

Spectrum and Scale

A Major Qualifying Project Report

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by

Baltasar Sanchez

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Prof. Frederick Bianchi, Major Advisor

Abstract

There exists a relationship between scale and spectrum. Using sensory dissonance curves this relationship can be observed and understood. In this project a timbre is chosen, its spectrum deduced and a scale is designed based on the spectrum; then this is done in reverse, a scale is chosen, a spectrum for it is deduced and a timbre is synthesized.

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III. List of Sound Examples

The sound examples can be found in the adjunct file "sound_examples.zip".

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3. 12-tet with the fm sound.....12-tet with fm sound.mp3
4. 9-tet using 7p harm timbreaug chord with p7 harm.mp3
5. 9-tet using C aug timbreaug chord with C aug.mp3
6. 9-tet with CDAG2..... 9-tet with CDAG2.mp3
7. 12-tet with CDAG2.....12-tet with CDAG2.mp3
8. 9-tet with 7p harm.....9-tet with 7p harm.mp3
9. 12-tet with 7p harm.....12-tet with 7p harm.mp3

Chapter 1. Introduction

The goal of this Major Qualifying Project (MQP) is to understand and be able to analyze the relationship between the scale in use and the timbre of a musical sound. The project applies some of the methods and ideas discussed in the book "*Tuning, Timbre, Spectrum, Scale*" (henceforth, TTSS) by William A. Sethares¹.

The personal motivation behind exploring this relationship (between scale and timbre) is twofold: a desire to create a new kind of sound for music which will take advantage of the relationship between the timbres of the sounds in a piece of music, and the scales and tunings used in the piece. In a way this has already been done by William Sethares in his 2008 proof-of-concept composition C2ShiningC².

The other motivation behind this MQP is to understand the development and evolution of other musical traditions, for example gamelan music from Indonesia.

The creation of a new sound for music is not the ultimate goal of this MQP but is one of the motives behind it; the goal of the project is to understand and be able to use the relationship between scale and timbre. The main tool which lets us analyze this relationship is the **dissonance curve**. This kind of analysis allows us to not only explain the *reason* behind the western musical scales (12-tet), but also the reason behind the scales of other traditions, namely gamelan. Thus, this lays the foundation for a music theory which is not tied to a specific musical tradition.

¹ William A. Sethares, *Tuning, Timbre, Spectrum, Scale* Second edn (Springer Verlag, 2005).

² <http://homepages.cae.wisc.edu/~sethares/C2ShiningC.mp3>

The relationship between scale and spectrum can be applied in two ways, starting from a timbre and coming up with an acceptable sounding scale; or starting with a scale and generating a possible spectrum that will create consonant, pleasant sounds. In this MQP the relationship is applied both ways.

A. Spectrum and Timbre

Although related, Timbre and Spectrum are not exactly the same. Spectrum refers to the result of a Fast Fourier Transform (FFT) on a waveform; i.e. the frequency domain of a periodic sound as opposed to the time domain.

Timbre on the other hand is *how* we perceive the spectrum. Timbre is a perceptual phenomenon. Whereas the spectrum is static — a snapshot of the frequencies frozen in time — the timbre is the spectrum in motion, it is our perception of a constantly changing spectrum.

Whenever referring to a set of partials defined by a sequence of ratios and a fundamental pitch, the word "spectrum" is preferred. The word "timbre", on the other hand, is used when referring to a synthesized spectrum.

Chapter 2. Background

A. Dissonance and Consonance

In this paper the terms consonance and dissonance do not refer to the traditional concepts associated with western music theory; but instead they refer to the concept of **tonal consonance** as developed by Plomp and Levelt³. Sethares refers to tonal consonance by its inverse *dissonance*, specifically he calls it **sensory dissonance**. The notion of sensory dissonance was originally introduced by Helmholtz; he concluded that dissonance is determined by the beats between two pure tones or between each pair of partials of two complex sounds. When there is no beating (or when the beating is slow and pleasant) the sound is consonant; but when the beating is extremely rapid, the sound becomes "*rough*". In a complex sound, the beating generated by each pair of partials will be most likely different; if so, the sound will have many different beats of different perceived frequencies resulting in a rougher more dissonant sound.

i. Sensory Dissonance

In the paper "Tonal Consonance and Critical Bandwidth" Plomp and Levelt describe the experiments in which they ask non-musically trained individuals to rank a harmonic interval of pure tones in terms of its pleasantness; using non-musicians guaranteed the results would not

³ R. Plomp and W. J. M. Levelt, "Tonal Consonance and Critical Bandwidth," *The Journal of the Acoustical Society of America*, ASA, 38 (1965), 548-560.

be influenced by traditional western theories. In experiments by M. Guernsey⁴ it was shown that trained musicians differentiate between consonance and pleasantness.

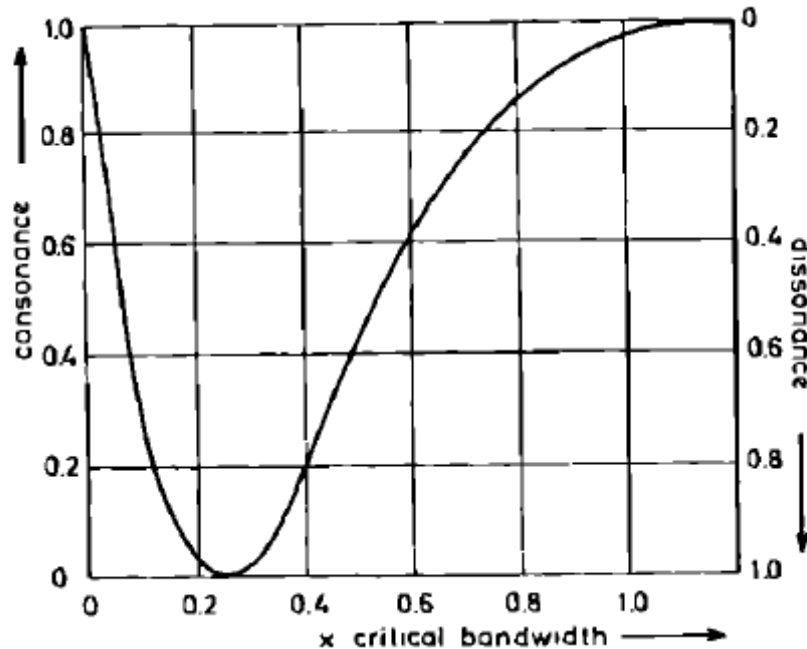


Figure 2.1. Results from Plomp and Levelt's experiments

In Plomp and Levelt's experiments several listeners were asked to rate the dissonance of a pure tone in a scale of 1 to 7. After discarding erratic results and other statistical manipulations they combined all their experimental results into Figure 2.1. They concluded (like Helmholtz) that dissonance is a result of the rough beating between two pure tones or between partials of a complex sound.

Plomp and Levelt also realized that the maximum dissonance depends on the frequencies in use and they suggested as a rule of thumb that the maximum roughness is at around 25% of the

⁴ M. Guernsey, "The role of consonance and dissonance in music," *The American Journal of Psychology*, JSTOR, 40 (1928), 173–204.

critical bandwidth (and this makes a lot of sense because the human ear is more sensitive at some frequencies than others).

B. Dissonance Curves

Dissonance curves are the principal tool used to analyze the spectrum-scale relationship. Figure 2.1 is an example of the dissonance curve of a pure tone (sine wave). A dissonance curve of a complex harmonic sound is shown below. It was generated by the dissonance curve calculation developed by Sethares; the complex tones used in Figure 2.2 have 7 harmonics.

