamma^n_{k-1}) \\
&\quad + \int_{\partial_N\Omega} f^n_k(u^n_k - u^n_{k-1}) - \int_{\partial_N\Omega} (f^n_k - f^n_{k-1})u^n_{k-1} - \int_{\partial_N\Omega} f^n_{k-1}u^n_{k-1} \\
&= \frac{1}{2}\int_\Omega |\nabla u^n_k|^2 - \int_{\partial_N\Omega} f^n_k u^n_k + 2\mathcal{H}^{N-1}(\Gamma^n_k \backslash \Gamma^n_{k-1}) + \int_{\partial_N\Omega} f^n_k(u^n_k - u^n_{k-1}),
\end{aligned}
\tag{3.43}
$$

which upon appealing to the definitions in (3.32) and (3.38) concludes the proof. □

3.2.3 Extension to $[0,1]$

We now extend the solution that we found in the previous section to the whole interval $[0,1]$. For each $t \in [0,1]\backslash I_\infty$, let $t_p \in I_\infty$ be an increasing sequence such that $t_p \nearrow t$ as $p \to \infty$. Note that the bounds in (3.18-3.20) give us an $SBV-$convergent subsequence (not relabeled) of $\{u_\infty(t_p) \ : \ p \in \mathbb{N}\}$, whose limit is denoted by $u_\infty(t)$ and is extended to $\overline{\Omega}$ by:

$$u_\infty(t)|_{\partial_D\Omega} := 0, \text{ and } u_\infty(t)|_{\partial_N\Omega} := T(u_\infty(t)).$$

Moreover, for $t \in [0,1]\backslash I_\infty$, we define

$$\Gamma_\infty(t) := \bigcup_{\substack{\tau\in I_\infty \\ \tau<t}} S_{u_\infty(\tau)}.$$

Notice that, alternatively, for $t \in [0,1]\backslash I_\infty$, we could use the uniform bounds (3.11) and (3.12) to extract a potentially $t-$dependent subsequence of $\{u_n(t) : n \in \mathbb{N}\}$, denoted by $u_{n_t}(t)$, which $SBV-$converges to some $\hat{u}(t)$ extended to $\overline{\Omega}$ by:

$$\hat{u}(t)|_{\partial_D\Omega} := 0, \text{ and } \hat{u}(t)|_{\partial_N\Omega} := T(\hat{u}(t)).$$

As we will see below, the choice of a $t-$dependent subsequence of $\{u_n(t) : n \in \mathbb{N}\}$ does not matter, and in fact $\nabla u_\infty(t) = \nabla \hat{u}(t)$ for a.e. t.

Moreover, notice that as a result of lower semicontinuity following from the $SBV-$convergence above and the subsequent convergence of traces we get that

$$u_\infty|_\Omega \in L^\infty\big([0,1]; L^\infty(\Omega)\big), \quad u_\infty|_{\partial\Omega} \in L^\infty\big([0,1]; L^\infty(\partial\Omega)\big), \quad \nabla u_\infty \in L^\infty\big([0,1]; L^2(\Omega)\big).$$

Now, we are in a position to state our main result. The proof will come in two steps: Lemma 3.2.3 and Lemma 3.3.1.

Theorem 3.2.1. *If the material does not fail under the boundary load $f \in W^{1,1}\big([0,1]; L^\infty(\partial_N\Omega)\big)$, then there exists a pair of displacement-crack $(u_\infty, \Gamma_\infty)$ that satisfies:*

- *For all $t \in [0,1]$, $\mathcal{H}^{N-1}(\Gamma_\infty(t)) < +\infty$, and for all $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, $\Gamma_\infty(t_1) \subset \Gamma_\infty(t_2)$;*
- *for all $t \in [0,1]$, $u_\infty(t)$ minimizes*

$$E_N[f](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N\Omega} f(t)v$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ S_v \subset \Gamma_\infty(t), \ v = 0 \text{ on } \partial_D\Omega\}$ and minimizes

$$E_D[\Gamma_\infty](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_v \backslash \Gamma_\infty(t)),$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = u_\infty(t) \text{ on } \partial\Omega\}$.

- *for all $t \in [0,1]$,*

$$\mathcal{E}(t) \leq \mathcal{E}(0) - \int_0^t \int_{\partial_N\Omega} \dot{f}(s) u_\infty(s) \ d\mathcal{H}^{N-1} ds,$$

where $\mathcal{E}(t) := \mathcal{E}[u_\infty, \Gamma_\infty, f](t)$, in view of the definition in (3.32).

Lemma 3.2.3. *For all $t \in [0,1]$, $\mathcal{H}^{N-1}(\Gamma_\infty(t)) < +\infty$, and $u_\infty(t)$ minimizes*

$$E_N[f](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N\Omega} f(t)v \tag{3.44}$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ S_v \subset \Gamma_\infty(t), \ v = 0 \text{ on } \partial_D\Omega\}$ *and minimizes*

$$E_D[\Gamma_\infty](v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_v \backslash \Gamma_\infty(t)), \tag{3.45}$$

over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = u_\infty(t) \text{ on } \partial\Omega\}$. *Furthermore,*

$$\mathcal{H}^{N-1}\big(S_{u_\infty(t)} \backslash \Gamma_\infty(t)\big) = 0,$$

and for a.e. $t \in [0,1]$, $\nabla u_\infty(t) \equiv \nabla \hat{u}(t)$, $u_n(t)$ *SBV−converges to* $u_\infty(t)$, *and* $\nabla u_n(t) \to \nabla u_\infty(t)$ *strongly in* $L^2(\Omega)$ *as* $n \to \infty$.

Proof. Notice that for each $t \in [0,1]\backslash I_\infty$, in view of (3.21) and (3.22) we have $\mathcal{H}^{N-1}(\Gamma_\infty(t)) \leq \mathcal{H}^{N-1}(\Gamma_\infty(1)) < +\infty$.

Next, the fact that $u_\infty(t)$ minimizes (3.44) and (3.45) follows from the stability of minimizers under $SBV-$convergence that we proved in Lemma 3.2.1. Also, proof of the fact that $\mathcal{H}^{N-1}\big(S_{u_\infty(t)} \backslash \Gamma_\infty(t)\big) = 0$ can be found in Lemma 3.8 in [10] and therefore we will not repeat it here.

