EFFECTIVE CHARACTERISTICS OF ELASTIC LAMINATES IN SPACE-TIME

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This paper is dedicated to my family. Specifically, it is dedicated to my father, my mother, and my little brother Jamie. Without their continued support, I would not be the person that I am today.

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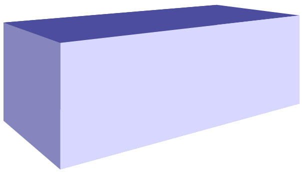
Abstract. The effective properties for a laminar elastic spatial-temporal composite were investigated. It was assumed that the composite is binary, that is, it is assembled of two original constituents, capable of changing (in space-time) their material density, as well as their material stiffness. The condition of plane strain was imposed on a bulk of an elastic material that possesses a periodic laminar microstructure in space-time. First, the effective elasticity tensor was derived for the static case. Next, the dynamic case was investigated, and expressions were found for average material properties. These expressions appeared to be diagonalizable in certain cases, but proved to be more complicated in others. An additional force, of Coriolis type, was found in the averaged equations of elastodynamics due to the presence of simultaneous change in both inertial and elastic properties of original material constituents. The appearance of a Coriolis force is a consequence of both dynamics and plane strain; it doesn't arise in the case of one dimensional strain that is typical for longitudinal dynamic disturbances that propagate along an elastic bar.

1. Introduction

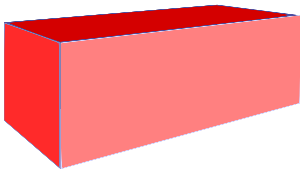
Advancements in materials science have made it possible to create what are known as "composite materials", a.k.a. materials with a microstructure. In many cases the effective material properties can be studied through an averaging process.

Consider two rods (rod 1 and rod 2) that have different electric permittivities (ϵ_1 and ϵ_2) and different magnetic permeabilities (μ_1 and μ_2), as in parts (a) and (b) of figure 1. It is clear that if identical electromagnetic waves are sent through rods 1 and 2, each rod will have a different effect on various wave properties (e.g. phase speed, and group velocity). Knowing this, imagine if these two rods are then connected to one another, or cut up into pieces, and reassembled into a "composite" rod, as in part (c) of figure 1. In this case, one may now ask, what is the effect on an electromagnetic wave which propagates through this new material? Is it possible to find "average" material properties experienced by long waves? or must we consider each piece of this microstructure individually? These are the questions addressed in the mathematical theory of homogenization, which has been studied at length for materials with what is termed a static (or unchanging) microstructure.

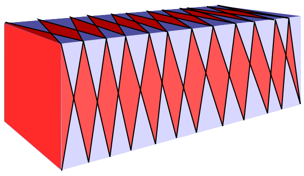
However, more recent advancements in material science and technology have shown that material parameters, such as dielectric permittivity and magnetic permeability [5], [14], or even elasticity and mass density [7], [6], can be changed both in space and in time. The rate at which these changes can happen range from slow to very fast (e.g. at the flip of a switch). For example, consider an array of LC-circuits (i.e. a simple transmission line) where the inductances and capacitances alternate in space (see figure 2), and can be switched at the "push of a button". This forms a material composite, and we can, in certain circumstances, use results from the theory of homogenization to study the effective properties of such material assemblages. These properties are perceived on a larger scale compared to the scale of material microstructure. Homogenization is particularly legitimate when the material structure is laminar in space and time [12]. Electromagnetic waves propagating through such structures can experience interesting properties. We see (Figure 3) that the "world-lines" of the laminar composite interfaces are characterized by straight lines in space-time; particularly when they are parallel to the t-axis, the picture characterizes what is called a "static" laminate (Figure 3



(a) Rod with electromagnetic paramaters ϵ_1 and μ_1



(b) Rod with electromagnetic paramaters ϵ_2 and μ_2



(c) Rod made by cutting up the first two, and then reassembling them into a composite rod.

FIGURE 1. Three rods with differing electromagnetic parameters.

(a)). However, suppose we start changing the capacitances and inductances using a periodic switching procedure, effectively creating a more general composite (the space-time diagram of such a material can be seen in 3 (d)); what is the affect on electromagnetic waves which propagate through this new structure in space-time?

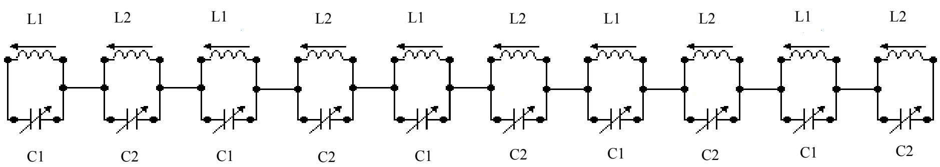
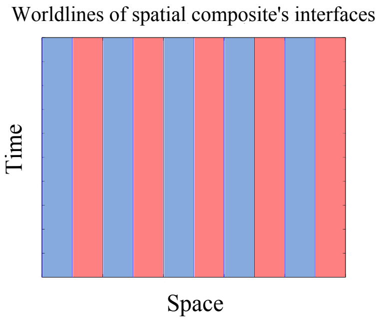


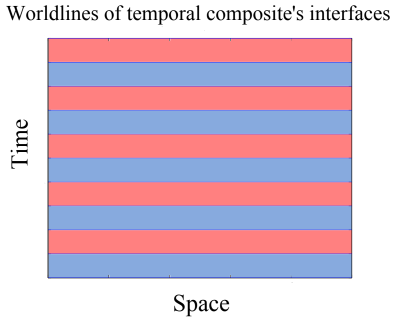
FIGURE 2. An array of LC-circuits alternating in space between L_1, C_1 and L_2, C_2.

As it turns out, the answer to the preceding questions is quite interesting, and has only recently been studied by a select few [12], [11], [16], [17], [9], [3]. This new area of research is termed "spatial-temporal composites", and is based on the argument that material composites can be dynamic as well as static!

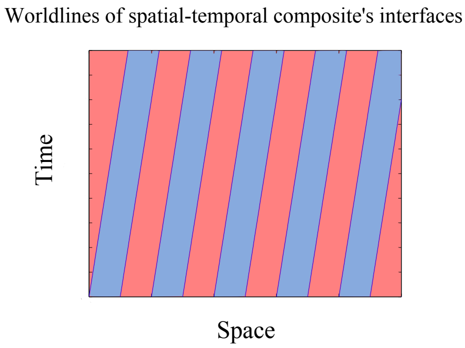
Electromagnetics is not the only area where material parameters can be changed in time. Many researchers are developing methods for changing elastic material parameters in time [7], [6]. As a result of this, it is becoming possible to create elastic spatial-temporal composites. With the possibility of the construction of these new composites, examining the effective properties of these new elastic spatial-temporal composites becomes very important.



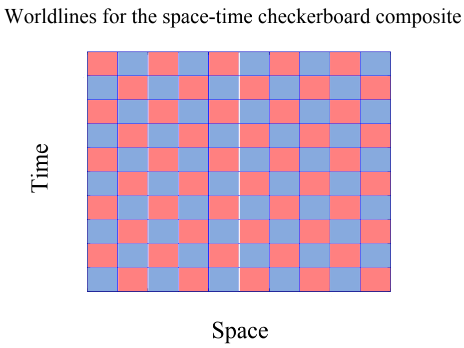
(a) Purely Spatial Material Composite



(b) Purely Temporal Material Composite



(c) General Spatial-Temporal Material Composite



(d) Space-Time Checkerboard Material Composite

FIGURE 3. Space-time diagrams for various types of spatial-temporal composites.

In this paper, we investigate the homogenization of material properties found in the theory of elasticity. We provide a brief background of the relevant theory surrounding elasticity and homogenization, as well as certain methods that will be used in the following calculations. We then derive the effective material properties for a static elastic laminate. After this procedure is established, the effective material properties for a general spatial-temporal elastic laminate under plane strain are found. Finally, the system's averaged Lagrangian is examined in order to check the previous derivations. In this examination, some interesting physical results are obtained, namely, the appearance of two new force terms (one of Coriolis type) which directly result from the moving property pattern and the physical assumptions placed on the system.

2. Background

2.1. Theory of Elasticity. The theory of elasticity is the study of materials which "bounce-back" to their original shape after an applied force has been removed from said material. This property is called *elasticity*. Almost all materials possess elastic tendencies, as long as they are not *deformed* past a certain limit [15]. A main assumption of elastic materials is that they obey what is known as Hooke's law.

Hooke's law, in its most basic form, can be seen when investigating the dynamics of a mass on a spring. We know that this relationship takes the following form:

$$F = -kx \quad \text{where,}$$

$$\begin{aligned} F &= ma \quad \text{net force on a mass m,} \\ x &= \text{displacement of the mass from it's equilibrium position,} \\ k &= \text{stiffness of the spring.} \end{aligned}$$

The above relationship gives us a differential equation for x that can be subsequently solved and the motion of the mass can be found. In a more general elastic material, we have essentially the same law; however, it is slightly more complicated.

The role of k in the general Hooke's law is played by a rank-4 tensor known as the stiffness, or elasticity tensor, which may be dependent on position. Simply put, this means that at every point in a general material, there is a set of coefficients that completely characterizes the stiffness of the material in every direction. This quantity is denoted as D. A particular element of D is denoted using subscripts, i.e., element D_{ijkl}.

The role of F in the general Hooke's law is played by a rank-2 tensor known as the stress tensor. This is simply a special symmetric 3×3 matrix which characterizes the force per unit area in every direction at every point of the material. In this paper, the stress tensor is denoted as p. A particular element of p is denoted by subscripting τ, i.e., the ij element of p is τ_{ij}.

The role of x in the general Hooke's law is played by a rank-2 tensor known as the strain tensor. This is a special 3×3 matrix that characterizes stretching of the material. The strain tensor is typically denoted as e, and a particular element of e is given by e_{ij}. The elements of the strain tensor are defined as follows:

Let the position of any point in the material be given by the vector $\underline{x} = x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$. Furthermore, define the displacement of the material from equilibrium at any point $\underline{x}$ be given by the vector $\underline{u} = \underline{u}(x_1, x_2, x_3) = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$, where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are unit vectors pointing in the x_1, x_2, and x_3 directions, respectively. With these assumptions, the strain tensor is defined as follows:

$$(1) \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{where } i, j = 1, 2 \text{ or } 3.$$

These three quantities are related through the double convolution operation:

$$p_{ij} = \sum_{k=0}^{3}\sum_{l=0}^{3} D_{ijkl}e_{kl} \quad .$$

Thus, we have shown the generalized Hooke's law.

2.1.1. *Static Elasticity.* The problem in static elasticity is as follows. It involves finding solution to the following set of partial differential equations for the material displacement from equilibrium. These are given as follows [1]:

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i = 0 \quad \text{where } i = 1,\ 2,\ \text{or } 3 \text{ and,}$$

$$\rho = \text{mass density as a function of position, and,}$$
$$f_i = \text{the sum of body forces in the relevant direction.}$$

2.1.2. *Dynamic Elasticity.* The problem in dynamic elasticity is similar to the previously discussed static case; in this case, however, the forces do not balance. Thus, there is material motion, and the vector $\underline{u}$ becomes dependent on time along with the coordinates. This means that velocity and acceleration vectors must be introduced. Let $\underline{\dot{u}} = \underline{\dot{u}}\,(x_1, x_2, x_3, t) = \dot{u}_1\underline{i} + \dot{u}_2\underline{j} + \dot{u}_3\underline{k} = \frac{\partial u_1}{\partial t}\underline{i} + \frac{\partial u_2}{\partial t}\underline{j} + \frac{\partial u_3}{\partial t}\underline{k}$ be the velocity vector of the material, and let $\underline{\ddot{u}} = \underline{\ddot{u}}\,(x_1, x_2, x_3, t) = \ddot{u}_1\underline{i} + \ddot{u}_2\underline{j} + \ddot{u}_3\underline{k} = \frac{\partial \dot{u}_1}{\partial t}\underline{i} + \frac{\partial \dot{u}_2}{\partial t}\underline{j} + \frac{\partial \dot{u}_3}{\partial t}\underline{k}$ be the acceleration vector. This case then becomes analogous to the simpler case of a mass on a spring. It involves finding the solution to the following set of partial differential equations for the material displacement $\underline{u}$ from equilibrium. These are given as follows [1]:

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i = \rho \underline{\ddot{u}_i} \quad \text{where } i = 1,\ 2,\ \text{or } 3.$$

Oftentimes it is convenient to look at specific cases of constraint for the elastic system that is being studied. For example, a basic dam experiences uniform stress with respect to the horizontal coordinate that is parallel to it. Another good example (and the one used later in this paper) is that of plane strain.

2.1.3. *Plane Strain.* In the case of a plane strain, all field variables are independent of x_3, and the displacement in the x_3 direction vanishes identically, [1]. Mathematically, this means that for any field quantity $f(\underline{x})$, the following conditions hold:

$$u_3 = 0 \quad \text{and,}$$
$$\frac{\partial}{\partial x_3} f = 0.$$

To physically achieve and maintain such strain, one has to apply opposing stresses in the positive and negative x_3-direction to constrain particle motion across the plane. Familiarity with plane strain is important, because it is central to the results of this paper.

2.2. **Homogenization.** Homogenization is the study of partial differential equations with rapidly oscillating coefficients. A main goal of homogenization is finding the effective values of these coefficients. This theory is readily applicable to static composite materials, and this field has been extensively studied [10], [13], and [8].

The intuitive idea behind homogenization is that there are different length scales: the microscale, and the macroscale, [13]. The microscale is a small length scale, i.e., it is characterized by lengths less than l_1, which is chosen to be greater than the maximum size of inhomogeneities in the microstructure. The macroscale is a larger length scale, characterized by some length l_2. At this scale, the composite appears homogeneous with respect to averaging. It is on this length scale that mathematical analysis and averaging are carried out. It is assumed that the length scales are well separated, i.e., $l_1 \ll l_2$.

In [10], [13], a very good example of homogenization is given. Consider the case of electroconductivity in a periodic microgeometry. Also, assume that there are no internal current sources. The microscale (l_1-scale) equations are given as follows:

$$\begin{aligned} \mathbf{j}(\underline{x}) &= \sigma(\underline{x})\,\mathbf{e}(\underline{x}) \quad , \\ \bigtriangledown \cdot \mathbf{j} &= 0 \quad , \\ \bigtriangledown \times \mathbf{e} &= 0 \quad \text{where,} \end{aligned}$$

$$\begin{aligned} \mathbf{j}(\underline{x}) &= \text{the current field,} \\ \mathbf{e}(\underline{x}) &= -\bigtriangledown \phi(\underline{x}) \text{ is the electric field} \\ \phi(\underline{x}) &= \text{the electrical potential,} \\ \sigma(\underline{x}) &= \text{the conductivity tensor field.} \end{aligned}$$

On the macroscopic level, the equations take the same basic form, but with average quantities.

$$\begin{aligned} \mathbf{j}_0(\underline{x}) &= \sigma_* \mathbf{e}_0(\underline{x}) \quad , \\ \bigtriangledown \cdot \mathbf{j}_0 &= 0 \quad , \\ \bigtriangledown \times \mathbf{e}_0 &= 0 \quad \text{where,} \end{aligned}$$

$$\begin{aligned} \mathbf{j}_0(\underline{x}) &= \text{the local average of } \mathbf{j} \text{ over a cube centered at } \underline{x}, \\ \mathbf{e}_0(\underline{x}) &= \text{the local average of } \mathbf{e} \text{ over a cube centered at } \underline{x}. \end{aligned}$$

First, it is important to note that the size of the cube in the above definition of $\mathbf{j_0}$ and $\mathbf{e}_0$ is large compared to the microstructure (i.e., it is on the l_2 scale). Secondly, we notice a new quantity σ_*. This quantity is the effective (average) conductivity tensor, and a main problem in the theory of composite materials is solving for this tensor.

3. Static Material Laminate

3.1. **Introduction.** We use the symbol p and e to represent, respectively, the stress and strain tensors

$$p = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix}, \; e = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}.$$

We know that, in the theory of elasticity, these quantities are related through Hooke's law. More specifically, the stress tensor and the strain tensor (both of rank 2) are related through a rank-4 material tensor called the stiffness (or elasticity) tensor; we denote it as D. This tensor completely characterizes the stiffness of the material. The following equation shows this relationship [1], [15]:

$$p = D \cdot \cdot e \quad ,$$

where the $\cdot\cdot$ operation denotes the double convolution. It is important to note that this notation is exactly the same as the index summation notation used in many texts on mathematical physics, i.e. $p = D \cdot \cdot e$ is equivalent to $p_{ij} = D_{ijkl} e_{kl}$.

