A New Class of Elliptically Contoured Random Fields and Its Stochastic Orderings

Boming Chen [*], Fangfang Wang [†], and Chunsheng Ma [‡]

October 25, 2020

Abstract

K-differenced vector random field, which belongs to the family of elliptically contoured vector random fields, has been preliminarily investigated in the literature. As a non-Gaussian vector random field, it has some good properties even better than the Guassian vector random field. In this paper, we have introduced a more genernal form of the K-differenced vector random field, whose density function is still constructed by the modified Bessel functions. Some properities of the K-differenced vector random field have been studied in this paper and some important stochastic orders have also been discussed. Through studying the stochastic orderings of the generating variable, we have understood how the parameters in density function of the generating variable influence the peakedness order of the K-differenced vector random field.

Keywords: convex order, elliptically contoured random field, modified Bessel function, peakedness order, usual stochastic order.

1 Introduction

Gaussian random fields are frequently used in modeling spatial dependence. In this paper, we would go beyond this and introduce a new non-Gaussian random field in the realm of

[*]bchen6@wpi.edu, Department of Mathematical Sciences, Worcester Polytechnic Institute.

[†]fwang4@wpi.edu, Department of Mathematical Sciences, Worcester Polytechnic Institute.

[‡]cma@math.wichita.edu, Department of Mathematics, Statistics, and Physics, Wichita State University .

elliptically contoured vector random fields. Such an m-variate random field is defined by

$$\mathbf{Z}(x) = U\mathbf{Y}(x) + \boldsymbol{\mu}(x), \qquad x \in \mathbb{D}, \tag{1}$$

where $\mathbb{D}$ represents a temporal, spatial, or spatio-temporal index domain and $\boldsymbol{\mu}(x)$ is a deterministic function with the range in $\mathbb{R}^m$. Moreover, $\{\mathbf{Y}(x), x \in \mathbb{D}\}$ is an m-variate Gaussian random field with mean $\mathbf{0}$ and covariance matrix function $\mathbf{C}(x_1, x_2)$, and is independent of a positive random variable U with density function

$$f_U(u) = \begin{cases} \frac{c_0}{u^{2\lambda\ell-1}+1} \prod_{k=1}^{\ell} \left(e^{-\nu_0 u^2} - e^{-\nu_k u^2}\right), & u \geq 0, \\ 0, & u < 0. \end{cases} \tag{2}$$

In (2), $0 \leq \lambda < 1$, c_0 is a constant such that $\int_0^\infty f_U(u)du = 1$, and ν_k $(k = 0, \ldots, \ell)$ are positive constants. We also write the density function $f_U(u)$ as $f_U(u|\lambda, \nu_0, \ldots, \nu_\ell)$ to emphasize the dependence on the parameters. What is appealing about the proposed random fields is that its finite-dimensional distributions have thin tails, even thinner than those of a Gaussian random field, and it has more parameters that affect its peakedness, besides the covariance matrix function.

The proposed vector random fields $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ encompass the K-differenced vector random field established in Alsultan and Ma (2019) as a particular case, i.e., when $\ell = 1$. The density and characteristic functions of the K-differenced random variable have first been derived, from which a natural extension to K-differenced vector random vector has been conducted. As a result, the density function of the random vector is a form of the difference of two modified Bessel functions and the characteristic function could become the characteristic function of the double exponential random field as an extreme case. In this paper, we have extended the density and characteristic functions of the K-differenced random vector to the general case, i.e, $\ell \geq 2$, and we have found that the the density function is still the form of sum or difference of some modified Bessel functions.

Closely related to the proposed random fields is the elliptical random vector defined by $\mathbf{Z}(x)$ for a fixed $x \in \mathbb{D}$. Some important definitions and facts about the stochastic orderings of random vectors are summarized in Pan et al. (2016) and it also gives some necessary and sufficient conditions for several important stochastic orders of elliptical random vectors . In this paper, we have provided the same conclusions for the usual stochastic ordering and the convex ordering with different ways of proof. Also, we have studied the usual stochastic ordering for the generating variable and how the generating variable influence the peakedness ordering of the elliptical random vector.

Considering the st, the cx, the icx, the uo and the dcx orderings, some necessary and sufficient conditions are discussed for the random vectors $\mathbf{X}$ and $\mathbf{Y}$ following the skew-normal(SN) distribution in Jamali et al. (2020). Amiri et al. (2020) discussed the sufficient and necessary conditions for the st, the cx and the icx ordering of random vectors $\mathbf{X}$ and $\mathbf{Y}$ following the multivariate scale mixtures of the skew-normal(SMSN) distribution. Further, linear stochastic orderings were discussed for the SMSN distributions in Amiri et al. (2020) and some equivalent conditions between the linear stochastic orderings and the integral stochastic orderings are also discussed. In sense of these results, we can characterize some integral stochastic orderings of most of the well-known elliptical distributions through their mean vectors and covariance matrices.

The rest of the paper is organized as follows. Section 2 provides background knowledge of the density function (2) and the elliptical vector random fields. The convex ordering, the usual stochastic ordering, and the peakedness ordering of the elliptical random vector are investigated in Section 3, and those of the elliptically random fields are discussed in Section 4. We conclude in Section 5. All the proofs are deferred to an Appendix.

Before closing this section, we introduce the notation and conventions that would be adopted throughout this paper. Denote by I_m an $m \times m$ identity matrix. We use $|A|$ to denote the determinant of a matrix A. The modified Bessel functions of the second type is defined by $K_\nu(x) = 2^{-1}(x/2)^\nu \int_0^\infty \exp(-u - u^{-1}x^2)u^{-1-\nu}dx$, for $x > 0$ (Watson, 1995).

2 Basic Distributional Properties

In this section, we will present some basic distributional properties of the newly introduced density (2) and the multivariate distribution associated with the random field.

Lemma 1. *Consider the density function defined in* (2) *with* $\ell \geq 1$. *A closed-form expression for* c_0 *is given by*

$$c_0 = \frac{2\lambda/\ell}{\Gamma(1-\frac{\lambda}{\ell})\left(-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{S=1}^{\ell}(-1)^{S+1}G_S(\frac{\lambda}{\ell},\nu_k)\right)},\ \ 0<\lambda<1,$$

with $G_S(\frac{\lambda}{\ell},\nu_k) = \sum_{\nu_k<\nu_{k+1}}^{\ell}(\sum_{k=1}^{S}\nu_k + (\ell - S)\nu_0)^{\frac{\lambda}{\ell}}$, *and*

$$c_0 = \frac{2}{-\ln(\ell\nu_0) + \sum_{S=1}^{\ell}(-1)^{S+1}H_S(\nu_k)},\ \ \lambda = 0,$$

with $H_S(\nu_k)=\sum_{\nu_k<\nu_{k+1}}^{\ell}\ln(\sum_{k=1}^{S}\nu_k+(\ell-S)\nu_0)$.

Boming: Is S a set? The expression $S\subset\{1,\cdots,\ell\}$ makes S appear to be a set. Please double check the notation you adopted.

When $\ell=1$, an application of Lemma 1 yields that $c_0=\frac{2\lambda}{\Gamma(1-\lambda)\left(-\nu_0^\lambda+\nu_1^\lambda\right)}$ for $0<\lambda<1$, and $c_0=\frac{2}{-\ln(\nu_0)+\ln(\nu_1)}$ for $\lambda=0$. Thus, the density function (2) becomes

$$f_U(u)=\frac{2\lambda}{(\nu_1^\lambda-\nu_0^\lambda)\Gamma(1-\lambda)}\frac{e^{-\nu_0u^2}-e^{-\nu_1u^2}}{u^{1+2\lambda}},\quad u\geq 0,$$

if $0<\lambda<1$, and

$$f_U(u)=\frac{2}{\ln\nu_1-\ln\nu_0}\frac{e^{-\nu_0u^2}-e^{-\nu_1u^2}}{u},\quad u\geq 0,$$

if $\lambda=0$. This is the density function studied in Alsultan and Ma (2019), so here we are considering a more general case.

Moreover, it follows directly from Lemma 1 that when $\ell=2$,

$$c_0=\begin{cases}\frac{\lambda}{\Gamma(1-\lambda/2)\left(-(2\nu_0)^{\frac{\lambda}{2}}+(\nu_1+\nu_0)^{\frac{\lambda}{2}}+(\nu_2+\nu_0)^{\frac{\lambda}{2}}-(\nu_1+\nu_2)^{\frac{\lambda}{2}}\right)}, & 0<\lambda<1,\\ \frac{2}{-\ln(2\nu_0)+\ln(\nu_0+\nu_1)+\ln(\nu_0+\nu_2)-\ln(\nu_1+\nu_2)}, & \lambda=0.\end{cases}\tag{3}$$

An appealing and important feature of the random variable U is that it has finite moments of any order, which is stated in the next lemma.

Lemma 2. *Consider the random variable U with the density function in* (2).
(1) If $j>2\lambda/\ell$, we have

$$\mathrm{E}U^j=\frac{\frac{\lambda}{\ell}\Gamma(\frac{j}{2}-\frac{\lambda}{\ell})}{\Gamma(1-\frac{\lambda}{\ell})}\frac{(\ell\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}}+\sum_{S=1}^{\ell}(-1)^SG_S(\frac{\lambda}{\ell}-\frac{j}{2},\nu_k)}{-(\ell\nu_0)^{\frac{\lambda}{\ell}}+\sum_{S=1}^{\ell}(-1)^{S+1}G_S(\frac{\lambda}{\ell},\nu_k)},$$

for $0<\lambda<1$, and

$$\mathrm{E}U^j=\Gamma(\frac{j}{2})\frac{(\ell\nu_0)^{-\frac{j}{2}}+\sum_{S=1}^{\ell}(-1)^SG_S(\frac{j}{2},\nu_k,k\in S)}{-\ln(\ell\nu_0)+\sum_{S=1}^{\ell}(-1)^{S+1}H_S(\nu_k,k\in S)},$$

for $\lambda=0$.

(2) If $j = 2\lambda/\ell$ and $0 < \lambda < 1$, then

$$\mathrm{E}U^j = \frac{-\ln(\ell\nu_0) + \sum_{S=1}^{\ell}(-1)^{S+1}H_S(\nu_k)}{-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{S=1}^{\ell}(-1)^{S+1}G_S(\frac{\lambda}{\ell},\nu_k)}.$$

(3) If $0 < j < 2\lambda/\ell$ and $0 < \lambda < 1$, then

$$\mathrm{E}U^j = \frac{\Gamma(1-\frac{\lambda}{2}+\frac{j}{2})}{\frac{\lambda}{2}-\frac{j}{2}} \frac{-(\ell\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} + \sum_{S=1}^{\ell}(-1)^{S+1}G_S(\frac{\lambda}{\ell}-\frac{j}{2},\nu_k)}{-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{S=1}^{\ell}(-1)^{S+1}G_S(\frac{\lambda}{\ell},\nu_k)}.$$

Because $0 \le \lambda < 1$, Lemma 2 (1) provides the moments $\mathrm{E}U^j$, $j = 1, 2, \ldots$, for $\ell \ge 2$. The case $\ell = 1$ has been discussed in Alsultan and Ma (2019). When $\ell = 2$, it follows from Lemma 2 that

$$\mathrm{E}U^j = -\frac{\lambda}{2}\frac{\Gamma(\frac{j-\lambda}{2})}{\Gamma(1-\frac{\lambda}{2})}\frac{(2\nu_0)^{\frac{\lambda-j}{2}} - (\nu_1+\nu_0)^{\frac{\lambda-j}{2}} - (\nu_2+\nu_0)^{\frac{\lambda-j}{2}} + (\nu_1+\nu_2)^{\frac{\lambda-j}{2}}}{(2\nu_0)^{\frac{\lambda}{2}} - (\nu_1+\nu_0)^{\frac{\lambda}{2}} - (\nu_2+\nu_0)^{\frac{\lambda}{2}} + (\nu_1+\nu_2)^{\frac{\lambda}{2}}},\ j = 1, 2, \ldots,$$

for $0 < \lambda < 1$, and

$$\mathrm{E}U^j = -\Gamma(\frac{j}{2})\frac{(2\nu_0)^{-\frac{j}{2}} - (\nu_1+\nu_0)^{-\frac{j}{2}} - (\nu_2+\nu_0)^{-\frac{j}{2}} + (\nu_1+\nu_2)^{-\frac{j}{2}}}{\ln(2\nu_0) - \ln(\nu_1+\nu_0) - \ln(\nu_2+\nu_0) + \ln(\nu_1+\nu_2)},\ j = 1, 2, \ldots,$$

for $\lambda = 0$.

