Option Pricing Using Monte Carlo Methods

A Directed Research Project

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in

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by

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Abstract

This project is devoted primarily to the use of Monte Carlo methods to simulate stock prices in order to price European call options using control variates, and to the use of the binominal model to price American put options. At the end, we can use the information to form a portfolio position using an Interactive Brokers paper trading account. This project was done as a part of the masters capstone course Math 573: Computational Methods of Financial Mathematics.

Acknowledgements

This project could not have been completed without the support of Professor Marcel Blais, whose support and enthusiasm throughout the process of finishing this project and profession made this a meaningful exercise.

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1. Introduction

The purpose of this project is to use Monte Carlo methods to price European Call options on equities and to use the binominal model to price American put options. The portfolio positions are formed in an interactive Brokers paper trading account[1] using the information obtained through the pricing process.

As the first step I choose ten stocks with historical data[2] attached and ten options. Second, I calculate the expected daily return for each stock and covariance matrix of daily returns. Third, based on the historical data I use a multidimensional geometric Brownian motion to simulate stock prices.

At last, but not least, in order to obtain option prices, I apply the variance reduction method of control variates for European call options, while implementing binominal model for American put options.

Finally, I compare the numerical prices with actual prices to make investment decisions.

[1] Interactive Brokers paper trading account is from http://www.interactivebrokers.com

[2] Data is downloaded from http://finance.yahoo.com

2. Process

2.1. Selection of Assets

I divide my portfolio into four main sections: financial services, Internet technology, motor industry, and retail sales.

Table 1

Financial Services	**Internet Technology**	**Motor Industry**	**Retail sales**
AMJ	MSFT	HMC	WMT
JPM	ORCL	GM	TESO
BAC	IBM		

I chose those companies by using some of the following criteria:

1. The company has a promising future.
2. The stock price of the company fluctuates slightly.
3. The company is one of the leading companies in its industry.
4. The option has a wide bid-ask spread.

2.2. Parameter

2.2.1. Stock price[3]

The stock price is represented by capital S. $S_{i,t}$ means the i^{th} stock price at time t. (0<= t<= N).

N is the number of days that stocks traded from 1/1/2011 to 4/22/2011.

i=1,2,3, , , d; d is the total number of stocks (in this case, d=10)   t=1,2,3,,,,N

[3] The data is downloaded from http://finance.yahoo.com/

2.2.2. Daily Return

$$r_{i,t} = （S_{i,t} - S_{i,t-1}） / （S_{i,t-1}）$$

2.2.3. Expected Return Vector

$$E(r_{i,t}) = \sum_t (r_{i,t})/N$$

2.2.4. Covariance Matrix

$$\mathrm{cov}(r_i, r_j) = E[(r_i - E[r_i])(r_j - E[r_j])]$$

where

N is the number of days that stocks traded from 1/1/2011 to 4/22/2011.

i=1,2,3, , , d. j=1,2,3,,,,d. d is the total number of stocks (in this case, d=10) t=1,2,3,,,,N

2.2.5. Risk Free Rate

The risk free rate in this project is the average of the three month Federal Interest Rate.[4]

2.2.6. Cholesky factorization of Σ

A is obtained from the Cholesky factorization of Σ [5]

[4] http://www.federalreserve.gov/releases/h15/update/ - The Interest Rate data source (May 2, 2011)

[5] Monte Carlo Methods in Financial Engineering –Paul Glasserman

2.3. Simulation of Stock Price

The first step uses multiple dimensional geometric Brownian motion to simulate the stock prices.

Algorithm:

$$S_i(t_{k+1}) = S_i(t_k)e^{(\mu_i - 0.5\sigma_i^2)(t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k \sum_{j=1}^{d} A_{ij} Z_{k+1,j}}}$$ [6]

i is the index of stocks

k is the index of steps

$\mathrm{Z_k} = (\mathrm{Z_{k1}}, \mathrm{Z_{k2,}} \ldots, \mathrm{Z_{kd}}) \sim \mathrm{N}(0, \mathrm{I}), \mathrm{Z_1}, \mathrm{Z_2}, \ldots, \mathrm{Z_n}$ are independent

d is the total number of stocks

n is the total number of steps.