3.2. **Physical Setup of System.** Consider a bulk of elastic material that is in static equilibrium. We pick a convenient origin and chose a rectangular coordinate system with unit vectors $\underline{i} = (1, 0, 0)$, $\underline{j} = (0, 1, 0)$, and $\underline{k} = (0, 0, 1)$. Thus, the position vector is given by $\underline{r}(x_1, x_2, x_3) = x_1\underline{i} + x_2\underline{j} + x_3\underline{k} = (x_1, x_2, x_3)$. The displacement of a particular piece of material at a given point is a vector given by $\underline{u} = \underline{u}(x_1, x_2, x_3) = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$.

We assume we are working under conditions of plane strain [1]. These conditions state that the displacement of any material particle is confined to the plane, so the x_3-component of the displacement is equal to zero. Also, the derivative of any field quantity with respect to the x_3-coordinate is equal to zero. These conditions are summarized as follows:

$$u_3 = 0 \quad ,$$
$$\frac{\partial}{\partial x_3} f = 0 \quad \text{for any field quantity } f \quad .$$

To maintain such strain, one has to apply opposing stresses in the positive and negative z-direction in order to constrain particle motion to the plane.

Furthermore, assume that this plane is equipped with a static lamination, in other words, assume the plane is divided into sections by lines parallel to the y-axis (interfaces), and assume that these interfaces both remain immovable and alternate periodically in space (see figure 4). The sections have different material properties, and thus, have different elasticity tensors. We denote the elasticity tensor of the $+$ sections as D_+, and the elasticity tensor of the $-$ sections as D_-. We term the strain tensor for the $+$ sides of the interface as e_+, and for the $-$ sections of the interface as e_-. It is thus apparent that the material will generally have different stress on different sections of the interface, $+$ or $-$, termed respectively, p_+ or p_-. On both sides, the material must obey Hooke's law , i.e., the following equations must hold:

$$p_+ = D_+ \cdot \cdot e_+ \quad ,$$
$$p_- = D_- \cdot \cdot e_- \quad .$$

3.3. **Compatibility Conditions on the Interface.** Note: Let $\underline{a}$ and $\underline{b}$ be two arbitrary vectors. In the following calculations, $\underline{ab}$ denotes the dyadic product of vectors $\underline{a}$ and $\underline{b}$ (i.e. $\underline{ab} = \underline{a} \otimes \underline{b}$):

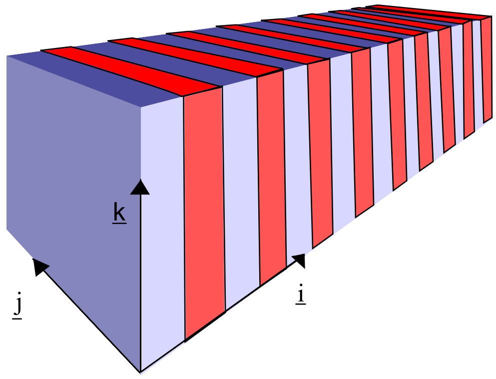


FIGURE 4. An elastic bulk of material with a periodic lamination in space.

$$\underline{a} \otimes \underline{b} = (a_1, a_2, a_3) \otimes (b_1, b_2, b_3) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} . \tag{2}$$

Similarly, we will assume that, in general, if A and B are tensors of arbitrary rank, then AB denotes A⊗B. Also, in the following calculations the symbol $[\cdot]_-^+ := (\cdot)_+ - (\cdot)_-$ will be used to denote the subtraction of quantities on the - side of the interface from quantities on the + side of the interface.

3.4. **Continuity Conditions.** The elastic material previously mentioned has interfaces that separate one type of elastic material from another type. At these interfaces, there are certain continuity conditions that are observed by both the stress and the strain. These conditions relate to the continuity of certain derivatives, namely, the derivatives of the displacement vector in the direction tangent to the interfaces of inhomogeneity. These conditions are as follows [1].

Recall, that in the static case, the material property pattern is not moving, and so at the interfaces we have that:

$$\left[\frac{\partial u_1}{\partial y}\right]_-^+ = 0 \quad ,$$

$$\left[\frac{\partial u_2}{\partial y}\right]_-^+ = 0 \quad .$$

From our previous definition of strain, i.e, equation (1), we see that only the e_{22} component of strain is continuous at the material interfaces. Thus, we have that e_{11} and e_{12} are discontinuous at the interface.

For stress, we have the following equation from [1]:

$$[\tau_{ij}n_j]_-^+ = 0 \quad \text{where} \tag{3}$$

$$n_j = \text{components of the vector normal to the interface.}$$

Thus, we know that as the interfaces are perpendicular to the $\underline{i}$-direction the vector normal to the interfaces is simply the unit vector $\underline{i}$. So, the components of stress τ_{11} and τ_{12} are continuous, while the component τ_{22} is discontinuous.

This results in the following conditions on strain and stress:

$$\begin{aligned} [e\cdot\cdot\underline{ii}]_-^+ &= (e_+ - e_-)\cdot\cdot\underline{ii} \\ &= e_{+11} - e_{-11} \neq 0 \quad , \end{aligned} \tag{4}$$

$$\begin{aligned} \left[e\cdot\cdot(\underline{ij}+\underline{ji})\right]_-^+ &= (e_+ - e_-)\cdot\cdot(\underline{ij}+\underline{ji}) \\ &= 2(e_{+12}) - 2(e_{-12}) \neq 0 \quad , \end{aligned} \tag{5}$$

$$\begin{aligned} \left[e\cdot\cdot\underline{jj}\right]_-^+ &= (e_+ - e_-)\cdot\cdot\underline{jj} \\ &= e_{+22} - e_{-22} = 0 \quad . \end{aligned} \tag{6}$$

and,

$$\begin{aligned} [p\cdot\cdot\underline{ii}]_-^+ &= (p_+ - p_-)\cdot\cdot\underline{ii} \\ &= \tau_{+11} - \tau_{-11} = 0 \quad , \end{aligned} \tag{7}$$

$$\begin{aligned} \left[p\cdot\cdot(\underline{ij}+\underline{ji})\right]_-^+ &= (p_+ - p_-)\cdot\cdot(\underline{ij}+\underline{ji}) \\ &= 2\tau_{+12} - 2\tau_{-12} = 0 \quad , \end{aligned} \tag{8}$$

$$\begin{aligned} \left[p\cdot\cdot\underline{jj}\right]_-^+ &= (p_+ - p_-)\cdot\cdot\underline{jj} \\ &= \tau_{+22} - \tau_{-22} \neq 0 \quad . \end{aligned} \tag{9}$$

Equations (4) - (6) and (7) - (9) represent a kind of duality between stress and strain. We see that if a particular component of strain is continuous, then the corresponding component of stress is discontinuous. Similarly, if a component of strain is discontinuous, then the corresponding component of stress is continuous.

3.5. **Effective (Average) Values of Properties.** Assume that the concentration of the + material in a laminate is given by m_1, and the concentration of the second material in a laminate is given by m_2. These values must add to unity (i.e. $m_1 + m_2 = 1$), as they represent the percentage of materials present. These can be used to represent both the average stress tensor and the average strain tensor through the arithmetic average. These values, the average stress and strain, are denoted by p_0 and e_0 respectively:

$$p_0 = m_1 p_+ + m_2 p_- \quad ,$$
$$e_0 = m_1 e_+ + m_2 e_- \quad .$$

3.6. **Problem Statement.** From the previous discussion of average values, we can also define an average material tensor $D_0 = m_1 D_+ + m_2 D_-$. A valid question to ask is whether or not there exists a linear relationship between the averaged stress and average strain, in the form of a fourth rank effective material tensor. Specifically, we ask if there exists a fourth rank tensor $D_{eff,s}$, that satisfies the following relation:

$$p_0 = D_{eff,s} \cdot \cdot e_0 \quad .$$

The problem addressed in the following sections is the derivation of this effective material tensor, in other words, we will attempt to derive $D_{eff,s}$. It will be seen that $D_{eff,s} \neq D_0$.

3.7. **Derivation of Effective Elasticity Tensor.** To simplify calculations, we use the following abbreviations:

$$a_1 = \underline{ii} \quad , \tag{10}$$
$$a_2 = \left(\underline{ij} + \underline{ji}\right) \quad , \tag{11}$$
$$a_3 = \underline{jj} \quad . \tag{12}$$

To derive the effective tensor, we apply a procedure similar to that introduced in [10]. From compatibility conditions (4), (5), and (6) across the interface, we see that the $\underline{jj}$-component of strain is continuous. Thus, the strain tensor (on both the $+$ and $-$ side of the material interface) can be expressed as a linear combination of e_0, a_1, and a_2. That is, $\exists \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that:

$$e_+ = e_0 + \alpha_1 a_1 + \beta_1 a_2 \quad , \tag{13}$$
$$e_- = e_0 + \alpha_2 a_1 + \beta_2 a_2 \quad . \tag{14}$$

By definition, we know that $e_0 = m_1 e_+ + m_2 e_-$, so by substituting the previous expansions of e_+ and e_- to this expression for e_0 we obtain the following result:

$$\begin{aligned} e_0 &= m_1 e_+ + m_2 e_- \\ &= m_1(e_0 + \alpha_1 a_1 + \beta_1 a_2) + m_2(e_0 + \alpha_2 a_1 + \beta_2 a_2) \\ &= (m_1 e_0 + m_2 e_0) + (m_1 \alpha_1 + m_2 \alpha_2) a_1 + (m_1 \beta_1 + m_2 \beta_2) a_2 \\ &= e_0 + (m_1 \alpha_1 + m_2 \alpha_2) a_1 + (m_1 \beta_1 + m_2 \beta_2) a_2 \quad . \end{aligned}$$

This implies that the following equations hold:

$$m_1 \alpha_1 + m_2 \alpha_2 = 0 \quad ,$$
$$m_1 \beta_1 + m_2 \beta_2 = 0 \quad ,$$

i.e., there exist constants α and β such that:

$$\begin{aligned} \alpha_1 &= m_2\alpha \quad , \\ \alpha_2 &= -m_1\alpha \ , \\ \beta_1 &= m_2\beta \quad , \\ \beta_2 &= -m_1\beta \ . \end{aligned}$$

3.7.1. *Simplification of Terms.* Using the relationships derived in the previous section, we will attempt to simplify some quantities which will be used in the following calculations. The first term we would like to simplify is the expression for the average strain tensor p_0.

$$\begin{aligned} p_0 &= m_1p_+ + m_2p_- \\ &= m_1(D_+ \cdot\cdot e_+) + m_2(D_- \cdot\cdot e_-) \\ &= m_1(D_+ \cdot\cdot(e_0 + \alpha_1 a_1 + \beta_1 a_2)) + m_2(D_- \cdot\cdot(e_0 + \alpha_2 a_1 + \beta_2 a_2)) \\ &= (m_1D_+ + m_2D_-) \cdot\cdot e_0 + (m_1\alpha_1D_+ + m_2\alpha_2D_-) \cdot\cdot a_1 + (m_1\beta_1D_+ + m_2\beta_2D_-) \cdot\cdot a_2 \quad . \end{aligned}$$

We now wish to apply the simplifications for $\alpha_1, \beta_1, \alpha_2$, and β_2 that were found in the previous section.

$$\begin{aligned} p_0 &= D_0 \cdot\cdot e_0 + (m_1(m_2\alpha)D_+ + m_2(-m_1\alpha)D_-) \cdot\cdot a_1 + (m_1(m_2\beta)D_+ + m_2(-m_1\beta)D_-) \cdot\cdot a_2 \\ &= D_0 \cdot\cdot e_0 + m_1m_2(\alpha(D_+ - D_-) \cdot\cdot a_1 + \beta(D_+ - D_-) \cdot\cdot a_2) \quad . \end{aligned}$$

For the remainder of the derivation, we shall denote $D_+ - D_-$ as ΔD. Thus, our final simplified expression for p_0 is as follows:

$$p_0 = D_0 \cdot\cdot e_0 + m_1m_2(\alpha\Delta D \cdot\cdot a_1 + \beta\Delta D \cdot\cdot a_2) \quad . \tag{15}$$

Our goal in the forthcoming sections will be to solve for the values of scalars α and β, these values depending linearly on e_0, and thus, allowing us to find an expression for $D_{eff,s}$.

To determine α and β, we shall use the compatibility conditions (7) - (9) for stress.

We will also simplify the difference between p_+ and p_-.

$$\begin{aligned} p_+ - p_- &= (D_+ \cdot\cdot e_+) - (D_- \cdot\cdot e_-) \\ &= (D_+ \cdot\cdot(e_0 + \alpha_1 a_1 + \beta_1 a_2)) - D_- \cdot\cdot(e_0 + \alpha_2 a_1 + \beta_2 a_2)) \\ &= (D_+ - D_-) \cdot\cdot e_0 + (\alpha_1D_+ - \alpha_2D_-) \cdot\cdot a_1 + (\beta_1D_+ - \beta_2D_-) \cdot\cdot a_2 \quad . \end{aligned}$$

Applying both the symbol ΔD for $D_+ - D_-$ and the simplifications for $\alpha_1, \beta_1, \alpha_2$, and β_2 found in the previous section, we have the following:

$$\begin{aligned} p_+ - p_- &= \Delta D \cdot \cdot e_0 + ((m_2 \alpha D_+) - (-m_1 \alpha D_-)) \cdot \cdot a_1 + ((m_2 \beta D_+) - (-m_1 \beta D_-)) \cdot \cdot a_2 \\ &= \Delta D \cdot \cdot e_0 + \alpha (m_2 \; D_+ + m_1 D_-) \cdot \cdot a_1 + \beta (m_2 D_+ + m_1 D_-) \cdot \cdot a_2 \quad . \end{aligned}$$

As a further means of simplification, we shall define $\overline{D}$ to be equal to the tensor $m_2 \; D_+ + m_1 D_-$. So for our final simplification of $p_+ - p_-$, we have the following:

$$p_+ - p_- \quad = \quad \Delta D \cdot \cdot e_0 + \alpha \overline{D} \cdot \cdot a_1 + \beta \overline{D} \cdot \cdot a_2 \quad .$$

3.7.2. *Applying the Compatibility Conditions for Stress.* We are now at a point where it is possible to apply compatibility conditions (7) and (8), and solve for α and β. Recall that these conditions are as follows:

$$\begin{aligned} (p_+ - p_-) \cdot \cdot a_1 &= 0 \quad , \\ (p_+ - p_-) \cdot \cdot a_2 &= 0 \quad . \end{aligned}$$

Using the simplified form of $p_+ - p_-$, we see that it is now just a matter of substituting the simplification for the compatibility conditions (7) and (8).