In parallel with the proposed random field, we define the following elliptical random vector

$$\mathbf{Z} = U\mathbf{Y} + \boldsymbol{\mu}, \tag{4}$$

where $\boldsymbol{\mu} \in \mathbb{R}^m$ is a constant, and $\mathbf{Y}$ is an m-variate Gaussian random vector with mean $\mathbf{0}$ and covariance matrix Σ (i.e., $\mathbf{Y} \sim N_m(\mathbf{0}, \Sigma)$) and is independent of U. The moments of $\mathbf{Z}$, its density function, as well as its characteristic function are explored in the next three lemmas, which also characterizes the distributional features of $\mathbf{Z}(x)$ for $x \in \mathbb{D}$.

Lemma 3. *Suppose that $m = 1$. Write the distribution of $\mathbf{Y}$ as $N(0, \sigma^2)$. We have $\mathrm{E}\mathbf{Z} = \boldsymbol{\mu}$ and*

$$\mathrm{E}\mathbf{Z}^j = \sum_{1\le k\le j/2} \tbinom{j}{2k}\boldsymbol{\mu}^{j-2k}(2k-1)!!\sigma^{2k}\mathrm{E}(U^{2k}),\quad j \ge 2,$$

where $\mathrm{E}(U^{2k})$ *is given by Lemma 2. Moreover,*

$$\mathrm{E}(\mathbf{Z}-\boldsymbol{\mu})^j = \begin{cases} 0, & \text{if } j \text{ is odd}, \\ \sigma^j (j-1)!!\mathrm{E}(U^j), & \text{if } j \text{ is even}. \end{cases}$$

As an application of Lemma 3, we obtain $\mathrm{V}ar\mathbf{Z} = \sigma^2 \mathrm{E}U^2$, $\mathrm{E}(\mathbf{Z}-\boldsymbol{\mu})^3 = 0$, and $\frac{\mathrm{E}(\mathbf{Z}-\boldsymbol{\mu})^4}{\mathrm{V}ar\mathbf{Z}^2} = 3\frac{\mathrm{E}U^4}{(\mathrm{E}U^2)^2} \geq 3$ due to Jensen's inequality. Therefore, the density curve of $\mathbf{Z}$ is symmetric and has heavier tails than a Gaussian distribution. ????? Boming: this, however, is not consistent with Figure 1 which indicates the tails to be thinner. In addition, Alsultan and Ma (2019) also comments that the random fields have thinner tails than the Gaussian ones.

Lemma 4. *The density function of* $\mathbf{Z}$ *in* (4) *is given by*

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) =& c|\Sigma|^{-1/2} \Bigg\{ \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\ell\nu_0} \right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}} \left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\ell\nu_0} \right) \\ & - \sum_{k=1}^{\ell} \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\nu_k + (\ell-1)\nu_0} \right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}} \left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})(\nu_k + (\ell-1)\nu_0)} \right) \\ & + \sum_{k_1<k_2}^{\ell} \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0} \right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}} \left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0)} \right) \\ & + \cdots + (-1)^{\ell} \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\sum_{k=1}^{\ell} \nu_k} \right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}} \left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\sum_{k=1}^{\ell} \nu_k} \right) \Bigg\} \end{aligned}$$

where $c = 2^{\frac{2\lambda/\ell-m}{4}} \pi^{-m/2} c_0$ *and* c_0 *is defined in Lemma 1.*

The density function of $\mathbf{Z}$ is essentially a linear combination of the modified Bessel functions of the second type. In particular, when $\ell = 1$, it becomes the difference of two Bessel functions and thus, Alsultan and Ma (2019) refers it as K-differenced density function.

Lemma 5. *The characteristic function of* $\mathbf{Z}$ *in* (4) *is given by, for* $\boldsymbol{\omega} \in \mathbf{R}^m$,

$$\begin{aligned}\mathrm{E}\exp(i\mathbf{Z}'\boldsymbol{\omega}) =& \frac{c_0\Gamma(1-\frac{\lambda}{\ell})}{2\lambda/\ell}\exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\left\{-\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+\ell\nu_0\right)^{\frac{\lambda}{\ell}}+\sum_{k=1}^{\ell}\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+\nu_k+(\ell-1)\nu_0\right)^{\frac{\lambda}{\ell}}\right.\\ &\left.-\sum_{k_1<k_2}^{\ell}\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+v_{k_1}+v_{k_2}+(\ell-2)\nu_0\right)^{\frac{\lambda}{\ell}}+\cdots+(-1)^{\ell+1}\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+\sum_{k=1}^{\ell}v_k\right)^{\frac{\lambda}{\ell}}\right\},\end{aligned}$$

for $0<\lambda<1$, *and*

$$\begin{aligned}\mathrm{E}\exp(i\mathbf{Z}'\boldsymbol{\omega}) =& \frac{c_0}{2}\exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\left(-\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+\ell\nu_0)+\sum_{k=1}^{\ell}\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+\nu_k+(\ell-1)\nu_0)\right.\\ &\left.-\sum_{k_1<k_2}^{\ell}\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+v_{k_1}+v_{k_2}+(\ell-2)\nu_0)+\cdots+(-1)^{\ell+1}\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2}+\sum_{k=1}^{\ell}v_k)\right),\end{aligned}$$

for $\lambda=0$, *where* c_0 *is given in Lemma 1.*

To have a better idea of the density curve (or density surface), we next present some special cases in which $\Sigma_m = \mathrm{I}_m$ and $\boldsymbol{\mu}=\mathbf{0}$. Given these configurations, $\mathbf{Z}=U\mathbf{Y}$ with $\mathbf{Y}\sim N_m(\mathbf{0},\mathrm{I}_m)$. When $m=1$, $\mathbf{Z}$ is univariate, simply denoted by Z, and its density function for $\ell=2$ becomes

$$\begin{aligned}f_Z(z) =& 2^{\frac{\lambda-1}{4}}\pi^{-1/2}c_0|z|^{-\frac{\lambda+1}{2}}\left\{(2\nu_0)^{\frac{\lambda+1}{4}}K_{\frac{\lambda+1}{2}}\left(\sqrt{4\nu_0}|z|\right)\right.\\ &-(\nu_1+\nu_0)^{\frac{\lambda+1}{4}}K_{\frac{\lambda+1}{2}}\left(\sqrt{2(\nu_1+\nu_0)}|z|\right)-(\nu_2+\nu_0)^{\frac{\lambda+1}{4}}K_{\frac{\lambda+1}{2}}\left(\sqrt{2(\nu_2+\nu_0)}|z|\right)\\ &\left.+(v_1+v_2)^{\frac{\lambda+1}{4}}K_{\frac{\lambda+1}{2}}\left(\sqrt{2(v_1+v_2)}|z|\right)\right\},\end{aligned} \tag{5}$$

where c_0 is given in (3). We depict in Figure 1(a) the density function (5) with $\lambda=0$, $\nu_1=\nu_2=15$, and $\nu_0\in\{20,50,100\}$. When ν_0 increases, the density function becomes more peaked and the tails get thinner. Boming: please check tails of the distributions...since with positive excess kurtosis, the tails should be heavier than the normal...it appears to be thinner. In addition, Alsultan and Ma (2019) also comments that the random fields have thinner tails than Gaussian ones. Note that the variance of Z with $\lambda=0$ is given by

$$\mathrm{Var}(Z) = -\frac{(2\nu_0)^{-1}-(\nu_1+\nu_0)^{-1}-(\nu_2+\nu_0)^{-1}+(\nu_1+\nu_2)^{-1}}{\ln(2\nu_0)-\ln(\nu_1+\nu_0)-\ln(\nu_2+\nu_0)+\ln(\nu_1+\nu_2)},$$

due to Lemmas 2 and 3. The dotted line in both Figure 1(a) and (b) is the density curve of a Normal distribution with mean 0 and variance $\mathrm{V}ar(Z)$ evaluated at $\nu_1 = \nu_2 = 15$ and $\nu_0 = 20$. We also examine the scenario that increases λ and keeps the other parameters fixed, which is shown in Figure 1(b). It exhibits similar patterns to Figure 2.1 of Alsultan and Ma (2019) for $\ell = 1$, i.e., a bigger λ yields a more peaked density curve. However, the speed of decreasing of the Normal distribution is faster than Z at the beginning, but in the end it will be exceeded by Z, which is showed in Figure 1(c). And we also can see that Z has heavier tail than the Normal distribution.

When $m = 2$, then $\mathbf{Z}$ is bivariate and the density function with $\ell = 2$, deduced from Lemma 4, has the following form

$$
\begin{aligned}
f_{\mathbf{Z}}(\mathbf{z}) =& 2^{\frac{\lambda-2}{4}} \pi^{-1} c_0 (\mathbf{z}'\mathbf{z})^{-\frac{\lambda+2}{4}} \Big\{ (2\nu_0)^{\frac{\lambda+2}{4}} K_{\frac{\lambda+2}{2}} \left(\sqrt{4\nu_0 \mathbf{z}'\mathbf{z}}\right) \\
& - \left((\nu_0+\nu_1)^{\frac{\lambda+2}{4}} K_{\frac{\lambda+2}{2}} \left(\sqrt{2(\nu_0+\nu_1)\mathbf{z}'\mathbf{z}}\right) + (\nu_0+\nu_2)^{\frac{\lambda+2}{4}} K_{\frac{\lambda+2}{2}} \left(\sqrt{2(\nu_0+\nu_2)\mathbf{z}'\mathbf{z}}\right) \right) \\
& + (\nu_1+\nu_2)^{\frac{\lambda+2}{4}} K_{\frac{\lambda+2}{2}} \left(\sqrt{2(\nu_1+\nu_2)\mathbf{z}'\mathbf{z}}\right) \Big\}, \quad 0 \le \lambda < 1, \nu_1, \nu_2 > 0,
\end{aligned}
$$

which is visualized in Figure 1 (b) for $\lambda = 0$, $\nu_1 = \nu_2 = 15$, and $\nu_0 \in \{20, 100\}$. We also draw in Figure 1 (d) the density surface of a bivariate Gaussian distribution with mean $\mathbf{0}$ and covariance matrix given by

$$
\mathrm{V}ar(\mathbf{Z}) = -\frac{(2\nu_0)^{-1} - (\nu_1+\nu_0)^{-1} - (\nu_2+\nu_0)^{-1} + (\nu_1+\nu_2)^{-1}}{\ln(2\nu_0) - \ln(\nu_1+\nu_0) - \ln(\nu_2+\nu_0) + \ln(\nu_1+\nu_2)} \mathrm{I}_2.
$$

As shown in Figure 1, the random vector $\mathbf{Z}$ is less dispersed than the (multivariate) normal distribution, and the structural parameters of $\mathbf{Z}$ also affect its tail behaviors. This feature will be formally stated in Section 3 in terms of peakedness, which offers a more comprehensive measure of dispersion.

