Exploring the Interplay of Euler, the Dodecahedron and Graph Theory

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Introduction

As with all other fields of study, mathematics has evolved over time. From the legends of old such as Pythagorus to the modern day masters such as Terrence Tao, many mathematicians leave their mark on a wide range of topics in their field. Yet as each before them, their work stands on the shoulders of giants; what each new genius discovers and creates must to some degree be built upon the foundation of what was previously discovered and created. These are the connections that bind, reinforce and then accelerate the evolution of the field.

Such is the case with Leonard Euler. Even those who consider themselves novices in the field of math know this man's name because of his profound influence. His interest in Plato's platonic solids helped move forward geometry and topology. This has led to many seeing what they can extrapolate out of these shapes, including myself. Euler's Seven Bridges of Königsberg problem sparked the interest in the topic of graph theory. Many years later, German mathematician, Victor Schlegel, introduced a model that could make the study of graph theory on 3-dimensional shapes immensely easier - including the platonic solids. Each concept, mathematician and breakthrough is connected in some way.

With this understanding, this paper will explore these three facets of mathematics, connected as all facets are. It will show the influence of Euler, the intricacy of the dodecahedron and the application of graph theory.

Section 1 - Leonard Euler and Platonic Solids

A Brief Biography of Leonhard Euler

Leonhard Euler was a Swiss mathematician, physicist and engineer. He was born on April 15, 1707 in Basel, Switzerland, and grew up in a family of scholars and theologians. He showed an early aptitude for mathematics and was admitted to the University of Basel at the age of 13, where he studied under the renowned mathematician, Johann Bernoulli.

He was a prolific mathematician, engineer, and physicist, who published over 850 papers and books through the course of his life. He made significant advances in various fields such as calculus, number theory and mechanics. For example, Euler developed the concept of a and many of the algebraic symbols used for it. Many notations and concepts in use today were also introduced by Euler, including the trigonometric functions and the letter *e* that is used for the base of the natural logarithm. In the field of number theory, Euler contributed to studies in prime numbers and managed to prove several important theorems, including the infinitude of primes as well as the law of quadratic reciprocity. Pertinently for this paper, he also made contributions to the study of graph theory and topology. In 1727, Euler became the physics professor at the St. Petersburg Academy of Sciences. He tackled a range of scientific and math issues over the next quarter-century. His pair of famous works, "Introductio in analysin infinitorum" (1735) and "Mechanica" (1736), revolutionized the study of calculus and classical mechanics respectively.

Euler's contributions were monumental, which led many to label him as one of history's greatest mathematicians. Not only did he belong to notable schools, he also received numerous awards and commendations from learned societies commemorating his achievements.

<u>A Brief History of Platonic Solids</u>

The Platonic Solids are a set of geometric shapes named after the ancient Greek philosopher, Plato, who wrote about them in his philosophical dialogues. By definition, these solids are regular polyhedral, or a convex 3-dimensional shape with all of its faces congruent regular polygons, and with the same number of faces at each vertex. There exist only five total Platonic Solids, including the Tetrahedron, Octahedron, Cube, Icosahedron, and Dodecahedron.

Plato believed that these shapes were the building blocks of the universe and that they represented the ‘five’ elements: earth, air, fire, water, and ether. He believed that each element was associated with a different Platonic solid: earth with the cube, air with the octahedron, fire with the tetrahedron, water with the icosahedron, and ether with the dodecahedron.

Mathematicians and philosophers scrutinized the Platonic solids across the centuries. For example, in the 16th century the German mathematician, Johannes Kepler, endeavored to comprehend the orbits of the planets in the solar system with the help of Platonic solids. He held a conviction that the solar system's configuration followed the Platonic solids' proportions, with the sun positioned at the center. He presented his novel conceptual framework and understanding in a work titled, "Mysterium Cosmographicum," that depicted his model. Using the model as evidence, he proved Nicholas Copernicus’ heliocentric theory. This not only marked a huge stride for modern astronomy but also laid the foundation for Isaac Newton’s theory of gravity with Kepler’s planetary motion laws. Though Mysterium Cosmographicum's insights were not ultimately upheld in contemporary astronomy, the model still garners respect for its striking beauty and innovative design.

One of the most interesting modern applications of Platonic solids is in the field of chemistry. Scientists have found that many molecules, including some of the most complex ones, can be modeled using Platonic solids. For example, the structure of the carbon molecule, which is the basis for all life on earth, can be represented using a tetrahedron. This allows scientists to better understand the properties of these molecules and how they interact with each other.

Platonic solids are also used to model the structures of crystals, which are solid materials made up of repeating patterns of atoms or molecules. By understanding the structures of these crystals, scientists can better understand the properties of the materials they are made of and this knowledge can in turn be used to develop new materials with useful properties.

Platonic solids are also influential in contemporary chemistry research, thanks to their versatility in modeling molecules. Even exceptionally intricate structures can be represented by these ancient shapes. For instance, carbon—the foundation of all earthly organisms—takes the form of a tetrahedron. This enables scientists to gain a deeper comprehension of various molecular properties and their interactivity. Solid materials, consisting of atoms or molecules arranged in repetitive patterns, are modeled using Platonic solids. Through studying these crystal structures, scientists gain a better understanding of the characteristics of the materials, enabling the development of novel materials with valuable properties.

Platonic solids have been studied for thousands of years and they continue to be a source of fascination for philosophers, mathematicians and scientists alike. Their symmetrical beauty and the way they can be used to model the structures of the universe make them a unique, fascinating and enduring subject of study.