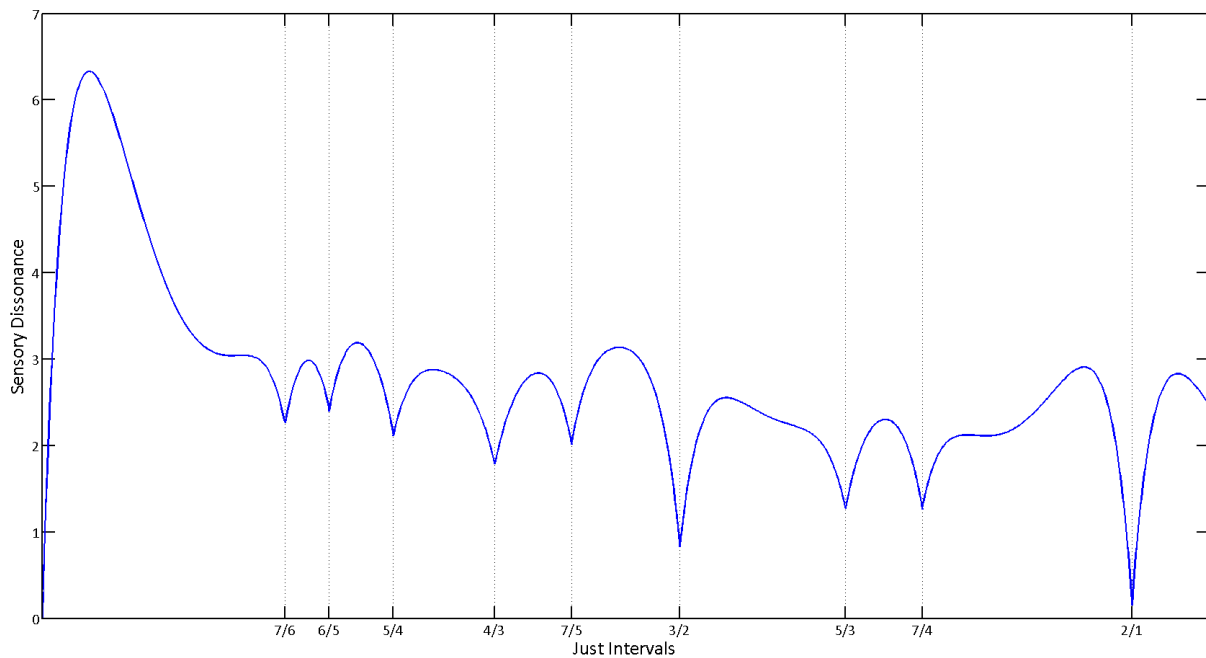


Figure 2.2. Dissonance curve of a complex harmonic tone; generated by software provided by TTSS

The dissonance curve for complex tones show a peak (point of maximum dissonance) at an interval very close to unison; it shows minima at the unison and at the octave. The other minima appear exactly at just intervals, the fifth being the second most consonant result.

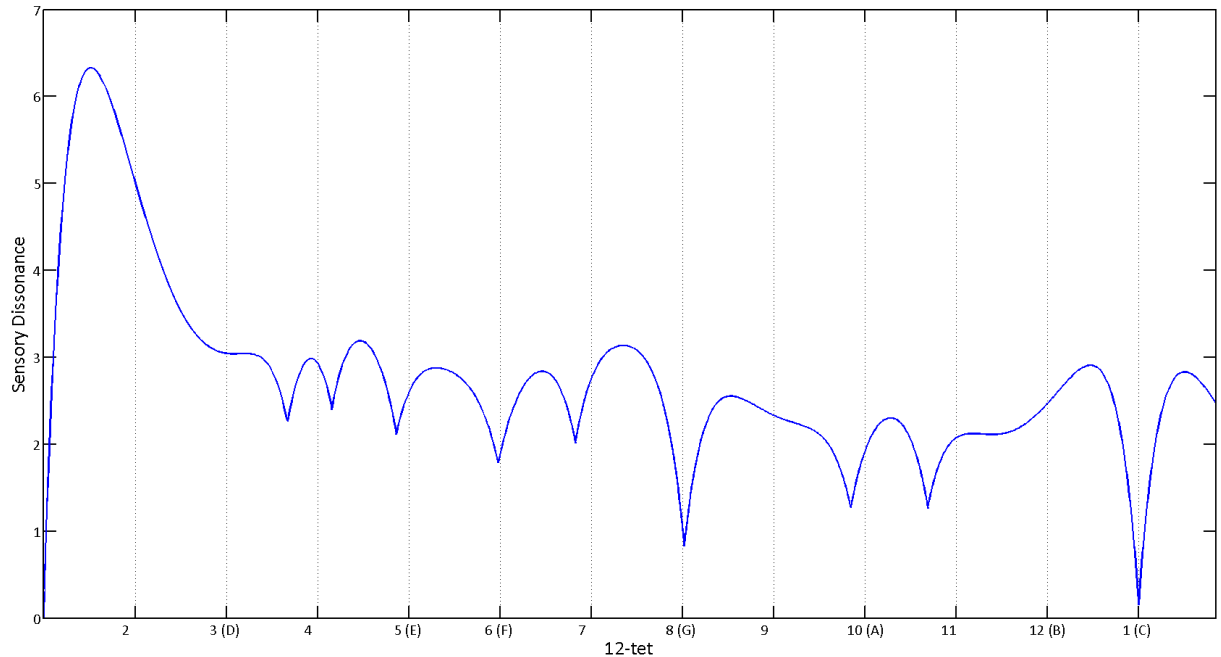


Figure 2.3. Same dissonance curve as Figure 2.2 but with grid line in 12-tet scale steps

Figure 2.3 shows how the most commonly used tuning in western music, the 12 tone equal temperament, lies very close to minima of the dissonance curve of a standard harmonic timbre. Thus, harmonic timbres and 12-tet are said to be **related**.

The fact that the dissonance curve of complex tone with harmonic partials is related to the 12 tone equal temperament scale is precisely the reason that western musicians developed this tradition. It has been shown by Sethares that musical traditions have developed scales related to the timbres of their instruments. Thus, this theory of dissonance curves can not only explain various musical traditions; but also aid the composer in the design of scales and timbres which take advantage of this knowledge to create new sonic landscapes and textures which have never been heard before. The rest of this paper will use this relationship in both directions: starting with a spectrum to design a scale, and selecting a scale and constructing a timbre.

i. Drawing Dissonance Curves

The dissonance curves used in this MQP are all based on a curve-mapping method; i.e. a method which utilizes a psychoacoustical curve of sensory dissonance to calculate the individual dissonance of every pair of partials; dissonances between pairs of partials are then added together in order to compute the total dissonance.