For (7), we have

$$\begin{aligned} (p_+ - p_-) \cdot \cdot a_1 &= (\Delta D \cdot \cdot e_0 + \alpha \overline{D} \cdot \cdot a_1 + \beta \overline{D} \cdot \cdot a_2) \cdot \cdot a_1 \\ &= a_1 \cdot \cdot \Delta D \cdot \cdot e_0 + \alpha \left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_1 \right) + \beta \left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_2 \right) = 0 \quad . \end{aligned}$$

For (8), we have

$$\begin{aligned} (p_+ - p_-) \cdot \cdot a_2 &= (\Delta D \cdot \cdot e_0 + \alpha \overline{D} \cdot \cdot a_1 + \beta \overline{D} \cdot \cdot a_2) \cdot \cdot a_2 \\ &= a_2 \cdot \cdot \Delta D \cdot \cdot e_0 + \alpha \left(a_2 \cdot \cdot \overline{D} \cdot \cdot a_1 \right) + \beta \left(a_2 \cdot \cdot \overline{D} \cdot \cdot a_2 \right) = 0 \quad . \end{aligned}$$

Thus, from the compatibility conditions for stress, we get the following linear system of equations:

$$\begin{aligned} a_1 \cdot \cdot \Delta D \cdot \cdot e_0 + \alpha \left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_1 \right) + \beta \left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_2 \right) = 0 \quad , \\ a_2 \cdot \cdot \Delta D \cdot \cdot e_0 + \alpha \left(a_2 \cdot \cdot \overline{D} \cdot \cdot a_1 \right) + \beta \left(a_2 \cdot \cdot \overline{D} \cdot \cdot a_2 \right) = 0 \quad . \end{aligned}$$

We notice that the quantities $(a_1 \cdot \cdot \Delta D \cdot \cdot e_0)$, $(a_2 \cdot \cdot \Delta D \cdot \cdot e_0)$, and $(a_s \cdot \cdot \overline{D} \cdot \cdot a_t$ where s, t are equal to 1 or 2) depend on the basis tensors (a_1, a_2), the values of the material tensors on both sides of the interface (D_+, D_-), and the average strain (e_0). To condense the linear system, we define the following:

$$P = (a_1 \cdot \cdot \Delta D \cdot \cdot e_0) \quad , \tag{16}$$

$$Q = (a_2 \cdot \cdot \Delta D \cdot \cdot e_0) \quad , \tag{17}$$

$$\overline{D}_{st} = a_s \cdot \cdot \overline{D} \cdot \cdot a_t \quad \text{where } s, t \text{ are equal to 1 or 2.} \tag{18}$$

Thus, our linear system takes the following form:

$$\begin{array}{c} P + \alpha \overline{D}_{11} + \beta \overline{D}_{12} = 0 \quad , \\ Q + \alpha \overline{D}_{21} + \beta \overline{D}_{22} = 0 \quad . \end{array}$$

These two simultaneous equations are equivalent to the matrix equation

$$A\underline{x} = \underline{b} \quad \text{where,}$$
$$A := \begin{pmatrix} \overline{D}_{11} & \overline{D}_{12} \\ \overline{D}_{21} & \overline{D}_{22} \end{pmatrix}, \ \underline{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \text{ and } \underline{b} = \begin{pmatrix} -P \\ -Q \end{pmatrix}.$$

For this system to have a solution, we require that the determinant of A is not equal to zero (i.e. $det(A) = \overline{D}_{11}\overline{D}_{22} - \overline{D}_{21}\overline{D}_{12} \neq 0$). If the material is isotropic, then this is automatically the case, as $\overline{D}_{12} = \overline{D}_{21} = 0$, and thus, $det(A) = \overline{D}_{11}\overline{D}_{22}$. We are assuming that the material has a stiffness which is non-zero, and so, $det(A) \neq 0$. Thus, we have the following unique solution for α and β:

$$\begin{array}{c} \alpha = \frac{1}{detA}(\overline{D}_{12}Q - \overline{D}_{22}P) \quad , \\ \beta = \frac{1}{detA}(\overline{D}_{21}P - \overline{D}_{11}Q) \quad . \end{array}$$

It is now possible to take the simplified expression for average stress (derived in previous section), and substitute the values derived for α and β into the expression for p_0.

$$\begin{aligned} p_0 &= D_0 \cdot \cdot e_0 + m_1 m_2 (\alpha \Delta D \cdot \cdot a_1 + \beta \Delta D \cdot \cdot a_2) \\ &= D_0 \cdot \cdot e_0 + \frac{m_1 m_2}{detA}(\overline{D}_{12}Q - \overline{D}_{22}P)\Delta D \cdot \cdot a_1 + \frac{m_1 m_2}{detA}(\overline{D}_{21}P - \overline{D}_{11}Q)\Delta D \cdot \cdot a_2 \quad . \end{aligned}$$

So, a simplified equation for p_0 is given by the following expression:

$$\begin{aligned} p_0 = \ & D_0 \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{detA}(\overline{D}_{12}(\Delta D \cdot \cdot a_1)Q + \overline{D}_{21}(\Delta D \cdot \cdot a_2)P) \\ & - \frac{m_1 m_2}{detA}(\overline{D}_{22}(\Delta D \cdot \cdot a_1)P + \overline{D}_{11}(\Delta D \cdot \cdot a_2)Q) \quad . \end{aligned}$$

Next, recall the definitions of P and Q as $(a_1 \cdot \cdot \Delta D \cdot \cdot e_0)$ and $(a_2 \cdot \cdot \Delta D \cdot \cdot e_0)$, respectively. Plug these values into the above expression for p_0; this gives us the following:

$$\begin{aligned} p_0 &= D_0 \cdot \cdot e_0 \\ &+ \frac{m_1 m_2}{detA}(\overline{D}_{12}(\Delta D \cdot \cdot a_1)(a_2 \cdot \cdot \Delta D \cdot \cdot e_0) + \overline{D}_{21}(\Delta D \cdot \cdot a_2)(a_1 \cdot \cdot \Delta D \cdot \cdot e_0)) \\ &- \frac{m_1 m_2}{detA}(\overline{D}_{11}(\Delta D \cdot \cdot a_2)(a_2 \cdot \cdot \Delta D \cdot \cdot e_0) + \overline{D}_{22}(\Delta D \cdot \cdot a_1)(a_1 \cdot \cdot \Delta D \cdot \cdot e_0)) \quad . \end{aligned}$$

Now, we can clearly see that p_0 is a linear function of e_0:

$$\begin{aligned} p_0 &= [D_0 + \frac{m_1 m_2}{detA}(\overline{D}_{12}(\Delta D \cdot \cdot a_1)(a_2 \cdot \cdot \Delta D) + \overline{D}_{21}(\Delta D \cdot \cdot a_2)(a_1 \cdot \cdot \Delta D)) \\ &- \frac{m_1 m_2}{detA}(\overline{D}_{11}(\Delta D \cdot \cdot a_2)(a_2 \cdot \cdot \Delta D) + \overline{D}_{22}(\Delta D \cdot \cdot a_1)(a_1 \cdot \cdot \Delta D))] \cdot \cdot e_0 \quad . \\ &= D_{eff,s} \cdot \cdot e_0 \end{aligned}$$

Thus, we have found a fourth rank tensor $D_{eff,s}$ that linearly relates the average stress p_0 to the average strain e_0, i.e.,

$$\begin{aligned} D_{eff,s} &= D_0 + \frac{m_1 m_2}{detA}\left[\overline{D}_{12}(\Delta D \cdot \cdot a_1) \otimes (a_2 \cdot \cdot \Delta D) + \overline{D}_{21}(\Delta D \cdot \cdot a_2) \otimes (a_1 \cdot \cdot \Delta D)\right] \\ &- \frac{m_1 m_2}{detA}\left[\overline{D}_{11}(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D) + \overline{D}_{22}(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D)\right] \quad . \end{aligned}$$

If we assume that D_+ and D_- are both isotropic, then $\overline{D}_{12} = \overline{D}_{21} = 0$, and so $det(A) = \overline{D}_{11}\overline{D}_{22}$, which means that the effective tensor takes the following form:

$$D_{eff,s} = D_0 - m_1 m_2 \left[\frac{(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D)}{\overline{D}_{22}} + \frac{(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D)}{\overline{D}_{11}}\right]$$

It is important to note that the above quantities $\overline{D}_{11}$ and $\overline{D}_{22}$ are simply components of the tensor $\overline{D}$, and as such, are linear combinations of components from D_+ and D_-. Because we are assuming that the materials (+ and −) are both elastic, they both have non-zero stiffness (i.e. $D_+ \neq 0$, $D_- \neq 0$). As a result, neither $\overline{D}_{11}$ or $\overline{D}_{22}$ are equal to zero.

4. Dynamic Material Laminate

4.1. **Introduction.** In the previous section, we derived the effective material tensor $D_{eff,s}$ for a static elastic material with a static (unchanging in time) laminate. We now wish to consider a laminate which is dynamic, in a material whose constituents are elastic. We shall attempt to derive the effective tensor $D_{eff,d}$ for a material that is equipped with a lamination that changes in time.

4.2. **Physical Setup of the System.** As before, consider a bulk of elastic material equipped with an alternating periodic lamination in space-time similar to the purely spatial lamination described in section 3.2. In each section of material the stiffness tensor (D) and the mass density (ρ) are different. Furthermore, assume that this lamination is dynamic (not static as in section 3.2), i.e., assume that the laminar property pattern is brought to motion at a velocity V, through the use of some external agent. In other words, assume that the sections of material with density ρ_+ and D_+ are changed into sections with parameters ρ_- and D_- and likewise, sections of the material with parameters ρ_- and D_- are changed into sections with parameters ρ_+ and D_+. It is important to note that there is no actual material motion; what is moving is the property pattern alone.

We will study the propagation of dynamic disturbances through the material environment changing its properties in space and time. Thus, the displacement of the material constituents produced by the travelling dynamic disturbances becomes a function of position as well as time, i.e., $\underline{u} = \underline{u}(x_1, x_2, x_3, t) = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$. The velocity of the constituents is then given by the vector $\underline{\dot{u}} = \underline{\dot{u}}\,(x_1, x_2, x_3, t) = \frac{\partial u_1}{\partial t}\underline{i} + \frac{\partial u_2}{\partial t}\underline{j} + \frac{\partial u_3}{\partial t}\underline{k}$, and furthermore, if the mass density is ρ, then the momentum of the constituents is given by $\underline{\sigma} = \underline{\sigma}(x_1, x_2, x_3) = \sigma_1\underline{i} + \sigma_2\underline{j} + \sigma_3\underline{k} = \rho\underline{\dot{u}} = \rho\left(\frac{\partial u_1}{\partial t}\underline{i} + \frac{\partial u_2}{\partial t}\underline{j} + \frac{\partial u_3}{\partial t}\underline{k}\right)$. As before, we notice that the stress and strain will be different in the plus and minus parts of the material, so we define p_+, p_-, e_+, and e_- respectively. Lastly, we realize that the momentum vectors on each side of the interface will also be different due to the varying inertial properties, and so we define $\underline{\sigma_+}$, and $\underline{\sigma_-}$ to be the momenta in the respective material sections, + and -.

4.3. **Compatibility Conditions on the Interface.** Now that we are considering a moving interface, there are two sets of compatibility conditions which must be observed on the interface. The first set of conditions are the kinematical compatibility conditions.

4.3.1. *Kinematical Compatibility Conditions.* For the kinematic compatibility conditions, we have the following equation from [1]:

$$[u_i n_j]_-^+ \quad = \quad -V\left[\frac{\partial u_i}{\partial x_j}\right]_-^+ \quad \text{where } i = 1 \text{ or } 2, \text{ and } j = 1 \text{ or } 2, \text{ and,}$$

$$n_j \quad = \quad \text{components of a unit normal to the interface moving at a velocity V.}$$

This gives us the following kinematical compatibility conditions:

$$\left[n_2\dot{u}_1 + V\frac{\partial u_1}{\partial x_2}\right]_-^+ \quad = \quad 0 \quad , \tag{19}$$

$$\left[n_2\dot{u}_2 + V\frac{\partial u_2}{\partial x_2}\right]_-^+ \quad = \quad 0 \quad , \tag{20}$$

$$\left[n_1\dot{u}_1 + V\frac{\partial u_1}{\partial x_1}\right]_-^+ = 0 \quad , \tag{21}$$

$$\left[n_1\dot{u}_2 + V\frac{\partial u_2}{\partial x_1}\right]_-^+ = 0 \quad . \tag{22}$$

4.3.2. *Another Method of Deriving KCC's.* Let $f(\underline{x}, t)$ denote a field quantity which may be discontinuous across the interface. Then

$$\begin{aligned}\nabla f \times \underline{N} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{\tau} \\ f_{x_1} & f_{x_2} & \dot{f} \\ n_1 & n_2 & -V \end{vmatrix} \\ &= -(f_{x_2}V + \dot{f}n_2)\underline{i} + (f_{x_1}V + \dot{f}n_1)\underline{j} + (f_{x_1}n_2 - f_{x_2}n_1)\underline{\tau} \quad .\end{aligned}$$

This cross product produces a vector which is perpendicular to the interface represented as a surface in space-time. As u_1 and u_2 are both field quantities, we obtain the following four kinematical continuity conditions:

$$\left[n_2\dot{u}_1 + V\frac{\partial u_1}{\partial x_2}\right]_-^+ = 0 \quad ,$$

$$\left[n_2\dot{u}_2 + V\frac{\partial u_2}{\partial x_2}\right]_-^+ = 0 \quad ,$$

$$\left[n_1\dot{u}_1 + V\frac{\partial u_1}{\partial x_1}\right]_-^+ = 0 \quad ,$$

$$\left[n_1\dot{u}_2 + V\frac{\partial u_2}{\partial x_1}\right]_-^+ = 0 \quad .$$

which match the kinematical continuity conditions (19)-(22).

4.3.3. *Dynamical Conditions.* For the dynamic compatibility conditions, we have the following equation from [1]:

$$[\tau_{ij}n_j]_-^+ = -V\,[\rho\dot{u}_i]_-^+ \quad .$$

This gives us the following dynamical compatibility conditions:

$$[\tau_{11}n_1 + \tau_{12}n_2 + V\rho\dot{u}_1]_-^+ = 0 \quad , \tag{23}$$

$$[\tau_{21}n_1 + \tau_{22}n_2 + V\rho\dot{u}_2]_-^+ = 0 \quad . \tag{24}$$

4.4. **Transformation of Kinematical Compatibility Conditions.** From the above discussion, we know that equations (19) - (22) must be obeyed on the interface; however, these can be greatly simplified. Previously, we assumed that the interface remains perpendicular to the x_1-axis through its motion, that is, n_1=1 and n_2=0. The kinematical compatibility equations then become:

$$\left[\frac{\partial u_1}{\partial x_2}\right]_-^+ = 0 \quad , \tag{25}$$

$$\left[\frac{\partial u_2}{\partial x_2}\right]_-^+ = 0 \quad , \tag{26}$$

$$\left[\dot{u}_1 + V\frac{\partial u_1}{\partial x_1}\right]_-^+ = 0 \quad , \tag{27}$$

$$\left[\dot{u}_2 + V\frac{\partial u_2}{\partial x_1}\right]_-^+ = 0 \quad . \tag{28}$$

The final simplification to be made to the kinematical compatibility conditions will be to apply the definition of strain (i.e., equation (1)) while combining equations (25) and (28). This gives us the following:

$$\begin{aligned}\left[\frac{\partial u_1}{\partial x_2}\right]_-^+ &= \left[2e_{12} - \frac{\partial u_2}{\partial x_1}\right]_-^+ \\ &= \left[2e_{12} + \frac{\dot{u_2}}{V}\right]_-^+ \\ &= \left[e\cdot\cdot a_2 + \frac{1}{V}\underline{\dot{u}}\cdot\underline{j}\right]_-^+ \\ &= 0 \quad ,\end{aligned} \tag{29}$$

$$\begin{aligned}\left[\dot{u}_1 + V\frac{\partial u_1}{\partial x_1}\right]_-^+ &= \left[e_{11} + \frac{\dot{u}_1}{V}\right]_-^+ \\ &= \left[e\cdot\cdot a_1 + \frac{1}{V}\underline{\dot{u}}\cdot\underline{i}\right]_-^+ \\ &= 0 \quad ,\end{aligned} \tag{30}$$

$$\begin{aligned}\left[\frac{\partial u_2}{\partial x_2}\right]_-^+ &= [e_{22}]_-^+ \\ &= [e\cdot\cdot a_3]_-^+ \\ &= 0 \quad .\end{aligned} \tag{31}$$

Due to the assumption that $n_1 = 1$ and $n_2 = 0$, the dynamical compatibility equations (23) and (24) become as follows:

$$\begin{aligned} [\tau_{11} + V\rho\dot{u}_1]_-^+ &= [p\cdot\cdot a_1 + V\rho\underline{\dot{u}}\cdot\underline{i}]_-^+ \\ &= 0 \quad , \end{aligned} \tag{32}$$

$$\begin{aligned} [\tau_{21} + V\rho\dot{u}_2]_-^+ &= \left[\frac{p\cdot\cdot a_2}{2} + V\rho\underline{\dot{u}}\cdot\underline{j}\right]_-^+ \\ &= 0 \quad . \end{aligned} \tag{33}$$

4.5. **Effective (Average) Values of Properties.** Assume that the concentration of the $+$ material is given by m_1, and the concentration of the $-$ material is given by m_2; this time, however, m_1 and m_2 denote concentrations in space-time, i.e., the portions of a spatial-temporal period occupied by the relevant material constituents. The average stress and strain are again defined as $p_0 = m_1 p_+ + m_2 p_-$ and $e_0 = m_1 e_+ + m_2 e_-$. The only new quantity we must introduce is the average momentum vector $\underline{\sigma_0}$. This is done naturally, i.e.:

$$\begin{aligned} \underline{\sigma_0} &= m_1\rho_+\underline{\dot{u}_+} + m_2\rho_-\underline{\dot{u}_-} \\ &= (m_1\rho_+\dot{u}_{1+} + m_2\rho_-\dot{u}_{1-})\underline{i} + (m_1\rho_+\dot{u}_{2+} + m_2\rho_-\dot{u}_{2-})\underline{j} \quad . \end{aligned} \tag{34}$$

4.6. **Problem Statement.** As before, a valid question to ask is whether or not there exists a linear relationship between the averaged stress and average strain in the form of a fourth rank effective material tensor. Specifically, we ask if there exists a fourth rank tensor $D_{eff,d}$, that satisfies the following relation:

$$p_0 = D_{eff,d}\cdot\cdot e_0 \quad .$$

The problem addressed in the following sections is the derivation of this effective material tensor, in other words, we will attempt to derive $D_{eff,d}$. It will be seen that the answer is more complex than in the static case. Instead of deriving an effective material tensor that shows a linear relationship between p_0 and e_0, we show that there is a linear relationship between p_0, e_0, and $\underline{\dot{u}_0}$. Later examination of the Euler equations (equations of motion) produced by an averaged Lagrangian will show that the relationship between these quantities is even more complex than originally expected.

