

**Optimal Design of Gradient Fields with Applications to
Electrostatics**

by

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Abstract

In this work we consider an optimal design problem formulated on a two dimensional domain filled with two isotropic dielectric materials. The objective is to find a design that supports an electric field which is as close as possible to a target field, under a constraint on the amount of the better dielectric. In the case of a zero target field, the practical purpose of this problem is to avoid the so called dielectric breakdown of the material caused due to a relatively large electric field.

In general, material layout problems of this type fail to have an optimal configuration of the two materials. Instead one must study the behavior of minimizing sequences of configurations. From a practical perspective, optimal or nearly optimal configurations of the two materials are of special interest since they provide the information needed for the manufacturing of optimal designs. Therefore in this work, we develop theoretical and numerical means to support a tractable method for the numerical computation of minimizing sequences of configurations and illustrate our approach through numerical examples.

The same method applies if we were to replace the electric field by electric flux, in our objective functional. Similar optimization design problems can be formulated in the mathematically identical contexts of electrostatics and heat conduction.

*To my Ph.D. advisor,
and inspiring professors and teachers throughout the years.*

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Contents

1	Introduction	1
1.1	Formulation of the Optimal Design Problem	1
1.2	Prior Work	3
1.3	Abstract Results	5
2	Relaxation of the Optimal Design Problem	10
2.1	An Explicit Formula for Layered Material	12
2.2	Formulation of the Relaxed Optimal Design Problem	14
3	Discretization of the Relaxed Optimal Design Problem	16
4	Optimal Designs	24
4.1	Minimizing Sequences of Configurations	24
4.2	Optimal Designs of the Relaxed Problem	29
5	Numerical Approach	34
5.1	Numerical Procedure	34
5.2	Numerical Examples	39
6	Abstract Results on Related Problems and Future Work	43
6.1	Optimal Design Problem on Flux Fields	43

6.2	Optimal Design Problem on Gradient Fields-General Design Space	
	Formulation	48
6.3	Future Work	51
A	Weak Convergence and G-Convergence	53
A.1	Weak Convergence	53
A.2	G -Convergence	54

Chapter 1

Introduction

1.1 Formulation of the Optimal Design Problem

Consider a two dimensional design domain Ω , with Lipschitz boundary, containing two isotropic dielectric materials. The dielectric permittivity is specified by ε and is piece-wise constant taking the values α and β where $0 < \alpha < \beta$. For a prescribed charge density f , the associated electric potential φ satisfies the Poisson equation given by,

$$-\operatorname{div}(\varepsilon \nabla \varphi) = f, \tag{1.1}$$

and $\varphi = 0$ on the boundary of Ω . In order to include the broadest class of charge densities we suppose that f lies in $W^{-1,2}(\Omega)$, ε lies in $L^\infty(\Omega, \{\alpha, \beta\})$ and that φ is a $W_0^{1,2}(\Omega)$ solution of the Poisson equation. The associated electric field in the domain is $E = \nabla \varphi$. We introduce a target electric field $\hat{\mathbf{E}} = \nabla \hat{\varphi}$, where $\hat{\varphi}$ is a potential in $W_0^{1,2}(\Omega)$. For a given charge density, our objective is to design a two phase dielectric that supports an electric field $\nabla \varphi$ that is as close as possible to $\nabla \hat{\varphi}$. Placing a constraint on the amount of the better dielectric β , the design problem is

to minimize the difference,

$$\int_{\Omega} |\nabla\varphi - \nabla\hat{\varphi}|^2 dx, \quad (1.2)$$

over all configurations of the two dielectrics.

In order to precisely formulate the problem, we introduce ω to be the subset of Ω containing the β dielectric. The characteristic function of this set is written as χ , and the dielectric permittivity associated with it is given by,

$$\varepsilon = \varepsilon(\chi) \equiv \beta\chi + \alpha(1 - \chi). \quad (1.3)$$

The space of admissible configurations is denoted by ad_{Θ} , and expresses the constraint on the amount of the β material,

$$ad_{\Theta} = \{\chi : \int_{\Omega} \chi dx \leq \Theta \text{ meas}(\Omega)\}, \quad (1.4)$$

where Θ is a constant, such that $0 < \Theta < 1$.

The objective functional is denoted by $F(\chi, \varepsilon(\chi), \nabla\hat{\varphi})$, and is given by,

$$F(\chi, \varepsilon(\chi), \nabla\hat{\varphi}) = \int_{\Omega} |\nabla\varphi - \nabla\hat{\varphi}|^2 dx, \quad (1.5)$$

where the state variable φ is a solution of (1.1).

As a result, we formulate our design problem as,

$$P = \inf_{\chi \in ad_{\Theta}} F(\chi, \varepsilon(\chi), \nabla\hat{\varphi}). \quad (1.6)$$

In general, material layout problems of this type fail to have an optimal design given by a configuration of the two materials. Instead one must study the behavior

of minimizing sequences of configurations. The purpose of the analysis given here is to provide the methodology for the recovery of optimal configurations when they exist and to identify minimizing sequences of configurations for (1.6) otherwise.