Euler's Characteristic

Leonard Euler has, perhaps, the most mathematical concepts named after him, many of which are pivotal to their respective topics. One of his namesake equations will prove to be very helpful as we dive into the dodecahedron. Euler's Characteristic, also known as Euler's Polyhedral Formula, is a fundamental equation of geometry and topology. It is represented by the following equation:

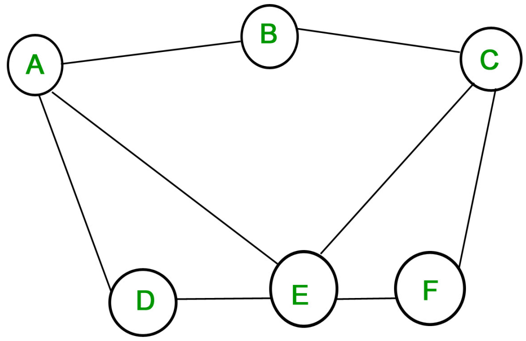
$$V - E + F = 2$$

Where for a given shape, V is the number of vertices on that shape, E is the number of edges on that shape, and F is the number of faces on that shape.

Many years prior to Euler, this formula had been stated in different ways by brilliant mathematicians such as Maurolico and Descartes. However, Euler rediscovered this equation and published it, which resulted in the concept being named for him. Euler's Characteristic is applicable to any polyhedron: a 3-dimensional shape with flat faces and edges. It has since been proven in a multitude of ways, including the following using induction on the edges.

Consider a graph G. Consider the base case when the graph has no edges. It can at most have a singular vertex and face. So $V - E + F = 1 - 0 + 1 = 2$. Now we can set our induction hypothesis. Any edge on graph G connects two vertices. Imagine contracting any such edge by removing the edge and connecting the two vertices it connected into one. This removes one E and V. If an edge still exists after combining the two edges, it must be connected to itself, so it can be removed, in this case reducing E and F by 1. In any case, $V - E + F = 2$ holds by induction for all polygons.

We can also prove this formula through planar graphs. A more rigorous description of planar graphs will be explained in section 3 of this paper. What is important to know now is that any polyhedron (3D shape) can be represented as a 2D simple graph. Therefore, if we can prove Euler's Characteristic works for simple graphs, it also works for all polyhedra. For clarity, the proof will be explained specifically for the graph below, but used to generalize for all scenarios.



Consider this simple graph. This contains 2 k_3 faces, 1 k_4 face, and 1 k_6 face. The subscript of k indicates how many sides are contained in each face. Notice how these k graphs add up to the total amount of faces on this graph - 4. It is also important to note that space outside of the graph is considered a face. To generalize this, we can say the following.

$$k_3 + k_4 + k_5 + k_6 + \ \dots \ = f \qquad \rightarrow \qquad 2 + 1 + 1 = f$$

We can also define the amount of edges using this notation. There will be the exact number of edges per face as there are sides to each face. However, they will each be counted twice. So we can say the following.

$$3k_3 + 4k_4 + 5k_5 + 6k_6 + \ \dots \ = 2e \qquad \rightarrow \qquad 6 + 4 + 6 = 2e$$

Last, we can define the vertices through the angles they create around the graph. Around each vertex, there exists 2π (a circle) worth of angle. So, we can multiply this number by the total amount of vertices.

$$\text{Sum of angles of vertices} = A = 2\pi v \qquad \rightarrow \qquad A = 2\pi 6 = 2\pi v$$

Now, we can use two helpful equations to combine all of these together: the sum of the interior angles of a polygon, $(n - 2)\pi$, and the sum of the exterior angles of a polygon, $(n + 2)\pi$. We can use the following equation to incorporate our graph.

$$\{k_3(3 - 2)\pi + k_4(4 - 2)\pi + k_5(5 - 2)\pi + \ldots\} + k_x(x + 2)\pi = k_x(x + 2)\pi + A$$

The $k_x(x + 2)\pi$ value represents the exterior angles of the graph, which we do not necessarily know. We can now manipulate this equation and plug in the values of $f,\ v,\ e$.

$$\{3k_3\pi + 4k_4\pi + 5k_5\pi + \ldots\} - k_3 2\pi - k_4 2\pi - k_5 2\pi + \pi - 2\pi = x\pi - 2\pi + 2v\pi$$

$$\pi(\{3k_3 + 4k_4 + 5k_5 + \ldots\} - k_3 2 - k_4 2 - k_5 2 + 2) = \pi(-2 + 2v)$$

$$\{3k_3 + 4k_4 + 5k_5 + \ldots\} - k_3 2 - k_4 2 - k_5 2 + 2 = -2 + 2v$$

$$2e - 2f + 2 = -2 + 2v$$

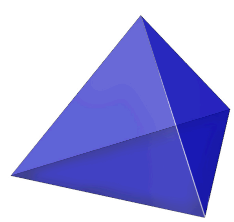
$$f + v - e = 2$$

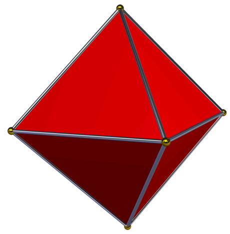
Thus, Euler's Characteristic applies to all simple graphs, and therefore all polyhedra.

