Probability for Applications
Students' Activity Book

Based on Elementary Probability for Applications (2009) by Rick Durrett

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Welcome to Your Probability Note-Taking Adventure!

Drawing from the esteemed ***"Elementary Probability for Application" textbook authored by Rick Durret***, this activity book serves as an invaluable supplement to traditional lecture notes.

What you can expect:

- **Interactive Blank Lecture Notes:**
  The book provides structured lecture notes that align with the core content of Durret's textbook, ensuring clarity and coherence in understanding fundamental probability principles. Gone are the days of passive note-taking! The activity book incorporates sections for students to fill in their own notes, encouraging active participation and reinforcing comprehension as they engage with the material directly.
- **Supplementary Exercises:**
  Practice makes perfect, and this book doesn't skimp on opportunities for practice. It features extra exercises to challenge students and reinforce their understanding of probability theory.
- **Detailed Solutions:**
  Learning from mistakes is just as important as getting things right the first time. That's why the book includes comprehensive solutions to all exercises, providing invaluable feedback and guidance for students as they work through the material.

This Student Activity Book on Probability is not just another resource; it's a dynamic tool for active learning, designed to empower students to master probability concepts with confidence. Whether used alongside lectures, as part of a study group, or for independent study, it's a must-have for anyone seeking to deepen their understanding of probability theory and its practical applications.

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Acknowledgment

I would like to express my deepest gratitude to everyone who contributed to the successful completion of the Women's Impact Network (WIN) - funded Open Educational Resources (OER) Development grant project at WPI, particularly in the creation of the Student Activity Book based on Lecture Notes and exercises in Probability for Applications.

First and foremost, I am immensely thankful to Prof. Buddika Peiris and Dr. Tharindu De Alwis for their unwavering support and invaluable guidance throughout the development of the lecture notes. Their expertise and encouragement were instrumental in shaping the content and structure of the materials, ensuring their relevance and educational value.

I am also indebted to Prof. Marja Bakermans and Ms. Lori Ostapowicz-Critz, the mentors of the OER project, whose constructive feedback and dedicated mentorship were pivotal in refining the educational resources. Their commitment to fostering open education initiatives has been truly inspiring. I extend my appreciation to Vikranth Vilas, our diligent computer assistant, for his technical expertise and assistance with the project.

Furthermore, I extend my appreciation to all the individuals who provided encouragement, feedback, and assistance at various stages of this endeavor. Your collective efforts have played an integral role in making this project a success.

Lastly, I would like to acknowledge the funding support provided by the EMPOwER (Engaging More Powerfully, Openly with Educational Resources) grant at WPI, which made this project possible and contributed to the broader goal of advancing open educational resources in our academic community.

Thank you all for your unwavering support, guidance, and commitment to enhancing educational resources for our students.

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Chapter 01: Basic Results

Outline 1. Overview:

- Random Experiment, Sample Space and Events
- Probability of an Event and Axioms
- Properties of Probability
- Conditional Probability
- Independence of Events
- Random Variable
- Probability Mass Function
- Geometric Distribution
- Expectation
- Variance

1.1 What Is Probability?

- The study of probability is the study of random phenomena. More specifically, in this course we will study **probabilistic models**: mathematical models of random phenomena.
- Such models can be used to quantify, explain, and to some extent at least, predict random phenomena.

The probability models we will consider are based on the notion of a **random experiment**.

1.2 Random Experiment

A **random experiment** is a process that results in exactly one of a collection of possible outcomes, but whose actual outcome cannot be known with certainty in advance.

1.2.1 Sample Space

The set of possible outcomes is called the **sample space**, and will be denoted Ω.

Example: 1. **Tossing a Coin Three Times:**
Consider tossing a coin three times and recording the side that ends facing up.

Example: 2. **Tossing a Coin until a Tail is Appeared:**
Consider tossing a coin until a tail is appeared and recording the all the results in order.

Example: 3: **Oil Leaks:**
In a long stretch of an oil pipeline, the number of leaks in a given year serious enough to require a special repair crew is considered a random experiment.

Example: 4. **Lifetime of a bulb**
Measuring the lifetime of a light bulb is considered a random experiment.

1.2.2 Events

> **Definition 1.1.** An **event** is **any subset** of the sample space Ω. Looked at another way, an event is a set of possible outcomes of the random experiment.

Here are some fundamental operations on events:

Complementation
If A is an event, so is its complement

$$A^c = \{\omega \in \Omega | \omega \notin A\}$$

.

Union
If A and B are events, so is their union

$$A \cup B = \{\omega \in \Omega | \omega \in A \text{ or } \omega \in B\}$$

Intersection
If A and B are events, so is their intersection

$$A \cap B = \{\omega \in \Omega | \omega \in A \text{ and } \omega \in B\}$$

.

Example: Tossing a coin Three Times

If we define two events, A - getting a head on the second toss, B - getting exactly two heads,

- $\Omega =$
- $A =$
- $B =$
- $A^c =$
- $B^c =$
- $A \cup B =$
- $A \cap B =$

NOTE: The union and intersection operations extend to any number of events:

If $A_1, A_2, \ldots A_m$ are events then so are

- $\bigcup_{n=1}^{m} A_n = \{\omega \in \Omega | \omega \in A_n \text{ for some } n \in \{1, \ldots, m\}\}$
- $\bigcap_{n=1}^{m} A_n = \{\omega \in \Omega | \omega \in A_n \text{ for all } n \in \{1, \ldots, m\}\}$

And if there are an infinite number of events, $A_1, A_2, \ldots$,

- $\bigcup_{n=1}^{\infty} A_n = \{\omega \in \Omega | \omega \in A_n \text{ for some } n\}$
- $\bigcap_{n=1}^{\infty} A_n = \{\omega \in \Omega | \omega \in A_n \text{ for all } n\}$

Here is some more terminology about events:

- An event A is contained in event B if every outcome $\omega \in A$ is also in B. We write $A \subset B$.

- Two events A and B are equal ($A = B$) if and only if $A \subset B$ and $B \subset A$.

- Ω, the entire sample space, and $\emptyset$, the set with no outcomes are events.

- Two events A and B are called **disjoint** if $A \cap B = \emptyset$.

- Events $A_1, A_2, \ldots$ are **pairwise disjoint** (i.e., **mutually exclusive**) if $A_i \cap A_j = \emptyset$ for each $i \neq j$.

- A collection of events $\{A_1, A_2, \ldots\}$ is **exhaustive** if $\bigcup_n A_n = \Omega$.

- A collection of events $\{A_1, A_2, \ldots\}$ is a **partition** of Ω if it is exhaustive and $A_1, A_2, \ldots$ are pairwise disjoint.

The Algebra of Events

Combining the operations of complementation, union and intersection results in algebraic relations on events. Assuming B and $\{A_1, A_2, \ldots\}$ are events, here are some examples:

- $B \cup B^c = \Omega$
- $B \cap B^c = \emptyset$

DeMorgan's Laws:

- $(\bigcup_n A_n)^c = \bigcap_n A_n^c$
- $(\bigcap_n A_n)^c = \bigcup_n A_n^c$

Example: Tossing a coin Three Times

Consider two events:
$A =$ getting a head on the second toss, $B =$ getting exactly two heads. Then,

$$\begin{aligned}
\Omega &= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \\
A &= \{HHH, HHT, THH, THT\} \\
A^c &= \{HTH, TTH, HTT, TTT\} \\
B &= \{HHT, HTH, THH\} \\
B^c &= \{HHH, HTT, THT, TTH, TTT\} \\
A \cup B &= \{HHH, HHT, HTH, THH, THT\} \\
A \cap B &= \{HHT, THH\}
\end{aligned}$$

- $(A \cup B)^c =$
- $(A \cap B)^c =$

1.3 Probability Function

Event Space

The event space $\mathcal{A}$ is the set of all events of the sample space Ω.

Probability of an Event

- The probability law (or the probability function), $P(\cdot)$, is a function that assigns to any event A a non-negative real number.
- $P(A)$, called the probability of event A.
- $P(A)$ measures how likely it is that the outcome of the random experiment is one of the outcomes in A.

The probability law must satisfy the following three axioms:

1. **(Nonnegativity)** $P(A) \geq 0$ for every event A.
2. **(Additivity)**
   (a) **(Finite Case)** If $\{A_1, A_2, \dots, A_m\}$ are pairwise disjoint events, then

   $$P\left(\bigcup_{n=1}^{m} A_n\right) = \sum_{n=1}^{m} P(A_n)$$

   (b) **(Infinite Case)** If $\{A_1, A_2, \dots, \}$ are pairwise disjoint events, then

   $$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

3. **(Normalization)** $P(\Omega) = 1$.

NOTE:

1. $P(\phi) = 0$.

2. If an event A consists of a finite or countably infinite set of outcomes $A = \{\omega_1, \omega_2, \ldots\}$, then the additive axiom results in
$$P(A) = \sum_n P(\omega_n).$$

> **Definition 1.2. (Probability of an event)**: If the sample space consists of a finite number of equally likely outcomes, the probability of an event A:
>
> $$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega} = \frac{n(A)}{n(\Omega)}$$

3. $0 \leq P(A) \leq 1$ for any event A.
   - 0: impossibility of the event (0% chance)
   - 1: certainty (100% chance)
   - Higher the probability of an event: the more likely it is that the event will occur

Example: Consider an experiment of rolling a pair of 4 sided fair dice. What is the probability of,

1. $A =$ sum of the rolls is even.

2. $B =$ the sum of the roll is odd.

3. $C =$ the first roll is equal to the second roll.

4. $D =$ the first roll is larger than the second roll.

5. $E =$ at least one roll is equal to 2.

1.3.1 Properties of Probability Laws

1. If A and B are events, $A \subset B$ implies $P(A) \leq P(B)$.

   NOTE:

   (a) $P(A \cap B) \leq \min\{P(A), P(B)\}$

   (b) $P(A \cup B) \geq \max\{P(A), P(B)\}$

2. $P(B) = P(A \cap B) + P(A^c \cap B)$ and $P(A) = P(A \cap B) + P(A \cap B^c)$.

   Proof:

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   Proof:

4. $P(A \cup B) \leq P(A) + P(B)$
   Proof:

   NOTE: This formula is used to approximate the probability of the union when the probability of the intersection unknown.

5. For any event A, $P(A^c) = 1 - P(A)$.

Example: Let $P(A) = 0.7, P(B) = 0.5, P(A \cap B) = 0.4$. Find $P(A \cup B)$.

Example: Let $P(A) = 0.7, P(B) = 0.2$. Find $P(A \cup B)$.

Probabilities of Unions

Recall: For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This can be extended to three events.

Definition 1.3. For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Example: Suppose that we roll 6-sided 3 dice. What is the probability that we get at least one 4?

General Union

Definition 1.4. For any events $A_1, A_2, A_3, \cdots, A_n$,

$$P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1} P((A_1 \cap A_2 \cap A_3 \cdots \cap A_n).$$

1.4 Conditional Probability

Additional information about a random experiment can change both the sample space and probability law. To see how, consider the following example.

Example: Consider tossing three coins experiment. Find the probability of getting exactly two heads **if the first toss is a head**.

Definition 1.5. (Conditional Probability) If A and B are events, with $P(B) > 0$, then the probability of A given B is defined as;

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 1.6. (Multiplication Rule): If A and B are events,

$$P(A \cap B) = P(A|B)P(B).$$

Example: Find the probability that a single toss of a balanced die results in a number less than 4 if,

1. no other information is given.
2. **it is given** that the toss resulted in an odd number.

1.5 Independence

Idea : Two events, A and B are said to be **independent** if the occurrence of one does not affect the probability that the other occurs.

Definition 1.7. (Independence): If $P(B) > 0$, A and B are said to be independent iff

$$P(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Another Result: A and B are are independent iff

$$P(A \cap B) = P(A)P(B)$$

Independence : General Results

There are two type of independence when we consider more than two events:

Mutually Independence

The events $\{A_i\}_{i=1}^n$ are **mutually independent** if

$$P\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S} P(A_i), \quad \text{for every subset } \ S \subset \{1, \dots, n\}$$

Pairwise Independence

The events $\{A_i\}_{i=1}^n$ are **pairwise independent** if

$$P\left(A_i \cap A_j\right) = P(A_i) \cdot P(A_j), \ \text{for all } i, j \in \{1, \dots, n\} \ \text{ and } i \neq j$$

Example: Tossing a Coin Three Times

If the coin is fair, the probability of a head on any one toss is 1/2, the same as the probability of a tail. Let H_j (T_j) be the event we get a head (tail) on toss j. If the tosses are done **independently**, the probability of getting the sequence HTT is:

$P(\{HTT\}) =$

This computation shows that each of the eight outcomes is equally likely, a result we obtained previously by assumption.

Example: Tossing a Coin Three Times Ctd...

Now suppose the coin is not necessarily fair, but has on each toss,

$$P(H_j) = p, \quad P(T_j) = 1 - p = q$$

Then we can compute,

$P(\{HTT\}) =$

Thus, the independence assumption allows us to model a wider range of random behavior.

Example: Weight/Height

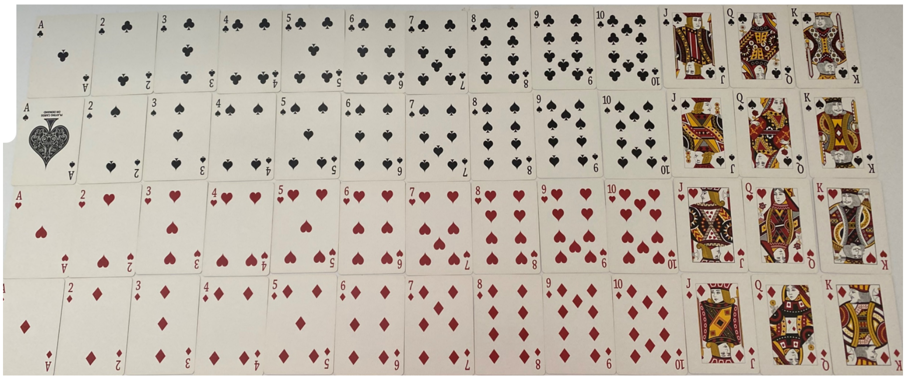
This table relates the weights and heights of a group of individuals participating in an observational study.

Weight/Height	Tall	Medium	Short	Total
Obese	18	28	14	60
Normal	20	51	28	99
Underweight	12	25	9	46
Total	50	104	51	205

Are the events Obese and Tall independent?

Example: Drawing Two Cards

Consider the random experiment of drawing two cards from a regular card deck. What is the probability that both are clubs?
(Regular card deck (52 cards): 13 -Spades, 13-Clubs, 13- Hearts, 13- Diamonds)



Example: Birthdays

Consider the three events:
A - Alice and Betty have the same birthday,
B - Betty and Carol have the same birthday,
C - Carol and Alice have the same birthday.

(a) Are A, B, C pairwise independent?

(b) Are A, B, C mutually independent?

1.6 Random Variables

Idea: You can think of a random variable as a **numerical measurement** assigned to each outcome of a random experiment.

Example: If you bet \$100 on red in a spin of a roulette wheel, the outcome of the random experiment (the spin of the wheel) is one of $\{$red, black, neutral$\}$, but you are probably more interested in X, the amount you gain:

$$X(\{red\}) = 100,\ X(\{black\}) \ = \ X(\{neutral\}) = -100.$$

Here, X is a random variable.

> **Definition 1.8.** Given a sample space Ω, **a random variable** X is a **function** that assigns to each outcome $\omega \in \Omega$ a real number $X(\omega)$.

NOTE:

1. A random variable is **discrete** if its possible values (its range) form a **finite or countably infinite** set of real numbers. For definiteness, let $\mathcal{R}(X)$ denote the range of X.

2. Random variable is **continuous** if it can takes values in an **interval**.

3. Random variables are denoted by capital letters while their values are denoted by lower case letters.

   E.g:- X- random variable, x- value; $X(\omega) = x$.

Examples:

1. **Tossing a coin 3 times**.
   Let X be the number of tails in each outcome.

2. **Rolling two six sided dice.**
   Let X - the sum of the two numbers.

3. **Rolling a die until a 3 appears.**
   Let X - the number of rolls.

4. **Time Length.**
   Let T be the length of time it takes a truck driver to go from Worcester to Boston.

1.6.1 The Probability Mass Function (PMF)

> **Definition 1.9.** Probability distribution of a discrete random variable is called as probability mass function (PMF) and it is defined as the probability of the event $\{X = x\}$.
>
> $$i.e., P(x) = P(X = x) = P(\{\omega \in \Omega | X(\omega) = x\})\ x \in \mathcal{R}(X).$$

Properties of PMFs:

- $0 \leq P(x) \leq 1$
- $\sum_{x \in \mathcal{R}(X)} P(x) = 1$
- $\sum_{x \in A \cap \mathcal{R}(X)} P(x) = P(X \in A)$, for every subset A of real numbers.

Example: Among a sample of 50 families, 10 families had 2 members, 13 families had 3 members, 21 families had 4 members and rest of the families had 5 members. Let X be the number of members in the family.

1. Find the PMF of X.

2. Compute the probability that a family has 3 or more members.

1.6.2 Geometric Distribution

Consider a sequence of **independent and identical trials such that each trial has only two outcomes** (Success and failure) with,

$$P(successs) = p \qquad \text{on each trial (i.e., binary trials)}$$

Definition 1.10. Let X be number of trials to come up (until) the first success. Then X follows a geometric distribution ($X \sim$ Geometric(p)) and its probability distribution (or PMF) is given by,

$$P(X = x) = P(x) = (1-p)^{x-1}p \ : \ x = 1, 2, 3, \cdots .$$

Example: A certain product is produced by a machine which has a 4% defective rate. **Let X be the number of items inspected until the first defective occurs**.

1. Find the PMF of X.

2. What is the probability that the first defective occurs **at the third** item inspected?

3. What is the probability that the first defective occurs **in the first three** inspections?

Example: In a football event, the first team to win three games wins the championship.(Suppose there are only two teams and each team has same chance to win any of the games). What is the probability distribution of number of games?

1.6.3 Expected Value

Idea: Expected value describes the long-term average level of a random variable based on its probability distribution. **Expectation measures the center of a distribution.**

Definition 1.11. The **expected value** of random variable X (mean of X or expectation of X or average of X) is denoted by $E(X)$ and defined as

$$E(X) = \sum_x xP(X = x) = \sum_x xP(x)$$

Example: 1. If you play roulette and bet $\$1$ on black then you win $\$1$ with probability $18/38$ and you lose $\$1$ with probability $20/38$. What is the expected win?

Example: 2. Find the expectation of Geometric distribution with parameter p.

NOTE: 1. Expectation of a function of a random variable.