4.7. **Relationship between Effective Stress and Strain in the Dynamic Case.** From kinematical compatibility conditions (29),(30), and (31) across the interface, we see that only the $\underline{jj}$-component of strain is continuous. Thus, the strain tensor (in both the $+$ and $-$ sections of the material) can be expressed as a linear combination of e_0, a_1, and a_2. That is, $\exists\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that (c.f. (13), (14))

$$\begin{aligned} e_+ &= e_0 + \alpha_1 a_1 + \beta_1 a_2 \quad , \\ e_- &= e_0 + \alpha_2 a_1 + \beta_2 a_2 \quad . \end{aligned}$$

Furthermore, conditions (29),(30), and (31) tell us that the velocity $\underline{\dot{u}_0}$ is also discontinuous across the interface, and so we have that the velocity vector (in both the + and − sections of the material) can be expressed as a linear combination of $\underline{\dot{u}_0}$, $\underline{i}$, and $\underline{j}$. That is, $\exists \gamma_1, \delta_1, \gamma_2, \delta_2 \in \mathbb{R}$ such that,

$$\begin{aligned} \underline{\dot{u}_+} &= \underline{\dot{u}_0} + \gamma_1 \underline{i} + \delta_1 \underline{j} \quad , \\ \underline{\dot{u}_-} &= \underline{\dot{u}_0} + \gamma_2 \underline{i} + \delta_2 \underline{j} \quad . \end{aligned}$$

By definition, we know that $e_0 = m_1 e_+ + m_2 e_-$ and that $\underline{\dot{u}_0} = m_1 \underline{\dot{u}_+} + m_2 \underline{\dot{u}_-}$, so by substituting the previous expansions of e_+, e_-, $\underline{\dot{u}_+}$, and $\underline{\dot{u}_-}$ to these expressions for e_0 and $\underline{\dot{u}_0}$, we obtain:

$$\begin{aligned} e_0 &= m_1 e_+ + m_2 e_- \\ &= m_1(e_0 + \alpha_1 a_1 + \beta_1 a_2) + m_2(e_0 + \alpha_2 a_1 + \beta_2 a_2) \\ &= (m_1 e_0 + m_2 e_0) + (m_1\alpha_1 + m_2\alpha_2)a_1 + (m_1\beta_1 + m_2\beta_2)a_2 \\ &= e_0 + (m_1\alpha_1 + m_2\alpha_2)a_1 + (m_1\beta_1 + m_2\beta_2)a_2 \quad , \end{aligned}$$

and

$$\begin{aligned} \underline{\dot{u}_0} &= m_1 \underline{\dot{u}_+} + +m_2 \underline{\dot{u}_-} \\ &= m_1(\underline{\dot{u}_0} + \gamma_1 \underline{i} + \delta_1 \underline{j}) + m_2(\underline{\dot{u}_0} + \gamma_2 \underline{i} + \delta_2 \underline{j}) \\ &= (m_1 \underline{\dot{u}_0} + m_2 \underline{\dot{u}_0}) + (m_1\gamma_1 + m_2\gamma_2)\underline{i} + (m_1\delta_1 + m_2\delta_2)\underline{j} \\ &= \underline{\dot{u}_0} + (m_1\gamma_1 + m_2\gamma_2)\underline{i} + (m_1\delta_1 + m_2\delta_2)\underline{j} \quad . \end{aligned}$$

As before, this implies:

$$\begin{aligned} m_1\alpha_1 + m_2\alpha_2 &= 0 \quad , \\ m_1\beta_1 + m_2\beta_2 &= 0 \quad , \\ m_1\gamma_1 + m_2\gamma_2 &= 0 \quad , \\ m_1\delta_1 + m_2\delta_2 &= 0 \quad . \end{aligned}$$

i.e., there exist constants α, β, γ, and δ such that

$$\begin{aligned} \alpha_1 &= m_2\alpha \quad , \\ \alpha_2 &= -m_1\alpha \; , \\ \beta_1 &= m_2\beta \quad , \\ \beta_2 &= -m_1\beta \; , \\ \gamma_1 &= m_2\gamma \quad , \\ \gamma_2 &= -m_1\gamma \; , \\ \delta_1 &= m_2\delta \quad , \\ \delta_2 &= -m_1\delta \; . \end{aligned}$$

4.7.1. *Simplification of Terms.* Using the relationships derived in the previous section, we will attempt to simplify some quantities which will be used in the following calculations. This will allow us to apply the continuity conditions in a clearer manner. First, recall the simplified expression for average stress (15), originally found in section 3.7.1.

$$p_0 = D_0 \cdot \cdot e_0 + m_1 m_2(\alpha \Delta D \cdot \cdot a_1 + \beta \Delta D \cdot \cdot a_2) \quad ,$$

As before, our goal in the forthcoming sections will be to solve for the values of the scalars α and β using conditions (29)-(33).

We will also simplify the difference between the stresses at the interface, i.e., p_+ and p_-. Recall from 3.7.1 that:

$$p_+ - p_- \quad = \quad \Delta D \cdot \cdot e_0 + \alpha \overline{D} \cdot \cdot a_1 + \beta \overline{D} \cdot \cdot a_2 \quad .$$

Next, we wish to simplify the difference between the strains at the interface, i.e., e_+ and e_-:

$$\begin{aligned} e_+ - e_- &= (e_0 + \alpha_1 a_1 + \beta_1 a_2) - (e_0 + \alpha_2 a_1 + \beta_2 a_2) \\ &= (e_0 + m_2 \alpha a_1 + m_2 \beta a_2) - (e_0 - m_1 \alpha a_1 - m_1 \beta a_2) \\ &= \alpha(m_1 a_1 + m_2 a_1) + \beta(m_1 a_2 + m_2 a_2) \quad . \end{aligned}$$

So, our simplification for $e_+ - e_-$ is as follows:

$$e_+ - e_- \quad = \quad \alpha a_1 + \beta a_2 \quad .$$

Now, we simplify the difference between the two velocities $\underline{\dot{u}_+}$ and $\underline{\dot{u}_-}$.

$$\begin{aligned} \underline{\dot{u}_+} - \underline{\dot{u}_-} &= (\underline{\dot{u}_0} + \gamma_1 \underline{i} + \delta_1 \underline{j}) - (\underline{\dot{u}_0} + \gamma_2 \underline{i} + \delta_2 \underline{j}) \\ &= (\underline{\dot{u}_0} + m_2 \gamma \underline{i} + m_2 \delta \underline{j}) - (\dot{u}_0 - m_1 \gamma \underline{i} - m_1 \delta \underline{j}) \\ &= \gamma \underline{i} + \delta \underline{j} \quad . \end{aligned}$$

We shall also simplify the difference between the respective momenta at the interface, i.e., $\underline{\sigma_+}$ and $\underline{\sigma_-}$:

$$\begin{aligned} \underline{\sigma_+} - \underline{\sigma_-} &= \rho_+ \underline{\dot{u}_+} - \rho_- \underline{\dot{u}_-} \\ &= \rho_+(\underline{\dot{u}_0} + \gamma_1 \underline{i} + \delta_1 \underline{j}) - \rho_-(\underline{\dot{u}_0} + \gamma_2 \underline{i} + \delta_2 \underline{j}) \\ &= (\rho_+ - \rho_-)\underline{\dot{u}_0} + (\rho_+ \gamma_1 - \rho_- \gamma_2)\underline{i} + (\rho_+ \delta_1 - \rho_- \delta_2)\underline{j} \\ &= (\rho_+ - \rho_-)\underline{\dot{u}_0} + \gamma(m_2 \rho_+ + m_1 \rho_-)\underline{i} + \delta(m_2 \rho_+ + m_1 \rho_-)\underline{j} \quad . \end{aligned}$$

We shall define $\overline{\rho} = m_2 \rho_+ + m_1 \rho_-$, and also $\Delta \rho = \rho_+ - \rho_-$. Thus, for our final simplification for $\underline{\sigma_+} - \underline{\sigma_-}$, we have the following:

$$\underline{\sigma_+} - \underline{\sigma_-} = (\Delta\rho)\,\underline{\dot{u}_0} + (\gamma\overline{\rho})\,\underline{i} + (\delta\overline{\rho})\,\underline{j} \quad .$$

And finally, we simplify the expression for average momentum $\underline{\sigma_0}$. Recall the expression (34) :

$$\begin{aligned}
\underline{\sigma_0} &= m_1\rho_+\underline{\dot{u}_+} + m_2\rho_-\underline{\dot{u}_-} \\
&= m_1\rho_+\left(\underline{\dot{u}_0} + \gamma_1\underline{i} + \delta_1\underline{j}\right) + m_2\rho_-\left(\underline{\dot{u}_0} + \gamma_2\underline{i} + \delta_2\underline{j}\right) \\
&= (m_1\rho_+ + m_2\rho_-)\,\underline{\dot{u}_0} + (m_1\rho_+\gamma_1 + m_2\rho_-\gamma_2)\,\underline{i} + (m_1\rho_+\delta_1 + m_2\rho_-\delta_2)\,\underline{j} \\
&= (m_1\rho_+ + m_2\rho_-)\,\underline{\dot{u}_0} + m_1m_2\,(\rho_+ - \rho_-)\,\underline{i} + m_1m_2\delta\,(\rho_+ - \rho_-)\,\underline{j} \\
&= \rho_0\underline{\dot{u}_0} + m_1m_2\left(\gamma\Delta\rho\underline{i} + \delta\Delta\rho\underline{j}\right) \qquad (35)
\end{aligned}$$

4.7.2. *Applying the Compatibility Conditions for Strain.* We can now apply compatibility conditions (29), (30), (32), and (33) to solve for α, β, γ, and δ. Recall that (29) and (30) state the following.

$$\begin{aligned}
(e_+ - e_-)\cdot\cdot a_2 + \frac{1}{V}\left(\underline{\dot{u}_+} - \underline{\dot{u}_-}\right)\cdot\underline{j} &= 0 \quad , \\
(e_+ - e_-)\cdot\cdot a_1 + \frac{1}{V}\left(\underline{\dot{u}_+} - \underline{\dot{u}_-}\right)\cdot\underline{i} &= 0 \quad .
\end{aligned}$$

Using the simplified form of $e_+ - e_-$ and $\underline{\dot{u}_+} - \underline{\dot{u}_-}$, we see that it is now just a matter of substituting these expressions into the equations (29) and (30).

For (29), we have

$$\begin{aligned}
(e_+ - e_-)\cdot\cdot a_2 + \frac{1}{V}\left(\underline{\dot{u}_+} - \underline{\dot{u}_-}\right)\cdot\underline{j} &= (\alpha a_1 + \beta a_2)\cdot\cdot a_2 + \frac{1}{V}\left(\gamma\underline{i} + \delta\underline{j}\right)\cdot\underline{j} \\
&= 2\beta + \frac{\delta}{V} \\
&= 0 \quad .
\end{aligned}$$

For (30), we have

$$\begin{aligned}
(e_+ - e_-)\cdot\cdot a_1 + \frac{1}{V}\left(\underline{\dot{u}_+} - \underline{\dot{u}_-}\right)\cdot\underline{i} &= (\alpha a_1 + \beta a_2)\cdot\cdot a_1 + \frac{1}{V}\left(\gamma\underline{i} + \delta\underline{j}\right)\cdot\underline{i} \\
&= \alpha + \frac{\gamma}{V} \\
&= 0 \quad .
\end{aligned}$$

Thus, conditions (29) and (30) give the expressions for γ and δ in terms of α and β, i.e.,

$$
\begin{aligned}
\gamma &= -V\alpha \quad , \\
\delta &= -2V\beta \quad .
\end{aligned}
$$

4.7.3. *Applying Compatibility Conditions for Stress.* It is now possible to apply the compatibility conditions (32) and (33), and solve for α and β. These conditions are as follows:

$$
\begin{aligned}
(p_+ - p_-) \cdot \cdot a_1 + V\left(\underline{\sigma_+} - \underline{\sigma_-}\right) \cdot \underline{i} &= 0 \quad , \\
\frac{1}{2}(p_+ - p_-) \cdot \cdot a_2 + V\left(\underline{\sigma_+} - \underline{\sigma_-}\right) \cdot \underline{j} &= 0 \quad .
\end{aligned}
$$

For (32), we have

$$
\begin{aligned}
(p_+ - p_-) \cdot \cdot a_1 + V\left(\underline{\sigma_+} - \underline{\sigma_-}\right) \cdot \underline{i} &= \left(\Delta D \cdot \cdot e_0 + \alpha \overline{D} \cdot \cdot a_1 + \beta \overline{D} \cdot \cdot a_2\right) \cdot \cdot a_1 \\
&\quad + V\left(\Delta\rho \underline{\dot{u}_0} + \gamma\overline{\rho}\underline{i} + \delta\overline{\rho}\underline{j}\right) \cdot \underline{i} \\
&= a_1 \cdot \cdot \Delta D \cdot \cdot e_0 + \alpha\left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_1\right) + \beta\left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_2\right) \\
&\quad + V\Delta\rho\underline{\dot{u}_0} \cdot \underline{i} + \gamma V\overline{\rho} \\
&= a_1 \cdot \cdot \Delta D \cdot \cdot e_0 + \alpha\left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_1\right) + \beta\left(a_1 \cdot \cdot \overline{D} \cdot \cdot a_2\right) \\
&\quad + V\Delta\rho\underline{\dot{u}_0} \cdot \underline{i} - V^2\alpha\overline{\rho} \\
&= 0 \quad .
\end{aligned}
$$

For (33), we have

$$
\begin{aligned}
(p_+ - p_-)\cdot\cdot a_2 + 2V\left(\underline{\sigma_+} - \underline{\sigma_-}\right)\cdot \underline{j} &= \left(\Delta D\cdot\cdot e_0 + \alpha\overline{D}\cdot\cdot a_1 + \beta\overline{D}\cdot\cdot a_2\right)\cdot\cdot a_2 \\
&\quad +2V\left(\Delta\rho\underline{\dot{u}_0} + \gamma\overline{\rho}\underline{i} + \delta\overline{\rho}\underline{j}\right)\cdot\underline{j} \\
&= a_2\cdot\cdot\Delta D\cdot\cdot e_0 + \alpha a_2\cdot\cdot\overline{D}\cdot\cdot a_1 + \beta a_2\cdot\cdot\overline{D}\cdot\cdot a_2 \\
&\quad +2V\Delta\rho\underline{\dot{u}_0}\cdot\underline{j} + 2\delta V\overline{\rho} \\
&= a_2\cdot\cdot\Delta D\cdot\cdot e_0 + \alpha a_2\cdot\cdot\overline{D}\cdot\cdot a_1 + \beta a_2\cdot\cdot\overline{D}\cdot\cdot a_2 \\
&\quad +2V\Delta\rho\underline{\dot{u}_0}\cdot\underline{j} - 4V^2\beta\overline{\rho} \\
&= 0 \quad .
\end{aligned}
$$

So, we have the following linear system for α and β:

$$
\begin{aligned}
-[(a_1\cdot\cdot\Delta D\cdot\cdot e_0) + V\Delta\rho(\underline{\dot{u}_0}\cdot\underline{i})] &= \alpha\left(a_1\cdot\cdot\overline{D}\cdot\cdot a_1 - V^2\overline{\rho}\right) + \beta(a_1\cdot\cdot\overline{D}\cdot\cdot a_2) \quad , \\
-[(a_2\cdot\cdot\Delta D\cdot\cdot e_0) + 2V\Delta\rho(\underline{\dot{u}_0}\cdot\underline{j})] &= \alpha\left(a_2\cdot\cdot\overline{D}\cdot\cdot a_1\right) + \beta(a_2\cdot\cdot\overline{D}\cdot\cdot a_2 - 4V^2\overline{\rho}) \quad .
\end{aligned}
$$

Using the notation from section 3.7.1, the system takes the following form

$$
\begin{aligned}
\alpha\left(\overline{D}_{11} - V^2\overline{\rho}\right) + \beta\overline{D}_{12} &= -\left[P + V\Delta\rho(\underline{\dot{u}_0}\cdot\underline{i})\right] \quad , \\
\alpha\overline{D}_{21} + \beta\left(\overline{D}_{22} - 4V^2\overline{\rho}\right) &= -\left[Q + 2V\Delta\rho(\underline{\dot{u}_0}\cdot\underline{j})\right] \quad .
\end{aligned}
$$

These two simultaneous equations are equivalent to the matrix equation

$A\underline{x} = \underline{b}$, where

$$
A := \begin{pmatrix} \overline{D}_{11} - V^2\overline{\rho} & \overline{D}_{12} \\ \overline{D}_{21} & \overline{D}_{22} - 4V^2\overline{\rho} \end{pmatrix}, \ \underline{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \text{ and}
$$

$$
\underline{b} = \begin{pmatrix} -\left[P + V\Delta\rho(\underline{\dot{u}_0}\cdot\underline{i})\right] \\ -\left[Q + 2V\Delta\rho(\underline{\dot{u}_0}\cdot\underline{j})\right] \end{pmatrix} \quad .
$$

For this system to have a solution, we require that the determinant of A is not equal to zero (i.e. $det(A) = (\overline{D}_{11} - V^2\overline{\rho})(\overline{D}_{22} - 4V^2\overline{\rho}) - \overline{D}_{21}\overline{D}_{12} \neq 0$). If the material is isotropic, then $\overline{D}_{12} = \overline{D}_{21} = 0$, which implies that for this system to have a solution we must have $V \neq \sqrt{\frac{\overline{D}_{11}}{\overline{\rho}}}$ and $V \neq \frac{1}{2}\sqrt{\frac{\overline{D}_{22}}{\overline{\rho}}}$. Using simplifications (40) and (41), these conditions take on the form: $V \neq \sqrt{\frac{\overline{\lambda+2\mu}}{\overline{\rho}}}$, and $V \neq \sqrt{\frac{\overline{\mu}}{\overline{\rho}}}$. In other words, the velocity V should be different from "averaged" phase velocities of dilatiation and shear waves, [1].

