An Investigation of Pòlya's Function

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Abstract

In 1913, George Pòlya published a paper describing an iterative geometric construction of a map, P, that maps an arbitrary $t \in [0,1]$ to $P(t)$ onto a non-isosceles right triangle, T. This mapping constructs $P(t)$ by producing a sequence of nested subtriangles by drawing the altitude of the current triangle at each step. This sequence has only one point in common, $P(t)$. In his paper, Pòlya proved that this mapping P is continuous and surjective. In 1973, Peter Lax faced the problem of finding the derivative of P and proved that the map's differentiability is dependent on the smallest angle of T. In this project, we built upon their research by investigating several properties of Pòlya's function.

We analytically proved that the trajectory of Pòlya's function in T is self-similar by constructing two pairs of contractive similitudes: ψ_0 and ψ_1 that apply to T and ϕ_0 and ϕ_1 that apply to $I = [0,1]$. Specifically, we proved that

$$P(\phi_{\mathbf{i}}(I)) = \psi_{\mathbf{i}}(T)$$

where $\mathbf{i}$ is an infinite sequence of 0's and 1's.

We plan to develop this study further and submit a paper with our own contributions to an appropriate journal.

Acknowledgements

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Contents

1 Pòlya's Function

Pòlya's function is a mapping from the interval $[0,1]$ onto a non-isosceles right triangle. Before we discuss the properties of this mapping, we must understand the mapping itself. Pòlya's function is not like more normal functions such as $f(x) = x^2$ or $sin(x)$. The ranges of these functions are found by specific calculations. For Pòlya's function, the point in a non-isosceles right triangle that corresponds to a specific value in the interval $[0,1]$ is found by a simple construction. This process is illustrated below.

Let t be a value between 0 and 1. Consider its binary expansion where $t = 0.d_1d_2d_3\ldots$ and $d_1, d_2, d_3\ldots$ are either 0 or 1. Let T be a non-isosceles right triangle. Pòlya's function P maps t to $P(t)$, a point inside T. This $P(t)$ is defined as the intersection of all of the triangles in a sequence of triangles $T_1, T_2, T_3, \ldots, T_n, \ldots$ The sequence of triangles is defined by following these steps:

1. Consider the triangle T. Divide T into two similar triangles by drawing the altitude of T. Since T is non-isosceles, these triangles must be of different sizes. Denote the smaller triangle by T_s and the larger triangle by T_l (See Figure 1).

2. Consider d_1, the first digit of the binary fraction representation of t. If $d_1 = 0$, let $T_1 = T_s$. If $d_1 = 1$, let $T_1 = T_l$

3. Repeat steps 1 and 2, replacing d_1 with d_n and T with T_{n-1} for $n = 1, 2, 3, \ldots$.

As $n \to \infty$, it is clear that the area of each T_n shrinks to a single point. Therefore, this sequence of triangles has one point in common. This point is $P(t)$.

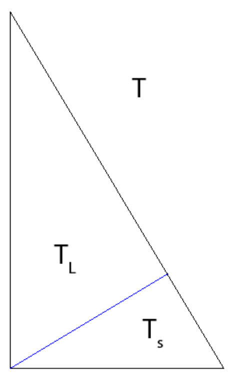
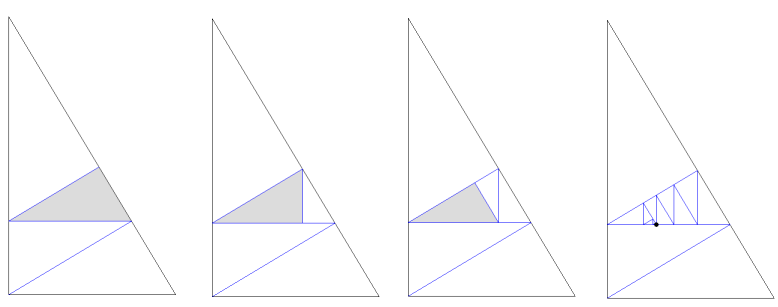


Figure 1: The larger and smaller subtriangles resulting from the decomposition of T by drawing its altitude.



(a) T_3 (b) T_4 (c) T_5 (d) T_n

Figure 2: Shaded triangles represent T_n after 3 iterations and 4 iterations. $P(t)$ is shown in (d) after many iterations.

Now that we know how to find $P(t)$ from a single t, we can investigate what happens when t changes. In fact, this mapping is onto: every point in the triangle T has a preimage in the interval $[0,1]$. This mapping is also continuous. George Pólya proved these results in his paper [5].

Pólya's Theorem. The function P maps the interval $[0,1]$ continuously onto the triangle T.

Proof: *To prove that the map P is onto*, we will show that for any point in the triangle T, we can find a corresponding binary fraction expansion $0.d_1d_2d_3\ldots$ that corresponds to that point. Let p be any point in the triangle T. Divide T into two subtriangles by drawing the altitude. Let T_1 be the subtriangle that contains p. If T_1 is the smaller of the two subtriangles, let $d_1 = 0$. If T_1 is the larger of the two triangles, let $d_1 = 1$. Next, divide T_1 by its altitude, and let $d_2 = 0$ or 1 in the same manner. We can continue in this way to obtain a sequence of digits $d_1, d_2, d_3, \ldots$ Let $t = 0.d_1d_2d_3\ldots$ Then it is clear that $p = P(t)$. Since we can perform this procedure for any p in T, the mapping P is onto. $\Box$

Now we prove that the function P is continuous. Let t and t' be binary fraction representations of two numbers in $[0,1]$. Suppose that these numbers are contained in an interval of size $1/2^N$, i.e., $|t-t'| \leq 1/2^N$. Then either (a) the first N digits of t and t' are the same, or (b) there exists a $t'' = k/2^N$ (where k is some integer) and that $t < t'' < t'$.

In case a), we assume that the first N digits of t and t' are the same. Let $T_N(t)$ denote the N^{th} subtriangle in the sequence of subtriangles that contains $P(t)$ and let $T_N(t')$ denote the N^{th} subtriangle that contains $P(t')$. Note that since the first N digits of t and t' are the same, $T_N(t) = T_N(t')$. Clearly, $P(t)$ and $P(t')$ are both contained in this subtriangle.