Lastly, to prove our final claim we first need to notice that the stability of minimizers under $SBV-$ convergence yields that $\hat{u}(t)$ minimizes $E_N[f]$ over $\{v \in SBV_2(\overline{\Omega}) \ : \ v = \hat{u}(t) \text{ on } \partial\Omega\}$. Now, our goal is to prove $\nabla u_\infty(t) \equiv \nabla \hat{u}(t)$. The key idea is to use minimality of u_{n_t} along with the monotonicity of the map $t \mapsto \Gamma_{n_t}(t)$. For $n \in \mathbb{N}$ and $t \in [0,1]$ let

$$l_n(t) := \mathcal{H}^{N-1}(\Gamma_n(t)), \tag{3.46}$$

and note that as a result of the bound (3.12), $\{l_n(t)\}$ is a sequence of uniformly bounded monotone increasing functions on $[0,1]$, by virtue of Helly's theorem we may extract a subsequence (not relabeled) such that pointwise converges to a monotone increasing function on [0,1], denoted by $\lambda(t)$. Also note that we can take the (potentially) $t-$dependent sequence $\{n_t\}$ above from this subsequence. Denote by H the (at most countable) set of discontinuity points of $\lambda(t)$. Then, for $t \notin H$, consider $t_p \in I_\infty \nearrow t$. Next, summing (3.43) up over all $t_k \in I_{n_t}$ with $t_p \leq t_k \leq t$ gives

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_{n_t}(t_p)|^2 \leq \frac{1}{2}\int_\Omega |\nabla u_{n_t}(t)|^2 + 2\mathcal{H}^{N-1}(\Gamma_{n_t}(t)) - 2\mathcal{H}^{N-1}(\Gamma_{n_t}(t_p))\\ - \sum_{\substack{t_k \in I_{n_t}\\ t_p \leq t_k \leq t}} \int_{\partial_N\Omega} \big(f_{n_t}(t_k) - f_{n_t}(t_{k-1})\big) u_{n_t}(t_{k-1}).\end{aligned}$$

where we have used the fact that $\Gamma_{n_t}(t_{p+1}) \supset \Gamma_{n_t}(t_p)$ for every $p \in \mathbb{N}$. Given that

$f \in W^{1,1}\big([0,1]; L^\infty(\partial_N\Omega)\big)$, we have that

$$f_{n_t}(t_k) - f_{n_t}(t_{k-1}) = \int_{t_{k-1}}^{t_k} \dot{f}(s)\ ds,$$

and hence, in view of the condition (i) of non-failure in Definition 3.2.1,

$$-\sum_{\substack{t_k \in I_\infty \\ t_p \le t_k < t}} \int_{\partial_N\Omega} \big(f_{n_t}(t_k) - f_{n_t}(t_{k-1})\big) u_{n_t}(t_{k-1}) \le M \int_{t_p}^{t} \|\dot{f}(s)\|_{L^\infty(\partial_N\Omega)}\ ds,$$

where

$$M := \mathcal{H}^{N-1}(\partial_N\Omega)\ \left(\sup_{n_t \in \mathbb{N}}\ \sup_{t \in I_\infty}\ \|u_{n_t}(t)\|_{L^\infty(\partial_N\Omega)} \right).$$

Now, passing to the limit as $n_t \to \infty$ and using the facts that $\nabla u_{n_t}(t_p) \to \nabla u_\infty(t_p)$ and $\nabla u_{n_t}(t) \to \nabla \hat{u}(t)$ strongly in $L^2(\Omega)$ (the former was proven in Lemma 3.2.1 and the latter can be proven with the same exact reasoning as the former) gives

$$\frac{1}{2}\int_\Omega |\nabla u_\infty(t_p)|^2 \le \frac{1}{2}\int_\Omega |\nabla \hat{u}(t)|^2 + 2\lambda(t) - 2\lambda(t_p) + M \int_{t_p}^{t} \|\dot{f}(s)\|_{L^\infty(\partial_N\Omega)}\ ds,$$

where we used the definition of λ above. So, letting $p \to \infty$, the last term on the right hand side vanishes and using the definition of $u_\infty(t)$ as well as the continuity of λ at t, we get

$$\int_\Omega |\nabla u_\infty(t)|^2 \le \int_\Omega |\nabla \hat{u}(t)|^2.$$

However, testing $u_\infty(t)$ against E_D-minimality of $\hat{u}(t)$ gives the reverse direction of the above inequality, which because of strict convexity of the Dirichlet integral implies that $\nabla u_\infty(t) \equiv \nabla \hat{u}(t)$ for $t \in [0,1]\backslash H$. Therefore, the limit $\hat{u}(t)$ does not depend upon the choice of a specific $t-$dependent subsequence and the whole (sub)sequence of $\{u_n\}$ determined by the convergence of the $u_n(t)$ for $t \in I_\infty$ (and by that of l_n to λ) is such that, for all $t \in [0,1]\backslash H$, $u_n(t)$ $SBV-$ converges to $u_\infty(t)$, while, as in Lemma 3.2.1, $\nabla u_n(t) \to \nabla u_\infty(t)$ strongly in $L^2(\Omega)$ as $n \to \infty$. Nonetheless, H is at most countable, which concludes the proof.

□

3.3 Energy Balance

In this section, we first show that the total energy associated to a quasi-static evolution does not increase in time, section 3.3.1. Notice that this is merely a consequence of the fact that the evolutions minimize energy at each time and the growth of cracks is irreversible. In other words, since whatever each evolution was doing earlier it could do now (it could not change the energy state), earlier states of energy are legitimate competitors at each time. This fact holds true among the global and local minimizers for the Dirichlet boundary conditions that we cited earlier in the background material, see section 1.1.

However, whether the later states of energy in the evolution can be compared to the earlier states is a question whose answer has not been so straightforward. In models based on global minimization for the case of Dirichlet boundary conditions, the answer is *yes*, since all the states of energy are accessible at all times, and any increment in the crack set can be offset with a decrease in the elastic energy so that the total energy remains balanced. Nevertheless, this is not true for local minimizers since things can happen at a later time that are not necessarily accessible at earlier times and thus there can be decreases in energy – particularly caused by jumps in time in the cracks whenever local energy wells disappear.

The interesting result of our variational formulation for boundary loads is that even though it is based on global minimization, there can be cases where the energy of an evolution decreases in time, as we discuss in section 3.3.2. Similar to the local minimization models, the accessibility of later states of energy is not guaranteed. Nevertheless, using the minimality properties of the evolutions, we give an estimate on the amount of drop in energy at any time, see proposition 3.3.1.

Let us emphasize that for quasi-static models, evolution of cracks in time may involve jumps and therefore it is in fact more realistic if there are drops in energy since the kinetic energy is not accounted for in the usual quasi-static energy equation. As a final note, at this time, it is not clear to us if assuming continuity for $t \mapsto \Gamma_\infty(t)$ is enough to conclude that the energy of an evolution under boundary load stays balanced.

