Taming Hypersingular Integrals
Using Dimensional Continuation

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ABSTRACT

We use the method of dimensional continuation to isolate singularities in integrals containing products of Green's functions or their derivatives. Rules for the extraction of the finite part of so-called hypersingular integrals are developed, which should be useful in methods based on boundary integral techniques in science and engineering. In applications to potential theory, electromagnetic scattering, and crack dynamics in continuum mechanics, boundary integrals now can be readily evaluated using computational techniques without recourse to complex analysis or contour distortions since the hypersingularities occurring in intermediate steps of the computations can be isolated and ignored while taking the finite parts of the integrals into account in a consistent manner. We have also identified new forms of the Dirac δ-function in D dimensions which are useful and convenient in the calculations. A summary of the integrable singular integrals is given in tabular form.

CONTENTS

I. INTRODUCTION

The properties of Green's functions and other generalized functions are defined[1] by the "company they keep," in the sense that their behavior is determined by an integration of such functions multiplied by well-behaved functions.[2] However, frequently in physical calculations in science and engineering we encounter derivatives of Green's functions as in the boundary integral method, or its numerical implementation in the boundary element method (BEM). This leads to non-integrable singularities that require careful attention in treating them.

In quantum field theory, we have an analogous situation in which products of Green's functions appearing in loop diagrams lead to infinities. Particularly lucid comments on this issue of the need to define new rules for the evaluation of products of singular functions have been given by Bogoliubov and Shirkov.[3] The method of analytic continuation in spatial dimension D of the integrals, to isolate the singular part and to identify the relevant finite values of the integrals, is used in relativistic field theory in perturbative evaluations of physically relevant quantities. In QFT, the nature of the divergences require "dimensional regularization" by which the infinities are absorbed into physically observable parameters through the process of renormalization.

Fortunately, in potential theory, electromagnetic field computations, and in the theory of crack dynamics and continuum mechanics, the singularities occurring in intermediate stages of the calculations can be shown to cancel out. Thus, while renormalization is not an issue in this case, managing the infinities in the theory and performing numerical analysis is an issue and it can be troublesome. Several investigations in the literature refer to the integrals appearing in the integral representation of potentials and fields and their evaluation by the BEM as hypersingular integrals.[4–6,17]

Here we wish to explore the use of dimensional continuation in the evaluation of integrals of the well known Green's functions in potential theory, in electromagnetic field calculations,[7] and in elasticity theory, and their derivatives. We identify the rules for obtaining consistent results through the use of such methods for the hypersingular integrals occurring in the BEM.[8] We provide a systematic approach to the identification of the singularities in typical integrals and show how to isolate them using the dimensional continuation method. These results are then used in examples of such integrals occurring in the above-mentioned physical applications. We have also identified Dirac δ-functions in D dimensions that are useful in

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simplifying and understanding the results. A summary of the integrable singular integrals is reported in tabular form in Tables I-III in Appendix A.

It is hoped that the present approach will provide an effective, practical method of evaluating the so-called hypersingular integrals in computational science and engineering applications, with an automated approach to accounting for these issues in a direct manner.

II. SINGULAR INTEGRALS

In this section we first classify a set of singular integrals that frequently occur in our discussion of integral equations. This is followed by identifying combinations of these integrals that are of physical interest.

We begin by considering singular integrals of the form

$$\int_{|\mathbf{r}|<R} \frac{f(\mathbf{r})}{|\mathbf{r}|^d} d^D\mathbf{r}, \tag{1}$$

where $f(\mathbf{r})$ has a Taylor series expansion around the origin. In Eq.(1), D is the dimension of space which we will take to be continuous. We will introduce a shift in the denominator[9,17] by substituting $|\mathbf{r}| \Rightarrow \sqrt{r^2 + \epsilon^2}$ in order to easily isolate the infinite part of the singular integral. At the end of the calculation, the limit $\epsilon \to 0$ will be used.

For convenience, we use the substitutions

$$\rho^2 = r^2 + \epsilon^2, \tag{2}$$

$$\mathbf{r} = (x_1, x_2, \ldots, x_D), \tag{3}$$

$$\delta = D - d, \tag{4}$$

$$A_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}. \tag{5}$$

We will always use d as the order of the singularity of the integrand, i.e. the power of r in the denominator of the integrand, and D as the dimension of the multi-dimensional integration. It will be shown below that whether an integral is singular and if so the type of the infinite part, is determined by $\delta = D - d$.

A. The basic integral $I_0(R; d, \delta) = \int_{|\mathbf{r}|<R} \frac{1}{|\mathbf{r}|^d} d^D\mathbf{r}$

Doing the "angular" integrations in D dimensions, we note that

$$d^D\mathbf{r} = A_D\, r^{D-1} dr. \tag{6}$$

We change $|\mathbf{r}|$ in the denominator to $\rho = \sqrt{r^2 + \epsilon^2}$ to write

$$I_0(R; d, \delta) \Rightarrow A_D \int_0^R \frac{r^{D-1}}{\rho^d}\, dr. \tag{7}$$

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The integral then becomes

$$\frac{I_0(R; d, \delta)}{A_D} = \int_0^R \rho^{-d/2} r^{D-1} dr$$

$$= \int_0^\infty \frac{r^{D-1}}{\rho^d} dr - \int_R^\infty \frac{r^{D-1}}{\rho^d} dr. \tag{8}$$

The first integral can be expressed in terms of Gamma functions,

$$\int_0^\infty \frac{r^{D-1}}{\rho^d} dr = \epsilon^\delta \left(\frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)} \right), \tag{9}$$

and the second integral can be expressed as a hypergeometric function

$$\int_R^\infty \frac{r^{D-1}}{\rho^d} dr = -\left(\frac{R^\delta}{\delta} \right) {}_2F_1\left(\frac{d}{2}, -\frac{\delta}{2}; 1 - \frac{\delta}{2}; -\frac{\epsilon^2}{R^2} \right), \tag{10}$$

where

$$_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{11}$$

with $\xi_n = [(a)_n, (b)_n, (c)_n]$ defined to be

$$(\xi)_n = \xi(\xi + 1)(\xi + 2) \cdots (\xi + n - 1); \qquad (\xi)_0 = 1. \tag{12}$$

From the series expansion of the hypergeometric function, to leading order in ϵ, we have

$$\int_R^\infty \frac{r^{D-1}}{\rho^d} dr = -\frac{R^\delta}{\delta}(1 + O(\epsilon^2)). \tag{13}$$

Therefore,

$$\frac{I_0(R; d, \delta)}{A_D} = \epsilon^\delta \frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)} + \frac{R^\delta}{\delta}. \tag{14}$$

We now consider three limits for the integral I_0:

(a) When $\delta > 0$, the first term vanishes when $\epsilon \to 0$. In fact, in this case I_0 is not a singular integral.

(b) When $\delta \to 0$, we have

$$\epsilon^\delta = 1 + \delta \ln \epsilon + O(\delta^2), \tag{15}$$

$$\frac{R^\delta}{\delta} = \delta^{-1} + \ln R + O(\delta), \tag{16}$$

$$\Gamma\left(\frac{D}{2} \right) = \Gamma\left(\frac{d + \delta}{2} \right) = \Gamma\left(\frac{d}{2} \right) + \frac{1}{2}\Gamma'\left(\frac{d}{2} \right) \delta + O(\delta^2), \tag{17}$$

$$\Gamma\left(-\frac{\delta}{2} \right) = -\frac{2}{\delta} - \gamma + O(\delta), \tag{18}$$

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where γ is Euler's constant $\gamma = 0.5772$. Combining these expressions together we have

$$\frac{I_0(R; d, \delta)}{A_D} = -\frac{\Gamma'(d/2)}{2\,\Gamma(d/2)} - \frac{\gamma}{2} + \ln\frac{R}{\epsilon} + O(\delta). \tag{19}$$

In this case, the integral has a logarithmic singularity.