1.2 Prior Work

The issue of nonexistence of optimal configurations for problems of material layout has been the object of much interest. The nonexistence of an optimal configuration for the design problem coincides with the appearance of minimizing sequences containing regions of finite measure where the dielectric permittivity becomes highly oscillatory. As one follows these minimizing sequences, the dielectric permittivity oscillates between the values α and β on progressively finer scales. The classic example is illustrated in the problem of minimizing the energy dissipation associated with configurations of two materials. Using the notation introduced in the previous section in the context of two dielectric materials, the energy dissipation for a configuration is given by,

$$\int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \varphi \, dx.$$

The problem of nonexistence is resolved in an elegant fashion in [5] and [9], by extending the design space to include all effective dielectric permittivities that could be obtained through oscillation. The crucial connection between minimizing sequences of configurations and optimal designs in the extended design space is then established through a continuity property of the energy dissipation in the context of weakly convergent sequences in $L^2(\Omega)^2$, given in [2]. Indeed, consider sequences $\{\varepsilon^\nu \nabla \varphi^\nu\}_{\nu=1}^\infty$ and $\{\nabla \varphi^\nu\}_{\nu=1}^\infty$, such that $-\operatorname{div}(\varepsilon^\nu \nabla \varphi^\nu) = f$. According to the G -convergence theory, see Appendix for definition, these sequences weakly converge to

the limits $\varepsilon^\infty \nabla \varphi^\infty$ and $\nabla \varphi^\infty$, where ε^∞ is an effective tensor in the extended space of designs satisfying, $-\operatorname{div}(\varepsilon^\infty \nabla \varphi^\infty) = f$, with φ^∞ in $W_0^{1,2}$. As a result one has,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} \varepsilon^\nu \nabla \varphi^\nu \cdot \nabla \varphi^\nu \, dx = \int_{\Omega} \varepsilon^\infty \nabla \varphi^\infty \cdot \nabla \varphi^\infty \, dx.$$

For the design problem treated here, one can attempt to resolve the nonexistence problem by extending the design space to include effective properties. However unlike the energy dissipation and other continuous functionals treated earlier, the objective functional given by (1.5) is not continuous with respect to the weak convergence. Thus additional theoretical work is required to provide the connection between an extended space of designs and minimizing sequences of configurations.

The goal of this thesis is to identify minimizing sequences of configurations for the P problem stated in (1.6), for any target potential $\hat{\varphi} \in W_0^{1,2}$.

Earlier work provides a way to characterize minimizing sequences of configurations for (1.6), but only for a special class of target potentials. This work is shown in [15], and for completeness is described below. The class of target potentials for which minimizing sequences of configurations can be found, is motivated by the following theorem stated in [15].

Theorem 1. *Let S be a non-empty strongly closed subset of a Hilbert space H . Then there exists a dense G_δ subset K of H such that for any $x \in K$, the minimizing sequences $\{c_\nu\}_{\nu=1}^\infty \in S$ of the function $c \rightarrow \|x - c\|$ are Cauchy sequences. In particular the subset of points of H with a unique projection on S contains a dense G_δ subset, as it contains K .*

In the context of optimal design, a relevant Hilbert space is $W_0^{1,2}(\Omega)$ with the inner product $(\phi, \psi) = \int_{\Omega} \nabla \phi \cdot \nabla \psi \, dx$. We apply Theorem 1 for the strong $W_0^{1,2}(\Omega)$

closure of the set S_Θ given by,

$$S_\Theta = \begin{cases} \varphi \mid \varphi \text{ is a } W_0^{1,2}(\Omega) \text{ solution of } -\operatorname{div}(\varepsilon(\chi)\nabla\varphi) = f, \\ \chi \in ad_\Theta. \end{cases} \quad (1.7)$$

The strong $W_0^{1,2}(\Omega)$ closure of the set S_Θ , denoted by \bar{S}_Θ , is established in [15] and [3], and is given by,

$$\bar{S}_\Theta = \begin{cases} \varphi \mid \varphi \text{ is a } W_0^{1,2}(\Omega) \text{ solution of } -\operatorname{div}(m_\theta\nabla\varphi) = f, \\ \text{for some } \theta \in L^\infty(\Omega, [0, 1]), \text{ such that } m_\theta = \alpha(1 - \theta) + \beta\theta. \end{cases} \quad (1.8)$$

From the definition of \bar{S}_Θ and the application of Theorem 1, it follows that there exists a dense G_δ subset K of target potentials $\hat{\varphi}$ in $W_0^{1,2}(\Omega)$, for which,

$$\begin{aligned} P &= \inf_{\chi \in ad_\Theta} \int_\Omega |\nabla\varphi - \nabla\hat{\varphi}|^2 dx = \inf_{\varphi \in S_\Theta} \int_\Omega |\nabla\varphi - \nabla\hat{\varphi}|^2 dx = \\ &= \min_{\varphi \in \bar{S}_\Theta} \int_\Omega |\nabla\varphi - \nabla\hat{\varphi}|^2 dx. \end{aligned} \quad (1.9)$$

As the above equalities suggest, the relaxation of the P problem for target potentials $\hat{\varphi} \in K$, can be done by extending the design space to include the scalar coefficients m_θ . But even so, there is no explicit representation for the set of the target potentials K , therefore the identification of the minimizing sequences of configurations for any choice of target potentials $\hat{\varphi} \in W_0^{1,2}(\Omega)$, still remains a problem and it is treated here.