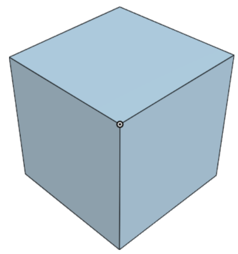
<u>The Platonic Solids</u>

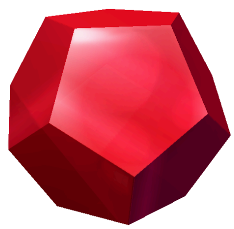
As mentioned earlier, there are five regular polyhedra that exist in geometry known as the platonic solids. Here they are with their corresponding models:

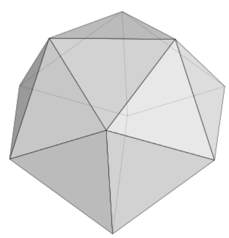
Tetrahedron:



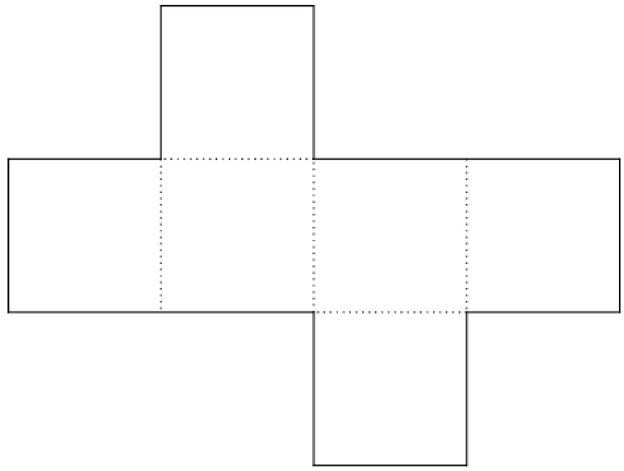
Octahedron: 

Cube: 

Dodecahedron: 

Icosahedron: 

In 2D, there is an unlimited amount of regular polygons because another side can always be added. In 3D, however, there are more constraints which only allow for the existence of these five shapes. The easiest way to explain this phenomenon is through interior angles. When folding up the 2D shapes to create the polyhedra, the interior angles cannot add up to 360° or more. Consider the cube for example.



This is a net of the cube. A net is a 2D pattern that can be folded and attached into a 3D shape. In this picture, the dotted lines represent where a fold would occur. Notice how at any given corner, or vertex, of the net there are only a maximum of three squares connected to it. Since we know the angle of a square is 90°, the maximum interior angle of the cube is 270°. Now imagine that a fourth square was placed in the top left corner of the net. Intuitively, it would then become impossible to fold the pattern without any overlap and this is because the interior angles here would now add up to 360°.

This phenomenon can also be explained with Euler's Characteristic. It is helpful to think about the faces of the polyhedra. We can set the number of sides on each face to be a, and we can set the number of faces that meet at any vertex b. We can determine the total number of face sides. Consider the fact that when two faces meet in a polyhedron, it creates one edge. If we expand this thinking to the whole shape, we can say that $aF = 2E$. This thinking also applies to the vertices. When two or more vertices meet, they combine to become one vertex. So we can say $bV = 2E$. Now we can combine these equations.

$F = 2E/a, \quad V = 2E/b \quad \rightarrow$ Now put these back into Euler's Characteristic.

$V - E + F = 2 \quad \rightarrow \quad 2E/b - E + 2E/a = 2$

Rearrange and divide each element by 2E.

$$1/b - 1/2 + 1/a = 1/E$$

E cannot be less than 0, so we can replace the 1/E element.

$$1/b - 1/2 + 1/a > 0 \quad \rightarrow \quad 1/b + 1/a > 1/2$$

Now we have found an equation that determines if a shape follows Euler's Characteristic based upon how many sides each face has (a) and the number of faces that meet at any vertex (b). Both of these values have to be at least 3. Observe the following table to see how this unfolds.

a	***b***	***1/b + 1/a > 0.5***	**Shape**
3	3	0.666… ✓	Tetrahedron
3	4	0.583… ✓	Octahedron
3	5	0.533… ✓	Icosahedron
3	6	0.5 X	N/A
4	3	0.583… ✓	Cube
4	4	0.5 X	N/A
5	3	0.533… ✓	Dodecahedron
5	4	0.45 X	N/A

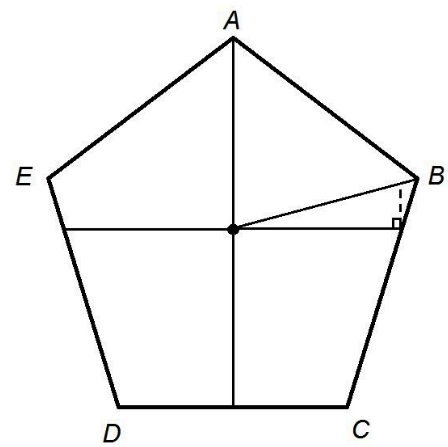
As you can see, as a and b increase past these values, it will never be less than 0.5, showing there only exist the five platonic solids. One of these, of course, is the dodecahedron which is the next focus of this paper. This shape is unique among the platonic solids, being that it is the only shape constructed with pentagons. We can begin by analyzing the pentagon before we move on to the dodecahedron.

Section 2 - The Dodecahedron

Analyzing the Pentagon

Finding the exact points of the vertices on the unit pentagon

Assume the origin of the unit pentagon is (0,0). This means we already know the exact coordinates of A, (0,1). From here, see that points E / B and points D / C are related. Each set of points are equidistant from y = 0, and also share the same y value. So we can label each of the remaining points as so:



$E = (m_1, -n_1)$

$B = (m_1, n_1)$

$D = (-m_2, -n_2)$

$C = (m_2, -n_2)$

Now we can find m_1, n_1, m_2, n_2. First, determining what the coordinates of B are. We can use trigonometry to find that B sits on (cos(18°), sin(18°)), and C sits on (cos(-54°), sin(-54°)). So first find the exact location of B. First, find sin(Θ) where Θ = 18°.

