Black Hole and Quantum Szilárd Heat Engines

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I Abstract

This report explores the properties of two unusual heat engines. The first, the black hole engine, operates between two thermal reservoirs consisting of black holes and extracts energy from the emitted Hawking radiation. The unusual features of this engine are explored. The second engine, the quantum Szilard engine, is used to understand why the violation of the second law of thermodynamics by a quantum Maxwell's demon turns out not to be the case.

II Acknowledgements

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1 Introduction

Heat engines are devices that extract energy from one or more thermal reservoirs and convert it partially into work. This report studies two unconventional heat engines: a black hole heat engine and a quantum Szilárd engine.

In the 1970s Bekenstein [7] realized that when an object falls into a black hole, the entropy of the hole increases as a result. He also found the entropy of the black hole to be proportional to its surface area and not its volume, as is the case in ordinary thermodynamics. Stephen Hawking [8,9] verified this prediction by applying quantum field theory to the region just outside a black hole. Hawking found that a black hole emits radiation (now termed "Hawking radiation") and behaves like blackbody at a temperature that is inversely proportional to its mass [2].

In chapter 2 we use these discoveries to show how a Carnot engine can be made to operate between two black hole heat reservoirs, following the treatment of Opatrný and Richterek [1]. We give derivations of some of the basic expressions in black hole thermodynamics, and also recall the operation of the idealized Carnot cycle. We then discuss the thermodynamics of a black hole in a box, and show how two such black holes could serve as the hot and cold reservoirs of a heat engine. The cycle of the black hole heat engine is described and analyzed, and it is shown to operate at the Carnot efficiency. We end the discussion of the black hole heat engine by describing its work output with and without any use of the cosmic microwave background radiation.

The idea of Maxwell's demon was first put forward by James Clerk Maxwell in his 1871 book, *Theory of Heat*, where he imagined a small creature, or demon, who was able to perform actions that led to a violation of the second law of thermodynamics [3]. The demon opens and closes a hatch between two sections of a box, allowing only faster particles to pass in one direction and slower particles in the other, creating a buildup of faster particles on one side of the box. The hatch is assumed to be massless and frictionless, so that the demon does no work in the process. However the demon succeeds in creating a temperature difference between the two halves of the box without doing any work, which is a violation of the second law of thermodynamics [4]. In 1929, 58 years after Maxwell's book, Leo Szilárd introduced a model of a single molecule heat engine and used it to analyze the paradox of Maxwell's demon. He reasoned that once the demon was considered to be a part of the system the paradox could be resolved by considering the demon's memory as a valid entropic cost. However, his analysis was limited since it was carried out entirely within the classical domain. Wojciech Zurek [10] later introduced a quantum version of Szilárd's engine to give a more complete resolution of the paradox.

In chapter 3, we discuss a variation of Zurek's engine given in a paper by Davies, Thomas, and Zahariade [5], which we refer to as the quantum harmonic

Szilárd engine. This engine has the advantage over Zurek's engine that it allows analytical calculations to be performed of all the stages of the engine's cycle and thus permits a more transparent resolution of the paradox of Maxwell's demon to be given.

A concluding chapter recapitulates the results of this work and indicates some areas in which new aspects of the problems studied here are being pursued.

2 Black Hole Heat Engine

This chapter explores the properties of a heat engine that operates in a Carnot cycle between two thermal reservoirs consisting of black holes. The laws of classical thermodynamics are used to analyze the properties of this engine and to gain an understanding of both its possibilities and its limitations. The account given below is based largely on the work of Opatrný and Richterek [1].

2.1 Black Hole Thermodynamics

In order to understand how black holes could be used in a heat engine, we must first explore their unique thermodynamic properties. For this analysis we will consider non-rotating, uncharged black holes. Black holes of this form are commonly referred to as a Schwarzschild black holes, and have a Schwarzschild radius of

$$R_S = \frac{2GM}{c^2}, \tag{1}$$

where M is the mass of the black hole, c is the speed of light and G is Newton's constant of gravitation. Black holes were once thought to be truly black, with no emission occurring from them. However, Beckenstein and Hawking discovered that black holes actually radiate at the Beckenstein-Hawking temperature

$$T_{BH} = \frac{\hbar c^3}{8\pi kGM}, \tag{2}$$

where h-bar is Planck's constant and k is Boltzmann's constant. We can use the Beckenstein-Hawking temperature to learn more about black holes by combining the first and second laws of thermodynamics in a joint form,

$$TdS = dU + dW. \tag{3}$$

If we consider our system to be an isolated black hole then it does no work on its surroundings, $dW = 0$. Expressing the energy of the black hole as $U = mc^2$, with a temperature $T = T_{BH}$, equation (3) becomes

$$T_{BH}dS = c^2 dm, \tag{4}$$

which can be converted into a differential equation

$$\frac{dS}{dm} = \frac{c^2}{T_{BH}} = \frac{8\pi Gk}{c\hbar}m. \tag{5}$$

Integrating this equation under the boundary condition $S = 0$ at m=0 gives the entropy of a black hole of mass M,

$$S_{BH} = \frac{4\pi Gk}{c\hbar}M^2. \tag{6}$$

The above equation can be written in terms of the surface area of the black hole, $A = 4\pi R_S^2$. This results in

$$S_{BH} = \frac{kc^3}{4G\hbar} A. \tag{7}$$

Notice how the entropy of the black hole depends on its surface area rather than its volume, diverging from the usual expectations of ordinary thermodynamics. This strange property is an expression of the holographic principle, which states that all the information about a black hole is encoded on its boundary.

From equations (2) and (7), we can infer a key feature of black hole thermodynamics; as a black hole emits radiation, its mass and surface area decrease, and its temperature increases. This can also be seen by combining equations (1) and (2) to get

$$R_S = \frac{\hbar c}{4\pi k T_{BH}}, \tag{8}$$

which further confirms that a black hole with a smaller Schwarzschild radius, and hence a smaller surface area, has a higher temperature. Thus, as a black hole radiates heat, its temperature increases. This implies that black holes have a negative heat capacity, C, which we can confirm through calculation,

$$C = \frac{dE}{dT_{BH}} = c^2 \frac{dM}{dT_{BH}} = -\frac{8\pi G k}{\hbar c} M^2. \tag{9}$$

This is a strange yet crucial result in understanding the thermodynamic properties of black holes. The more heat black holes give off, the smaller and hotter they become. Hence, black holes with smaller a surface area have a higher temperature, and black holes with a larger surface area have a lower temperature. This principle will be crucial in exploring the use of black holes in a heat engine, with the goal of constructing a heat engine with maximal efficiency. But in order to do that we must first define a specific type of engine to emulate.