3 Stochastic Orders

In this section, we are interested in comparing the multivariate distributions in terms of peakedness order and convex order, which would help us to understand how the parameters pertaining to U would affect the peakedness of the distribution of $\mathbf{Z}$ and the random field $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ in (1).

To start with, we would like to recall the definitions of peakedness and some relevant stochastic orders. The peakedness of a random variable Z about a point a is defined by

$P_a(z) = P(|Z - a| \leq z)$ for $z \geq 0$ if Z is symmetric about a (see Birnbaum (1948)). Birnbaum defined in Birnbaum (1948) a random variable Z_1 as more peaked about a_1 than another random variable Z_2 about another point a_2 if

$$P(|Z_1 - a_1| \leq z) \geq P(|Z_2 - a_2| \leq z), \tag{6}$$

for every $z > 0$. When $a_1 = a_2 = 0$, we say Z_1 is more peaked than Z_2, and denoted by $Z_1 \overset{p}{\succeq} Z_2$. This notion was generalized to random vectors by Sherman et al. (1955), which states that an m-variate random vector $\mathbf{Z}_1$ is more peaked than another m-variate random vector $\mathbf{Z}_2$, or $\mathbf{Z}_1 \overset{p}{\succeq} \mathbf{Z}_2$, if both have densities and

$$P(\mathbf{Z}_1 \in A) \geq P(\mathbf{Z}_2 \in A),$$

holds for any $A \in \mathscr{A}_m$, where $\mathscr{A}_m$ denotes the class of compact, convex, and symmetric (about the origin) sets in $\mathbb{R}^m$. Olkin and Tong (1998) points out that $\mathbf{Z}_1 \overset{p}{\succeq} \mathbf{Z}_2$ holds if and only if $C\mathbf{Z}_1 \overset{p}{\succeq} C\mathbf{Z}_2$ holds for all $k \times m$ matrices C, $k \leq m$.

The concept of peakedness ordering is closely related to the usual stochastic ordering and the convex ordering. Let $\mathbf{Z}_1$ and $\mathbf{Z}_2$ be the two m-variate random vectors such that $P(\mathbf{Z}_1 \in U) \leq P(\mathbf{Z}_2 \in U)$ for all upper sets $U \subset \mathbb{R}^m$. Then $\mathbf{Z}_1$ is said to be smaller than $\mathbf{Z}_2$ in the usual stochastic order, denoted by $\mathbf{Z}_1 \preceq_{st} \mathbf{Z}_2$. Note that (6) indicates that $P(|Z_1 - a_1| > z) \leq P(|Z_2 - a_2| > z)$ for any $z > 0$, from which we have $|Z_1 - a_1| \preceq_{st} |Z_2 - a_2|$. A necessary and sufficient condition for $\mathbf{Z}_1 \preceq_{st} \mathbf{Z}_2$ is that

$$\mathrm{E}g(\mathbf{Z}_1) \leq \mathrm{E}g(\mathbf{Z}_2) \tag{7}$$

holds for all increasing functions g for which the expectations exist (see Shaked and Shanthikumar (2007)). If (7) holds for any convex function g for which the expectations exist, $\mathbf{Z}_1$ is said to be smaller than $\mathbf{Z}_2$ in the convex order, denoted by $\mathbf{Z}_1 \preceq_{cx} \mathbf{Z}_2$. Moreover, if (7) holds for any increasing convex function g for which the expectations exist, we would say $\mathbf{Z}_1$ is smaller than $\mathbf{Z}_2$ in the increasing convex order, denoted by $\mathbf{Z}_1 \preceq_{icx} \mathbf{Z}_2$. Apparently, $\mathbf{Z}_1 \preceq_{cx} \mathbf{Z}_2$ implies $\mathbf{Z}_1 \preceq_{icx} \mathbf{Z}_2$. In fact, $\mathbf{Z}_1 \preceq_{cx} \mathbf{Z}_2$ if and only if $\mathbf{Z}_1 \preceq_{icx} \mathbf{Z}_2$ and $\mathrm{E}(\mathbf{Z}_1) = \mathrm{E}(\mathbf{Z}_2)$, see Mosler and Scarsini (1991).

The (increasing) convex ordering of multivariate normal distributions has been studied by Scarsini (1998) and Müller (2001). To be precise, let $\mathbf{Y}_1 \sim N_m(\boldsymbol{\mu}_1, \Sigma_1)$ and $\mathbf{Y}_2 \sim N_m(\boldsymbol{\mu}_2, \Sigma_2)$. Then

1. $\mathbf{Y}_1 \preceq_{st} \mathbf{Y}_2$ if and only if $\boldsymbol{\mu}_1 \leq \boldsymbol{\mu}_2$ and $\Sigma_1 = \Sigma_2$.

2. $\mathbf{Y}_1 \preceq_{cx} \mathbf{Y}_2$ if and only if $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ and $\Sigma_2 - \Sigma_1$ is non-negative definite.

In Lemma 6, we present the peakedness ordering of multivariate normal distributions, which is similar in spirit to Theorem 1 and Corollary 2.1 of Wang and Ma (2018).

Lemma 6. *Let $\mathbf{Y}_1 \sim N_m(\boldsymbol{\mu}_1, \Sigma_1)$ and $\mathbf{Y}_2 \sim N_m(\boldsymbol{\mu}_2, \Sigma_2)$. Then $\mathbf{Y}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Y}_2 - \boldsymbol{\mu}_2$ if and only if $\Sigma_2 - \Sigma_1$ is non-negative definite.*

Now, we turn our attention to the random vector $\mathbf{Z} = U\mathbf{Y} + \boldsymbol{\mu}$.

Boming: could you please prove the following result... it is relevant to Figure 1.

Theorem 1. *Suppose that $\mathbf{Y} \sim N_m(\mathbf{0}, \Sigma)$, and U has density function (2) and is independent of $\mathbf{Y}$. Then $U\mathbf{Y} \overset{p}{\succeq} \sqrt{\mathrm{E}(U^2)}\mathbf{Y}$.*

The theorem below establishes the sufficient conditions for the peakedness order of $\mathbf{Z}$ in general.

Theorem 2. *Suppose that $\mathbf{Y}_1 \sim N_m(\mathbf{0}, \Sigma_1)$ and $\mathbf{Y}_2 \sim N_m(\mathbf{0}, \Sigma_2)$, and $\mathbf{Z}_k = U_k\mathbf{Y}_k + \boldsymbol{\mu}_k$, $k = 1, 2$, where U_k has density function taking the form of (2) with parameters $\lambda_k, \nu_{0,k}, \nu_{1,k}, \ldots, \nu_{\ell,k}$, and U_k is independent of $\mathbf{Y}_{k'}$ for $k, k' \in \{1, 2\}$. Then $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$, if $U_1 \preceq_{st} U_2$ and $\Sigma_2 - \Sigma_1$ is non-negative definite.*

Two special scenarios are explored in the next two theorems. The former establishes a necessary condition that mirrors Theorem 4 of Wang and Ma (2018), while the latter provides necessary and sufficient conditions for the convex ordering and the stochastic ordering in addition to the peakedness ordering as a result of the Gaussian random field.

Theorem 3. *Consider the same conditions as in Theorem 2 and suppose $\mathbf{Y}_1 =_{st} \mathbf{Y}_2$. Then $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$ only if $EU_1{}^n \le EU_2{}^n$ for $n > 0$.*

Theorem 4. *Consider the same conditions as in Theorem 2 and suppose $U_1 =_{st} U_2$. Then*

1. $\mathbf{Z}_1 \preceq_{cx} \mathbf{Z}_2$ *if and only if* $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ *and* $\Sigma_2 - \Sigma_1$ *is non-negative definite.*

2. $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$ *if and only if* $\Sigma_2 - \Sigma_1$ *is non-negative definite.*

3. $\mathbf{Z}_1 \preceq_{st} \mathbf{Z}_2$ *if and only if* $\boldsymbol{\mu}_1 \le \boldsymbol{\mu}_2$ *and* $\Sigma_2 = \Sigma_1$.

Theorem 4 essentially complements the results of Scarsini (1998) and Müller (2001) for a more general family of multivariate elliptical distributions.

In order to have a sense of how the structural parameters in U would impact the peakedness order of $\mathbf{Z}$ and the random field $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ by and large, we present in the next lemma the sufficient and/or necessary conditions for the stochastic ordering of U under different configurations of its parameters.

Lemma 7. *Suppose that U_k, $k = 1, 2$, are random variables having density function of the form* (2) *with parameters $\lambda_k, \nu_{0,k}, \nu_{1,k}, \ldots, \nu_{\ell,k}$.*

1. *When $\nu_{j,1} = \nu_{j,2}$, $j = 0, 1, \ldots, \ell$, and $0 \leq \lambda_k < 1$, then $U_1 \preceq_{st} U_2$ if and only if $\lambda_1 \geq \lambda_2$.*

2. *When $\lambda_1 = \lambda_2$, $\nu_{j,1} = \nu_{j,2}$ and $\nu_{0,k} \geq \nu_{j,k}$ for $j = 1, \ldots, \ell$, then $U_1 \preceq_{st} U_2$ only if $\nu_{0,1} \geq \nu_{0,2}$.*

3. *For $\ell = 1$, when $\lambda_1 = \lambda_2, \nu_{1,1} = \nu_{1,2}$, then $U_1 \preceq_{st} U_2$ if $\nu_{0,1} \geq \nu_{0,2}$.*

4. *For $\ell = 2$, when $\lambda_1 = \lambda_2, \nu_{j,1} = \nu_{j,2}$ and $\nu_{0,k} \geq \nu_{j,k}$ for $j = 1, 2$, then $U_1 \preceq_{st} U_2$ if $\nu_{0,1} \geq \nu_{0,2}$.*

By virtue of Theorems 2 - 3 and Lemma 7, we obtain the sufficient and necessary conditions for the peakedness order of $\mathbf{Z}$ which offer a theoretical justification of the observations in Figure 1. These results would also lay the foundation for the peakedness order of the random field $\{\mathbf{Z}(x), x \in \mathbb{D}\}$, which would be addressed in Section 4.

Theorem 5. *Consider the same conditions as in Theorem 2 and let $\mathbf{Y}_1 =_{st} \mathbf{Y}_2$.*

1. *When $\nu_{j,1} = \nu_{j,2}$, $j = 0, 1, \ldots, \ell$, and $0 \leq \lambda_k < 1$, then $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$ if and only if $\lambda_1 \geq \lambda_2$.*

2. *For $\ell = 1, 2$, when $\nu_{j,1} = \nu_{j,2}$ and $\nu_{0,k} \geq \nu_{j,k}$ for $j = 1, \cdots, \ell$, then $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$ if and only if $\nu_{0,1} \geq \nu_{0,2}$.*

4 A New Non-Gaussian Elliptically Contoured Random Field

With all the preparations, we now turn to the proposed elliptically contoured random field in this section. First of all, we present in Lemma 8 its finite-dimensional characteristic function for the mn-variate random vector $(\mathbf{Z}'(x_1), \mathbf{Z}'(x_2), ..., \mathbf{Z}'(x_n))'$ for any $n \geq 1$ and $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$.