Let $h_N(t)$ denote the length of the hypotenuse of $T_N(t)$

Then it follows that:

$$|P(t) - P(t')| \leq h_N(t) \tag{1}$$

In case b), we assume that there exists a $t'' = k/2^N$ such that $t < t'' < t'$. In this case, t'' is a rational number, so it has two binary fraction expansions (Example: $1/2 = 0.1000\ldots$ or $0.0111\ldots$). In one expansion, the first N digits of t'' are the same as those of t. In the second expansion, the first N digits of t'' are the same as those of t'. So by (1) we have that

$$|P(t) - P(t'')| \leq h_N(t)$$
$$|P(t') - P(t'')| \leq h_N(t')$$

And, by the triangle inequality,

$$|P(t) - P(t')| \leq h_N(t) + h_N(t')$$

Since

$$\lim_{N\to\infty} h_N = 0$$

in either case, we have that

$$\lim_{N\to\infty} |P(t) - P(t')| = 0.$$

This proves the continuity of P. Thus we have proven that the function P maps the interval $[0,1]$ continuously onto the triangle T. $\Box$

Now that we have shown that this map is a continuous function, we can see if this curve is differentiable. In his paper [3], Peter Lax investigated the differentiability properties of Polya's function. We will present his results in the following chapter.

2 Borel's Theorem

One of the key ideas introduced in the proof of the differentiability of Polya's function is a theorem introduced by Émile Borel in 1909. A summary of the proof can be found in *An Introduction to the Theory of Numbers* by Hardy and Wright [2].

First we must discuss the concept of "almost everywhere." This is a concept related to Lebesgue measure. The Lebesgue measure, denoted μ, of any interval $[a, b]$ is equal to the length of the interval. So,

$$\mu([a, b]) = b - a.$$

The Lebesgue measure of an isolated point is 0. The Lebesgue measure also has an additive property, where for any disjoint intervals $I_1, I_2, \ldots I_n$,

$$\mu\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} \mu(I_i) \tag{2}$$

Therefore, if a set I is composed of isolated points $p_1, p_2, \ldots, p_n$,

$$\mu(I) = \mu(p_1) + \mu(p_2) + \ldots + \mu(p_n) = 0 + 0 + \ldots + 0 = 0.$$

We will now introduce the following definition, adapted from Bauer [1].

Definition: Let η be a property of points $p \in P$: every p either has property η or not. We say that *"almost all points of P have property η"* if there is a set of measure 0, denoted Q, such that all points of the complement of Q have property η.

We will now introduce two definitions. Note that in the following definitions, we will consider x to be in base 2, but these definitions can be applied to a number in any base r.

Let x be a number expressed in base 2. Let n_b be the number of a times a particular digit $b \in \{0, 1\}$ occurs in the first n digits of x.

Definition: The number x is said to be simply normal in base 2 if

$$\frac{n_b}{n} \to \frac{1}{2} \tag{3}$$

for each possible value of b.

Definition: The number x is said to be normal in base 2 if x is simply normal in all of the bases $2, 2^2, 2^3, \ldots$

Borel's Theorem: *Almost all numbers are normal.*
The proof of Borel's Theorem can be found in "An Introduction to the Theory of Numbers" [2].

Now, we introduce important notation that we will use in the following chapter. Let $\mathcal{N}$ be the set of all normal numbers in $[0, 1]$. Let $\mathcal{N}^c$ be the set of all non-normal numbers in $[0, 1]$. Then $\mu(\mathcal{N}) = 1$ and $\mu(\mathcal{N}^c) = 0$ and is an uncountable set. Clearly, $\mathcal{N} \cup \mathcal{N}^c = [0, 1]$.

Let $\delta(j)$ be an arbitrary sequence of 1, 0, and -1. Let K be an integer greater than 1. Let $\mathcal{M}$ be the set of all t where the digits d_n of t are defined by:

$$\begin{cases} d_n & = 1 \text{ whenever } n \text{ is of the form } n = jK + \delta(j), j = 1, 2, 3, \ldots \\ d_n & = 0 \text{ otherwise.} \end{cases}$$

Clearly, $\mathcal{M} \subset \mathcal{N}^c$ and $\mu(\mathcal{M}) = 0$. Moreover, $\mathcal{M}$ is an uncountable set.

3 Differentiability Properties

In his paper [3], Peter Lax proved that the differentiability of the Pólya curve depends on the smallest angle of T. The statement of his theorem is as follows:

Differentiability Theorem: *Let θ be the smaller angle of a non-isoceles right triangle T. Then,*

1. *If $30° < \theta < 45°$, then P is nowhere differentiable.*
2. *If $15° < \theta < 30°$, then P has no derivative for all $t \in \mathcal{N}$, but has derivative zero for $t \in \mathcal{M}$, where $\mathcal{M} \subset \mathcal{N}^c$ and $\mathcal{M}$ is an uncountable set.*
3. *If $\theta < 15°$, then P has derivative zero for all $t \in \mathcal{N}$.*

We will summarize this proof as presented in Lax's paper. First, we recall important notation. $h(T)$ denotes the hypotenuse of the triangle T. $h_N(t)$ denotes the hypotenuse of the triangle containing t produced by N iterations of the construction for $P(t)$.

Lemma 1: Suppose that

- t and t_N have the same first $N-1$ digits.
- t and t_N have different Nth digits.
- The Nth, $N+1$ and $N+2$ digits of t_N are the same.

Then,

$$|P(t) - P(t_N)| > \text{const } h_N(t) \ , \tag{4}$$

where const is a constant.

Comment on Lemma 1:

An alternative way to state this lemma is that if t and t_N satisfy the above conditions, then the distance between the two points $P(t)$ and $P(t_N)$ in the triangle T is greater than a positive constant that is related to the hypotenuse of T_N. This concept is demonstrated in the following example.

Example: For N=3, Let $t = 0.100\ldots$ and $t_3 = 0.10111\ldots$

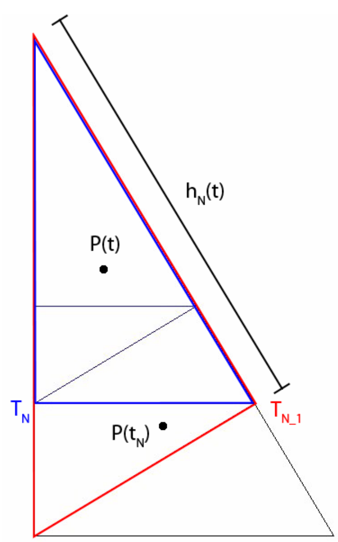


Figure 3: Clearly, the minimum distance between $P(t)$ and $P(t_N)$ is greater than some constant multiplied by $h_N(t)$.