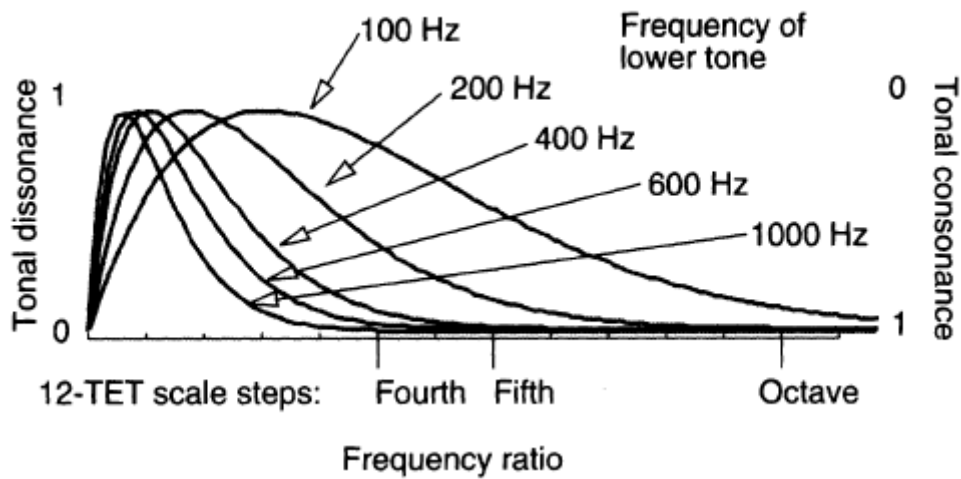


Figure 2.4. Dissonance vs. ratio of different frequencies (from "Consonance-Based Spectral Mappings" by Sethares⁵)

In order to draw a dissonance curve, one must be able to calculate the dissonance between two simple tones (one pair of partials). The function $d(f_1, f_2, \ell_1, \ell_2)$ takes the frequencies f_1 and f_2 and their corresponding loudness ℓ_1 and ℓ_2 ; and returns the sensory dissonance value based on the data of the curve in Figure 2.4 (which was parameterized by Sethares). A useful way to think about it is that $d(f_1, f_2, \ell_1, \ell_2)$ contains the same information as Figure 2.4.

⁵ William A. Sethares, "Consonance-based spectral mappings," *Computer Music Journal*, 22 (1998), 56-72.

The parameterization was done by first turning the curve in Figure 2.1 (on page 11) into the equation:

$$d(x) = e^{-3.51x} - e^{-5.75x}$$

The curve described by this equation shows that the point of maximum dissonance is at $x^* = 0.24$.

The dissonance function $d(x)$ can be refined to return a different value depending on the base frequency (like Figure 2.4); this refinement is the function $d(f_1, f_2, \ell_1, \ell_2)$ which is defined as

$$d(f_1, f_2, \ell_1, \ell_2) = \ell_{12} [e^{-3.51s(f_2-f_1)} - e^{-5.75s(f_2-f_1)}]$$

where

$$\ell_{12} = \min(\ell_1, \ell_2)$$

and

$$s = \frac{x^*}{0.0207f_1 + 18.96}$$

The numerical constants were determined by Sethares using a least square fit on Figure 2.4.

In order to draw the dissonance curve of a complex sound with a spectrum $F = [f_1, f_2, \dots, f_n]$

the intrinsic dissonance D_F of the complex sound must be calculated. The intrinsic dissonance is the dissonance between every combination of partials of the complex spectrum F .

Assume for example that the spectrum F has 3 partials: f_n with equally indexed loudness ℓ_n

($n = 1,2,3$); the intrinsic dissonance D_F is the sum of $d(f_i, f_j, \ell_i, \ell_j)$ as i and j take all the values

of n . The intrinsic dissonance is formally defined as:

$$D_F = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m d(f_i, f_j, \ell_i, \ell_j)$$

Similarly to what Helmholtz did in his violin experiment — playing a violin on a given note while playing another violin from that note to the octave — the dissonance between F and cF (with $cF = [cf_1, cf_2, \dots, cf_n]$) has to be calculated for enough values of c to make a smooth curve; the function $D_F(c)$ draws this curve:

$$D_F(c) = D_F + D_{cF} + \sum_{i=1}^n \sum_{j=1}^n d(f_i, cf_j, \ell_i, \ell_j)$$

To make dissonance curves between two different complex spectra, use:

$$D_{F,G}(c) = D_F + D_{cG} + \sum_{i=1}^n \sum_{j=1}^m d(f_i, cg_j, \ell_{f_i}, \ell_{g_j})$$

ii. Properties of Dissonance Curves

There are a few things that can be said about every dissonance curve. The proofs of these properties are omitted in this report but appear originally published in a paper by Sethares⁶ and later refined in Appendix F of TTSS. They can also be found on Sethares' webpage⁷.

Assume the spectrum $F = [f_1, f_2, \dots, f_n]$:

- **Property 1:** The unison is the point of most consonance, i.e. the deepest minima of the curve.
- **Property 2:** As the interval grows larger, the dissonance must approach a value that is no more than the intrinsic dissonance of the timbre.
- **Property 3:** The dissonance curve of a spectrum F has at most $2n(n - 1)$ local minima

⁶ William A. Sethares, "Local consonance and the relationship between timbre and scale," *The Journal of the Acoustical Society of America*, 94 (1993), 1218.

⁷ <http://eceserv0.ece.wisc.edu/~sethares/consemi.html>

which are located symmetrically (on a logarithmic scale) so that half occur for intervals between 0 and 1, and half occur for intervals between 1 and infinity.

- **Property 4: The principle of coinciding partials:** Up to half of the local minima occur at intervals $c = f_j/f_i$ where f_i and f_j are any partials of F . Thus minima at interval c are created when $f_i = cf_j$ except when $c < x^*$.

Basically, the term $d(f_i, cf_j, \ell_i, \ell_j)$ disappears from $D_F(c)$ because $d(f_i, f_i, \ell_i, \ell_j) = 0$.

In the case when $c < x^*$, the intrinsic dissonance is greater so the minimum does not appear.

Chapter 3. From Spectrum to Scale

Starting with a spectrum and designing a scale can be a rather straightforward procedure using dissonance curves, just choosing the points of maximum consonance sufficed for this project.

At first it was attempted to use the sampled spectrum of WPI's gong, however due to its complexity it was decided to instead generate a less complicated — but not trivial — sound generated using FM synthesis with MOTU's Proton synthesizer⁸.

Figure 3.1 shows the frequency spectrum of 32K samples of a portion of the entire gong sound, it can be easily observed that this spectrum is complex; what cannot be seen in the figure below (but can be heard on the sound) is that the spectrum also changes over time, with different peaks as the sound decays. This were the reasons a simpler FM sound was made.

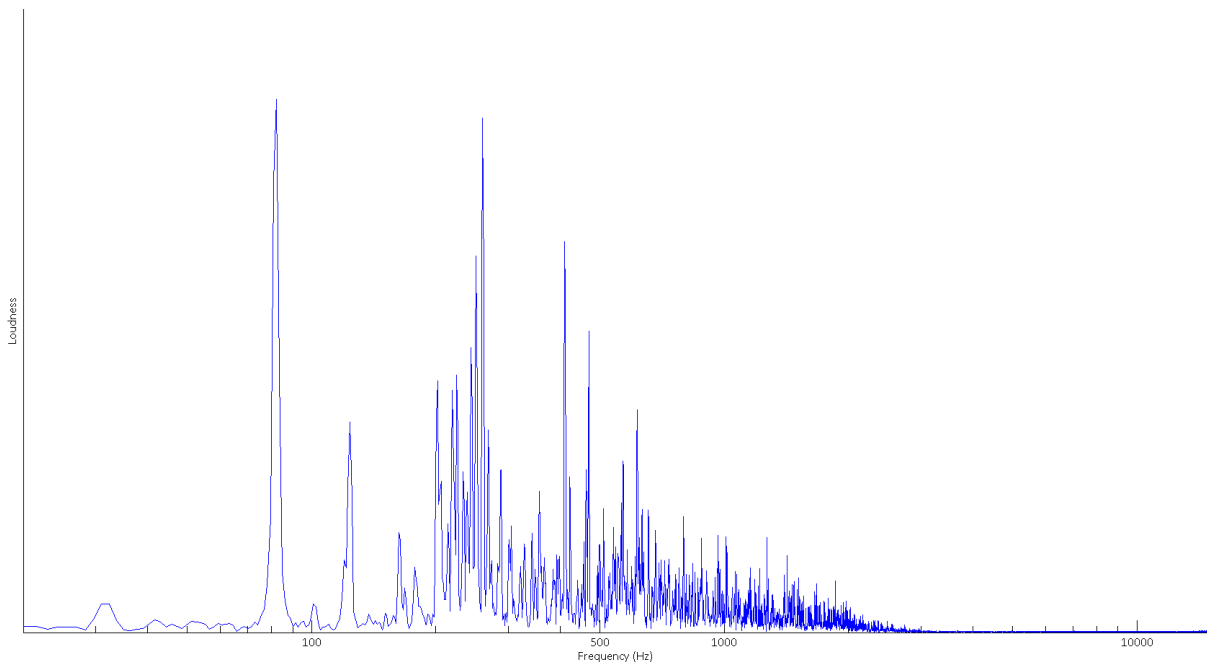


Figure 3.1. Spectrum of a small portion of the sound of the sampled gong

⁸ Part of Digital Performer (<http://www.motu.com/products/software/dp/virtual-instruments/proton.html>)