(c) When $\delta < 0$, we have

$$\frac{I_0(R; d, \delta)}{A_D} = \frac{R^\delta}{\delta} + \epsilon^{-|\delta|}\left(\frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)}\right). \tag{20}$$

In this case, the singular integral has an $\epsilon^{-|\delta|}$ type infinity.

Therefore, we separate the infinite part of the singular integral $I_0(R; d, \delta)$ as follows. We have

$$\frac{I_0(R; d, \delta)}{A_D} = \begin{cases} \dfrac{R^\delta}{\delta}, & \text{for } \delta > 0, \text{ no infinity;} \\[2ex] -\dfrac{\Gamma'(d/2)}{2\,\Gamma(d/2)} - \dfrac{\gamma}{2} + \ln\dfrac{R}{\epsilon}, & \text{for } \delta = 0, \text{ log infinity;} \\[2ex] \dfrac{R^\delta}{\delta} + \epsilon^{-|\delta|}\left(\dfrac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)}\right), & \text{for } \delta < 0, \ \epsilon^{-|\delta|} \text{ infinity.} \end{cases} \tag{21}$$

Notice that the nature of the infinite part is determined only by $\delta = D - d$.

B. Integrals of the type $I_1(R; d, \delta, n) = \displaystyle\int_{|\mathbf{r}|<R} \frac{x^n}{|\mathbf{r}|^{d+n}}\, d^D\mathbf{r}$

In this case, we make the substitutions

$$x = r\cos\theta, \tag{22}$$

$$d^D\mathbf{r} = A_{D-1} r^{D-1} \sin^{D-2}\theta\, dr\, d\theta, \tag{23}$$

and shift the denominator from $|\mathbf{r}|$ to $\rho = \sqrt{r^2 + \epsilon^2}$ to write

$$I_1(R; d, \delta, n) \Rightarrow A_{D-1} \int_0^\pi \int_0^R \frac{r^n \cos^n\theta}{\rho^{d+n}}\, r^{D-1} \sin^{D-2}\theta\, dr\, d\theta$$

$$= A_{D-1} \int_0^R \rho^{-(d+n)}\, r^{[(D+n)-1]}\, dr \int_0^\pi \cos^n\theta\, \sin^{D-2}\theta\, d\theta$$

$$= I_0(R; d+n, \delta) \cdot \frac{A_{D-1}}{A_D} \int_0^\pi \cos^n\theta\, \sin^{D-2}\theta\, d\theta. \tag{24}$$

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The last integral is a Beta-function and we have

$$\frac{A_{D-1}}{A_D} \int_0^\pi \cos^n \theta \sin^{D-2} \theta \, d\theta = \begin{cases} 0, & \text{for } n \text{ odd;} \\ \dfrac{\Gamma(\frac{D}{2}) \Gamma(\frac{n+1}{2})}{\sqrt{\pi} \, \Gamma(\frac{D+n}{2})}, & \text{for } n \text{ even.} \end{cases} \tag{25}$$

When n is even, the Gamma functions can be simplified further to obtain

$$I_1(R; d, \delta, n) = \frac{(n-1)(n-3)\cdots 1}{(D+n-2)(D+n-4)\cdots D} I_0(R; d+n, \delta). \tag{26}$$

Since the type of infinity just depends on δ, $I_1(R; d, \delta, n)$ has the same singular behavior as $I_0(R; d, \delta)$. For example, the most commonly occurring non-zero case in typical applications is when $n = 2$, for which we obtain

$$I_1(R; d, \delta, 2) = \int_{|\mathbf{r}|<R} \frac{x^2}{|\mathbf{r}|^{d+2}} \, d^D\mathbf{r} = \frac{1}{D} I_0(R; d+2, \delta). \tag{27}$$

C.   Integrals of the type $I_2(R; d, \delta, m, n) = \displaystyle\int_{|\mathbf{r}|<R} \frac{x_1^m x_2^n}{|\mathbf{r}|^{d+m+n}} \, d^D\mathbf{r}$

For such integrals we make the substitutions

$$x_1 = r \cos \theta_1,$$
$$x_2 = r \sin \theta_1 \cos \theta_2,$$
$$d^D\mathbf{r} = A_{D-2} r^{D-1} \sin^{D-2} \theta_1 \sin^{D-3} \theta_2 \, dr \, d\theta_1 \, d\theta_2, \tag{28}$$

and as usual change $|\mathbf{r}|$ in the denominator to $\rho = \sqrt{r^2 + \epsilon^2}$ to obtain

$$I_2(R; d, \delta, m, n) \Rightarrow A_{D-2} \int_0^\pi \int_0^\pi \int_0^R \frac{r^{m+n} \cos^m \theta_1 \sin^n \theta_1 \cos^n \theta_2}{\rho^{d+m+n}} \times$$

$$r^{D-1} \sin^{D-2} \theta_1 \sin^{D-3} \theta_2 \, dr \, d\theta_1 \, d\theta_2$$

$$= I_0(R; d+m+n, \delta) \times$$
$$\frac{A_{D-2}}{A_D} \int_0^\pi \cos^m \theta_1 \sin^{D+n-2} \theta_1 \, d\theta_1 \int_0^\pi \cos^n \theta_2 \sin^{D-3} \theta_2 \, d\theta_2$$

$$= \begin{cases} \dfrac{\Gamma(\frac{D}{2}) \Gamma(\frac{m+1}{2}) \Gamma(\frac{n+1}{2})}{\pi \, \Gamma(\frac{D+m+n}{2})} I_0(R; d+m+n, \delta), & \text{both } m \text{ and } n \text{ are even;} \\ \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

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Thus I_2 also has the same singular behavior as $I_0(R; d, \delta)$.

D. Integrals of the type $I_k(R; d, \delta, \{n_i\}_{i=1}^k) = \int_{|\mathbf{r}|<R} \dfrac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{|\mathbf{r}|^{d+N}} d^D\mathbf{r}, \ N = \sum_{i=1}^k n_i$

Using the same approach as above, we can obtain a general formula

$$I_k(R; d, \delta, \{n_i\}_{i=1}^k) = \int_{|\mathbf{r}|<R} \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{|\mathbf{r}|^{d+N}} d^D\mathbf{r}$$

$$= \begin{cases} \dfrac{\prod_{i=1}^k (n_i - 1)!!}{(D+N-2)(D+N-4)\cdots D} I_0(R; d+N, \delta), & \text{all } n_i \text{ are even;} \\[6mm] 0, & \text{otherwise;} \end{cases} \tag{30}$$

where $(n_i - 1)!! = (n_i - 1)(n_i - 3) \cdots 1$, and $(-1)!! = 1$.