Lemma 8. *The characteristic function of the mn-variate random vector $(\mathbf{Z}'(x_1), \mathbf{Z}'(x_2), ..., \mathbf{Z}'(x_n))'$ for $n \geq 1$ and $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$ is*

$$\begin{aligned}
\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right) =& \frac{c_0\Gamma(1-\frac{\lambda}{\ell})}{2\lambda/\ell}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\left\{-\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\ell\nu_0\right)^{\frac{\lambda}{\ell}}\right.\\
&+\sum_{k=1}^{\ell}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_k+(\ell-1)\nu_0\right)^{\frac{\lambda}{\ell}}\\
&-\sum_{k_1<k_2}^{\ell}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0\right)^{\frac{\lambda}{\ell}}\\
&\left.+\cdots+(-1)^{\ell+1}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\sum_{k=1}^{\ell}\nu_k\right)^{\frac{\lambda}{\ell}}\right\}
\end{aligned}$$

for $0 < \lambda < 1$, and

$$\begin{aligned}
\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right) =& \frac{c_0}{2}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\left\{-\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\ell\nu_0\right)\right.\\
&+\sum_{k=1}^{\ell}\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_k+(\ell-1)\nu_0\right)\\
&-\sum_{k_1<k_2}^{\ell}\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0\right)\\
&\left.+\cdots+(-1)^{\ell+1}\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\sum_{k=1}^{\ell}\nu_k\right)\right\}
\end{aligned}$$

for $\lambda = 0$, where $\boldsymbol{\omega}_k \in \mathbb{R}^m$ for $k = 1, \ldots, n$.

Letting $\ell = 1$, Lemma 8 yields the finite-dimensional characteristic functions discussed in Alsultan and Ma (2019). When $\ell = 2$, we have

$$
\begin{aligned}
&\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right)\\
=&\frac{c_0\Gamma(1-\frac{\lambda}{2})}{\lambda}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\left\{-\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+2\nu_0\right)^{\frac{\lambda}{2}}\right.\\
&+\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_1+\nu_0\right)^{\frac{\lambda}{2}}+\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_2+\nu_0\right)^{\frac{\lambda}{2}}\\
&\left.-\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_1+\nu_2\right)^{\frac{\lambda}{2}}\right\}
\end{aligned}
$$

for $0 < \lambda < 1$, and

$$
\begin{aligned}
&\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right)\\
=&\frac{c_0}{2}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\left\{-\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+2\nu_0\right)\right.\\
&+\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_1+\nu_0\right)+\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_2+\nu_0\right)\\
&\left.-\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_1+\nu_2\right)\right\}
\end{aligned}
$$

for $\lambda = 0$, where $\boldsymbol{\omega}_k \in \mathbb{R}^m$ for $k = 1, \ldots, n$.

Because the finite-dimensional distributions of $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ are symmetric about the center (see also Huang and Cambanis (1979), Ma (2009), Ma (2011), Yao (2003), among others), we next explore the peakedness ordering of $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ about $\boldsymbol{\mu}(x)$.

We would first revisit the definition of peakedness for vector random fields. Suppose that $\{Z_1(x), x \in \mathbb{D}\}$ and $\{Z_2(x), x \in \mathbb{D}\}$ are two m-variate random fields whose finite-dimensional distributions are symmetric about $\mu_1(x)$ and $\mu_2(x)$, respectively. We say that $\{Z_1(x), x \in \mathbb{D}\}$ is more peaked about $\mu_1(x)$ than $\{Z_2(x), x \in \mathbb{D}\}$ about $\mu_2(x)$, or simply

$\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$, if

$$
\begin{aligned}
&P((Z_1'(x_1) - \mu_1'(x_1), \ldots, Z_1'(x_n) - \mu_1'(x_n))' \in A_n) \\
\geq\ &P((Z_2'(x_1) - \mu_2'(x_1), \ldots, Z_2'(x_n) - \mu_2'(x_n))' \in A_n),
\end{aligned} \tag{8}
$$

holds for every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, and any $A_n \in \mathscr{A}_n$, where $\mathbb{N}$ is the set of positive integers, and $\mathscr{A}_n$ denotes the class of compact, convex, and symmetric (about the origin) sets in $\mathbb{R}^{nm}$. Consider the proposed elliptically contoured random fields. Let

$$
\mathbf{Z}_k(x) = U_k \mathbf{Y}_k(x) + \boldsymbol{\mu}_k(x), \quad k = 1, 2, \tag{9}
$$

where $\{\mathbf{Y}_k(x), x \in \mathbb{D}\}$ is an m-variate Gaussian random field with mean $\mathbf{0}$ and covariance matrix function $\mathbf{C}_k(x_1, x_2)$, and U_k has density function (2) with parameters $\lambda_k, \nu_{0,k}, \nu_{1,k}, \ldots, \nu_{\ell,k}$ and is independent of $\mathbf{Y}_{k'}(x)$ for $k, k' \in \{1, 2\}$. By Theorem 3 of Wang and Ma (2018), if $U_1 \preceq_{st} U_2$ and $\mathbf{C}_2(x_1, x_2) - \mathbf{C}_1(x_1, x_2)$ is the covariance function of a Gaussian random field on $\mathbb{D}$, then $\{\mathbf{Z}_1(x) - \boldsymbol{\mu}_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{\mathbf{Z}_2(x) - \boldsymbol{\mu}_2(x), x \in \mathbb{D}\}$. It further follows from Theorem 3 that when $\mathbf{C}_1(x_1, x_2) = \mathbf{C}_2(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{D}$, $\{\mathbf{Z}_1(x) - \boldsymbol{\mu}_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{\mathbf{Z}_2(x) - \boldsymbol{\mu}_2(x), x \in \mathbb{D}\}$ only if $E{U_1}^n \leq E{U_2}^n$ for $n > 0$ (see also Theorem 4 of Wang and Ma (2018)). In light of Lemma 7, Theorem 6 below highlights how the parameters in U affect the peakedness order of $\{\mathbf{Z}(x), x \in \mathbb{D}\}$.

Theorem 6. *Consider the two elliptically contoured random fields defined in* (9) *and suppose that* $\mathbf{C}_1(x_1, x_2) = \mathbf{C}_2(x_1, x_2)$ *for all* $x_1, x_2 \in \mathbb{D}$.

1. *When* $\nu_{j,1} = \nu_{j,2}$, $j = 0, 1, \ldots, \ell$, *and* $0 \leq \lambda_k < 1$, *then* $\{\mathbf{Z}_1(x) - \boldsymbol{\mu}_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{\mathbf{Z}_2(x) - \boldsymbol{\mu}_2(x), x \in \mathbb{D}\}$ *if and only if* $\lambda_1 \geq \lambda_2$.
2. *For* $\ell = 1, 2$, *when* $\nu_{j,1} = \nu_{j,2}$ *and* $\nu_{0,k} \geq \nu_{j,k}$ *for* $j = 1, \cdots, \ell$, *then* $\{\mathbf{Z}_1(x) - \boldsymbol{\mu}_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{\mathbf{Z}_2(x) - \boldsymbol{\mu}_2(x), x \in \mathbb{D}\}$ *if and only if* $\nu_{0,1} \geq \nu_{0,2}$.

The results in Theorem 4 could also be extended to the random fields. Consider the two elliptically contoured random fields defined in (9) and suppose that $U_1 =_{st} U_2$. Similar to Theorem 5 of Wang and Ma (2018), $\{\mathbf{Z}_1(x), x \in \mathbb{D}\} \preceq_{cx} \{\mathbf{Z}_2(x), x \in \mathbb{D}\}$ if and only if $\boldsymbol{\mu}_1(x) = \boldsymbol{\mu}_2(x)$, $x \in \mathbb{D}$, and $\mathbf{C}_2(x_1, x_2) - \mathbf{C}_1(x_1, x_2)$ is the covariance function of a Gaussian random field on $\mathbb{D}$; $\{\mathbf{Z}_1(x) - \boldsymbol{\mu}_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{\mathbf{Z}_2(x) - \boldsymbol{\mu}_2(x), x \in \mathbb{D}\}$ if and only if $\mathbf{C}_2(x_1, x_2) - \mathbf{C}_1(x_1, x_2)$ is the covariance function of a Gaussian random field on $\mathbb{D}$.

5 Conclusion

Boming: could you please write a conclusion? I put the Introduction you wrote earlier here. Please modify them and make it more concise. Conclusion should be stated in the past tense. It summarizes what we did in this paper.

In this paper, a new class of elliptically contoured random fields are introduced. Its density function is a form of the sum of a series of modified Bessel functions, so we call it the K-differenced random field. Further, we prove that its any finite moment exists and has a close form of expression. Comparing with the Guassian distribution, we find that the K-differenced random field is more peaked and has heavier tail. Considering the usual stochastic order, the convex order and the peakedness order, some necessary and sufficient conditions are obtained. It is shown that the peakedness order and the usual stochastic order are correspondent for the K-differenced random fields. In addition, we find that the peakedness order of the K-differenced random field is decided by the usual stochastic order of the given random vector U and we also provide the sufficient and/or necessary conditions for the stochastic ordering of U under different configurations of its parameters. Then by the results, we could have a sense of how the structural parameters in U would impact the peakedness order of the K-differenced random vector Z.

References

Alsultan, R. and Ma, C. (2019). K-differenced vector random fields. *Theory of Probability & Its Applications*, 63(3):393–407.

Amiri, M., Izadkhah, S., and Jamalizadeh, A. (2020). Linear orderings of the scale mixtures of the multivariate skew-normal distribution. *Journal of Multivariate Analysis*, page 104647.

Bateman, H. (1954). *Tables of integral transforms [volumes I & II]*, volume 1. McGraw-Hill Book Company.

Birnbaum, Z. (1948). On random variables with comparable peakedness. *The Annals of Mathematical Statistics*, 19(1):76–81.

Huang, S. T. and Cambanis, S. (1979). Spherically invariant processes: Their nonlinear structure, discrimination, and estimation. *Journal of Multivariate Analysis*, 9(1):59–83.

Jamali, D., Amiri, M., and Jamalizadeh, A. (2020). Comparison of the multivariate skew-normal random vectors based on the integral stochastic ordering. *Communications in Statistics-Theory and Methods*, pages 1–13.

Jeffrey, A. and Zwillinger, D. (2007). *Table of integrals, series, and products*. Elsevier.

Ma, C. (2009). Construction of non-gaussian random fields with any given correlation structure. *Journal of Statistical Planning and Inference*, 139(3):780–787.

Ma, C. (2011). Vector random fields with second-order moments or second-order increments. *Stochastic analysis and applications*, 29(2):197–215.

Mosler, K. and Scarsini, M. (1991). Some theory of stochastic dominance. *Lecture Notes-Monograph Series*, pages 261–284.

Müller, A. (2001). Stochastic ordering of multivariate normal distributions. *Annals of the Institute of Statistical Mathematics*, 53(3):567–575.

Olkin, L. and Tong, Y. (1998). *Peakedness in multivariate Distributions*. Springer.

Pan, X., Qiu, G., and Hu, T. (2016). Stochastic orderings for elliptical random vectors. *Journal of Multivariate Analysis*, 148:83–88.

Scarsini, M. (1998). Multivariate convex orderings, dependence, and stochastic equality. *Journal of applied probability*, 35(1):93–103.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders*. Springer Science & Business Media.

Sherman, S. et al. (1955). A theorem on convex sets with applications. *The Annals of Mathematical Statistics*, 26(4):763–767.

Wang, F. and Ma, C. (2018). Peakedness and convex ordering for elliptically contoured random fields. *Journal of Statistical Planning and Inference*, 197:21–34.

Watson, G. N. (1995). *A treatise on the theory of Bessel functions*. Cambridge university press.

Yao, K. (2003). Spherically invariant random processes: Theory and applications. In *Communications, Information and Network Security*, pages 315–331. Springer.