A. Synthesizing a Sound

Using MOTU's Proton synthesizer a Frequency Modulation based sound was made. The attack and decay were deliberately reduced for simplicity.

The FM sound was recorded twice, once playing the note A3 (220 Hz) and once playing A5 (880 Hz). The original analysis was mistakenly made extracting 4096 samples from the beginning of the sound file; in spite of having an extremely sharp attack, the synthesizer still put a very slight attack in the beginning of the sound sample.

The second analysis — used to make the dissonance curves in Figure 3.4 — took 4096 samples from A3 and 1024 samples from A5 from the end of the file. The results have the same contour but differ only in the average dissonance (see Figure 3.4). The reason is because they have different intrinsic dissonances due to being in different frequency ranges.

B. Analyzing the Sound

The first analysis of the original sounds (the ones with a small attack) was done using Audacity to plot the spectrum, this resulted in problems when extracting the peaks and when converting the loudness into a value that the software provided by TTSS could use to correctly calculate the dissonance curve.

The second analysis used MATLAB to compute the FFT and plot the power spectrum; in MATLAB it was much simpler to automatically detect the loudness and frequency of the peaks.

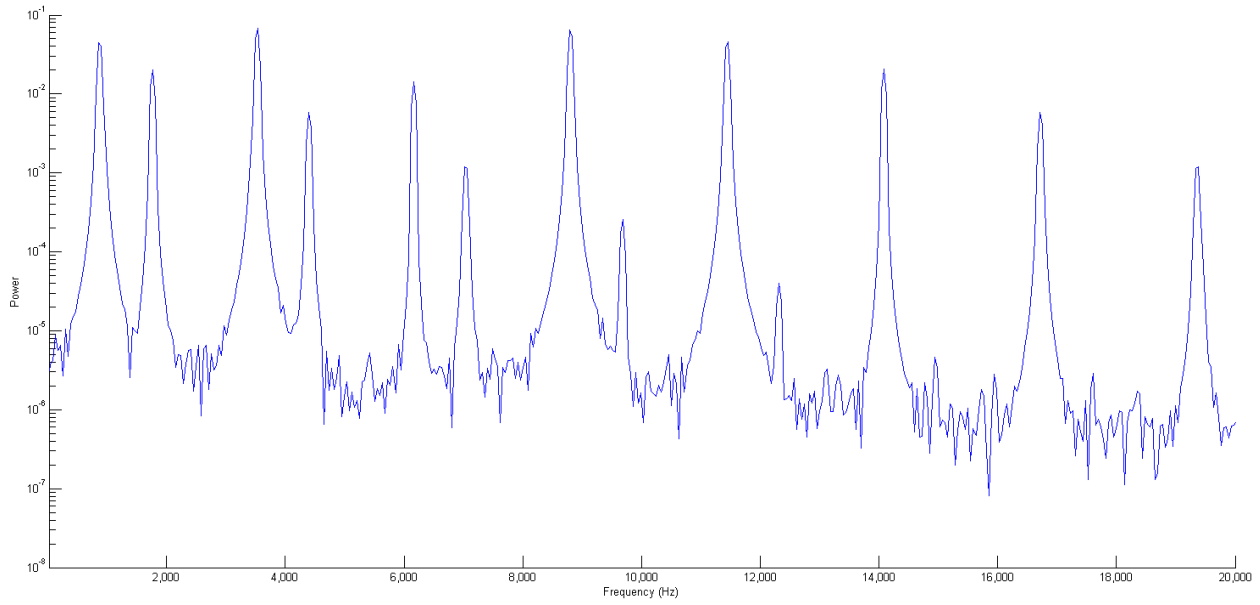


Figure 3.2. Power spectrum generated by applying a FFT to sound sample A5

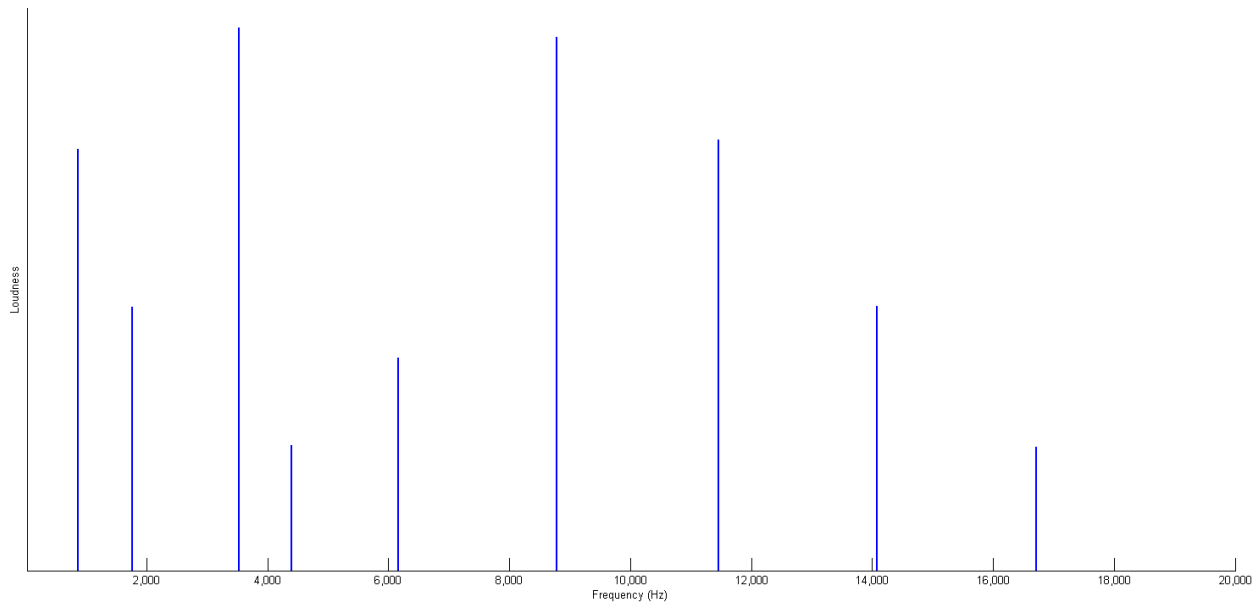


Figure 3.3 Line spectrum of sound sample A5 automatically detected by MATLAB

Using the data represented by Figure 3.2 as input, an automatic algorithm detected the local maxima, i.e. the peaks in the spectrum plot. The results are shown in Table 3.1 and plotted as a line spectrum in Figure 3.3. The same methods were used to plot the power spectrum and find the peaks of the sound A3 (also in Table 3.1).