We see that t and t_N have the same first two digits, they differ at the third digit, and the third, fourth, and fifth digits of t_N are equal. So t and t_N satisfy the above criteria. Figure 3 demonstrates the position of $P(t)$ and $P(t_N)$ in the triangle T. It is clear that the shortest possible distance between $P(t)$ and $P(t_N)$ is positive.

Proof of Lemma 1:

Since the first $N-1$ digits of t and t_N are the same, $P(t)$ and $P(t_N)$ are in the same triangle T_{N-1}, but since the $Nth, (N+1)$, and $(N+2)$ digits of t_N are different from that of the Nth digit of t, $P(t_N)$ lies in a triangle $T_{N+2}(t_N)$ which has no point in common with $T_N(t)$. Therefore, the minimum distance between $T_{N+2}(t_N)$ and $T_N(t)$ is greater than a constant multiplied by $h_N(t)$, denoted by const $h_N(t)$. This completes the proof of the Lemma. $\square$

Proof of Differentiability Theorem:

$$\textbf{\textit{Goal 1:}}\text{ Prove that } h_N(t) \geq h(T)(sin\theta)^N \tag{5}$$

We start by denoting the number of zeros in the first N digits of t by $Z_N = Z_N(t)$ and the number of ones in the first N digits of t by $V_N = V_N(t)$. Note that

$$Z_N + V_N = N \ . \tag{6}$$

If we consider an appropriate triangle, T_i, then the scaling factor from T_i to T_s is $sin(\theta)$ and the scaling factor from T_i to T_l is $cos(\theta)$. Therefore, we can find the length of the hypotenuse of T_N, $h_N(t)$, by multiplying the length of the hypotenuse of the original triangle, $h(T)$, by $sin\theta$ $Z_N(t)$ times and $cos\theta$ $V_N(t)$ times. So

$$h_N(t) = h(T)(sin\theta)^{Z_N(t)}(cos\theta)^{V_N(t)} \ . \tag{7}$$

Since θ is the smaller of the two acute angles, $sin\theta < cos\theta$, and it follows from (6) and (7) that

$$h_N(t) \geq h(T)(sin\theta)^{Z_N(t)+V_N(t)} = h(T)(sin\theta)^N \ .$$

$$\textbf{\textit{Goal 2:}}\text{ Prove that } \frac{|P(t)-P(t_N)|}{|t-t_N|} > \text{const } (2sin\theta)^N$$

First, we choose a t_N that satisfies the conditions defined in Lemma 1. Since (4) states that

$$|P(t) - P(t_N)| > \text{const } h_N(t)$$

and (5) states that

$$h_N(t) \geq h(T)s^N$$

We have that

$$|P(t) - P(t_N)| > \text{const } (sin\theta)^N \tag{8}$$

Additionally, since t and t_N have the same first $N-1$ digits,

$$|t - t_N| < 1/2^{N-1} \tag{9}$$

So if we divide (8) by (9), we get that

$$\frac{|P(t)-P(t_N)|}{|t-t_N|} > \text{const } (2sin\theta)^N \tag{10}$$

Note that the form of the left side of this inequality is the difference quotient. Since the difference quotient is related to a constant that depends on the angle θ, we will look at how the value of θ affects the difference quotient in each case as stated above.

We will now consider the three different cases presented in the theorem.

Case 1: $30^\circ < \theta < 45^\circ$

Note that t_N tends to t as N tends to infinity. So we can take the limit of (10) as $N \to \infty$ to find $P'(t)$. Since $\theta > 30^\circ$, $sin\theta > 1/2$. Therefore, the right side of (10) tends to infinity with N as does the difference quotient of P. Therefore, P *is nowhere differentiable when* $30^\circ < \theta < 45^\circ$.

Case 2a: $15^\circ < \theta < 30^\circ$, $t \in \mathcal{N}$

To analyze this case, we consider both normal and non-normal t separately. First, we let t be a normal number, meaning that t will, on average, have the same number of 0's and 1's in its binary expansion. More formally,

$$\lim_{N\to\infty} \left(\frac{Z_N}{N}\right) = \lim_{N\to\infty} \left(\frac{V_N}{N}\right) = \frac{1}{2} \tag{11}$$

We will denote $Z_N - V_N = 2D_N$. Since $Z_N + V_N = N$, we have that

$$Z_N = \frac{N}{2} + D_N, \quad V_N = \frac{N}{2} - D_N \tag{12}$$

Introducing this notation into (11) we have

$$\lim_{N\to\infty} \left(\frac{D_N}{N}\right) = 0. \tag{13}$$

Similarly, (7) becomes

$$h_N = h(T)\ (sin\theta\ cos\theta)^{N/2} \left(\frac{sin\theta}{cos\theta}\right)^{D_N}. \tag{14}$$

Now we choose a t_N that satisfies Lemma 1. Then if we combine (14) with (4) we have that

$$|P(t) - P(t_N)| > \text{const}\ (sin\theta\ cos\theta)^{N/2} \left(\frac{sin\theta}{cos\theta}\right)^{D_N} \tag{15}$$

Now we divide (15) by (9) to get

$$\frac{|P(t) - P(t_N)|}{|t - t_N|} > \text{const}\ (4sin\theta\ cos\theta)^{N/2} \left(\frac{sin\theta}{cos\theta}\right)^{D_N} \tag{16}$$

Recall that in this case, $\theta > 15^\circ$, so $2\theta > 30^\circ$ and $4sin\theta\ cos\theta = 2\sin 2\theta > 1$. As before, we take the limit of (16) as $N \to \infty$. We can see that the factor $(4sin\theta\ cos\theta)^{N/2}$ increases exponentially. But we know from (13) that the second factor $\left(\frac{sin\theta}{cos\theta}\right)^{D_N}$ decreases at a rate slower that exponential. Therefore, the right side of (16) tends to infinity with N. Clearly, the difference quotient in (16) tends to ∞ as well. We have shown that P is not differentiable at any normal t. According to Borel's Theorem, P *has no derivative for all* $t \in \mathcal{N}$.