2.2 The Carnot Cycle

According to the second law of thermodynamics, no heat engine can have 100% efficiency, as not all heat can be fully converted into work. The highest possible efficiency is attained with an idealized engine that operates in a reversible manner between two heat reservoirs. The cycle of this hypothetical engine is known as a Carnot cycle. To ensure that the engine achieves maximum efficiency, it must avoid all irreversible processes. Hence, every process in the Carnot cycle must be carried out quasi-statically (i.e. with infinite slowness) to prevent any irreversible heat flow. Thermodynamic equilibrium must also be maintained at every stage of the process. There are four reversible processes in the Carnot cycle:

1. Isothermal Expansion
2. Adiabatic Expansion

3. Isothermal Compression
4. Adiabatic Compression

After adiabatic compression, the system is back in its initial state and the process repeats. The efficiency of a Carnot engine is given by

$$e_{Carnot} = 1 - \frac{T_C}{T_H}. \tag{10}$$

Hence, the maximal efficiency of a heat engine only depends on the temperatures of the two reservoirs [6]. A conventional Carnot engine has an ideal gas as its working substance. However, we will explore the properties of a Carnot engine using black holes as the heat reservoirs and the photon gas emitted by them as the working substance. Now that we have familiarized ourselves with the thermodynamic properties of black holes and the properties of the Carnot cycle, we are ready to construct a theoretical model of a black hole heat engine.

2.3 Black Hole in a Box

The heat engine we will discuss consists of two Schwarzschild black holes as heat reservoirs and the gas of photons they are in equilibrium with as the working substance. First we will consider a single one of these black holes inside a box filled with the radiation it emits, and take it to be in thermal equilibrium with this radiation at the temperature T. The total energy of the box, or thermal reservoir, is the sum of the energy of the black hole and the radiation,

$$E = Mc^2 + aVT^4, \tag{11}$$

where the first term on the right is the energy of the black hole and the second term is the energy of the blackbody radiation in a box of volume V at temperature T, and

$$a = \frac{\pi^2 k^4}{15c^3\hbar^3} \tag{12}$$

The entropy of the system is

$$S_{tot} = S_{BH} + S_{rad}, \tag{13}$$

with S_{BH} given by equation (7) and the radiation entropy

$$S_{rad} = \frac{4}{3}aVT^3. \tag{14}$$

By using equation (11), we can solve for the radiation temperature in terms of the total energy and black hole mass, and plug this into equation(14) to get an expression for the total entropy as a function of M:

$$S_{tot} = k\left[\frac{4\pi G}{\hbar c}M^2 + \frac{4}{3}\sqrt[4]{\frac{\pi^2 Vc^3}{15\hbar}}(M_{tot} - M)^{\frac{3}{4}}\right] \tag{15}$$

where $M_{tot} \equiv \frac{E_{tot}}{c^2}$. We now want to determine the conditions under which the system is stable. The system is stable at the equilibrium entropy, which also happens to be the maximum entropy. Hence, we need to find local maximums of the total entropy. One local maximum of equation (15) is at the boundary M=0, which is when the black hole has completely evaporated, leaving only radiation in the box. In order to find the other local extrema we set the partial derivative of the entropy equal to zero,

$$\left(\frac{\partial S_{tot}}{\partial M}\right) = 0, \tag{16}$$

and obtain

$$M^4 \left(M_{tot} - M\right) = \frac{\pi^2 c^7 \hbar V}{15 \left(8\pi G\right)^4}. \tag{17}$$

This equation has real solutions between 0 and M_{tot} only for V sufficiently small V,

$$V \leq V^* = \frac{2^{20} 3\pi^2 G^4 M_{tot}^5}{5^4 c^7 \hbar}. \tag{18}$$

For larger volumes, the black hole will evaporate completely by the emission of Hawking radiation before it has the time to absorb radiation reflected by the walls and come to equilibrium with it. Condition (18) determines the length of the box required to maintain the black hole in thermodynamic equilibrium as

$$l_{box} \leq 40 l_p \left(\frac{M}{m_p}\right)^{2/3} \tag{19}$$

where $l_p = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length and $m_p = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass. In the black hole heat engine we will study, the thermal reservoirs are taken to be black holes in equilibrium with cavity radiation in boxes whose dimensions obey the constraint (19).

2.4 Black Hole Carnot Cycle

Consider a black hole heat engine with two thermal reservoirs of the kind described above. The larger black hole has a lower temperature and will serve as the cold reservoir, while the smaller black hole has a higher temperature and will serve as the hot reservoir. This is due to the thermodynamic properties of black holes described in equations (2), (8), and (9). In between the two reservoirs there is a cylinder with a movable piston. The cylinder's walls and piston reflect radiation perfectly, and the volume of the cylinder is very small in comparison to the two boxes. During the process, the cylinder can be completely isolated or open to one of the two reservoirs.

The engine operates in a four-stage cycle that differs slightly from that in the usual Carnot engine. The cycle begins with the piston close to the hot

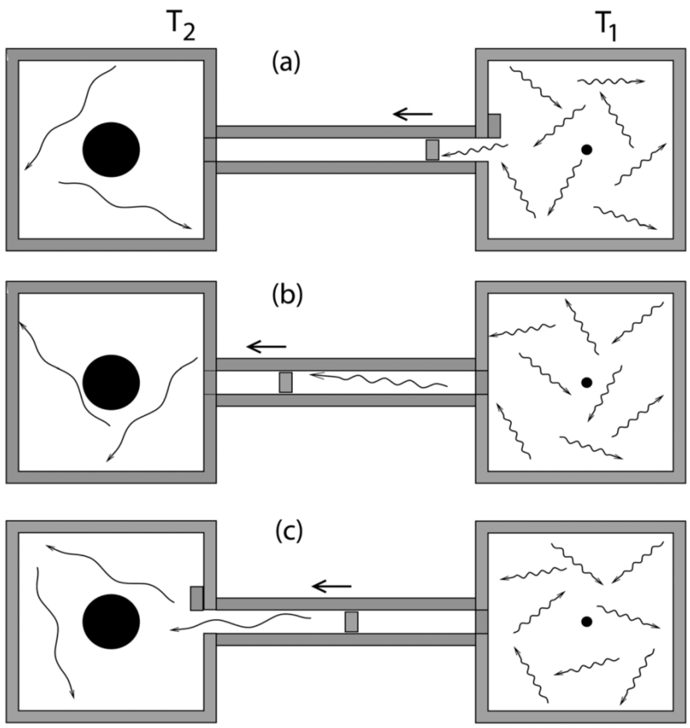

Figure 1: Black Hole Heat engine with the small black hole serving as the hot reservoir and the large black hole as the cold reservoir.