A is obtained from Cholesky factorization of Σ [6]

[6] Monte Carlo Methods in Financial Engineering –Paul Glasserman

2.4. Price European Call Options

After obtaining the simulated prices of the targeted stocks, I can calculate the European call option price with various strike prices.

First of all, in order to estimate the option prices on May 20, 2011, I set the stock prices at April 22, 2011 as initial stock prices. Secondly, I count the number of days until maturity in terms of year. Thus, T equals to 0.1111(28/252). Using the same method we recalculate the maturity T, I transform the covariance matrix of the daily returns to the one of yearly returns. {Σ = covariance matrix/sqrt(252)}

For European call option, the payoff is:

$$\mathrm{P} = \max(\mathrm{S}_\mathrm{T} - \mathrm{K}, 0)$$

The second step uses a variance reduction method to control variance of option prices.

Algorithm:

$$\mathrm{E}\big(\bar{Y}(\mathrm{b})\big) = \mathrm{E}(\bar{Y} - \mathrm{b}(\bar{X} - \mathrm{E(X)}))$$ [7]

Where

$$\mathrm{b} = \frac{cov(X, Y)}{Var(X)}$$

$\bar{Y}$ contains the mean value of discounted payoffs which are simulated using multiple dimensional geometric Brownian motion. $\bar{X}$ is a control variate estimator (stock prices in this case). X includes the stock prices simulated using a multiple dimensional geometric Brownian motion. [7]

According to simulation process mentioned above, I have obtained the results below:

[7] Monte Carlo Methods in Financial Engineering –Paul Glasserman

Table 2

	S0 (aprl.22)	ANS	K	BID	ASK
BAC	14.19	0.1921	13	0.08	0.09
AMJ	36.48	3.4819	35	3.6	4.2
JPM	43.58	3.5808	42	3.25	3.3
MSFT	27.98	0.9866	27	0.32	0.34
ORCL	31.62	0.6241	31	0.52	0.66
IBM	147.48	2.4811	145	2.1	2.54
HMC	40.02	0.0343	40	0.02	0.03
GM	37.06	0.0690	37		0.05
WMT	54.56	4.5646	50	3.9	4
TESO	16.02	1.0175	15	3.7	4.4

2.5. Pricing American Put Options

Comparing to the valuation of a European option, the valuation of an American option is a difficult problem in pricing because it involves the determination of optimal exercise timing due to the fact that the option can be exercised at any time prior to its own maturity. The option holder will face a choice of either exercising the option immediately or holding the option for a better position. Therefore in order to produce the optimal value at current point in time, we have to compare the exercising payoff value with the value of holding the option.

The parameters are obtained as following:

$$u = \exp((r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t})$$

$$d = \exp((r - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t})$$

$$p_u = 0.5$$

The option price will be

$$V_{j,l} = \max\left[max\left(K - S_{j,l}, 0\right), exp(-r\Delta t)\left(p_u V_{j+1,l+1} + (1 - p_u)V_{j+1,l}\right)\right]$$ [8]

u is the upward factor.

d is the downward factor.

p_u is the probability of an upward movement.

j is the time step.

l is the position variable at j^{th} time.

V is the American put option price.

K is the strike price.

S is the stock price.

According to the algorithm and equations above, I need to calculate the option price backwards, which means tracking the option price from the maturity date back to the initial date in order to obtain the optimal option price that we seek. [8]

We have the following result:

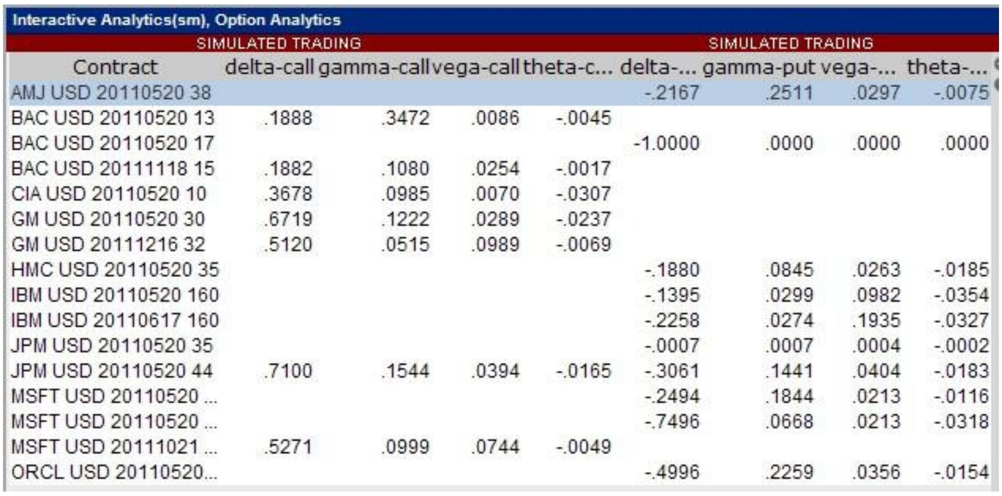
[8] Implementing Models in Quantitative Finance: Methods and Cases - Gianluca Fusai and Andrea Roncoroni

Table 3

	S0 (aprl.22)	Ans	K	BID	ASK
BAC	14.19	2.8053	17	4.7	4.8
AMJ	36.48	1.5149	38	1.5	1.6
JPM	43.58	0.4079	44	0.48	0.5
MSFT	27.98	2.0138	30	3.8	4.2
ORCL	31.62	3.3709	35	3.2	3.3
IBM	147.48	12.4884	160	11	12
HMC	40.02	4.9657	45	7	7.8
GM	37.06	2.9271	40	8.6	9
WMT	54.56	0.4307	55	1.75	1.77
TESO	16.02	1.4712	17.5	1.4	1.6

3. Performance

My portfolio did not performance very well as the following graph illustrates:



Interactive Analytics(sm), Option Analytics

SIMULATED TRADING					SIMULATED TRADING			
Contract	delta-call	gamma-call	vega-call	theta-c...	delta-...	gamma-put	vega-...	theta-...
AMJ USD 20110520 38					-.2167	.2511	.0297	-.0075
BAC USD 20110520 13	.1888	.3472	.0086	-.0045				
BAC USD 20110520 17					-1.0000	.0000	.0000	.0000
BAC USD 20111118 15	.1882	.1080	.0254	-.0017				
CIA USD 20110520 10	.3678	.0985	.0070	-.0307				
GM USD 20110520 30	.6719	.1222	.0289	-.0237				
GM USD 20111216 32	.5120	.0515	.0989	-.0069				
HMC USD 20110520 35					-.1880	.0845	.0263	-.0185
IBM USD 20110520 160					-.1395	.0299	.0982	-.0354
IBM USD 20110617 160					-.2258	.0274	.1935	-.0327
JPM USD 20110520 35					-.0007	.0007	.0004	-.0002
JPM USD 20110520 44	.7100	.1544	.0394	-.0165	-.3061	.1441	.0404	-.0183
MSFT USD 20110520 ...					-.2494	.1844	.0213	-.0116
MSFT USD 20110520 ...					-.7496	.0668	.0213	-.0318
MSFT USD 20111021 ...	.5271	.0999	.0744	-.0049				
ORCL USD 20110520...					-.4996	.2259	.0356	-.0154