1.3 Abstract Results

With the final goal to identify minimizing sequences of configurations for the P problem (1.6), we take our first step in Chapter 2 and relax the problem. In order

to accomplish this, we relax the design space ad_{Θ} described in (1.4), and choose the new objective functional to be a continuous extension of the original functional (1.5) for piecewise oscillating sequences of layered designs. This relaxation is motivated by oscillating sequences of layered designs and is explained below. We describe an oscillatory sequence of layered material by $\chi^\nu(x) = \mu(\nu x \cdot n)$, $\nu = 1, 2, \dots, \infty$, where $\mu(t)$ is a periodic function on the real line of period unity taking the values 1 for $0 \leq t \leq \theta$ and 0 for $\theta < t \leq 1$. Here θ is a constant, $0 < \theta < 1$, representing the relative thickness of the layers of the β material while $n = (\cos \gamma, \sin \gamma)$ represents the normal vector to the layers. The sequence of permittivities $\{\varepsilon(\chi_\nu)\}_{\nu=1}^\infty$ G -converges to an effective permittivity tensor $\varepsilon(\theta, \gamma)$, related to a composite material called a rank one laminate, and expressed as,

$$\varepsilon(\theta, \gamma) = R(\gamma)\Lambda(\theta)R^T(\gamma).$$

Here $R(\gamma)$ is the rotation matrix with angle γ , $\Lambda(\theta) = \begin{pmatrix} h_\theta & 0 \\ 0 & m_\theta \end{pmatrix}$, where $h_\theta = (\frac{1-\theta}{\alpha} + \frac{\theta}{\beta})^{-1}$ and $m_\theta = \alpha(1-\theta) + \beta\theta$ are the harmonic and arithmetic means of the two dielectric permittivities α and β . On the other hand, the corresponding sequence of the potentials, $\{\varphi^\nu\}_{\nu=1}^\infty$ satisfying the equilibrium equation (1.1), weakly converges to $\varphi \in W_0^{1,2}$, satisfying the homogenized equilibrium equation,

$$-\operatorname{div}(\varepsilon(\theta, \gamma)\nabla\varphi) = f.$$

In general for a weakly converging sequence of potentials $\{\varphi^\nu\}_{\nu=1}^\infty$ to φ in $W_0^{1,2}$, one writes,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} F(\chi^\nu, \varepsilon(\chi^\nu), \nabla\hat{\varphi}) dx = \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\hat{\varphi}|^2 dx =$$

$$= \int_{\Omega} |\nabla\varphi - \nabla\hat{\varphi}|^2 dx + \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\varphi|^2 dx.$$

However in this particular case, we are able to calculate the closed form expression given by,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} F(\chi^\nu, \varepsilon(\chi^\nu), \nabla\hat{\varphi}) &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\hat{\varphi}|^2 dx = \\ &= \int_{\Omega} |\nabla\varphi - \nabla\hat{\varphi}|^2 dx + \int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla\varphi \cdot \nabla\varphi dx, \end{aligned}$$

$$\text{where } H(\theta) = \begin{pmatrix} (\frac{1}{\alpha} - \frac{1}{\beta})^2\theta(1-\theta)h_\theta^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Motivated by this result, we extend our design space to include generalized designs associated with rank one laminate materials,

$$\begin{aligned} D_\Theta &= \{ (\theta, \gamma, \varepsilon(\theta, \gamma)) : \theta \in L^\infty(\Omega; [0, 1]); \gamma \in L^\infty(\Omega; [0, 2\pi]); \\ &\quad \varepsilon(\theta, \gamma) = R(\gamma)\Lambda(\theta)R^T(\gamma) ; \int_{\Omega} \theta dx \leq \Theta \text{ meas}(\Omega) \}. \end{aligned}$$

One easily sees that the extended space of designs D_Θ contains the original space of pure material designs ad_Θ . We also propose a new objective functional given by,

$$RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla\hat{\varphi}) = \int_{\Omega} |\nabla\varphi - \nabla\hat{\varphi}|^2 dx + \int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla\varphi \cdot \nabla\varphi dx.$$

Here we point out that $F(\chi, \varepsilon(\chi), \nabla\hat{\varphi}) = RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla\hat{\varphi})$ when $\theta = \chi \in ad_\Theta$.

The relaxed design problem, referred to as the RP problem, is formulated as,

$$RP = \inf_{(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_\Theta} RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla\hat{\varphi}).$$

For a special class of target potentials, we show in Section 4.2 that the RP problem has an optimal design attainable by minimizing sequences of configurations. However in general, we focus on identifying minimizing sequences for the RP problem. We establish that for every $\hat{\varphi} \in W_0^{1,2}(\Omega)$ and $f \in W^{-1,2}(\Omega)$,

$$P = RP.$$

Our approach is to first identify a minimizing sequence of generalized designs in D_Θ for the RP problem, and next to apply the continuity property of the extended functional to generate a minimizing sequence of pure material designs for the P problem. We accomplish this in Chapters 3 and 4, by following the three steps given below.