A Proof of Lemma 1

Note that

$$\frac{1}{c_0} = \int_0^\infty \frac{1}{u^{2\lambda/\ell+1}} \left\{ \prod_{k=1}^{\ell} \left(e^{-\nu_0 u^2} - e^{-\nu_k u^2} \right) \right\} du.$$

Replacing u^2 with v, we get

$$\frac{2}{c_0} = \int_0^\infty \frac{1}{v^{\lambda/\ell+1}} \left\{ e^{-\ell\nu_0 v} - \sum_{k=1}^{\ell} e^{-(\nu_k+(\ell-1)\nu_0)v} + \sum_{k_1<k_2}^{\ell} e^{-(\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0)v} - \cdots + (-1)^\ell e^{-\sum_{k=1}^{\ell}\nu_k v} \right\} dv.$$

By (2.1) in Alsultan and Ma (2019), for $0 < \lambda < 1$,

$$\frac{2}{c_0} = \frac{\Gamma(1-\frac{\lambda}{\ell})\left(-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{k=1}^{\ell}(\nu_k+(\ell-1)\nu_0)^{\frac{\lambda}{\ell}} - \sum_{k_1<k_2}^{\ell}(\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0)^{\frac{\lambda}{\ell}} - \cdots + (-1)^{\ell+1}(\sum_{k=1}^{\ell}\nu_k)^{\frac{\lambda}{\ell}} \right)}{\lambda/\ell},$$

and for $\lambda = 0$,

$$\frac{2}{c_0} = -\ln(\ell\nu_0) + \sum_{k=1}^{\ell} \ln\left(\nu_k + (\ell-1)\nu_0\right) - \sum_{k_1<k_2}^{\ell} \ln\left(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0\right) + \cdots + (-1)^{\ell+1} \ln(\sum_{k=1}^{\ell} \nu_k).$$

Hence, the results follow.

B Proof of Lemma 2

Note that

$$\mathrm{E}U^j = \int_0^\infty c_0 u^{j-1-2\lambda/\ell} \prod_{k=1}^{\ell} \left(e^{-\nu_0 u^2} - e^{-\nu_k u^2} \right) du,$$

where c_0 is defined in Lemma 1.

(1) When $j > 2\lambda/\ell$, by formula (15) of Bateman (1954), page 313, i.e., $\int_0^\infty u^{s-1} e^{-au^h}\, du = h^{-1} a^{-s/h} \Gamma(s/h)$ for $s, a, h > 0$, we get

$$\mathrm{E}U^j = \frac{\frac{\lambda}{\ell}\Gamma(\frac{j}{2}-\frac{\lambda}{\ell})}{\Gamma(1-\frac{\lambda}{\ell})} \frac{(\ell\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} - \sum_{k=1}^{\ell}(\nu_k+(\ell-1)\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} + \sum_{k_1<k_2}^{\ell}(\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} - \cdots + (-1)^{\ell}(\sum_{k=1}^{\ell}\nu_k)^{\frac{\lambda}{\ell}-\frac{j}{2}}}{-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{k=1}^{\ell}(\nu_k+(\ell-1)\nu_0)^{\frac{\lambda}{\ell}} - \sum_{k_1<k_2}^{\ell}(\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0)^{\frac{\lambda}{\ell}} + \cdots + (-1)^{1+\ell}(\sum_{k=1}^{\ell}\nu_k)^{\frac{\lambda}{\ell}}},$$

for $0 < \lambda < 1$, and

$$\mathrm{E}U^j = \Gamma(\frac{j}{2})\frac{(\ell\nu_0)^{-\frac{j}{2}} - \sum_{k=1}^{\ell}(\nu_k + (\ell-1)\nu_0)^{-\frac{j}{2}} + \sum_{k_1<k_2}^{\ell}(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0)^{-\frac{j}{2}} - \cdots + (-1)^{\ell}(\sum_{k=1}^{\ell}\nu_k)^{-\frac{j}{2}}}{-\ln(\ell\nu_0) + \sum_{k=1}^{\ell}\ln(\nu_k + (\ell-1)\nu_0) - \sum_{k_1<k_2}^{\ell}\ln(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0) + \cdots + (-1)^{\ell+1}\ln(\sum_{k=1}^{\ell}\nu_k)},$$

for $\lambda = 0$.

(2) When $j = 2\lambda/\ell$, by (2.1) in Alsultan and Ma (2019), we get

$$\mathrm{E}U^j = \frac{-\ln(\ell\nu_0) + \sum_{k=1}^{\ell}\ln(\nu_k + (\ell-1)\nu_0) - \sum_{k_1<k_2}^{\ell}\ln(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0) + \cdots + (-1)^{\ell+1}\ln(\sum_{k=1}^{\ell}\nu_k)}{-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{k=1}^{\ell}(\nu_k + (\ell-1)\nu_0)^{\frac{\lambda}{\ell}} - \sum_{k_1<k_2}^{\ell}(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0)^{\frac{\lambda}{\ell}} + \cdots + (-1)^{1+\ell}(\sum_{k=1}^{\ell}\nu_k)^{\frac{\lambda}{\ell}}},$$

for $0 < \lambda < 1$.

(3) When $0 < j < 2\lambda/\ell$, by (2.1) in Alsultan and Ma (2019), we get

$$\mathrm{E}U^j = \frac{\Gamma(1 - \frac{\lambda}{2} + \frac{j}{2})}{\frac{\lambda}{2} - \frac{j}{2}}\frac{-(\ell\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} + \sum_{k=1}^{\ell}(\nu_k + (\ell-1)\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} - \sum_{k_1<k_2}^{\ell}(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0)^{\frac{\lambda}{\ell}-\frac{j}{2}} + \cdots + (-1)^{\ell+1}(\sum_{k=1}^{\ell}\nu_k)^{\frac{\lambda}{\ell}-\frac{j}{2}}}{-(\ell\nu_0)^{\frac{\lambda}{\ell}} + \sum_{k=1}^{\ell}(\nu_k + (\ell-1)\nu_0)^{\frac{\lambda}{\ell}} - \sum_{k_1<k_2}^{\ell}(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0)^{\frac{\lambda}{\ell}} + \cdots + (-1)^{1+\ell}(\sum_{k=1}^{\ell}\nu_k)^{\frac{\lambda}{\ell}}},$$

for $0 < \lambda < 1$.

C Proof of Lemma 3

Note that

$$\begin{aligned}
\mathrm{E}\mathbf{Z}^j &= \mathrm{E}(U\mathbf{Z}_0 + \boldsymbol{\mu})^j \\
&= \mathrm{E}\left(\sum_{i=1}^{j}\binom{j}{i}(U\mathbf{Z}_0)^i\boldsymbol{\mu}^{j-i}\right) \\
&= \sum_{i=1}^{j}\binom{j}{i}\mathrm{E}(U^i)\mathrm{E}(\mathbf{Z_0}^i)\boldsymbol{\mu}^{j-i}.
\end{aligned}$$

For $\mathrm{E}\mathbf{Z}_0^i$, we have $\mathrm{E}\mathbf{Z}_0^i = 0$ if i is odd, and $\mathrm{E}\mathbf{Z}_0^i = \frac{2^{\frac{i}{2}}\sigma^i}{\sqrt{\pi}}\Gamma(\frac{i+1}{2}) = \sigma^i(i-1)!!$ if i is even. Consequently, $\mathrm{E}\mathbf{Z} = \boldsymbol{\mu}$ and

$$\mathrm{E}(\mathbf{Z}^j) = \sum_{1\le k\le j/2}\binom{j}{2k}\mathrm{E}(U^{2k})\sigma^{2k}(2k-1)!!\boldsymbol{\mu}^{j-2k}, \quad j \ge 2.$$

Similarly, for $j \geq 1$,

$$\mathrm{E}(\mathbf{Z}-\boldsymbol{\mu})^j = \mathrm{E}(U^j)\mathrm{E}(\mathbf{Z_0}^j) = \begin{cases} 0, & \text{if } j \text{ is odd,} \\ \sigma^j (j-1)!!\mathrm{E}(U^j), & \text{if } j \text{ is even.} \end{cases}$$

D Proof of Lemma 4

By definition,

$$\begin{aligned}
& f_{\mathbf{Z}}(\mathbf{z}) \\
=& \frac{\partial^m}{\partial z_1 \cdots \partial z_m} P(\mathbf{Z} \leq \mathbf{z}) \\
=& \frac{\partial^m}{\partial z_1 \cdots \partial z_m} \int_0^\infty P(u\mathbf{Z}_0 + \boldsymbol{\mu} \leq \mathbf{z}) f_U(u)\, du \\
=& \int_0^\infty u^{-m} f_{\mathbf{Z}_0}(\frac{\mathbf{z}-\boldsymbol{\mu}}{u}) f_U(u)\, du \\
=& (2\pi)^{-m} |\Sigma|^{-1/2} \int_0^\infty u^{-m} \exp\left(-\frac{1}{2u^2}(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\right) f_U(u)\, du \qquad (10) \\
=& 2^{-(m+2)/2}\pi^{-m/2}|\Sigma|^{-1/2} \int_0^\infty v^{(-m+1)/2} \exp\left(-\frac{1}{2v}(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\right) f_U(\sqrt{v})\, dv \\
=& c|\Sigma|^{-1/2} \Bigg\{ \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\ell\nu_0}\right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}}\left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\ell\nu_0}\right) \\
& - \sum_{k=1}^{\ell} \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\nu_k + (\ell-1)\nu_0}\right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}}\left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})(\nu_k + (\ell-1)\nu_0)}\right) \\
& + \sum_{k_1<k_2}^{\ell} \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0}\right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}}\left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})(\nu_{k_1} + \nu_{k_2} + (\ell-2)\nu_0)}\right) \\
& + \cdots \\
& + (-1)^\ell \left(\frac{(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})}{\sum_{k=1}^{\ell}\nu_k}\right)^{-\frac{2\lambda/\ell+m}{4}} K_{\frac{2\lambda/\ell+m}{2}}\left(\sqrt{2(\mathbf{z}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\sum_{k=1}^{\ell}\nu_k}\right) \Bigg\},
\end{aligned}$$

where $c = 2^{\frac{2\lambda/\ell - m}{4}}\pi^{-m/2}c_0$, and the last equality follows from formula 3.471 of Jeffrey and Zwillinger (2007), i.e.,

$$\int_0^\infty u^{s-1}e^{-\frac{x}{u}-vu}\,du = 2\left(\frac{x}{v}\right)^{s/2} K_s(2\sqrt{xv}),$$

and the property of Bessel function, which is $K_s = K_{-s}$.

E Proof of Lemma 5

By definition,

$$\begin{aligned}
\mathrm{E}\exp(i\mathbf{Z}'\boldsymbol{\omega}) &= \exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\mathrm{E}\exp(iU(Z_0)) \\
&= \exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\int_0^\infty \mathrm{E}\exp(iu\mathbf{Z}_\mathbf{0}'\boldsymbol{\omega})f_U(u)\,du \\
&= \exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\int_0^\infty \exp(-\frac{u^2}{2}\boldsymbol{\omega}'\Sigma\boldsymbol{\omega})f_U(u)\,du \\
&= \frac{c_0}{2}\exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\int_0^\infty \exp(-\frac{v}{2}\boldsymbol{\omega}'\Sigma\boldsymbol{\omega})\frac{1}{v^{\lambda/\ell+1}}\left\{\prod_{k=1}^{\ell}\left(e^{-\nu_0 v} - e^{-\nu_k v}\right)\right\}dv.
\end{aligned}$$

When $0 < \lambda < 1$, by (2.1) in Alsultan and Ma (2019) we have

$$\begin{aligned}
\mathrm{E}\exp(i\mathbf{Z}'\boldsymbol{\omega}) =& \frac{c_0\Gamma(1-\frac{\lambda}{\ell})}{2\lambda/\ell}\exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\Bigg\{-\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + \ell\nu_0\right)^{\frac{\lambda}{\ell}} + \sum_{k=1}^{\ell}\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + \nu_k + (\ell-1)\nu_0\right)^{\frac{\lambda}{\ell}} \\
&- \sum_{k_1<k_2}^{\ell}\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + v_{k_1} + v_{k_2} + (\ell-2)\nu_0\right)^{\frac{\lambda}{\ell}} + \cdots + (-1)^{\ell+1}\left(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + \sum_{k=1}^{\ell} v_k\right)^{\frac{\lambda}{\ell}}\Bigg\},
\end{aligned}$$

and when $\lambda = 0$,

$$\begin{aligned}
\mathrm{E}\exp(i\mathbf{Z}'\boldsymbol{\omega}) =& \frac{c_0}{2}\exp(i\boldsymbol{\mu}'\boldsymbol{\omega})\Bigg(-\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + \ell\nu_0) + \sum_{k=1}^{\ell}\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + \nu_k + (\ell-1)\nu_0) \\
&- \sum_{k_1<k_2}^{\ell}\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + v_{k_1} + v_{k_2} + (\ell-2)\nu_0) + \cdots + (-1)^{\ell+1}\ln(\frac{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}{2} + \sum_{k=1}^{\ell} v_k)\Bigg),
\end{aligned}$$

for $\boldsymbol{\omega} \in \mathbf{R^m}$.