Let $g(X)$ be a function of random variable X, then

$$E(g(x)) = \sum_x g(x)P(x)$$

Example :

Consider $$P(X) = \begin{cases} 1/16, & : x = 0, \\ 6/16, & : x = 1, \\ 9/16 & : x = 2. \end{cases}$$ Find$E(X^2)$.

2) If a and b are constants,

$$E(aX + b) = a \cdot E(X) + b$$

Proof:

3) If $X_1, X_2, X_3, \cdots, X_n$ are random variables and $a_1, a_2, a_3, \cdots, a_n$ are constants. Then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

Specially,

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

1.6.4 Variance

Idea: Variance measures the spread of a distribution. It determines the degree to which the values of a random variable differ from the expected value.

> **Definition 1.12.** The variance of random variable X is denoted by $Var(X)$ or $V(X)$, and defined as
> $$Var(X) = E\left[(X - E(X))^2\right].$$

NOTE:

1) $Var(X) = E(X^2) - [E(X)]^2$

Proof:

2) $Var(aX + b) = a^2 \cdot Var(X)$

Proof:

1.6.5 Standard Deviation

Definition 1.13. Standard deviation of random variable X is denote by $\sigma(X)$, and defined as

$$\sigma(X) = \sqrt{Var(X)}.$$

Example :

Let X be the amount of memory of purchased in a flash drive (GB). The PMF of X is given in the below. Find the mean and the standard deviation of the following distribution.

$X = x$	1	2	4	8	16
$P(X = x)$	0.05	0.10	0.35	0.40	0.10

NOTE:

If $X \sim Geometric(p)$, then

$$E(X) = \frac{1}{p} \qquad\qquad Var(X) = \frac{1-p}{p^2}.$$

Example : Let $X \sim Geometric(p = 0.2)$. Then,

1. $E(X) =$
2. $Var(X)$

1.7 Summary

Summary 1. ***Chapter 01: Basic Results***

- Experiment: an action whose outcome cannot be predicted with certainty.
- Event: Some specified result that may or may not occur when an experiment is performed. Can be denoted as capital letters, $A, B, \ldots$

  Suppose an experiment has $N = n$ possible outcomes, all equally likely. An event that can occur in f ways

  $$\text{Probability of an event A} = \frac{\text{Number of ways event occur}}{\text{Total number of outcomes}} = \frac{n(A)}{n(\Omega)} = \frac{f}{N}$$

- Relations among events
  - A^c : (not A) : the event A does not occur
  - $A \cap B$: $(A\&B)$: the event both A and B occur
  - $A \cup B$: (A or B) : the event either A or B or both occur
- Two events are disjoint if $A \cap B = \emptyset$
- Events $A_1, A_2, \ldots, A_n$ are pairwise disjoint if $A_i \cap A_j = \emptyset$ for each pair $i, j = 1, 2, 3, \ldots, n$
- A is a subset of B if every outcome is also in B, $A \subset B$.
- De-Morgan's Law
  - $(\bigcup_n A_n)^c = \bigcap_n A_n^c$
  - $(\bigcap_n A_n)^c = \bigcup_n A_n^c$
- Probability
  - **(Nonnegativity)** $P(A) \geq 0$ for every event A.
  - **(Additivity)**
    - **(Finite Case)** If $\{A_1, A_2, \ldots, A_m\}$ are pairwise disjoint events, then

      $$P\left(\bigcup_{n=1}^{m} A_n\right) = \sum_{n=1}^{m} P(A_n)$$

    - **(Infinite Case)** If $\{A_1, A_2, \ldots, \}$ are pairwise disjoint events, then

      $$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

  - **(Normalization)** $P(\Omega) = 1$.

- Some rules on Probability
  - Complement Rule: $P(A) = 1 - P(A)$
  - $P(A \cap B) \leq \text{min}\{P(A), P(B)\}$
  - $P(A \cup B) \geq \text{max}\{P(A), P(B)\}$
  - $P(B) = P(A \cap B) + P(A^c \cap B)$.
  - Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - $P(A \cup B) \leq P(A) + P(B)$

Conditional Probability

- The conditional probability of A given B is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

- The conditional probability of B given A is,

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, \quad P(A) > 0$$

- The Multiplication rule

$$P(A \cap B) = P(A|B)P(B) \text{ or } P(A \cap B) = P(B|A)P(A)$$

Independence of events

- Events are independent of each other if and only if

$$P(A \cap B \cap C \cap \ldots) = P(A) \cdot P(B) \cdot P(C) \cdot \ldots$$

- The events $\{A_i\}_{i=1}^n$ are **mutually independent** if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i), \quad \text{for every subset } \ S \subset \{1, \ldots, n\}$$

- Events $\{A_i\}_{i=1}^n$ are **pairwise independent** if

$$P\left(A_i \cap A_j\right) = P(A_i) \cdot P(A_j), \text{ for all } i, j \in \{1, \ldots, n\} \text{ and } i \neq j$$

Random Variables

- Random variable: a numerical description of any outcome of an experiment.

$$X(\omega) = x$$

- Bernoulli Random variable: random variable whose only possible values are 0 and 1.
- Types: Discrete random variables and Continuous random variables.

Probability Mass Function (PMF)

- Definition:

$$P(x) = P(X = x) = P(\{\omega \in \Omega | X(\omega) = x\}) \; x \in \mathcal{R}(X).$$

- Properties:
  - $0 \leq P(x) \leq 1$
  - $\sum_{x \in \mathcal{R}(X)} P(x) = 1$
  - $\sum_{x \in A \cap \mathcal{R}(X)} P(x) = P(X \in A)$, for every subset A of real numbers.

Geometric Distribution

(Can use if we have infinite outcomes in the sample space)

- Each trail is binary (2 outcomes: success or failure)
- Each trial is independent.
- X be number of trials to come up the first success and $p = P(success)$.

Then X **follows a geometric distribution**. ($X \sim$ Geometric(p)) and its probability distribution is given by

$$P(X = x) = P(x) = (1-p)^{x-1}p \; : \; x = 1, 2, 3, \cdots .$$

Let $X \sim Geometric(p)$. Then,

$$E(X) = \frac{1}{p}, \qquad Var(X) = \frac{1-p}{p^2}$$

Expected Value

- Expected Value
$$E(X) = \sum_x xP(X = x) = \sum_x xP(x)$$
- Let $g(X)$ be a function of random variable X, then
$$E(g(x)) = \sum_x g(x)P(x)$$
- If a and b are constants,
$$E(aX + b) = aE(X) + b$$
- If $X_1, X_2, X_3, \cdots, X_n$ are random variables and $a_1, a_2, a_3, \cdots, a_n$ are constants. Then
$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

Variance and Standard Deviation

- Variance:
$$Var(X) = E\left[(X - E(X))^2\right].$$
- Shortcut Formula
$$Var(X) = E(X^2) - [E(X)]^2$$
- Properties:
$$Var(aX + b) = a^2 Var(X)$$
- Standard Deviation:
$$\sigma(X) = \sqrt{Var(X)}.$$

1.8 Exercises

1. Two six-sided dice are rolled. What is the probability that
   (a) the two numbers will differ by 1 or less and
   (b) the maximum of the two numbers will be 5 or larger? *[Page 26; Durret, 2009]*

2. In a group of students, 25% smoke cigarettes, 60% drink alcohol, and 15% do both. What fraction of students have at least one of these bad habits? *[Page 27; Durret, 2009]*

3. Two students, Alice and Betty, are registered for a statistics class. Alice attends 80% of the time, Betty 60% of the time, and their absences are independent. On a given day, what is the probability that
   (a) at least one of these students is in class and
   (b) exactly one of them is there? *[Page 28; Durret, 2009]*

4. How many times should a coin be tossed so that the probability of at least one head is $\geq$ 99%? *[Page 29; Durret, 2009]*

5. The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying joint probability table gives the proportions of individuals in the various ethnic group–blood group combinations. Suppose that an individual is randomly selected from the population, and define events by $A-$ type A selected, $B-$ type B selected, and $C-$ ethnic group 3 selected.
   (a) Calculate $P(A), P(C)$, and $P(A \cap C)$.
   (b) Calculate both $P(A|C)$ and $P(C|A)$, and explain in context what each of these probabilities represents.
   (c) If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1?

6. Let X has PMF given by $P(X = x) = \frac{1}{5}$ for $x = -2, -1, 0, 1, 2$ and 0 for otherwise. Find the PMF of $Y = 2X + 1$.

7. Determine the constant k, so that the following PMF of the random variable is a valid probability mass function:

$$P(x) = \begin{cases} k \cdot (7x + 3) & ; x = 1, 2, 3 \\ 0 & ; \text{otherwise} \end{cases}$$

8. You play a game of chance that you can either win or lose (there are no other possibilities) until you lose. Your probability of losing is $p = 0.57$. Let X be the number of games you play until you lose (includes the losing game).
   (a) What is the probability that it takes five games until you lose?
   (b) What is the probability that it takes at least three games until you lose?
   (c) Find $E(X)$ and $Var(X)$.

9. A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose the PMF of X is as given in the accompanying table. Calculate the probability of each of the following events.

x	0	1	2	3	4	5	6
$P(x)$	0.10	0.15	0.20	0.25	0.20	0.06	0.04

   (a) {at most three lines are in use}
   (b) {fewer than three lines are in use}
   (c) {at least three lines are in use}
   (d) {between two and five lines, inclusive, are in use}
   (e) {between two and four lines, inclusive, are not in use}
   (f) {at least four lines are not in use}

10. Use the PMF of X in the previous question. Then find the following.

   (a) $E(X)$
   (b) $Var(X)$
   (c) $\sigma(X)$
   (d) $E(2X+1)$
   (e) $Var(3X-1)$

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Chapter 02: Combinatorial Probability

Outline 2. Overview:

- Recall: Probability of an events with equally likely outcomes
- Multiplication Rule
- Factorial
- Permutation
- Combination
- Binomial Theorem
- Partition
- Binomial Distribution
- Multinomial Distribution
- Poisson Distribution
- Poisson Approximation of Binomial

Recall from Chapter 01:

- If the sample space Ω consists of **finite equally likely outcomes**, then the probability of event A is,
$$P(A) = \frac{n(A)}{n(\Omega)}.$$
- In order to compute probabilities, it is often necessary to count the number of possible outcomes in a random experiment or an event.

We use some counting principles and formulas as below.

2.1 Multiplication Rule

Definition 2.1. If a process has r stages with n_1 results for stage 1, n_2 results for stage 2 and so on, then the total number of results for the entire process is

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i.$$

Example: If each license plate in a state consists of three letters followed by four integers, how many different license plates are possible?

Example:
How many ways can 5 people stand in line?

Note: General Formula:

Definition 2.2. Total number of ways to arrange n different objects on a line is: $n!$.

Factorial

Definition 2.3. Let n be a non negative integer. Then the factorial n is,

$$n! = \begin{cases} 1 & : n = 0, \\ n \times (n-1) \times (n-2) \times \cdots \times 1 & : n > 0. \end{cases}$$

Note:

$$
\begin{aligned}
0! &=1\\
1! &=1\\
2! &=2 \times 1 = 2\\
3! &=3 \times 2 \times 1 = 6\\
n! &=n \times (n-1) \times (n-2) \times \cdots \times 1\\
n! &=n \times (n-1)!\\
n! &=n \times (n-1) \times (n-2)!\\
&\vdots\\
n! &=n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) \times (n-k)!
\end{aligned}
$$

2.2 Permutations (Order matters/ordered subset)

- Act of arranging objects or numbers in order.

> **Definition 2.4.** Number of different ways to pick k objects out of n different objects and arrange them on a line is denoted by ${}^nP_k$ where,
>
> $$ {}^nP_k = \frac{n!}{(n-k)!} \ : \ k \leq n. $$

Note:

- When $k = 0$, ${}^nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
- When $k = n$, ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$
- ${}^nP_k = \frac{n!}{(n-k)!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) \times (n-k)!}{(n-k)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$

Example: A little league baseball team has 15 players on its roaster. Home many ways are there to select 9 players to form a starting lineup?

Example: A student activity club at a college has 24 members. In how many different ways can the club select a president, a vice president, a treasurer, and a secretary?

Example: In how many ways can 3 novels, 2 mathematics books and a chemistry book be arranged on a shelf if,

1. the books can be arranged in any order?
2. the mathematics books must be together and the novels must be together?

2.3 Combinations (Order does not matter/unordered subset)

- A way of selecting objects or numbers from a group of objects or collections.

> **Definition 2.5.** Number of different ways to pick k objects out of n objects (do not arrange) is denoted by ${}^{n}C_{k}$ or $\binom{n}{k}$ where,
>
> $$ {}^{n}C_{k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} \ : \ k \leq n. $$

Note:

- ${}^{n}C_{0} ={}^{n} C_{n} = 1$, for any positive integer n

- ${}^{n}C_{m} ={}^{n} C_{n-m}$, for any positive integer $m \leq n$

  Proof:

- ${}^{n-1}C_{k-1} +{}^{n-1} C_{k} ={}^{n} C_{k}$

  Proof:

- ${}^{n}C_{r}$ values can be obtained from the Pascal's triangle.

Pascal's Triangle

$$\begin{array}{cccccccccccc}
 & & & & & 1 & & & & & & \leftarrow n=0 \\
 & & & & 1 & & 1 & & & & & \leftarrow n=1 \\
 & & & 1 & & 2 & & 1 & & & & \leftarrow n=2 \\
 & & 1 & & 3 & & 3 & & 1 & & & \leftarrow n=3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & \leftarrow n=4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \leftarrow n=5 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Example: A group of 4 students is to be chosen from 15 member class to represent the class on the student council. How many ways can this be done?

Example: Consider flipping 4 fair coins. What is the probability distribution of random variable X, where X be the number of tails out of 4 flips?

Binomial Theorem

- A method of expanding an expression that has been raised to any finite power.
- For any real numbers x, y, and positive integer n,

$$(x+y)^n = \sum_{m=0}^{n} {}^nC_m \cdot x^{n-m}y^m =^n C_0 \cdot x^n +^n C_1 \cdot x^{n-1}y + \cdots +^n C_{n-1} \cdot xy^{n-1} +^n C_n \cdot y^n.$$

Example: $(x+y)^3 =$

Example: From a group of 16 graduates and 20 undergraduates, a researcher wants to randomly select 5 graduates and 6 undergraduates for a study. In how many ways can the study group be selected?

2.4 Partitions

Suppose a set of n elements is divided into r disjoint subsets such that i^{th} subset has n_i elements, $i = 1, 2, 3, \cdots, r$, and $n_1 + n_2 + \cdots + n_r = n$.

Then the total number of different choices

$$
\begin{aligned}
&=^n C_{n_1} \times^{n-n_1} C_{n_2} \times^{n-(n_1+n_2)} C_{n_3} \times^{n-(n_1+n_2+n_3)} C_{n_4} \times \cdots \times^{n-(n_1+n_2+\cdots+n_{r-1})} C_{n_r} \\
&= \frac{n!}{n_1! \times (n-n_1)!} \times \frac{(n-n_1)!}{n_2! \times (n-(n_1+n_2))!} \times \frac{(n-(n_1+n_2))!}{n_3! \times (n-(n_1+n_2+n_3))!} \times \\
&\qquad \cdots \times \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{n_r! \times (n-n_1-n_2-\cdots-n_{r-1}-n_r)!} \\
&= \frac{n!}{n_1! \times n_2! \times n_3! \times \cdots \times n_r!} \\
&= \binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}
\end{aligned}
$$

NOTE :

- ${}^nC_r = \binom{n}{r}$ is called the **binomial coefficient**.
- $\binom{n}{n_1, n_2, \ldots, n_r}$ is called the **multinomial coefficient**.
- Partition: used to find the number of arrangements when all the objects are not different (i.e., when some of them are same).

Definition 2.6. Suppose there are n objects but n_1 of them are same kind, n_2 of them are another kind and so on (r groups).

Then the total number of arrangements is:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Example: There are 39 students in a class. In how many ways can a professor give out 9 A's, 13 B's, 12 C's and 5 F's?

Example: A house has 10 rooms. We want to paint 2 yellow, 3 blue, and 5 pink. How many ways can this be done?

Example: How many different words can be obtained by rearranging the word "MASSACHUSETTS"?.

Example: A class consisting of 4 graduate students and 12 undergraduate students randomly divide into four groups of 4. What is the probability that each group includes a graduate student?

2.5 Binomial Distribution

Binomial Setting: A random experiment consists of n Bernoulli trails such that,

1. the trials are independent and identical.
2. each trial results in only two possible outcomes, success and failure.
3. $P(Success) = p$.

called a binomial experiment.

Definition 2.7. Let X be number of successes out of n trials.

- Then X follows a **binomial** distribution with parameters n and p.
- The probability distribution (PMF) is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n.$$

Note: If $X \sim Bin(n, p)$, then,

1. $\sum_x P(X = x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1.$

   Proof:

2. $E(X) = np$

   Proof:

3. $Var(X) = np \cdot (1 - p)$

Example: Suppose that a 6-sided die is rolled 4 times. Let X be the number of threes.

1. Find the probability distribution of X.

2. What is the probability of getting **exactly 2** threes?

3. What is the probability of getting **at least one** three?

4. What is the probability of getting **at most one** three?

5. Find the expected value and the variance of the number of threes out of four trials.

Example: Recall the random experiment of tossing a coin 3 times. Suppose the probability of getting a head is p. Let X is the number of heads.

Then $X \sim Bin(3, p)$ and has PMF,

$$P(x) = \begin{cases} (1-p)^3, & x = 0, \\ 3p(1-p)^2, & x = 1, \\ 3p^2(1-p), & x = 2, \\ p^3, & x = 3. \end{cases}$$

This is same as,

$$p(x) = \binom{3}{x} p^x (1-p)^{3-x} \qquad x = 0, 1, 2, 3$$

Example: A multiple-choice test has 10 questions, each with 5 choices. What is the probability that by purely guessing, a student gets 70% of the questions correct?

2.6 Multinomial Distribution

Definition 2.8. Consider n identical and independent trials such that each trial has "k"(≥ 2) outcomes with probabilities $p_1, p_2, p_3, \cdots, p_k$. Then probability of getting n_i outcomes of type i with $n = n_1 + n_2 + n_3 + \cdots + n_k$ is

$$P(n_1, n_2, n_3, \cdots, n_k) = \frac{n!}{n_1!n_2!n_3! \cdots n_k!} \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}.$$

NOTE:

- Binomial distribution is a special case of Multinomial distribution where

$$k = 2, \quad p_1 = p, \quad p_2 = 1 - p.$$

Example: A baseball player gets a hit with probability 0.3, a walk with probability 0.1 and an out with probability 0.6. If he bats four times during a game, what is the probability that he will get 1 hit, 1 walk and 2 outs?