I) We consider a partition of Ω into disjoint subdomains of diameter less than κ and introduce a discrete approximation of the design space D_Θ , denoted by D_Θ^κ . The design space D_Θ^κ consists of designs of rank one laminates with constant effective properties in each subdomain of the partition. We show that the design problem given by,

$$RP^\kappa = \inf_{(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa)) \in D_\Theta^\kappa} RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}),$$

and referred to as the RP^κ problem, has an optimal design denoted by $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$. Moreover, as indicated in Chapter 3, one can approach this design by a piecewise oscillatory sequence of layered designs of α and β materials.

II) As we further refine the partitions of Ω , we show in Chapter 4 that the sequence of optimal designs $\{(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))\}_{\kappa>0}$ for the discrete problems $\{RP^\kappa\}_{\kappa>0}$, is a minimizing sequence for the RP problem.

III) Finally, by approaching each design $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$, by fine layers of α and β materials, we construct a minimizing sequence for the P problem.

We point out that the discrete RP^κ problem for a fixed κ , is of interest on its own right. From a practical perspective there is a prohibitive manufacturing cost incurred when attempting to make a material with possibly different anisotropic dielectric properties at every point. Instead there is a smallest scale κ over which the dielectric properties change. The scale is set by the manufacturing cost. Practically speaking one partitions the design domain into subdomains of diameter κ and inside these subdomains one optimizes the dielectric properties. This approach is naturally incorporated in the formulation of the discrete problem given here and it is used in the context of the numerical procedures and examples in Chapter 5.

As we numerically compute the optimal design of the discrete design problem for a fine partition of Ω , we conclude that such a design, approached by fine layers of α and β materials is a nearly optimal design for the P problem. In this way we provide all the information needed for the manufacturing of nearly optimal designs made of α and β materials. The numerical examples included in Chapter 5 provide optimal designs for several settings and illustrate how electrostatic fields can be controlled by using functionally graded materials.

Abstract results on other problems which can be solved by the same procedures used to solve the P problem and the relaxed RP problem, as well as future work, are given in Chapter 6. Finally in the Appendix we provide the definitions for the concepts of weak convergence, G -convergence and its more generalized concept of H -convergence.

Chapter 2

Relaxation of the Optimal Design

Problem

Our methodology to identify minimizing sequences of configurations is based on a careful extension of the design space ad_{Θ} and the replacement of the objective functional (1.5) by a suitable relaxed functional, associated with the extended design space. As discussed in Section 1.2, any attempt to identify minimizing sequences of configurations must account for the possibility of oscillations in the associated sequence of gradients. We use the weak $L^{\infty}(\Omega)$ star convergence of the sequence of characteristic functions $\{\chi^{\nu}\}_{\nu=1}^{\infty}$ to describe the oscillation of the sequence of configurations, and the weak $L^2(\Omega)^2$ convergence to characterize the oscillation of the associated sequence of gradients. To fix ideas, we denote by $\{\chi^{\nu}\}_{\nu=1}^{\infty}$, the sequence of characteristic functions weak $L^{\infty}(\Omega)$ star converging to some density θ in $L^{\infty}(\Omega)$ where $0 \leq \theta \leq 1$. We denote by $\{\nabla\varphi^{\nu}\}_{\nu=1}^{\infty}$ the weakly converging sequence of gradients related to the sequence of configurations through the equilibrium condition,

$$-\operatorname{div} (\varepsilon(\chi^{\nu})\nabla\varphi^{\nu}) = f. \tag{2.1}$$

The weak limit of the sequence of gradients, denoted by $\nabla\tilde{\varphi}$, satisfies the homogenized equilibrium equation given by,

$$-\operatorname{div}(\varepsilon^e \nabla\tilde{\varphi}) = f. \quad (2.2)$$

The tensor ε^e is called the effective tensor or the G -limit of the sequence of dielectric tensors $\{\varepsilon(\chi^\nu)\}_{n=1}^\infty$. Related to the sequence of gradients, one can write the following,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \int_{\Omega} F(\chi^\nu, \varepsilon(\chi^\nu), \nabla\hat{\varphi}) &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\hat{\varphi}|^2 dx = \\ &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |(\nabla\varphi^\nu - \nabla\tilde{\varphi}) + (\nabla\tilde{\varphi} - \nabla\hat{\varphi})|^2 dx = \\ &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\tilde{\varphi}|^2 dx + \\ &+ 2 \lim_{\nu \rightarrow \infty} \int_{\Omega} (\nabla\varphi^\nu - \nabla\tilde{\varphi})(\nabla\tilde{\varphi} - \nabla\hat{\varphi}) dx + \int_{\Omega} |\nabla\tilde{\varphi} - \nabla\hat{\varphi}|^2 dx. \end{aligned}$$