reservoir, with the volume of the working substance being equal to zero. In the first stage of the cycle, the cylinder is opened to the hot reservoir and the hot radiation enters the cylinder and pushes the piston toward the cold reservoir. The radiation in the cylinder expands isothermally and the volume of the working medium increases to V_1. Radiation pressure is given by $p = \frac{aT^4}{3}$, which only depends on temperature, implying that an isothermal process is also an isobaric process. The work done by the system during this first stage is

$$W_1 = p_H V_1 = \frac{1}{3} a T_H^4 V_1, \tag{20}$$

where p_H and T_H are the pressure and temperature of the hot reservoir respectively. The energy extracted from the hot reservoir is

$$Q_1 = T_H \Delta S_1 = \frac{4}{3} a V_1 T_H^4. \tag{21}$$

In the second stage, the cylinder is isolated from the two reservoirs, and the radiation expands adiabatically, by pushing the piston to volume V_2, cooling the temperature to T_C, the temperature of the cold reservoir. Since the radiation expands adiabatically, the radiation entropy in equation (14) is constant. Thus, the resulting volume is $V_2 = V_1 \left(T_H/T_C\right)^3$. During this process, pressure is not constant, so we must do an integral to calculate the work done during this stage, which is

$$W_2 = \int_{V_1}^{V_2} p dV = \frac{a}{3} \int_{V_1}^{V_2} T^4 dV. \tag{22}$$

We can use the relation $V = V_1 \left(T_H/T\right)^3$ to convert the integral over V into one over T to get

$$W_2 = -a V_1 T_H^3 \int_{T_H}^{T_C} dT. \tag{23}$$

which results in,

$$W_2 = a T_H^4 V_1 \left(1 - \frac{T_C}{T_H}\right). \tag{24}$$

In the third stage, the cold radiation is pushed out of the cylinder isothermally into the cold reservoir. Work is done on the system in this process, since the radiation is pushed into a region of nonzero pressure. The work done on the system is given by

$$W_3 = -p_C V_2 = -\frac{1}{3} a T_H^3 T_C V_1, \tag{25}$$

and the heat energy pushed into the cold reservoir is

$$Q_3 = T_C \Delta S_3 = \frac{4}{3} a V_2 T_C^4 = \frac{4}{3} a V_1 T_H^3 T_C. \tag{26}$$

In the last stage of the cycle, the piston is returned to its original position through an empty cylinder by a mechanism in the heat engine that performs no work. Therefore, the net work in one cycle is

$$W_{net} = W_1 + W_2 + W_3 = \frac{4}{3}aV_1T_H^3\left(T_H - T_C\right) = Q_1 - Q_3. \tag{27}$$

We have thus described a complete cycle of our black hole heat engine. Using equation (27), we can calculate the efficiency of the engine to be

$$e = \frac{W_{net}}{Q_1} = \frac{Q_1 - Q_3}{Q_1} = 1 - \frac{Q_3}{Q_1}. \tag{28}$$

By dividing equation (27) by equation (21), we obtain the relationship

$$\frac{Q_3}{Q_1} = \frac{T_C}{T_H}. \tag{29}$$

Therefore, the efficiency becomes

$$e = 1 - \frac{T_C}{T_H}, \tag{30}$$

which matches equation (10) exactly, showing that our black hole heat engine operates with the same efficiency as an ideal Carnot engine between the same reservoirs.

2.5 Work Extraction and Power

Now that we have a working model of a black hole Carnot engine, we must discuss how much work can actually be extracted from it. So far, we have only considered a single cycle during which the reservoirs remain relatively unchanged. However, each cycle extracts heat from the hot reservoir and transfers it to the cold reservoir. Hence, over several cycles, the temperatures of the two heat reservoirs will change. By using equation (29), and the relationships $dQ_1 = -dM_1c^2$ and $dQ_3 = dM_2c^2$, we can write

$$\frac{dM_1}{dM_2} = -\frac{T_H}{T_C}. \tag{31}$$

This equation describes how the change in the black hole masses resulting from the energy transfer is constrained by the temperatures of the two reservoirs. From equation (2), we know that the temperature is inversely proportional to mass, allowing equation (31) to be expressed as

$$M_1dM_1 = -M_2dM_2. \tag{32}$$

This can easily be integrated to find the solution

$$M_1^2 + M_2^2 = M_{1,0}^2 + M_{2,0}^2 \tag{33}$$

where $M_{1,0}$ and $M_{2,0}$ are the initial masses of the small (hot) and large (cold) black hole, respectively. During the entire Carnot process the hot reservoir loses mass and becomes hotter, while the cold reservoir gains mass and becomes colder. This strange behavior is again due to the black hole's negative heat capacity, which causes its temperature to increase when energy is extracted. This process will continue until the black hole in the hot reservoir completely evaporates, leaving the cold black hole with the final mass

$$M_f = \sqrt{M_{1,0}^2 + M_{2,0}^2} \tag{34}$$

We can use the decrease in mass in conjunction with Einstein's mass-energy equivalence to express the total work extracted over the entire Carnot process as

$$W_{tot} = \left(M_{1,0} + M_{2,0} - \sqrt{M_{1,0}^2 + M_{2,0}^2} \right) c^2 \tag{35}$$