3.3.1 Energy Does Not Increase

Now that we have extended the solution to the whole time interval, we can express the result of Lemma 3.2.2 for all $t \in [0, 1]$ as follows. Note that following the definition in (3.32), the corresponding energy to u_∞ is defined by

$$\mathcal{E}(t) := \mathcal{E}[u_\infty, \Gamma_\infty, f](t). \tag{3.47}$$

Lemma 3.3.1. *For all $t \in [0,1]$,*

$$\mathcal{E}(t) \leq \mathcal{E}(0) - \int_0^t \int_{\partial_N \Omega} \dot{f}(s) u_\infty(s) \; d\mathcal{H}^{N-1} ds. \tag{3.48}$$

Proof. Let us assume first that $t \in I_\infty$, and note that there exists $n_0 \in \mathbb{N}$ such that $t \in I_n$ for all $n \geq n_0$. Therefore, for any $n \geq n_0$, summing (3.33) for all $t_k \in I_n$ with $0 < t_k \leq t$ gives

$$\begin{aligned} \mathcal{E}_n(t) &\leq \mathcal{E}_n(0) - \sum_{\substack{t_k \in I_n \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} \big(f_n(t_k) - f_n(t_{k-1})\big) u_n(t_{k-1}) \; d\mathcal{H}^{N-1} \\ &= \mathcal{E}_n(0) - \sum_{\substack{t_k \in I_n \\ 0 < t_k \leq t}} \int_{t_{k-1}}^{t_k} \int_{\partial_N \Omega} \dot{f}(s) u_n(s) \; d\mathcal{H}^{N-1} ds = \mathcal{E}_n(0) - \int_0^t \int_{\partial_N \Omega} \dot{f}(s) u_n(s) \; d\mathcal{H}^{N-1} ds, \end{aligned}$$

where we used the facts that $f \in W^{1,1}\big([0,1]; L^\infty(\partial_N \Omega)\big)$, which gives $f_n(t_k) - f_n(t_{k-1}) = \int_{t_{k-1}}^{t_k} \dot{f}(s) \; ds$, and that by definition, $u_n(s) = u_n(t_{k-1})$ for all $s \in [t_{k-1}, t_k)$. Notice that by Lemma 3.2.1 and from (3.24), for $t \in I_\infty$ we have that

$$\|\nabla u_\infty(t)\|_{L^2(\Omega)} = \lim_{n \to \infty} \|\nabla u_n(t)\|_{L^2(\Omega)},$$

$$\int_{\partial_N \Omega} f(t) u_\infty(t) = \lim_{n \to \infty} \int_{\partial_N \Omega} f_n(t) u_n(t).$$

Moreover, since u_n $SBV-$converges to u_∞ as $n \to \infty$, from Lemma 3.1 in [10] we conclude that,

$$\mathcal{H}^{N-1}(\Gamma_\infty(t)) \leq \liminf_{n \to \infty} \mathcal{H}^{N-1}(\Gamma_n(t)). \tag{3.49}$$

Hence, it follows from the definitions of $\mathcal{E}(t)$ and $\mathcal{E}_n(t)$ and above that,

$$\mathcal{E}(t) \leq \liminf_{n \to \infty} \mathcal{E}_n(t). \tag{3.50}$$

Moreover, it follows from Lemma 3.2.3 that for a.e. $s \in [0,1]$, $u_n(s) \to u_\infty(s)$ in $L^2(\partial_N \Omega)$, and therefore,

$$\int_{\partial_N \Omega} \dot{f}(s) u_\infty(s) \; d\mathcal{H}^{N-1} = \lim_{n \to \infty} \int_{\partial_N \Omega} \dot{f}(s) u_n(s) \; d\mathcal{H}^{N-1}.$$

Also, the sequence of maps $s \mapsto \int_{\partial_N \Omega} \dot{f}(s) u_n(s) \; d\mathcal{H}^{N-1}$ is bounded on $[0,t]$, and hence, by the bounded convergence theorem

$$\int_0^t \Big(\int_{\partial_N \Omega} \dot{f}(s) u_\infty(s) \, d\mathcal{H}^{N-1} \Big) \, ds = \lim_{n \to \infty} \int_0^t \Big(\int_{\partial_N \Omega} \dot{f}(s) u_n(s) \, d\mathcal{H}^{N-1} \Big) \, ds.$$

Therefore, the above, (3.50) and the fact that $\mathcal{E}_n(0) = \mathcal{E}(0)$ for all $n \in \mathbb{N}$ (since $f_n(0) \equiv f(0)$) prove the lemma for the case $t \in I_\infty$. Nevertheless, if $t \in [0,1] \backslash I_\infty$, we can take increasing sequences $t_p \in I_\infty$ such that $t_p \nearrow t$. Notice that since $u_\infty(t_p)$ $SBV-$converges to $u_\infty(t)$, the traces converge strongly $L^2(\partial_N \Omega)$ (similar to (3.24)) and since they are minimizers, the gradients converge strongly in $L^2(\Omega)$. Moreover, by monotonicity of $t \mapsto \Gamma_\infty(t)$, the $\mathcal{H}^{N-1}-$measures converge and therefore,

$$\mathcal{E}(t) = \lim_{p \to \infty} \mathcal{E}(t_p), \tag{3.51}$$

which together with the continuity of the map

$$t \mapsto \int_0^t \int_{\partial_N \Omega} \dot{f}(s) u_\infty(s) \, d\mathcal{H}^{N-1} ds,$$

finishes the proof. □

Proposition 3.3.1. *For all $t \in [0,1]$, there exists $\Delta(t) \geq 3\mathcal{H}^{N-1}(\Gamma_\infty(t) \backslash \Gamma_\infty(0))$, defined below, such that*

$$\mathcal{E}(t) + \Delta(t) \geq \mathcal{E}(0) - \limsup_{M \to \infty} \sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} \big(f(t_k) - f(t_{k-1}) \big) u_\infty(t_{k-1}) \, d\mathcal{H}^{N-1}. \tag{3.52}$$

Proof. Similar to before let us first assume that $t \in I_\infty$, which means that there exists $M_0 \in \mathbb{N}$ such that $t \in I_M$ for all $M \geq M_0$. Notice that we can repeat the same minimization procedure as in the proof of (3.37) for u_∞ and get that for any $t_i, t_j \in I_\infty$ with $t_i \leq t_j$

$$\mathcal{E}(t_j) + \Delta^{i,j} \geq \mathcal{E}(t_i) - \int_{\partial_N \Omega} \big(f(t_j) - f(t_i) \big) u_\infty(t_i),$$

where

$$\Delta^{i,j} := \mathcal{H}^{N-1}(\Gamma_\infty(t_j) \backslash \Gamma_\infty(t_i)) + \int_{\partial_N \Omega} f(t_j) \big(u_\infty(t_j) - u_\infty(t_i) \big) - \int_{\partial_N \Omega} \big(f(t_j) - f(t_i) \big) u_\infty(t_i).$$

Now, for a fixed $M \geq M_0$, if we sum the above inequality for $t_i = t_{k-1}$ and $t_j = t_k$

over all $t_k \in I_M$ with $0 < t_k \leq t$ we get that

$$\mathcal{E}(t) + \Delta_M(t) \geq \mathcal{E}(0) - \sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} \big(f(t_k) - f(t_{k-1})\big) u_\infty(t_{k-1})\, d\mathcal{H}^{N-1}, \tag{3.53}$$

where

$$\begin{aligned}\Delta_M(t) := \mathcal{H}^{N-1}(\Gamma_\infty(t) \backslash \Gamma_\infty(0)) &+ \sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} f(t_k)\big(u_\infty(t_k) - u_\infty(t_{k-1})\big)\, d\mathcal{H}^{N-1} \\ &- \sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} \big(f(t_k) - f(t_{k-1})\big) u_\infty(t_{k-1})\, d\mathcal{H}^{N-1}.\end{aligned}$$