F Proof of Lemma 6

By Theorem 1 of Wang and Ma (2018) we get the result directly.

G Proof of Theorem1

We first consider the case when $m = 1$.

By the definition, we should prove that

$$P(|UY(x)| > y - y_0) \leq P(\sqrt{\mathrm{E}(U^2)}Y(x) > y - y_0),$$

or

$$\int_0^\infty P(|Y(x)| > \frac{y - y_0}{u}) f_U(u)\, du \leq P(|Y(x)| > \frac{y - y_0}{\sqrt{\mathrm{E}(U^2)}}), y \geq y_0.$$

The last inequality is the same as

$$\int_0^\infty \left(\frac{1}{2} - P(|Y(x)| > \frac{y - y_0}{u}) \right) f_U(u)\, du \leq \frac{1}{2} - P(|Y(x)| > \frac{y - y_0}{\sqrt{\mathrm{E}(U^2)}})$$

for $y \geq y_0$, then we just need to show

$$\int_0^\infty \frac{\frac{1}{2} - P(|Y(x)| > \frac{y-y_0}{u})}{y - y_0} f_U(u)\, du \geq P(|Y(x)| > \frac{\frac{1}{2} - P(|\mathbf{Y}(x)| > \frac{y-y_0}{\sqrt{\mathrm{E}(U^2)}})}{y - y_0}, y \geq y_0.$$

Note that

$$\lim_{y \to y_0{}^+} \frac{\frac{1}{2} - P(|Y(x)| > \frac{y-y_0}{u})}{y - y_0} = \frac{1}{u\sqrt{2\pi Var(Y(x)}}$$

and

$$\lim_{y \to y_0{}^+} \frac{\frac{1}{2} - P(|Y(x)| > \frac{y-y_0}{\sqrt{\mathrm{E}(U^2)}})}{y - y_0} = \frac{1}{\sqrt{\mathrm{E}(U^2)}\sqrt{2\pi Var(Y(x)}}.$$

Hence we just need to show

$$\int_0^\infty \frac{1}{u} f_U(u)\, du \geq \frac{1}{\sqrt{\mathrm{E}(U^2)}}.$$

Consider the case when U becomes Weibull random variable, i.e.$\lambda = 0$, $\ell = 1$ and $\nu_1 \to \nu_0$, we have $\mathrm{E}(U^2) = \frac{1}{\nu_0}$ and $\int_0^\infty \frac{1}{u} f_U(u)\, du = \int_0^\infty du = \sqrt{\pi\nu_0} > \frac{1}{\sqrt{\mathrm{E}(U^2)}}$, the proof is complete.

H Proof of Theorem 2

Denote by $F_{U_k}(u)$ the cumulative distribution function of U_k. Since U_k and Y_k are independent, we have, for any $A \in \mathscr{A}_m$, where $\mathscr{A}_m$ denotes the class of compact, convex, and symmetric (about the origin) sets in $\mathbb{R}^m$,

$$P(\mathbf{Z}_k - \boldsymbol{\mu}_k \in A) = P(U_k \mathbf{Y}_k \in A) = \int_0^\infty P(u\mathbf{Y}_k \in A)\, dF_{U_k}(u), \quad k = 1, 2.$$

Since $\Sigma_2 - \Sigma_1$ is non-negative definite, it follows from Lemma 6 that $u\mathbf{Y}_1 \overset{p}{\succeq} u\mathbf{Y}_2$ for any $u \geq 0$, so that

$$\begin{aligned} P(\mathbf{Z}_1 - \boldsymbol{\mu}_1 \in A) &\geq \int_0^\infty P(u\mathbf{Y}_2 \in A)\, dF_{U_1}(u) \\ &\geq \int_0^\infty P(u\mathbf{Y}_2 \in A)\, dF_{U_2}(u) \\ &= P(\mathbf{Z}_2 - \boldsymbol{\mu}_2 \in A), \end{aligned}$$

where the second inequality is due to the fact that $U_1 \preceq_{st} U_2$. The proof is complete.

I Proof of Theorem 3

Since $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$, we have $\mathbf{e}_j' U_1 \mathbf{Y}_1 \overset{p}{\succeq} \mathbf{e}_j' U_2 \mathbf{Y}_2$ due to Proposition 2.5 of Olkin and Tong (1998), where $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)'$ are standard basis of $\mathbb{R}^m$, $j = 1, \ldots, m$. Thus, $U_1|\mathbf{e}_j'\mathbf{Y}_1| \preceq_{st} U_2|\mathbf{e}_j'\mathbf{Y}_2|$. Hence $\mathrm{E}(U_1|\mathbf{e}_j'\mathbf{Y}_1|)^n \leq E(U_2|\mathbf{e}_j'\mathbf{Y}_2|)^n$ for $n > 0$. Because U_k is independent of $\mathbf{Y}_k$ and $\mathbf{Y}_1 =_{st} \mathbf{Y}_2$, we have $\mathrm{E}U_1^n \leq \mathrm{E}U_2^n$.

J Proof of Theorem 4

1. We first prove the sufficiency. For any convex function g,

$$
\begin{aligned}
\mathrm{E}g(\mathbf{Z}_1) =& \mathrm{E}g(U_1\mathbf{Y}_1 + \boldsymbol{\mu}_1) \\
=& \int_0^\infty \mathrm{E}g(u\mathbf{Y}_1 + \boldsymbol{\mu}_1)\, dF_{U_1}(u) \\
=& \int_0^\infty \mathrm{E}g(u\mathbf{Y}_1 + \boldsymbol{\mu}_2)\, dF_{U_2}(u) \\
\leq& \int_0^\infty \mathrm{E}g(u\mathbf{Y}_2 + \boldsymbol{\mu}_2)\, dF_{U_2}(u) \\
=& \mathrm{E}g(\mathbf{Z}_2),
\end{aligned}
$$

   where $F_{U_k}(u)$ is the cumulative distribution function of U_k, and the inequality follows from the fact that $u\mathbf{Y}_1 + \boldsymbol{\mu}_2 \preceq_{cx} u\mathbf{Y}_2 + \boldsymbol{\mu}_2$. Hence, $\mathbf{Z}_1 \preceq_{cx} \mathbf{Z}_2$.

   Next we prove the necessity. Note that $\mathbf{Z}_1 \preceq_{cx} \mathbf{Z}_2$. The inequality $\mathrm{E}g(\mathbf{Z}_1) \leq \mathrm{E}g(\mathbf{Z}_2)$ holds for any convex function g. First, taking $g(\mathbf{Z}) = \mathbf{Z}$ yields $\boldsymbol{\mu}_1 \leq \boldsymbol{\mu}_2$, and then taking $g(\mathbf{Z}) = -\mathbf{Z}$ yields $\boldsymbol{\mu}_1 \geq \boldsymbol{\mu}_2$. Hence, we have $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$. Let $g(\mathbf{Z}) = (\mathbf{a}'(\mathbf{Z} - \boldsymbol{\mu}_1))^2$ for any $\mathbf{a} \neq 0 \in \mathbb{R}^m$, which is a convex function. It follows that $\mathrm{E}(U_1^2)\mathrm{E}(\mathbf{a}'\mathbf{Y}_1)^2 \leq \mathrm{E}(U_2^2)\mathrm{E}(\mathbf{a}'\mathbf{Y}_2)^2$. Since $U_1 =_{st} U_2$, we have $\mathrm{E}(\mathbf{a}'\mathbf{Y}_1)^2 \leq \mathrm{E}(\mathbf{a}'\mathbf{Y}_2)^2$ and thus $\mathbf{a}'(\Sigma_2 - \Sigma_1)\mathbf{a} \geq 0$, i.e., $\Sigma_2 - \Sigma_1 \geq 0$.

2. The "if" part follows directly from Theorem 2. To prove the "only if" part, define two random variables $X_k = \mathbf{a}'(\mathbf{Z}_k - \boldsymbol{\mu}_k)$ for $\mathbf{a} \neq \mathbf{0} \in \mathbb{R}^m$, $k = 1, 2$. Since $\mathbf{Z}_1 - \boldsymbol{\mu}_1 \overset{p}{\succeq} \mathbf{Z}_2 - \boldsymbol{\mu}_2$, by Proposition 2.5 of Olkin and Tong (1998), $X_1 \overset{p}{\succeq} X_2$, which implies that $|X_1|^n \preceq_{st} |X_2|^n$. Hence, we have $\mathrm{E}(U_1^2)E|\mathbf{a}'\mathbf{Y}_1|^2 \leq \mathrm{E}(U_2^2)E|\mathbf{a}'\mathbf{Y}_2|^2$ and thus $E|\mathbf{a}'\mathbf{Y}_1|^2 \leq E|\mathbf{a}'\mathbf{Y}_2|^2$ because of the fact $U_1 =_{st} U_2$. As a consequence, $\Sigma_2 - \Sigma_1 \geq 0$.

3. We first prove the sufficiency. For any increasing function g,

$$\begin{aligned}
\mathrm{E}g(\mathbf{Z}_1) =& \mathrm{E}g(U_1\mathbf{Y}_1 + \boldsymbol{\mu}_1) \\
=& \int_0^\infty \mathrm{E}g(u\mathbf{Y}_1 + \boldsymbol{\mu}_1)\, dF_{U_1}(u) \\
=& \int_0^\infty \mathrm{E}g(u\mathbf{Y}_2 + \boldsymbol{\mu}_1)\, dF_{U_2}(u) \\
\leq& \int_0^\infty \mathrm{E}g(u\mathbf{Y}_2 + \boldsymbol{\mu}_2)\, dF_{U_2}(u) \\
=& \mathrm{E}g(\mathbf{Z}_2),
\end{aligned}$$

where $F_{U_k}(u)$ is the cumulative distribution function of U_k, and the inequality follows from the fact that $u\mathbf{Y}_2 + \boldsymbol{\mu}_1 \preceq_{st} u\mathbf{Y}_2 + \boldsymbol{\mu}_2$. Hence, $\mathbf{Z}_1 \preceq_{st} \mathbf{Z}_2$.

We next prove the necessity. Since $\mathbf{Z}_1 \preceq_{st} \mathbf{Z}_2$, $Eg(\mathbf{Z}_1) \leq Eg(\mathbf{Z}_2)$ for all increasing function g. With $g(\mathbf{Z}) = \mathbf{Z}$, we have $\boldsymbol{\mu}_1 \leq \boldsymbol{\mu}_2$.