Assuming that $det(A) \neq 0$, we have the following unique solutions for α and β:

$$\alpha = \frac{1}{detA}(\overline{D}_{12}(Q + 2V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{j})) - (\overline{D}_{22} - 4V^2\overline{\rho})(P + V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{i}))) \ ,$$
$$\beta = \frac{1}{detA}(\overline{D}_{21}(P + V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{i})) - (\overline{D}_{11} - V^2\overline{\rho})(Q + 2V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{j}))) \ .$$

It is now possible to take the simplified expression for average stress (15), and substitute the values derived for α and β into the expression for p_0.

$$\begin{aligned} p_0 &= D_0 \cdot \cdot e_0 + m_1 m_2(\alpha \Delta D \cdot \cdot a_1 + \beta \Delta D \cdot \cdot a_2) \\ &= D_0 \cdot \cdot e_0 + \frac{m_1 m_2}{detA}\left[\overline{D}_{12}(Q + 2V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{j})) - (\overline{D}_{22} - 4V^2\overline{\rho})(P + V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{i}))\right]\Delta D \cdot \cdot a_1 \\ &\quad + \frac{m_1 m_2}{detA}\left[\overline{D}_{21}(P + V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{i})) - (\overline{D}_{11} - V^2\overline{\rho})(Q + 2V\Delta\rho(\underline{\dot{u}_0} \cdot \underline{j}))\right]\Delta D \cdot \cdot a_2 \quad . \end{aligned}$$

So, a simplified equation for p_0 is given by the following expression:

$$\begin{aligned} p_0 \ = \ & D_0 \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{det(A)}\left[\overline{D}_{21}(\Delta D \cdot \cdot a_2) + (4V^2\overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1)\right] P \\ & + \frac{m_1 m_2}{det(A)}\left[\overline{D}_{12}(\Delta D \cdot \cdot a_1) + (V^2\overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2)\right] Q \\ & + \frac{m_1 m_2 V\Delta\rho}{det(A)}\left[(4V^2\overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) + \overline{D}_{21}(\Delta D \cdot \cdot a_2)\right](\underline{i} \cdot \underline{\dot{u}_0}) \\ & + \frac{2m_1 m_2 V\Delta\rho}{det(A)}\left[(V^2\overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) + \overline{D}_{12}(\Delta D \cdot \cdot a_1)\right](\underline{j} \cdot \underline{\dot{u}_0}) \quad . \end{aligned}$$

Next, recall the definitions of P and Q as $(a_1 \cdot \cdot \Delta D \cdot \cdot e_0)$ and $(a_2 \cdot \cdot \Delta D \cdot \cdot e_0)$, respectively. Plug these values into the above expression for p_0; this gives us the following:

$$\begin{aligned} p_0 \ = \ & D_0 \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{det(A)}\left[\overline{D}_{21}(\Delta D \cdot \cdot a_2) + (4V^2\overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1)\right](a_1 \cdot \cdot \Delta D \cdot \cdot e_0) \\ & + \frac{m_1 m_2}{det(A)}\left[\overline{D}_{12}(\Delta D \cdot \cdot a_1) + (V^2\overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2)\right](a_2 \cdot \cdot \Delta D \cdot \cdot e_0) \\ & + \frac{m_1 m_2 V\Delta\rho}{det(A)}\left[(4V^2\overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) + \overline{D}_{21}(\Delta D \cdot \cdot a_2)\right](\underline{i} \cdot \underline{\dot{u}_0}) \\ & + \frac{2m_1 m_2 V\Delta\rho}{det(A)}\left[(V^2\overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) + \overline{D}_{12}(\Delta D \cdot \cdot a_1)\right](\underline{j} \cdot \underline{\dot{u}_0}) \quad . \end{aligned}$$

This can be expressed as follows:

$$\begin{aligned} p_0 \quad = \quad & D_0 \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{21} (\Delta D \cdot \cdot a_2) \otimes (a_1 \cdot \cdot \Delta D) + (4V^2 \overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D) \right] \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{12} (\Delta D \cdot \cdot a_1) \otimes (a_2 \cdot \cdot \Delta D) + (V^2 \overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D) \right] \cdot \cdot e_0 \\ & + \frac{m_1 m_2 V \Delta\rho}{det(A)} \left[(4V^2 \overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) \otimes \underline{i} + \overline{D}_{21} (\Delta D \cdot \cdot a_2) \otimes \underline{i} \right] \cdot \underline{\dot{u}_0} \\ & + \frac{2 m_1 m_2 V \Delta\rho}{det(A)} \left[(V^2 \overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) \otimes \underline{j} + \overline{D}_{12} (\Delta D \cdot \cdot a_1) \otimes \underline{j} \right] \cdot \underline{\dot{u}_0} \quad . \\ & (36) \end{aligned}$$

To simplify the above equation, we define the following quantities:

$$\begin{aligned} D_{eff,d} \quad = \quad & D_0 \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{21} (\Delta D \cdot \cdot a_2) \otimes (a_1 \cdot \cdot \Delta D) + (4V^2 \overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D) \right] \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{12} (\Delta D \cdot \cdot a_1) \otimes (a_2 \cdot \cdot \Delta D) + (V^2 \overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D) \right] \quad , \end{aligned}$$

and,

$$\begin{aligned} \Lambda \quad = \quad & \frac{m_1 m_2 V \Delta\rho}{det(A)} \left[(4V^2 \overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) \otimes \underline{i} + \overline{D}_{21} (\Delta D \cdot \cdot a_2) \otimes \underline{i} \right] \\ & + \frac{2 m_1 m_2 V \Delta\rho}{det(A)} \left[(V^2 \overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) \otimes \underline{j} + \overline{D}_{12} (\Delta D \cdot \cdot a_1) \otimes \underline{j} \right] \quad . \end{aligned}$$

With these simplifications, equation (36) becomes more clear.

$$p_0 \quad = \quad D_{eff,d} \cdot \cdot e_0 + \Lambda \cdot \underline{\dot{u}_0} \tag{37}$$

This expression shows Hooke's law for a dynamic elastic laminate. We see that when there is a moving property pattern, the average stress p_0 is not just linearly related to the average stress, but it is also linearly related to the average velocity of disturbances propagating through the material. This relationship is due to the rank 3 tensor Λ. We term (37) the dynamic Hooke's law.

If we assume that D_+ and D_- are both isotropic, then $\overline{D}_{12} = \overline{D}_{21} = 0$, and so $det(A) = \left(\overline{D}_{11} - V^2\overline{\rho}\right)\left(\overline{D}_{22} - 4V^2\overline{\rho}\right) = \left(V^2\overline{\rho} - \overline{D}_{11}\right)\left(4V^2\overline{\rho} - \overline{D}_{22}\right)$, which means that $D_{eff,d}$ and Λ take the following form:

$$D_{eff,d} \quad = \quad D_0 + m_1 m_2 \left[\frac{(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D)}{V^2 \overline{\rho} - \overline{D}_{11}} + \frac{(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D)}{4V^2 \overline{\rho} - \overline{D}_{22}} \right]$$

and,

$$\Lambda = m_1 m_2 V \Delta\rho \left[\frac{(\Delta D \cdot \cdot a_1) \otimes \underline{i}}{V^2 \overline{\rho} - \overline{D}_{11}} + 2 \frac{(\Delta D \cdot \cdot a_2) \otimes \underline{j}}{(4V^2 \overline{\rho} - \overline{D}_{22})} \right]$$

Further examination will reveal some interesting facts about the physics involved. To see this, we examine various possibilities for the material parameters, namely, when $\Delta D \neq 0$ and $\Delta\rho = 0$ (i.e. when there is a change in stiffness across the lamination, but no change in density parameters) and when $\Delta D = 0$ and $\Delta\rho \neq 0$ (i.e. when there is no change in stiffness across the lamination, but there is a change in density). Before these are investigated though, it will prove beneficial to simplify expressions for certain terms involved in the calculations.

4.8. **Specific Cases.**

4.8.1. *Simplification of Terms.* Previously, we assumed that the tensors D_+ and D_- are both isotropic. This implies that $D_{\pm}$ takes the following form:

$$D_{\pm} = \left(\lambda_{\pm} + \mu_{\pm}\right) \alpha_1 \alpha_1 + \mu_{\pm} \left(\alpha_{12}\alpha_{12} + \alpha_2\alpha_2\right) \quad \text{where } \lambda_{\pm} \text{ and } \mu_{\pm} \text{ are the Lamé moduli and,}$$

$$\begin{aligned} \alpha_1 &= \underline{ii} + \underline{jj} \quad , \\ \alpha_2 &= \underline{ii} - \underline{jj} \quad \text{and,} \\ \alpha_{12} &= \underline{ij} + \underline{ji} \quad . \end{aligned}$$

So, we have that:

$$\begin{aligned} \Delta D &= D_+ - D_- \\ &= (\Delta\lambda + \Delta\mu)\, \alpha_1\alpha_1 + \Delta\mu\, (\alpha_{12}\alpha_{12} + \alpha_2\alpha_2) \quad \text{where,} \end{aligned}$$

$$\begin{aligned} \Delta\mu &= \mu_+ - \mu_- \quad \text{and,} \\ \Delta\lambda &= \lambda_+ - \lambda_- \quad . \end{aligned}$$

Also,

$$\begin{aligned} \overline{D} &= m_2 D_+ + m_1 D_- \\ &= (m_2(\lambda_+ + \mu_+) + m_1(\lambda_- + \mu_-))\, \alpha_1\alpha_1 + (m_2\mu_+ + m_1\mu_-)\, (\alpha_{12}\alpha_{12} + \alpha_2\alpha_2) \quad \text{where,} \\ &= \left(\overline{\lambda} + \overline{\mu}\right) \alpha_1\alpha_1 + \overline{\mu}\, (\alpha_{12}\alpha_{12} + \alpha_2\alpha_2) \quad \text{where,} \end{aligned}$$

$$\begin{aligned} \overline{\lambda} &= m_2\lambda_+ + m_1\lambda_- \quad \text{and,} \\ \overline{\mu} &= m_2\mu_+ + m_1\mu_- \quad . \end{aligned}$$

Thus, to simplify the application of the above equations, we wish to express the above basis tensors $\alpha_1\alpha_1$, $\alpha_{12}\alpha_{12}$, and $\alpha_2\alpha_2$ in terms of the previously used tensors $a_1 = \underline{ii}$, $a_2 = \underline{ij} + \underline{ji}$, and $\underline{jj}$:

$$\begin{aligned}
\alpha_1\alpha_1 &= \left(\underline{ii} + \underline{jj}\right) \otimes \left(\underline{ii} + \underline{jj}\right) \\
&= a_1a_1 + a_1a_3 + a_3a_1 + a_3a_3 \quad , \\
\\
\alpha_{12}\alpha_{12} &= \left(\underline{ij} + \underline{ji}\right) \otimes \left(\underline{ij} + \underline{ji}\right) \\
&= a_2a_2 \quad , \\
\alpha_2\alpha_2 &= \left(\underline{ii} - \underline{jj}\right) \otimes \left(\underline{ii} - \underline{jj}\right) \\
&= a_1a_1 - a_1a_3 - a_3a_1 + a_3a_3 \quad .
\end{aligned}$$

Thus, the expressions for ΔD and $\overline{D}$ become as follows:

$$\begin{aligned}
\Delta D &= (\Delta\lambda + \Delta\mu)\,\alpha_1\alpha_1 + \Delta\mu\,(\alpha_{12}\alpha_{12} + \alpha_2\alpha_2) \\
&= (\Delta\lambda + \Delta\mu)\,(a_1a_1 + a_1a_3 + a_3a_1 + a_3a_3) + \Delta\mu\,(a_2a_2 + (a_1a_1 - a_1a_3 - a_3a_1 + a_3a_3)) \\
&= (\Delta\lambda + 2\Delta\mu)\,a_1a_1 + \Delta\lambda\,(a_1a_3 + a_3a_1) + (\Delta\lambda + 2\Delta\mu)\,a_3a_3 + \Delta\mu a_2a_2 \quad ,
\end{aligned}$$

and

$$\begin{aligned}
\overline{D} &= \left(\overline{\lambda} + \overline{\mu}\right)\alpha_1\alpha_1 + \overline{\mu}\,(\alpha_2\alpha_2 + \alpha_2\alpha_2) \\
&= \left(\overline{\lambda} + \overline{\mu}\right)(a_1a_1 + a_1a_3 + a_3a_1 + a_3a_3) + \overline{\mu}\,(a_2a_2 + (a_1a_1 - a_1a_3 - a_3a_1 + a_3a_3)) \\
&= \left(\overline{\lambda} + 2\overline{\mu}\right)a_1a_1 + \overline{\lambda}\,(a_1a_3 + a_3a_1) + \left(\overline{\lambda} + 2\overline{\mu}\right)a_3a_3 + \overline{\mu}a_2a_2 \quad .
\end{aligned}$$

So, we can introduce the following simplifications to the quantities (i.e. P, Q, and $\overline{D_{st}}$ for s, t=0, 1) found in the above equations.

$$\begin{aligned}
P &= (a_1 \cdot\cdot \Delta D \cdot\cdot e_0) \\
&= a_1 \cdot\cdot ((\Delta\lambda + 2\Delta\mu)\,a_1a_1 + \Delta\lambda\,(a_1a_3 + a_3a_1) + (\Delta\lambda + 2\Delta\mu)\,a_3a_3 + \Delta\mu a_2a_2) \cdot\cdot e_0 \\
&= ((\Delta\lambda + 2\Delta\mu)\,a_1 + \Delta\lambda a_3) \cdot\cdot e_0 \\
&= (\Delta\lambda + 2\Delta\mu)\,[e_{11}]_0 + \Delta\lambda\,[e_{22}]_0 \\
&= (\Delta\lambda + 2\Delta\mu)\left[\frac{\partial u_1}{\partial x_1}\right]_0 + \Delta\lambda\left[\frac{\partial u_2}{\partial x_2}\right]_0 \quad ,
\end{aligned}$$