Next, we show that $\Delta_M(t) \geq 3\mathcal{H}^{N-1}(\Gamma_\infty(t) \backslash \Gamma_\infty(0))$. Summing up (3.40) over all $t_k \in I_M$ with $0 < t_k \leq t$ we get that

$$\begin{aligned}2\mathcal{H}^{N-1}(\Gamma_n(t) \backslash \Gamma_n(0)) \leq &\sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} f_n(t_k)\big(u_n(t_k) - u_n(t_{k-1})\big)\, d\mathcal{H}^{N-1} \\ &- \sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} \big(f_n(t_k) - f_n(t_{k-1})\big) u_n(t_{k-1})\, d\mathcal{H}^{N-1}.\end{aligned}$$

On the other hand, in view of (3.49) and strong convergence of f_n to f and u_n to u_∞, the above yields

$$\begin{aligned}2\mathcal{H}^{N-1}(\Gamma_\infty(t) \backslash \Gamma_\infty(0)) \leq &\sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} f(t_k)\big(u_\infty(t_k) - u_\infty(t_{k-1})\big)\, d\mathcal{H}^{N-1} \\ &- \sum_{\substack{t_k \in I_M \\ 0 < t_k \leq t}} \int_{\partial_N \Omega} \big(f(t_k) - f(t_{k-1})\big) u_\infty(t_{k-1})\, d\mathcal{H}^{N-1}.\end{aligned}$$

Appealing to the definition of $\Delta_M(t)$ above, we conclude from the above that $\Delta_M(t) \geq 3\mathcal{H}^{N-1}(\Gamma_\infty(t) \backslash \Gamma_\infty(0))$. Then, we let

$$\Delta(t) := \liminf_{M \to \infty} \Delta_M(t) \geq 3\mathcal{H}^{N-1}(\Gamma_\infty(t) \backslash \Gamma_\infty(0)),$$

which upon passing to the lim inf as $M \to \infty$ in (3.53) proves the proposition for the case where $t \in I_\infty$. Finally, if t belongs to $[0,1]\backslash I_\infty$, similar to the previous lemma, we can take an increasing sequence in I_∞ that converges to t, and use continuity of each of the terms in (3.52) at t. For the energy terms, look at (3.51), and for the terms involving summations note that since $f \in W^{1,1}\big([0,1]; L^\infty(\partial_N\Omega)\big)$ and u_∞ is bounded, the summations can be made arbitrarily small if we get close enough to t. $\square$

3.3.2 Counter-example to Energy Balance

In this section, we give an example of a crack evolution under a boundary load where the energy balance is violated. Consider the domain $\Omega = (-1,1) \times (0,1) \subset \mathbb{R}^2$ made up of a material with unit stiffness and fracture toughness, except near the lateral boundary:

$$G_\varepsilon(x_1, x_2) = G_\varepsilon(x_2) = \begin{cases} 1 & \varepsilon \le x_2 \le 1-\varepsilon \\ +\infty & \text{otherwise} \end{cases}.$$

Suppose we apply the following boundary conditions

$$\partial_{x_1} u = \partial_{x_2} u = 0 \;\text{ on }\; x_2 = 0, 1, \quad \partial_{x_3} u = f\mathbf{e_3} \;\text{ on }\; x_1 = 1, \quad \text{and} \quad u = 0 \;\text{ on }\; x_1 = -1,$$

where the load f is given by $f(x,t) \equiv t$. Now, consider the 1-D problem along horizontal slices parallel to x_1-axis with $u(-1) = 0$ and $u'(1) = f = t$. For $t \le 1$ there exists a solution, and for $t > 1$ there does not, since minimizing the Dirichlet energy gives $\nabla u \equiv 0$, $\mathcal{H}^0(S_u) = 1$, and hence $\inf E_N(u) = -\infty$.

Returning to the 2-D problem, a slicing argument shows that for $t < 1$, there exists a unique solution, whose slices are solutions to the 1-D problem. Now consider $t > 1$, with solution $u_\varepsilon(t)$. Then the Neumann energy of $u_\varepsilon(t)$, in view of E_N-minimality of $u_\varepsilon(t)$, is

$$E_N(u_\varepsilon(t)) = -\frac{1}{2}\int_\Omega |\nabla u_\varepsilon(t)|^2 = -\frac{1}{2}\int_{\partial_N\Omega} t u_\varepsilon(t) = -\frac{1}{2} t \|u_\varepsilon(t)\|_{L^1(\partial_N\Omega)} \to -\infty$$

as $\varepsilon \to 0$. This limit follows since if the energy stays bounded, we can take the limit of $u_\varepsilon(t)$ as $\varepsilon \to 0$, and get a solution of a Neumann problem which does not have a solution for $\varepsilon = 0$, with a reasoning similar to the 1-D case.

Next, note that $\tau \mapsto \|u_\varepsilon(\tau)\|_{L^1(\partial_N\Omega)}$ is increasing, since $\tau \mapsto f(\tau)$ and $\tau \mapsto \Gamma_\varepsilon(\tau)$ are increasing.

Now for $t_- < 1$, suppose

$$\mathcal{E}[u_\varepsilon, \Gamma_\varepsilon](t) = \mathcal{E}[u_\varepsilon](t_-) - \int_{t_-}^{t} \int_{\partial_N \Omega} u_\varepsilon(\tau) \; d\mathcal{H}^{N-1} d\tau.$$

Note that the last term, in absolute value, is less than or equal to $(t-t_-)\|u_\varepsilon(t)\|_{L^1(\partial_N\Omega)}$, and $\mathcal{E}[u_\varepsilon](t_-)$ stays bounded as $\varepsilon \to 0$. Finally, recall $\mathcal{E}[u_\varepsilon, \Gamma_\varepsilon](t) = -\frac{1}{2}t\|u_\varepsilon(t)\|_{L^1(\partial_N\Omega)} + \mathcal{H}^1(\Gamma_\varepsilon(t)) \to -\infty$ as $\varepsilon \to 0$. This gives a contradiction for ε and $t > 1$ small enough.

Chapter 4

$\Gamma-$convergence for Interface Cracks

In this chapter, we show $\Gamma-$convergence of a regularized model for fracture in layered structures with interfaces to a sharp interface model. In layered structures, the mechanical properties of the material bonding the layers together is different from that of the material that the layers are comprised of. The interesting result of our analysis is that when a crack starts growing along the interface it faces an *effective* fracture toughness that is equal to an (in a sense) average of the fracture toughnesses of the bulk and interface materials.

4.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ and $I \subset \Omega$ be a material interface that we assume is a finite union of (topologically) closed C^1 curves (with finite length). The fracture toughness of the bulk, $\Omega \backslash I$, is equal to g_b and the interface, I, is equal to g_i. We define $E_\varepsilon : H^1(\Omega) \times H^1(\Omega) \to [0, +\infty]$ by

$$E_\varepsilon(u,d) := \int_\Omega (1-d)^2 |\nabla u|^2 \, dx + \int_\Omega g_\varepsilon \Big(\varepsilon |\nabla d|^2 + \frac{1}{4\varepsilon} d^2 \Big) dx \tag{4.1}$$

where

$$g_\varepsilon(x) := \begin{cases} g_i & \text{if } \operatorname{dist}(x, I) \leq m\varepsilon \\ \\ g_b & \textit{otherwise}, \end{cases} \tag{4.2}$$

and $m, g_i, g_b > 0$.