$5(\theta) = 5(18) = 90$

$2\theta = 90 - 3\theta$

$\sin(\theta) = \sin(90 - 3(\theta))$

$\sin(\theta) = \cos(3\theta)$

$2\sin(\theta) * \cos(\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ $\quad\rightarrow\quad$ from $\cos^3(\theta) = 4\cos^3(\theta) - 3\cos(\theta)$

$4\cos^3(\theta) - 3\cos(\theta) - 2\sin(\theta) * \cos(\theta) = 0$

$\cos(\theta) * (4\cos^2(\theta) - 3 - 2\sin(\theta)) = 0$

$(4\cos^2(\theta) - 3 - 2\sin(\theta)) = 0$

$4 - 4\sin^2(\theta) - 2\sin\theta) - 3 = 0$ → from $\cos^2(\theta) + \sin^2(\theta) = 1$

$\sin(\theta) = -2 \pm \sqrt{}[(-22) - 4(4)(1)]/(2 + 4)$ → Apply the quadratic formula

$\sin(\theta) = 1/4 * \sqrt{5} - 1$

Now find find $\cos(\Theta)$ where $\Theta = 18°$.

We know $\cos2(\theta) + \sin2(\theta) = 1$

$\cos(\theta) = \sqrt{1} - \sin2(\theta)$ → plug in $\sin(\theta)$

$\cos(\theta) = 1/4 * \sqrt{}[10 + 2\sqrt{5}]$

So we can say the exact coordinates of B and E are

$B = (1/4 * \sqrt{5} - 1,\ 1/4 * \sqrt{}[10 + 2\sqrt{5}])$, and

$E = (-\{1/4 * \sqrt{5} - 1\},\ 1/4 * \sqrt{}[10 + 2\sqrt{5}])$

We can use a similar process to determine that the coordinates of C and D are

$C = (1/4 * \sqrt{5} + 1,\ -\{1/4 * \sqrt{}[10 - 2\sqrt{5}]\})$

$D = (-\{1/4 * \sqrt{5} + 1\},\ -\{1/4 * \sqrt{}[10 - 2\sqrt{5}]\})$.

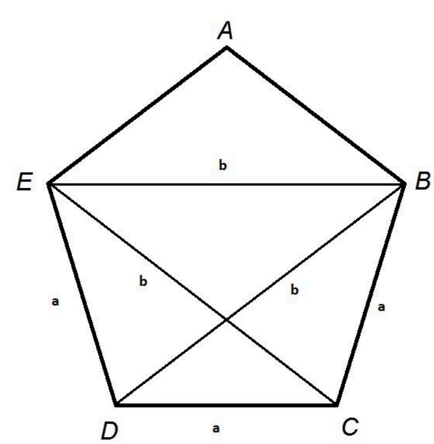
The pentagon's relationship to the golden ratio

Observe the pentagon on the right and notice how each side of the pentagon is the same length a, and that the distances between points across the face of the pentagon are the same length b. This creates a quadrilateral across the bottom half of the shape. This means we can use Ptolemy's theorem, which states that the product of the diagonals in a quadrilateral always equals the sum of the products of the opposite sides. Knowing this, we can say that $ab + a * a = b * b$. So,

$b^2 - ab - a^2 = 0$

$(b^2/a^2) - b/a - 1 = 0$

Set the ratio of b/a to x and apply the quadratic formula.



$$x^2 - x - 1 = 0$$

$$x = [1 \pm \sqrt{(1^2 + 4 * 1)}] / 2$$

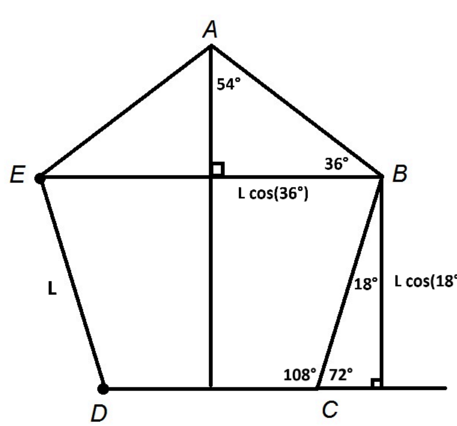
$$x = [1 \pm \sqrt{5}] / 2 = \varphi$$

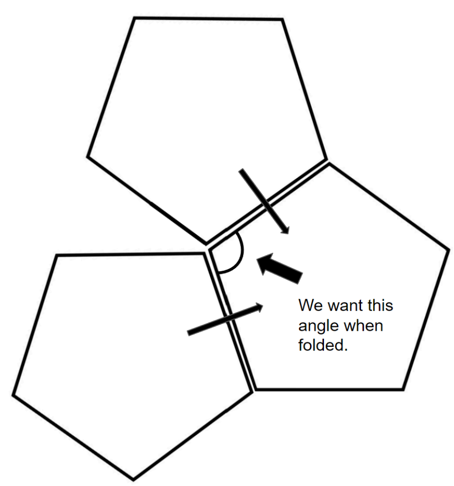
Math lovers will quickly realize that this value is the golden ratio. Dividing any side by any diagonal will result in this value. This directly ties this famous ratio to the pentagon. This will be useful information as we now dive into the dodecahedron.

Analyzing the Dodecahedron

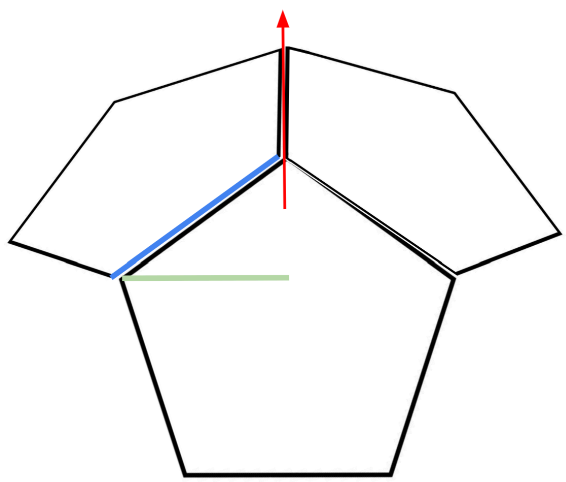
Finding the dihedral angle of the dodecahedron

To begin finding the dihedral angle of the dodecahedron, we can first begin looking at what it is made up of. Observe the pentagon to the left. Specific angles and lengths have been laid out that will be useful later, specifically $Lcos(18°)$, $Lcos(36°)$. Notice that the length of any edge of the pentagon is L, as the specific length is irrelevant as long as the shape is regular. On the right is a partial net of a dodecahedron. This illustrates the angle we are looking for in two dimensions. The two thinner arrows indicated how the outer pentagons will fold in.