E. Integrals of the type $I_k^\epsilon(R; d, \delta, \{n_i\}_{i=0}^k) = \int_{|\mathbf{r}|<R} \dfrac{\epsilon^{n_0} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{|\mathbf{r}|^{d+n_0+N}} d^D\mathbf{r}, \ N = \sum_{i=1}^k n_i$

We can have a singular integral that has ϵ in the numerator. We assume that ϵ is a constant when performing the integration. Hence we will simply have

$$I_k^\epsilon(R; d, \delta, \{n_i\}_{i=0}^k) = \epsilon^{n_0} I_k(R; d+n_0, \delta-n_0, \{n_i\}_{i=1}^k). \tag{31}$$

We only need consider the case when all $n_i, i \geq 1$, are even, since the integral vanishes otherwise. For non-zero cases, we have

$$I_k^\epsilon(R; d, \delta, \{n_i\}_{i=0}^k) = \frac{\prod_{i=1}^k (n_i - 1)!!}{(D+N-2)(D+N-4)\cdots D} \epsilon^{n_0} I_0(R; d+N+n_0, \delta-n_0). \tag{32}$$

To simplify the notation we define

$$I_0^\epsilon(R; d, \delta, n_0) = \epsilon^{n_0} I_0(R; d+n_0, \delta-n_0), \tag{33}$$

so

$$I_k^\epsilon(R; d, \delta, \{n_i\}_{i=0}^k) = \frac{\prod_{i=1}^k (n_i - 1)!!}{(D+N-2)(D+N-4)\cdots D} I_0^\epsilon(R; d+N, \delta, n_0). \tag{34}$$

Therefore, I_k^ϵ is transformed to I_0^ϵ, so that we need discuss the property of I_0^ϵ. This is done in the following.

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F. The integral $I_0^\epsilon(R; d, \delta, n_0) = \epsilon^{n_0} I_0(R; d + n_0, \delta - n_0) = \int_{|\mathbf{r}| < R} \dfrac{\epsilon^{n_0}}{|\mathbf{r}|^{d + n_0}} d^D \mathbf{r}$

With the result for I_0 derived above, we have

$$I_0^\epsilon(R; d, \delta, n_0) = \epsilon^\delta \left(\frac{\Gamma\left(-\frac{\delta - n_0}{2}\right) \Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{d + n_0}{2}\right)} \right) + \epsilon^{n_0} \left(\frac{R^{\delta - n_0}}{\delta - n_0} \right). \tag{35}$$

Because n_0 is always greater than 0, the second term vanishes in the limit $\epsilon \to 0$. Notice that we make the above definitions for I_k^ϵ and I_0^ϵ because we want to make them also to have the factor of ϵ^δ. Here we consider the three limits:

(a) When $\delta > 0$, we simply get $I_0^\epsilon(R; d, \delta, n_0) = 0$.

(b) When $\delta \to 0$, we have

$$I_0^\epsilon(R; d, \delta, n_0) = \frac{\Gamma\left(\frac{n_0}{2}\right) \Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{d + n_0}{2}\right)} + O(d), \tag{36}$$

which is finite.

(c) When $\delta < 0$, we have

$$I_0^\epsilon(R; d, \delta, n_0) = \epsilon^{-|\delta|} \left(\frac{\Gamma\left(-\frac{\delta - n_0}{2}\right) \Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{d + n_0}{2}\right)} \right), \tag{37}$$

which has a singularity arising from the $\epsilon^{-|\delta|}$ factor.

III.   INTEGRABLE SINGULAR INTEGRALS

If two singular integrals have the same infinite part their difference is a finite number. More generally, a linear combination of singular integrals may sum to a finite number when their infinite parts cancel. We call such combinations as *integrable singular integrals* (ISI).[10] As will be shown below, most of the singular integrals in physics applications of potential theory and engineering analysis using Green's functions are ISI's.[11]

For the non-zero cases, the integrals $I_k(R; d, \delta, \{n_i\}_{i=1}^k)$ are always a multiple of $I_0(R; d + N, \delta)$, so that both types of integrals have the same type of infinity, logarithmic infinity when $\delta = 0$, and $\epsilon^{-|\delta|}$-type infinity when $\delta < 0$. Therefore, we can take the linear combination of $I_k(R; d, \delta, \{n_i\}_{i=1}^k)$ and $I_0(R; d, \delta)$ to cancel the singular parts and obtain ISI's. (We don't want to use $I_0(R; d + N, \delta)$ because this is an integral in $D + N$ dimension). Such ISI's are given by

$$I_0(R; d, \delta) - \frac{(d + N - 2)(d + N - 4) \cdots d}{\prod_{i=1}^k (n_i - 1)!!} I_k(R; d, \delta, \{n_i\}_{i=1}^k)$$

$$= \begin{cases} \left[1 - \dfrac{(d + N - 2)(d + N - 4) \cdots d}{(D + N - 2)(D + N - 4) \cdots D} \right] \dfrac{A_D}{\delta} R^\delta, & \text{for } \delta < 0; \\[4mm] \dfrac{1}{2} \left[\Psi \left(\dfrac{d + N}{2} \right) - \Psi \left(\dfrac{d}{2} \right) \right] A_D, & \text{for } \delta = 0, \end{cases} \tag{38}$$

where all n_i are even, and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. $\Psi((d+N)/2) - \Psi(d/2)$ can be written as

$$\Psi \left(\frac{d + N}{2} \right) - \Psi \left(\frac{d}{2} \right) = \frac{2}{d} + \frac{2}{d + 2} + \cdots + \frac{2}{d + N - 2}. \tag{39}$$

Another type of ISI's includes I_k^ϵ. Because I_k^ϵ can always be transformed to I_0^ϵ, we just need to consider I_0^ϵ. We note that I_0^ϵ is finite when $\delta = 0$, and is integrable. When $\delta < 0$, we have

$$I_0(R; d, \delta) - \left(\frac{\Gamma\left(-\frac{\delta}{2}\right)}{\Gamma\left(-\frac{\delta}{2} + \frac{n_0}{2}\right)} \frac{\Gamma\left(\frac{d}{2} + \frac{n_0}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \right) I_0^\epsilon(R; d, \delta, n_0) = \frac{A_D}{\delta} R^\delta. \tag{40}$$

We call these the *fundamental ISI* because all the other ISI's can be written as linear combinations of these ISI's. Some simple examples of fundamental ISI's are given as follows.

(*a*) By setting $k = 1$, $n_1 = 2$ in Eq.(38), we obtain *the simplest ISI* given by

$$I_0(R; d, \delta) - d \, I_1(R; d, \delta, 2) = \int_{|\mathbf{r}| < R} \left(\frac{1}{|\mathbf{r}|^d} - d \, \frac{x^2}{|\mathbf{r}|^{d+2}} \right) d^D \mathbf{r} = \frac{A_D}{D} R^\delta. \tag{41}$$

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It can be checked that this formula holds for all $\delta \geq 0$, or $\delta < 0$.

(b) By setting $n_0 = 2$ in Eq.(40), we have another ISI obtained from I_0 and I_0^ϵ, which we call ϵ^2-ISI, for which

$$I_0(R; d, \delta) - \frac{d}{d-D} \ I_0^\epsilon(R; d, \delta, 2) = \int_{|\mathbf{r}|<R} \left(\frac{1}{|\mathbf{r}|^d} - \frac{d}{d-D} \ \frac{\epsilon^2}{|\mathbf{r}|^{d+2}} \right) d^D \mathbf{r} = \frac{A_D}{\delta} R^\delta. \quad (42)$$

IV.   EXAMPLES OF ISI IN PHYSICS APPLICATIONS

We now illustrate the above considerations with three applications. The first is the case of calculating the potential and its derivatives using Poisson's solution of the electrostatic problem. The second is the evaluation of electromagnetic fields emitted by a conducting surface, where again integrable singularities occur. The final example is from the field of fracture dynamics.