Chart 1

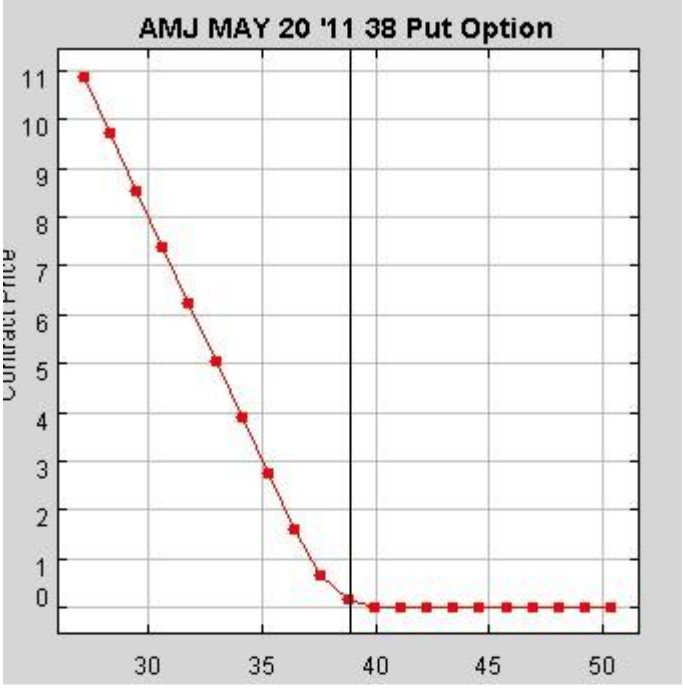


Chart 2

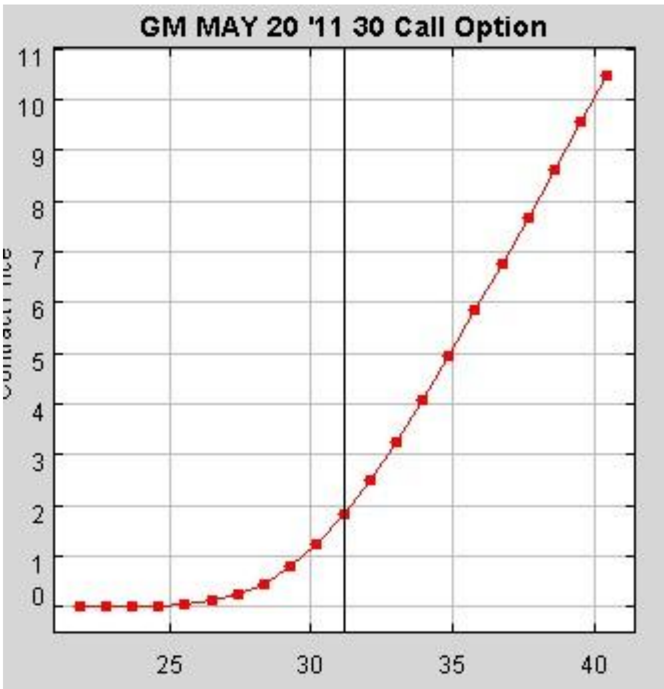


Chart 3

----graphs come from IB Trader Workstation

4. Conclusion

Comparing the numerical option prices with actual market prices, we find that the numerical option prices do not match very well with the real market prices. This may be due to the fact that the estimation of stocks' volatility is not accurate.

During the programming process, I encountered several difficulties. First, I was wondering which is the appropriate discount rate between the federal fund interest rate and US Treasury bill rate. Secondly, I had a problem choosing suitable strike prices.

As a result, if I had chance to do this project again, I would like to collect more data in order to make my simulation process more reliable and to choose more reasonable strike prices. Furthermore, in order to fix the inaccurate estimation problem of volatility I mentioned above, I would like to try industry factor model to gauge the covariance matrix instead of calculating it directly.

5. References and Data

References - Literature

1. Paul Glasserman, 2003, Monte Carlo Methods in Financial Engineering, Springer, New York City, 596P

2. Gianluca Fusai and Andrea Roncoroni, 2008, Implementing Models in Quantitative Finance: Methods and Cases, Springer, New York City, 631P.

Data

1. Interactive Brokers http://www.interactivebrokers.com - The online discount brokerage firm in the United States, founded by Thomas Peterffy, 1977.