However a real engine would not be able to operate in a perfectly reversible fashion, and as a result some some of the energy would be lost to irreversible processes. So equation (35) is really an upper limit to how much work can be extracted from our black hole heat engine. Truly reversible processes are infinitely slow. Thus, for any practical heat engine maximal efficiency and maximal power cannot coexist. There must be some sacrifice of efficiency if we want to have an engine that produces a sufficient amount of energy in a reasonable amount of time. For the black hole Carnot engine, the power is limited by the rate at which the hot reservoir releases energy and also by the rate at which the cold reservoir absorbs energy. The total power of the engine is the difference between these two powers,

$$P = P_H - P_C. \tag{36}$$

The power radiated by a black hole that emits only photons is given by the classical thermodynamic equation

$$P_{BH} = e\sigma A T^4, \tag{37}$$

where e is the emissivity, $\sigma = ac/4$ is the Stefan-Boltzmann constant. On using $A = 4\pi R_S^2$ along with (1) and (12), we can write (37) as

$$P_{BH} = \frac{\pi}{240} \frac{k^2 T^2}{\hbar}, \tag{38}$$

where the emissivity has been set to one for a black hole. Using this equation for both P_H and P_C allows us to express the efficiency of the black hole heat engine as

$$e = \frac{P_H - P_C}{P_H} = 1 - \frac{T_C^2}{T_H^2}. \tag{39}$$

Notice that this efficiency exceeds that for an ideal Carnot engine given in (10) (and represents a situation that cannot be achieved in practice).To get around this difficulty, we can reduce the power P_H by the factor T_C/T_H, while keeping P_C unchanged as this would allow us to operate an engine at the ideal efficiency.

In order for the two black holes to remain in thermal equilibrium, energy must be extracted from the hot reservoir at a much lower rate than the black hole emits it. This is also true for radiation absorption by the cold reservoir. Therefore,

$$P_H \ll \frac{\pi k^2 T_H}{240\hbar} \tag{40}$$

and

$$P_C \ll \frac{\pi k^2 T_C}{240\hbar}. \tag{41}$$

Using equations (40) and (41), and reducing the hot reservoir's power by T_C/T_H, we find that the maximum power that can be extracted from the heat engine must obey the inequality

$$P \ll \frac{\pi}{240}\frac{k^2 T_C\left(T_H - T_C\right)}{\hbar}. \tag{42}$$

In order to assess the practicality of this engine, let us consider a numerical example. We shall consider a hot black hole with a mass of $M_1 = 5 \times 10^{12}$ kg, and a cold black hole of mass $M_2 = 4 \times 10^{25}$ kg. Using equation (2), we can compute the corresponding black hole temperatures to be $T_C = 3.07 \times 10^{-3}$ K and $T_H = 2.46 \times 10^{10}$ K. Using these temperatures, we find from equation (42) that the output power must be much less than 2×10^{-6} W. The maximum energy that can be extracted from this system can be calculated to be 4.5×10^{29} J, implying that it would take over 7×10^{27} years to extract it all. This example shows that this type of heat engine would be practically useless, even if it could be realized.

2.6 Considering a Universe With Background Radiation

If the universe outside the heat engine was empty, its temperature would be zero, $T_C = 0$. Then the universe could be used as the cold reservoir instead of the cold black hole. In that case we could modify our original heat engine to only include one black hole; the hot black hole would remain in the hot reservoir, and waste radiation would be pumped out into the empty universe, which now serves as the cold reservoir. In the ideal case, the total extractable work is equal to the total energy of the black hole.

In the real world we must consider background radiation, which is ever-present in the universe. If the background radiation is colder than any of the black holes, it would still be simpler to pump the waste photons into space than a black hole. Using this alternative version of our heat engine, we must account for the fact that space can absorb heat at an unlimited rate. Now, in equation (39), P_C becomes the problematic factor. To prevent the Carnot limit from being exceeded, we must increase the emission rate into the cold reservoir by the factor T_H/T_C. On doing this, equation (39) reduces to the proper Carnot efficiency, and we can express the total power of the heat engine as

$$P \ll \frac{\pi}{240}\frac{k^2 T_H\left(T_H - T_C\right)}{\hbar}. \tag{43}$$

If the black hole is allowed to evaporate entirely during the engine's use, we can calculate the maximum amount of work that can be extracted by assuming that no increase in the entropy of the engine or the environment occurs in the process. We know from equation (6) that the entropy of a black hole with mass M is

$$S_{BH} = \frac{4\pi Gk}{c\hbar} M^2. \tag{44}$$

If the entire black hole were to evaporate, the entropy of the universe would go down by this amount. In order for the entropy change to be zero, the entropy of the cold reservoir, or the universe in this case, must go up by this amount. Let's assume that the engine emits a fraction f of the energy Mc^2 of a black hole as heat into the cold reservoir. The remainder is then converted into useful work,

$$W = (1 - f)\, Mc^2 \tag{45}$$

Therefore, the entropy of the cold reservoir goes up by fMc^2/T_C. For zero entropy change, we must have

$$\frac{fMc^2}{T_C} = \frac{4\pi Gk}{c\hbar} M^2. \tag{46}$$

By solving this equation for f and using equation (2), we find that

$$f = \frac{T_C}{2T_{BH}}. \tag{47}$$

Using this value for f, along with equation (45), allow us to calculate the work that can be extracted from this engine when the entire black hole evaporates as

$$W = \left(1 - \frac{T_C}{2T_{BH}}\right) Mc^2. \tag{48}$$

Notice that $Mc^2/2$ of useful work can still be extracted, even when the background radiation and black hole start out at the same temperature.

In the event that the black hole is much colder than the background radiation, we can flip this process and now use the radiation as the hot reservoir and deposit the waste energy into the black hole.

2.7 Comparison to Conventional Carnot Engine

Despite the unconventional nature of its working substance, a black hole heat engine shares some of the same properties as the conventional Carnot engine. It consists of two thermal reservoirs that regulate the heat flow, and it follows a reversible set of operations consisting of isothermal and adiabatic stages like the usual Carnot engine. Its efficiency also matches that of a Carnot engine operating between reservoirs at the same temperatures.

However, there are some key differences from the standard Carnot engine. We have seen that black holes have a negative heat capacity, and thus increase in temperature as they give off radiation. As a result of this strange behavior, the hot reservoir gets hotter until the hot black hole completely disappears. Meanwhile, the cold reservoir gets colder as it absorbs the radiation and the temperature difference between the reservoirs therefore diverges. This is in stark contrast to a conventional Carnot engine, in which the temperatures of the two reservoirs gradually become equal, and the operation of the engine grinds to a halt.

It is interesting that the old field of thermodynamics is perfectly up to the task of working out the properties of as exotic a heat engine as one based on black holes.