Example: The output of a machine is graded excellent 70% of the time, good 20% of the time, defective 10% of the time. What is the probability that a sample size 20 has 12 excellent, 5 good, and 3 defective items?

2.7 Poisson Distribution

- Helps to predict the probability of certain events happening when we know how often the event has occurred.
- It gives us the probability of a given number of events happening in a fixed interval of time.

Definition 2.9. Random variable X is said to have a **Poisson distribution** with parameter λ (or $X \sim Poisson(\lambda)$) if the PMF of X is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \ : \ x = 0, 1, 2, 3, \cdots .$$

Note:

- Poisson distribution is used to find number of successes when the average number of successes is given.
- λ - average number of successes (parameter),
- X - actual number of successes (random variable),
- $\sum_{x=0}^{\infty} P(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right] = 1$
- $E(X) = \lambda, Var(X) = \lambda.$

  Proof:

Example: Consider a computer system of job-arrival stream at an average of 2 per one minute (in any one-minute interval). **Let X be the number of jobs in any one-minute interval.**

1. Find the probability distribution (PMF) of X.
2. Determine the probability that there are **exactly two** jobs in any one-minute interval.
3. Determine the probability that there are **at most three** jobs in any one-minute interval.
4. **Let Y be the number of jobs in any two-minute interval.** Then, find the probability distribution of Y.

Example: In an oil pipeline, the number of leaks serious enough to require a special repair crew is assumed to follow a Poisson distribution with mean of one leak every 10 miles. In a 50 mile section of pipe, what is the probability there are at least three such leaks?

2.8 Poisson Approximation of Binomial

Theorem 2.10. Suppose S_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \lambda$ as $n \to \infty$ then,

$$P(S_n = x) \to \frac{e^{-\lambda}\lambda^x}{x!}$$

Idea:

For large n and small p, binomial probabilities can be approximated using Poisson distribution, with $\lambda = np$.

Example: Suppose we roll two dies 12 times.

Let D be the number of times a double six appears. Find the exact and approximation values of $P(D = k)$ for $k = 0, 1, 2$.

Example: Suppose there is a room full of 30 people. What is the probability that no one else has your birthday? Calculate the exact and approximate probabilities.

2.9 Summary

Summary 2. ***Chapter 02: Combinatorial Probability***

Counting Techniques:

- **Multiplication Rule** (All objects are different!)

  If a process has r stages with n_1 results for stage 1, n_2 results for stage 2 and so on, then the total number of results for the entire process is

  $$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i.$$

  Total number of ways to arrange n different objects on a line is: $n!$.

- **Permutation** (Doing arrangements and order is matter)

  Number of different ways to pick k objects out of n different objects and arrange them on a line is denoted by ${}^{n}P_k$ and

  $${}^{n}P_k = \frac{n!}{(n-k)!} \ : \ k \leq n$$

- **Combinations** (No arrangements, order does not matter!)

  Number of different ways to pick k objects out of n objects (do not arrange) is denoted by ${}^{n}C_k$ or $\binom{n}{k}$ and

  $${}^{n}C_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} \ : \ k \leq n$$

- **Partitions** (All are NOT different, there are some same elements!)

  Suppose a set of n elements is divided into r disjoint sbsets such that i^{th} subset has n_i elements, $i = 1, 2, 3, \cdots, r$, and $n_1 + n_2 + \cdots + n_r = n$,

  $$\text{Total number of ways } = \frac{n!}{n_1! n_2! \cdots n_r!}$$

The Binomial Distribution

- If a random experiment consist of n Bernoulli trails such that,
  - trials are independent and identical,
  - each trail has only two outcomes, ”success” and ”failure”,
  - the probability of a success in each trail is p,

  then, it is called a Binomial experiment and $X \sim Binomial(n, p)$. It has a probability distribution,

$$p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad ; x = 0, 1, 2, \ldots, n$$

  where

$$\binom{n}{x} = \frac{n!}{(n-x)!x!} \text{ for } x = 0, 1, 2, \ldots, n$$

- Mean: $E(X) = np$.
- Variance: $Var(X) = np(1-p)$.

Multinational Distribution

- Gives the probability of any particular combination of numbers of successes for the various categories.
- Consider n identical and independent trials such that each trial has “n”(≥ 2) outcomes with probabilities $p_1, p_2, p_3, \cdots, p_k$.
- Then probability of getting n_i outcomes of type i with $n = n_1 + n_2 + n_3 + \cdots + n_k$ is

$$P(n_1, n_2, n_3, \cdots, n_k) = \frac{n!}{n_1!n_2!n_3! \cdots n_k!} \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}.$$

The Poisson Distribution

- Useful for **counting the number of occurrences of an event** over a specified region or specified period of time.
- A random variable X has a Poisson distribution with parameter λ, $X \sim Poisson(\lambda)$ and the probability mass function of X,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad \text{for } x = 0, 1, 2, \ldots$$

- Mean and variance: $E(X) = Var(X) = \lambda$

2.10 Exercises

1. Find the number of words, with or without meaning, that can be formed with the letters of the word “CHAIR”.
2. Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,
    (a) do the words start with P
    (b) do all the vowels always occur together
    (c) do the vowels never occur together
    (d) do the words begin with I and end in P?
3. In how many ways can a committee of 1 man and 3 women can be formed from a group of 3 men and 4 women?
4. Find the number of words, with or without meaning, that can be formed with the letters of the word “SWIMMING”?
5. 18 mice were placed in two experimental groups and one control group, with all groups equally large. In how many ways can the mice be placed into three groups?
6. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?
7. A particular telephone number is used to receive both voice calls and fax messages. Suppose that 25% of the incoming calls involve fax messages, and consider a sample of 25 incoming calls. What is the probability that,
    (a) At most 6 of the calls involve a fax message?
    (b) Exactly 6 of the calls involve a fax message?
    (c) At least 6 of the calls involve a fax message?
    (d) More than 6 of the calls involve a fax message?
    (e) Find $E(X)$ and $Var(X)$.
8. A bowl has 2 maize marbles, 3 blue marbles and 5 white marbles. A marble is randomly selected and then placed back in the bowl. You do this 5 times. What is the probability of choosing 1 maize marble, 1 blue marble and 3 white marbles?
9. The article “Expectation Analysis of the Probability of Failure for Water Supply Pipes” proposed using the Poisson distribution to model the number of failures in pipelines of various types. Suppose that for cast-iron pipe of a particular length, the expected number of failures is 1 (very close to one of the cases considered in the article). Then X, the number of failures, has a Poisson distribution with $\lambda = 1$.
    (a) Obtain $P(X \leq 5)$.
    (b) Determine $P(X = 2)$.
    (c) Determine $P(2 \leq X \leq 4)$.

(d) What is the probability that X exceeds its mean value by more than one standard deviation?

10. The article “Reliability-Based Service-Life Assessment of Aging Concrete Structures” suggests that a Poisson process can be used to represent the occurrence of structural loads over time. Suppose the mean time between occurrences of loads is 0.5 year.

    (a) How many loads can be expected to occur during a 2-year period?

    (b) What is the probability that more than five loads occur during a 2-year period?

    (c) How long must a time period be so that the probability of no loads occurring during that period is at most 0.1?

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Chapter 03: Conditional Probability

Outline 3. Overview:

- Conditional Probability: Definition
- Tree Diagrams
- Total Probability Theorem
- Bayes' Theorem
- Discrete Joint Distributions
- Discrete Marginal Distributions
- Functions of Multiple Random Variables (Discrete)
- Independence of Random Variables

3.1 Conditional Probability

Definition: For any event A and B with $P(A) > 0$, the probability B will occur given that A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \;:\; P(A) > 0.$$

Conditional Probability also follows probability axioms.

1. $P(\Omega|B) = 1.$

   Proof:

2. If A_1 and A_2 are disjoint (i.e., $A_1 \cap A_2 = \phi$)

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B).$$

Proof:

3. $P(A^c|B) = 1 - P(A|B)$.

Example: Suppose that five good fuses and three defective ones have been mixed up. To find the defective fuses, we test them one by one, at random and without replacement. What is the probability that we are lucky and find both of the defectives in the first two test?

3.2 Multiplication Rule

Definition 3.1. For any events $A_1, A_2, A_3, \cdots, A_n$,

$$P(\cap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n| \cap_{i=1}^{n-1} A_i).$$

Example: Three cards are drawn from an ordinary 52-card deck without replacement. Find the probability that none of the three card is a diamond.

3.3 Two Stage Experiments

Definition 3.2. A collection of events $A_1, A_2, \cdots, A_n$ is called a **partition** of Ω if $\cup_{i=1}^{n} A_i = \Omega$ and $A_i \cap A_j = \phi$ for all $i, j = 1, 2, \cdots, n$ and $i \neq j$.

3.3.1 Total Probability Theorem

Theorem 3.3. Let $A_1, A_2, \cdots, A_n$ be a partition of Ω. Then for any event B,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Proof:

NOTE:

- If $n = 2$

- If $n = 3$

Example:
You enter to a chess tournament where your probability of winning a game is 0.3 against half of the players (call them type 1), 0.4 against the quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Example:
We roll a four-sided fair die. If the result is one or two, we roll once more but otherwise, we stop. What is the probability that the sum of the rolls is at least four?

3.3.2 Bayes' Theorem

Theorem 3.4. (Bayes' Theorem) Let $A_1, A_2, \cdots, A_n$ be a partition of Ω such that $P(A_i) > 0$ for all $i = 1, 2, \cdots, n$. Then for any event B,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} = \frac{P(B|A_j)P(A_j)}{P(B)},\ j = 1, 2, \cdots, n.$$

Proof:

NOTE:

- If $n = 2$

- If $n = 3$

Example: (Recall the Chess Tournament Problem!)

Consider the chess tournament problem: (A_i- had a type i opponent ($i = 1, 2, 3$), B- won the game).

$$P(A_1) = 0.5, P(A_2) = 0.25, P(A_3) = 0.25$$

$$P(B|A_1) = 0.3, P(B|A_2) = 0.4, P(B|A_3) = 0.5$$

Assume that $P(B) = 0.375$ (from the previous example). Suppose you won the game, what is the probability that you had a type 1 opponent?

Example:

We roll a four-sided fair die. If the result is one or two, we roll once more but otherwise, we stop. (A_i- The first roll is i ($i = 1, 2, 3, 4$), B- the sum is at least four)

$$P(A_i) = 1/4 \; (i = 1, 2, 3, 4)$$

$$P(B|A_1) = 1/2, P(B|A_2) = 3/4, P(B|A_3) = 0, P(B|A_4) = 1$$

Assume that $P(B) = 9/16$.

1. Suppose the sum is at least four, what is the probability that the first roll is two?

2. Suppose the sum is less than four, what is the probability that the first roll is one?

Example:
Three factories (label 1, 2, and 3) make 20%, 30%, and 50% of the computer chips for a company respectively. The probability of a defective chip is 0.04, 0.03, and 0.02 for three factories. We have a defective chip. What is the probability that it came from factory 2?

3.4 Discrete Joint Distributions

Definition 3.5. Let X and Y are two random variables associated with the same random experiment. Then the joint distribution of X and Y is given by

$$P(x,y) = P(X = x, Y = y) = P(\{X = x\} \cap \{Y = y\})$$

NOTE:

- $P(x,y) \geq 0$ for any x **and** y.
- $\sum_{x,y} P(x,y) = 1$.

Example:

Roll pair of four sided pair dice. Let X be the maximum of the two numbers and Y be the sum.
a) Find the joint distribution of X and Y.
b) Find $P(X \leq 2, Y \leq 3)$.

3.5 Marginal Distributions

Definition 3.6. Suppose $P(x, y)$ is the joint distribution of random variables X and Y. Then,

- The marginal distribution of X is given by

$$P(x) = P(X = x) = \sum_{y} P(x, y).$$

- The marginal distribution of y is given by

$$P(y) = P(Y = y) = \sum_{x} P(x, y).$$

Example:
Suppose we draw 2 balls out of an urn with 6 red, 5 blue and 4 green balls.

Let $X =$ # of red balls and $Y =$ # of blue balls.

1. Find the joint distribution function of X and Y.

2. Find the marginal distributions of X and Y.

3.6 Functions of Multiple Random Variables

Definition 3.7. Let X and Y be two random variables and g is a function of X and Y. Then $Z = g(X, Y)$ is also a random variable and probability distribution of Z is given by

$$P(z) = P(Z = z) = \sum_{\{(x,y)/g(x,y)=z\}} P(x, y),$$

NOTE:

- $E(Z) = E(g(X, Y)) = \sum_x \sum_y g(x, y) \cdot P(x, y)$.
- $E(aX + bY + c) = a \cdot E(X) + b \cdot E(Y) + c,\ a, b, c$ be constants.
- More generally, for any sequence of random variables, $X_1, X_2, \cdots, X_n$ and constants $a_1, a_2, \cdots, a_n$,

  $$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1 \cdot E(X_1) + a_2 \cdot E(X_2) + \cdots a_n \cdot E(X_n).$$

  Especially,

  $$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

Example: Consider the joint distribution function of X and Y.

$X\ Y$	1	2	
1	1/5	1/5	
2	2/5	1/5	

Marginal Distributions:

$$P(X = x) = \begin{cases} 2/5 & ; x = 1, \\ 3/5 & ; x = 2. \end{cases}$$

$$P(Y = y) = \begin{cases} 3/5 & ; y = 1, \\ 2/5 & ; y = 2. \end{cases}$$

a) Let $Z = X + 2Y$, find the PMF of Z.

b) Find $E(Z)$.

3.7 Independence of Random Variables

> **Definition 3.8.** Two random variables X and Y are independent (i.e., $X \perp Y$) iff
>
> $$P(X = x, Y = y) = P(X = x) \cdot P(Y = y), \text{ for all } x \text{ and } y.$$

NOTE:

1. If X and Y are independent,

$$E(XY) = E(X)E(Y).$$

Proof:

2. If X and Y are independent,

$$Var(aX + bY + c) = a^2 \cdot Var(X) + b^2 \cdot Var(Y).$$

3. More generally, if $X_1, X_2, \cdots X_n$ are independent events and $a_1, a_2, \cdots, a_n$ are constants,

$$Var(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2 \cdot Var(X_1) + a_2^2 \cdot Var(X_2) \cdots + a_n^2 \cdot Var(X_n)$$

Especially,

$$Var(X_1 + X_2 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n).$$

Example: Suppose we have the following joint distribution of X and Y. Are X and Y independent?

$X\ Y$	$Y = 0$	$Y = 1$	$P(X = x)$
$X = 0$	0.5	0.3	
$X = 1$	0.2	0	
$P(Y = y)$			

Now consider the new distribution. Are X and Y independent?

$X\ Y$	$Y = 0$	$Y = 1$	$P(X = x)$
$X = 0$	0.42	0.28	
$X = 1$	0.18	0.12	
$P(Y = y)$			

Example: Find the expectation and the variance of the Binomial Distribution with parameters n and p.

3.8 Summary

Summary 3. ***Chapter 03: Conditional Probability***

Conditional Probability

For any event A and B with $P(A) > 0$, the probability B will occur given that A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \; : \; P(A) > 0.$$

- $P(B^c|A) = 1 - P(B|A)$
- $P(\Omega|B) = 1$
- If A_1 and A_2 are disjoint (i.e., $A_1 \cap A_2 = \phi$)

  $$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B).$$

- **Partition:** A collection of events $A_1, A_2, \cdots, A_n$ is called a partition of Ω if $\cup_{i=1}^{n} A_i = \Omega$ and $A_i \cap A_j = \phi$ for all $i, j = 1, 2, \cdots, n$ and $i \neq j$.
- **Total Probability Theorem:**
  Let $A_1, A_2, \cdots, A_n$ be a partition of Ω. Then for any event B,

  $$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

- **Bayes' Theorem**
  Let $A_1, A_2, \cdots, A_n$ be a partition of Ω such that $P(A_i) > 0$ for all $i = 1, 2, \cdots, n$. Then for any event B,

  $$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} = \frac{P(B|A_j)P(A_j)}{P(B)}, \; j = 1, 2, \cdots, n.$$

Discrete Joint Probability

- Let X and Y are two random variables associated with the same random experiment. Then the joint distribution of X and Y is given by

  $$P(x, y) = P(X = x, Y = y) = P(\{X = x\} \cap \{Y = y\})$$

- $P(x, y) \geq 0$ for any x **and** y.
- $\sum_{x,y} P(x, y) = 1.$
- Suppose $P(x, y)$ is the joint distribution of random variables X and Y. Then,

- The marginal distribution of X is given by

$$P(x) = P(X = x) = \sum_y P(x, y).$$

- The marginal distribution of y is given by

$$P(y) = P(Y = y) = \sum_x P(x, y).$$

- Functions of Multiple Random Variables
  - Let X and Y be two random variables and g is a function of X and Y.
  - Then $Z = g(X, Y)$ is also a random variable and probability distribution of Z is given by

$$P(z) = P(Z = z) = \sum_{\{(x,y)/g(x,y)=z\}} P(x, y),$$

  - $E(Z) = E(g(X, Y)) = \sum_x \sum_y g(x, y) \cdot P(x, y)$.
  - $E(aX + bY + c) = aE(X) + bE(Y) + c, \;\; a, b, c$ be constants.
  - For any sequence of random variables, $X_1, X_2, \cdots, X_n$ and constants $a_1, a_2, \cdots, a_n$,

$$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \cdots a_nE(X_n).$$

- Independence of Random Variables
  - Two random variables X and Y are independent (i.e., $X \perp Y$) iff

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y), \quad \text{for all } \; x \text{ and } y.$$

If at least one joint probability does not hold above property, two random variables are **NOT** independent.

  - If X and Y are independent,

$$E(XY) = E(X)E(Y).$$

  - If X and Y are independent,

$$Var(aX + bY + c) = a^2Var(X) + b^2Var(Y).$$

  - If $X_1, X_2, \cdots X_n$ are independent events and $a_1, a_2, \cdots, a_n$ are constants,

$$Var(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2Var(X_1) + a_2^2Var(X_2) \cdots + a_n^2Var(X_n)$$

3.9 Exercises

1. Suppose you draw 5 cards out of a deck of 52 and get 2 spades and 3 hearts. What is the probability that the first card drawn was a spade? *[Page 105; Durret, 2009]*

2. Suppose 60% of the people subscribe to newspaper A, 40% to newspaper B, and 30% to both. If we pick a person at random who subscribes to at least one newspaper, what is the probability that she subscribes to newspaper A? *[Page 106; Durret, 2009]*

3. You are going to meet a friend at the airport. Your experience tells you that the plane is late 70% of the time when it rains, but is late only 20% of the time when it does not rain. The weather forecast that morning calls for a 40% chance of rain. What is the probability that the plane will be late? *[Page 108; Durret, 2009]*

4. Imagine you are a financial analyst at an investment bank. According to your research of publicly-traded companies, 60% of the companies that increased their share price by more than 5% in the last three years replaced their CEOs during the period. At the same time, only 35% of the companies that did not increase their share price by more than 5% in the same period replaced their CEOs. Knowing that the probability that the stock prices grow by more than 5% is 4%, find the probability that the shares of a company that fires its CEO will increase by more than 5%.