We consider the class $\mathcal{C}$ of closed sets $K \subset \overline{\Omega}$ with $\mathcal{H}^1(K) < +\infty$ such that $K \cap I$ is a finite union of closed uniformly C^1 curves (with finite length), together with a finite collection of points.

The set of admissible displacements $\mathcal{A}[K]$ is $\{u \in H^1(\Omega \setminus K) : K \in \mathcal{C}\}$. We make the additional assumption that these u are discontinuous across their corresponding K, except possibly at the endpoints of the C^1 curves making up $K \cap I$.

For $K \in \mathcal{C}$ and $u \in \mathcal{A}[K]$, we define

$$E(u, K) := \int_\Omega |\nabla u|^2 dx + g_{\text{eff}} \mathcal{H}^1(K \cap I) + g_b \mathcal{H}^1(K \backslash I).$$

The *effective toughness*, g_{eff}, is defined by

$$g_{\text{eff}} := \inf_{d \in \mathcal{V}} \int_{-\infty}^{\infty} \bar{g}\Big((d')^2 + \frac{1}{4} d^2\Big) dt, \tag{4.3}$$

where $\mathcal{V} := \{d \in H^1(\mathbb{R}) : \max d = 1\}$, and,

$$\bar{g}(t) := \begin{cases} g_i & \text{if } |t| \leq m \\ g_b & \textit{otherwise.} \end{cases}$$

The main result of this chapter is that

Theorem 4.1.1. *E_ε Γ−converges to E. That is, for every $\varepsilon_n \to 0$,*

$$u_n, d_n \in H^1(\Omega), K \in \mathcal{C}, u \in \mathcal{A}[K], u_n \to u \text{ in } L^2(\Omega) \implies \liminf E_{\varepsilon_n}(u_n, d_n) \geq E(u, K) \tag{4.4}$$

and

$$\forall K \in \mathcal{C}, \forall u \in \mathcal{A}[K], \exists u_n, d_n \in H^1(\Omega) \text{ such that}$$
$$u_n \to u \text{ in } L^2(\Omega) \text{ and } \limsup E_{\varepsilon_n}(u_n, d_n) \leq E(u, K). \tag{4.5}$$

The proof of the above comes in the ensuing sections, statement (4.4) will be proved in section 4.2 and statement (4.5) will be proved in section 4.3.

Remark 4.1.1. Note that in the definition of g_ε, if we consider positive powers of ε, for the scaling of the diffused interface toughness, then

$$\bar{g}_\varepsilon(t) := \begin{cases} g_i & \text{if } |t| \leq m(\varepsilon)\varepsilon \\ g_b & \textit{otherwise.} \end{cases}$$

where $m(\varepsilon) := m\varepsilon^{\alpha}$ and $\alpha > -1$. So, with the change of variable $t \mapsto t/\varepsilon$ we have

$$\bar{g}^{\varepsilon}(t) := \bar{g}_{\varepsilon}(t/\varepsilon) = \begin{cases} g_i & \text{if } |t| \leq m(\varepsilon) \\ g_b & \textit{otherwise.} \end{cases}$$

Since as $\varepsilon \to 0$, for $-1 < \alpha < 0$, $m(\varepsilon) \to \infty$, for $\alpha = 0$, $m(\varepsilon) = m$, and for $\alpha > 0$, $m(\varepsilon) \to 0$, we have that

$$\begin{cases} \bar{g}^{\varepsilon} \to g_i & \text{if } -1 < \alpha < 0, \\ \bar{g}^{\varepsilon} = \bar{g} & \text{if } \alpha = 0, \\ \bar{g}^{\varepsilon} \to g_b & \text{if } \alpha > 0. \end{cases}$$

Therefore, for the cases $-1 < \alpha < 0$ and $\alpha > 0$, g_{eff} defined in (4.3) will be replaced by g_i and g_b, respectively, in the $\Gamma-$limit.

Remark 4.1.2. Notice that

$$\inf_{d \in \mathcal{V}} \int_{-\infty}^{\infty} \bar{g}\Big((d')^2 + \frac{1}{4}d^2\Big)dt = \inf_{d \in \mathcal{V}} \int_{-\infty}^{\infty} \bar{g}_{\varepsilon}\Big(\varepsilon(d')^2 + \frac{1}{4\varepsilon}d^2\Big)dt,$$

where

$$\bar{g}_{\varepsilon}(t) := \begin{cases} g_i & \text{if } |t| \leq m\varepsilon \\ g_b & \textit{otherwise.} \end{cases}$$

This follows from the fact that given $d \in \mathcal{V}$, we can define $d_\varepsilon \in \mathcal{V}$ by $d_\varepsilon(t) := d(t/\varepsilon)$, and if we consider the energy on the left for d, we get the same value as the energy on the right for d_ε (and similarly in the other direction). So, the infima are the same, and $\bar{d}$ minimizes the energy on the left if and only if $\bar{d}_\varepsilon$ minimizes the energy on the right.

Remark 4.1.3. If $g_i \leq g_b$, the infimum defined in (4.3) is achieved and if $g_i < g_b$, the minimizer $\bar{d} \in \mathcal{V}$ is unique, and satisfies $\bar{d}(0) = 1$. If $g_i > g_b$, there does not exist a minimizer, but the infimum is the same as the infimum in (4.3) with $\bar{g}$ replaced by g_b. Indeed, if $\{d_n\}$ is a minimizing sequence and x_n is the point at which d_n equals 1, then $|x_n| \to \infty$ (and so g_i becomes irrelevant).

To see this, we consider the one dimensional energy considered above,

$$E_s(d) := \int_{-\infty}^{\infty} \bar{g}\Big((d')^2 + \frac{1}{4}d^2\Big)dt,$$

now minimized over $d \in \mathcal{V}_x$, the space $\mathcal{V}$ with the additional constraint that $d(x) = 1$, at a point $x \in \mathbb{R}$. We label the unique minimizer $\bar{d}_x$. From the Euler-Lagrange equations, a straightforward calculation gives, for $0 < x < m$,

$$\frac{d}{dx}E_s(\bar{d}_x) = \frac{g_i(g_b - g_i)(g_b + g_i)\sinh(x)\big((g_b^2 + g_i^2)\sinh(m) + 2g_b g_i \cosh(m)\big)}{\big((g_b^2 + g_i^2)\cosh(m) - (g_b^2 - g_i^2)\cosh(x) + 2g_b g_i \sinh(m)\big)^2},$$

and for $x > m$,

$$\frac{d}{dx}E_s(\bar{d}_x) = \frac{g_b(g_b - g_i)(g_b + g_i)\sinh(m)\big((g_b^2 + g_i^2)\sinh(m) + 2g_b g_i \cosh(m)\big)}{L},$$

where

$$\begin{aligned} L := \Big[(g_b - g_i)\Big(g_b \sinh\Big(\frac{3m - x}{2}\Big) + g_i \cosh\Big(\frac{3m - x}{2}\Big)\Big) \\ + (g_b + g_i)\Big(g_b \sinh\Big(\frac{m + x}{2}\Big) + g_i \cosh\Big(\frac{m + x}{2}\Big)\Big)\Big]^2. \end{aligned}$$

Due to symmetry, we need only consider the case $x \in [0, \infty)$. Since all of the terms except $(g_b - g_i)$ are positive, if $g_i < g_b$, $x \mapsto E_s(\bar{d}_x)$ is strictly increasing, and the energy is minimum at $x = 0$. However, if $g_i > g_b$, the function $x \mapsto E_s(\bar{d}_x)$ is strictly decreasing, and a minimizer does not exist.