3 Harmonic Quantum Szilárd Engine

In the previous chapter, we used classical thermodynamics to analyze an unusual type of heat engine involving black holes. In this chapter we show how the analysis of a type of heat engine proposed by Szilard [3] through the lens of quantum mechanics can help to save the second law of thermodynamics from the attack launched on it by Maxwell's demon. Our account below is based on the recent paper of Davies, Thomas, and Zahariade [5].

3.1 Maxwell's Demon and the Classical Szilárd Engine

Maxwell's demon is a thought experiment that was proposed in 1867 by theoretical physicist James Clerk Maxwell. In this thought experiment, a demon is in control of a massless door, positioned between two chambers of gas. The demon quickly opens and closes the door, conspiring to only allow fast moving particles to pass in one direction and slow moving particles in the other. After this action is repeated for awhile, one chamber will be full of faster moving particles, while the other will contain slower moving particles. Hence, the demon would have caused one chamber to heat up and the other to cool down without doing any work. The demon has therefore succeeded in making the entropy of the system decrease without doing any work, seemingly violating the second law of thermodynamics.

This apparent contradiction was addressed in 1929 by Leo Szilárd [3], who sought to resolve the issue by introducing a theoretical device that we now know as the Szilárd engine. Szilárd imagined an engine consisting of a single classical particle confined to a rigid box, immersed in a thermal bath at temperature T. In order to extract work from this engine, the demon needs to determine which half of the box the particle is located and then insert a movable barrier in the middle of the box. The barrier slides like a piston without friction and is used to extract work through the isothermal expansion of the single particle ideal gas. Szilárd showed that the demon would need a method of measuring the speed of the particle to determine where it was, and then use this information to make the engine do work. However, the act of obtaining information would demand an expenditure of energy, which would cause an increase in the demon's own entropy, that would be larger than the loss in the gas's entropy. Thus Szilárd showed that the violation of the second law could be resolved if the demon was considered to be a part of the system, so that his entropy increase was taken into account. However,Szilard's analysis is carried out within the framework of classical mechanics and is not definitive.

In this chapter, we will focus on a quantum Szilárd engine, in which a particle is confined by a harmonic oscillator potential while being in contact with a heat reservoir at a fixed temperature with which it can exchange energy. The use of such a model was recently suggested by Davies, Thomas, and Zahariade [5], and we will see how it can be used to give a better resolution of the paradox

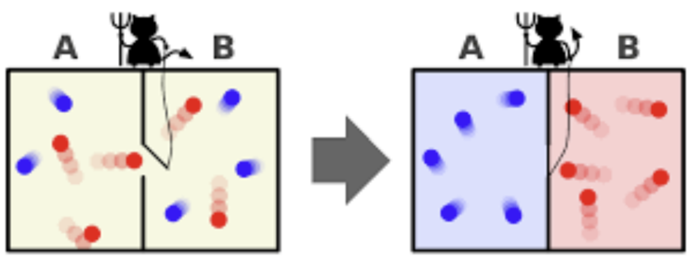

Figure 2: Maxwell's demon allows the faster moving particles to enter the right side of the box and the slower particles to enter the left side of the box.

than many of the other approaches that have preceded it.

3.2 Quantum Szilárd Cycle

Before we can examine the cycle of this heat engine, we first need to characterize its initial state. The Hamiltonian of the engine at the start of the cycle is the standard Hamiltonian for a quantum harmonic oscillator,

$$\hat{H}_{in} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2, \tag{49}$$

with energy eigenvalues

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \tag{50}$$

for $n = 0, 1, 2, 3, ...,$ etc. However, the engine is in contact with a thermal reservoir at temperature T, so the probability of finding the system in any given energy level is

$$P_n = Ae^{-\beta E_n}, \tag{51}$$

where $\beta = \frac{1}{kT}$. Due to the normalization condition,

$$\sum_{n=0}^{\infty} P_n = 1, \tag{52}$$

we can write

$$A\sum_{n=0}^{\infty} e^{-\beta E_n} = 1, \tag{53}$$

and solve for A to find

$$A = \frac{1}{\sum_{n=0}^{\infty} e^{-\beta E_n}}. \tag{54}$$

where the quantity in the denominator is the so called partition function

$$Z := \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}. \tag{55}$$

The partition function is useful because it can be used to derive many other properties of a system in thermodynamic equilibrium. Using the partition function, we can express the average energy as

$$E = -\frac{1}{Z}\frac{dZ}{d\beta}, \tag{56}$$

and the useful work that can be extracted from the system, otherwise known as the Helmholtz free energy, can be written as

$$A = -\frac{1}{\beta}\ln Z. \tag{57}$$

We now apply these equations to determine the initial state of the quantum harmonic Szilárd engine. The initial Hamiltonian is given in equation (43) and because the particle is in a thermal state at a temperature T, its initial density matrix is

$$\hat{\rho}_{in} = \frac{1}{Z_{in}} \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega} |\psi_n\rangle\langle\psi_n|, \tag{58}$$

with $|\psi_n\rangle$ being the eigenstates of the oscillator with the energy given by (50). The initial partition function is then of the same form as equation (49),

$$Z_{in} = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}. \tag{59}$$

This equation can be expressed as

$$Z_{in} = \sum_{n=0}^{\infty} e^{-\frac{1}{2}\beta\hbar\omega} e^{-n\beta\hbar\omega} = \sum_{n=0}^{\infty} e^{-\frac{1}{2}\beta\hbar\omega} \left(e^{-\beta\hbar\omega}\right)^n, \tag{60}$$

which is a geometric series. Therefore, equation (60) can be summed to give

$$Z_{in} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}, \tag{61}$$

which can be rewritten using hyperbolic trigonometry as

$$Z_{in} = \frac{1}{2} csch\left(\frac{\beta\hbar\omega}{2}\right). \tag{62}$$

Now that we have the initial partition function we can find the Helmholtz free energy,

$$A_{in} = -\frac{1}{\beta} \ln Z_{in} = \frac{1}{\beta} \ln\left[2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right], \tag{63}$$

and the average energy,

$$E_{in} = -\frac{1}{Z_{in}} \frac{dZ_{in}}{d\beta} = \frac{1}{2}\hbar\omega\coth\left(\frac{\beta\hbar\omega}{2}\right). \tag{64}$$

Finally, we calculate the initial entropy of the system

$$S_{in} = -\frac{dA_{in}}{dT} = k\left[\frac{\beta\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right) - \ln\left(2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right)\right]. \tag{65}$$

We can check that (64) has the expected limiting behaviors at both low and high temperatures. In the low temperature limit, where $\beta \gg 1$, the average energy becomes $1/2\hbar\omega$, which is the ground state energy of the simple harmonic oscillator. This result agrees with equation (44) for n=0. For high temperatures, where $\beta \ll 1$, the average energy is kT. This is consistent with the equipartition theorem, which states that each degree of freedom adds $1/2kT$ energy to the system. Our harmonic well has two degrees of freedom; the momentum p, and the displacement from equilibrium q, which agrees with the energy value of kT.