It is important to know that while the length of the edges of the pentagons do not change physically, they do change length based upon perspective. For a more clearer indication of this, observe the next image of the three pentagons folded up. The red arrow represents our line of intersection, which contains the set of the points where the two faces meet on the plane. The green line represents our $Lcos(36°)$. In this perspective of the shape, the blue line represents our $Lcos(18°)$.



Now consider the triangle that the three colored lines create across the face of the front pentagon. We know the lengths of two of the sides, so we can find the angle that is created between the blue line and the line of intersection. Using trigonometry, we can say this angle equates to the following.

$sin\Theta \ = \ Lcos(36°)/\ Lcos(18°)$ → Length L cancels out.

$sin\Theta \ = \ cos(36°)/\ cos(18°)$

$\Theta \ = arcsin(cos(36°)/\ cos(18°)) \ = \ 58.28252559$

This value of Θ represents half of the angle, so multiply this by 2 to determine the dihedral angle.

$\Theta \ = 2arcsin(cos(36°)/\ cos(18°)) \ = \ \sim 116.57°$

The dodecahedron's relationship to the golden ratio

The golden ratio, often denoted by the Greek letter phi (Φ), is a mathematical constant approximately equal to 1.618, or $(1 + \sqrt{5})/2$. This can be expressed mathematically as $(a + b)/a = a/b = \Phi$. It appears in nature, art, and design, which makes it such a unique mathematical property. This ratio was derived from the Fibonacci Sequence, the series of numbers where each term is the sum of the two preceding ones. The golden ratio is often used in art and architecture for its perceived ability to create harmony and balance.

Interestingly, mathematicians have found that there are multiple ways the golden ratio is incorporated into the structure of the dodecahedron. It appears in many mundane, but also unusual aspects of the shape. We can begin with the surface area and volume of the shape. It will be easier to derive these answers by using the pentagon as a baseline. The surface area of a pentagon follows this equation.

$A = 1/4\sqrt{[5(5 + 2\sqrt{5})]}a^2$ Where a is the length of any edge.

Given that we know the dodecahedron consists of 12 pentagons, we can say that the surface area of the dodecahedron is $12A$. Thus, the surface area of a dodecahedron is the following.

$A = 3\sqrt{[25 + 10\sqrt{5}]}a^2$ Where a is the length of any edge.

We can set $a = 1$ and manipulate this equation further with the golden ratio in mind. First we can begin with what is under the square root, as this part most closely resembles Φ . We can divide this by the golden ratio.

$25 + 10\sqrt{5}$

$(25 + 10\sqrt{5})/[(1 + \sqrt{5})/2]$

$2(25 + 10\sqrt{5})/(1 + \sqrt{5})$ → Factor out the conjugate.

$2(1 - \sqrt{5})(25 + 10\sqrt{5})/ - 4$

$(25 + 15\sqrt{5})/2$ → Put back into the equation.

$A = 3\sqrt{[\Phi(25 + 15\sqrt{5})/2]}$

Naturally, this phenomenon also happens when considering the volume of the dodecahedron. For simplicity, this equation describes the volume of a dodecahedron.

$V = 1/4(15 + 7\sqrt{5}))a^3$ Where a is the length of any edge.

We can do a similar process of manipulating the equation with the golden ratio in mind. We can again set $a = 1$. We can again look at the term inside the parentheses as it most closely resembles Φ.

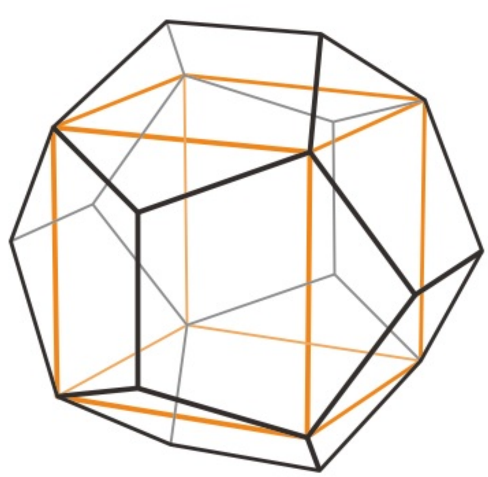
$2(15 + 7\sqrt{5})/[1 + \sqrt{5}]$ → Factor out the conjugate.

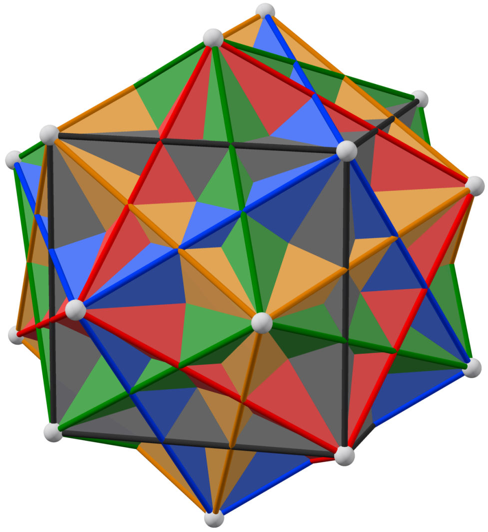
$(2 - 2\sqrt{5})(15 + 7\sqrt{5})/ - 4$

$10 + 4\sqrt{5}$ → Put back into the equation

$V = \Phi(10 + 4\sqrt{5})/4$

One of the more intriguing forms it appears is when observing an embedded cube inside the dodecahedron. Observe the graphics below which follows this description. The first image shows a single cube inside while the second image shows how 5 different cubes can fit along the vertices of the dodecahedron.