5. You are about to have an interview for Harvard Law School. 60% of the interviewers are conservative and 40% are liberal. 50% of the conservatives smoke cigars but only 25% of the liberals do. Your interviewer lights up a cigar. What is the probability that he is a liberal?

6. A cab was involved in a hit-and-run accident at night. Two cab companies green and blue operate 85% and 15% of the cabs in the city respectively. A witness identified the cab as blue. However, in a test only 80% of witnesses were able to correctly identify the cab color. Given this what is the probability that the cab involved in the accident was blue? *[Page 109; Durret, 2009]*

7. An undergraduate student has asked a professor for a letter of recommendation. He estimates that the probability he will get the job is 0.8 with a strong letter, 0.4 with a medium letter, and 0.1 with a weak letter. He also believes that the probabilities that the letter will be strong, medium, or weak are 0.5, 0.3, and 0.2. What is the probability that the letter was strong given that he got the job?

8. Suppose we draw two tickets from a hat that contains tickets numbered 1, 2, 3, 4. Let X be the first number drawn and Y be the second.

   (a) Find the joint distribution of X and Y.

   (b) Find the marginal distribution of X and Y.

   (c) Are X and Y independent?

9. Using the clues given below, fill in the rest of the joint distribution of X and Y.

   - $P(Y = 2|X = 0) = 1/4$.
   - X and Y are independent. **(There is only one answer)**

Y	$X = 0$	2	4
1	a	b	c
2	0.1	0.05	d

10. The table below shows the joint distribution of two discrete random variables X and Y.

Y	$X = 1$	2	3	4
1	$6c$	$3c$	$2c$	$4c$
2	$4c$	$2c$	$4c$	0
3	$2c$	c	0	$2c$

   (a) Find c.
   (b) Find $P(X = 1, Y = 3)$.
   (c) Compute the marginal distribution of X and Y.
   (d) Find $E(X)$ and $E(Y)$.
   (e) Find $P(X \leq 2, Y < 3)$.
   (f) Find $P(X = 2|Y = 1)$.
   (g) Find $P(Y = 1|X = 3)$
   (h) Are X and Y independent?

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Chapter 05-06: Continuous Random Variables & Limit Theorems

Outline 4. Overview:

- Probability Density Function (pdf) (Sec 5.1)
- Uniform Distribution (Sec 5.1)
- Exponential Distribution (Sec 5.1)
- Cumulative Distribution Function (cdf) (Sec 5.2)
- Normal Distribution (Sec 6.4)
- Central Limit Theorem (CLT) (Sec 6.5)

4.1 Probability Density Function (pdf)

- When a random variable can take any value in an interval, it is called a **continuous** random variable.
- Probability distribution of a continuous random variable X is called a **probability density function (pdf)**.

Definition 4.1. (pdf) A continuous random variable X is said to have a **probability density function (pdf)** if and only if for all $a \leq b$

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

NOTE:

1. $P(X = a) = 0$, for any constant a.
2. For any constants a and b

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b).$$

Proof: $P(a \leq X \leq b) = P(X = a) + P(a < X < b) + P(X = b) = P(a < X < b)$

3. $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$ **(Total probability!)**.

Example: Let $f(x) = \begin{cases} \frac{a}{\sqrt{x}} & : 0 \leq x \leq 1 \\ 0 & : \text{ otherwise} \end{cases}$, is a pdf. Find the constant a.

Example: Let $f(x) = \begin{cases} c & : a \leq x \leq b \\ 0 & : \text{ otherwise} \end{cases}$, is a pdf. Find the constant, c.

4.2 Expectated Value

Definition 4.2. Expected value of a continuous random variable X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

Definition 4.3. If g is a function of X, then ,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

NOTE:

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx \qquad k^{th} \text{ moment of } X.$$

4.3 Variance

Definition 4.4. Variance of a continuous random variable X is

$$Var(X) = E[(X - E(X))^2] = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx.$$

NOTE: Properties of expectation & variance

For any constant a and b,

- $Var(X) = E(X^2) - [E(X)]^2$
- $E(aX + b) = a \cdot E(X) + b$
- $Var(aX + b) = a^2 \cdot Var(X)$

Example: Find the mean, variance of the random variable X if the pdf is given by:

$$f(x) = \begin{cases} \frac{a}{\sqrt{x}} & : 0 \leq x \leq 1 \\ 0 & : \text{ otherwise} \end{cases}$$

4.4 Uniform Distribution

- Describes an experiment where there is an arbitrary outcome that lies between certain bounds.
- Defines equal probability over a given range for a continuous distribution.

Definition 4.5. X follows a uniform distribution with parameters a and b if the pdf is given by

$$f(x) = \frac{1}{b-a} \ : a \leq x \leq b,$$

Note: Let $X \sim Uniform(a, b)$. Then, $E(X) = \frac{a+b}{2}$.

Proof:

Note: Let $X \sim Uniform(a, b)$. Then, $Var(X) = \frac{(b-a)^2}{12}$.

Proof:

4.5 Exponential Distribution

Exponential distribution is used to model **time between two successive events**.

E.g:

1. Time between two calls.
2. Time between two hurricanes.

Definition 4.6. Random variable X follows an exponential distribution with parameter λ if the pdf is given by:

$$f(X) = \begin{cases} \lambda \cdot e^{-\lambda x} & : x \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

where

- X is the time between two events,
- $\frac{1}{\lambda}$ is the average time between two events.

NOTE: If $X \sim Exp(\lambda)$,

1. Probability Density Function (PDF) is legitimate.

   Proof:

2. $E(x) = \frac{1}{\lambda}$

   Proof:

3. $Var(X) = \dfrac{1}{\lambda^2}$.

   Proof:

4. $P(X > a) = e^{-\lambda a},\ a > 0$ constant.

   Proof:

Example: The time until a small meteorite first lands anywhere in the Sahara desert is modeled as an exponential random variable with a mean of 10 days. The time is currently midnight. What is the probability that a meteorite first lands sometime between 6 a.m. and 6 p.m. of the first day?

4.6 Cumulative Distribution Function

Definition 4.7. Cumulative distribution function (cdf) of a random variable X is given as

$$F(x) = \begin{cases} \sum_{k \leq x} F(k) & : X \text{ is discrete,} \\ \int_{-\infty}^{x} f(t)dt & : X \text{ is continuous.} \end{cases}$$

Properties of cdfs

1. $\lim_{x \to -\infty} F(x) = 0,$ and $\lim_{x \to \infty} F(x) = 1.$
2. $F(x)$ is non-decreasing (constant or increasing).
3. $F(X)$ is right continuous (i.e., $\lim_{x \to -a^+} F(x) = F(a)$)
4. $P(X < a) = P(X \leq a) = F(a).$
5. $P(X > a) = P(X \geq a) = 1 - F(a).$
6. $P(a < X < b) = P(a \leq X \leq b) = F(b) - F(a).$
7. For a discrete random variable, the cdf is always a step function.

8. Usually for a continuous random variable, cdf is an increasing function.

Example : Let $P(x) = 1/3 \ : \ x = 1, 2, 3$. Find the cdf of X.

Example : Let X be a continuous random variable with PDF,

$$f(x) = \begin{cases} \frac{3}{8} \cdot x^2 & ; 0 < x < 2 \\ 0 & ; \text{otherwise} \end{cases}$$

1. Compute the CDF of X.

2. Find $P(X = 2)$.

3. Find $P(X < 1)$.

4. Find $P(X > 1.5)$.

5. Find $P(0.5 \leq X \leq 1)$.

Example: Let $X \sim Geometric(p)$. Find the cdf of X.

Example: Let $X \sim Exp(\lambda)$. Find the cdf of X.

4.7 Normal Distribution

Definition 4.8. A continuous random variable X is said to have normal distribution (i.e., Gaussian distribution) with parameters μ and σ^2 (i.e.,$X \sim N(\mu, \sigma^2)$) if

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0.$$

NOTE: If $X \sim N(\mu, \sigma^2)$,

1. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$
2. $E(X) = \mu$ and $Var(X) = \sigma^2$ (standard deviation =$\sigma(X) = \sigma$)
3. Graphs:
4. If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ (a,b -constants), then $Y \sim N(a\mu + b, a^2\sigma^2)$.
   Proof:

4.8 The Standard Normal Distribution

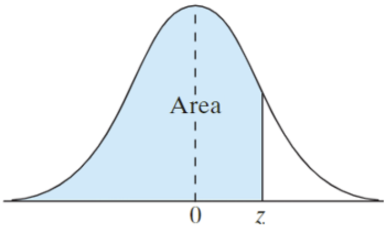
Definition 4.9. A normal random variable Z has a mean 0 and a variance of 1.

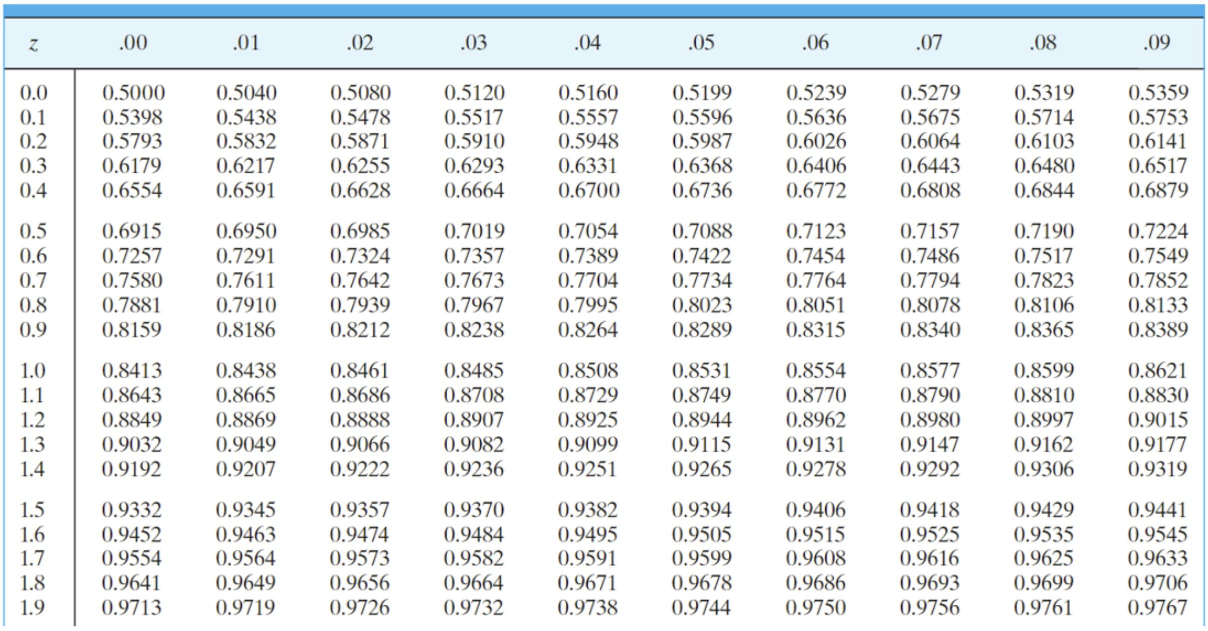
$$i.e., E(Z) = 0 \text{ and } Var(Z) = 1$$

The distribution of Z is called the standard normal distribution.

NOTE:

- The pdf of Z : $f(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty,$
- The cdf of Z : $F(z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \quad -\infty < z < \infty,$
- Values of $\Phi(z)$ are tabulated for different values of z.





z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Use of the Standard Normal Table:

1. Z value needs to be rounded to 2 decimal places.
2. The first 2 numbers tells us the row of the standard normal table.

   The third number tells us the column of the table.
3. The Z value is positive, so we go to the positive z value page, then find the row. Then you can find the corresponding probability.

Example: Find the following probabilities using the standard normal table.

1. $P(Z \leq 1.76352)$
2. $P(Z \leq 1.06)$

3. $P(Z > 2.32)$

4. $P(Z < -2.56)$

5. $P(-2.56 < Z < 1.06)$

Standardize

Let $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$, then

$$\begin{aligned} E(Z) &= E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}(E(X)-\mu) = \frac{1}{\mu}(\mu-\mu) = 0 \\ Var(Z) &= Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}(Var(X)) = \frac{1}{\sigma^2}\sigma^2 = 1, \end{aligned}$$

(Normality will be proved later).

Then

- $P(X < a) = P\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = P\left(Z < \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right)$

- $P(X > a) = P\left(\frac{X-\mu}{\sigma} > \frac{a-\mu}{\sigma}\right) = P\left(Z > \frac{a-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{a-\mu}{\sigma}\right)$

- For constants a and b,

$$\begin{aligned} P(a < X < b) &= P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) \\ &= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$

Example: 1. The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of 60 inches, and a standard deviation of 20 inches. What is the probability that this year's snowfall will be **at least** 80 inches?

Example: 2. Suppose that the blood chloride concentration ($mmol/L$) has a normal distribution with mean 104 and standard deviation 5. Find the probability of the blood concentration is **less than** 100.36 $mmol/L$.

4.9 Central Limit Theorem

Definition 4.10. (Random Sample) A sequence of independent and identically distributed random variables $X_1, X_2, \cdots, X_n$ is called a random sample.

Idea:

- Suppose we take a sample of k students from our probability class and assume there are n students in the class.
- Then our sample is one of the $\binom{n}{k}$ possible samples.
- Suppose we consider all the possible samples.
- Then the random sample is the variable for all the possible samples.

4.9.1 Sample Total

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with mean μ and variance σ^2.

Let $S_n = X_1 + X_2 + \cdots + X_n$, then

$$\begin{aligned} E(S_n) &= E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= \mu + \mu + \cdots + \mu \; (\because \text{ identically distributed}) \\ &= n\mu. \end{aligned}$$

$$\begin{aligned} Var(S_n) &= Var(X_1 + X_2 + \cdots + X_n) \\ &= Var(X_1) + Var(X_2) + \cdots + Var(X_n) \; (\because \text{ independent}) \\ &= \sigma^2 + \sigma^2 + \cdots + \sigma^2 \; (\because \text{ identically distributed}) \\ &= n\sigma^2. \end{aligned}$$

4.9.2 Sample Mean

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with mean μ and variance σ^2.

Let $\bar{X} = M_n = \frac{X_1+X_2+\cdots+X_n}{n} = \frac{S_n}{n}$, then

- the expected value:

$$E(\bar{X}) = E(M_n) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = E\left(\frac{S_n}{n}\right) = \frac{1}{n} \cdot E(S_n) = \frac{1}{n} \cdot n\mu = \mu.$$

- the variance:

$$Var(S_n) = Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2} \cdot Var(S_n) = \frac{1}{n^2} \cdot n\sigma^2 = \sigma^2/n.$$

Theorem 4.11. Let $X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$. (i.e.,$X_1, X_2, \cdots, X_n$ be a random sample from a normally distributed population with mean μ and variance σ^2). Then

$$a)\ S_n \sim N(n\mu, n\sigma^2) \ \Rightarrow \frac{S_n - n\mu}{\sqrt{n}\sigma} \ \sim \ N(0,1) \text{ Standard Normal}$$

$$b)\ M_n \sim N(\mu, \sigma^2/n) \ \Rightarrow \frac{M_n - \mu}{\sigma/\sqrt{n}} \ \sim \ N(0,1) \text{ Standard Normal}$$

Problem: How do we find the distribution of the sample total or the sample mean when the distribution of the population is unknown (or not normal)?

Answer: Use Central Limit Theorem (CLT).

4.9.3 Central Limit Theorem

Theorem 4.12. Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with mean μ and variance σ^2. Assume n is large (i.e, $n \geq 35$, Rule of Thumb for our class). Then,

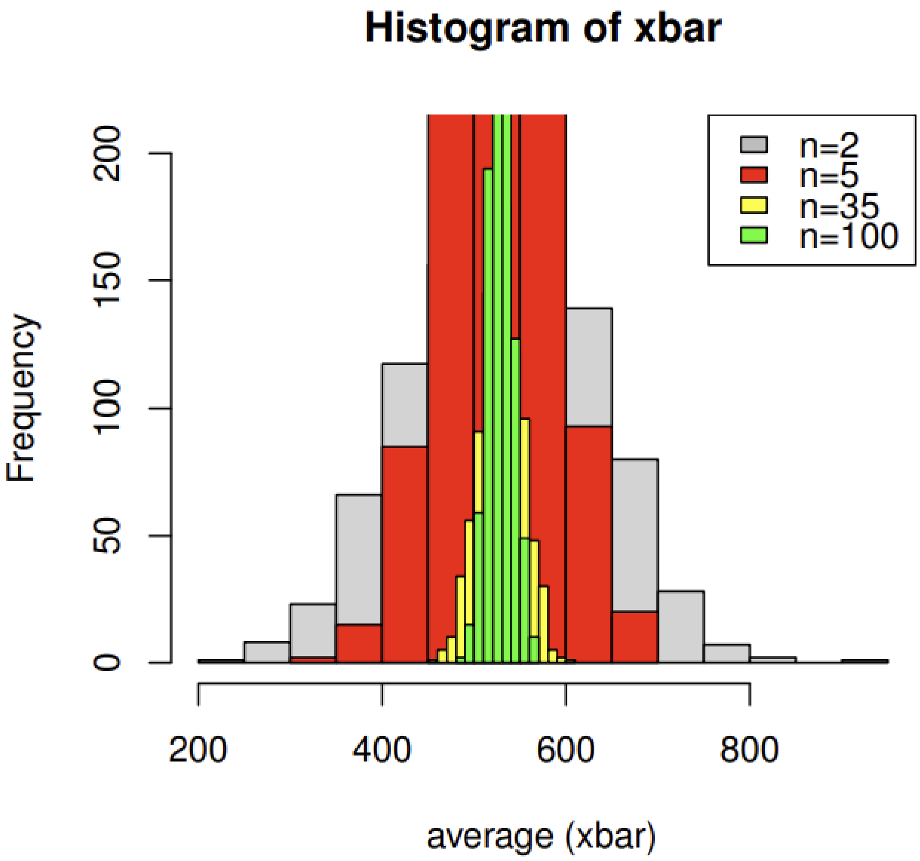
1. If $S_n = X_1 + X_2 + \cdots + X_n$ is the sample total, then