Remark 4.1.4. Note that approximation of free discontinuity functionals with elliptic functionals was first done by Ambrisio and Tortorelli in [3]. Although they proved the Γ−convergence result below for a more general class of approximating functionals, with minor changes in their proof, we have that $F_\varepsilon : H^1(\Omega) \times H^1(\Omega) \to [0, +\infty]$ defined by

$$F_\varepsilon(u, z) := \int_\Omega z^2|\nabla u|^2 \, dx + \int_\Omega \Big(\varepsilon|\nabla z|^2 + \frac{\alpha^2}{4\varepsilon}(z - 1)^2\Big)dx$$

Γ−converges to $F : SBV(\Omega) \to [0, +\infty]$ defined by

$$F(u) := \int_\Omega |\nabla u|^2 \, dx + \alpha\mathcal{H}^{N-1}(S_u).$$

4.2 The Lower Bound Inequality

We now prove the first inequality in Theorem 4.1.1, (4.4). We suppose $u_n, d_n \in H^1(\Omega), K \in \mathcal{C}, u \in \mathcal{A}[K], u_n \to u$ in $L^2(\Omega)$, and seek to show that $\liminf E_{\varepsilon_n}(u_n, d_n) \geq E(u, K)$.

We first choose rectangles that almost cover K. We label the components of $K \cap I$ by C_i, $i = 1, \dots, N$, and their endpoints by X_1^i, X_2^i. We begin by choosing $r, h > 0$ and a C^1 function $\gamma_{X_1^1} : [0, r] \to \mathbb{R}$ such that with appropriate choice of coordinate axes we have, up to a translation,

$$(K \cap I) \cap R(X_1^1, r, h) = Graph(\gamma_{X_1^1}),$$

where $R(x, r, h) := \{y \in \mathbb{R}^2 : x_1 \leq y_1 \leq x_1 + r,\ |y_2 - x_2| \leq h\}$ is a rectangle with side lengths r and $2h$, oriented so that the normal to C_1 at X_1^1 is e_2 and $\gamma'_{X_1^1}(0) = 0$. By continuity of $\gamma'_{X_1^1}$, for given $\eta > 0$ we can choose $r > 0$ such that

$$|\gamma'_{X_1^1}(t)| < \eta,$$

for all $t \in [0, r]$. Note that by the uniform continuity of $K \cap I$, It follows that for these t,

$$|\gamma_{X_1^1}(t_1) - \gamma_{X_1^1}(t_2)| < \eta |t_1 - t_2| < \eta r.$$

Moreover,

$$\mathcal{H}^1(K \cap R(X_1^1, r, h)) = \int_0^r \sqrt{1 + |\gamma'_{X_1^1}(t)|^2}\ dt < (1 + \eta) r. \tag{4.6}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, which for a C^1 curve equals its length. Note further that if $h > \eta r$, then we are guaranteed that the graph of $\gamma_{X_1^1}$ does not intersect the top or bottom of the rectangle $R(X_1^1, r, h)$.

Since $K \cap I$ is a finite union of uniformly C^1 curves, the r chosen above can in fact be chosen uniformly. We can then consider a next rectangle of the same dimensions, chosen to begin at a point $x^2 \in C_1$, such that $R(x^2, r, h)$ does not intersect $R(X_1^1, r, h)$, and the length of C_1 between these rectangles is less than $\eta r h$. Note that if this second rectangle contains the other endpoint X_2^1 of C_1 in its interior, then we can reduce the dimensions of the second rectangle until X_2^1 lies on a side of the rectangle.

We now show that within each rectangle R,

$$\liminf_{n \to \infty} E_{\varepsilon_n}(u_n, d_n, R) \geq g_{\text{eff}}\, r, \tag{4.7}$$

where

$$E_{\varepsilon_n}(u_n, d_n, R) := \int_R (1-d_n)^2 |\nabla u_n|^2 \, dx + \int_R g_{\varepsilon_n}\Big(\varepsilon_n |\nabla d_n|^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) dx.$$

Set (ξ, ζ) to be a local coordinate system for R, so that $R = \{(\xi,\zeta) : 0 \le \xi \le r, -h \le \zeta \le h\}$. By the convergence of u_n to u, we have that, at least for a subsequence, for a.e. $\xi \in [0, r]$, we have $u_n \to u$ in $L^2(l_\xi)$, where l_ξ is the line segment $l_\xi := \{(\xi,\zeta) \in \mathbb{R}^2 : -h \le \zeta \le h\}$.

Now, since u has a discontinuity on every l_ξ, $\partial_\zeta u_n$ must be unbounded in $L^2(l_\xi)$ for a.e. ξ. So, in order for the first term in $E_{\varepsilon_n}(u_n, d_n, R)$ to be bounded, for a.e. ξ, $\max d_n|_{l_\xi} \to 1$. Denoting $(d_n|_{l_\xi})'$ by d_n', using Fubini's theorem, Fatou's lemma and passing to a subsequence such that the lim inf is achieved, we get

$$\begin{aligned}
\liminf_{n\to\infty} \int_R g_{\varepsilon_n}\Big(\varepsilon_n |\nabla d_n|^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) dx &\ge \liminf_{n\to\infty} \int_0^r \Big[\int_{l_\xi} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta\Big] d\xi \\
&\ge \int_0^r \Big[\liminf_{n\to\infty} \int_{l_\xi} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta\Big] d\xi \\
&\ge \int_0^r \Big[\lim_{n\to\infty} \int_{-h}^{h} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta\Big] d\xi.
\end{aligned}$$

For $E_{\varepsilon_n}(u_n, d_n, R)$ to stay bounded, we must have $\int_{-h}^{h} d_n^2 d\zeta \to 0$, which up to a subsequence gives $d_n \to 0$ a.e. on $(-h, h)$. Thus, we may choose points $h_1 \in (-h, -\eta r)$ and $h_2 \in (\eta r, h)$ such that $d_n(h_1) \to 0$ and $d_n(h_2) \to 0$. Furthermore, we can extract a subsequence such that $|d_n(h_1)| < \varepsilon_n$ and $|d_n(h_2)| < \varepsilon_n$. Next, for every $n \in \mathbb{N}$, define $D_n : \mathbb{R} \to \mathbb{R}$ by

$$D_n(t) := \begin{cases} d_n(t) & h_1 < t < h_2, \\ \dfrac{d_n(h_1)}{\varepsilon_n}(t - h_1 + \varepsilon_n) & h_1 - \varepsilon_n \le t \le h_1, \\ \dfrac{d_n(h_2)}{\varepsilon_n}(-t + h_2 + \varepsilon_n) & h_2 \le t \le h_2 + \varepsilon_n, \\ 0 & t < h_1 - \varepsilon_n, t > h_2 + \varepsilon_n. \end{cases}$$