Case 2b: $15^\circ < \theta < 30^\circ$, $t \in \mathcal{M}$

Lemma 2: Let t and t' be any two numbers whose first $N-1$ digits are identical and differ in the Nth digit. Denote $M = M(N)$ as the smallest integer $M > N$ such that $d_M(t) = d_N(t)$. Then,

$$|t - t'| > 1/2^M . \tag{17}$$

The Proof of Lemma 2 follows directly from the definition of M. To investigate this case, we construct an uncountable set with measure zero of non-normal t such that t has more 0's than 1's. Note that this same proof holds when we construct a set of t such that t has more 1's than 0's. Let K denote an integer to be fixed later. Then define $\tilde{\mathcal{M}} \subset \mathcal{M}$ as the set of all t where the digits of t are defined as follows:

$$d_n(t) = \begin{cases} 1 & \text{if } n \text{ is a multiple of } K \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Goal 3: Prove that $P'(t) = 0$ in case 2b for $t \in \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}} \subset \mathcal{N}^c$.

Now let t' be any number and N the smallest integer such that $d_N(t) \neq d_N(t')$. From our definition of t, it follows that M, as defined in Lemma 2, is either $N+1, N+2,$ or $N+K$; so by (17),

$$|t - t'| > 1/2^{N+K} \tag{19}$$

Since t and t' have the same first $N-1$ digits, $P(t)$ and $P(t')$ are in the same triangle T_{N-1}, and

$$|P(t) - P(t')| < h_{N-1}(t) \leq (1/s)h_N \tag{20}$$

since $h_N(t) = sin\theta\ h_{N-1}(t)$ if $N = 0$ and $h_N(t) = cos\theta\ h_{N-1}(t)$ if $N = 1$, and in this case $(1/cos\theta) < (1/sin\theta)$, $h_{N-1}(t) \leq (1/sin\theta)h_N$. For the special t defined by (18), we have with an error < 1, approximately

$$Z_N(t) = \left(\frac{K-1}{K}\right)N, \quad V_N(t) = \frac{N}{K}$$

Therefore, from (7), we know that

$$h_N = (sin\theta)^{\left(\frac{K-1}{K}\right)N}(cos\theta)^{\frac{N}{K}} < (sin\theta)^{\left(\frac{K-1}{K}\right)N}$$

Combining this result with (20) we get that

$$|P(t) - P(t')| < \text{const } (sin\theta)^{\left(\frac{K-1}{K}\right)N}$$

So by dividing by (19) we can look at our familiar difference quotient of P at this special point t.

$$\frac{|P(t) - P(t_N)|}{|t - t_N|} < \text{const } 2^K \left(2(sin\theta)^{\left(\frac{K-1}{K}\right)}\right)^N \tag{21}$$

Since $\theta < 30°$, $sin\theta < 1/2$. Fix K such that

$$2(sin\theta)^{\left(\frac{K-1}{K}\right)} < 1$$

It follows that for such K the right hand side of (21) is bounded and tends to zero as N tends to infinity. Therefore, $P'(t) = 0$ at these points.

The same analysis shows that $P'(t_\delta) = 0$ for any $t_\delta \in \mathcal{M}$. So for case 2, when $15° < \theta < 30°$ and $t \in \mathcal{M}$, we have shown that *If* $15° < \theta < 30°$*, then* P *has no derivative for all* $t \in \mathcal{N}$*, but has derivative zero for* $t \in \mathcal{M}$*, where* $\mathcal{M} \subset \mathcal{N}^c$.

Case 3: $\theta < 15°, t \in \mathcal{N}$

Lemma 3: For normal t,

$$\lim_{N\to\infty} \frac{M(N)}{N} = 1 \tag{22}$$

where $M(N)$ is defined as in Lemma 2.

Proof of Lemma 3:

Suppose $d_N = 1$; then by the definition of M,

$$d_n = 0 \text{ for } N < n < M$$

so that

$$Z(M) = Z(N) + M - N.$$

Divide both sides by N. The resulting equation can be written as follows:

$$\left(\frac{Z(M)}{M} - 1\right)\left(\frac{M}{N}\right) = \left(\frac{Z(N)}{N} - 1\right)$$

By the definition of normality, $\frac{Z(N)}{N}$ and $\frac{Z(M)}{M}$ both tend to 1/2 as M, N tend to infinity and (22) follows from this relation. □

For Case 3 we let t be normal, t' any number not equal to t and let the Nth digit be the first digit where t and t' differ. By dividing (20) by (17) and combining the result in (14), we can now look at our difference quotient:

$$\frac{|P(t) - P(t')|}{|t - t'|} \leq \text{const } (sin\theta\ cos\theta)^{N/2} \left(\frac{sin\theta}{cos\theta}\right)^{D_N} 2^M \tag{23}$$

$$= \text{const } (4sin\theta\ cos\theta)^{N/2} 2^{M-N} \left(\frac{sin\theta}{cos\theta}\right)^{D_N} \tag{24}$$

Since in this case $\theta < 15°$, $4sin\theta\ cos\theta < 1$ and the factor $(4sin\theta\ cos\theta)^{N/2}$ on the right hand side of (24) tends to 0 exponentially as N tends to infinity. It follows from (22) and (13) that the factors 2^{M-N} and $\left(\frac{sin\theta}{cos\theta}\right)^{D_N}$ tend to infinity at a rate which is slower than exponential. This shows that the right hand side of (24) tends to zero as N tends to infinity as it does when t' tends to t. Thus $P'(t) = 0$ for t normal. Since almost every t is normal and t is in the interval $[0, 1]$, we have shown that *when* $\theta < 15°$, $P'(t) = 0$ *for all* $t \in \mathcal{N}$.

This completes the proof of the theorem.

We can interpret the Pólya map as a trajectory. What does this trajectory look like? It will certainly be more complicated than elementary curves such as a parabola or sine curve. In the next chapter, we provide an intuitive interpretation of this trajectory so that we can begin to reveal this trajectory's other interesting properties.

4 The Firefly's Journey: An Illustration of Polya's Function

Fireflies are arguably among the most enchanting of insects. Memories of pleasant summer evenings are often accompanied by images of fields lit up by countless intermittent lights produced by the insects. Fireflies generate a glow from their lower abdomen in special light-emitting organs through a chemical reaction called bioluminescence. The glow from a firefly larvae serves to warn predators of their bad taste, but as adults, fireflies light up to attract mates. In this way, the firefly's nebulous glow serves the species' most basic evolutionary purpose.