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$$

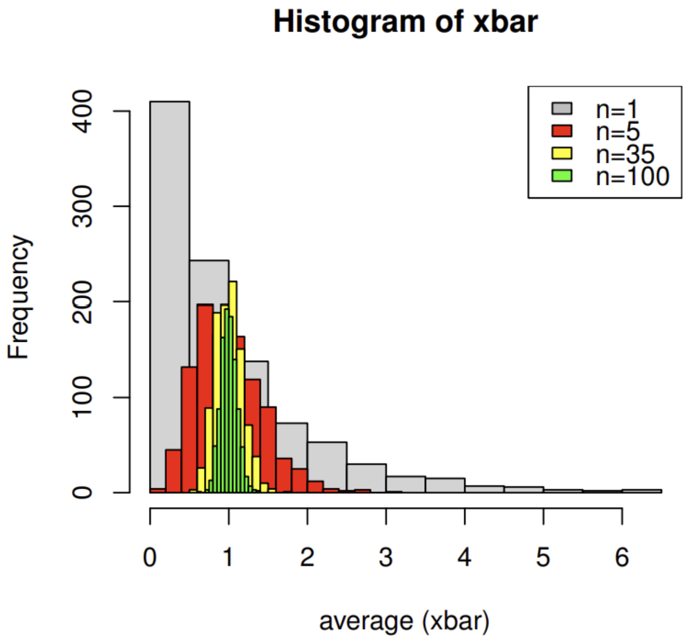
2. If $M_n = \frac{X_1+X_2+\cdots+X_n}{n} = \frac{S_n}{n}$ is the sample mean, then

$$\frac{M_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

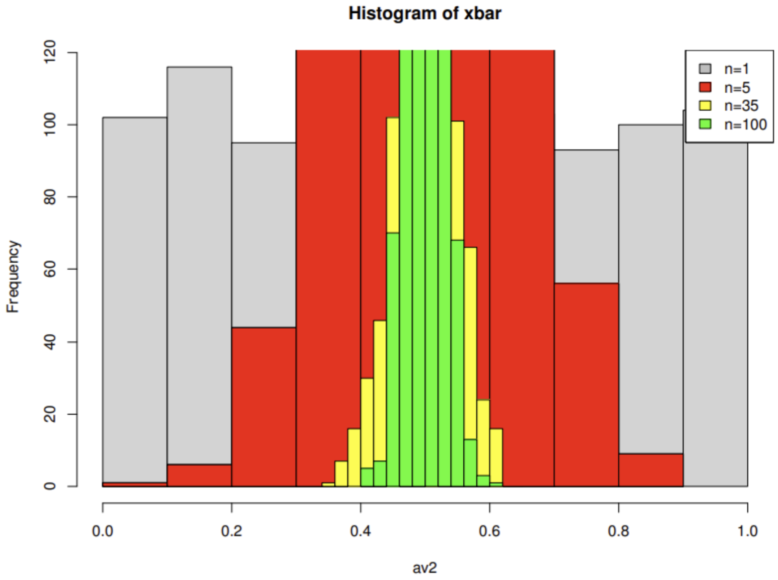
Example 1: Sampling from $N(528.8, 137.2^2)$



Example 2: Sampling from $Exp(1)$



Example 3: Sampling from $Uni(0,1)$



Example: We load on a plane 100 packages whose weights are independent random variables that are uniformly distributed between 5 and 50 pounds. What is the probability that the total weight exceeds 3000 pounds?

Example: The income of college students is distributed with a mean income per year is $\$12,000$ and a standard deviation of $\$6,000$. If we randomly sample 50 college students,

1. What is the expected average income of our sample?

2. What is the variance of the average income of our sample?

3. What is the probability that the average income of our sample is less than $\$10,000$?

4.10 Summary

Summary 4. ***Chapter 05-06: Continuous Random variables and Limit Theorems***

Continuous Random Variables

- Probability Density Function (PDF):

  For any pdf $f(x)$, then,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

- Total probability:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Cumulative Distribution Function (CDF)

- Probability that observed value of X will be at most x:
  - X is continuous:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

  - X is discrete:

$$F(x) = P(X \leq x) = \sum_{k}^{x} p(k)$$

- In general, for a continuous random variable, X,

$$\begin{aligned} P(X < a) &= P(X \leq a) = F(a) \\ P(X > a) &= 1 - P(X \leq a) = 1 - F(a) \\ P(a \leq x \leq b) &= P(X \leq b) - P(X \leq a) = F(b) - F(a) \end{aligned}$$

Expected Values

- Expected value of X: $E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$.
- Expected value of $g(X)$: $E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)dx$.
- Variance of X.

$$V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx = E[(x - E(X))^2]$$

  Shortcut formula (easy method)

$$V(X) = E(X^2) - [E(X)]^2$$

- Standard Deviation, $\sigma(X) = \sqrt{Var(X)}$

Continuous Distributions

Uniform Distribution: $X \sim Uniform(a, b)$

- The probability density function:

$$f(x) = \begin{cases} \frac{1}{b-a} & ; a \leq x \leq b \\ 0 & ; \text{otherwise} \end{cases}$$

- The cumulative distribution function:

$$F(x) = \begin{cases} 0 & ; x \leq a \\ \frac{x-a}{b-a} & ; a \leq x \leq b \\ 1 & ; x \geq b \end{cases}$$

- $E(X) = \dfrac{a+b}{2}$ and $Var(X) = \dfrac{(b-a)^2}{12}$

Exponential Distribution: $X \sim Exp(\lambda)$

- The probability density function:

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda \cdot x} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

- The cumulative distribution function:

$$F(x) = \begin{cases} 1 - e^{-\lambda \cdot x} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

- $E(X) = \dfrac{1}{\lambda}$ and $Var(X) = \dfrac{1}{\lambda^2}$
- $P(X > a) = e^{-\lambda \cdot a}$ for constant a.

Normal Distribution

- The probability density function, $X \sim N(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty$$

- The probability density function for the standard normal distribution, $X \sim N(0, 1)$

$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-z^2}{2}\right), \quad -\infty < z < \infty$$

- The cumulative distribution for standard normal distribution:

$$\Phi(x) = P(Z \leq z)$$

  Refer to the Standard Normal Table to find **left probabilities**, $P(Z < z) = P(Z \leq z)$.

  - Write z value with 2 decimal places.
  - Consider first two numbers including the sign as the row and the third number with decimal as the column.
  - Then, find the intersect value which gives from the row and the column in the body of the table.

- Standardizing normal distribution
  - Use the transformation,

$$Z = \frac{X-\mu}{\sigma}$$

  - To find the probability,

$$P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X \leq a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X \geq b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$$

- If $X \sim N(\mu, \sigma^2)$, then the distribution of,
  - Sample Total: $S_n \sim N(n\mu, n\sigma^2)$
  - Sample Mean: $M_n = \bar{X} \sim N(\mu, \sigma^2/n)$

Central Limit Theorem

Consider the population mean is μ, the population standard deviation is σ, and sample size is n.

- Use Normal approximation for large $n \geq 35$ (Rule of Thumb!).
- Mean of sample mean, $\bar{X}$ is: $\mu_{\bar{X}} = \mu$
- Variance of sample mean, $\bar{X}$ is: $\sigma^2_{\bar{X}} = \frac{\sigma^2}{n}$
- Standard Deviation of sample mean, $\bar{X}$ is: $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$
- The distribution of sample total, S_n and sample mean, $\bar{X}$:
    - If X is normally distributed, then

$$S_n \sim N(n\mu.n\sigma^2), \qquad \bar{X} \sim N\left(\mu, \sigma^2_{\bar{X}} = \frac{\sigma^2}{n}\right)$$

    - If X is not normally distributed, then
        - If $n \geq 35$, then, $S_n, \bar{X}$ is approximately Normal.

$$S_n \dot{\sim} N(n\mu.n\sigma^2), \qquad \bar{X} \dot{\sim} N\left(\mu, \sigma^2_{\bar{X}} = \frac{\sigma^2}{n}\right)$$

        - If $n < 35$, we can't find the distribution of $\bar{X}$.
- Standardizing:

$$Z = \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}$$

Then, use standard normal table to find the probabilities.

4.11 Exercises

1. The time between buses on Elm Street is 12 minutes. Therefore the wait time of a passenger who arrives randomly at a bus stop is uniformly distributed between 0 and 12 minutes.
    (a) Find the probability that a person randomly arriving at the bus stop to wait for the bus has a wait time of at most 5 minutes.
    (b) Find the mean and standard deviation of waiting time.
2. On the average, a certain computer part lasts ten years. The length of time the computer part lasts is exponentially distributed.
    (a) What is the probability that a computer part lasts more than 7 years?
    (b) Find the cdf of the time that a computer part last.
    (c) Using part (b), answer the part (a).
    (d) Calculate the mean and the standard deviation of the time that a computer part last.
3. Suppose X is a discrete random variable. Let the pmf of X be equal to:

$$P(X = x) = \frac{5-x}{10}; x = 1, 2, 3, 4$$

Find the cumulative distribution function (CDF) of X.

4. The length of a telephone call made to a company is denoted by the continuous random variable T. It is modeled by the probability density function,

$$f(t) = \begin{cases} kt & ; 0 \leq t \leq 10; \\ 0 & ; \text{otherwise} \end{cases}$$

    (a) Find the constant k such that the given pdf is legitimate.
    (b) Find $P(T > 6)$.
    (c) Calculate the expect value, $E(T)$ and and the variance, $Var(T)$.
    (d) Find the cumulative distribution of X, $F(t)$.
    (e) Using part (d), find the value of part (b).
    (f) Using part (d), find $P(3 < T < 8)$.
5. Determine the following standard normal probabilities.
    (a) $P(Z \leq 2.4378)$
    (b) $P(Z > 2.44)$
    (c) $P(Z \leq -0.2534)$
    (d) $P(-0.3222 < Z < 1.2523)$
    (e) $P(Z < 0)$

6. The average number of acres burned by forest and range fires in a large New Mexico county is 4,300 acres per year, with a standard deviation of 750 acres. The distribution of the number of acres burned is normal.
   (a) What is the probability that more than 4,200 acres will be burned in any given year?
   (b) What is the probability that between 2,500 and 4,200 acres will be burned in any given year?

7. Suppose that the data concerning the first-year salaries of WPI graduates is normally distributed with the population mean $60000 and the population standard deviation $15000.
   (a) Find the probability of a randomly selected WPI graduate earning less than $45000 annually.
   (b) Find the probability of randomly selecting a WPI graduate that makes more than $80000 a year.

8. A population has mean 72 and standard deviation 6.
   (a) Find the mean and standard deviation of the mean for samples of size 45. Also determine the probability distribution of mean.
   (b) Find the probability that the mean of a sample of size 45 will differ from the population mean 72 by at least 2 units, that is, is either less than 70 or more than 74.

9. An unknown distribution has a mean of 90 and a standard deviation of 15. A sample of size 80 is drawn randomly from the population. Find the probability that the sum of the 80 values (or the total of the 80 values) is more than 7,500.

10. Survey found that the American family generates an average of 17.2 pounds of glass garbage each year. Assume the standard deviation of the distribution is 2.5 pounds.
    (a) Find the probability that the mean of a sample of 55 families will be between 17 and 18 pounds.
    (b) Why the central limit theorem can be applied?

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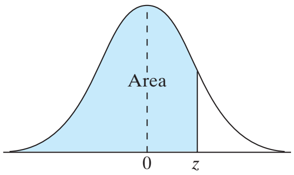
Appendix

5.1 Standard Normal Table

Table 3 Areas Under the Normal Curve



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table 3 Areas Under the Normal Curve *(continued)*

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Solutions for Exercises

6.1 Solutions: Chapter 01

Solutions

(01) Two six-sided dice are rolled.

$$\Omega = \left\{ \begin{array}{l}
\overset{1}{(1,1)}^{0}, \overset{2}{(1,2)}^{1}, \overset{3}{(1,3)}^{2}, \overset{4}{(1,4)}^{3}, \overset{5}{(1,5)}^{4}, \overset{6}{(1,6)}^{5} \\
\overset{2}{(2,1)}^{1}, \overset{2}{(2,2)}^{0}, \overset{3}{(2,3)}^{1}, \overset{4}{(2,4)}^{2}, \overset{5}{(2,5)}^{3}, \overset{6}{(2,6)}^{4} \\
\overset{3}{(3,1)}^{2}, \overset{3}{(3,2)}^{1}, \overset{3}{(3,3)}^{0}, \overset{4}{(3,4)}^{1}, \overset{5}{(3,5)}^{2}, \overset{6}{(3,6)}^{3} \\
\overset{4}{(4,1)}^{3}, \overset{4}{(4,2)}^{2}, \overset{4}{(4,3)}^{1}, \overset{4}{(4,4)}^{0}, \overset{5}{(4,5)}^{1}, \overset{6}{(4,6)}^{2} \\
\overset{5}{(5,1)}^{4}, \overset{5}{(5,2)}^{3}, \overset{5}{(5,3)}^{2}, \overset{5}{(5,4)}^{1}, \overset{5}{(5,5)}^{0}, \overset{6}{(5,6)}^{1} \\
\overset{6}{(6,1)}^{5}, \overset{6}{(6,2)}^{4}, \overset{6}{(6,3)}^{3}, \overset{6}{(6,4)}^{2}, \overset{6}{(6,5)}^{1}, \overset{6}{(6,6)}^{0}
\end{array} \right\}$$

(a) Need to get the difference of two numbers.

Difference of two numbers is in red color

So, let A = two numbers will differ by 1 or less

$$= \left\{ \begin{array}{l} (1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4) \\ (4,3), (4,4), (4,5), (5,4), (5,5), (5,6), (6,5), (6,6) \end{array} \right\}$$

$$P(A) = \frac{n(A)}{n(\Omega)} = \frac{16}{36} = \boxed{\frac{4}{9}}$$

(b) Need to consider the maximum of two numbers.

Maximum of two numbers are denoted by green color.

Let B = maximum of two numbers will be 5 or larger. (5 or 6)

$$= \left\{ \begin{array}{l} (1,5), (2,5), (3,5), (4,5), (5,1), (5,2), (5,3), (5,4) \\ (5,5), (5,6), (1,6), (2,6), (3,6), (4,6), (6,1), (6,2) \\ (6,3), (6,4), (6,5), (6,6) \end{array} \right\}$$

So, $P(B) = \frac{n(B)}{n(\Omega)} = \frac{20}{36} = \boxed{\frac{5}{9}}$

(02) Let A - smoke cigaretts, B - drink alcohol

$P(A) = 0.25$, $P(B) = 0.60$, $P(A \cap B) = 0.15$

Need to find,

P(at least one of these bad habits) = $P(A \cup B) = ?$

So, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.25 + 0.60 - 0.15$

$= \boxed{0.7}$

(03) Let A- Alice attends the class ; B – Betty attends the class

$P(A) = 0.80$, $P(B) = 0.60$

Their absences are independent. $\Rightarrow$ A^c and B^c are independent.
$\Rightarrow$ A and B are independent.

(a) P(at least one of these students is in the class)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A) \cdot P(B)$ ($\because$ A and B are independent)

$= 0.80 + 0.60 - 0.80 \cdot 0.60$

$= \boxed{0.92}$

OR

$P(A) = 0.8 \Rightarrow P(A^c) = 1 - 0.8 = 0.2$

$P(B) = 0.6 \Rightarrow P(B^c) = 1 - 0.6 = 0.4$

Since A^c and B^c are independent,

$P(A^c \cap B^c) = P(A^c) \cdot P(B^c) = 0.2 \cdot 0.4 = 0.08$

So,

P(at least one attends the class) $= 1 - P($both are absent$)$

$= 1 - P(A^c \cap B^c)$

$= 1 - 0.08$

$= \boxed{0.92}$

(b) P(exactly one of them is there)

$P(B) = P(A \cap B) - P(A^c \cap B)$

$P(A) = P(A \cap B) - P(A \cap B^c)$

$= P(A \cap B^c) + P(A^c \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2\,P(A \cap B)$

$= P(A) + P(B) - 2\,P(A) \cdot P(B)$ ($\because$ A, B are independent)

$= 0.8 + 0.6 - 2\,(0.8 \cdot 0.6)$

$= \boxed{0.44}$

(04) $P(\text{Head}) = P(H) = 0.5$, Let X = number of heads.

Let n = number of times should a coin be tossed

Probability of getting at least one head $= P(X \geq 1)$.

$$= 1 - P(X<1)$$
$$= 1 - P(X=0)$$
$$= 1 - P(\text{getting } n \text{ tails})$$
$$= 1 - \left(\tfrac{1}{2}\right)^n \quad (\because \text{tosses are independent})$$

So, we need to find n such that

$$1-\left(\tfrac{1}{2}\right)^n \geq 0.99$$

$$\Rightarrow \quad 1-0.99 \geq \left(\tfrac{1}{2}\right)^n$$

$$\Rightarrow \quad 0.01 \geq \left(\tfrac{1}{2}\right)^n$$

$$\Rightarrow \quad \frac{1}{2^n} \leq 0.01$$

$$\Rightarrow \quad \frac{1}{0.01} \leq 2^n$$

$$\Rightarrow \quad 100 \leq 2^n$$

$$\Rightarrow \quad \log(100) \leq n \log(2)$$

$$\Rightarrow \quad \frac{\log(100)}{\log(2)} \leq n$$

$$\Rightarrow \quad 6.6438 \leq n$$

So, the smallest possible value which satisfies the inequality is $\boxed{n=7}$ ($\because$ # of tosses should be an integer).

(05)

Blood Group.

Ethnic Group	O	A	B	AB	
1	0.082	0.106	0.008	0.004	0.2
2	0.135	0.141	0.018	0.006	0.3
3	0.215	0.200	0.065	0.020	0.5
	0.432	0.447	0.091	0.030	$\boxed{1}$

Let A - type A selected, B - type B selected, C - ethnic group 3 selected)

(a) $P(A) = 0.106 + 0.141 + 0.200 = \boxed{0.447}$

$P(C) = 0.215 + 0.200 + 0.065 + 0.020 = \boxed{0.5}$

$P(A \cap C) = \boxed{0.2}$

(b) $P(A|C) = \dfrac{P(A\cap C)}{P(C)} = \dfrac{0.2}{0.5} = \dfrac{2}{5} = \boxed{0.4}$

$\Rightarrow$ If we know that the individual came from ethnic group 3, the probability that he/she has type A blood is 0.40. //

$P(C|A) = \dfrac{P(A\cap C)}{P(A)} = \dfrac{0.2}{0.447} = \boxed{0.4474}$

$\Rightarrow$ If a person has Type A blood, then the probability that he/she is from ethnic group 3 is 0.4474. //

(c) Need to find, P(from ethnic group 1 | does not have type B blood)

Let D - ethnic group 1 selected.

we need to find : $P(D|B^c) = \dfrac{P(D\cap B^c)}{P(B^c)}$

From table, $P(D\cap B^c) = 0.082 + 0.106 + 0.004 = 0.192$

$P(B^c) = 1 - (0.008 + 0.018 + 0.065) = 0.909$.