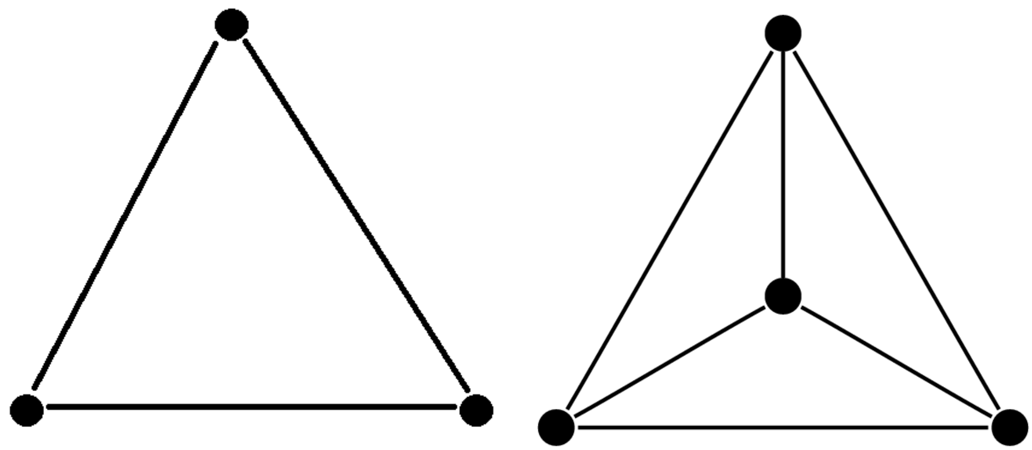
https://en.wikipedia.org/wiki/Compound_of_five_cubes#/media/File:Compound_of_five_cubes,_perspective.png

These cubes reveal information about the dodecahedron. As we learned from the pentagon, the ratio of any diagonal to any side is the golden ratio. These cubes follow along the diagonals of the 12 pentagons that make up the dodecahedron. So if we decide to set each edge length to 1, then we can immediately infer that the length of any edge of any cube embedded inside the dodecahedron is Φ.

Section 3 - Graph Theory and the Dodecahedron

A Brief Explanation of Graphs

Graph Theory is a mathematical field centered around the study of graphs. Graphs, often portrayed as planar graphs (or Schlegel Diagrams), are mathematical structures that represent polygons in 2D space. They are made up of a set of vertices and edges, arranged in a specific way depending on what they represent. This concept is straightforward when considering 2D shapes, but becomes much more applicable and useful when analyzing a 3D object in 2D space. To show this distinction, compare the graphs of a triangle (left) and a tetrahedron (right).



Even though these shapes consist of different dimensions, they are both able to be represented here on the 2D plane. A triangle has 3 vertices and 3 edges, and a tetrahedron has 4 vertices and 6 edges, both of which are represented in their respective diagrams. These are accurate descriptions of both shapes, especially since these graphs follow Euler's Characteristic.

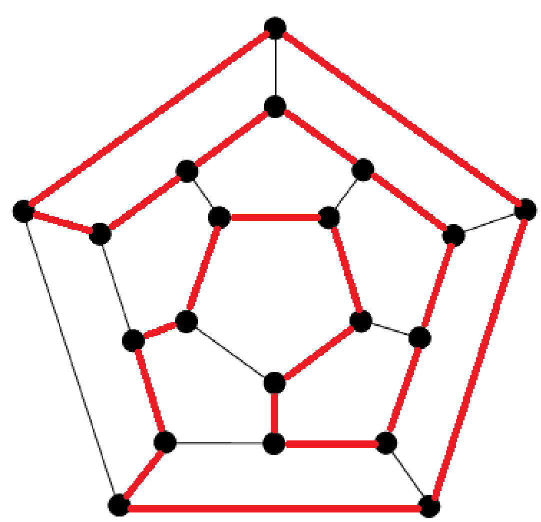
Hamiltonian Paths/Cycles

In graph theory, a Hamiltonian path is a path along a graph that visits each vertex exactly once. This concept was formulated by and named after the Irish mathematician, Sir William Rowan Hamilton. In 1857, he devised a puzzle named the Icosian Game, in which the object of the game was to create what is now known as a Hamiltonian cycle (a Hamiltonian path that ends where the path started) along the vertices of the dodecahedron.

A similar concept called the Eulerian trail was formulated by Leonard Euler in 1763. A Eulerian trail is a path along a graph that visits each edge exactly once. Euler, just as Hamilton did 94 years later, devised a puzzle named the Seven Bridges of Königsberg, where the goal of the puzzle was to devise a path that would cross each of the Prussian city's seven bridges exactly once. Despite its obvious similarities to the Hamiltonian path, the dodecahedron graph does not contain any Eulerian trails. We can prove both statements to be true. First, we will show that the graph of the dodecahedron is Hamiltonian (has a Hamiltonian path/cycle).

One way we can prove this is with Tait's theorem which states that a 3-regular graph (a graph in which every vertex has degree 3) is Hamiltonian if and only if its dual graph is planar. We know that every vertex in the graph of the dodecahedron has a degree of 3, as there are 3 edges that stem from each one, so we can use this theorem. A dual graph is created by setting every face of the graph as a vertex, and then connecting each vertex with an edge wherever two faces share an edge. Incidentally, the dual graph of the dodecahedron is the graph of the icosahedron. Now we just have to show the icosahedron graph is planar; a graph that cannot be disconnected by removing two vertices. Since the icosahedron is a platonic solid, we know that it is planar, as all platonic solids are. Therefore, the graph of the dodecahedron is Hamiltonian.