By Theorem 6.B.16(c) of Shaked and Shanthikumar (2007), we also have $\mathbf{e}_j'\mathbf{Z}_1 \preceq_{st} \mathbf{e}_j'\mathbf{Z}_2$ where $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)'$ are standard basis of $\mathbb{R}^m$, $j = 1, \ldots, m$. Recall that, as in (10),

$$f_{\mathbf{e}_j'\mathbf{Z}_k}(z) = (2\pi)^{-1}|{\sigma^2}_{k,jj}|^{-1/2} \int_0^\infty u^{-1} \exp\left(-\frac{1}{2u^2{\sigma^2}_{k,jj}}(z - \mu_{k,j})^2\right) f_U(u)\, du,$$

where ${\sigma^2}_{k,jj} = \mathbf{e}_j'\Sigma_k\mathbf{e}_j$, $\mu_{k,j} = \mathbf{e}_j'\boldsymbol{\mu}_k$, and $f_U(u)$ is the density function of U defined in (2). Because $\mathbf{e}_j'\mathbf{Z}_1 \preceq_{st} \mathbf{e}_j'\mathbf{Z}_2$,

$$\int_{-\infty}^{z} (f_{\mathbf{e_j'Z_2}}(z) - f_{\mathbf{e_j'Z_1}}(z))dz \leq 0, \tag{11}$$

$$\int_{z}^{\infty} (f_{\mathbf{e_j'Z_2}}(z) - f_{\mathbf{e_j'Z_1}}(z))dz \geq 0, \tag{12}$$

for all z. We will show that ${\sigma^2}_{1,jj} = {\sigma^2}_{2,jj}$. If ${\sigma^2}_{1,jj} < {\sigma^2}_{2,jj}$, then

$$\begin{aligned}
&\lim_{z\to-\infty} \left(\frac{z-\mu_{1,j}}{\sigma_{1,jj}}\right)^2 - \left(\frac{z-\mu_{2,j}}{\sigma_{2,jj}}\right)^2 \\
=& \lim_{z\to-\infty} \left[\left(\frac{1}{\sigma_{1,jj}} + \frac{1}{\sigma_{2,jj}}\right)(-z) + \left(\frac{\mu_{1,j}}{\sigma_{1,jj}} + \frac{\mu_{2,j}}{\sigma_{2,jj}}\right)\right] \left[\left(\frac{1}{\sigma_{1,jj}} - \frac{1}{\sigma_{2,jj}}\right)(-z) + \left(\frac{\mu_{1,j}}{\sigma_{1,jj}} - \frac{\mu_{2,j}}{\sigma_{2,jj}}\right)\right] \\
=& \infty.
\end{aligned}$$

Thus, for any large $M > 0$, we can always find a $z_0 = -(aM + b)$, where $a, b > 0$, such that $(\frac{z-\mu_{1,jj}}{\sigma_{1,jj}})^2 - (\frac{z-\mu_{2,jj}}{\sigma_{2,jj}})^2$ is greater than $2M^2$ for any $z < z_0$. Then we have

$$\left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right| \exp\left\{\frac{1}{2u^2}(\frac{z-\mu_{1,jj}}{\sigma_{1,jj}})^2 - \frac{1}{2u^2}(\frac{z-\mu_{2,jj}}{\sigma_{2,jj}})^2\right\} - 1 \geq \left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right| \exp\left\{\frac{M^2}{u^2}\right\} - 1.$$

Next, we want to show that

$$\int_{-\infty}^{z_0}\int_0^{\infty}(2\pi u)^{-1}\left(\frac{\left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right|\exp\left\{\frac{M^2}{u^2}\right\}-1}{|{\sigma^2}_{1,jj}|^{1/2}\exp\left(\frac{1}{2u^2{\sigma^2}_{1,jj}}(z-\mu_{1,j})^2\right)}\right)f_U(u)\,dudz$$

is positive, if ${\sigma^2}_{1,jj} < {\sigma^2}_{2,jj}$. To prove this, re-express the integral as $T_1 - T_2$, where

$$T_1 = \int_{-\infty}^{z_0}\int_0^{u_M}(2\pi u)^{-1}\left(\frac{\left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right|\exp\left\{\frac{M^2}{u^2}\right\}-1}{|{\sigma^2}_{1,jj}|^{1/2}\exp\left(\frac{1}{2u^2{\sigma^2}_{1,jj}}(z-\mu_{1,j})^2\right)}\right)f_U(u)\,dudz > 0$$

and

$$T_2 = \int_{-\infty}^{z_0}\int_{u_M}^{\infty}(2\pi u)^{-1}\left(\frac{1-\left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right|\exp\left\{\frac{M^2}{u^2}\right\}}{|{\sigma^2}_{1,jj}|^{1/2}\exp\left(\frac{1}{2u^2{\sigma^2}_{1,jj}}(z-\mu_{1,j})^2\right)}\right)f_U(u)\,dudz > 0$$

with $u_M = M(\log(\sigma_{2,jj}/\sigma_{1,jj}))^{-1/2}$. By Fubini's theorem we can rewrite T_1 and T_2 as

$$T_1 = \int_0^{u_M}\int_{-\infty}^{z_0}\frac{1}{\sqrt{2\pi}|\sigma_{1,jj}|u}\exp\left(-\frac{1}{2u^2{\sigma^2}_{1,jj}}(z-\mu_{1,j})^2\right)dz\frac{1}{\sqrt{2\pi}}\left(\left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right|\exp\left\{\frac{M^2}{u^2}\right\}-1\right)f_U(u)\,du$$

and

$$T_2 = \int_{u_M}^{\infty}\int_{-\infty}^{z_0}\frac{1}{\sqrt{2\pi}|\sigma_{1,jj}|u}\exp\left(-\frac{1}{2u^2{\sigma^2}_{1,jj}}(z-\mu_{1,j})^2\right)dz\frac{1}{\sqrt{2\pi}}\left(1-\left|\frac{\sigma_{1,jj}}{\sigma_{2,jj}}\right|\exp\left\{\frac{M^2}{u^2}\right\}\right)f_U(u)\,du.$$

Note that for $W \sim N(0,1)$, its tail has a lower bound and an upper bound

$$\left(-\frac{1}{x}+\frac{1}{x^3}\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}} \leq P\left\{W \leq x\right\} \leq -\frac{1}{x}\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

for all $x < 0$. Replace $\frac{z_0-\mu_{1,jj}}{\sigma_{1,jj}}$ by z', and choose proper a, b such that $u_M < -z'$. Then we have

$$T_1 \geq \int_0^{u_M} (2\pi)^{-1} \left(\left| \frac{\sigma_{1,jj}}{\sigma_{2,jj}} \right| \exp\left\{ \frac{M^2}{u^2} \right\} - 1 \right) \left(-\frac{u}{z'} + (\frac{u}{z'})^3 \right) \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)\, du$$

and

$$\begin{aligned} T_2 &\leq \int_{u_M}^{\infty} (2\pi)^{-1} \left(1 - \left| \frac{\sigma_{1,jj}}{\sigma_{2,jj}} \right| \exp\left\{ \frac{M^2}{u^2} \right\} \right) \frac{u}{-z'} \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)\, du \\ &\leq \int_{u_M}^{\infty} (2\pi)^{-1} \frac{u}{-z'} \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)\, du. \end{aligned}$$

Choose $0 < M_0 < u_M$ such that

$$\left(\left| \frac{\sigma_{1,jj}}{\sigma_{2,jj}} \right| \exp\left\{ \frac{M^2}{{M_0}^2} \right\} - 1 \right) \left(1 - (\frac{M_0}{z'})^2 \right) > 1.$$

Then we have

$$\begin{aligned} T_1 > &\int_0^{M_0} (2\pi)^{-1} \frac{u}{-z'} \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)\, du \\ &+ \int_{M_0}^{u_M} (2\pi)^{-1} \left(\left| \frac{\sigma_{1,jj}}{\sigma_{2,jj}} \right| \exp\left\{ \frac{M^2}{u^2} \right\} - 1 \right) \left(-\frac{u}{z'} + (\frac{u}{z'})^3 \right) \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)\, du. \end{aligned}$$

Let

$$S = \int_0^{\infty} (2\pi)^{-1} \frac{u}{-z'} \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)\, du.$$

Define a new random variable $\tilde{U}$ with density function $h(u) = (2\pi S)^{-1} \frac{u}{-z'} \exp\left\{ -\frac{z'^2}{2u^2} \right\} f_U(u)$. Note that the mean of $\tilde{U}$ is greater than the median. To see this, first consider a simple case: $\ell = 1$, $\lambda = 0$, and $\nu_0 \to \nu_1 = 2$. Then $f_U(u) = 4u \exp(-2u^2)$. We have $\mathrm{E}(\tilde{U}) = z'^{\frac{1}{2}} \frac{K_2(-2z')}{K_{\frac{3}{2}}(-2z')}$. Note that $K_2(-2z') - K_{\frac{3}{2}}(-2z') \to 0$ as $M \to \infty$. So $\lim_{M\to\infty} \frac{\mathrm{E}(\tilde{U})}{\sqrt{\frac{aM+b+\mu_{1,jj}}{|\sigma_{1,jj}|}}} = 1$, which indicates $\mathrm{E}(\tilde{U}) < M_0$ as $M \to \infty$. For general cases, using a similar argument, one could show that $\mathrm{E}(\tilde{U})$ is smaller than M_0 when

M is sufficiently large. Hence, there exists a $\epsilon > 0$ such that

$$\int_0^{M_0} (2\pi)^{-1} \frac{u}{-z'} \exp\left\{-\frac{z'^2}{2u^2}\right\} f_U(u)\, du \geq \frac{S}{2} + \epsilon$$

and

$$\begin{aligned} &\int_{u_M}^{\infty} (2\pi)^{-1} \frac{u}{-z'} \exp\left\{-\frac{z'^2}{2u^2}\right\} f_U(u)\, du \\ &< \int_{M_0}^{\infty} (2\pi)^{-1} \frac{u}{-z'} \exp\left\{-\frac{z'^2}{2u^2}\right\} f_U(u)\, du \\ &\leq \frac{S}{2} - \epsilon. \end{aligned}$$

We get $T_1 - T_2 > (\frac{S}{2} + \epsilon) - (\frac{S}{2} - \epsilon) = 2\epsilon > 0$, which contradicts (11). So we must have ${\sigma^2}_{1,jj} \geq {\sigma^2}_{2,jj}$. However, if ${\sigma^2}_{1,jj} > {\sigma^2}_{2,jj}$, using a similar argument, we have ${\sigma^2}_{1,jj} \leq {\sigma^2}_{2,jj}$. As a result, ${\sigma^2}_{1,jj} = {\sigma^2}_{2,jj}$ holds for all $j = 1, \ldots, m$.

Next we consider the covariance. Note that the variance of $\mathbf{e}_i'\mathbf{Z}_k + \mathbf{e}_j'\mathbf{Z}_k$ is ${\sigma_{k,ii}}^2 + {\sigma_{k,jj}}^2 + 2\sigma_{k,ij}$, and by Theorem 6.B.16(c) of Shaked and Shanthikumar (2007), we have $\mathbf{e}_i'\mathbf{Z}_1 + \mathbf{e}_j'\mathbf{Z}_1 \preceq_{st} \mathbf{e}_i'\mathbf{Z}_2 + \mathbf{e}_j'\mathbf{Z}_2$. Using a similar argument, we can show that ${\sigma^2}_{1,ii} + {\sigma^2}_{1,jj} + 2\sigma_{1,ij} = {\sigma^2}_{2,ii} + {\sigma^2}_{2,jj} + 2\sigma_{2,ij}$, which implies $\sigma_{1,ij} = \sigma_{2,ij}$ for all i and j. Hence $\Sigma_1 = \Sigma_2$.