Then, $P(D|B^c) = \dfrac{P(D\cap B^c)}{P(B^c)} = \dfrac{0.192}{0.909} = \boxed{0.2112}$

(06) The PMF of X :

X	-2	-1	0	1	2
$P(X=x)$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

Let $Y = 2X+1$

x	y
-2	$2(-2)+1 = -3$
-1	$2(-1)+1 = -1$
0	$2(0)+1 = 1$
1	$2(1)+1 = 3$
2	$2(2)+1 = 5$

$\Rightarrow$ Possible values of Y ; $-3, -1, 1, 3, 5$.

So,

$P(Y=-3) = P(X=-2) = 1/5$

$P(Y=-1) = P(X=-1) = 1/5$

$P(Y=1) = P(X=0) = 1/5$

$P(Y=3) = P(X=1) = 1/5$

$P(Y=5) = P(X=2) = 1/5$

So, the PMF of Y:

$Y=y$	-3	-1	1	3	5
$P(Y=y)$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

//

(07). We have,

$$P(x) = \begin{cases} K(7x+3) & ; \ x = 1,2,3 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Since P(X) is a valid probability mass function,

$$\sum_{x} P(x) = 1 \Rightarrow P(X=1)+P(X=2)+P(X=3) = 1$$

$$\Rightarrow K(7\cdot1+3) + K(7\cdot2+3) + K.(7\cdot3+1) = 1$$

$$\Rightarrow K(10+17+22) = 1$$

$$\Rightarrow \boxed{K = \frac{1}{49}}$$

(08) Let X - number of games you play until you lose (includes the losing game).

$\Omega = \{L, WL, WWL, WWWL, \dots\}$ $\Rightarrow$ there are infinite number of posibilities for X.

- each game is independent
- each game has 2 outcomes $<$ win / lose
- $P(\text{success}) = P = P(\text{lose}) = 0.57$

$\Rightarrow$ X has a geometric distribution. , $X \sim \text{Geometric}(p=0.57)$

So, PMF (or probability distribution) of X :

$$P(X=x) = (1-0.57)^{x-1} \cdot 0.57 \quad ; \quad x = 1, 2, 3, \ldots$$

(a). $P(X=5) = (1-0.57)^{5-1} \cdot 0.57 = \boxed{0.01949}$

(b) $P(X \geq 3) = 1 - P(X<3)$

$= 1 - [P(X=1) + P(X=2)]$

$= 1 - \left[(1-0.57)^{1-1} \cdot 0.57 + (1-0.57)^{2-1} \cdot 0.57\right]$

$= \boxed{0.1849}$

(c) $E(X) = \frac{1}{p} = \frac{1}{0.57} = 1.754 \approx \boxed{2 \text{ games}}$

$Var(X) = \frac{1-p}{p^2} = \frac{1-0.57}{(0.57)^2} = 1.32348 \approx \boxed{2 \text{ games}}$

$X \sim \text{Geometric}(p) \Rightarrow P(X=x) = (1-p)^{x-1} \cdot p \; ; \; x \geq 1 \; , \; E(X) = 1/p \; , \; Var(X) = \frac{1-p}{p^2}$

(09) Let X - number of lines in use at a specific time.

The PMF of X :

X	0	1	2	3	4	5	6
P(X)	0.10	0.15	0.20	0.25	0.20	0.06	0.04

(a) P(at most 3 lines are in use) $= P(X \leq 3)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0.10 + 0.15 + 0.20 + 0.25$

$= \boxed{0.7}$

(b) P(fewer than three lines are in use)

$= P(X<3)$

$= P(X=0) + P(X=1) + P(X=2)$

$= 0.10 + 0.15 + 0.20$

$= \boxed{0.45}$

(c) P(at least three lines are in use) $= P(X \geq 3)$

$= 1 - P(X<3)$

$= 1 - 0.45$

$= \boxed{0.55}$

(d). P(between two and five lines, inclusive, are in use)

$$= P(2 \leq X \leq 5)$$
$$= P(X=2) + P(X=3) + P(X=4) + P(X=5)$$
$$= 0.20 + 0.25 + 0.20 + 0.06$$
$$= \boxed{0.71}$$

(e) P(between two and four lines, inclusive, are not in use)

$0, 1, \boxed{\cancel{2}, \cancel{3}, \cancel{4}}, 5, 6.$

$$= P(X=0) + P(X=1) + P(X=5) + P(X=6)$$
$$= 0.10 + 0.15 + 0.06 + 0.04$$
$$= \boxed{0.35}$$

OR

$$= 1 - P(2 \leq X \leq 4)$$
$$= 1 - [P(X=2) + P(X=3) + P(X=4)]$$
$$= 1 - [0.20 + 0.25 + 0.20]$$
$$= \boxed{0.35}$$

(f). P(at least four lines are not in use) $= 1 - P(X \geq 4)$

$$= 1 - [P(X=4) + P(X=5) + P(X=6)]$$
$$= 1 - [0.20 + 0.06 + 0.04]$$
$$= \boxed{0.7}$$

OR $\boxed{\cancel{0}, \cancel{1}, \cancel{2}, \cancel{3}}, 4, 5, 6.$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= 0.10 + 0.15 + 0.20 + 0.25$$
$$= \boxed{0.7}$$

(10). (a) $E(X) = \sum_x x \cdot P(X=x)$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 P(X=3) + 4 \cdot P(X=4) + 5 \cdot P(X=5) + 6 \cdot P(X=6)$$
$$= 0(0.10) + 1(0.15) + 2 \cdot (0.20) + 3 \cdot (0.25) + 4 \cdot (0.20) + 5(0.06) + 6(0.04)$$
$$= \boxed{2.64}$$

(b) $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \sum_x x^2 P(X=x)$$
$$= 0^2 \cdot (0.10) + 1^2 (0.15) + 2^2 \cdot (0.20) + 3^2 \cdot (0.25) + 4^2 \cdot (0.20) + 5^2 \cdot (0.06) + 6^2 (0.04) = 9.34$$

So, $Var(x) = E(x^2) - (E(x))^2$

$= 9.34 - (2.64)^2$

$\doteq \boxed{2.3704}$

(c) $\sigma(X) = \sqrt{Var(X)}$

$= \sqrt{2.3704} = \boxed{1.5396}$

(d) $E(2X+1) = 2E(X)+1 = 2(2.64)+1 = \boxed{6.28}$

(e) $Var(3X-1) = 3^2 Var(X) = 9(2.3704) = \boxed{21.3336}$

Use: $E(aX+b) = aE(X)+b$ (a, b are constants)

$Var(aX+b) = a^2 Var(X)$

6.2 Solutions: Chapter 02

Solutions

(01) CHAIR : all letters are different.

$\underset{5}{\_\_}\ \underset{4}{\_\_}\ \underset{3}{\_\_}\ \underset{2}{\_\_}\ \underset{1}{\_\_}$ # of arrangements = $5 \times 4 \times 3 \times 2 \times 1 = \boxed{5!}$

(02) INDEPENDENCE

I-1 , N-3 , D-2 , E-4 , P-1 , C-1 Total = n = 12.

Here, all letters are NOT different.

(a). Start with P.

P _ _ _ _ _ _ _ _ _ _ _

We need to arrange (12-1) = 11 letters.

I, N, D, E, E, N, D, E, N, C, E ⇒ we have,
I-1 , N-3 , D-2 , E-4
C-1

Since letters are repeating, we use the formula:

$$\# \text{ of arrangements} = \frac{n!}{n_1!\, n_2!\, n_3!\, n_4!\, n_5!}$$

$$= \frac{11!}{1!\, 3!\, 2!\, 4!\, 1!} = \boxed{\frac{11!}{3!\, 2!\, 4!} = 138600}$$

//

(b) all the vowels always occur together

There are 5 vowels in the given word "INDEPENDENCE".
E-4 , I-1

They have occur together we treat them as a single object
we treat $\boxed{EEEEI}$

So, letters become $\boxed{EEEEI}$ N D P N D N C

we arrange them now,

- Arranging 5 vowels : Total = 5 , E-4 , I-1
  Since letters are repeating,
  $\text{Total arrangements} = \frac{5!}{4!\, 1!} = 5$

- Arranging remaining letters $(\_)\_\,\_\,\_\,\_\,\_\,\_\,\_$

Numbers we need to arrange $7+1=8$

(Single unit together with 7 letter)

$$(\quad)\ \underline{N}\ \underline{D}\ \underline{P}\ \underline{N}\ \underline{D}\ \underline{N}\ \underline{C}$$

↑ bundle

N-3, D-2, P-1, C-1 $n=8$

$n_1=3$ $n_2=2$ $n_3=1$ $n_4=1$

$$\text{Number of arrangements} = \frac{8!}{3!\,2!\,1!\,1!} = \frac{8!}{3!\,2!}$$

Hence, the required number of arrangements

= # of ways of arrangement of vowels × # of arrangements remaining letters

$$= \frac{5!}{4!} \times \frac{8!}{3!\,2!}$$

$$= \boxed{16800}$$

(c) <u>vowels never occur together</u>

Number of arrangements where vowel never occur together

= Total numbers of arrangements − Number of arrangements when all vowels occur together

$$= \frac{12!}{3!\,2!\,4!} - 16800$$

$$= 1663200 - 16800$$

$$= \boxed{1646400}$$

(d) <u>words begin with I and end in P</u>

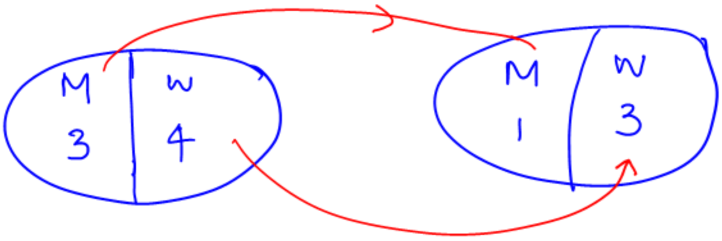
We need to fix I and P at extreme ends

$$\underline{I}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{10 letters in which D-2, E-4, N-3}}\ \underline{P}$$

Since letters are repeating,

$$\text{required number of arrangement} = \frac{10!}{2!\,4!\,3!} = \boxed{12600}$$

(03)



$$\# \text{ of ways} = \left(\begin{array}{c}\# \text{ of ways to} \\ \text{select } 1 \text{ man}\end{array}\right) \times \left(\begin{array}{c}\# \text{ of ways to} \\ \text{select } 3 \text{ women}\end{array}\right)$$

$$= {}^{3}C_{1} \times {}^{4}C_{3}$$

$$= \frac{3!}{2!\,1!} \times \frac{4!}{3!\,1!}$$

$$= \frac{4!}{2!} = \boxed{12}$$

(04) S W I M M I N G

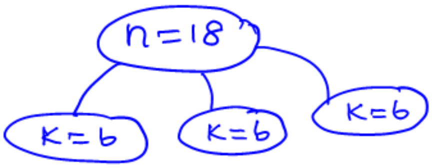
Total = n = 8 , S-1 , W-1 , I-2 , M-2 , N-1 , G-1

Since some letters are repeating,

$$\# \text{ of arrangements} = \frac{8!}{1!\,1!\,2!\,2!\,1!\,1!} = \boxed{\frac{8!}{2!\,2!} = 10080}$$

(05) Total number of mice = 18.

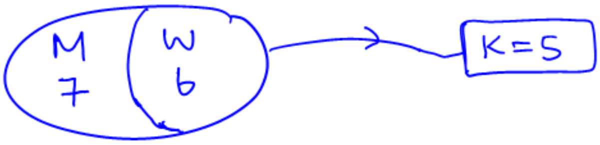
Number of groups = 3



Since the groups are equally large, the possible number of mice in each group = $18/3 = 6$

For each group, the placement of mice = 6!

Hence, number of total arrangements = $\frac{18!}{6!\,6!\,6!} = \boxed{\frac{18!}{(6!)^3}}$

(06) 

← at least 3 men are on the committee
≥ 3.

Men	Women	
3	2	$\Rightarrow {}^{7}C_{3} \cdot {}^{6}C_{2}$
4	1	$\Rightarrow {}^{7}C_{4} \cdot {}^{6}C_{1}$
5	0	$\Rightarrow {}^{7}C_{5} \cdot {}^{6}C_{0}$

$$\text{Total \# of ways} = {}^{7}C_{3} \cdot {}^{6}C_{2} + {}^{7}C_{4} \cdot {}^{6}C_{1} + {}^{7}C_{5} \cdot {}^{6}C_{0}$$

$$= \frac{7!}{3!\,4!} \cdot \frac{6!}{4!\,2!} + \frac{7!}{4!\,3!} \cdot \frac{6!}{5!\,1!} + \frac{7!}{2!\,5!} \cdot \frac{6!}{0!\,6!}$$

$$= \boxed{756}$$

(07) $n = 25$, Let X - number of calls involve a fax message.

- each call is independent.
- each call has 2 outcomes. < involve a fax message ← success / NOT
- P(success) = P(invovle a fax message) = 0.25 = p

So, X has a Binomial Distribution. $X \sim \text{Binomial}(n=25, p=0.25)$

$\Rightarrow$ PMF of X : $P(X=x) = {}^{25}C_{x} (0.25)^{x} (1-0.25)^{10-x}$; $x = 0, 1, \ldots, 25$

(a) P(at most 6 of the calls involve a fax message) = $P(X \leq 6)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= {}^{25}C_{0} (0.25)^{0} (0.75)^{25-0} + {}^{25}C_{0} (0.25)^{1} (0.75)^{24} + {}^{25}C_{2} (0.25)^{2} . (0.75)^{23}$
$+ {}^{25}C_{3} (0.25)^{3} (0.75)^{22} + {}^{25}C_{4} (0.25)^{4} (0.75)^{21} + {}^{25}C_{5} (0.25)^{5} (0.75)^{20}$
$+ {}^{25}C_{6} (0.25)^{6} (0.75)^{19}$

$= \boxed{0.5611}$

(b) P(exactly 6 of the calls involve a fax message = $P(X=6)$
$= {}^{25}C_{6} (0.25)^{6} (0.75)^{19} = \boxed{0.1828}$

(c) P(at least 6 of the calls involve a fax message) = $P(X \geq 6)$

$$= 1 - P(X<6)$$

$$= 1 - [P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)]$$

$$= 1 - \left[{}^{25}C_0 (0.25)^0 (0.75)^{25} + \dots + {}^{25}C_5 (0.25)^5 (0.75)^{20} \right]$$

$$= 1 - 0.3783$$

$$= \boxed{0.6217}$$

(d) P(more than 6 of the calls involve a fax message)

$$= P(X>6)$$

$$= 1 - P(X \leq 6)$$

$$= 1 - 0.5611$$

$$= \boxed{0.4389}$$

(e) $E(X) = np = 25 \times 0.25 = 6.25 \approx \boxed{7 \text{ calls}}$

$Var(X) = np(1-p) = 25 \times 0.25 \times 0.75 = 4.6875 \approx \boxed{5 \text{ calls}}$

$X \sim \text{Binomial}(n,p) \Rightarrow P(X=x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$; $x = 0, 1, \dots, n$

$E(X) = np$

$Var(X) = np(1-p)$

(08)

Maize - 2
Blue - 3
White - 5

Select
n=5

$P(\text{Maize marble}) = 2/10 = 0.2 = p_1$

$P(\text{Blue marble}) = 3/10 = 0.3 = p_2$

$P(\text{white marble}) = 5/10 = 0.5 = p_3.$

1 trial < Maize
Blue
white

⇒ each trial has 3 outcomes ⇒ Use Multinomial Distribution

Need to find P(1 Maize, 1 Blue, 3 white) = $P(n_1=1, n_2=1, n_3=3)$.

So, $P(n_1=1, n_2=1, n_3=3) = P(1,1,3) = \dfrac{n!}{n_1!\, n_2!\, n_3!}\; p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3}$

$$= \frac{5!}{1!\,1!\,3!} (0.2)^1 \cdot (0.3)^1 \cdot (0.5)^3 = \boxed{0.15}$$

(09) Let X = the number of failures in pipelines of various type

$X \sim \text{Poisson}(\lambda = 1)$

Then the PMF of X : $P(X=x) = \frac{e^{-1} \cdot 1^{x}}{x!}$; $x = 0, 1, 2, \ldots$

(a) $P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \frac{e^{-1} \cdot 1^{0}}{0!} + \frac{e^{-1} \cdot 1^{1}}{1!} + \frac{e^{-1} \cdot 1^{2}}{2!} + \frac{e^{-1} \cdot 1^{3}}{3!} + \frac{e^{-1} \cdot 1^{4}}{4!} + \frac{e^{-1} \cdot 1^{5}}{5!}$$

$$= \boxed{0.9994}$$

(b) $P(X=2) = \frac{e^{-1} \cdot 1^{2}}{2!} = \boxed{0.18394}$

(c) $P(2 \le X \le 4) = P(X=2) + P(X=3) + P(X=4)$

$$= \frac{e^{-1} \cdot 1^{2}}{2!} + \frac{e^{-1} \cdot 1^{3}}{3!} + \frac{e^{-1} \cdot 1^{4}}{4!}$$

$$= \boxed{0.26058}$$

(d) mean of $X = E(X) = \lambda = 1$

Standard deviation of $X = \sigma(X) = \sqrt{\text{var}(X)} = \sqrt{\lambda} = 1$

Need to find $P(X > E(X) + \sigma(X)) = P(X > 1+1) = P(X > 2)$

So, $P(X>2) = 1 - P(X \le 2)$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[\frac{e^{-1} \cdot 1^{0}}{0!} + \frac{e^{-1} \cdot 1^{1}}{1!} + \frac{e^{-1} \cdot 1^{2}}{2!}\right]$$

$$= \boxed{0.08030}$$

$X \sim \text{Poisson}(\lambda) \Rightarrow P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$; $x = 0, 1, 2, \ldots$

$E(X) = \lambda$

$\text{Var}(X) = \lambda$

(10) Let X = number of loads occur during <u>one year</u> period

$X \sim \text{Poisson}(\lambda = 0.5)$.

(a) Need to find the expected number of loads occur during <u>2-year</u> period.

↑ Need to adjust λ.

$$1 \text{ year} \longrightarrow 0.5$$
$$2 \text{ year} \longrightarrow \underset{1}{\underline{0.5}} \times 2 = 1 \quad \Rightarrow \quad \boxed{\text{Expected Number} = 1}$$

(b) Let's define, Y = number of loads occur during 2-year period.

Then, $Y \sim \text{Poisson}(\lambda = 1)$

$$\Rightarrow \quad P(Y=y) = \frac{e^{-1} \cdot 1^{y}}{y!} \; ; \quad y = 0, 1, 2, \dots$$

Need to find $P(Y>5)$.