A.   Poisson's Equation

1.   Poisson's equation in 3D

The solution of Poisson equation,

$$\nabla^2 \varphi(\mathbf{r}) = -4\pi \rho(\mathbf{r}), \tag{43}$$

is given by[7]

$$\varphi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}'. \tag{44}$$

Here the potential is represented by $\varphi(\mathbf{r})$, and our use of $\rho(\mathbf{r})$ for the charge density, in order to conform to the usual notation, should cause no confusion. The Green's function for the Poisson problem is $G(\mathbf{r}, \mathbf{r}') = 1/|\mathbf{r} - \mathbf{r}'|$.

The potential's first and second order derivatives are

$$\partial_i \varphi(\mathbf{r}) = -\int \frac{(r_i - r_i')\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3\mathbf{r}', \tag{45}$$

and

$$\partial_i \partial_j \varphi = \int \left(-\frac{\rho(\mathbf{r}')\delta_{ij}}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3(r_i - r_i')(r_j - r_j')\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^5} \right) d^3\mathbf{r}'. \tag{46}$$

When $i = j$, we should have for Eq.(46) the result

$$\sum_{i=1}^{3} \partial_i \partial_i \varphi(\mathbf{r}) = \nabla^2 \varphi(\mathbf{r}) = -4\pi \rho(\mathbf{r}), \tag{47}$$

from the standard identity $\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$. Thus we should be able to carry out the above integral explicitly, and we expect to have

$$\sum_{i=1}^{3} \int \left(-\frac{1}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3(r_i - r_i')^2}{|\mathbf{r} - \mathbf{r}'|^5} \right) \rho(\mathbf{r}') \, d^3\mathbf{r}' = -4\pi \rho(\mathbf{r}). \tag{48}$$

To verify the above we take a series expansion of $\rho(\mathbf{r}')$

$$\rho(\mathbf{r}') = \rho(\mathbf{r}) + (\mathbf{r}' - \mathbf{r}) \cdot \nabla\rho(\mathbf{r}) + O((\mathbf{r}' - \mathbf{r})^2). \tag{49}$$

The leading term of the expansion gives

$$\sum_{i=1}^{3} \int \left(-\frac{1}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3(r_i - r_i')^2}{|\mathbf{r} - \mathbf{r}'|^5} \right) \rho(\mathbf{r}) \, d^3\mathbf{r}' = -3\rho(\mathbf{r}) \int \left(\frac{1}{|\mathbf{s}|^3} - \frac{3s_i^2}{|\mathbf{s}|^5} \right) d^3\mathbf{s}, \tag{50}$$

where $\mathbf{s} = \mathbf{r}' - \mathbf{r}$. This is the simplest ISI with $d = D = 3$. So from Eq. (41) we have

$$\sum_{i=1}^{3} \int \left(-\frac{1}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3(r_i - r_i')^2}{|\mathbf{r} - \mathbf{r}'|^5} \right) \rho(\mathbf{r}) \, d^3\mathbf{r}' = -3\rho(\mathbf{r}) \frac{A_3}{3} = -4\pi\rho(\mathbf{r}). \tag{51}$$

We can show that the further terms in the series expansion are zero. Actually, any integral of the following form can be expressed as

$$\sum_{i=1}^{3} \int_{|\mathbf{s}|<R} \left(\frac{1}{|\mathbf{s}|^3} - \frac{3s_i^2}{|\mathbf{s}|^5} \right) s_x^a s_y^b s_z^c \, d^3\mathbf{s} = \lambda \, I_0(R; d, \delta), \tag{52}$$

where $d = 3 - a - b - c$, $\delta = D - d = a + b + c > 0$, and λ is a constant that is obtained by doing the angular integration and is given by Eq.(30). Because $\delta > 0$, we know this is a regular integral with no singularity, and from Eq.(21) with $D = 3$ we obtain

$$\sum_{i=1}^{3} \int_{|\mathbf{s}|<R} \left(\frac{1}{|\mathbf{s}|^3} - \frac{3s_i^2}{|\mathbf{s}|^5} \right) s_x^a s_y^b s_z^c \, d^3\mathbf{s} = 4\pi \, \lambda \, \frac{R^\delta}{\delta}. \tag{53}$$

On the other hand, by evaluating the difference between two such integrals over the ranges $[0, R_1]$ and $[0, R_2]$ with $R_2 > R_1$ we have

$$\sum_{i=1}^{3} \int_{R_1<|\mathbf{s}|<R_2} \left(\frac{1}{|\mathbf{s}|^3} - \frac{3s_i^2}{|\mathbf{s}|^5} \right) s_x^a s_y^b s_z^c \, d^3\mathbf{s} = 4\pi \, \lambda \, \frac{R_2^\delta - R_1^\delta}{\delta}. \tag{54}$$

We note that for $\mathbf{s} \neq 0$

$$\sum_{i=1}^{3} \left(\frac{1}{|\mathbf{s}|^3} - \frac{3s_i^2}{|\mathbf{s}|^5} \right) = \frac{3}{|\mathbf{s}|^3} - \frac{3s^2}{|\mathbf{s}|^5} = 0. \tag{55}$$

Hence the left side of Eq.(54) is zero, so that $\lambda = 0$. Therefore the integral in Eq.(53) vanishes. Combined with Eq.(51), we reconstruct the relation

$$\sum_{i=1}^{3} \int \left(-\frac{1}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3(r_i - r_i')^2}{|\mathbf{r} - \mathbf{r}'|^5} \right) \rho(\mathbf{r}') \, d^3\mathbf{r}' = -4\pi\rho(\mathbf{r}). \tag{56}$$

This can be generalized to higher dimensions.

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We note that

$$\delta^{(D)}(\mathbf{r}) = \frac{1}{A_D} \sum_{i=1}^{D} \left(\frac{1}{|\mathbf{r}|^D} - \frac{D \, r_i^2}{|\mathbf{r}|^{D+2}} \right), \tag{57}$$

is a Dirac δ-function in D dimension in the sense that[12]

$$\frac{1}{A_D} \sum_{i=1}^{D} \int \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|^D} - \frac{D(r_i - r_i')^2}{|\mathbf{r} - \mathbf{r}'|^{D+2}} \right) f(\mathbf{r}') \, d^D\mathbf{r}' = f(\mathbf{r}). \tag{58}$$

Also, we can write the δ-function as a limit

$$\begin{aligned}
\delta^{(D)}(\mathbf{r}) &= \frac{D}{A_D} \lim_{\epsilon \to 0} \left(\frac{1}{(\sqrt{r^2 + \epsilon^2})^D} - \frac{r^2}{(\sqrt{r^2 + \epsilon^2})^{D+2}} \right) \\
&= \frac{D}{A_D} \lim_{\rho \to r^+} \left(\frac{1}{\rho^D} - \frac{r^2}{\rho^{D+2}} \right).
\end{aligned} \tag{59}$$

2. Poisson's equation in 2D

In the 2D Poisson problem, cast in terms of the boundary integral method, we have[8]