Both A3 and A5 have the same timbre, thus the frequencies of the peaks can be re-written in terms of the fundamental frequency f_1 . Both A3 and A5 have the same spectrum $F = [f_1, 2.05f_1, 4.1f_1, 5.1f_1, 7.15f_1, 10.2f_1, 13.3f_1, 16.35f_1, 19.4f_1]$. However, to plot the dissonance curve the values of frequencies are needed, not the ratios.

C. Computing the Dissonance Curve

The dissonance curve computation method here applied uses a simplified model to approximate the loudness ℓ of the partials of the sound. The main shortcoming of this simplification is that real sensory dissonance depends on the loudness of the tones⁹. This simplistic model also doesn't take into account the differences in the perception of loudness across the audible frequency range.

With these considerations in mind, a reasonable dissonance curve can still be made. TTSS provides a MATLAB function to generate such curves using the functions defined in Chapter 2.B.i. Such software slightly modified¹⁰ by this MQP and used to plot the dissonance curves of sound samples A3 and A5:

⁹ A. Kameoka and M. Kuriyagawa, "Consonance theory part I: consonance of dyads.," *The Journal of the Acoustical Society of America*, 45 (1969), 1451-9.

¹⁰ The MATLAB script provided by TTSS was turned into a MATLAB function.

Partial No.	A3		A5	
	f (Hz)	ℓ	f (Hz)	ℓ
1	215.3	5.2	861.3	5.2
2	441.4	3.3	1765.7	3.3
3	882.9	6.7	3531.4	6.7
4	1098.2	1.6	4392.8	1.6
5	1539.6	2.6	6158.5	2.6
6	2196.4	6.6	8785.5	6.6
7	2863.9	5.4	11455.7	5.4
8	3520.7	3.3	14082.7	3.3
9	4177.4	1.5	16709.8	1.5

Table 3.1. Peaks from the spectra of sound samples A3 and A5, detected automatically by a MATLAB algorithm

The reason the loudness in both A3 and A5 is the same, is because the only difference in the midi file which generated the sound samples A3 and A5 is the pitch. Using these 9 partials the dissonance curve $D_{A3}(r)$ and $D_{A5}(r)$ were calculated with values of r between 1 and 2.1.

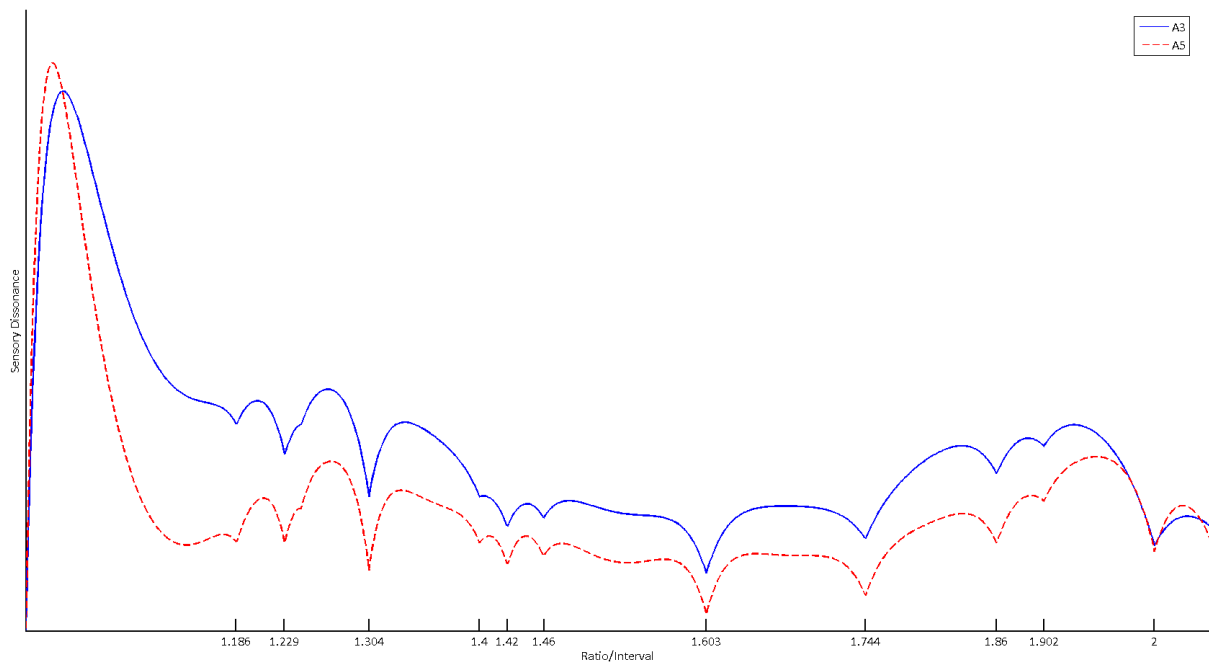


Figure 3.4. Dissonance curve of A3(solid blue line) and A5(red dashed line); the both have the same contour

The reason the only difference between these curves is their average height should be obvious at this point; it is because A3 has a higher intrinsic dissonance than A5. The reason the valleys have the same magnitudes is due to the simplification of the loudness model and the fact that

both sound samples were generated from the same midi sequence (they had equal MIDI velocities).

D. Finding Minima and Choosing a Scale

The minima are already shown on Figure 3.4, their values are:

11-fm	A	B	C	D	E	F	G	H	I	J	K	A'
Ratio	1	1.186	1.229	1.304	1.402	1.427	1.459	1.603	1.744	1.86	1.902	2
Cents	0	295	357	460	585	616	654	817	963	1074	1113	1200

Table 3.2. Ratios and cents of the minima on the dissonance curves in Figure 3.4 with consecutive letter names. The 11-fm scale

They were found by flipping the dissonance curve so it was a *consonance* curve; and then finding the point of maximum consonance by using the same method used to find the peaks of the spectra using MATLAB.

The simplest scale is made by simply taking the values in Table 3.2. However, like in 12-tet and harmonic dissonance curves (see Figure 2.3), the ratios used by a related scale can be approximations of the real minima of the dissonance curve. In order to create a scale which better musical properties we can deviate from the minima by a little bit. Since 12 tone equal tempered scale differs from a just intonation scale by up to 30 cents¹¹, we could deviate by up to that amount and still have a related scale and timbre.

The first thing to notice is the consonance of each interval, the deepest minima in Figure 3.4 not including the octave is at the interval 817 cents, we call it H, thus this is the most important note in our scale (as is the fifth in 12-tet); and so the notes in order of consonance are H, A', I, F, H, E, D, J, C, K, B.