So, $P(Y>5) = 1 - P(Y \leq 5)$

$$= 1 - [P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5)]$$

$$= 1 - \left[\frac{e^{-1} \cdot 1^{0}}{0!} + \frac{e^{-1} \cdot 1^{1}}{1!} + \frac{e^{-1} \cdot 1^{2}}{2!} + \frac{e^{-1} \cdot 1^{3}}{3!} + \frac{e^{-1} \cdot 1^{4}}{4!} + \frac{e^{-1} \cdot 1^{5}}{5!} \right]$$

$$= 1 - 0.99941$$

$$= \boxed{0.00059}$$

(c) $\lambda = 1$, t is unknown, $\Rightarrow$ New $\lambda = t$

$$P(\text{no loads occur during the period of length } t) = P(X=0) = \frac{e^{-t} \cdot t^{0}}{0!} = e^{-t}$$

So, $P(X=0) \leq 0.1$

$$e^{-t} \leq 0.1$$

$$-2t \leq \ln(0.1)$$

$$t \geq \frac{\ln(0.1)}{-2} = 1.15129 \quad \Rightarrow$$

$\Rightarrow$

6.3 Solutions: Chapter 03

Extra Problems Chapters 05-06 Solutions

(01)

$$\Omega = \left\{ \begin{matrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{matrix} \right\}$$

Let A - getting at least one 6 $= \left\{ \begin{matrix} (1,6), (2,6), (3,6), (4,6), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{matrix} \right\}$

$\Rightarrow P(A) = \frac{11}{36}$

B - sum is at least 9 $= \left\{ \begin{matrix} (3,6), (4,5), (4,6), (5,4), (5,5), (5,6) \\ (6,3), (6,4), (6,5), (6,6) \end{matrix} \right\}$

$A \cap B = \{ (3,6), (4,6), (5,6), (6,3), (6,4), (6,5), (6,6) \} \Rightarrow P(A \cap B) = \frac{7}{36}$

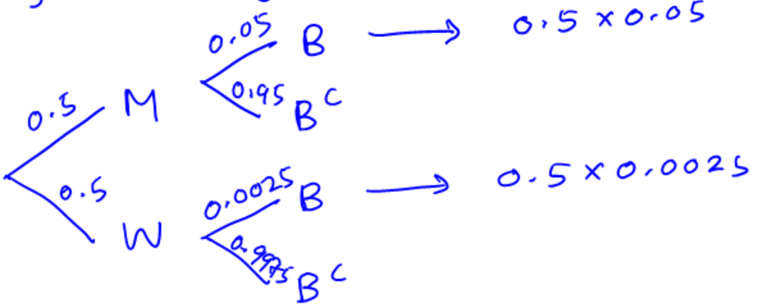
Need to find $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{7/36}{11/36} = \boxed{\frac{7}{11}}$

(02). Let M - men , W - women , B - color blind.

$P(M) = 0.5 = P(W)$, $P(B|M) = 0.05$, $P(B|W) = 0.0025$

Need to find P(color blind) = P(B).

Using Tree Diagram



$\oplus \Rightarrow P(B) = \boxed{0.02625}$

OR

Using Total Probability Theorem,

$$P(B) = P(B|M) \cdot P(M) + P(B|W) \cdot P(W)$$
$$= 0.05 \times 0.5 + 0.0025 \times 0.5 = \boxed{0.02625}$$

Need to find P(a color-blind person is a man) = $P(M|B)$

By Baye's Theorem,

$$P(M|B) = \frac{P(B|M) \cdot P(M)}{P(B)}$$

$$= \frac{0.05 \times 0.5}{0.02625}$$

$$= \boxed{0.9524}$$

(03) Let A - the stock price increase by 5%.
B - that the CEO is replaced.

We have,

$P(A) = 0.04$, $P(B|A) = 0.60$, $P(B|A^c) = 0.35$.

Need to find $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

By using Total Probability Theorem,

$$P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$$
$$= 0.60 \cdot (0.04) + 0.35 \cdot (1 - 0.04)$$
$$\doteq 0.36.$$

Then, by Baye's Theorem,

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{0.60 \times 0.04}{0.36} = \boxed{0.0667}$$

(04) Let C - conservative, L - Liberal, S - Smoke cigars.

$P(C) = 0.60$, $P(L) = 0.40$, $P(S|C) = 0.50$, $P(S|L) = 0.25$

Need to find $P(L|S)$.

So, by Total Probability Theorem,

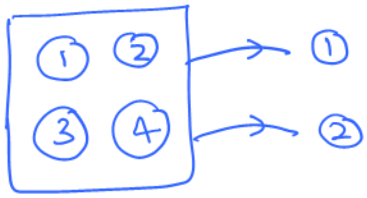
$$P(S) = P(S|C) \cdot P(C) + P(S|L) \cdot P(L)$$
$$= 0.50 \times 0.60 + 0.25 \times 0.40$$
$$= 0.95 //$$

Then, by Baye's Theorem,

$$P(L|S) = \frac{P(S|L) \cdot P(L)}{P(S)} = \frac{0.25 \times 0.40}{0.95} = \boxed{0.1053}$$

So, the probability of that he is a liberal if he lights up a cigar is 0.1053. //

(05)



With replacement

Let X – the first number drawn
Y – the second number drawn.

Possible values for X : 1, 2, 3, 4
Possible values for Y : 1, 2, 3, 4

All possible values for (X,Y) = (1,1), (1,2), (1,3), (1,4)
(2,1), (2,2), (2,3), (2,4)
(3,1), (3,2), (3,3), (3,4)
(4,1), (4,2), (4,3), (4,4)

Since 2 draws are done with replacement, there are $4 \times 4 = 16$ possible outcomes. Since each outcome will be equally likely, so probability of each outcome is $1/16$.

Hence, the joint distribution of X and Y:

X \ Y	1	2	3	4	P(X=x)
1	1/16	1/16	1/16	1/16	4/16
2	1/16	1/16	1/16	1/16	4/16
3	1/16	1/16	1/16	1/16	4/16
4	1/16	1/16	1/16	1/16	4/16
P(Y=y)	4/16	4/16	4/16	4/16	1

← Marginal of X

↗ Marginal of Y

Without Replacement

$P(1,1) = 0$, $P(2,2) = 0$, $P(3,3) = 0$, $P(4,4) = 0$

$P(1,2) = P(X=1) \cdot P(Y=2 \mid X=1) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

All other outcomes have the same probability.

So, the joint distribution of X and Y:

X \ Y	1	2	3	4	$P(X=x)$
1	0	$1/12$	$1/12$	$1/12$	$3/12$
2	$1/12$	0	$1/12$	$1/12$	$3/12$
3	$1/12$	$1/12$	0	$1/12$	$3/12$
4	$1/12$	$1/12$	$1/12$	0	$3/12$
$P(Y=y)$	$3/12$	$3/12$	$3/12$	$3/12$	

//

(b) With replacement

- Marginal distribution of X

$X=x$	1	2	3	4
$P(X=x)$	$1/4$	$1/4$	$1/4$	$1/4$

- Marginal distribution of Y.

$Y=y$	1	2	3	4
$P(Y=y)$	$1/4$	$1/4$	$1/4$	$1/4$

Without replacement

- Marginal distribution of X

$X=x$	1	2	3	4
$P(X=x)$	$1/4$	$1/4$	$1/4$	$1/4$

- Marginal distribution of Y

$Y=y$	1	2	3	4
$P(Y=y)$	$1/4$	$1/4$	$1/4$	$1/4$

(c) To check for the independence, we need to consider the all joint probabilities.

With replacement

We have $P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$

for all $x=1,2,3,4$, $y=1,2,3,4$.

So, X and Y are independent //

Without replacement

We have $P(X=x, Y=y) \neq P(X=x) \cdot P(Y=y)$

for all $x = 1, 2, 3, 4$, $y = 1, 2, 3, 4$.

So, X and Y are not independent //

If one joint probability does not satisfied the property of independence, then X and Y are not independent.

(06). Consider the joint distribution of X and Y.

Y	X = 1	2	3	4
1	6c	3c	2c	4c
2	4c	2c	4c	0
3	2c	c	0	2c

(a). Using Total Joint probability = 1 ,

$$6C + 3C + 2C + 4C + 4C + 2C + 4C + 0 + 2C + C + 0 + 2C = 1$$

$$30C = 1 \Rightarrow \boxed{C = 1/30}$$

(b) $P(X=1, Y=3) = 2C = 2(1/30) = \boxed{1/15}$

(c) The joint distribution of X and Y becomes:

Y	X = 1	2	3	4	P(Y=y)
1	6/30	3/30	2/30	4/30	15/30
2	4/30	2/30	4/30	0	10/30
3	2/30	1/30	0	2/30	5/30
P(X=x)	12/30	6/30	6/30	6/30	

← Marginal of Y (Sum of over x)

Marginal of X → (Sum over y)

So, the marginal distribution of X :

$X=x$	1	2	3	4
$P(X=x)$	$\frac{12}{30}$	$\frac{6}{30}$	$\frac{6}{30}$	$\frac{6}{30}$

The marginal distribution of Y :

$Y=y$	1	2	3
$P(Y=y)$	$\frac{15}{30}$	$\frac{10}{30}$	$\frac{5}{30}$

(d) • $E(X) = \sum_{x} x \cdot p(x)$

$$= 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4)$$

$$= 1\left(^{12}/_{30}\right) + 2\left(^{6}/_{30}\right) + 3\left(^{6}/_{30}\right) + 4\left(^{6}/_{30}\right)$$

$$\boxed{E(X) = 2.2}$$

• $E(Y) = \sum_{y} y \cdot p(y)$

$$= 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$

$$= 1\left(^{15}/_{30}\right) + 2 \cdot \left(^{10}/_{30}\right) + 3 \cdot \left(^{5}/_{30}\right)$$

$$= \boxed{1.66667}$$

(e) $P(X \leq 2, Y < 3) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=2)$

$X = 1, 2$
$Y = 1, 2$

$$= \frac{6}{30} + \frac{4}{30} + \frac{3}{30} + \frac{2}{30}$$

$$= \frac{15}{30} = \frac{1}{2} = \boxed{0.5}$$

(f) $P(X=2 \mid Y=1) = \dfrac{P(X=2, Y=1)}{P(Y=1)} = \dfrac{3/30}{15/30} = \dfrac{3}{15} = \boxed{0.2}$

(g) $$P(Y=1 \mid X=3) = \frac{P(X=3, Y=1)}{P(X=3)}$$

$$= \frac{2/30}{6/30}$$

$$= 1/3 = \boxed{0.3333}$$

(h) Since $P(X=3, Y=3) = 0 \neq \frac{6}{30} \times \frac{5}{30} = P(X=3) \cdot P(Y=3)$,

X and Y are <u>NOT</u> independent. //

6.4 Solutions: Chapter 05-06

Extra Problems Chapters 5-6 Solutions

(01) Let x - waiting time of a passenger (in minutes)

$X \sim \text{Uniform}(0, 12) \Rightarrow f(x) = \frac{1}{12-0} = \frac{1}{12} \;;\; 0 \leq x \leq 12$

(a) P(wait time of at most 5 minutes) $= P(X \leq 5)$

$$= \int_0^5 f(x)\,dx$$

$$= \int_0^5 \frac{1}{12} \cdot dx$$

$$= \frac{1}{12} \quad x \Big|_0^5$$

$$= \frac{1}{12}(5-0) = \boxed{0.4167}$$

(b) • Mean waiting time $= E(X) = \frac{0+12}{2} = \boxed{6 \text{ minutes}}$

• Variance of waiting time $= \text{Var}(X) = \frac{(12-0)^2}{12} = 12$

$\Rightarrow$ Standard deviation $= \sqrt{\text{Var}(X)}$

$$= \sqrt{12} = \boxed{3.4641 \text{ minutes}}$$

If $X \sim \text{Uniform}(a, b)$

$f(x) = \frac{1}{b-a} \;;\; a \leq x \leq b$

$E(X) = \frac{a+b}{2}$

$\text{Var}(X) = \frac{(b-a)^2}{12}$

(02) Let X - length of time the computer part lasts.

$X \sim Exp(\lambda = 1/10)$.

average $= 10 = \frac{1}{\lambda}$

$\Rightarrow \lambda = 1/10$

$$\Rightarrow f(x) = \frac{1}{10} e^{-1/10\, x} \; ; \; x > 0$$

(a) P(part lasts more than 7 years) $= P(X > 7)$

$$= \int_7^\infty f(x)\, dx$$

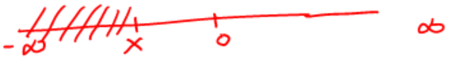
$$= \int_7^\infty \frac{1}{10} \cdot e^{-\frac{1}{10} x}\, dx$$

$$= \frac{1}{10} \cdot \frac{e^{-1/10\, x}}{-1/10} \Big|_7^\infty$$

$$= -1\left(0 - e^{-7/10}\right)$$

$$= e^{-0.7} = \boxed{0.49659}$$

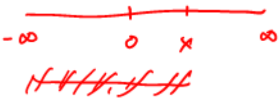
(b) CDF of X



- $x \leq 0$

$$F(X) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} 0 \cdot dt = 0$$

- $x > 0$



$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{0} \cancel{0\, dt}^{\,0} + \int_0^x \frac{1}{10} \cdot e^{-1/10\, t}\, dt$$

$$= \frac{1}{10} \cdot \frac{e^{-1/10\, t}}{-1/10} \Big|_0^x$$

$$= -\left(e^{-1/10\, x} - 1\right) = 1 - e^{-1/10\, x}$$

So, the cdf of x:

$$F(x) = \begin{cases} 0 & ; \; x \leq 0 \\ 1 - e^{-\frac{1}{10} x} & ; \; x > 0 \end{cases}$$

(c) $P(X>7) = 1-P(X\leq 7)$

$$= 1-F(7)$$

$$= 1-(1-e^{-\frac{1}{10}(7)})$$

$$= e^{-\frac{7}{10}} = \boxed{0.49659}$$

(e) • Mean of the time $= E(X) = 1/\lambda = 1/(1/10) = \boxed{10 \text{ years}}$

• Variance of time $= Var(X) = 1/\lambda^2$

• Standard deviation of time $= \sqrt{Var(X)} = \sqrt{1/\lambda^2} = 1/\lambda = \boxed{10 \text{ yrs}}$

If $X \sim Exp(\lambda)$, then

- pdf of x: $f(x) = \lambda e^{-\lambda x}$; $x>0$
- $E(X) = 1/\lambda$
- $Var(X) = 1/\lambda^2$
- $P(a<x<b) = \int_a^b \lambda . e^{-\lambda x} dx$

(03) X - discrete

The PMF of x : $P(x) = \frac{5-x}{10}$; $x = 1, 2, 3, 4.$

$$P(x) = \begin{cases} 0.4 & ; x = 1 \\ 0.3 & ; x = 2 \\ 0.2 & ; x = 3 \\ 0.1 & ; x = 4 \end{cases}$$

CDF of x:

$$F(x) = P(X \leq x) = \sum_{K=1}^{x} P(K).$$

- $x < 1$

$$F(x) = 0$$

- $1 \leq x < 2$

$$F(x) = \sum_{K=1}^{x} P(K) = P(1) = 0.4$$

- $2 \leq x < 3$

$$F(x) = \sum_{K=1}^{x} P(K) = P(1) + P(2) = 0.4 + 0.3 = 0.7$$

- $3 \leq x < 4$

$$F(x) = \sum_{K=1}^{x} P(K) = P(1) + P(2) + P(3) = 0.4 + 0.3 + 0.2 = 0.9$$

- $4 \leq x$

$$F(x) = \sum_{K=1}^{x} P(K) = P(1) + P(2) + P(3) + P(4) = 0.4 + 0.3 + 0.2 + 0.1 = 1$$

So, cdf of x ;

$$\boxed{F(x) = \begin{cases} 0 & ; x < 1 \\ 0.4 & ; 1 \leq x < 2 \\ 0.7 & ; 2 \leq x < 3 \\ 0.9 & ; 3 \leq x < 4 \\ 1 & ; x \geq 4 \end{cases}}$$

(04) The pdf of T : $f(t) = \begin{cases} Kt & ; \ 0 \le t \le 10 \\ 0 & ; \ \text{otherwise.} \end{cases}$

(a) Since f is legitimate, $\int_{-\infty}^{\infty} f(t)\,dt = 1.$

$$\int_{-\infty}^{\infty} f(t)\,dt = 1 \Rightarrow \cancelto{0}{\int_{-\infty}^{0} 0\,dt} + \int_{0}^{10} Kt\,dt + \cancelto{0}{\int_{10}^{\infty} 0 \cdot dt} = 1$$

$$\Rightarrow K \int_{0}^{10} t\,dt = 1$$

$$\Rightarrow K \cdot \frac{t^2}{2} \Big|_{0}^{10} = 1$$

$$\Rightarrow \frac{K}{2} \left(10^2 - 0^2\right) = 1$$

$$\Rightarrow 50k = 1$$

$$\Rightarrow \boxed{K = 1/50}$$

(b) $$P(T>6) = \int_{6}^{\infty} f(t)\,dt = \int_{6}^{10} \frac{1}{50} t\,dt + \cancelto{0}{\int_{10}^{\infty} 0 \cdot dt}$$

$$= \frac{1}{50} \frac{t^2}{2} \Big|_{6}^{10}$$

$$= \frac{1}{100} \left(10^2 - 6^2\right) = \boxed{0.64}$$

(c) · $$E(T) = \int_{-\infty}^{\infty} t \cdot f(t)\,dt$$

$$= \cancelto{0}{\int_{-\infty}^{0} t \cdot 0\,dt} + \int_{0}^{10} t \cdot \frac{1}{50} t\,dt + \cancelto{0}{\int_{10}^{\infty} 0\,dt}$$

$$= \frac{1}{50} \int_{0}^{10} t^2\,dt$$

$$= \frac{1}{50} \frac{t^3}{3} \Big|_{0}^{10} = \frac{1}{150} \left(10^3 - 0^3\right) = \boxed{\frac{20}{3}}$$

• $\text{Var}(T) = \int_{-\infty}^{\infty} (t - E(T))^2 \cdot f(t)\, dt$ or $\text{Var}(T) = E(T^2) - (E(T))^2$

So,

$$E(T^2) = \int_{-\infty}^{\infty} t^2 \cdot f(t)\, dt = \int_{0}^{10} t^2 \cdot \frac{1}{50} \cdot t\, dt$$

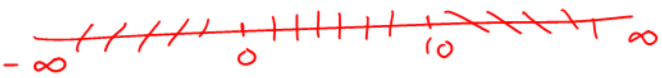
$$= \frac{1}{50} \int_{0}^{10} t^3\, dt$$

$$= \frac{1}{50} \frac{t^4}{4}\Big|_0^{10} = \frac{1}{200}\left(10^4 - 0\right) = 50$$