(38)

$$
\begin{aligned}
Q &= (a_2 \cdot\cdot \Delta D \cdot\cdot e_0) \\
&= a_2 \cdot\cdot ((\Delta\lambda + 2\Delta\mu)\, a_1 a_1 + \Delta\lambda\, (a_1 a_3 + a_3 a_1) + (\Delta\lambda + 2\Delta\mu)\, a_3 a_3 + \Delta\mu a_2 a_2) \cdot\cdot e_0 \\
&= (2\Delta\mu a_2) \cdot\cdot e_0 \\
&= 4\Delta\mu\, [e_{12}]_0 \\
&= 2\Delta\mu \left(\left[\frac{\partial u_1}{\partial x_2}\right]_0 + \left[\frac{\partial u_2}{\partial x_1}\right]_0 \right) \quad ,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\overline{D_{11}} &= a_1 \cdot\cdot \overline{D} \cdot\cdot a_1 \\
&= a_1 \cdot\cdot \left[\left(\overline{\lambda} + 2\overline{\mu}\right) a_1 a_1 + \overline{\lambda}\, (a_1 a_3 + a_3 a_1) + \left(\overline{\lambda} + 2\overline{\mu}\right) a_3 a_3 + \overline{\mu} a_2 a_2 \right] \cdot\cdot a_1 \\
&= \overline{\lambda} + 2\overline{\mu} \quad ,
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\overline{D_{22}} &= a_2 \cdot\cdot \overline{D} \cdot\cdot a_2 \\
&= a_2 \cdot\cdot \left[\left(\overline{\lambda} + 2\overline{\mu}\right) a_1 a_1 + \overline{\lambda}\, (a_1 a_3 + a_3 a_1) + \left(\overline{\lambda} + 2\overline{\mu}\right) a_3 a_3 + \overline{\mu} a_2 a_2 \right] \cdot\cdot a_2 \\
&= 4\overline{\mu} \quad ,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\overline{D_{12}} &= a_1 \cdot\cdot \overline{D} \cdot\cdot a_2 \\
&= a_1 \cdot\cdot \left[\left(\overline{\lambda} + 2\overline{\mu}\right) a_1 a_1 + \overline{\lambda}\, (a_1 a_3 + a_3 a_1) + \left(\overline{\lambda} + 2\overline{\mu}\right) a_3 a_3 + \overline{\mu} a_2 a_2 \right] \cdot\cdot a_2 \\
&= 0 \quad ,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\overline{D_{21}} &= a_2 \cdot\cdot \overline{D} \cdot\cdot a_1 \\
&= a_2 \cdot\cdot \left[\left(\overline{\lambda} + 2\overline{\mu}\right) a_1 a_1 + \overline{\lambda}\, (a_1 a_3 + a_3 a_1) + \left(\overline{\lambda} + 2\overline{\mu}\right) a_3 a_3 + \overline{\mu} a_2 a_2 \right] \cdot\cdot a_1 \\
&= 0 \quad .
\end{aligned}
\tag{43}
$$

So, we can apply equations (40)-(43)

$$
\begin{aligned}
det(A) &= (\overline{D}_{11} - V^2\overline{\rho})(\overline{D}_{22} - 4V^2\overline{\rho}) - \overline{D}_{21}\overline{D}_{12} \\
&= (\overline{\lambda} + 2\overline{\mu} - V^2\overline{\rho})(4\overline{\mu} - 4V^2\overline{\rho}) \\
&= 4(\overline{\lambda} + 2\overline{\mu} - V^2\overline{\rho})(\overline{\mu} - V^2\overline{\rho}) \quad .
\end{aligned}
$$

Lastly, define the following for simplification:

$$\begin{aligned}\Delta_d &= V^2\overline{\rho} - (\overline{\lambda} + 2\overline{\mu}) \quad , \\ \Delta_s &= V^2\overline{\rho} - \overline{\mu} \quad .\end{aligned}$$

and thus,

$$det(A) = 4\Delta_s\Delta_d \quad .$$

4.8.2. *Case 1.* When $\Delta\rho = 0$ and $\Delta D \neq 0$, we have that $\Lambda = 0$, and thus there is no inertial term in the dynamic Hooke's law, and so it becomes as follows:

$$\begin{aligned}p_0 = \; & D_0 \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{21}(\Delta D \cdot \cdot a_2) \otimes (a_1 \cdot \cdot \Delta D) + (4V^2\overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D)\right] \cdot \cdot e_0 \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{12}(\Delta D \cdot \cdot a_1) \otimes (a_2 \cdot \cdot \Delta D) + (V^2\overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D)\right] \cdot \cdot e_0 \quad .\end{aligned}$$

It now becomes possible to solve for an effective elasticity tensor ($D_{eff,d}$):

$$\begin{aligned}D_{eff,d} = \; & D_0 \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{21}(\Delta D \cdot \cdot a_2) \otimes (a_1 \cdot \cdot \Delta D) + (4V^2\overline{\rho} - \overline{D}_{22})(\Delta D \cdot \cdot a_1) \otimes (a_1 \cdot \cdot \Delta D)\right] \\ & + \frac{m_1 m_2}{det(A)} \left[\overline{D}_{12}(\Delta D \cdot \cdot a_1) \otimes (a_2 \cdot \cdot \Delta D) + (V^2\overline{\rho} - \overline{D}_{11})(\Delta D \cdot \cdot a_2) \otimes (a_2 \cdot \cdot \Delta D)\right] \quad .\end{aligned}$$

So by applying the above assumption of isotropy, this becomes as follows:

$$\begin{aligned}D_{eff,d} = \; & D_0 \\ & + \frac{m_1 m_2}{4\Delta_s\Delta_d} \left[4\Delta_s((\Delta\lambda + 2\Delta\mu)\, a_1 + \Delta\lambda a_3) \otimes ((\Delta\lambda + 2\Delta\mu)\, a_1 + \Delta\lambda a_3)\right] \\ & + \frac{m_1 m_2}{4\Delta_s\Delta_d} \left[\Delta_d(2\Delta\mu a_2) \otimes (2\Delta\mu a_2)\right] \\ = \; & D_0 \\ & + \frac{m_1 m_2}{\Delta_d} \left[(\Delta\lambda + 2\Delta\mu)^2\, a_1 a_1 + (\Delta\lambda + 2\Delta\mu)\, \Delta\lambda(a_1 a_3 + a_3 a_1) + (\Delta\lambda)^2 a_3 a_3\right] \\ & + \frac{m_1 m_2}{\Delta_s} (\Delta\mu)^2 a_2 a_2 \quad .\end{aligned}$$

We will represent this in terms of α_1, α_2, and α_{12}. From before, we have that $a_1 = \frac{1}{2}(\alpha_1 + \alpha_2)$, $a_2 = \frac{1}{2}(\alpha_1 - \alpha_2)$, and $a_2 = \alpha_{12}$. This changes the effective tensor as follows:

$$\begin{aligned}
D_{eff,d} &= D_0 \\
&+\frac{m_1 m_2}{4\Delta_d}\left[(\Delta\lambda+2\Delta\mu)^2\left(\alpha_1\alpha_1+\alpha_1\alpha_2+\alpha_2\alpha_1+\alpha_2\alpha_2\right)\right] \\
&+\frac{m_1 m_2}{4\Delta_d}\left[2\left(\Delta\lambda+2\Delta\mu\right)\Delta\lambda(\alpha_1\alpha_1+\alpha_2\alpha_2)\right] \\
&+\frac{m_1 m_2}{4\Delta_d}\left[(\Delta\lambda)^2(\alpha_1\alpha_1-\alpha_1\alpha_2-\alpha_2\alpha_1+\alpha_2\alpha_2)\right] \\
&+\frac{m_1 m_2}{\Delta_s}(\Delta\mu)^2\alpha_{12}\alpha_{12} \quad . \\
&= D_0 \\
&+\frac{m_1 m_2}{4\Delta_d}\left[4\left(\Delta(\lambda+\mu)\right)^2\left(\alpha_1\alpha_1+\alpha_2\alpha_2\right)+4\Delta\mu\Delta(\lambda+\mu)(\alpha_1\alpha_2+\alpha_2\alpha_1)\right] \\
&+\frac{m_1 m_2}{\Delta_s}(\Delta\mu)^2\alpha_{12}\alpha_{12}
\end{aligned}$$

Thus, our final simplified expression for the effective tensor is the following:

$$\begin{aligned}
D_{eff,d} &= D_0 \\
&+\frac{m_1 m_2}{\Delta_d}\left[\left(\Delta(\lambda+\mu)\right)^2\left(\alpha_1\alpha_1+\alpha_2\alpha_2\right)+\Delta\mu\Delta(\lambda+\mu)(\alpha_1\alpha_2+\alpha_2\alpha_1)\right] \\
&+\frac{m_1 m_2}{\Delta_s}(\Delta\mu)^2\alpha_{12}\alpha_{12} \quad .
\end{aligned}$$

4.8.3. *Diagonalization of Case 1.* From the derivations in the previous section, we have shown that when there is no change in the mass density across the interface (i.e. $\Delta\rho = 0$) but there is a change in the stiffness tensor ($\Delta D \neq 0$) the effective stiffness tensor is as follows:

$$\begin{aligned}
D_{eff,d} &= D_0+\frac{m_1 m_2}{\Delta_d}\left[\left(\Delta(\lambda+\mu)\right)^2\left(\alpha_1\alpha_1+\alpha_2\alpha_2\right)+\Delta(\lambda+\mu)\Delta\mu(\alpha_1\alpha_2+\alpha_2\alpha_1)\right] \\
&+\frac{m_1 m_2}{\Delta_s}(\Delta\mu)^2\alpha_{12}\alpha_{12} \\
&= (\lambda_0+\mu_0)\alpha_1\alpha_1+\mu_0(\alpha_2\alpha_2+\alpha_{12}\alpha_{12}) \\
&+\frac{m_1 m_2}{\Delta_d}\left[\left(\Delta(\lambda+\mu)\right)^2\left(\alpha_1\alpha_1+\alpha_2\alpha_2\right)+\Delta(\lambda+\mu)\Delta\mu(\alpha_1\alpha_2+\alpha_2\alpha_1)\right] \\
&+\frac{m_1 m_2}{\Delta_s}(\Delta\mu)^2\alpha_{12}\alpha_{12} \quad .
\end{aligned}$$

The above equation contains cross terms which we wish to eliminate for simplification. We introduce a new coordinate basis, with two free parameters (ξ and κ) whose value we will choose so that the cross terms are eliminated.

$$\begin{aligned}\overline{\alpha_1} &= \xi(\alpha_1 + \kappa\alpha_2) \quad , \\ \overline{\alpha_2} &= \xi(-\kappa\alpha_1 + \alpha_2) \quad ,\end{aligned}$$

$$\begin{aligned}\overline{\alpha_1} \cdot\cdot \overline{\alpha_1} &= \left[(\alpha_1 \cdot\cdot \alpha_1) + k^2(\alpha_2 \cdot\cdot \alpha_2)\right]\xi^2 \\ &= (1+k^2)\xi^2 \quad \text{, and thus,}\end{aligned}$$

$$\xi = \frac{1}{\sqrt{1+k^2}} \quad .$$

As such, the new basis is given by the following:

$$\begin{aligned}\overline{\alpha_1} &= \frac{1}{\sqrt{1+k^2}}\alpha_1 + \frac{\kappa}{\sqrt{1+k^2}}\alpha_2 \quad , \\ \overline{\alpha_2} &= -\frac{\kappa}{\sqrt{1+k^2}}\alpha_1 + \frac{1}{\sqrt{1+k^2}}\alpha_2 \quad .\end{aligned}$$

Let us introduce the parameter ϕ in the following manner:

$$\begin{aligned}\overline{\alpha_1} &= \cos\phi\ \alpha_1 + \sin\phi\ \alpha_2 \quad , \\ \overline{\alpha_2} &= -\sin\phi\ \alpha_1 + \cos\phi\ \alpha_2 \quad .\end{aligned}$$

And thus, the equations for a_1, and a_2 are as follows:

$$\begin{aligned}\alpha_1 &= \cos\phi\ \overline{\alpha_1} + \sin\phi\ \overline{\alpha_2} \quad , \\ \alpha_2 &= -\sin\phi\ \overline{\alpha_1} + \cos\phi\ \overline{\alpha_2} \quad .\end{aligned}$$

So we have the following expression for the dyadics $\alpha_1\alpha_1$, $\alpha_2\alpha_2$, and $\alpha_1\alpha_2+\alpha_2\alpha_1$:

$$\begin{aligned}\alpha_1\alpha_1 &= \cos^2\phi\ \overline{\alpha_1\alpha_1} - \cos\phi\sin\phi\ (\overline{\alpha_1\alpha_2} + \overline{\alpha_2\alpha_1}) + \sin^2\phi\ \overline{\alpha_2\alpha_2} \quad , \\ \alpha_2\alpha_2 &= \sin^2\phi\ \overline{\alpha_1\alpha_1} + \cos\phi\sin\phi\ (\overline{\alpha_1\alpha_2} + \overline{\alpha_2\alpha_1}) + \cos^2\phi\ \overline{\alpha_2\alpha_2} \quad , \\ \alpha_1\alpha_2 + \alpha_2\alpha_1 &= \left[\cos\phi\sin\phi\ (\overline{\alpha_1\alpha_1} - \overline{\alpha_2\alpha_2}) + \cos^2\phi\ \overline{\alpha_1\alpha_2} - \sin^2\phi\overline{\alpha_2\alpha_1}\right] \\ &\quad + \left[\cos\phi\sin\phi\ (\overline{\alpha_1\alpha_1} - \overline{\alpha_2\alpha_2}) - \sin^2\phi\ \overline{\alpha_1\alpha_2} + \cos^2\phi\overline{\alpha_2\alpha_1}\right] \\ &= 2\cos\phi\sin\phi(\overline{\alpha_1\alpha_1} - \overline{\alpha_2\alpha_2}) + (\cos^2\phi - \sin^2\phi)(\overline{\alpha_1\alpha_2} + \overline{\alpha_2\alpha_1}) \\ &= \sin 2\phi\ (\overline{\alpha_1\alpha_1} - \overline{\alpha_2\alpha_2}) + \cos 2\phi\ (\overline{\alpha_1\alpha_2} + \overline{\alpha_2\alpha_1}) \quad .\end{aligned}$$

Applying this new representation for the basis, the expression for $D_{eff,d}$ changes in the following manner:

$$\begin{aligned} D_{eff,d} &= \left[\lambda_0 + \mu_0 + \frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right]\left[\cos^2\phi\,\overline{\alpha_1\alpha_1} - \cos\phi\sin\phi\,(\overline{\alpha_1\alpha_2}+\overline{\alpha_2\alpha_1}) + \sin^2\phi\,\overline{\alpha_2\alpha_2}\right] \\ &+ \left[\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right]\left[\sin^2\phi\,\overline{\alpha_1\alpha_1} + \cos\phi\sin\phi\,(\overline{\alpha_1\alpha_2}+\overline{\alpha_2\alpha_1}) + \cos^2\phi\,\overline{\alpha_2\alpha_2}\right] \\ &+ \frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\left[\sin 2\phi\,(\overline{\alpha_1\alpha_1}-\overline{\alpha_2\alpha_2}) + \cos 2\phi\,(\overline{\alpha_1\alpha_2}+\overline{\alpha_2\alpha_1})\right] \\ &+ \left[\mu_0 + \frac{m_1 m_2}{\Delta_s}\Delta\mu^2\right]\alpha_{12}\alpha_{12} \quad . \end{aligned}$$

Rearranging gives the following:

$$\begin{aligned} D_{eff,d} &= \left[\left(\lambda_0 + \mu_0 + \frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right)\cos^2\phi + \left(\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right)\sin^2\phi\right]\overline{\alpha_1\alpha_1} \\ &+ \left[\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\sin 2\phi\right]\overline{\alpha_1\alpha_1} \\ &+ \left[\left(\lambda_0 + \mu_0 + \frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right)\sin^2\phi + \left(\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right)\cos^2\phi\right]\overline{\alpha_2\alpha_2} \\ &- \left[\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\sin 2\phi\right]\overline{\alpha_2\alpha_2} \\ &+ \left[\mu_0 + \frac{m_1 m_2}{\Delta_s}\Delta\mu^2\right]\alpha_{12}\alpha_{12} \\ &- \left[\left(\lambda_0 + \mu_0 + \frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right)\cos\phi\sin\phi\right](\overline{\alpha_1\alpha_2}+\overline{\alpha_2\alpha_1}) \\ &+ \left[\left(\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right)\cos\phi\sin\phi + \left(\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\right)\cos 2\phi\right](\overline{\alpha_1\alpha_2}+\overline{\alpha_2\alpha_1}) \quad . \end{aligned}$$

Thus, elimination of cross terms from the above equation is equivalent to having $D_{eff,d}\cdot\cdot(\overline{a_1 a_2}+\overline{a_2 a_1}) = 0$, i.e.,

$$\begin{aligned} D_{eff,d}\cdot\cdot(\overline{\alpha_1\alpha_2}+\overline{\alpha_2\alpha_1}) &= -\left(\lambda_0 + \mu_0 + \frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right)\cos\phi\sin\phi \\ &\quad + \left(\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right)\cos\phi\sin\phi \\ &\quad + \left(\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\right)\cos 2\phi \\ &= 0 \quad . \end{aligned}$$

Thus,

$$\begin{aligned}\lambda_0 \cos\phi \sin\phi &= \left(\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\right)\cos 2\phi \quad ,\\ \lambda_0 \frac{\sin 2\phi}{2} &= \left(\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\right)\cos 2\phi \quad ,\end{aligned}$$

and so, ϕ is expressed as follows:

$$\tan 2\phi = \frac{2m_1 m_2 \Delta(\lambda+\mu)\Delta\mu}{\Delta_d \lambda_0} \quad ,$$

$$\phi = \frac{1}{2}\tan\left(\frac{2m_1 m_2 \Delta(\lambda+\mu)\Delta\mu}{\Delta_d \lambda_0}\right) \quad .$$

Thus, the diagonalized expression for $D_{eff,d}$ is the following:

$$\begin{aligned}D_{eff,d} = &\left[\left(\lambda_0+\mu_0+\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right)\cos^2\phi + \left(\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right)\sin^2\phi\right]\overline{\alpha_1\alpha_1}\\ &\left[\left(\lambda_0+\mu_0+\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)^2\right)\sin^2\phi + \left(\mu_0 + \frac{m_1 m_2}{\Delta_d}\left(\Delta(\lambda+\mu)\right)^2\right)\cos^2\phi\right]\overline{\alpha_2\alpha_2}\\ &\left(\frac{m_1 m_2}{\Delta_d}\Delta(\lambda+\mu)\Delta\mu\sin 2\phi\right)\left(\overline{\alpha_1\alpha_1}-\overline{\alpha_2\alpha_2}\right)\\ &+\left[\mu_0+\frac{m_1 m_2)}{\Delta_s}\Delta\mu^2\right]\alpha_{12}\alpha_{12} \quad .\end{aligned}$$

4.8.4. *Case 2.* When $\Delta\rho \neq 0$ and $\Delta D = 0$, we have that $\Lambda = 0$ and that $D_{eff,d} = D_+ = D_- = D$ and thus there is no inertial term in the Dynamic Hooke's Law, and so it becomes as follows:

$$\begin{aligned}p_0 &= D_{eff,d}\cdot\cdot e_0 + \Lambda\cdot\underline{\dot{u}_0}\\ &= D\cdot\cdot e_0 \quad .\end{aligned}$$

This shows that once there is no change in stiffness across the interface, then the material obeys Hooke's law throughout, however, we're interested in learning more about the inertial term $\left(\Lambda\cdot\cdot\underline{\dot{u}_0}\right)$ that appears above. Thus, we look to the averaged Lagrangian to see what more we can learn about this system, and it's equations of motions.