From the fact that $\nabla\tilde{\varphi}$ is the weak limit of the sequence of gradients, it follows that,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} (\nabla\varphi^\nu - \nabla\tilde{\varphi})(\nabla\tilde{\varphi} - \nabla\hat{\varphi}) dx = 0,$$

and therefore,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \int_{\Omega} F(\chi^\nu, \varepsilon(\chi^\nu), \nabla\hat{\varphi}) &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\hat{\varphi}|^2 dx = \\ &= \int_{\Omega} |\nabla\tilde{\varphi} - \nabla\hat{\varphi}|^2 dx + \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\tilde{\varphi}|^2 dx. \end{aligned} \quad (2.3)$$

The oscillatory behavior of minimizing sequences is naturally linked to the dependence of the limit,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\tilde{\varphi}|^2 dx, \quad (2.4)$$

on the weak limits $\nabla\tilde{\varphi}$, θ , and on other weak limits of other geometric quantities. Our methodology for identifying minimizing sequences is based upon writing (2.4) as an explicit function of the relevant weak limits.

2.1 An Explicit Formula for Layered Material

Although at the present time we are unable to write a formula for every type of oscillation, we show that,

Theorem 2. *For the case of oscillating layers of two materials, with θ representing the relative thickness of the β layer and γ representing the normal direction to the layers, there exists a closed form expression for (2.4), given by,*

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^{\nu} - \nabla\tilde{\varphi}|^2 dx = \int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla\tilde{\varphi} \cdot \nabla\tilde{\varphi} dx. \quad (2.5)$$

Here $R(\gamma)$ is the rotation matrix with angle γ , while the matrix $H(\theta)$ is a function of the density θ given by,

$$H(\theta) = \begin{pmatrix} (\frac{1}{\alpha} - \frac{1}{\beta})^2\theta(1-\theta)h_{\theta}^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.6)$$

where $h_{\theta} = (\frac{1-\theta}{\alpha} + \frac{\theta}{\beta})^{-1}$ is the harmonic mean of the two dielectric permittivities α and β .

Proof. The theorem follows from the corrector theory of homogenization given in [1] and [8]. We describe an oscillatory sequence of layered material by $\chi^{\nu}(x) = \mu(\nu x \cdot n)$, $\nu = 1, 2, \dots, \infty$, where $\mu(t)$ is a periodic function on the real line of period unity taking the values 1 for $0 \leq t \leq \theta$ and 0 for $\theta < t \leq 1$. Here θ is a constant, $0 < \theta < 1$, representing the relative thickness of the layers of the β material while

$n = (\cos \gamma, \sin \gamma)$ represents the normal vector to the layers. The oscillatory sequence $\{\chi^\nu\}_{\nu=1}^\infty$ described this way, weak $L^\infty(\Omega)$ star converges to the density θ in $L^\infty(\Omega)$. In this case, the effective tensor appearing in the homogenized equation (2.2) and related to the sequence of dielectric tensors $\{\varepsilon(\chi^\nu)\}_{\nu=1}^\infty$, represents a so-called rank one laminate material and is given by the following formula,

$$\varepsilon(\theta, \gamma) = R(\gamma)\Lambda(\theta)R^T(\gamma), \quad (2.7)$$

where $R(\gamma)$ is the rotation matrix with a rotation angle of γ radians, $0 \leq \gamma \leq 2\pi$, $\Lambda(\theta)$ is the diagonal tensor $\Lambda(\theta) = \begin{pmatrix} h_\theta & 0 \\ 0 & m_\theta \end{pmatrix}$, while $h_\theta = (\frac{1-\theta}{\alpha} + \frac{\theta}{\beta})^{-1}$ and $m_\theta = \alpha(1-\theta) + \beta\theta$ are the harmonic and the arithmetic means of the two dielectric permittivities. Applying the corrector theory of homogenization given in [8], one has that $\nabla\varphi^\nu = P^\nu\nabla\tilde{\varphi} + z^\nu$, where P^ν is the corrector matrix associated with χ^ν and is given by,

$$P^\nu = R(\gamma) \begin{pmatrix} \frac{h_\theta}{[\alpha(1-\chi^\nu) + \beta\chi^\nu]} & 0 \\ 0 & 1 \end{pmatrix} R^T(\gamma).$$

Here $P^\nu \rightharpoonup I$ weakly in $L^2(\Omega)^{2 \times 2}$, where I represents the 2-by-2 identity matrix. Since $P^\nu \in L^\infty(\Omega)^{2 \times 2}$, the corrector theorem given in [8], implies that $z^\nu \rightarrow 0$ strongly in $L^2(\Omega)^2$ as $\nu \rightarrow \infty$. As a consequence we obtain the following sequence of equalities,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla\varphi^\nu - \nabla\tilde{\varphi}|^2 dx = \lim_{\nu \rightarrow \infty} \int_{\Omega} |(P^\nu - I)\nabla\tilde{\varphi} + z^\nu|^2 dx = \quad (2.8)$$

$$= \lim_{\nu \rightarrow \infty} \int_{\Omega} R(\gamma) \begin{pmatrix} \left(\frac{h_{\theta}}{[\alpha(1-\chi^{\nu}) + \beta\chi^{\nu}]} - 1 \right)^2 & 0 \\ 0 & 0 \end{pmatrix} R^T(\gamma) \nabla \tilde{\varphi} \cdot \nabla \tilde{\varphi} \, dx = \quad (2.9)$$

$$= \int_{\Omega} R(\gamma) H(\theta) R^T(\gamma) \nabla \tilde{\varphi} \cdot \nabla \tilde{\varphi} \, dx, \quad (2.10)$$

and the theorem follows.