Now suppose that one summer evening a firefly is traveling in an enclosed field; looking for a mate. Unfortunately for the firefly, summer is almost over and he only has 16 minutes left to find a mate. In order to ensure the survival of his species, he must search every point in the field within this time period. Since it's reaching the end of the evening and getting dark, we can only see the firefly when it lights up. This firefly happens to light up at evenly-spaced intervals in time. We happen to notice another firefly flying in another enclosed field, but flashing four times as frequently. We decide to take note of where these fireflies light up. Then we drew lines connecting these dots to indicate the order in which we saw the fireflies flashing.

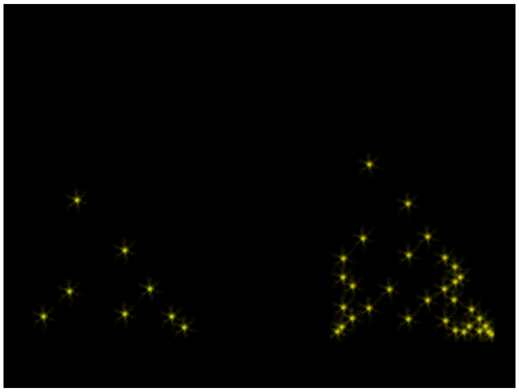


Figure 4: The lights of two fireflies in two enclosed fields.

Obviously, we don't know where the fireflies were when they weren't flashing, so these lines represent an interpolated path of the firefly.

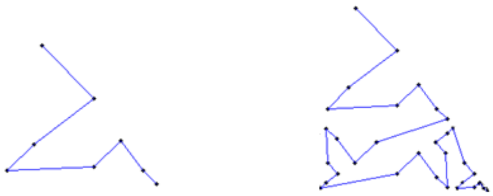


Figure 5: The lights of two fireflies in two enclosed fields.

After much deliberation, we realized that these fireflies are following the trajectory of Pólya's function. Now we can think about the path of the firefly in terms of Pólya's function.

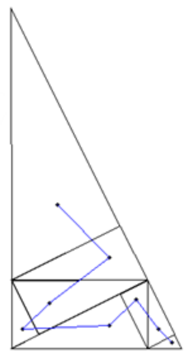


Figure 6: The lights of two fireflies in two enclosed fields.

Perhaps it would be useful to compare the first firefly's set of flashes with the second firefly's set of flashes. When we made this comparison,, we noticed that the first firefly flashed only once in the largest subtriangle of the triangular field (its last flash) and that the second firefly flashed four times in the respective subtriangle in its field. Since we know that the second firefly flashes four times as often as the first firefly, the two fireflies must have taken the same amount of time to cover the same area.

There is another conclusion we can draw from this comparison. Notice that the interpolated trajectory of the last eight flashes of the second firefly is identical to the interpolated trajectory of the eight total flashes of the first firefly. In fact, we can split the second firefly's trajectory into four sets of eight flashes and notice the the same pattern exists in each. This is an intrinsic property of Pólya's function.

The image below demonstrates this property further. Consider an interpolated trajectory of Pólya's function and take note of the pattern. If we chose any subtriangle and examine the interpolated trajectory within it, we see that this trajectory follows the same pattern. We can do this as many times as we like.

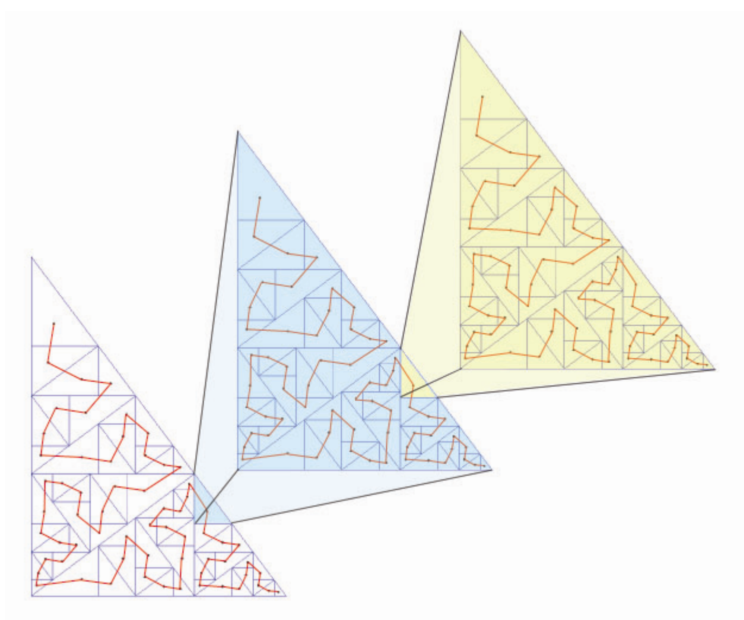


Figure 7: Zooming in on smaller subtriangles demonstrates the self-similarity of the Polya curve.

We believe that this occurs due to the special properties of Polya's function.

5 Similarity Properties

When we divide T by its altitude, we create two similar triangles. This division can also be performed by applying certain functions, which we will call ψ_0 and ψ_1, to T. These functions are similitudes. A similitude is any transformation that preserves distances up to scaling. More formally,

Definition: Let $\mathbb{R}^2$ be a metric space with $d(x,y) = |x-y| = ((x_1-y_1)^2+(x_2-y_2)^2)^{\frac{1}{2}}$. A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a similitude if $d(f(x), f(y)) = rd(x,y)$ for all $x, y \in R^2$ and some fixed $r \in R$.

It can be shown that any composition of an orthonormal transformation, a homothethy, and a translation is a similitude. The orthonormal transformation will be denoted by O. We will denote a homothety, or scaling, by $\mu_r(x) = rx, r \geq 0$. The translation will be denoted by $\tau_b(x) = (x-b)$.

In our case, we will consider the map $\Psi : R^2 \to R^2$ where $\Psi = \psi_0 \cup \psi_1$ and

In the following definitions of ψ_0 and ψ_1, we will refer to θ as the smallest angle of T, $a = (a_1, a_2)$ as the vertex of T at the angle θ, $b = (b_1, b_2)$ as the vertex of T at the angle $\pi/2 - \theta$, and $c = (c_1, c_2)$ as the vertex of T at the right angle of T.