Then,

$$\text{Var}(T) = E(T^2) - (E(T))^2 = 50 - \left(\frac{20}{3}\right)^2 = \boxed{\frac{50}{9}}$$

(d) T is continuous



• $t<0$

$$F(t) = \int_{-\infty}^{t} 0 \cdot dr = 0$$

• $0 \leq t \leq 10$

$$F(t) = \int_{-\infty}^{t} f(x)\, dx = \cancelto{0}{\int_{-\infty}^{0} 0 \cdot dx} + \int_{0}^{t} \frac{1}{50}\, x\, dx = \frac{1}{50} \frac{x^2}{2}\Big|_0^{t} = \frac{1}{100}\left(t^2 - 0\right)$$

$$= \frac{t^2}{100}$$

• $t > 10$

$$F(t) = \int_{-\infty}^{t} f(x)\, dx = \cancelto{0}{\int_{-\infty}^{0} 0 \cdot dx} + \int_{0}^{10} \frac{1}{50} x\, dx + \cancelto{0}{\int_{10}^{t} 0 \cdot dx} = 1$$

So, the cdf of T,

$$F(t) = \begin{cases} 0 & ; \quad t<0 \\ \frac{t^2}{100} & ; \quad 0 \leq t \leq 10 \\ 1 & ; \quad t > 10 \end{cases}$$

(e) $P(T>6) = 1 - P(T \leq 6)$

$= 1 - F(6)$

$= 1 - \frac{6^2}{100}$

$= \boxed{0.64}$

(f) $P(3<T<8) = P(T<8) - P(T<3)$

$= F(8) - F(3)$

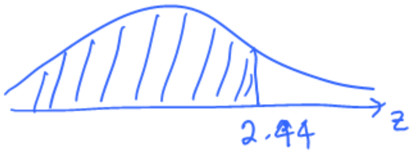
$= \frac{8^2}{100} - \frac{3^2}{100}$

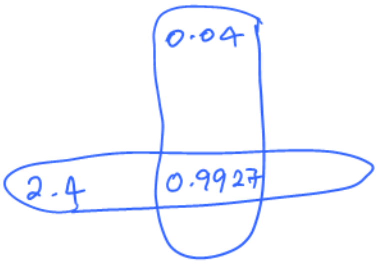
$= \boxed{0.55}$

(05) (a) $P(Z \le 2.4378) \approx P(Z \le 2.44)$

$= \boxed{0.9927}$

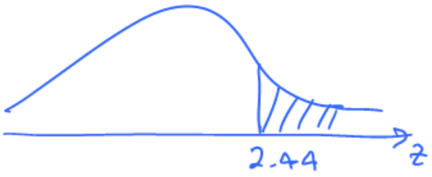
$Z = \underbrace{2.4}_{\text{row}}4$ | 0.04 col





(b) $P(Z > 2.44) = 1 - P(Z \le 2.44)$

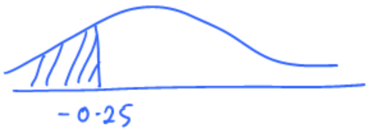
$= 1 - 0.9927 = \boxed{0.0073}$

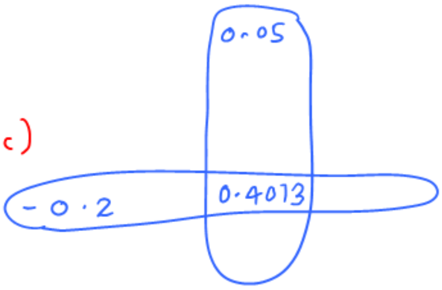


(c) $P(Z \le -0.2534) \approx P(Z \le -0.25)$

$= \boxed{0.4013}$

$Z = \underbrace{-0.2}_{\text{row}}5$ \ 0.05 col





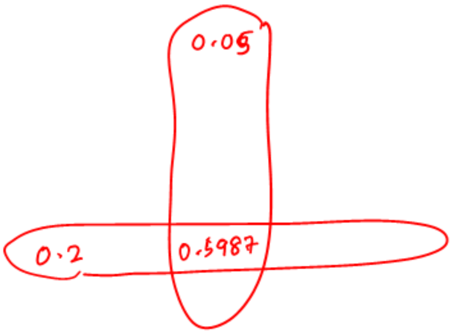
OR $P(Z \le -0.2534) = P(Z \ge 0.2534)$ ($\because$ Symmetric)

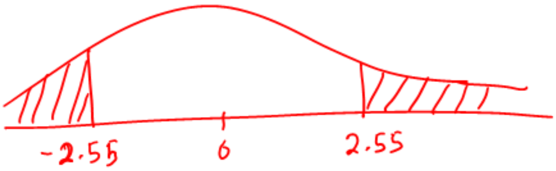
$\approx P(Z \ge 0.25)$

$= 1 - P(Z < 0.25)$

$= 1 - 0.5987$

$= \boxed{0.4013}$





(d) $P(-0.3222 < Z < 1.2523)$

$= P(Z < 1.2523) - P(Z < -0.3222)$

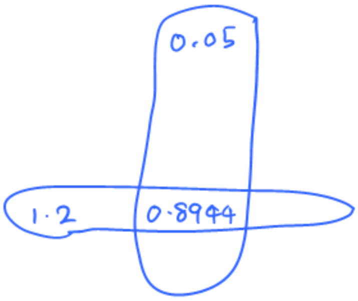
$\approx P(Z < 1.25) - P(Z < -0.32)$

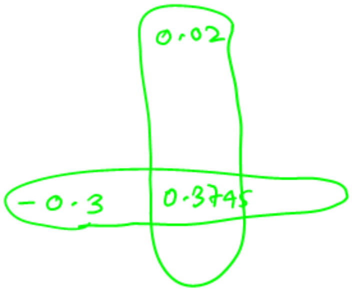
$= 0.8944 - 0.3745$

$\approx \boxed{0.5199}$

$\underbrace{-0.3}_{\text{row}}2$ | 0.02 col

$\underbrace{1.2}_{\text{row}}5$ | 0.05 col





(e) $P(Z < 0) = \boxed{0.5}$

(06) Let X - number of acres. $\mu = 4300$, $\sigma = 750$

$X \sim N(\mu = 4300, \sigma^2 = (750)^2)$

(a) $P(X > 4200) = P\left(\frac{X-\mu}{\sigma} > \frac{4200-\mu}{\sigma}\right)$

$= P\left(Z > \frac{4200-4300}{750}\right)$

$\approx P(Z >$

$= 1 - P(Z \leq$

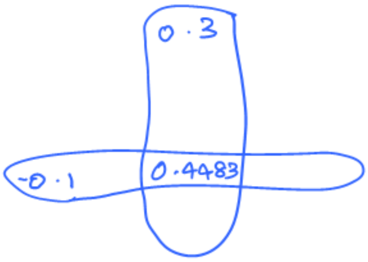
$= 1 - 0.4483$

$\approx \boxed{0.5517}$

$-0.13 = 0.1 + 0.03$

row

0.03 col



(b) $P(2500 < X < 4200)$

$= P\left(\frac{2500-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{4200-\mu}{\sigma}\right)$

$= P\left(\frac{2500-4300}{750} < Z < \frac{4200-4300}{750}\right)$

$\approx P(-2.4 < Z < -0.13)$

$\approx P(Z < -0.13) - P(Z < -2.4)$

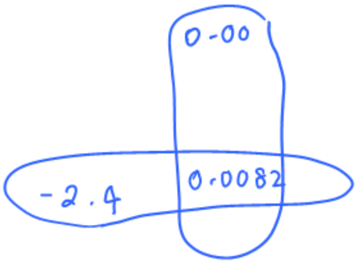
$\approx 0.4483 - 0.0082$

$= \boxed{0.4401}$

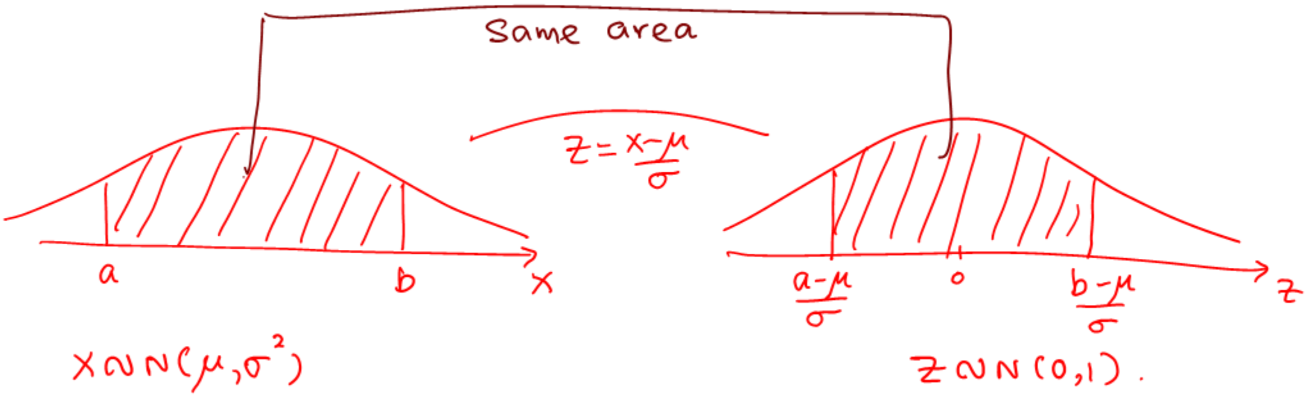
-2.40

row

0.00 col



Use transformation : $Z = \frac{X-\mu}{\sigma}$



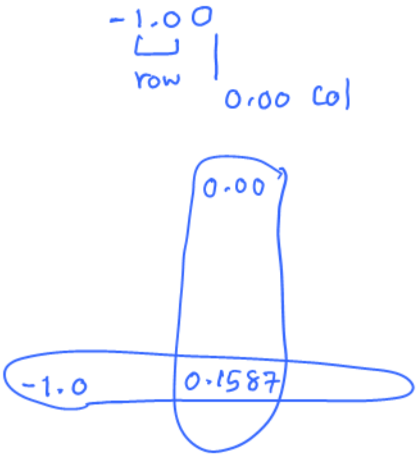
(07) Let X- first year salary of WPI graduates.

$X \sim N(\mu = 60000, \sigma^2 = (15000)^2)$.

(a) $P(X < 45000) = P\left(\frac{X-\mu}{\sigma} < \frac{45000-60000}{15000}\right)$

$= P(Z < -1)$

$= \boxed{0.1587}$



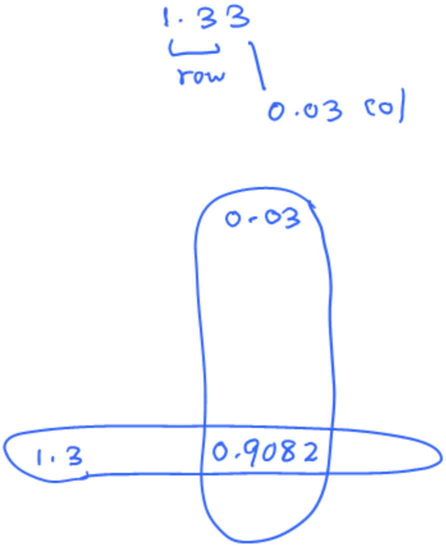
(b) $P(X > 80000) = P\left(\frac{X-\mu}{\sigma} > \frac{80000-60000}{15000}\right)$

$= P(Z > 1.33)$

$= 1 - P(Z \leq 1.33)$

$= 1 - 0.9082$

$= \boxed{0.0918}$



(08) $\mu = 72$, $\sigma = 6$, $n = 45$

(a) • Mean of sample mean $= \mu_{\bar{x}} = \mu = 72 \Rightarrow \boxed{\mu_{\bar{x}} = 72}$

• Variance of Sample mean $= \sigma^2_{\bar{x}} = \frac{\sigma^2}{n} = \frac{6^2}{45} = 0.8$

$\Rightarrow$ Standard deviation of sample mean $\Big\} = \sigma_{\bar{x}} = \sqrt{\sigma^2_{\bar{x}}} = \sqrt{0.8} = \boxed{0.8944}$

Since $n = 45 > 35$, (Large sample) by using Central Limit Theorem,

$\bar{X} \,\dot{\sim}\, N(\mu_{\bar{x}}, \sigma^2_{\bar{x}})$.

i.e, $\boxed{\bar{X} \,\dot{\sim}\, N(\mu_{\bar{x}} = 72, \sigma^2_{\bar{x}} = 0.8)}$

(b) Sample mean is differ from population mean 72 by at least 2 units means , $\bar{x} < 70$ or $\bar{x} > 74$

So, we need to find, $P(\bar{x} > 70) + P(\bar{x} > 74)$. $= 1 - P(70 < \bar{x} < 74)$

Then, $P(70 < \bar{x} < 74) = P\left(\frac{70 - 72}{\sqrt{0.8}} < \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} < \frac{74 - 72}{\sqrt{0.8}}\right)$

$= P(-2.24 < Z < 2.24)$

$= P(Z < 2.24) - P(Z < -2.24)$

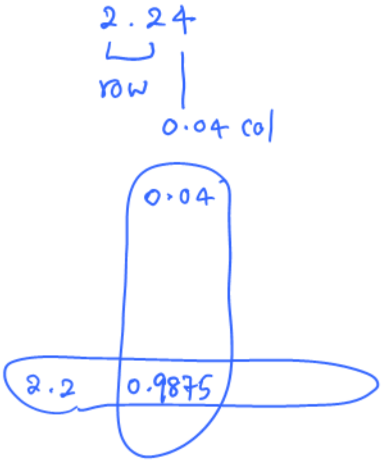
$= P(Z < 2.24) - P(Z > 2.24)$

($\because$ using symmetric property)

$= P(Z < 2.24) - (1 - P(Z \leq 2.24))$

$= 2P(Z < 2.24) - 1$

$= 0.9875 \times 2 - 1 = 0.975$



So, $P(X < 70) + P(X > 74) = 1 - 0.975$
$= \boxed{0.025}$

(09) $X \sim ?\,(\mu = 90, \sigma^2 = (15)^2)$, $n = 80$

Need to find $P(\text{sum} > 7500)$.

Let S_{80} - total sum. Then,

$$\mu_{S_{80}} = E(S_{80}) = n\mu = 80 \times 90 = 7200$$

$$\sigma^2_{S_{80}} = Var(S_{80}) = n\sigma^2 = 80 \times (15)^2 = 18000$$

Since $n = 80 > 35$ (Large sample), by Central Limit Theorem,

$$S_{80} \mathrel{\dot\sim} N(\mu_{S_{80}} = 7200, \sigma^2_{S_{80}} = 18000)$$

Need to find, $P(S_{80} > 7500) = ?$

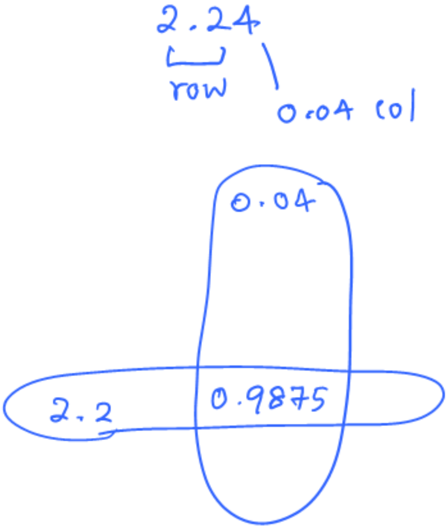
$$P(S_{80} > 7500) = P\left(\frac{S_{80} - 7200}{\sqrt{18000}} > \frac{7500 - 7200}{\sqrt{18000}}\right)$$

$$= P(Z > 2.24)$$

$$= 1 - P(Z \le 2.24)$$

$$= 1 - 0.9875$$

$$= \boxed{0.0125}$$



If X is not Normal and $n \geq 35$, we can use Central Limit Theorem.

$$S_n \mathrel{\dot\sim} N(n\mu, n\sigma^2)$$

(10) $\mu = 17.2$, $\sigma = 2.5$.

(a) $n = 55 > 35$ (Large Sample), by Central Limit Theorem,

$$\bar{X} \overset{\cdot}{\sim} N\left(\mu_{\bar{x}} = 17.2,\ \sigma^2_{\bar{x}} = 2.5^2/55\right)$$

Need to find $P(17 < \bar{X} < 18) = ?$

$$P(17 < \bar{X} < 18) = P\left(\frac{17-17.2}{\sqrt{(2.5)^2/55}} < \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} < \frac{18-17.2}{\sqrt{(2.5)^2/55}}\right)$$

$$= P(-0.5933 < Z < 2.373)$$

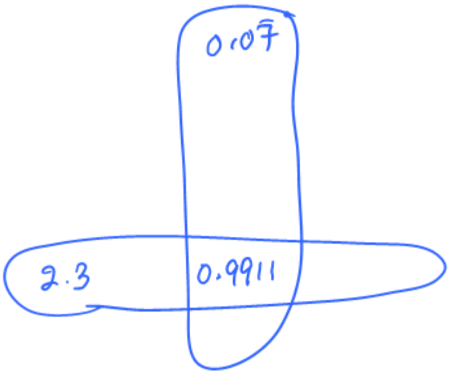
$$= P(-0.59 < Z < 2.37)$$

$$= P(Z < 2.37) - P(Z < -0.59)$$

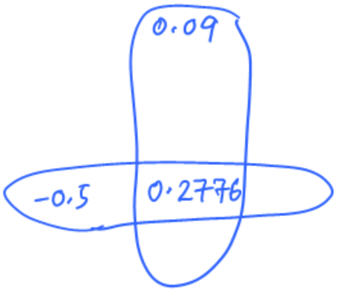
$$= 0.9911 - 0.2776$$

$$= \boxed{0.7135}$$

$\underbrace{2.3}_{\text{row}}\underbrace{7}_{0.07 \text{ col}}$



$\underbrace{-0.5}_{\text{row}}\underbrace{9}_{0.09 \text{ col}}$



(b) Here, $n = 55 > 35$ (Large Sample).

So, we can apply Central Limit Theorem).

$X \sim N(\mu, \sigma^2) \longrightarrow \bar{X} \sim N(\mu, \sigma^2/n)$

X is <u>not</u> Normal $\longrightarrow$ • If $n \geq 35$, then $\bar{X} \overset{\cdot}{\sim} N(\mu, \sigma^2/n)$

• If $n < 35$, then $\bar{X}$??

Central Limit Theorem

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Bibliography

[1] Rick Durrett. *Elementary probability for applications*. Cambridge university press, 2009.

“The only certainty is that nothing is certain.” – Pliny the Elder

“Probability is not just a math concept; it’s a way of thinking about the world.”

“In every situation, you have a choice—what you do with it is your probability of success.”

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