For the initial entropy in equation (59), the low temperature limit causes the entropy to become zero, which is in agreement with the third law of thermodynamics. In the high energy limit, the entropy approaches $S_{in} = k \ln (kT/\hbar\omega)$, which agrees with the prediction of classical statistical mechanics. Thus the expressions we have derived for the average energy and the entropy make sense.

3.3 Barrier Insertion

We now introduce Maxwell's demon into the process. His goal is to convert information about the particle's whereabouts into work, and attempt to violate the second law. As in the classical Szilárd engine, the demon inserts a barrier into the system to gain the information he needs. However, due to the subtleties involved at the quantum level, a new approach for barrier insertion is required.

In order to localize the particle to the left or right of the potential well's center, an infinitely thin potential barrier is quasi-statically inserted at q=0. This modifies the initial Hamiltonian to

$$\hat{H}_{bar}\left(t\right) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 + \alpha\left(t\right)\delta\left(\hat{q}\right). \tag{66}$$

The strength of this delta function is given by the time-dependent function $\alpha(t)$, which satisfies the conditions $\alpha(-\infty) = 0$ and $\alpha(+\infty) = \infty$. These conditions ensure that the barrier gets gradually inserted over a long period of time, with the second condition ensuring that the particle is unable to tunnel through the (infinitely high) barrier at late times. We shall also impose a slowness condition $|\dot{\alpha}/\alpha| \ll \omega$, which allows us to treat the evolution of the wave function adiabatically. Essentially, by ensuring that the strength of the delta function increases very slowly compared to the frequency of the oscillator, we are able to treat the Hamiltonian as fixed at any given moment. This also ensures that the system is in thermal equilibrium with the bath at a temperature T at every moment. The evolution must be very slow in order to retain this equilibrium, hence we call it a quasi-static evolution. During the said evolution, the eigenstates $|\psi_n\rangle$ corresponding to the initial Hamiltonian $\hat{H}_{in}$ slowly vary and continuously adjust themselves to be instantaneous eigenstates $|\psi_n^\alpha\rangle$ of $\hat{H}_{bar}$. This causes the density matrix in equation (58) to change. In order to calculate the density matrix after the barrier has been fully inserted at $t = +\infty$ we need to compute the instantaneous eigenstates $|\psi_n^\alpha\rangle$ and their eigenvalues E_n^α at this time. Their wave functions Ψ_n^α are solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dq^2}\Psi^\alpha\left(q\right) + \frac{1}{2}m\omega^2q^2\Psi^\alpha\left(q\right) + \alpha\delta\left(q\right)\Psi^\alpha\left(0\right) = E^\alpha\Psi^\alpha\left(q\right). \tag{67}$$

We can observe that, for odd n, the initial eigenstates $|\psi_n\rangle$ and their energy eigenvalues $E_n = (n+1/2)\hbar\omega$ will not be affected by the insertion of the barrier. This is because, for a simple quantum harmonic oscillator, the odd wave functions vanish at $q = 0$, which is exactly where the barrier is inserted in order to divide the well into two equal halves. We can express this as $|\psi^{\alpha}_{2k+1}\rangle = |\psi_{2k+1}\rangle$ and $E^{\alpha}_{2k+1} = E_{2k+1}$ for all α. This gives odd solutions $\Psi^{\alpha}_{2k+1} = \Psi_{2k+1}$ to the above Schrödinger equation. For even n, the initial eigenstates and eigenvalues are affected by the inserted barrier. As α increases from 0 to ∞, the initial quantized energy levels, $(2k+1/2)\hbar\omega$, converge to $(2k+3/2)\hbar\omega$. Thus, $E^{\alpha}_{2k} = E_{2k+1}$ and the even wave functions are

$$\Psi^{\alpha}_{2k}(q) = \begin{cases} \Psi_{2k+1}(q), & \text{for } q > 0, \\ -\Psi_{2k+1}(q), & \text{for } q < 0. \end{cases} \tag{68}$$

These are the even solutions to equation (67) in the limit $\alpha \to \infty$, and completes our description of the eigenstates $|\psi^{\infty}_{n}\rangle$ after the barrier is fully inserted. Notice how the new energy solutions for odd and even n both equal E_{2k+1}, since $E^{\alpha}_{2k} = E_{2k+1}$ and $E^{\alpha}_{2k+1} = E_{2k+1}$. This means that each energy level is now twofold degenerate, which effectively produces a spectrum of the simple harmonic oscillator that has a frequency 2ω and is shifted up by $\hbar\omega$. The lowest energy for $n = 0$ is now $3/2\hbar\omega$ which is $\hbar\omega/2$ higher than the initial ground state energy.