2.2 Formulation of the Relaxed Optimal Design Problem

The methodology presented here uses the explicit formula given by (2.5). Motivated by the layer case, our approach is to replace χ and $\varepsilon(\chi)$ with the new design variables θ , γ , and $\varepsilon(\theta, \gamma)$ given by (2.7). The admissible space of designs for the new design problem is given by,

$$D_{\Theta} = \{ (\theta, \gamma, \varepsilon(\theta, \gamma)) \mid \theta \in L^{\infty}(\Omega; [0, 1]); \gamma \in L^{\infty}(\Omega; [0, 2\pi]); \\ \varepsilon(\theta, \gamma) = R(\gamma) \Lambda(\theta) R^T(\gamma); \int_{\Omega} \theta \, dx \leq \Theta \, \text{meas} \Omega \}. \quad (2.11)$$

In addition we introduce the new objective functional given by,

$$RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}) = \int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^2 \, dx + \int_{\Omega} R(\gamma) H(\theta) R^T(\gamma) \nabla \varphi \cdot \nabla \varphi \, dx, \quad (2.12)$$

where the state variable φ is the $W_0^{1,2}$ solution of,

$$-\text{div} (\varepsilon(\theta, \gamma) \nabla \varphi) = f. \quad (2.13)$$

We formulate the relaxed design problem as,

$$RP = \inf_{(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_\Theta} RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}). \quad (2.14)$$

Remark 1. *The extended space of designs D_Θ contains the original space ad_Θ of the pure material designs.*

Remark 2. $P \geq RP$.

The above remarks follow from the fact that for $\theta = \chi$, we have $\varepsilon(\theta, \gamma) = \varepsilon(\chi)$, $H(\theta) = 0$, and $F(\chi, \varepsilon(\chi), \nabla \hat{\varphi}) = RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi})$. Later in Chapter 4, we show that in fact $P = RP$.

Chapter 3

Discretization of the Relaxed Optimal Design Problem

In this chapter we pose the relaxed design problem (2.14), on a discrete approximation of the design space D_Θ . We first establish in the discretized space the existence of an optimal design, then we apply the corrector theory in [8], to approach such design by a piecewise oscillatory sequence of layered designs of α and β materials.

We consider any partition T^κ of Ω consisting of a finite number of pair-wise disjoint subdomains $\Omega_i \subset \Omega$, $i = 1, \dots, N(\kappa)$, such that,

$$\Omega = \bigcup_{i=1}^{N(\kappa)} \Omega_i \quad \text{and} \quad \max_{i=1, \dots, N(\kappa)} (\text{diam}(\Omega_i)) \leq \kappa. \quad (3.1)$$

For a fixed partition T^κ , the discretized space of designs denoted by D_Θ^κ , is given in terms of piece-wise constant functions as follows,

$$D_{\Theta}^{\kappa} = \{ (\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa})) \mid \theta^{\kappa}|_{\Omega_i} = \theta_i^{\kappa} = \text{constant}; 0 \leq \theta_i^{\kappa} \leq 1; \quad (3.2)$$

$$\gamma^{\kappa}|_{\Omega_i} = \gamma_i^{\kappa} = \text{constant}; 0 \leq \gamma_i^{\kappa} \leq 2\pi; \quad (3.3)$$

$$\varepsilon(\theta^{\kappa}, \gamma^{\kappa})|_{\Omega_i} = R(\gamma^{\kappa})\Lambda(\theta^{\kappa})R^T(\gamma^{\kappa})|_{\Omega_i} = R(\gamma_i^{\kappa})\Lambda(\theta_i^{\kappa})R^T(\gamma_i^{\kappa}); \quad (3.4)$$

$$\sum_{i=1}^{N(\kappa)} \theta_i^{\kappa} \text{meas}(\Omega_i) = \Theta \text{meas}(\Omega) \}. \quad (3.5)$$

It is clear that D_{Θ}^{κ} is contained in D_{Θ} and the relaxed design problem posed on this smaller set of designs is written as,

$$RP^{\kappa} = \inf_{(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa})) \in D_{\Theta}^{\kappa}} RF(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa}), \nabla \hat{\varphi}), \quad (3.6)$$

where the state variable φ^{κ} associated with the piece-wise constant dielectric permittivity tensor $\varepsilon(\theta^{\kappa}, \gamma^{\kappa})$, solves the Poisson equation,

$$-\text{div} (\varepsilon(\theta^{\kappa}, \gamma^{\kappa}) \nabla \varphi^{\kappa}) = f. \quad (3.7)$$

We establish the existence of an optimal design for the RP^{κ} problem, by using the direct method of the calculus of variations. We start by introducing the type of convergence relevant to the discrete problem. Based on (3.2-3.3), a design $(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa}))$ in D_{Θ}^{κ} can be identified with the vector $(\theta_1^{\kappa}, \gamma_1^{\kappa}, \dots, \theta_i^{\kappa}, \gamma_i^{\kappa}, \dots, \theta_{N(\kappa)}^{\kappa}, \gamma_{N(\kappa)}^{\kappa})$ in $R^{2N(\kappa)}$. Thus D_{Θ}^{κ} can be identified with a compact subset of $R^{2N(\kappa)}$ and convergence of designs in D_{Θ}^{κ} is given by sequential convergence in $R^{2N(\kappa)}$. The existence of an optimal design will follow once we show that the functional $RF(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa}), \nabla \hat{\varphi})$ is continuous with respect to sequential convergence in $R^{2N(\kappa)}$.