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \oint dl' \left(G(\mathbf{r}, \mathbf{r}') \frac{\partial \phi(\mathbf{r}')}{\partial n'} - \phi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \right), \tag{60}$$

where $G(\mathbf{r}, \mathbf{r}') = -2 \ln |\mathbf{r} - \mathbf{r}'|$. we can assume ϕ and $\partial_{n'}\phi$ to be constants, as a worst case scenario, over a small line element from ℓ_a to ℓ_b, so that we need to evaluate the singular integrals

$$I_1 = \int_{\ell_a}^{\ell_b} \ln s \, dl', \tag{61}$$

$$I_2 = \int_{\ell_a}^{\ell_b} \frac{\mathbf{s} \cdot \mathbf{n}'}{s^2} \, dl', \tag{62}$$

where $\mathbf{s} = \mathbf{r} - \mathbf{r}'$. I_1 is a well-defined integrable end-point singular integral typified by

$$\int_0^R \ln x \, dx = \lim_{\epsilon \to 0} (x \ln x - x)|_\epsilon^R = R \ln R - R. \tag{63}$$

The integral I_2 can be evaluated using the point-shifting technique used earlier. We make use of the geometry displayed in Fig. 1 and write

$$dl' = \frac{s \, d\theta}{\cos \alpha} = \frac{s \, d\theta}{\hat{\mathbf{s}} \cdot \mathbf{n}'}. \tag{64}$$

Then,

$$\begin{aligned}
I_2 &= \int_{\ell_a}^{\ell_b} \frac{\mathbf{s} \cdot \mathbf{n}'}{s^2} \frac{s \, d\theta}{\hat{\mathbf{s}} \cdot \mathbf{n}'} \\
&= \int_{\ell_a}^{\ell_b} d\theta.
\end{aligned} \tag{65}$$

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This integral in the limit $\epsilon \to 0$ corresponds to an angle subtended by the contour at the singular point, so that for a straight contour (See Fig. 2a) we have $I_2 = \pi$, while for a corner, as shown in Fig. 2b, we have $I_2 = 3\pi/2$, as an exterior angle.

B. Electromagnetic Scattering

The technique of the integrable singular integral and dimensional continuation developed here can also be used in calculations with Helmholtz equation[13] and for electromagnetic scattering.[14]

1. 3D Scattering

In 3D, the electric field radiated by a conducting surface takes the form[9,15]

$$\mathbf{E} = -ikZ_0 \int_S \left[G(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}') + \frac{1}{k^2}\nabla\nabla G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') \right] dS', \tag{66}$$

where the Green's function is given by $G(\mathbf{r}, \mathbf{r}') = e^{ik\varrho}/4\pi\varrho$, with $\varrho = |\mathbf{r}-\mathbf{r}'|$, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.[13] The second term in the integral involves a second derivative of the Green's function, and therefore the corresponding integral is a hypersingular integral. If we invoke the finite element method to evaluate the integral and discretize the surface into small elements, we can assume that the current $\mathbf{J}(\mathbf{r}')$ is essentially a constant $\mathbf{J_0}$ over a suitably small element. We can then write the second term of the integral explicitly as

$$\int_{\Delta S} \nabla\nabla G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J_0} \, dS = \mathbf{J_0} \cdot \sum_{i,j} \hat{\varrho}_i \hat{\varrho}_j \int_{\Delta S} G_{ij} \, dS, \tag{67}$$

where ΔS is an element containing the singularity and

$$G_{ij} = \left[\frac{(3 - 3ik\varrho - k^2\varrho^2)\varrho_i\varrho_j}{\varrho^5} - \frac{\delta_{ij}(1 - ik\varrho)}{\varrho^3} \right] e^{ik\varrho}. \tag{68}$$

The singular integrals in $\int G_{ij} \, dS$ are

$$I_1 = \int_{\Delta S} \left(\frac{3\varrho_i^2}{\varrho^5} - \frac{1}{\varrho^3} \right) dS, \tag{69}$$

$$I_2 = \int_{\Delta S} \frac{\varrho_i\varrho_j}{\varrho^5} \, dS, \qquad (i \neq j), \tag{70}$$

where we have expanded the exponential $\exp(ik\varrho) \simeq (1 + ik\varrho)$ for small ϱ to isolate the singular terms. If we take the region of integration to be a circle around the singularity, we find $I_1 = \pi/R$ is the simplest ISI with $d = 3, D = 2$, and $I_2 = 0$ as is evident from Eq.(29). In the full calculation, we have to take the integral within and outside the circular region

separately; we then note that the integral over the exterior of the circle is a regular integral and can be computed directly. These identifications of the finite parts should substantially simplify the computational modeling of electromagnetic scattering.

2. 2D Scattering

In 2D scattering, the Green's function is given by[13]

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{ik\varrho}}{\sqrt{\varrho}}, \tag{71}$$

where $\varrho = |\mathbf{r} - \mathbf{r}'|$. Similarly, the components of $\nabla\nabla G$ are

$$\partial_i \partial_j G = \left[\left(\frac{5}{4} \varrho^{-\frac{9}{2}} - 2ik\,\varrho^{-\frac{7}{2}} - k^2\,\varrho^{-\frac{5}{2}} \right) \varrho_i \varrho_j + \left(-\frac{1}{2}\varrho^{-\frac{5}{2}} + ik\,\varrho^{-\frac{3}{2}} \right) \delta_{ij} \right] e^{ik\varrho}$$

$$= \frac{5}{4}\,\varrho_i \varrho_j\, \varrho^{-\frac{9}{2}} - \frac{1}{2}\delta_{ij}\varrho^{-\frac{5}{2}} + ik \left(-\frac{3}{4}\,\varrho_i \varrho_j\, \varrho^{-\frac{7}{2}} + \frac{1}{2}\delta_{ij}\varrho^{-\frac{3}{2}} \right) + O\left(\varrho^{-\frac{1}{2}} \right). \tag{72}$$

It can be easily checked that the singular terms in $\int G_{ij}dS$ also sum to ISI's. For example, the leading terms of $\int G_{ij}dS$ given by

$$I_3 = \int_{\varrho<R} \left(\frac{1}{\varrho^{5/2}} - \frac{5}{2}\frac{\varrho_i^2}{\varrho^{9/2}} \right) dS, \tag{73}$$

where $I_3 = \pi R^{-1/2}$ is the simplest ISI with $d = 5/2, D = 2$, and

$$I_4 = \int_{\varrho<R} \left(\frac{1}{\varrho^{3/2}} - \frac{3}{2}\frac{\varrho_i^2}{\varrho^{7/2}} \right) dS, \tag{74}$$

is of the form of the simplest ISI with $d = 3/2, D = 2$ and $I_4 = \pi R^{1/2}$.