5. Examination of Euler Equations

We introduce the averaged Lagrangian L_0 (double action density) of the system, and subsequently solve for the system's Euler equations in an attempt to better interpret the dynamic terms.

For the ease of the following calculations, we will make substitutions for the relevant variables.

$$\begin{aligned} x_1 &= x \quad , \\ x_2 &= y \quad , \\ u_1 &= u \quad , \\ u_2 &= v \quad . \end{aligned}$$

Thus, the Lagrangian takes the following form (subscript denotes differentiation with respect to the variable): $L = L(t, x, y, u_t, v_t, u_x, u_y, v_x, v_y)$. We have:

$$L = \underline{\sigma} \cdot \underline{\dot{u}} - p \cdot \cdot e \quad ,$$

or, equivalently,

$$\begin{aligned} L &= \sigma_1 \dot{u}_1 + \sigma_2 \dot{u}_2 - \tau_{11} e_{11} - 2\tau_{12} e_{12} - \tau_{22} e_{22} \\ &= \sigma_1 \dot{u}_1 + \sigma_2 \dot{u}_2 - (\tau_{11} + V\sigma_1) e_{11} + V\sigma_1 e_{11} - 2(\tau_{12} + V\sigma_2) e_{12} + 2V\sigma_2 e_{12} - \tau_{22} e_{22} \\ &= -(\tau_{11} + V\sigma_1) e_{11} - 2(\tau_{12} + V\sigma_2) e_{12} + \sigma_1(\dot{u}_1 + Ve_{11}) + \sigma_2(\dot{u}_2 + 2Ve_{12}) - \tau_{22} e_{22} \\ &= -\left[p \cdot \cdot a_1 + V\underline{\sigma} \cdot \underline{i}\right] e_{11} - \left[p \cdot \cdot a_2 + 2V\underline{\sigma} \cdot \underline{j}\right] e_{12} \\ &\quad + \left[\underline{\dot{u}} \cdot \underline{i} + Ve \cdot \cdot a_1\right] \sigma_1 + \left[\underline{\dot{u}} \cdot \underline{j} + Ve \cdot \cdot a_2\right] \sigma_2 - \left[e \cdot \cdot a_3\right] \tau_{22} \quad . \end{aligned}$$

When we introduce L_+ and L_-, then from the continuity conditions on the interface (29)-(33), we see that the bracketed terms in the above equation for the Lagrangian remain continuous across the material interface, so in the weak limit, we have.

$$L_0 = \underline{\sigma_0} \cdot \underline{\dot{u}_0} - p_0 \cdot \cdot e_0 \quad , \tag{44}$$

By applying simplification (35) found for $\underline{\sigma}_0$, along with expression for p_0, we find the following:

$$\begin{aligned} L_0 &= \rho_0 \underline{\dot{u}_0} \cdot \underline{\dot{u}_0} - e_0 \cdot \cdot D_0 \cdot \cdot e_0 - m_1 m_2 \left[\frac{(P + V\Delta\rho\dot{u})^2}{\Delta_d} + \frac{(Q + 2V\Delta\rho\dot{v})^2}{4\Delta_s}\right] \\ &= \rho_0 \underline{\dot{u}_0} \cdot \underline{\dot{u}_0} - e_0 \cdot \cdot D_0 \cdot \cdot e_0 - m_1 m_2 \left[\frac{P^2 + (V\Delta\rho\dot{u})^2}{\Delta_d} + \frac{\frac{Q^2}{4} + (V\Delta\rho\dot{v})^2}{\Delta_s}\right] \\ &\quad - m_1 m_2 \left[\frac{2PV\Delta\rho\dot{u}}{\Delta_d} + \frac{QV\Delta\rho\dot{v}}{\Delta_s}\right] \quad , \end{aligned}$$

$$
\begin{aligned}
L_0 &= \rho_0 \underline{\dot{u}_0} \cdot \underline{\dot{u}_0} - e_0 \cdot\cdot D_0 \cdot\cdot e_0 - m_1 m_2 \left[\frac{P^2 + (V\Delta\rho\dot{u})^2}{\Delta_d} + \frac{\frac{Q^2}{4} + (V\Delta\rho\dot{v})^2}{\Delta_s} \right] \\
&\quad - m_1 m_2 \left[\frac{2PV\Delta\rho\dot{u}}{\Delta_d} + \frac{QV\Delta\rho\dot{v}}{\Delta_s} \right] \quad , \\
&= \underline{\dot{u}_0} \cdot \rho_0 \left(a_1 + a_3\right) \cdot \underline{\dot{u}_0} - m_1 m_2 \left[\frac{(V\Delta\rho\dot{u})^2}{\Delta_d} + \frac{(V\Delta\rho\dot{v})^2}{\Delta_s} \right] \\
&\quad - e_0 \cdot\cdot D_0 \cdot\cdot e_0 - m_1 m_1 \left[\frac{P^2}{\Delta_d} + \frac{Q^2}{4\Delta_s} \right] \\
&\quad - m_1 m_2 \left[\frac{2PV\Delta\rho\dot{u}}{\Delta_d} + \frac{QV\Delta\rho\dot{v}}{\Delta_s} \right] \quad ,
\end{aligned}
$$

$$
\begin{aligned}
L_0 &= \underline{\dot{u}_0} \cdot \left[\rho_0 (a_1 + a_3) - m_1 m_2 (V\Delta\rho)^2 \left(\frac{1}{V\overline{\rho} - \overline{D}_{11}} a_1 + \frac{1}{V\overline{\rho} - \overline{D}_{22}} a_3 \right) \right] \cdot \underline{\dot{u}_0} \\
&\quad - e_0 \cdot\cdot \left[D_0 + m_1 m_2 \left(\frac{(\Delta D \cdot\cdot a_1) \otimes (a_1 \cdot\cdot \Delta D)}{V\overline{\rho} - \overline{D}_{11}} + \frac{(\Delta D \cdot\cdot a_2) \otimes (a_2 \cdot\cdot \Delta D)}{V\overline{\rho} - \overline{D}_{22}} \right) \right] \cdot\cdot e_0 \\
&\quad - m_1 m_2 \left[\frac{2(a_1 \cdot\cdot \Delta D \cdot\cdot e_0) V\Delta\rho\dot{u}}{V\overline{\rho} - \overline{D}_{11}} + \frac{(a_2 \cdot\cdot \Delta D \cdot\cdot e_0) V\Delta\rho\dot{v}}{V\overline{\rho} - \overline{D}_{22}} \right] \quad .
\end{aligned}
$$

So, we have the following form for the averaged Lagrangian

$$
L_0 = \underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0} - e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0 - L_{cross} \quad , \tag{45}
$$

where,

$$
M_{eff} = \left[\rho_0 (a_1 + a_3) - m_1 m_2 (V\Delta\rho)^2 \left(\frac{1}{V\overline{\rho} - \overline{D}_{11}} a_1 + \frac{1}{V\overline{\rho} - \overline{D}_{22}} a_3 \right) \right] \quad ,
$$

and

$$
L_{cross} = m_1 m_2 \left[\frac{2(a_1 \cdot\cdot \Delta D \cdot\cdot e_0) V\Delta\rho\dot{u}_0}{V\overline{\rho} - \overline{D}_{11}} + \frac{(a_2 \cdot\cdot \Delta D \cdot\cdot e_0) V\Delta\rho\dot{v}_0}{V\overline{\rho} - \overline{D}_{22}} \right] \quad .
$$

Equation (45) tells us much about this composite. It again shows the effective stiffness tensor ($D_{eff,d}$) for the dynamic case, however, it is possible to deduce more from these equations.

5.1. **Case 2 Revisited.** Again, consider when $\Delta\rho \neq 0$ and $\Delta D = 0$. Then, we have that $D_{eff,d} = D_0$ and $L_{cross} = 0$, and thus equation (45) becomes:

$$L_0 = \underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0} - e_0 \cdot \cdot D_0 \cdot \cdot e_0 \quad .$$

and thus, the only term of interest is M_{eff}, but what is this quantity?

M_{eff} can be interpreted as a tensor of effective (or attached) masses. This concept shows up in hydrodynamics [2].

When a body is immersed in a fluid flow, it has what is known as an effective mass with respect to the direction of fluid flow. Depending on its orientation with respect to the flow, it has what can be interpreted as attached masses in certain directions. Likewise, in the theory of spatial-temporal composites, we can interpret M_{eff} to be the tensor of attached masses with respect to the moving property pattern.

5.2. **Investigation of Cross Terms.** It is clear that in the above formulation for the homogenized Lagrangian there are cross terms that appear. Let us introduce new notation to distinguish the diagonalized terms from the cross terms.

$$\begin{aligned} L_{diag} &= \underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0} - e_0 \cdot \cdot D_{eff,d} \cdot \cdot e_0 \quad \text{and,} \\ L_{cross} &= m_1 m_2 \left[\frac{2PV\Delta\rho\dot{u}_0}{\Delta_d} + \frac{QV\Delta\rho\dot{v}_0}{\Delta_s} \right] \quad \text{so,} \\ L_0 &= L_{diag} - L_{cross} \quad . \end{aligned} \tag{46}$$

Thus, we now wish to use the Euler equations as a means of interpreting the cross terms that appear in the above Lagrangian. The Euler equations take the following form [4]:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial L_0}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_0}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_0}{\partial u_y} &= 0 \quad , \\ \frac{\partial}{\partial t}\frac{\partial L_0}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_0}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_0}{\partial v_y} &= 0 \quad . \end{aligned}$$

Introducing equation (46), this becomes the following:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial L_{diag}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_{diag}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_{diag}}{\partial u_y} - \left(\frac{\partial}{\partial t}\frac{\partial L_{cross}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_{cross}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_{cross}}{\partial u_y} \right) &= 0 \quad , \\ \frac{\partial}{\partial t}\frac{\partial L_{diag}}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_{diag}}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_{diag}}{\partial v_y} - \left(\frac{\partial}{\partial t}\frac{\partial L_{cross}}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_{cross}}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_{cross}}{\partial v_y} \right) &= 0 \quad . \end{aligned}$$

We introduce some simplifications

$$E^1_{diag} = \frac{\partial}{\partial t}\frac{\partial L_{diag}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_{diag}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_{diag}}{\partial u_y} \quad , \tag{47}$$

$$E^2_{diag} = \frac{\partial}{\partial t}\frac{\partial L_{diag}}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_{diag}}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_{diag}}{\partial v_y} \quad , \tag{48}$$

$$E^1_{cross} = \frac{\partial}{\partial t}\frac{\partial L_{cross}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_{cross}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_{cross}}{\partial u_y} \quad , \tag{49}$$

$$E^2_{cross} = \frac{\partial}{\partial t}\frac{\partial L_{cross}}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_{cross}}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_{cross}}{\partial v_y} \quad . \tag{50}$$

Thus, our Euler equations become as follows:

$$E^1_{diag} - E^1_{cross} = 0 \quad , \tag{51}$$

$$E^2_{diag} - E^2_{cross} = 0 \quad . \tag{52}$$

The diagonal parts of (51) and (52) have been previously shown to have effective tensors. However, we must evaluate the cross terms to see what they represent, and what kind of effect they have on the system.

$$\begin{aligned} E^1_{cross} &= \frac{\partial}{\partial t}\frac{\partial L_{cross}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_{cross}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_{cross}}{\partial u_y} \\ &= m_1 m_2 V \Delta\rho \left(\frac{2}{\Delta_d}\frac{\partial P}{\partial t} + \frac{2}{\Delta_d}\frac{\partial \dot u}{\partial x}\frac{\partial P}{\partial u_x} + \frac{1}{\Delta_s}\frac{\partial \dot v}{\partial y}\frac{\partial Q}{\partial u_y} \right) \\ E^2_{cross} &= \frac{\partial}{\partial t}\frac{\partial L_{cross}}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_{cross}}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_{cross}}{\partial v_y} \\ &= m_1 m_2 V \Delta\rho \left(\frac{1}{\Delta_s}\frac{\partial Q}{\partial t} + \frac{1}{\Delta_s}\frac{\partial \dot v}{\partial x}\frac{\partial Q}{\partial v_x} + \frac{2}{\Delta_d}\frac{\partial \dot u}{\partial y}\frac{\partial P}{\partial v_y} \right) \end{aligned}$$

Recall the above simplifications for P and Q,

$$\begin{aligned} P &= (\Delta\lambda + 2\Delta\mu)\left[\frac{\partial u_1}{\partial x_1}\right]_0 + \Delta\lambda\left[\frac{\partial u_2}{\partial x_2}\right]_0 \quad , \\ Q &= 2\Delta\mu\left(\left[\frac{\partial u_1}{\partial x_2}\right]_0 + \left[\frac{\partial u_2}{\partial x_1}\right]_0\right) \quad . \end{aligned}$$

In what follows, we will omit zero indices in u_0 and v_0, and apply just u and v (for example u_x will be used instead of $\left[\frac{\partial u_1}{\partial x_1}\right]_0$, etc.):

$$\begin{aligned} E^1_{cross} &= m_1 m_2 V \Delta\rho \left(\frac{2}{\Delta_d} \frac{\partial \left(\left(\Delta\lambda + 2\Delta\mu \right) u_x + \Delta\lambda v_y \right)}{\partial t} + \frac{2}{\Delta_d} \frac{\partial \dot{u}}{\partial x} \frac{\partial \left(\left(\Delta\lambda + 2\Delta\mu \right) u_x + \Delta\lambda v_y \right)}{\partial u_x} \right) \\ &+ \left(\frac{1}{\Delta_s} \frac{\partial \dot{v}}{\partial y} \frac{\partial \left(2\Delta\mu \left(u_y + v_x \right) \right)}{\partial u_y} \right) \quad , \end{aligned}$$

$$\begin{aligned} E^2_{cross} &= m_1 m_2 V \Delta\rho \left(\frac{1}{\Delta_s} \frac{\partial \left(2\Delta\mu \left(u_y + v_x \right) \right)}{\partial t} + \frac{1}{\Delta_s} \frac{\partial \dot{v}}{\partial x} \frac{\partial \left(2\Delta\mu \left(u_y + v_x \right) \right)}{\partial v_x} \right) \\ &+ \left(\frac{2}{\Delta_d} \frac{\partial \dot{u}}{\partial y} \frac{\partial \left(\left(\Delta\lambda + 2\Delta\mu \right) u_x + \Delta\lambda v_y \right)}{\partial v_y} \right) \quad , \end{aligned}$$

$$E^1_{cross} = 2 m_1 m_2 V \Delta\rho \left(\frac{1}{\Delta_d} \left(\left(\Delta\lambda + 2\Delta\mu \right) \frac{\partial u_x}{\partial t} + \Delta\lambda \frac{\partial v_y}{\partial t} \right) + \frac{\left(\Delta\lambda + 2\Delta\mu \right)}{\Delta_d} \frac{\partial \dot{u}}{\partial x} + \frac{\Delta\mu}{\Delta_s} \frac{\partial \dot{v}}{\partial y} \right) \quad ,$$

$$E^2_{cross} = 2 m_1 m_2 V \Delta\rho \left(\frac{\Delta\mu}{\Delta_s} \left(\frac{\partial u_y}{\partial t} + \frac{\partial v_x}{\partial t} \right) + \frac{\Delta\mu}{\Delta_s} \frac{\partial \dot{v}}{\partial x} + \frac{\Delta\lambda}{\Delta_d} \frac{\partial \dot{u}}{\partial y} \right) \quad .$$