We can also show the graph is Hamiltonian by providing an example. Observe the following Hamiltonian Cycle. The path reaches every single vertex once while indeed creating a cycle.

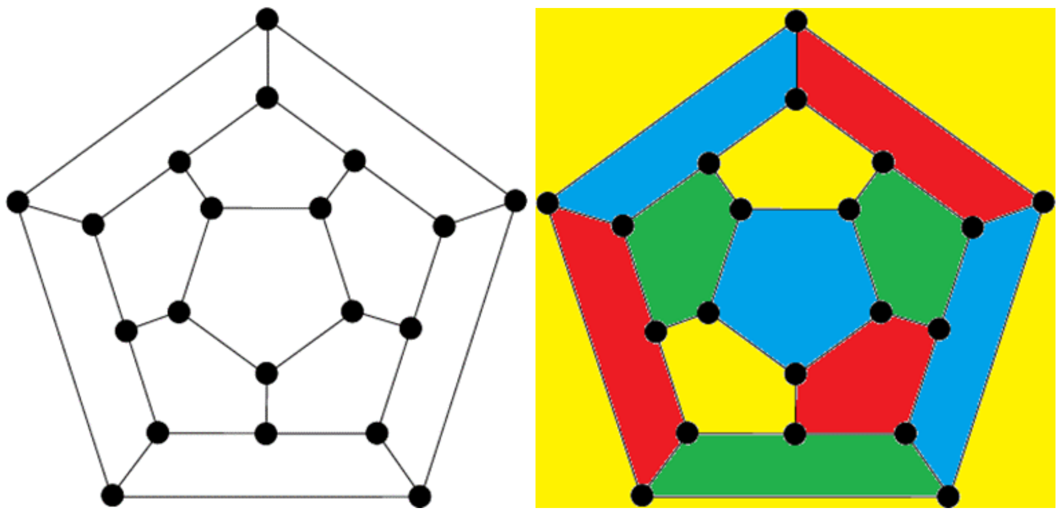


There is one very easy way to show that the graph of the dodecahedron is not Eulerian (Eulerian trail/circuit). Consider Euler's Graph Theorem, which states that a connected graph has an Eulerian cycle if and only if every vertex has even degree. A connected graph has a path between every pair of vertices. We know that this is true as every platonic solid has a connected graph. However, we know that every vertex on the graph of the dodecahedron has a degree of 3, an odd number. This contradiction shows that the dodecahedron is not Eulerian.

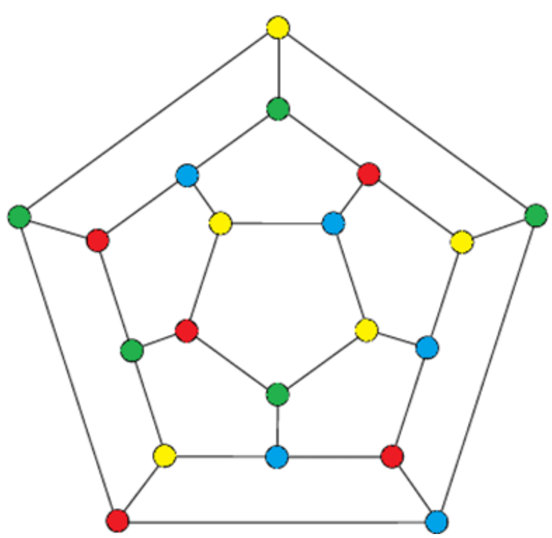
Graph Coloring and Coloring the Dodecahedron

In graph theory, the concept of graph coloring refers to the assignment of colors to either the faces, edges or vertices of a graph. Specifically, no two colors can be next to each other. Usually, the goal of graph coloring is to see what the fewest number of colors is to completely fill the graph while following this rule. The history of this problem came primarily from cartography in the 1800s. Cartographers wanted to color in their maps in a way where no two adjacent countries, counties, etc. share the same color. By simplifying the maps of the world to simple planar graphs, mathematicians began tackling this problem. In 1890, Percy John Heawood developed and proved the five-color theorem. This theorem proved that every planar graph could be colored with no more than 5 colors. This advancement led mathematicians to see if they could lower this number, which proved to be a challenging task. After more than 80 years, Kenneth Appel and Wolfgang Haken were able to prove the four-color theorem for planar graphs. This proof was not only significant for the field of graph theory, but also was one of the first major mathematical proofs that was computer aided.

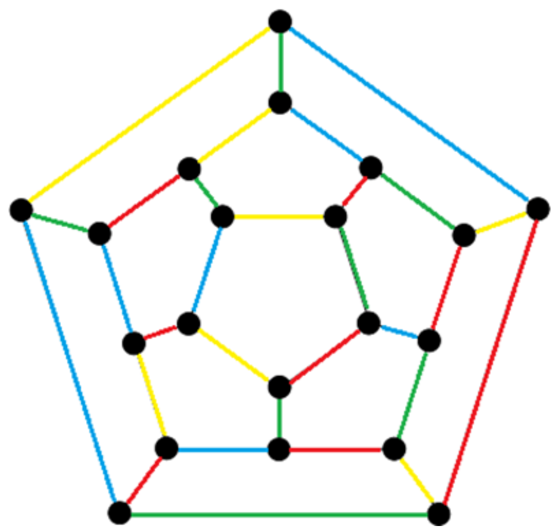
Given that we know that all planar graphs can be colored with at most 4 colors, we can apply this to the dodecahedron. Consider the following planar graph of the regular dodecahedron:



On the left is an uncolored planar dodecahedral graph and on the right is a 4-colored planar dodecahedral graph. As you can see, each face of the graph (including the face outside of the graph) is not adjacent to any of its partner colors. Given that there are 4 colors and 12 total faces, each color takes up 3 faces. This can also be done in a similar way with the graph's vertices:



As you can see, this follows the rules necessary for graph 4-coloring. Since there are 4 colors and 20 vertices, each color takes up 5 vertices. The last iteration of this concept is with edges, as you can see below:



While this does work like the other two examples, it is not as clean. Since there are 4 colors and 30 edges, there must be two colors with one less edge. In this example, there are 8 red and green edges and 7 blue and yellow edges. Still, no edge color is adjacent to the same edge color.