Using these solutions, we can write the final thermal density matrix after the barrier has been fully inserted as

$$\hat{\rho}_{\perp} = \frac{1}{Z_{\perp}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+3/2)} \left(|\psi^{\infty}_{2n}\rangle\langle\psi^{\infty}_{2n}| + |\psi^{\infty}_{2n+1}\rangle\langle\psi^{\infty}_{2n+1}| \right). \tag{69}$$

The partition function after barrier insertion is

$$Z_{\perp} = 2\sum_{n=0}^{\infty} e^{-\beta(2n+3/2)\hbar\omega}, \tag{70}$$

where we have multiplied the infinite sum by two to account for the twofold degeneracy of each of the energy levels. As before, we can sum the geometric series to express the partition function as

$$Z_{\perp} = \frac{2e^{-\frac{3}{2}\beta\hbar\omega}}{1 - e^{-2\beta\hbar\omega}} = e^{-\frac{\beta\hbar\omega}{2}} csch\left(\beta\hbar\omega\right). \tag{71}$$

Now that we have the modified partition function, we can again calculate all the relevant thermodynamic quantities of the system such as the Helmholtz free energy and the average energy, respectively, as

$$A_{\perp} = -\frac{1}{\beta}\ln Z_{\perp} = \frac{1}{\beta}\ln\left[\sinh\left(\beta\hbar\omega\right)\right] + \frac{\hbar\omega}{2} \tag{72}$$

and

$$E_{\perp} = -\frac{1}{Z_{\perp}}\frac{dZ_{\perp}}{d\beta} = \hbar\omega\coth\left(\beta\hbar\omega\right) + \frac{1}{2}\hbar\omega. \tag{73}$$

Notice that the insertion of the barrier requires an energy cost,

$$A_{\perp} - A_{in} = \frac{\hbar\omega}{2} + \frac{1}{\beta}\ln\left[\cosh\left(\frac{\beta\hbar\omega}{2}\right)\right]. \tag{74}$$

Therefore, the demon must provide energy to the system in the form of work. The entropy of the system now becomes

$$S_{\perp} = -\frac{dA_{\perp}}{dT} = k\beta\hbar\omega \coth\left(\beta\hbar\omega\right) - k\ln\left[\sinh\left(\beta\hbar\omega\right)\right], \tag{75}$$

which makes sense because the adiabatic insertion of the barrier alters the spectrum of the simple harmonic oscillator to have double the frequency, $\omega \to 2\omega$. So the modified entropy is of the same form as the initial entropy in equation (65), but with the factors of one half canceled out by the doubled frequency. In the high temperature limit, $S_{in} = S_{\perp}$, meaning the barrier insertion has no effect at very high temperatures.

3.4 Quantum Localization and Measurement

Now that the barrier has been inserted, the demon needs to localize the particle to the left or right side of the potential well. We introduce the left and right eigenstates

$$|L_n\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{2n}^{\infty}\rangle - |\psi_{2n+1}^{\infty}\rangle\right) \tag{76}$$

and

$$|R_n\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{2n}^{\infty}\rangle + |\psi_{2n+1}^{\infty}\rangle\right). \tag{77}$$

These eigenstates are defined in such a way as to ensure that the particle is localized to only the left or the right of the barrier. We can rewrite the density matrix after barrier insertion in terms of these newly defined eigenstates,

$$\hat{\rho}_{\perp} = \frac{1}{Z_{\perp}}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+3/2)}\left(|L_n\rangle\langle L_n| + |R_n\rangle\langle R_n|\right). \tag{78}$$

Now the demon must determine whether the particle is on the left or right of the barrier. This can be done by projecting the state of the particle onto one of the two eigenspaces of the observable

$$\hat{\Pi} = \sum_{n=0}^{\infty}\left(|L_n\rangle\langle L_n\rangle - |R_n\rangle\langle R_n|\right), \tag{79}$$

with associated projection operators

$$\hat{P}_L = \sum_{n=0}^{\infty}|L_n\rangle\langle L_n\rangle, \tag{80}$$

$$\hat{P}_R = \sum_{n=0}^{\infty}|R_n\rangle\langle R_n|. \tag{81}$$

A successful projective measurement of $\hat{\Pi}$ on the state of the particle reduces the density matrix $\hat{\rho}_{\perp}$ to

$$\hat{\rho}_{L} = \frac{1}{Z_L} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+3/2)} |L_n\rangle\langle L_n|, \tag{82}$$

or

$$\hat{\rho}_{R} = \frac{1}{Z_R} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+3/2)} |R_n\rangle\langle R_n|. \tag{83}$$

Here, the left and right partition functions can be expressed as

$$Z_{L,R} = \frac{1}{2} Z_{\perp} = \frac{e^{-3/2\beta\hbar\omega}}{1 - e^{-2\beta\hbar\omega}} = \frac{1}{2} e^{-\beta\hbar\omega/2} csch\left(\beta\hbar\omega\right). \tag{84}$$

Once again, we can use the partition function to calculate all of the important thermodynamic quantities once the particle has been localized to the left or right of the potential well. The Helmholtz free energy is found to be

$$A_{L,R} = -\frac{1}{\beta} \ln Z_{L,R} = \frac{1}{\beta} \ln\left[\sinh\left(\beta\hbar\omega\right)\right] + \frac{\hbar\omega}{2} + \frac{1}{\beta} \ln\left(2\right), \tag{85}$$

and the average energy is

$$E_{L,R} = -\frac{1}{Z_{L,R}} \frac{dZ_{L,R}}{d\beta} = \frac{1}{2}\hbar\omega + \hbar\omega \coth\left(\beta\hbar\omega\right). \tag{86}$$

Although the measurement process leaves the average energy unchanged, the entropy of the system does change,

$$S_{L,R} = -\frac{A_{L,R}}{dT} = k\beta\hbar\omega \coth\left(\beta\hbar\omega\right) - k\ln\left[\sinh\left(\beta\hbar\omega\right)\right] - k\ln\left(2\right) \tag{87}$$

The change in entropy of the system after measurement is

$$\Delta S = S_{L,R} - S_{\perp} = -k\ln\left(2\right). \tag{88}$$

As a result of the demon acquiring information on the localization of the particle, the entropy of the system has actually decreased. Meanwhile the Helmholtz free energy increases by the quantity

$$\Delta A = A_{L,R} - A_{\perp} = \frac{1}{\beta} \ln\left(2\right). \tag{89}$$

Now that the demon has accrued a surplus of energy, he will attempt to turn this into useful work.

3.5 The Violation of The Second Law

Once the quantum particle has been projected onto one side of the barrier, the barrier will experience a force that pushes it in the opposite direction. For example, if the particle is localized to the left side of the barrier, then there will be a rightward force. This force will do work on the barrier as it slides it to the right end of the box. This could be used to lift an external weight of some sort. Eventually, the barrier will be pushed all the way to the end of the box, and the system will have returned to its initial state, thus completing one cycle. During this last step, the change in free energy is

$$A_{in} - A_L = -\frac{1}{2}\hbar\omega - \frac{1}{\beta}\ln\left[\cosh\left(\frac{\beta\hbar\omega}{2}\right)\right] - \frac{1}{\beta}\ln 2 \tag{90}$$

The first two terms of equation (90) correspond to the work the demon supplied to the system during the partition insertion, while the last term corresponds to energy that the demon is able to extract from the system. This excess energy, as previously predicted, could be used by the demon to lift a weight or load. This amount of extractable energy will emerge every cycle, meaning the demon can do this indefinitely. The demon can therefore reason that he has accomplished his goal of violating the second law of thermodynamics since he was able to decrease the entropy of the universe each Szilárd cycle by $k\ln 2$, whilst also converting information into work. However, there is a flaw in the demon's reasoning. The apparent violation of the second law is a result of not considering the demon as a part of the system. In order to save the second law of thermodynamics, we must introduce a new concept into our analysis.