The result of applying ψ_0 on T is the smaller of the two similar triangles formed by the altitude of T. For an arbitrary point $x = (x_1, x_2) \in T$,

$$\psi_0(x) = \psi_0(x_1, x_2) = \begin{bmatrix} b_1 + sin\theta[(x_1 - b_1)sin\theta + (b_2 - x_2)cos\theta] \\ b_2 + sin\theta[(b_1 - x_1)cos\theta + (b_2 - x_2)sin\theta] \end{bmatrix}$$

More formally, $\psi_0 : T \to T_s$ through the following similtude:

$$\psi_0 : \tau_0' \circ \mu_0 \circ O_0 \circ \tau_0$$

Where

$$\begin{aligned} \tau_0(x) &= x - b \\ O_0 &= \begin{bmatrix} -sin\theta & cos\theta \\ -cos\theta & -sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \mu_0(x) &= sin\theta(x) \\ \tau_0'(x) &= x + b \end{aligned}$$

The result of applying ψ_1 on T is the larger of the two similar triangles formed by the altitude of T. For an arbitrary point $x = (x_1, x_2) \in T$,

$$\psi_1(x) = \psi_1(x_1, x_2) = \begin{bmatrix} a_1 + cos\theta[(a_1 - x_1)cos\theta + (a_2 - x_2)sin\theta] \\ a_2 + cos\theta[(a_1 - x_1)sin\theta + (x_2 - a_2)cos\theta] \end{bmatrix}$$

Formally,

$$\psi_1 : \tau_1' \circ \mu_1 \circ O_1 \circ \tau_1$$

and

$$\begin{aligned} \tau_1(x) &= x - a \\ O_1 &= \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \mu_1(x) &= cos\theta(x) \\ \tau_1'(x) &= x + a \end{aligned}$$

Now we define two contractive similitudes that operate on $I = [0, 1]$.

Now consider $\Phi : \mathbb{R} \to \mathbb{R} = \phi_0 \cup \phi_1$, where for any number $x \in I$,

$$\phi_0(x) = \frac{x}{2}$$
$$\phi_1(x) = \frac{x+1}{2}$$

"**Definition:**" A fractal is "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole," according to Mandelbrot [4].

Fractals often have self-similar structure. We will show that the trajectory generated by Pòlya's function is self-similar in a sense that is specified below.

Let $S_0 \subset \mathbb{R}^2$ be nonempty. Let χ_0 and χ_1 be contractive similitudes. Define $S_1, S_2, \ldots$ by

$$\begin{aligned} S_1 &= \chi_0(S_0) \cup \chi_1(S_0) \\ S_2 &= \chi_0(S_1) \cup \chi_1(S_1) \\ &\vdots \\ S_n &= \chi_0(S_{n-1}) \cup \chi_1(T_{n-1}) \\ &\vdots \end{aligned}$$

Now let S be defined by

$$S = \overline{\bigcup_{n=0}^{\infty} S_n}$$

Then

$$S = \chi_0(S) \cup \chi_1(S) \tag{25}$$

and S is called the invariant set of $\{\chi_0, \chi_1\}$. Now we introduce important notation:

$$\chi_{\mathbf{i}/n}(S) = \chi_{i_1} \circ \chi_{i_2} \circ \cdots \circ \chi_{i_n}(S)$$

where $\mathbf{i/n} = i_1 i_2 \ldots$ is a sequence of 0's and 1's of fixed length n.

For the invariant set S, we have

$$\begin{aligned} S &= \chi_0(S) \cup \chi_1(S) && \text{all possible } i/n \text{ for } n=1 \\ &= \chi_0\left(\chi_0(S) \cup \chi_1(S)\right) \;\cup\; \chi_1\left(\chi_0(S) \cup \chi_1(S)\right) \\ &= \chi_{00}(S) \;\cup\; \chi_{01}(S) \;\cup\; \chi_{10}(S) \;\cup\; \chi_{11}(S) && \text{all possible } i/n \text{ for } n=2 \end{aligned}$$

Clearly, we can continue using (25) to construct S out of subsets of S in the following manner:

$$\begin{aligned} S &= \chi_0(S) \cup \chi_1(S) && n=1 \\ &= \chi_{00}(S) \;\cup\; \chi_{01}(S) \;\cup\; \chi_{10}(S) \;\cup\; \chi_{11}(S) && n=2 \\ &= \chi_{000}(S) \;\cup\; \chi_{001}(S) \;\cup\; \chi_{010}(S) \;\cup\; \chi_{011}(S) \cup \chi_{100}(S) \;\cup\; \chi_{101}(S) \;\cup\; \chi_{110}(S) \;\cup\; \chi_{111}(S) && n=3 \\ &\vdots && \vdots \end{aligned}$$

So, these subsets that make up S are created by all possible compositions of χ_0 and χ_1 applied to S.

Proposition 1: In general, for every n

$$S = \bigcup_{\mathbf{i}/n} \chi_{\mathbf{i}/n}(S)$$

Then we say that S is self-similar with respect to χ_0 and χ_1.

Now we define what we mean by a self-similar map.

Definition: Let S be, as above, self-similar with respect to χ_0 and χ_1. Let $S' \in \mathbb{R}^2$ be another set which is self-similar with respect to two other contractive similitudes ω_0 and ω_1. Let F be a surjective mapping $F : S \to S'$ such that

$$F(\chi_{\mathbf{i}/n}(S)) = \omega_{\mathbf{i}/n}(S') \tag{26}$$

Then we say that the mapping F is self-similar.

Proposition 2: Let $F : S \to S'$ be surjective and continuous. Let χ_0, χ_1, ω_0, and ω_1 be contractive similitudes. Then when $n \to \infty$, we have

$$\lim_{n\to\infty} F(\chi_{\mathbf{i}/n}(S)) = F(\lim_{n\to\infty} \chi_{\mathbf{i}/n}(S)) = F(\chi_{\mathbf{i}}(S)) \tag{27}$$

and

$$\lim_{n\to\infty} \omega_{\mathbf{i}/n}(S') = \omega_{\mathbf{i}}(S') \tag{28}$$

Then by (26),

$$F(\chi_{\mathbf{i}}(S)) = \omega_{\mathbf{i}}(S') \tag{29}$$

By the contraction principle, we know that $\chi_{\mathbf{i}}(S)$ and $\omega_{\mathbf{i}}(S')$ are specific points.

Now we want to show that the map P is a self-similar.