Notice that $D_n \in H^1(\mathbb{R})$, $\max D_n \to 1$ and moreover,

$$\begin{aligned}\int_{-\infty}^{\infty} g_{\varepsilon_n}\Big(\varepsilon_n (D_n')^2 + \frac{1}{4\varepsilon_n} D_n^2\Big) d\zeta &= \int_{h_1-\varepsilon_n}^{h_1} g_{\varepsilon_n}\Big[\varepsilon_n\Big(\frac{d_n(h_1)}{\varepsilon_n}\Big)^2 + \frac{1}{4\varepsilon_n}\Big(\frac{d_n(h_1)}{\varepsilon_n}(t-h_1+\varepsilon_n)\Big)^2\Big] d\zeta \\ &+ \int_{h_1}^{h_2} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta \\ &+ \int_{h_2}^{h_2+\varepsilon_n} g_{\varepsilon_n}\Big[\varepsilon_n\Big(\frac{d_n(h_2)}{\varepsilon_n}\Big)^2 + \frac{1}{4\varepsilon_n}\Big(\frac{d_n(h_2)}{\varepsilon_n}(-t+h_2+\varepsilon_n)\Big)^2\Big] d\zeta \\ &\le \int_{h_1}^{h_2} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta + 2\|g\|_{L^\infty}\Big(1+\frac{1}{12}\Big)\varepsilon_n^2 \\ &\le \int_{-h}^{h} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta + C\varepsilon_n^2,\end{aligned}$$

where C is a constant. Hence,

$$\begin{aligned}\liminf_{n\to\infty} \int_R g_{\varepsilon_n}\Big(\varepsilon_n |\nabla d_n|^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) dx &\ge \int_0^r \Big\{\lim_{n\to\infty}\Big[\int_{-\infty}^{\infty} g_{\varepsilon_n}\Big(\varepsilon_n (D_n')^2 + \frac{1}{4\varepsilon_n} D_n^2\Big) d\zeta - C\varepsilon_n^2\Big]\Big\} d\xi \\ &\ge \int_0^r \Big[\lim_{n\to\infty}\int_{-\infty}^{\infty} \bar{g}\Big((D_n')^2 + \frac{1}{4} D_n^2\Big) d\zeta\Big] d\xi \\ &\ge r g_{\text{eff}},\end{aligned} \tag{4.8}$$

where the last inequality follows from the fact that the uniform bound in $H^1(\mathbb{R})$ of the sequence $\{D_n\}$ gives us a weak in $H^1(\mathbb{R})$ and strong in $L^2(\mathbb{R})$ convergent subsequence to $d \in H^1(\mathbb{R})$; the lower semi-continuity of the weak limit and the a.e. convergence of a subsequence of $\{D_n\}$ to d which implies $\max d = 1$, together with (4.3), conclude (4.8), which gives (4.7).

Now, note that the Γ-convergence of E_ε to E on $\Omega \setminus I$ follows from Ambrosio-Tortorelli (see Remark 4.1.4). This, together with the arbitrariness of $\eta > 0$ gives (4.4).

4.3 The Upper Bound Inequality

We now show that there exist $\{u_n\} \subset H^1(\Omega)$ and $\{d_n\} \subset H^1(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ and the upper bound inequality (4.5) for $\{E_{\varepsilon_n}\}$ and E holds.

We define $\eta_n := \varepsilon_n^2$ and N_n to be

$$N_n := \{x \in \Omega : \operatorname{dist}(x, K \cap I) < \eta_n\}.$$

Again, from Ambrosio-Tortorelli (see Remark 4.1.4), we have that there exist u_m^n, d_m^n defined on $\Omega \setminus N_n$ with the desired convergence, as $m \to \infty$, and (4.5) holds on $\Omega \setminus N_n$. For each $n \in \mathbb{N}$, choose $m(n)$ such that

$$\|u_{m(n)}^n - u\|_{L^2(\Omega\setminus N_n)} < \frac{1}{n}$$

and the inequality in (4.5) holds on $\Omega \setminus N_n$ to within $\frac{1}{n}$. We set $u_n := u_{m(n)}^n$ and extend arbitrarily to $H^1(\Omega)$ subject to $\|u_n\|_{L^2(N_n)} \to 0$. Below, we will define d_n to be 1 on N_n, so this extension of u_n will play no role in the energy E_{ε_n}.

We now show how to define $d_n \in H^1(\Omega)$. First, we assume that $g_i < g_b$ and let $\bar{d} \in \mathcal{V}$ be the minimizer in (4.3). For $0 < \beta < 1$, choose $T_\beta > 0$ such that $\bar{d}(t) < \beta$ for all $|t| > T_\beta$; and thus define,

$$\bar{d}_\beta := \frac{1}{1-\beta}\big(\bar{d}\chi_{[-T_\beta, T_\beta]} - \beta\big). \tag{4.9}$$

For $\varepsilon > 0$, define $d_\varepsilon : (-h, h) \to \mathbb{R}$ by

$$d_\varepsilon(\zeta) := \begin{cases} 1 & |\zeta| \le \eta_\varepsilon \\ \bar{d}_\beta\Big(\dfrac{|\zeta| - \eta_\varepsilon}{\varepsilon}\Big) & \eta_\varepsilon < |\zeta| < \eta_\varepsilon + \varepsilon T_\beta \\ 0 & |\zeta| \ge \eta_\varepsilon + \varepsilon T_\beta. \end{cases}$$

As we did in proving (4.4), we choose rectangles R_i beginning with $R(X_1^1, \eta, 2h)$. The difference is that in choosing subsequent rectangles, we allow overlap. In particular, we ensure that all line segments normal to $K \cap I$ extending a distance h on each side of $K \cap I$ are included in at least one rectangle, while the length of $K \cap I$ in the overlap of any two rectangles is no more than $\eta r h$.

For the first rectangle $R = R_1$, set (ξ, ζ) to be the local coordinate system as before, with R given by $0 \le \xi \le r$, $-2h \le \zeta \le 2h$. Here, $r > 0$ is chosen for $\eta_n = \varepsilon_n^2$. Define functions $d_n^R : R \to \mathbb{R}$ by

$$d_n^R|_{l_\xi}(\zeta) := d_{\varepsilon_n}(\zeta),\ \forall \zeta \in l_\xi,\ \forall l_\xi \subset R,$$

where l_ξ's are one-dimensional slices of R defined previously. We repeat this for each rectangle R_i.