These are simply a few examples of a wide range of colorings that begs the question, how many unique colorings of a particular graph are there given n colors? To answer this, we need to explore a theorem from discrete mathematics and combinatorics: Polya's Theorem.

Using Pólya's Method for Dodecahedral Colorings

Pólya's method, sometimes known as Pólya's enumeration theorem, is used to find the total amount of colorings given a graph. By using the different permutations of a graph obtained by rotating and or reflecting it, we can determine the amount of times certain colors will appear on the shape. We can use a simple equilateral triangle in 3-space as an example.

Consider a triangle with each vertex labeled 1, 2 and 3 respectively. We want to find the total number of colorations for the vertices of this triangle with n colors. We can write the different permutations that the vertices will end up in after reflections or rotations in cycle notation.

0° rotation, π_0 : (1)(2)(3)

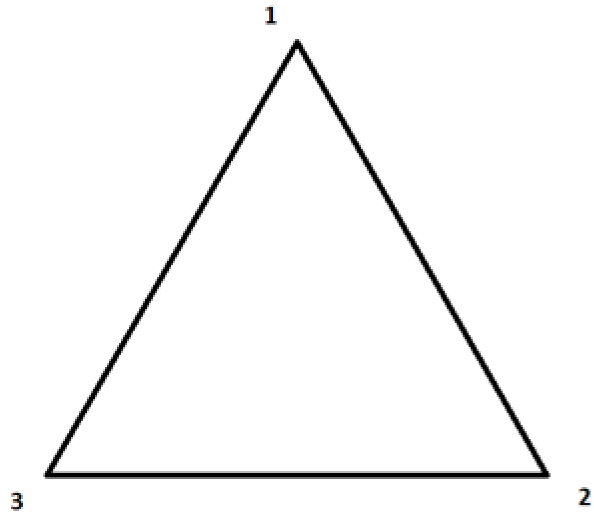
120° rotation, π_1 : (1 2 3)

180° rotation, π_2 : (1 3 2)

Symmetry over 1, r_1 : (1) (2 3)

Symmetry over 2, r_2 : (2) (1 3)

Symmetry over 3, r_3 : (3) (1 2)



We can now describe these in terms of x. The cycle length is described as the subscript and the number of that number of cycles is described as the superscript.

0° rotation, π_0 : (1)(2)(3)	-	x_1^3
120° rotation, π_1 : (1 2 3)	-	x_3^1
180° rotation, π_2 : (1 3 2)	-	x_3^1
Symmetry over 1, r_1 : (1) (2 3)	-	$x_1^1 * x_2^1$
Symmetry over 2, r_2 : (2) (1 3)	-	$x_1^1 * x_2^1$
Symmetry over 3, r_3 : (3) (1 2)	-	$x_1^1 * x_2^1$

Following Polya's method, we combine like terms from these permutations and divide them by the total amount of actions we made, giving us the following:

$$Z = 1/6\ (\ x_1^3 + 2x_3^1 + 3(x_1^1 * x_2^1))$$

Now, we simply replace all the x's with n. Move some terms around and we obtain:

$$Z = 1/6\ (n^3 + 3n^2 + 2n)$$

So, for example, if we want to color the vertices of the triangle with 3 colors, we would be able to do this in 10 possible ways. We can now apply the same principles and methodology to the dodecahedron.

Another way we can approach this problem is simply with different types of rotations, especially since we are working with a platonic solid. First, account for the identity rotation with x_1^{12}, one cycle for each face. Now, we can think about rotations in terms of the faces of the shape. There are six pairs of opposite faces and four rotations for each of these, all 72° each time. The two opposite faces create two five-cycles, giving $24x_1^2x_5^2$. There are ten pairs of opposite vertices each with two rotations about 120°, each creating three cycles which gives $20x_3^4$. Finally, there are fifteen pairs of opposite edges with a 180° rotation, giving two cycles, leaving us with $15x_2^6$. With a total of 60 actions, our final equation is:

$$Z = 1/60\ (x_1^{12} + 24x_1^2x_5^2 + 20x_3^4 + 15x_2^6)$$

We can simplify this and sub in n for x and we finally get:

$$Z = 1/60(n^{12}) + 1/4(n^6) + 11/15(n^4)$$

For n = 1 - 5 we get the following values:

1, 96, 9099, 280832, 4073375, …

Conclusion

This paper discussed multiple topics. We began with Leonard Euler, Platonic Solids, and certain properties of geometry and topology. We then analyzed the dodecahedron, one of these solids and unveiled many interesting concepts. We then translated this shape into the realm of graph theory, analyzing it in a completely new way. Euler's formulas frequently apply to geometry (like the platonic solids), which are extremely relevant in graph theory. It became clear that each of these topics is related in some way, as is the case with all of mathematics. The beauty of this field then, in my opinion, is how everything is connected through the language that great mathematicians have created across the centuries. Knowledge is iterative, continuously building upon what we have learned before, and these connections demonstrate this perfectly. Through mathematics, we have found a way to describe the world around us in a practical language, useful for countless endeavors. Having gained considerable fluency in this language during my time at WPI, I hope to evaluate and explore new challenges and contribute to this fascinating field of inquiry.

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