Theorem 3. *For a given a sequence of designs $\{(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}, \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}))\}_{\nu=1}^{\infty}$ in D_{Θ}^{κ} ,*

there exists a design $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$ in D_Θ^κ , for which,

$$\lim_{\nu \rightarrow \infty} (\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) = (\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \quad \text{as elements of } R^{2N(\kappa)}, \quad (3.8)$$

$$\lim_{\nu \rightarrow \infty} \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) = \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \quad \text{pointwise in } \Omega, \quad (3.9)$$

and,

$$\lim_{\nu \rightarrow \infty} RF(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}, \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}), \nabla \hat{\varphi}) = RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}). \quad (3.10)$$

Proof. The convergence of the sequence given by (3.8) follows from the compactness property in $R^{2N(\kappa)}$ and it further implies the pointwise convergence of the sequences $\{\varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu})\}_{\nu=1}^\infty$ and $\{R(\gamma^{\kappa, \nu})H(\theta^{\kappa, \nu})R^T(\gamma^{\kappa, \nu})\}_{\nu=1}^\infty$ to $\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$ and $R(\bar{\gamma}^\kappa)H(\bar{\theta}^\kappa)R^T(\bar{\gamma}^\kappa)$ respectively, as $\nu \rightarrow \infty$, establishing this way (3.9). From here, the continuity property given by (3.10), will be established once we show that as $\nu \rightarrow \infty$, the sequence of potentials $\{\varphi^{\kappa, \nu}\}_{\nu=1}^\infty$ converges strongly in $W_0^{1,2}(\Omega)$. Indeed, using the theory of G -convergence, one derives that the sequence of tensors $\{\varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu})\}_{\nu=1}^\infty$ G -converges to $\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$, and therefore the associated sequence of potentials $\{\varphi^{\kappa, \nu}\}_{\nu=1}^\infty$ converges weakly in $W_0^{1,2}(\Omega)$ to the state variable $\bar{\varphi}^\kappa$, associated with $\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$ through the homogenized equilibrium equation,

$$-\text{div} (\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa) \nabla \bar{\varphi}^\kappa) = f. \quad (3.11)$$

With these facts in mind and the following estimate,

$$0 < \alpha I \leq \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) \leq \beta I,$$

where I is the 2-by-2 identity matrix, we write,

$$\begin{aligned}
\int_{\Omega} \alpha |\nabla \varphi^{\kappa, \nu} - \nabla \bar{\varphi}^{\kappa}|^2 dx &\leq \int_{\Omega} \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) (\nabla \varphi^{\kappa, \nu} - \nabla \bar{\varphi}^{\kappa}) \cdot (\nabla \varphi^{\kappa, \nu} - \nabla \bar{\varphi}^{\kappa}) dx = \quad (3.12) \\
&= \int_{\Omega} \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) \nabla \varphi^{\kappa, \nu} \cdot \nabla \varphi^{\kappa, \nu} dx - 2 \int_{\Omega} \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) \nabla \varphi^{\kappa, \nu} \cdot \nabla \bar{\varphi}^{\kappa} dx + \\
&+ \int_{\Omega} \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}) \nabla \bar{\varphi}^{\kappa} \cdot \nabla \bar{\varphi}^{\kappa} dx. \quad (3.13)
\end{aligned}$$

Passing to the limit as $\nu \rightarrow \infty$ in (3.12-3.13), we apply the well known properties of G -convergence together with the pointwise convergence of $\{\varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu})\}_{\nu=1}^{\infty}$ and the Lebesgue convergence theorem to find that,

$$\lim_{\nu \rightarrow \infty} \|\nabla \varphi^{\kappa, \nu} - \nabla \bar{\varphi}^{\kappa}\|_{L^2}^2 = 0, \quad (3.14)$$

and strong convergence of $\varphi^{\kappa, \nu}$ to $\bar{\varphi}^{\kappa}$ in $W_0^{1,2}$ follows.