For the end points of the curves, again we call a generic point x^*, define a semi-circular region with center at x^* and radius h, and set a local polar coordinate system, (r,θ) at x^*. For every $\theta \in [0,\pi]$ define

$$d_n^R(r,\theta) := d_{\varepsilon_n}(r), \ \forall r \in [0,h].$$

We then define R' to be the extension of R by $\eta_\varepsilon + \varepsilon T_\beta$ at each end that contains an endpoint of a component of $K \cap I$, so the region in which $d_n^R \neq 0$ is contained in some R_i'. Finally, extend d_n^R by zero to the whole domain and define $d_n : \Omega \to \mathbb{R}$ by

$$d_n := \max\{\max_{R_i} d_n^{R_i}, d_{m(n)}^n\}. \tag{4.10}$$

Notice that by construction, $d_n \in H^1(\Omega;[0,1])$, for every $n \in \mathbb{N}$.

Now, we plug the constructed sequences $\{u_n\}$ and $\{d_n\}$ into the functionals $\{E_{\varepsilon_n}\}$, introduced in (4.1), and proceed with the calculations as follows. Notice that inside of the $\eta_{\varepsilon_n}-$strips in the rectangles and the end points triangular regions, $d_n \equiv 1$, and outside of the mentioned region, $\nabla u_n = \nabla u_{m(n)}^n$ and $(1-d_n)^2 \leq 1$; thus, for the first integral term of the functionals we have,

$$\limsup_{n\to\infty} \int_\Omega (1-d_n)^2 |\nabla u_n|^2 dx \leq \limsup_{n\to\infty} \int_{\Omega\setminus N_n} (1-d_{m(n)}^n)^2 |\nabla u_{m(n)}^n|^2 dx \leq \int_\Omega |\nabla u|^2 dx. \tag{4.11}$$

Letting $S := \bigcup_{R\in\mathcal{R}_{\eta_{\varepsilon_n}}} R$, by construction of the sequence $\{d_n\}$ we know that on the set $\Omega\setminus S$, $d_n \equiv 0$; and so it is enough to work with S instead of Ω. Further, we decompose the set S into two disjoint subsets G and B, where $G := \bigcup_{k=1}^K R_k'$ is composed of rectangles R_k' that are achieved by removing the overlaps from the original rectangles, R; and $B := S\setminus G$ which includes the overlapping regions. Hence,

$$\begin{aligned}
\int_{\{d_n \neq d_{m(n)}^n\}} g_{\varepsilon_n}\Big(\varepsilon_n |\nabla d_n|^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) dx &\leq \sum_{i=1}^N \int_{R_i'} g_{\varepsilon_n}\Big(\varepsilon_n |\nabla d_n^{R_i}|^2 + \frac{1}{4\varepsilon_n}(d_n^{R_i})^2\Big) dx \\
&\leq \sum_{i=1}^N \int_0^{r_i} \int_{-h}^{h} g_{\varepsilon_n}\Big(\varepsilon_n (d_n')^2 + \frac{1}{4\varepsilon_n} d_n^2\Big) d\zeta d\xi + N\eta r h \\
&\leq \Big(\sum_{i=1}^N r_i\Big) \int_{-T_\beta}^{T_\beta} \bar{g}\big((\bar{d}_\beta')^2 + \frac{1}{4}\bar{d}_\beta^2\big) d\zeta + N\eta r h,
\end{aligned} \tag{4.12}$$

where the second equality follows from the fact that the derivative of d_n's in $\xi-$direction vanishes. Since Nr is controlled by $\mathcal{H}^1(K \cap I)$, $N\eta rh$ can be chosen to be arbitrarily small. Similarly, $\left(\sum_{i=1}^N r_i\right)$ can be made arbitrarily close to $\mathcal{H}^1(K \cap I)$, by choosing η small. Finally, by choosing β small, $\int_{-T_\beta}^{T_\beta} \bar{g}\big((\bar{d}_\beta')^2 + \frac{1}{4}\bar{d}_\beta^2\big)d\zeta$ can be made arbitrarily close to g_{eff}. Therefore,

$$\limsup_{n\to\infty} \int_{\{d_n \neq d^n_{m(n)}\}} g_{\varepsilon_n}\Big(\varepsilon_n|\nabla d_n|^2 + \frac{1}{4\varepsilon_n}d_n^2\Big)dx \leq g_{\text{eff}}\mathcal{H}^1(K \cap I).$$

On the other hand, it is immediate from the Ambrosio-Tortorelli Γ-convergence (see Remark 4.1.4) that

$$\limsup_{n\to\infty} \int_{\{d_n = d^n_{m(n)}\}} g_{\varepsilon_n}\Big(\varepsilon_n|\nabla d_n|^2 + \frac{1}{4\varepsilon_n}d_n^2\Big)dx \leq g_b\mathcal{H}^1(K \setminus I).$$

Combining these with (4.11) gives (4.5).

List of Notations

Notation	Description
$\mathrm{dist}(x, A)$	distance function, the inf of $\lvert x-y\rvert$ over $y \in A$
$\mathrm{Cap}(A)$	$2-$capacity of A, the inf of $\int \lvert\nabla\phi\rvert^2$ over $\phi \in C^1(\mathbb{R}^N)$ with $\phi \geq 1$ on A
$\mathcal{N}_\varepsilon(A)$	$\varepsilon-$neighborhood of A, $\{x \in \overline{\Omega} : \mathrm{dist}(x, A) < \varepsilon\}$
$\partial_D\Omega$ and $\partial_N\Omega$	Dirichlet and Neumann parts of the boundary
$\partial_\nu u$	normal derivative of u at the boundary
f	Neumann data
u, v, w	displacement functions
K, Γ	crack sets
$\mathcal{W}$ (W)	stored elastic energy (density)
$E_D[\Gamma]$	Dirichlet energy, $\frac{1}{2}\int_\Omega \lvert\nabla u\rvert^2 + \mathcal{H}^{N-1}(S_u \backslash \Gamma)$
$E_N[f]$	Neumann energy, $\frac{1}{2}\int_\Omega \lvert\nabla u\rvert^2 - \int_\Omega fu$
$\mathcal{H}^{N-1}$	$(N-1)-$dimensional Hausdorff measure
$\mathcal{L}^N$	$N-$dimensional Lebesgue measure
$SBV(\Omega)$	Special Functions of Bounded Variation on Ω
$SBV(\overline{\Omega})$	$u\vert_\Omega \in SBV(\Omega)$ and $u\vert_{\partial\Omega} \in L^1(\partial\Omega; \mathcal{H}^{N-1}\lfloor\partial\Omega)$
$SBV_2(\overline{\Omega})$	$u \in SBV(\overline{\Omega})$ and $\nabla u\vert_\Omega \in L^2(\Omega)$
S_u	set of approximate discontinuities of $u \in SBV(\overline{\Omega})$
$[u](x)$	jump of u at x
ν	approximate unit normal to S_u
Tu	boundary trace of u
$u_n \overset{SBV}{\rightharpoonup} u$	$\{u_n\}$ $SBV-$converges to u
$\lVert u\rVert_\infty$	$\max\{\lVert u\rVert_{L^\infty(\Omega)}, \lVert u\rVert_{L^\infty(\partial\Omega)}\}$

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