We can interchange the preceding derivatives to obtain:

$$E^1_{cross} = 2 m_1 m_2 V \Delta\rho \left(\frac{1}{\Delta_d} \left(\left(\Delta\lambda + 2\Delta\mu \right) \dot{u}_x + \Delta\lambda \dot{v}_y \right) + \frac{\left(\Delta\lambda + 2\Delta\mu \right)}{\Delta_d} \dot{u}_x + \frac{\Delta\mu}{\Delta_s} \dot{v}_y \right) \quad ,$$

$$E^2_{cross} = 2 m_1 m_2 V \Delta\rho \left(\frac{\Delta\mu}{\Delta_s} \left(\dot{u}_y + \dot{v}_x \right) + \frac{\Delta\mu}{\Delta_s} \dot{v}_x + \frac{\Delta\lambda}{\Delta_d} \dot{u}_y \right) \quad .$$

5.2.1. *First Euler Equation: Cross Terms.* To interpret the preceding Euler equations, we notice that the mixed derivatives in the above Euler equations as components of the "rate of strain" tensor (i.e. velocity of strain). This is defined in the following manner:

$$\begin{aligned} \dot{e}_0 &= \left[\dot{e}_{11} \right]_0 a_1 + \left[\dot{e}_{12} \right]_0 a_2 + \left[\dot{e}_{22} \right]_0 a_3 \\ &= \dot{u}_x a_1 + \frac{1}{2} \left(\dot{u}_y + \dot{v}_x \right) a_2 + \dot{v}_y a_3 \quad . \end{aligned}$$

With this knowledge, we can interpret E^1_{cross}. Define F^1_{eff} as follows:

$$F^1_{eff} = 2 m_1 m_2 V \Delta\rho \left[\left(\frac{2(\Delta\lambda + 2\Delta\mu)}{\Delta_d} \right) a_1 + \left(\frac{\Delta\lambda}{\Delta_d} + \frac{\Delta\mu}{\Delta_s} \right) a_3 \right] \quad .$$

And so, the cross term for the first Euler equation becomes:

$$E^1_{cross} = F^1_{eff} \cdot\cdot \dot{e}_0 \quad .$$

5.2.2. *Second Euler Equation: Cross Terms.* Recall the cross term for the second Euler equation:

$$E^2_{cross} = 2m_1m_2V\Delta\rho\left(\frac{\Delta\mu}{\Delta_s}(\dot{u}_y+\dot{v}_x)+\frac{\Delta\mu}{\Delta_s}\dot{v}_x+\frac{\Delta\lambda}{\Delta_d}\dot{u}_y\right) \quad .$$

Using the definition of the rate of strain, we notice the following:

$$\begin{aligned} \dot{u}_y &= [\dot{e}_{12}]_0+\omega \quad , \\ \dot{v}_x &= [\dot{e}_{12}]_0-\omega \quad \text{where,} \\ \omega &= \frac{1}{2}(\dot{u}_y-\dot{v}_x) \quad . \end{aligned}$$

Substituting these values into E^2_{cross} gives the following:

$$\begin{aligned} E^2_{cross} &= 2m_1m_2V\Delta\rho\left(\frac{\Delta\mu}{\Delta_s}([\dot{e}_{12}]_0+\omega+[\dot{e}_{12}]_0-\omega)+\frac{\Delta\mu}{\Delta_s}([\dot{e}_{12}]_0-\omega)+\frac{\Delta\lambda}{\Delta_d}([\dot{e}_{12}]_0+\omega)\right) \\ &= 2m_1m_2V\Delta\rho\left(\left(3\frac{\Delta\mu}{\Delta_s}+\frac{\Delta\lambda}{\Delta_d}\right)[\dot{e}_{12}]_0+\frac{\Delta\mu}{\Delta_s}(-\omega)+\frac{\Delta\lambda}{\Delta_d}(\omega)\right) \quad . \end{aligned} \tag{53}$$

Thus, define the following tensors:

$$F^2_{eff} = m_1m_2V\Delta\rho\left(3\frac{\Delta\mu}{\Delta_s}+\frac{\Delta\lambda}{\Delta_d}\right)a_2 \quad ,$$

and,

$$C_{eff} = m_1m_2V\Delta\rho\left(\frac{\Delta\mu}{\Delta_s}\underline{ij}+\frac{\Delta\lambda}{\Delta_d}\underline{ji}\right) \quad .$$

So, we have the following expression for the second Euler equation:

$$\begin{aligned} E^2_{cross} &= F^2_{eff}\cdot\cdot\dot{e}+C_{eff}\cdot\cdot\Omega \quad \text{, where} \\ \Omega &= \omega\left(\underline{ij}-\underline{ji}\right) \quad . \end{aligned}$$

5.2.3. *Specifying Forces.* It is now possible to rearrange the Euler equations to obtain the equations of motion for this system.

$$\begin{aligned}
E^1_{diag} - E^1_{cross} &= E^1_{diag} - F^1_{eff} \cdot\cdot \dot{e}_0 \\
&= \frac{\partial}{\partial t}\frac{\partial L_{diag}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L_{diag}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L_{diag}}{\partial u_y} - F^1_{eff} \cdot\cdot \dot{e}_0 \\
&= \frac{\partial}{\partial t}\frac{\partial\left(\underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0}\right)}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial\left(-e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial\left(-e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial u_y} \\
&\quad - F^1_{eff} \cdot\cdot \dot{e}_0 \\
&= 0 \quad ,
\end{aligned}$$

$$\begin{aligned}
E^2_{diag} - E^2_{cross} &= E^2_{diag} - F^2_{eff} \cdot\cdot \dot{e}_0 - C_{eff} \cdot\cdot \Omega \\
&= \frac{\partial}{\partial t}\frac{\partial L_{diag}}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial L_{diag}}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial L_{diag}}{\partial v_y} - F^2_{eff} \cdot\cdot \dot{e}_0 - C_{eff} \cdot\cdot \Omega \\
&= \frac{\partial}{\partial t}\frac{\partial\left(\underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0}\right)}{\partial v_t} + \frac{\partial}{\partial x}\frac{\partial\left(-e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial\left(-e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial v_y} \\
&\quad - F^2_{eff} \cdot\cdot \dot{e}_0 - C_{eff} \cdot\cdot \Omega \\
&= 0 \quad .
\end{aligned}$$

This gives the averaged equations of motion for the system (we return to zero indices):

$$\frac{\partial}{\partial t}\frac{\partial\left(\underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0}\right)}{\partial u_t} = \frac{\partial}{\partial x}\frac{\partial\left(e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial\left(e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial u_y} + F^1_{eff} \cdot\cdot \dot{e}_0 \quad ,$$

$$\frac{\partial}{\partial t}\frac{\partial\left(\underline{\dot{u}_0} \cdot M_{eff} \cdot \underline{\dot{u}_0}\right)}{\partial v_t} = \frac{\partial}{\partial x}\frac{\partial\left(e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial v_x} + \frac{\partial}{\partial y}\frac{\partial\left(e_0 \cdot\cdot D_{eff,d} \cdot\cdot e_0\right)}{\partial v_y} + F^2_{eff} \cdot\cdot \dot{e}_0 + C_{eff} \cdot\cdot \Omega \quad .$$

It is now clear that there are two additional forces that appear in the averaged equations of elastodynamics. The first terms of interest ($F^1_{eff} \cdot\cdot \dot{e}_0$, $F^2_{eff} \cdot\cdot \dot{e}_0$) correspond to a force that is due to the previously defined rate of strain tensor. The second term ($C_{eff} \cdot\cdot \Omega$) corresponds to a Coriolis type force. We shall examine both terms.

However, before this interpretation, we shall introduce a special tensor that will help us interpret the above results. We define the following:

$$\frac{d\underline{\underline{\dot{u}_0}}}{d\underline{r}} = \begin{pmatrix} \frac{\partial \dot{u}}{\partial x} & \frac{\partial \dot{u}}{\partial y} & \frac{\partial \dot{u}}{\partial z} \\ \frac{\partial \dot{v}}{\partial x} & \frac{\partial \dot{v}}{\partial y} & \frac{\partial \dot{v}}{\partial z} \\ \frac{\partial \dot{w}}{\partial x} & \frac{\partial \dot{w}}{\partial y} & \frac{\partial \dot{w}}{\partial z} \end{pmatrix}$$

Due to plane strain, we notice that the above z-derivatives and z-displacements vanish. This tensor can be split into a symmetric and antisymmetric terms:

$$\begin{aligned}\frac{d\underline{\underline{\dot{u}_0}}}{d\underline{r}} &= \begin{pmatrix} \frac{\partial \dot{u}}{\partial x} & \frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) & \frac{\partial \dot{v}}{\partial y} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} - \frac{\partial \dot{v}}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} - \frac{\partial \dot{v}}{\partial x}\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} [\dot{e}_{11}]_0 & [\dot{e}_{12}]_0 \\ [\dot{e}_{12}]_0 & [\dot{e}_{22}]_0 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \\ &= \dot{e}_0 + \Omega \quad .\end{aligned} \tag{54}$$

This observation will help us in interpreting the Euler equations.

5.2.4. *Interpretation of $F^1_{eff} \cdot \cdot \dot{e}_0$ and $F^2_{eff} \cdot \cdot \dot{e}_0$.* Let us solve for $F^1_{eff} \cdot \cdot \dot{e}_0$ and $F^2_{eff} \cdot \cdot \dot{e}_0$ explicitly:

$$\begin{aligned} F^1_{eff} \cdot \cdot \dot{e}_0 &= m_1 m_2 V \Delta\rho \left(\left(\frac{4(\Delta\lambda + 2\Delta\mu)}{\Delta_d} \right) a_1 + 2\left(\frac{\Delta\lambda}{\Delta_d} + \frac{\Delta\mu}{\Delta_s} \right) a_3 \right) \cdot \cdot \dot{e}_0 \\ &= m_1 m_2 V \Delta\rho \left[\left(\frac{4(\Delta\lambda + 2\Delta\mu)}{\Delta_d} \right) [\dot{e}_{11}]_0 + 2\left(\frac{\Delta\lambda}{\Delta_d} + \frac{\Delta\mu}{\Delta_s} \right) [\dot{e}_{22}]_0 \right] \quad , \end{aligned}$$

$$\begin{aligned} F^2_{eff} \cdot \cdot \dot{e}_0 &= m_1 m_2 V \Delta\rho \left(\left(3\frac{\Delta\mu}{\Delta_s} + \frac{\Delta\lambda}{\Delta_d} \right) a_2 \right) \cdot \cdot \dot{e}_0 \\ &= m_1 m_2 V \Delta\rho \left[2\left(3\frac{\Delta\mu}{\Delta_s} + \frac{\Delta\lambda}{\Delta_d} \right) [\dot{e}_{12}]_0 \right] \quad . \end{aligned}$$

As can be seen from the above equation and (54), it appears that this force term arises due to the symmetric portion (i.e., $\dot{e}_0$) of $\frac{d\underline{\underline{\dot{u}_0}}}{d\underline{r}}$. Define the force vector as $\underline{F_{\dot{e}}}$.

Thus, this takes the following form:

$$\begin{aligned}\underline{F_{\dot{e}}} &= \left(F^2_{eff}\cdot\cdot\dot{e}_0\right)\underline{i} + \left(F^2_{eff}\cdot\cdot\dot{e}_0\right)\underline{j}\\ &= m_1m_2V\Delta\rho\left[\left(\frac{4(\Delta\lambda+2\Delta\mu)}{\Delta_d}\right)[\dot{e}_{11}]_0 + 2\left(\frac{\Delta\lambda}{\Delta_d}+\frac{\Delta\mu}{\Delta_s}\right)[\dot{e}_{22}]_0\right]\underline{i}\\ &\quad + m_1m_2V\Delta\rho\left[2\left(3\frac{\Delta\mu}{\Delta_s}+\frac{\Delta\lambda}{\Delta_d}\right)[\dot{e}_{12}]_0\right]\underline{j}\end{aligned}$$

The simultaneous change in ρ and D, along with the motion of the laminar property pattern produce this force (i.e., it disappears if either one of the quantities $\Delta\rho$, ΔD, or V is equal to zero). It is important to note that this force does not appear in the case of one-dimensional strain [12].

5.2.5. *Interpretation of $C_{eff}\cdot\cdot\Omega$.* Let us solve for $C_{eff}\cdot\cdot\Omega$ explicitly:

$$\begin{aligned}C_{eff}\cdot\cdot\Omega &= \left[m_1m_2V\Delta\rho\left(\frac{\Delta\mu}{\Delta_s}\underline{ij}+\frac{\Delta\lambda}{\Delta_d}\underline{ji}\right)\right]\cdot\cdot\left[\omega\left(\underline{ij}-\underline{ji}\right)\right]\\ &= m_1m_2V\Delta\rho\left(\frac{\Delta\lambda}{\Delta_d}-\frac{\Delta\mu}{\Delta_s}\right)\omega \quad .\end{aligned}$$

Thus, this corresponds to a force in the $\underline{j}$-direction. We term this force $\underline{F_\Omega}$, it is as follows:

$$\begin{aligned}\underline{F_\Omega} &= \left(C_{eff}\cdot\cdot\Omega\right)\underline{j}\\ &= m_1m_2V\Delta\rho\left(\frac{\Delta\lambda}{\Delta_d}-\frac{\Delta\mu}{\Delta_s}\right)\omega\underline{j} \quad .\end{aligned}$$

As can be seen from the above equation and (54), it appears that this force term arises due to the anti-symmetric portion (i.e. Ω) of $\frac{d\underline{\underline{\dot{u}_0}}}{d\underline{r}}$. This anti-symmetric portion generates an accompanying vector, call it $\underline{\omega}=\omega\underline{k}$, such that the above force is equal to $\underline{\omega}\times\underline{a}$, where $\underline{a}=a\underline{i}=m_1m_2V\Delta\rho(\frac{\Delta\lambda}{\Delta_d}-\frac{\Delta\mu}{\Delta_s})\underline{i}$, i.e.,

$$\begin{aligned}\underline{\omega}\times\underline{a} &= \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 0 & 0 & \omega\\ a & 0 & 0\end{vmatrix}\\ &= \omega a\underline{j}\\ &= m_1m_2V\Delta\rho\left((\frac{\Delta\lambda}{\Delta_d}-\frac{\Delta\mu}{\Delta_s}\right)\omega\underline{j} \quad .\end{aligned}$$

Like the previous force $\underline{F_{\dot{e}}}$, it appears that this force arises strictly due to the presence of simultaneous change in both inertial and elastic properties in a dynamical pattern of original material constituents (i.e., it disappears if either $\Delta\rho=0$, or

$\Delta D = 0$, or $V = 0$). The physical reason for this is the motion of the main dynamic disturbance relative to the background motion of the material pattern. The plane strain situation results in the appearance of transverse displacement that creates rotation of every elementary material volume. The appearance of the Coriolis type force is a consequence of both dynamics and plane strain; it doesn't arise in the case of one dimensional strain that is typical for longitudinal dynamic disturbances that propagate along an elastic bar [12], this is why it is so important.

6. Conclusion

Two special forces were discovered as a direct result of homogenization. The first force $\underline{F_{\dot{e}}}$ that arises (due to the symmetric portion of $\frac{d\dot{u}}{d\underline{r}}$) takes care of deformational motion similar to that of a liquid particle in classical hydrodynamics. As to the second force $\underline{F_{\Omega}}$, this one is perceived as the Coriolis force that always emerges when rotation occurs as a relative motion. It is a kind of pseudo-Coriolis effect associated with the motion of the material pattern interpreted as a background motion. The appearence of the two force terms is interesting, and perhaps a more general approach (i.e. one without the constraint of plane strain) will allow for a better algebraic characterization of the relevant quantities.

The research undertaken has great implications in the design of future composite materials and nano-arrays. The two forces that were discovered clearly must be taken into account when constructing such devices. Future work should focus on computational modeling and experimental verification of the forces, in order to better understand them. Though the research conducted in this project was theoretical, it is more than feasible that, in time, there will be a wide implimentation to mechanical devices.

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