C. Fracture Analysis

A final example, from the theory of crack energetics, again illustrates the issue of resolving hypersingular integrals using dimensional continuation. For the sake of completeness we briefly describe the relation appearing in fracture analysis. The relation between surface displacements $u_i(P)$ and tractions $\tau_i(P)$ for a smooth crack is given by the integral equation[16–18]

$$u_j(P) = 2 \int_{\partial C} [U_{ij}(P,Q)\tau_i(Q) - T_{ij}(P,Q)u_i(Q)]ds_Q, \tag{75}$$

where ∂C is the crack surface. A sum over repeated indices is implied. The displacement $U_{ij}(P,Q)$ and traction $T_{ij}(P,Q)$ at the observation point P due to source point Q are given by Kelvin's solution,

$$U_{ij} = \frac{1}{16\pi \, r \, (1-\nu) \, G}[(3-4\nu)\delta_{ij} + \partial_i r \, \partial_j r], \tag{76}$$

and

$$T_{ij} = -\frac{1}{8\pi \, r^2 \, (1-\nu)} \left\{ [(1-2\nu)\,\delta_{ij} + 3\,\partial_i r \, \partial_j r]\frac{\partial r}{\partial n} + (1-2\nu)(n_j\,\partial_i r - n_i\,\partial_j r) \right\}, \tag{77}$$

where $r = |\mathbf{r}_P - \mathbf{r}_Q|$, ν is Poisson's ratio, and G is the shear modulus. With the normal force $\mathbf{N} = N_i\,\mathbf{e}_i$, the traction τ is given by

$$\tau_i(P) = G\left[(\partial_j u_i + \partial_i u_j)N_j + \frac{2\nu}{1-2\nu}\,N_i\,\partial_k u_k\right] \tag{78}$$

The derivative of u_i can be obtained from Eq.(75) to be substituted here, and we have

$$\begin{aligned}
\tau_i(P) = \;&2GN_j \int_{\partial C} \{\,[\,\partial_j U_{mi}(P,Q) + \partial_i U_{mj}(P,Q)\,]\,\tau_m(Q) \\
&\qquad -[\,\partial_j T_{mi}(P,Q) + \partial_i T_{mj}(P,Q)\,]\,u_m(Q)\,\}\,ds_Q \\
&+\frac{4\nu}{1-2\nu}\,GN_i \int_{\partial C}[\,\tau_m(Q)\,\partial_k U_{mk}(P,Q) + u_m(Q)\,\partial_k T_{mk}(P,Q)\,]\,ds_Q.
\end{aligned} \tag{79}$$

We assume the boundary condition that the traction $\tau_m(Q) = 0$ on the crack, so the above integral is simplified to

$$\begin{aligned}
0 = \;&-2GN_j \int_{\partial C}[\,\partial_j T_{mi}(P,Q) + \partial_i T_{mj}(P,Q)\,]\,u_m(Q)\,ds_Q \\
&-\frac{4\nu}{1-2\nu}\,GN_i \int_{\partial C} u_m(Q)\,\partial_k T_{mk}(P,Q)\,ds_Q,
\end{aligned} \tag{80}$$

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and $\partial_k T_{ij}$ is given by

$$\partial_k T_{ij}(P,Q) = \frac{1}{8\pi\,(1-\nu)\,r^3} \times$$

$$\left\{ 3(\delta_{jk}\,\partial_i r + \delta_{ik}\,\partial_j r - 5\,\partial_i r\,\partial_j r\,\partial_k r)\frac{\partial r}{\partial n} + 3n_k\,\partial_i r\,\partial_j r \right.$$

$$+ (1-2\nu)\left[\delta_{ij}n_k - \delta_{jk}n_i + \delta_{ik}n_j \right.$$

$$\left.\left. + 3\left(n_i\,\partial_j r\,\partial_k r - n_j\,\partial_i r\,\partial_k r - \delta_{ij}\,\partial_k r\,\frac{\partial r}{\partial n} \right) \right] \right\}. \qquad (81)$$

We will show that the first integral of Eq.(80) is a singular integral and can be resolved by the ISI method. The same technique can be applied for the second integral. With the finite element method approach, we assume the crack surface is flat over a small element ΔS, and choose the local coordinate system so that the normal direction of ΔS is $\mathbf{e}_3$. Here ΔS contains the singular point, so that P and Q are points in ΔS. On this element we have $\mathbf{n} = \mathbf{e}_3$ and the normal force $\mathbf{N} = N_3\mathbf{e}_3$. Hence the first integral in Eq.(80) becomes

$$-2GN_3 \int_{\Delta S} \left[\, \partial_3 T_{mi}(P,Q) + \partial_i T_{m3}(P,Q) \,\right] u_m(Q)\,ds_Q, \qquad (82)$$

and $\partial_k T_{ij}$ becomes

$$\partial_k T_{ij}(P,Q) = \left(\frac{1}{8\pi\,(1-\nu)\,r^3} \right) \left\{ 3(\delta_{jk}\,\partial_i r + \delta_{ik}\,\partial_j r - 5\,\partial_i r\,\partial_j r\,\partial_k r)\,\partial_3 r + 3\delta_{3k}\,\partial_i r\,\partial_j r \right.$$

$$+ (1-2\nu)\left[\delta_{ij}\delta_{3k} - \delta_{jk}\delta_{3i} + \delta_{ik}\delta_{3j} \right.$$

$$\left.\left. + 3\left(\delta_{3i}\,\partial_j r\,\partial_k r - \delta_{3j}\,\partial_i r\,\partial_k r - \delta_{ij}\,\partial_k r\,\partial_3 r \right) \right] \right\}. \qquad (83)$$

We further assume that $u_m(Q)$ is a constant u_m over the small element ΔS, and consider the integral in Eq.(80) to be over a small circle centered at P. We then have the singular integral

$$I_{im} = 8\pi\,(1-\nu) \int_{r<R} \left[\, \partial_3 T_{mi}(P,Q) + \partial_i T_{m3}(P,Q) \,\right] ds_Q$$

$$= \delta_{im} \int_{r<R} \left[\frac{(3+12\delta_{3m})(\partial_3 r)^2 + 3(\partial_m r)^2 - 30(\partial_3 r)^2(\partial_m r)^2}{r^3} \right.$$

$$\left. + (1-2\nu)\,\frac{2 - 3(\partial_3 r)^2 - 3(\partial_m r)^2}{r^3} \right] d^2\mathbf{r}, \qquad (84)$$

with no sum over m. Notice that we take the integral to be on the xy-plane, and the z direction is actually the direction along which we shift the origin. We therefore write

$$\partial_3 r = \frac{z}{r} = \frac{\epsilon}{r}. \qquad (85)$$

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Since I_{im} is zero for $i \neq m$ we are left with

$$I_{mm} = \int_{r<R} \left[\frac{3r_m^2}{r^5} + \frac{3\epsilon^2}{r^5} - \frac{30\epsilon^2 r_m^2}{r^7} + (1-2\nu)\left(\frac{2}{r^3} - \frac{3r_m^2}{r^5} - \frac{3\epsilon^2}{r^5} \right) \right] d^2\mathbf{r}, \quad \text{for } m \neq 3, \quad (86)$$

and

$$I_{33} = \int_{r<R} \left[\frac{18\epsilon^2}{r^5} - \frac{30\epsilon^4}{r^7} + (1-2\nu)\left(\frac{2}{r^3} - \frac{6\epsilon^2}{r^5} \right) \right] d^2\mathbf{r}. \quad (87)$$

When $m \neq 3$, I_{mm} is a linear combination of the integrals

$$J_1 = \int_{r<R} \left(\frac{1}{r^3} - \frac{3r_m^2}{r^5} \right) dS, \quad (88)$$

$$J_2 = \int_{r<R} \left(\frac{1}{r^3} - \frac{3\epsilon^2}{r^5} \right) dS, \quad (89)$$

$$J_3 = \int_{r<R} \left(\frac{1}{r^5} - \frac{5r_m^2}{r^7} \right) dS. \quad (90)$$

Here, J_1, J_2 and J_3 are all ISI's with no singularities. In the above, $J_1 = \pi R^{-1}$ is the simplest ISI with $d = 3, D = 2$, $J_2 = -2\pi R^{-1}$ is an ϵ^2-ISI with $d = 3, D = 2$, and $J_3 = \pi R^{-3}$ is the simplest ISI with $d = 5, D = 2$. In fact, we have

$$I_{mm} = -J_1 + J_2 + 6\epsilon^2 J_3 + (1-2\nu)(J_1 + J_2) = 2(\nu - 2)\pi R^{-1}. \quad (91)$$

Returning to I_{33} we see that it is a linear combination of J_2 and J_4 given by

$$J_4 = \int_{r<R} \left[\frac{1}{r^5} - \frac{5}{3}\frac{\epsilon^2}{r^5} \right] dS. \quad (92)$$

Here $J_4 = -2\pi R^{-3}/3$ is an ϵ^2-ISI with $d = 5, D = 2$. So we have

$$I_{33} = 18\epsilon^2 J_4 + 2(1-2\nu)J_2 = -4(1-2\nu)\pi R^{-1}. \quad (93)$$

In all the above integrals the finite parts are explicitly determined by the ISI method. We thus see again that dimensional continuation provides a unified approach to all hypersingular integrals making it easy to isolate the singularities, which actually cancel, leaving a well-defined finite part.