3.6 Rescuing the Second Law

In order to demonstrate that the second law of thermodynamics is not violated by the quantum Szilárd engine, we must consider the demon itself as part of the system. The demon can perform one of two possible actions. It can either attach a string to the left or right of the partition, depending on on the results of the quantum measurement. As stated before, the motion of the partition would pull the string, allowing the piston to perform work on some mass. Since the side of the partition that the demon attaches the string to depends on which side the particle has been projected to, the state of the demon becomes entangled with the particle as a result of his measurement and subsequent action. We now proceed to give the detailed argument that leads to this conclusion. The initial state of the demon is taken to be

$$|D_0\rangle = \frac{1}{\sqrt{2}}|D_L\rangle + \frac{1}{\sqrt{2}}|D_R\rangle, \tag{91}$$

where $|D_L\rangle$ and $|D_R\rangle$ are the states of the demon for measuring the particle on the left and right side of the harmonic well, respectively. In other words, the demon is taken to be a two-state quantum object in an equally likely superposition of the "left" and "right" states. The initial density matrix of the engine and the demon is

$$\hat{\rho}_{in} \otimes |D_0\rangle\langle D_0|. \tag{92}$$

After the barrier is inserted, the state of the system becomes

$$\hat{\rho}_{\perp} \otimes |D_0\rangle\langle D_0|. \tag{93}$$

The Hamiltonian then becomes $\hat{H}_\alpha \otimes I_D$, where I_D is the identity operator for the demon's Hilbert space. Notice how the analysis we did for the barrier insertion previously still holds as the demon and the quantum particle have not yet interacted. During the quantum measurement process however, the demon and the particle are coupled, with a coupling strength λ. Thus, the interaction Hamiltonian can be expressed as

$$\hat{H}_{int} = i\lambda\hat{\Pi} \otimes (|D_L\rangle\langle D_R| - |D_R\rangle\langle D_L|)\,. \tag{94}$$

Notice how this Hamiltonian is time independent, meaning that the evolution operator for the measurement is represented by

$$\hat{U}_{int} = e^{-i/\hbar \hat{H}_{int}\delta t} = \frac{1}{\sqrt{2}}\left[I_P \otimes I_D + \hat{\Pi} \otimes (|D_L\rangle\langle D_R| - |D_R\rangle\langle D_L|)\right] \tag{95}$$

where I_P is the identity operator on the particle Hilbert space. After the measurement process is completed, the density matrix that describes the system is given to be

$$\hat{U}_{int}\,(\hat{\rho}_{\perp} \otimes |D_0\rangle\langle D_0|)\,\hat{U}^{\dagger}_{int} = \frac{1}{2}\,(\hat{\rho}_L \otimes |D_L\rangle\langle D_L| + \hat{\rho}_R|D_R\rangle\langle D_R|)\,. \tag{96}$$

Therefore, as a result of the interaction between the particle and the demon, the two systems become correlated. They are no longer independent of each other. The particle measurement process causes the state of the demon to become either $|D_L\rangle$ or $|D_R\rangle$, for a left or right projection of the particle respectively. Once the barrier is quasi-statically pushed all the way to one side of the box, the final state of the system is

$$\frac{1}{2}\hat{\rho}_{in} \otimes (|D_L\rangle\langle D_L| + |D_R\rangle\langle D_R|)\,. \tag{97}$$

Notice how, despite the particle returning to its initial state, the demon has been put into a mixed state, which has an entropy associated with it (which counteracts the entropy decrease of the system, and therefore saves the second law). In order to restart the cycle, the demon must be made pure again, which requires some input of energy. The energy cost required to return the state to $|D_0\rangle$ would be at least $kT\ln 2$ in terms of free energy, and this would also nullify the work done by the engine in the course of a cycle. Therefore, the second law of thermodynamics is not violated by the Szilárd engine. Once the demon is described as a part of the total system, one can see that the apparent failure of the second law is actually a misconception. In conclusion, we have shown that if the demon is treated as a quantum two-state system and considered as a part of the engine in analyzing its performance, the net entropy increase of the system plus demon is at best equal to zero and that the useful work that can be extracted from the system is also at best zero. Because of non-idealities in both the engine and the demon, the net entropy increase is usually greater than zero and a positive amount of work must also be done on the system plus demon to restore it to its initial state at the start of the cycle.

4 Conclusion

This report has investigated two unconventional heat engines. First we used classical thermodynamics to describe the properties of a black hole heat engine that operated in an ideal Carnot cycle. We reviewed the basic features of black hole thermodynamics and then discussed the operation of the engine under a variety of conditions. In the second part of the study we used quantum statistical mechanics to rescue the second law of thermodynamics from Maxwell's demon by using a quantum harmonic Szilárd engine. Once the demon is considered a part of the system, the resolution to the paradox lies in realizing that the entropy of the demon goes up as he acquires information he needs to extract work from the engine.

While the black hole engine is certainly possible, at least as a thought experiment, it is very difficult to realize practically and so is mainly of academic interest. It is nevertheless interesting to see what the laws of physics will allow in these strange circumstances. Much of the physics discussed in connection with this engine is relevant for an understanding of primordial black holes and their evaporation, and was in fact developed by Hawking [8] and others for this purpose.

The quantum Szilard engine, by contrast, is well within the reach of what can be done experimentally nowadays. Physicists and biologists have come up with a number of examples of microscopic engines in which a Maxwell's demon can exploit the information he gains about the system to do more work than allowed by the second law of thermodynamics [11,12]. However there is always then an increase in the entropy of the environment of the engine and the demon. This is a fascinating and ongoing area of activity in which we can expect to see significant progress in the years to come.

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