¹¹ http://en.wikipedia.org/wiki/Equal_temperament#Comparison_to_just_intonation

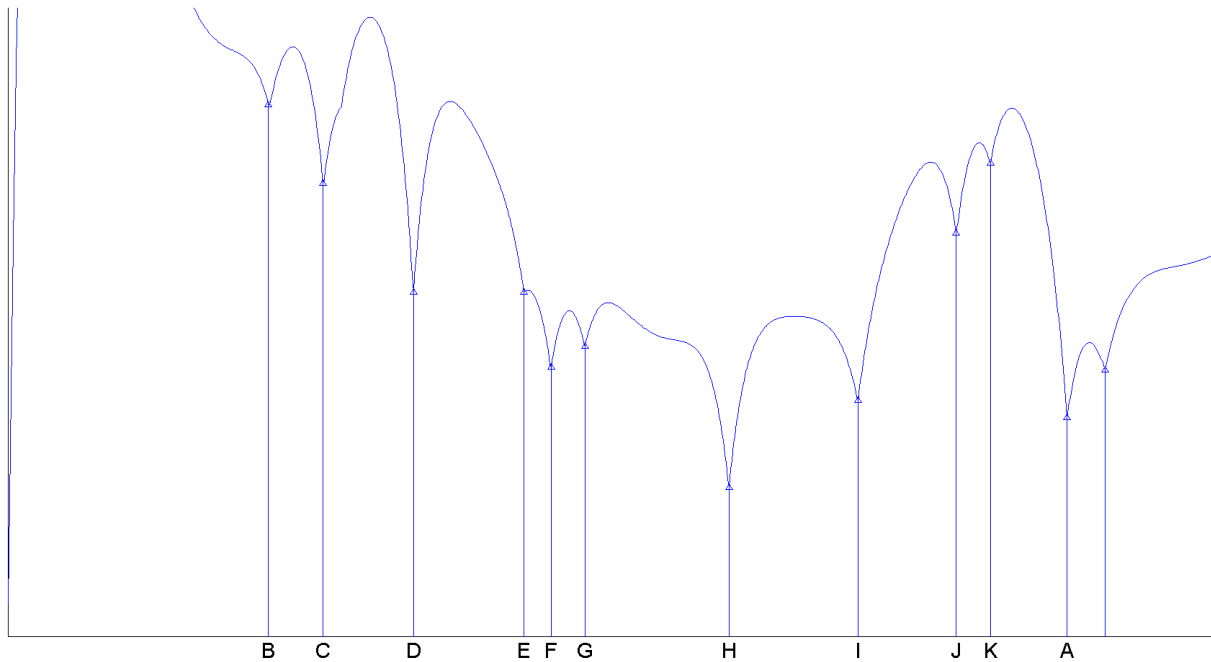


Figure 3.5. Points of minimum dissonance by assigned scale letter (zoomed in dissonance curve of A3)

Table 3.3 contains all the intervals possible between the notes in the scale:

11-fm	A	B	C	D	E	F	G	H	I	J	K	A'
A	0	295	357	460	585	616	654	817	963	1074	1113	1200
B	295	0	62	165	290	321	359	522	668	779	818	905
C	357	62	0	103	228	259	297	460	606	717	756	843
D	460	165	103	0	125	156	194	357	503	614	653	740
E	585	290	228	125	0	31	69	232	378	489	528	615
F	616	321	259	156	31	0	38	201	347	458	497	584
G	654	359	297	194	69	38	0	163	309	420	459	546
H	817	522	460	357	232	201	163	0	146	257	296	383
I	963	668	606	503	378	347	309	146	0	111	150	237
J	1074	779	717	614	489	458	420	257	111	0	39	126
K	1113	818	756	653	528	497	459	296	150	39	0	87
A'	1200	905	843	740	615	584	546	383	237	126	87	0

Table 3.3. All possible intervals in 11-fm scale (cents)

By examining this table, and listening to the scale, it was noticed that the subset of the scale containing the notes A, B, D, E, G, I was remarkably similar to the blues scale.

E. Testing the Scale

Having decided on the simplest scale (Table 3.2), named 11-fm, a program called Scala¹² was used to define the scale and calculate the frequencies of every note. The scale was exported as a `.tun` file accepted by a few software synthesizers and samplers.

Using a sampler with support for micro-tonality the sound sample A3 was used to generate a few examples. Reverb was added so the sensation of each note lingers a little more in each example and a bass note in an octave below was added in some examples.

In order to test the scales against a benchmark sound, an ideal harmonic spectrum with 7 partials of equal amplitudes — named "7p harm" — was rendered in MATLAB; similarly to the other synthesized spectra (as in Chapter 4.C), it was thereafter looped and sampled.

Sound Examples

All these sound examples play a scale ascending then descending, each file uses a different tuning and different timbre:

1. 11-fm scale with the fm sound
2. 11-fm scale with 7p harm sound
3. 12-tet with the fm sound

The most remarkable result is the contrast of the 11-fm scale when played with the harmonic timbre and when played with the fm-sound from which the scale was deduced. Notice as well that the fm sound is slightly more consonant with the 11-fm scale than with 12-tet; however, the enormous familiarity of 12-tet (or of harmonic timbres) makes this a difficult judgment.

¹² <http://www.huygens-fokker.org/scala/>

Chapter 4. From Scale to Spectrum

Starting with a scale and finding a spectrum that will complement the scale nicely in a musical setting is a more complicated task than its reverse; for some scales it might be impossible to find perfectly related spectra. Unlike when deducing a scale from a spectrum, where there is only one dissonance curve and a related scale can be easily deduced, when attempting to determine a suitable spectrum there are many possible solutions (or none) and thus many decisions need to be made.

In TTSS and in the paper "Specifying spectra for musical scales"¹³, Sethares demonstrates a symbolic method to determine appropriate spectra for non-equal, but still regular, tunings. For simplicity an equally tempered scale was chosen for this project. And so, the symbolic method is not examined in detail in this report.

The spectra designed for this MQP were made in general using a hit or miss approach with a some constraints and directions in mind. The amplitudes were constrained to the same value for all partials in all spectra; this constraint simplified the process of designing the spectra but made all the timbres sound organ-like. Also, the goal of the spectra designed wasn't to have consonance at all scale steps, but to maximize the consonance only of certain intervals.

The method used did not involve searching for overtones over the entire frequency range, due to the principle of induction of spectra (Chapter 4.B.i) and the use of an equal temperament, it was only a matter of deciding which of the many possible overtones to use.

¹³ William A. Sethares, "Specifying spectra for musical scales," *The Journal of the Acoustical Society of America*, 102 (1997), 2422.

The hit or miss method utilized consisted of quickly defining a spectrum in MATLAB, then plotting its dissonance curve, and then tweaking it in an attempt to make certain minima (at the desired scale steps) deeper. With very few partials this was straightforward and simple; however, as the number of partials involved increases, the number of minima and the intrinsic dissonance of the spectrum tend to increase as well.

A. Choosing a Scale

An equally tempered scale m -tet always divides the octave in m equal parts. So for any equal temperament scale based on the octave, the ratios from f to $2f$ are divided in m parts; Thus the scale steps are $s = \sqrt[m]{2}$. And so, the ratio (or interval) between notes separated by k scale steps is $s^k = 2^{k/m}$. For example, in 9-tet (Table 4.1) the ratio between E and G[#], which are 3 scale steps apart, is $s^3 = 2^{3/9} = \sqrt[3]{2} = 1.2599$ which is approximately 400 cents. This property of m -tet scales makes it easier to design perfectly related spectra for them.

The equally tempered scale of 9 tones, 9-tet, was arbitrarily chosen and is shown in Table 4.1.

9-tet	C	C [#]	D	E	F	G	G [#]	A	B	C'
Ratio	$\sqrt[9]{2}^0$	$\sqrt[9]{2}^1$	$\sqrt[9]{2}^2$	$\sqrt[9]{2}^3$	$\sqrt[9]{2}^4$	$\sqrt[9]{2}^5$	$\sqrt[9]{2}^6$	$\sqrt[9]{2}^7$	$\sqrt[9]{2}^8$	$\sqrt[9]{2}^9$
Cents	0	133.3	266.6	400	533.3	666.6	800	933.3	1066.6	1200

Table 4.1. Equally tempered scale with 9 notes

Table 4.1 and Table 4.2 show that two familiar intervals from 12-tet exist in 9-tet: 400 cents (major third) and 800 cents (minor sixth); these two intervals together make a standard augmented chord. However, Table 4.2 also shows that in some cases these intervals are merely approximated; 399.7 instead of 400 or 799.7 instead of 800 cents.