Lemma 1: *The full triangle T is self-similar with respect to ψ_0 and ψ_1. In particular,*

$$T = \bigcup_{\mathbf{i}/n} \psi_{\mathbf{i}/n}(T) \text{ for every } n \tag{30}$$

Proof of Lemma 1: Let ψ_0 and ψ_1 be contractive similitudes as defined above. Recall that if we draw the altitude of a triangle T_n, this triangle is decomposed into two subtriangles, one larger than the other. ψ_0 maps T_n to the smaller subtriangle and ψ_1 maps T_n to the larger subtriangle. Therefore,

$$T = \psi_0(T) \cup \psi_1(T)$$

Then by Proposition 1, we have (30) and T is self-similar with respect to ψ_0 and ψ_1. $\square$

Lemma 2: *The interval $I = [0, 1]$ is self-similar with respect to ϕ_0 and ϕ_1. In particular,*

$$I = \bigcup_{\mathbf{i}/n} \phi_{\mathbf{i}/n}(I) \text{ for every } n \tag{31}$$

Proof of Lemma 2: Let ϕ_0 and ϕ_1 be contractive similitudes as defined above. If we divide any interval I_n into two halves, we see that ϕ_0 maps I_n to first half of I_n and ϕ_1 maps I_n to the second half of I_n. Therefore,

$$I = \phi_0(I) \cup \phi_1(I)$$

Then by Proposition 1, we have (31) and I is self-similar with respect to ϕ_0 and ϕ_1. $\square$

Theorem: Let P be Pólya's Function. Then $P : I \to T$ is continuous and surjective and we have that

$$P(\phi_{\mathbf{i}/n}(I)) = \psi_{\mathbf{i}/n}(T). \tag{32}$$

Moreover,

$$P(\phi_{\mathbf{i}}(I)) = \psi_{\mathbf{i}}(T). \tag{33}$$

In particular, P is self-similar according to (26).

Proof of Theorem: Let

$$\psi_{\mathbf{i}}(T) = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}(T)$$
$$\phi_{\mathbf{i}}(I) = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}(I)$$

where $\mathbf{i} = i_1 i_2 \ldots$ is an infinite sequence of 0's and 1's.
Let $\mathbf{i/n} = i_1 i_2 \ldots i_n$ be the sequence $\mathbf{i}$ truncated after n elements. Then let

$$I_{\mathbf{i/n}} = \phi_{\mathbf{i/n}}([0,1]) = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$$

Then, since every point in $I_{\mathbf{i/n}}$ has the same first n digits, these points will all be mapped to the same subtriangle of T. We can find which subtriangle of T this is if we take $\psi_{\mathbf{i/n}}(T)$. ψ_0 and ψ_1 model the division of T by the altitude and choosing the smaller or larger triangle, which is the process carried out by Pòlya's function. Then, we have that

$$P(\phi_{\mathbf{i/n}}(I)) = \psi_{\mathbf{i/n}}(T)$$

We have now proven (32).

To prove (33), we use Proposition 2. We have

$$P(\phi_{\mathbf{i}}(I)) = \psi_{\mathbf{i}}(T).$$

and we know that $\phi_{\mathbf{i}}(I)$ and $\psi_{\mathbf{i}}(T)$ are specific points. Thus we have proven (33).

Now we observe that from Lemma 1 and Lemma 2, both T and I are self-similar. Since $P : I \to T$, and P satisfies our definition of a self-similar map, we have shown that the map P is self-similar. $\square$

We can simplify (33) by introducing the following notation:

$$t_{\mathbf{i}} = \phi_{\mathbf{i}}(I)$$
$$P_{\mathbf{i}} = \psi_{\mathbf{i}}(T)$$

Then,

$$P_{\mathbf{i}} = P(t_{\mathbf{i}}) \tag{34}$$

Recall the original construction of Pòlya's function in Chapter 1, where $P(t)$ was determined by the digits $d_1, d_2, d_3, \ldots$ of t. Note that

$$\text{If } \mathbf{i} = d_1 d_2 d_3 \ldots, \text{ then } t_{\mathbf{i}} = \phi_{\mathbf{i}}(I) = t. \tag{35}$$

Equation (34) is actually a parametric equation for P. If $\mathbf{i}$ depends on t as in (35), then $P_{\mathbf{i}}$ produces an x and y coordinate determined by t. Then we have that $P_{\mathbf{i}}$ is a parametric equation not of a simple curve, but of the area of a triangle.

6 Future Work

We suspect that Lax's proof of the differentiability of Pòlya's Function can be simplified by using the parametric equation $P_{\mathbf{i}}$. We intend to revisit the proof using our new ideas.

We also developed an idea for a three-dimensional analog of Polya's function. If we take the trajectory $P_{\mathbf{i}}$ and rotate it around the vertical leg of T, we have a solid cone (as seen in Figures 9 - 12 below). We propose that this process can be formulated as a constructive map $Q : \mathbb{R}^2 \to \mathbb{R}^3$. Specifically, Q maps from a rectangle in the $t\alpha$-plane onto a solid cone, where $t \in [0, 1]$ and $\alpha \in [0, 2\pi]$. Moreover, we propose that this map is surjective, continuous, and differentiable at certain points. Additionally, we think it is possible to show that the rectangle and cone have self-similar structures and that the map Q is self-similar. Therefore, we could create a parametric equation for Q as well, presumably $Q_{\mathbf{i}}$. Similar to $P_{\mathbf{i}}$, $Q_{\mathbf{i}}$ would be a parametric equation of a cone.

We plan to finish and submit this work to a suitable journal.

Below are multiple images that help provide a basic understanding of Q.

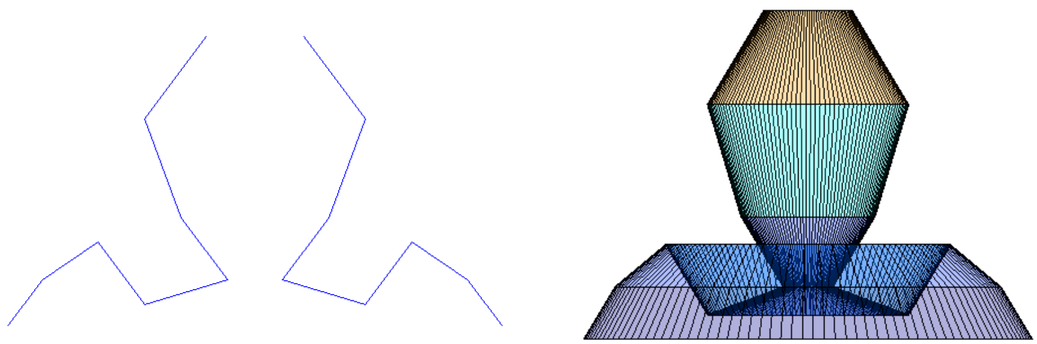


Figure 8: Different views of the object generated by rotating the approximate trajectory of Pòlya's Function after 3 iterations through the third dimension. Left, a view of the object on the plane z = 0. Right, a translucent view of the object.