2. http://finance.yahoo.com/ - The stock price data source (May 2, 2011)

3. https://www.interactivebrokers.com/ibg/main.php - The portfolio combination came source (May 2, 2011)

4. http://www.federalreserve.gov/releases/h15/update/ - The Interest Rate data source (May 2, 2011)

6. Appendix:

This is the main code, which includes the call option pricing and American put option pricing codes.

```
matrix=csvread('stockprices.csv');
T=28/252;N=200;mt=100;          % T is the matruity days counted as year
r=0.001;                        % federal fund rate
Kprice=[13,35,42,27,31,145,40,37,50,15];    % set strike price
[n,m]=size(matrix);
retprice=zeros(n-1,m);

%calculate the relative daily reutrn
for i=2:n
    retprice((i-1),:)=(matrix(i,:)-matrix(i-1,:))./matrix(i,:);
end;
muret=mean(retprice);
% make covariance matrix measures the cov as years.
covmatrix=cov(retprice);
covmatrix=covmatrix/sqrt(252);
yhat=zeros(10,(mt+1));yy=[];Stm=zeros(10,mt);

%replicate GBM 100 times, get option price=max(st-k,0)
for ii=1:mt
  temp=zeros(10,(N+1));opt=zeros(10,1);
  temp=multipleGeoBrownianMotion(muret,covmatrix,T,N,matrix(n,:));
  Stm(:,ii)=temp(:,length(temp));
  opt=max((Stm(:,ii)-Kprice'),0);
  y=opt*exp(-r*T);
  yy=[yy,y];
end;
% use variance reduction method
for ii=1:10
    cov22=cov(Stm(ii,:),yy(ii,:));
    b(ii)=cov22(1,2)/var(Stm(ii,:));
end;
meanstm=mean(Stm,2);
Stemp=zeros(10,mt);
```

```
for ii=1:mt
    Stemp(:,ii)=Stm(:,ii)-meanstm;
end;
b=b';
yhat=[];
for ii=1:10
    yhat(ii,:)=yy(ii,:)-b(ii)*Stemp(ii,:);
end;
mean(yhat,2)                % output the option price.




% the following is the american put option

Kprice2=[17,38,44,30,35,160,45,40,55,17.5];    % set strike price for put option
[n,m]=size(matrix);
retprice=zeros(n-1,m);
%calculate the relative daily reutrn
for i=2:n
    retprice((i-1),:)=(matrix(i,:)-matrix(i-1,:))./matrix(i,:);
end;
muret=mean(retprice);
covmatrix=cov(retprice);
outcome=[];
for kk=1:m
sigma=std(retprice(:,kk));     %get the standard divation of relative daily return.
deltat=T/N;                 %  dt
u=exp((r-1/2*sigma^2)*deltat+sigma*sqrt(deltat));   % the upward mutiplier
d=exp((r-1/2*sigma^2)*deltat-sigma*sqrt(deltat));   % the downward mytiplier
p=1/2;                                % possibility of up and down.
s0=matrix(n,kk);                            % initial stock price
bioput=[];
strike=Kprice2(kk);                         % set strike price for certain stock
for i=N:(-1):1                            % we backtrack the option from N downto 1
   for j=1:i
      root=s0;
    root=root*u^j*d^(i-j);
    bioput(i,j)=max(strike-root,0);               % calculate the option price at current step
  end;
  if i<N
    for j=1:i
      bioput(i,j)=max(bioput(i,j),exp(-
r*T/N)*(0.5*bioput(i+1,j)+0.5*bioput(i+1,j+1)));
      % compare current option price and the discouted, expected option
      % of forward option price, and choose the better one.
    end;
  end;
end;
output(kk)=bioput(1,1);
end;
output'
```

This is the GMB mode needed in the main file.

```
function[S]=multipleGeoBrownianMotion(mu,Sigma,T,N,S0)
  d=size(Sigma,1);
  A=chol(Sigma)';
  St=zeros(d,N+1);
  St(:,1)=S0;
  for ii=1:N
      random=randn(d,1);
      for jj=1:d
          St(jj,ii+1)=St(jj,ii)*exp((mu(jj)-
Sigma(jj,jj)^2/2)*(T/N)+sqrt(T/N)*A(jj,:)*random);
      end;
  end;
  S=St;
```