9-tet	C	C [#]	D	E	F	G	G [#]	A	B	C'
C	0.0	133.3	266.6	400.0	533.3	666.6	800.0	933.3	1066.3	1200.0
C [#]	133.3	0.0	133.3	266.7	400.0	533.3	666.7	800.0	933.0	1066.7
D	266.6	133.3	0.0	133.4	266.7	400.0	533.4	666.7	799.7	933.4
E	400.0	266.7	133.4	0.0	133.3	266.6	400.0	533.3	666.3	800.0
F	533.3	400.0	266.7	133.3	0.0	133.3	266.7	400.0	533.0	666.7
G	666.6	533.3	400.0	266.6	133.3	0.0	133.4	266.7	399.7	533.4
G [#]	800.0	666.7	533.4	400.0	266.7	133.4	0.0	133.3	266.3	400.0
A	933.3	800.0	666.7	533.3	400.0	266.7	133.3	0.0	133.0	266.7
B	1066.3	933.0	799.7	666.3	533.0	399.7	266.3	133.0	0.0	133.7
C'	1200.0	1066.7	933.4	800.0	666.7	533.4	400.0	266.7	133.7	0.0

Table 4.2. All possible intervals in 9-tet (cents)

B. Designing a Spectrum

There is more than one possible spectrum which can render dissonance curves with minima in scale steps of 9-tet. So instead of attempting to minimize dissonance at all scale steps, two distinct spectra were created which try to create deeper minima at different scale steps. An intuitive method based on the principle of *induction* of spectra guided the design of the spectra; one of which is a simple spectrum designed to maximize consonance at scale steps 1, 3 and 6 (CEG[#]). A second more complicated spectrum with a larger number of partials was designed by trying to make scale steps 0, 2, 5 and 7 (CDGA) more consonant.

It is reasonable to assume that the dissonance of a certain interval at a minimum will depend on the relative loudness ℓ of each partial; but it seems that the dissonance of a minimum at a certain interval is also related to the number of times which different ratios of partials equal that same interval (or its octaves).

i. Inducing a Spectrum

All the intervals of a m-tet scale are powers of the scale step (see why in Chapter 4.A) and the principle of coinciding partials (introduced in Chapter 2.B.ii) states that the dissonance curve of

a spectrum F with partials f_i and f_j has up to half of its minima at ratios f_i/f_j . Thus, when there is a spectrum which has ratios between partials equal to powers of the scale step for m -tet, such a spectrum is said to be *induced* by m -tet. In general, when the ratios between the partials of a spectrum equal the steps of some scale, the a spectrum is *induced* by that scale.

For example, consider a spectrum which has partials at ratios of powers of $s = \sqrt[9]{2}$. For instance, $F = [f, s^3f, s^9f, s^{15}f]$. This spectrum has a minimum at $s^{15}/s^9 = s^6$ which is a scale step in 9-tet ($\sqrt[9]{2}^6$ is G^\sharp). If the spectrum F could generate every scale step in 9-tet in any octave, then it could be said that F was induced by 9-tet. Sometimes it might be necessary to transpose the intervals down to the same octave. For example in the spectrum F above, it is necessary to transpose the ratio $s^{15}/s^3 = s^{12}$ down one octave in order to notice that it is scale step 3 (E) but an octave above¹⁴.

In general, when using m -tet to induce a spectrum whose dissonance curve needs minima at step σ , there must be a ratio $f_i/f_j = s^{\sigma+m\omega}$ where ω is the octave number (starting from 0) and $s = \sqrt[m]{2}$.

ii. Creating a Spectrum

Several spectra were created, two are used in this report and in the example sound files, they were named CDGA2 and C aug. Table 4.3 shows the values of the ratios of f for these spectra in terms of $s = \sqrt[9]{2}$. Remember that any $x^0 = 1$.

¹⁴ exponential math shows: $s^{12} = \sqrt[9]{2}^{12} = \sqrt[9]{2}^9 \cdot \sqrt[9]{2}^3 = 2 \cdot \sqrt[9]{2}^3$

Partial No.	CDGA2 f	C aug f
1	s^0	s^0
2	s^{14}	s^3
3	s^{18}	s^9
4	s^{29}	s^{12}
5	s^{38}	s^{15}
6	s^{45}	s^{21}
7	s^{50}	s^{24}
8	s^{52}	
9	s^{61}	
10	s^{63}	

Table 4.3. Spectra CDGA2 and C aug designed for 9-tet

It is worth mentioning that the first synthesis attempt for the spectra was made without a fundamental frequency, while lacking a fundamental doesn't really change the contour of the dissonance curve, it certainly influences the intrinsic dissonance. Also, sounds made without the fundamental don't fuse very well, that is, they don't sound like just one instrument.

Spectrum C aug

The most familiar interval encountered in 9-tet is the major third (400 cents). In order to exploit this familiar interval the dissonance curve was made to have minima at the scale step E or its octaves and at scale step G[#] or its octaves; these consonant intervals correspond to the common intervals between 9-tet and 12-tet; they make up an standard augmented C chord. Every ratio of the spectrum C aug chosen had the form $s^{\sigma+9\omega}$ with $\sigma = 0,3,6$ and $\omega = 0,1,2$. This choice of sigma guarantees minima at all the possible differences of every possible value of σ ; as these differences are always a multiple of 3, it was a rather trivial choice of ratios for the spectrum.

Spectrum CDGA2

This spectrum was designed to maximize consonance at scale steps C (0), D (2), G (5) and A (7).

This spectrum was more complicated to determine. Because unlike in spectrum C aug where the possible differences between scale steps σ were always a multiple of 3, in this spectrum the differences between scale steps are not only our scale steps. For this spectrum the form $s^{\sigma+9\omega}$ is also used with $\sigma = 0,2,5,7$. The differences amongst all of these are not only 2, 5, and 7 (the scale steps for which higher consonance was sought) but also 3. This 3 is a "side-effect" ratio.

The intuitive idea consists of repeating the number of times in which the differences between values of σ equal scale steps for which lesser dissonance is desired. This is done by using octaves to repeat such steps, while trying to minimize the number of times that "side-effect" ratios appear. An ordered chart of all $s^{\sigma+9\omega}$ with $\sigma = 0,2,5,7$ and $\omega = 0,1,2,3,4,5,6,7$ was made and from this ordered list it was easy to pick out the values which determined the spectrum, shown in Table 4.3. The original spectrum CDGA was a the initial selection of values, which were tweaked using the hit or miss method described in the introduction of Chapter 4, and which resulted in spectrum CDGA2

iii. Analyzing a Spectrum

Using the same methodology and MATLAB functions from Chapter 3.C the dissonance curves below were plotted. A frequency of 440 Hz was used as a fundamental in these curves.