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V.   CONCLUDING REMARKS

We have used dimensional continuation in the evaluation of integrals of the well known Green's functions and their derivatives. We have identified the rules for obtaining consistent results through the use of such methods for the hypersingular integrals occurring in the BEM, potential theory, electromagnetic scattering, and in crack dynamics. We have provided a systematic approach to the identification of the singularities in typical integrals and shown how to isolate them using the dimensional continuation method. We have identified representations for the Dirac δ-function in D dimensions that are not stated in the standard literature. These results are then used in the calculation of examples of such integrals occurring in physical applications. A summary of the integrable singular integrals is given in tabular form in Appendix A.

It is hoped that the present approach will provide an effective, practical method of evaluating the so-called hypersingular integrals in computational science and engineering applications. Our tabulated ISI will lead to an automated computation of the physical quantities of interest without having to recalculate finite parts of integrals for each specific occurrence.
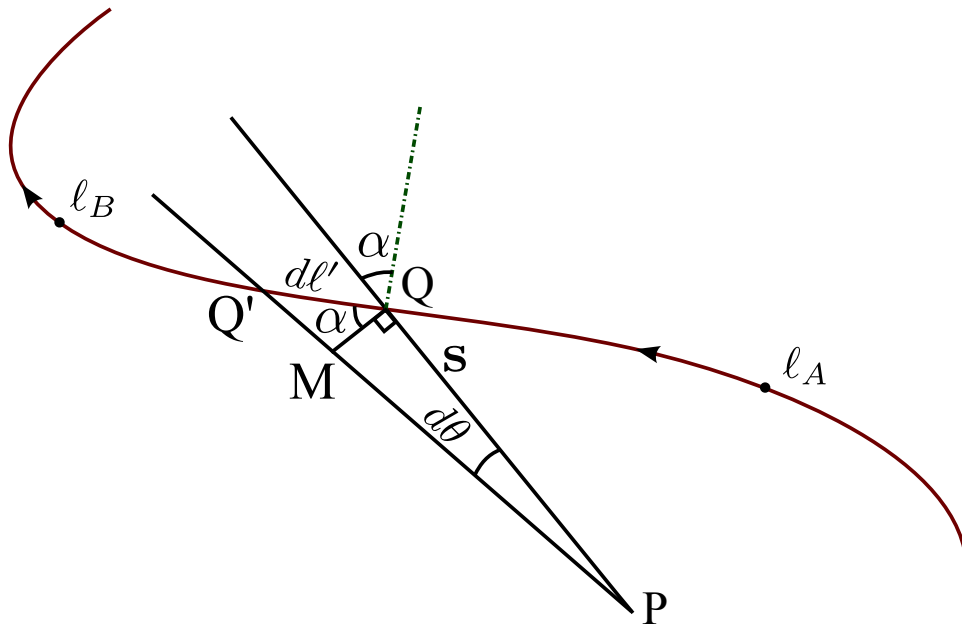
FIG. 1. The contour used to identify the terms in the integrand of the boundary integral approach for evaluating the 2D Poisson potential.
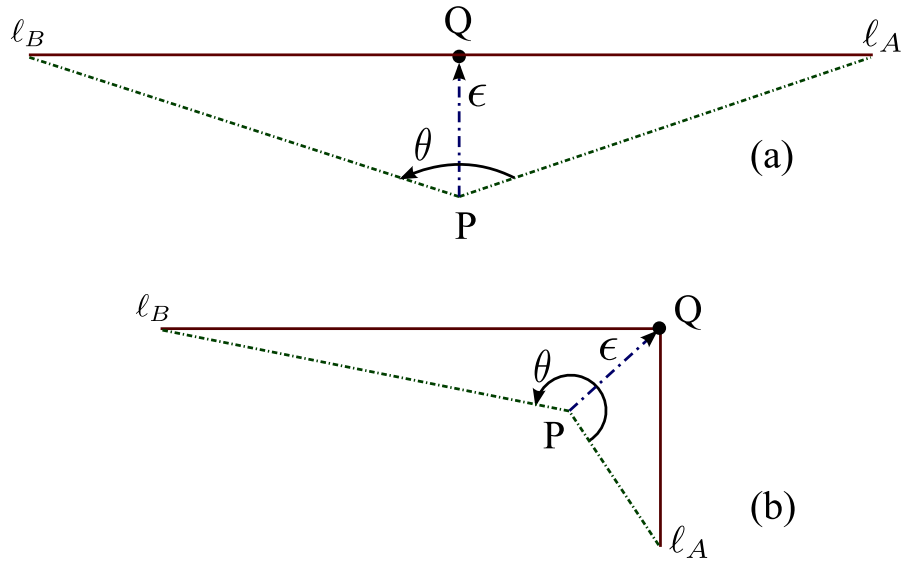
FIG. 2. The geometry used in evaluating the 2D Poisson contour integral in the boundary element method, (a) for a straight contour and (b) for an angular edge.

Appendix A: Tables of Integrable Singular Integrals

1. A Table of the Simplest ISI: $\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{x^2 d}{r^{d+2}}\right)d^D\mathbf{r}=\frac{A_D}{D}\,R^{-(d-D)}$

Integral	D	d	Value
$\displaystyle\int_{r<R}\left(\frac{1}{r}-\frac{x^2}{r^3}\right)dx$	1	1	2
$\displaystyle\int_{r<R}\left(\frac{1}{r^2}-\frac{2x^2}{r^4}\right)dx$	1	2	$2R^{-1}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)dx$	1	3	$2R^{-2}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d\,x^2}{r^{d+2}}\right)dx$	1	d	$2R^{-(d-1)}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^2}-\frac{2x^2}{r^4}\right)dS$	2	2	π
$\displaystyle\int_{r<R}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)dS$	2	3	πR^{-1}
$\displaystyle\int_{r<R}\left(\frac{1}{r^4}-\frac{4x^2}{r^6}\right)dS$	2	4	πR^{-2}
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d\,x^2}{r^{d+2}}\right)dS$	2	d	$\pi R^{-(d-2)}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)dV$	3	3	$4\pi/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^4}-\frac{4x^2}{r^6}\right)dV$	3	4	$4\pi R^{-1}/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^5}-\frac{5x^2}{r^7}\right)dV$	3	5	$4\pi R^{-2}/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d\,x^2}{r^{d+2}}\right)dV$	3	d	$4\pi R^{-(d-3)}/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d\,x^2}{r^{d+2}}\right)d^4\mathbf{r}$	4	d	$\pi^2 R^{-(d-4)}/2$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d\,x^2}{r^{d+2}}\right)d^5\mathbf{r}$	5	d	$8\pi^2 R^{-(d-5)}/15$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d\,x^2}{r^{d+2}}\right)d^D\mathbf{r}$	D	d	$A_D R^{-(d-D)}/D$

TABLE I. A table of Integrable Singular Integrals for spatial dimension $D=\{1,\ldots 5\}$, with singular denominators r^{-d}, with $d=\{1,\ldots 5,d\}$.