K Proof of Lemma 7

1. We first prove the "only if" part. By (7) and Lemma 2, it is necessary that $\mathrm{E}{U_1}^2 \leq \mathrm{E}{U_2}^2$ for all ν_j, $j \leq \ell$. By Taylor's theorem,

$$\lim_{\nu_j \to \nu_0, j=1,\ldots,\ell} \mathrm{E}U_k^2 = \frac{\int_0^\infty u^{-\lambda_k/\ell+\ell} e^{-\ell\nu_0 u} du}{\int_0^\infty u^{-1-\lambda_k/\ell+\ell} e^{-\ell\nu_0 u} du} = \frac{\Gamma(1+\ell-\lambda_k/\ell)}{\ell\nu_0\Gamma(\ell-\lambda_k/\ell)} = \frac{\ell-\lambda_k/\ell}{\ell\nu_0},$$

for $0 \leq \lambda_k < 1$. We must have $\frac{\ell-\lambda_1/\ell}{\ell\nu_0} \leq \frac{\ell-\lambda_2/\ell}{\ell\nu_0}$ and thus, $\lambda_1 \geq \lambda_2$.

We next show the "if" part. As $\nu_{j,1} = \nu_{j,2}$ for $j = 0, 1, \ldots, \ell$, the ratio of two density functions

$$\frac{f_{U_2}(u)}{f_{U_1}(u)} = \frac{c_{02}}{c_{01}} u^{2(\lambda_1-\lambda_2)} \tag{13}$$

is increasing in $u > 0$, which implies that U_1 is smaller than U_2 in the likelihood ratio order, and thus $U_1 \preceq_{st} U_2$ (see Theorems 1.C.1 and 1.B.1 of Shaked and

Shanthikumar (2007)).

2. We just need to show that the ratio $\prod_{j=1}^{\ell} \frac{e^{-\nu_{0,2}u}-e^{-\nu_j u}}{e^{-\nu_{0,1}u}-e^{-\nu_j u}}$ is increasing in u, which is equivalent to the fact that $\frac{\nu_{0,2}-\nu_j}{1-\exp((\nu_{0,2}-\nu_j)u)} \leq \frac{\nu_{0,1}-\nu_j}{1-\exp((\nu_{0,1}-\nu_j)u)}$ holds for every j. Let $v_k = \nu_{0,k} - \nu_j \geq 0$. Then $v_1 \geq v_2 \geq 0$. So we need to show $\frac{v}{1-e^{uv}}$ is increasing in $v \geq 0$. Since $\frac{d}{dv}\frac{v}{1-e^{uv}} = \frac{1+uve^{uv}-e^{uv}}{(1-e^{uv})^2} \geq 0$ always holds for $v \geq 0$, the result follows.

3. By (7) and Lemma 2, it is necessary that $\mathrm{E}{U_1}^2 \leq \mathrm{E}{U_2}^2$. Consider $\nu_{1,1} = \nu_{1,2} = 1$ and $\lambda = 0$. Then $EU^2 = \frac{1-\nu_0{}^{-1}}{\ln \nu_0}$. Since $\frac{d}{d\nu_0}\mathrm{E}U^2 = \frac{\ln \nu_0-(\nu_0-1)}{(\nu_0 \ln \nu_0)^2} \leq 0$, $\mathrm{E}U^2$ is a decreasing function of ν_0. Hence $\nu_{0,1} \geq \nu_{0,2}$.

4. By (7) and Lemma 2, it is necessary that $\mathrm{E}{U_1}^2 \leq \mathrm{E}{U_2}^2$. For $\ell = 2$, consider $\nu_{j,1} = \nu_{j,2} = 1$ and $\lambda = 0$. Then we have $EU^2 = \frac{\frac{1}{2\nu_0}-\frac{2}{\nu_0+1}+\frac{1}{2}}{\ln(\frac{(\nu_0+1)^2}{4\nu_0})}$. Since $\ln(\frac{(\nu_0+1)^2}{4\nu_0}) = \ln(\frac{1}{4}(\nu_0 + \frac{1}{\nu_0} + 2))$ is increasing in $\nu_0 \geq 1$ and $\frac{d}{d\nu_0}\left(\frac{1}{2\nu_0} - \frac{2}{\nu_0+1}\right) = \frac{(1-\nu_0)(1+3\nu_0)}{2\nu_0{}^2(\nu_0+1)^2} \leq 0$, EU^2 is a decreasing function of ν_0. So $\nu_{0,1} \geq \nu_{0,2}$.

L Proof of Lemma 8

By definition,

$$
\begin{aligned}
&\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right)\\
=&\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\mathrm{E}\exp\left(i\sum_{k=1}^{n}U\mathbf{Z}_{\mathbf{0}}'(x_k)\boldsymbol{\omega}_k\right)\\
=&\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\int_0^{\infty}\mathrm{E}\exp\left(i\sum_{k=1}^{n}u\mathbf{Z}_{\mathbf{0}}'(x_k)\boldsymbol{\omega}_k\right)f_U(u)\,du\\
=&\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\int_0^{\infty}\exp\left(-\frac{u^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j\right)f_U(u)\,du,\\
=&\frac{c_0}{2}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\\
&\times\int_0^{\infty}\exp\left(-\frac{v}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j\right)\frac{1}{v^{\lambda/\ell+1}}\left\{\prod_{k=1}^{\ell}\left(e^{-\nu_0 v}-e^{-\nu_k v}\right)\right\}dv,
\end{aligned}
$$

for $\boldsymbol{\omega}_k \in \mathbb{R}$, $x_k \in \mathbb{D}$, $k = 1, \ldots, n$, where the last equality follows by replacing u^2 with v. When $0 < \lambda < 1$, by (2.1) of Alsultan and Ma (2019) we have

$$
\begin{aligned}
\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right) =& \frac{c_0\Gamma(1-\frac{\lambda}{\ell})}{2\lambda/\ell}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\left\{-\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\ell\nu_0\right)^{\frac{\lambda}{\ell}}\right.\\
&+\sum_{k=1}^{\ell}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_k+(\ell-1)\nu_0\right)^{\frac{\lambda}{\ell}}\\
&-\sum_{k_1<k_2}^{\ell}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0\right)^{\frac{\lambda}{\ell}}\\
&\left.+\cdots+(-1)^{\ell+1}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\sum_{k=1}^{\ell}\nu_k\right)^{\frac{\lambda}{\ell}}\right\}.
\end{aligned}
$$

When $\lambda = 0$,

$$
\begin{aligned}
\mathrm{E}\exp\left(i\sum_{k=1}^{n}\mathbf{Z}'(x)\boldsymbol{\omega}_k\right) =& \frac{c_0}{2}\exp\left(i\sum_{k=1}^{n}\boldsymbol{\mu}'(x_k)\boldsymbol{\omega}_k\right)\left\{-\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\ell\nu_0\right)\right.\\
&+\sum_{k=1}^{\ell}\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_k+(\ell-1)\nu_0\right)\\
&-\sum_{k_1<k_2}^{\ell}\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\nu_{k_1}+\nu_{k_2}+(\ell-2)\nu_0\right)\\
&\left.+\cdots+(-1)^{\ell+1}\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j}{2}+\sum_{k=1}^{\ell}\nu_k\right)\right\}
\end{aligned}
$$

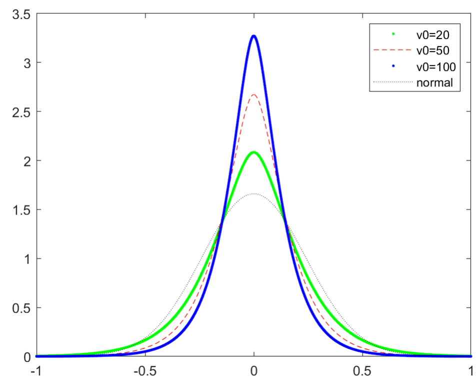
for $\boldsymbol{\omega}_k \in \mathbb{R}$, $x_k \in \mathbb{D}$, $k = 1, \ldots, n$, where c_0 is given in 1.

M Proof of Theorem 6

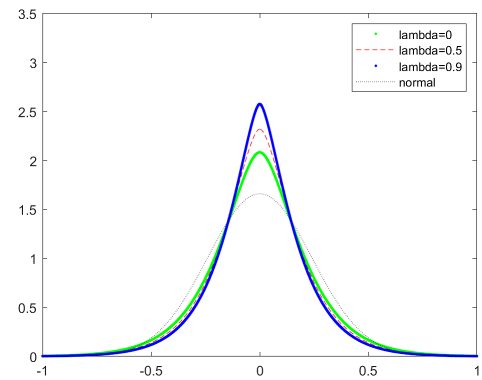
1. For the "only if" part, because $\{\mathbf{Z}_1(x) - \boldsymbol{\mu}_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{\mathbf{Z}_2(x) - \boldsymbol{\mu}_2(x), x \in \mathbb{D}\}$, by Theorem 4 in Wang and Ma (2018) we have $E(U_1)^j \leq E(U_2)^j$ holds for any positive j. Recall the proof of the first conclusion of 7 then we get $\lambda_1 \geq \lambda_2$. For the "if" part, because $\lambda_1 \geq \lambda_2$, then by 7 we have $U_1 \preceq_{st} U_2$. Thus, by Corollary 3.2 in

Wang and Ma (2018), the result follows.

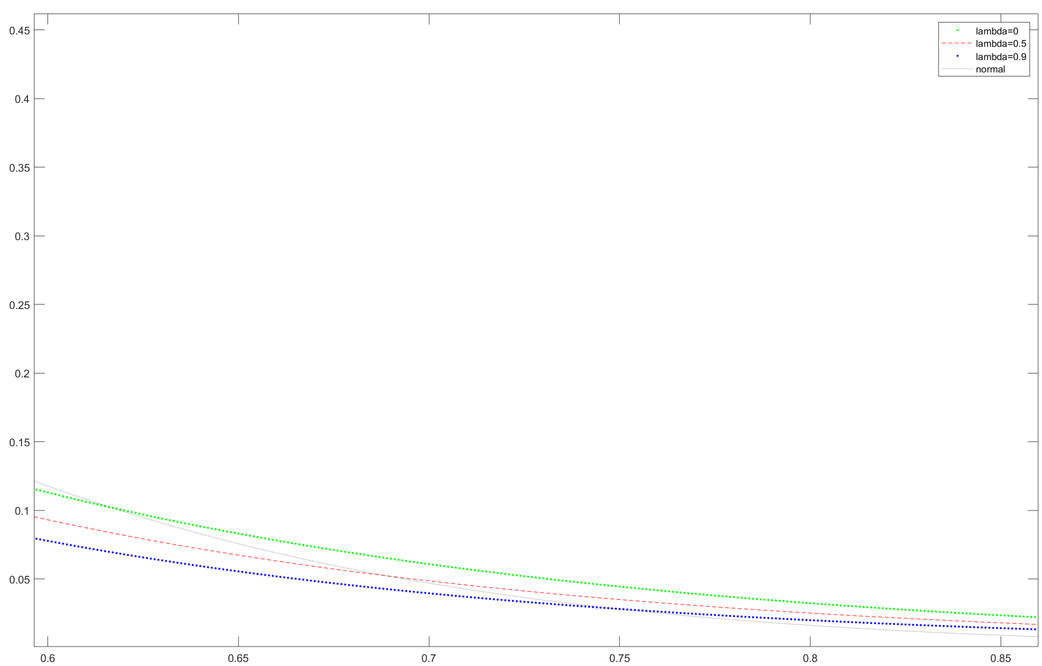
2. Similarly, the result directly follows from Corollary 3.2 and Theorem 4 in Wang and Ma (2018) and Lemma 7.



(a) $m = 1$, $\lambda = 0$, and $\nu_0 = 20, 50, 100$



(b) $m = 1$, $\nu_0 = 20$, and $\lambda = 0, 0.5, 0.9$



(c) $m = 1$, $\nu_0 = 20$, and $\lambda = 0, 0.5, 0.9$