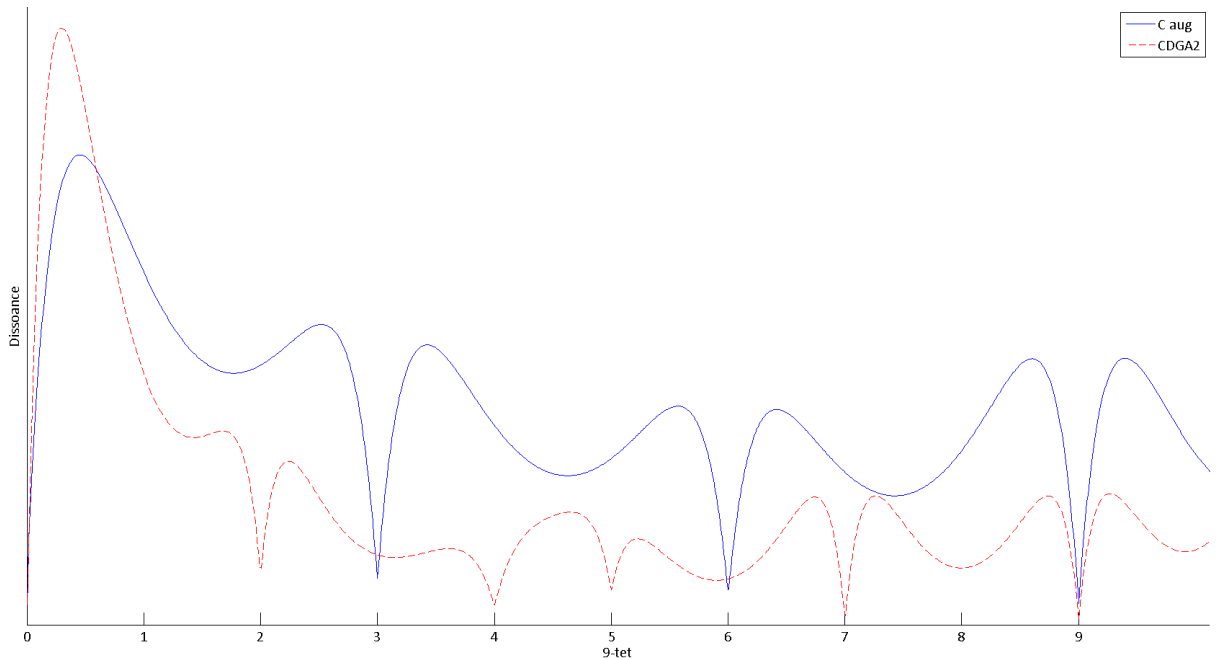


Figure 4.1. Dissonance curve for spectra C aug (blue above) and CDGA2 (red dashed below)

The intrinsic dissonance of spectrum CDGA2 is lower because its partials are more widespread (see Table 4.3). Also, the spectrum C aug only has partials at multiples of s^3 ; and so, it is a rather simple spectrum perfectly induced by the scale 3-tet. 3-tet is a subset of 9-tet and of 12-tet.

Also, spectrum CDGA seems to be induced by a regular scale based around two different intervals: $s^2 = b$ and $s = a$ in the following sequence: b, b, a, b, b .

C. Synthesizing a Spectrum

Using a modified additive synthesis function for MATLAB based on code bundled with TTSS, the spectra were generated based on the specifications in Table 4.3 using a base frequency of 220 Hz. These .wav files were later sampled as in Chapter 3.E. Again Scala¹² was used to generate the 9-tet .tun file.

For the C aug spectrum, it was possible to create a seamlessly looping file which was easily sampled and mapped to the correct pitch. For the more complicated spectrum a sufficiently long (60 seconds) sample was made to avoid having to create a seamless loop for it.

D. Testing the Timbres

The same timbre 7p harm (see Chapter 3.E on page 25) was used in these examples.

Sound Examples for C aug

These examples use 9-tet. They only use the notes C, A, G and G#, the note G is a flat fifth. (666.6 cents instead of 702 cents). A most remarkable result is the comparison of these two sound examples. Both use the same midi sequence: The notes C, E and G are introduced in layers, then G is raised to the next scale step G#, finally the top note descends back down to G.

4. 9-tet using 7p harm timbre

In sound example 4, notice that when the somewhat familiar major chord CEG becomes the standard augmented chord CEG#, the dissonance increases

5. 9-tet using C aug timbre

On the other hand, in sound example 5, the C aug spectrum sounds dissonant at first, then it becomes extremely consonant after the G is raised to G#, and returns to being dissonant in the end, when the notes being played are once again C, E and G.

Sound Examples for CDGA2

All the examples are generated by the exact same midi sequence. Thus, the abstract structural elements rhythm and pitch (but not frequency) are consistent on all examples 6 to 9. However,

they are all different because of differences in the textural elements of the sound which are the tuning (9-tet or 12-tet) and the timbre (7p harm or CDAG2).

6. 9-tet with CDAG2

Example 6 is a simple melody in 9-tet. The scale and timbre used are related. The final chord is the notes CDGA in all examples.

7. 12-tet with CDAG2

Example 7 uses the timbre specifically designed for a few scale steps of 9-tet; and so, this sound examples is *out-of-tune*. However, the example uses the very familiar 12 tone equal temperament.

8. 9-tet with 7p harm

Example 8 is perhaps the stranger of all examples, as it uses a harmonic timbre and a 9-tet scale; this example is being played with the wrong instrument, it is *out-of-spectrum*.

9. 12-tet with 7p harm

Example 9 like the sound 6, is both in-tune and in-spectrum, so it sounds more consonant thorough. However, the scale steps which the midi sequence uses are more important scale steps for spectrum CDGA2 in conjunction with 9-tet than for 12-tet with a harmonic timbre.

Chapter 5. Conclusions

A. From Spectrum to Scale

Designing a scale based on the dissonance curve can be a more than simply looking for the minimum points of the dissonance curve and using those. Approximating the points of minimum dissonance may be a way to come up with similar scales that are still be related (although not as perfectly) to a certain timbre.

If the dissonance curve does not have many minima, it is also possible to make intermediate scale steps at points between minima; these intermediate points could be passing scale steps. If the dissonance curve has too many minima, not all of them have to be used in the scale.

This project was constrained to dissonance curves generated by a single spectrum; however, dissonance curves can be made for multiple spectra. When attempting to understand music from other musical traditions, it is often necessary to make dissonance curves between different timbres.

B. From Scale to Spectrum

Designing a spectrum for a specific scale involves many decisions. Since the principle of coinciding partials can only help us in determining the frequencies of the overtones needed, the amplitudes were constrained in order to reduce the number of decisions that needed to be made. Every spectra synthesized was constrained to the same loudness for all of its partials; this constrain gave all the timbres made an organ-like quality. Constraining the amplitude reduced the amount of control over the depth of the minima.

Perhaps the 11-fm scale defined in Chapter 3.D could have been used in Chapter 4 as the starting scale; however, the problem of finding suitable spectra for an irregular scale is not trivial. If the scale has only irregular steps, the only way to come up with a scale might be by using a computer optimization algorithm. If, however, the scale repeats intervals, Sethares' symbolic method might be used to come up with a spectrum.

In the case of this MQP, a perfect spectrum for 9-tet was not made because the scale step for 9-tet (133.3 cents) is too small to achieve reasonable consonance at every scale step (at least without changing the amplitudes of the overtones). And so the scales for which spectra were designed were subsets of 9-tet. The scale was tweaked and a subset was used instead.

C. Spectrum and Scale

Whenever the goal is to make music, it is not sufficient to choose a scale and blindly stick to it until a spectrum is decided upon. It is often necessary (like in this project) to tweak the scale in order to produce reasonable results. Likewise, when constructing a scale based on a timbre, it might be rather useful to re-adjust the loudness of the partials so that the scale steps which are more important become more consonant. Thus, the sound design process should be done in both ways; relating a timbre and a scale is not a one way street when new music is intended.

D. Improving the Loudness Model

The dissonance curve model used by this MQP is simplified (as discussed in Chapter 3.C). A more advanced model would take into account the perception of loudness and the relationship between Sound Pressure Level (SPL) and most dissonant frequency difference.

The paper by Kameoka and Kuriyagawa⁹ discusses the relationship between the SPL of two tones and the frequency differences at which more dissonance is perceived. Also, they report a different perceived dissonance depending on whether the higher frequency tone is louder than the lower tone or quieter. Thus they conclude that the maximum dissonance depends not only on the critical bandwidth, but also on the loudness of the involved tones.

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