2. A Table of the Integrals: ϵ^2-ISI $\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-D}\frac{\epsilon^2}{r^{d+2}}\right)d^D\mathbf{r} = -\frac{A_D}{d-D}R^{-(d-D)}$

Integral	D	d	Value
$\displaystyle\int_{r<R}\left(\frac{1}{r^2}-2\frac{\epsilon^2}{r^4}\right)dx$	1	2	$-2R^{-1}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^3}-\frac{3}{2}\frac{\epsilon^2}{r^5}\right)dx$	1	3	$-2R^{-2}/2$
$\displaystyle\int_{r<R}\left(\frac{1}{r^4}-\frac{4}{3}\frac{\epsilon^2}{r^6}\right)dx$	1	4	$-2R^{-3}/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-1}\frac{\epsilon^2}{r^{d+2}}\right)dx$	1	d	$-2R^{-(d-1)}/(d-1)$
$\displaystyle\int_{r<R}\left(\frac{1}{r^3}-3\frac{\epsilon^2}{r^5}\right)dS$	2	3	$-2\pi R^{-1}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^4}-2\frac{\epsilon^2}{r^6}\right)dS$	2	4	$-\pi R^{-2}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^5}-\frac{5}{3}\frac{\epsilon^2}{r^7}\right)dS$	2	5	$-2\pi R^{-3}/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-2}\frac{\epsilon^2}{r^{d+2}}\right)dS$	2	d	$-2\pi R^{-(d-2)}/(d-2)$
$\displaystyle\int_{r<R}\left(\frac{1}{r^4}-4\frac{\epsilon^2}{r^6}\right)dV$	3	4	$-4\pi R^{-1}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^5}-\frac{5}{2}\frac{\epsilon^2}{r^7}\right)dV$	3	5	$-2\pi R^{-2}$
$\displaystyle\int_{r<R}\left(\frac{1}{r^6}-2\frac{\epsilon^2}{r^8}\right)dV$	3	6	$-4\pi R^{-3}/3$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-3}\frac{\epsilon^2}{r^{d+2}}\right)dV$	3	d	$-4\pi R^{-(d-3)}/(d-3)$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-4}\frac{\epsilon^2}{r^{d+2}}\right)d^4\mathbf{r}$	4	d	$-2\pi^2 R^{-(d-4)}/(d-4)$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-5}\frac{\epsilon^2}{r^{d+2}}\right)d^5\mathbf{r}$	5	d	$-8\pi^2 R^{-(d-5)}/3(d-5)$
$\displaystyle\int_{r<R}\left(\frac{1}{r^d}-\frac{d}{d-D}\frac{\epsilon^2}{r^{d+2}}\right)d^D\mathbf{r}$	D	d	$-A_D R^{-(d-D)}/(d-D)$

TABLE II. A table of ϵ^2-Integrable Singular Integrals for spatial dimension $D = \{1,\ldots 5\}$, with singular denominators r^{-d}, with $d = \{1,\ldots 5, d\}$.

3. **The Dirac δ-function in ISI:** $\delta^{(D)}(\mathbf{r}) = \dfrac{D}{A_D} \lim\limits_{\rho \to r^+} \left(\dfrac{1}{\rho^D} - \dfrac{r^2}{\rho^{D+2}} \right), \ (\rho^2 = r^2 + \epsilon^2)$

Dimension	δ-function
1	$\delta^{(1)}(\mathbf{r}) = \dfrac{1}{2} \lim\limits_{\rho \to r^+} \left(\dfrac{1}{\rho} - \dfrac{r^2}{\rho^3} \right)$
2	$\delta^{(2)}(\mathbf{r}) = \dfrac{1}{\pi} \lim\limits_{\rho \to r^+} \left(\dfrac{1}{\rho^2} - \dfrac{r^2}{\rho^4} \right)$
3	$\delta^{(3)}(\mathbf{r}) = \dfrac{3}{4\pi} \lim\limits_{\rho \to r^+} \left(\dfrac{1}{\rho^3} - \dfrac{r^2}{\rho^5} \right)$
4	$\delta^{(4)}(\mathbf{r}) = \dfrac{2}{\pi^2} \lim\limits_{\rho \to r^+} \left(\dfrac{1}{\rho^4} - \dfrac{r^2}{\rho^6} \right)$

TABLE III. A table of Dirac δ-functions in spatial dimension $D = \{1, \ldots 4\}$ arising in integrable singular integrals.

REFERENCES

[1] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[2] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vols. I & II (Academic Press, New York, 1964); M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1968); H. Bremermann, *Distributions, Complex Variables, and Fourier Transforms* (Addison-Wesley, Reading, MA, 1965).

[3] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley-Interscience, NY, 1959); p177.

[4] P. A. Martin and F. J. Rizzo *On Boundary Integral Equations for Crack Problems*, Proceedings of the Royal Society A **421**, 341–355 (1989).

[5] W. L. Wendland and E. P. Stephan *A Hypersingular Boundary Integral Method for Two-Dimensional Screen and Crack Problems*, Archive for Rational Mechanics and Analysis **112** 363–390 (1990).

[6] Q. Huang and T. A. Cruse *Some Notes on Singular Integral Techniques in Boundary Elements Analysis*, International Journal for Numerical Methods in Engineering **36**, 2643–2659 (1993).

[7] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1975).

[8] L. R. Ram-Mohan, *Finite Element and Boundary Element Applications in Quantum Mechanics* (Oxford U.P., Oxford, UK, 2002); Chs. 16-18 and 408.

[9] M. S. Tong and W. C. Chew, *Super-Hyper Singularity Treatment for Solving 3D Electric Field Integral Equations*, Microwave and Optical Technology Letters **49**, 1383–1388 (2007).

[10] The nomenclature, integrable singular integral, is in analogy with integrable end-point singularities. For example, in $\int_0^R \ln(x)dx$ the argument has a singularity at $x = 0$, but is integrable.

[11] As mentioned earlier, only in quantum field theory do we encounter a need for renormalization of physical parameters such as mass, charge or coupling constants entering the calculations. This is not the issue in engineering applications and in potential theory.

[12] The results here may be compared with those given in Ref. 2;Vol. I, p 29.

[13] P. M. Morse and H. Feshbach *Methods of Theoretical Physics* Part I & II (McGraw-Hill, 1953); p1770 and p1374.

[14] J. Jin *The Finite Element Method in Electromagnetics* (Wiley, New York, 1993); p314–314.

[15] J. L. Volakis, A. Chatterjee, and L. C. Kempel *Finite Element Method for Electromagnetics* (IEEE Press, New York, 1998); p10–11.

[16] F. J. Rizzo *An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics*, Quarterly of Applied Mathematics **25**, 83–95 (1967).

[17] L. J. Gray, L. F. Martha, and A. R. Ingraffea *Hypersingular Integral in Boundary Element Fracture Analysis*, International Journal for Numerical Methods in Engineering **29**, 1135–1158 (1990).

[18] T. L. Anderson *Fracture Mechanics, Fundamentals and Applications*, (CRC Press, Texas, 1995); p570.