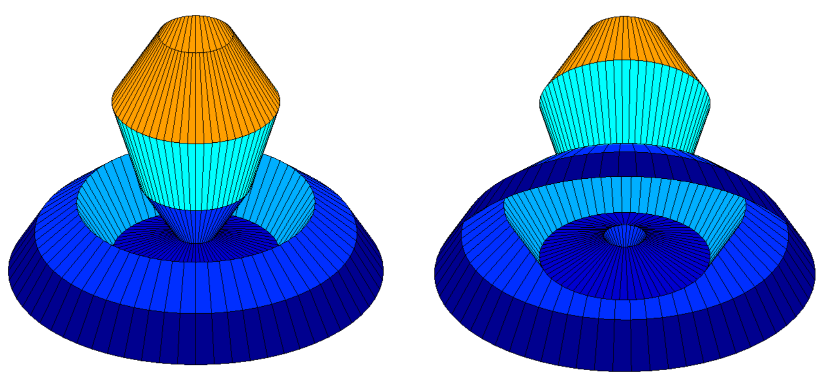


Figure 9: Different views of the solid object generated by rotating the approximate trajectory of Pòlya's Function after 3 iterations through the third dimension

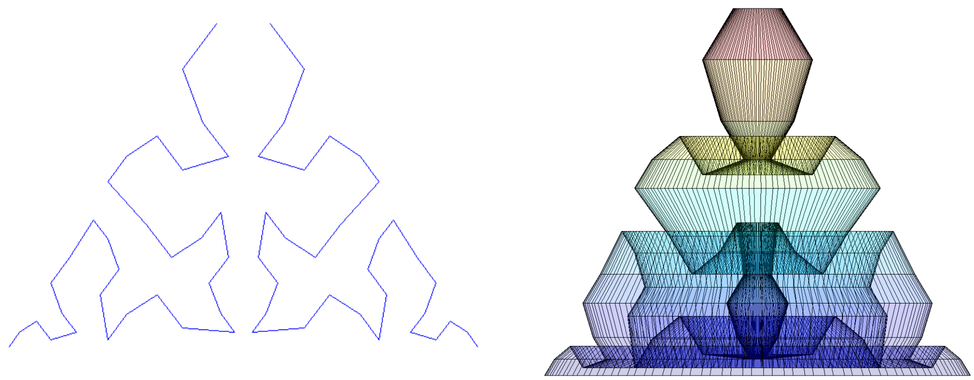


Figure 10: Different views of the object generated by rotating the approximate trajectory of Pòlya's Function after 5 iterations through the third dimension. Left, a view of the object on the plane z = 0. Right, a translucent view of the object.

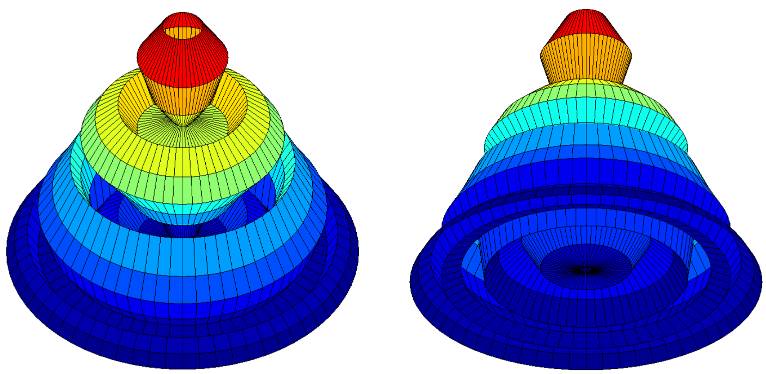


Figure 11: Different views of the object generated by rotating the approximate trajectory of Pòlya's Function after 5 iterations through the third dimension

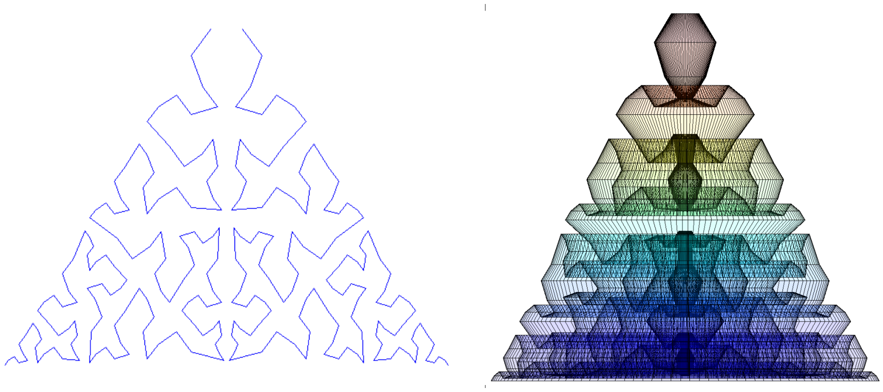


Figure 12: Different views of the object generated by rotating the approximate trajectory of Pòlya's Function after 7 iterations through the third dimension. Left, a view of the object on the plane z = 0. Right, a translucent view of the object.

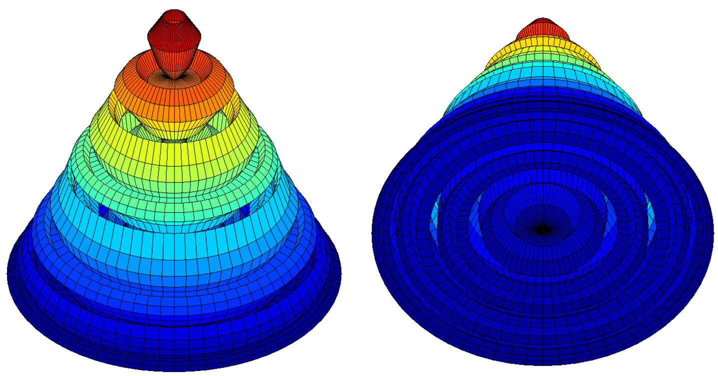


Figure 13: A view of the object generated by rotating the approximate trajectory of Pòlya's Function after 7 iterations through the third dimension

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