From (2.6) we easily obtain the following estimate,

$$\forall \eta \in \mathbb{R}^2, \quad R(\gamma)H(\theta)R^T(\gamma)\eta \cdot \eta \leq \frac{\beta^2}{4} \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 |\eta|^2. \quad (3.15)$$

Finally we put everything together, and we apply (3.15) and the Lebesgue convergence theorem to conclude that,

$$\begin{aligned}
&\lim_{\nu \rightarrow \infty} RF(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}, \varepsilon(\theta^{\kappa, \nu}, \gamma^{\kappa, \nu}), \nabla \hat{\varphi}) = \\
&= \lim_{\nu \rightarrow \infty} \left(\int_{\Omega} |\nabla \varphi^{\kappa, \nu} - \nabla \hat{\varphi}|^2 dx + \int_{\Omega} R(\gamma^{\kappa, \nu})H(\theta^{\kappa, \nu})R^T(\gamma^{\kappa, \nu}) \nabla \varphi^{\kappa, \nu} \cdot \nabla \varphi^{\kappa, \nu} dx \right) \\
&= \int_{\Omega} |\nabla \bar{\varphi}^{\kappa} - \nabla \hat{\varphi}|^2 dx + \int_{\Omega} R(\bar{\gamma}^{\kappa})H(\bar{\theta}^{\kappa})R^T(\bar{\gamma}^{\kappa}) \nabla \bar{\varphi}^{\kappa} \cdot \nabla \bar{\varphi}^{\kappa} dx = \\
&= RF(\bar{\theta}^{\kappa}, \bar{\gamma}^{\kappa}, \varepsilon(\bar{\theta}^{\kappa}, \bar{\gamma}^{\kappa}), \nabla \hat{\varphi}), \quad (3.16)
\end{aligned}$$

which proves the theorem.

Applying Theorem 3 for a minimizing sequence of designs, it follows that,

Theorem 4. *There exists an optimal design $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$ in D_Θ^κ for the discrete problem, i.e.,*

$$\begin{aligned} RP^\kappa &= RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}) = \\ &= \min_{(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa)) \in D_\Theta^\kappa} RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}). \end{aligned} \quad (3.17)$$

Now we show how to construct a sequence of configurations which approaches an optimal design, i.e. a sequence of configurations described by the sequence of characteristic functions $\{\chi^\nu\}_{\nu=1}^\infty$, which satisfies,

$$\lim_{\nu \rightarrow \infty} F(\chi^\nu, \varepsilon(\chi^\nu), \nabla \hat{\varphi}) = RP^\kappa. \quad (3.18)$$

In view of Theorem 4, it is sufficient to consider any design in D_Θ^κ given by $(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa))$, and show how to construct a sequence $\{\chi^{\kappa, \nu}\}_{\nu=1}^\infty$ for which,

$$\lim_{\nu \rightarrow \infty} F(\chi^{\kappa, \nu}, \varepsilon(\chi^{\kappa, \nu}), \nabla \hat{\varphi}) = RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}). \quad (3.19)$$

We start by observing that for $\theta = 0$ or $\theta = 1$ that $\varepsilon(\theta, \gamma) = \alpha I$ or βI respectively. For a design specified by $(\theta^\kappa, \gamma^\kappa)$ we proceed to construct the sequence $\{\chi^{\kappa, \nu}\}_{\nu=1}^\infty$ as an oscillatory sequence of locally layered material. Thus in the subdomains Ω_i for which $\theta_i^\kappa = 0$, we set $\chi^{\kappa, \nu} = 0$, $\nu = 1, 2 \dots \infty$ and in the subdomains Ω_i for which $\theta_i^\kappa = 1$, we set $\chi^{\kappa, \nu} = 1$, $\nu = 1, 2 \dots \infty$. Next we consider the subdomains Ω_i where $0 < \theta_i^\kappa < 1$. In these subdomains, as previously described in Chapter 2, we set $\chi^{\kappa, \nu} = \mu(\nu x \cdot n(\gamma_i^\kappa))$, where $\mu(t)$ is a periodic function on the real line of period unity taking the values 1 for $0 \leq t \leq \theta_i^\kappa$ and 0 for $\theta_i^\kappa < t \leq 1$ and $n(\gamma_i^\kappa) = (\cos \gamma_i^\kappa, \sin \gamma_i^\kappa)$

where $0 \leq \gamma_i^k \leq 2\pi$. Summarizing our construction we write,

$$\chi^{\kappa,\nu} = \begin{cases} 0, & \text{in } \Omega_i \text{ for which } \theta_i^\kappa = 0, \\ 1, & \text{in } \Omega_i \text{ for which } \theta_i^\kappa = 1, \\ \mu(\nu x \cdot n(\gamma_i^\kappa)), & \text{in } \Omega_i \text{ for which } 0 < \theta_i^\kappa < 1. \end{cases} \quad (3.20)$$

We point out the following facts related with the construction:

- The associated dielectric permittivity $\varepsilon(\chi^{\kappa,\nu})$ corresponds to pure α dielectric in the subdomains Ω_i where $\theta_i^\kappa = 0$, pure β dielectric in the subdomains Ω_i where $\theta_i^\kappa = 1$, and layers of α and β dielectric with layer normal in the direction $n(\gamma_i^\kappa) = (\cos \gamma_i^\kappa, \sin \gamma_i^\kappa)$ in the subdomains Ω_i where $0 < \theta_i^\kappa < 1$.
- The sequence $\{\varepsilon(\chi^{\kappa,\nu})\}_{\nu=1}^\infty$ G -converges to $\varepsilon(\theta^\kappa, \gamma^\kappa)$, hence the sequence of the associated state variables $\varphi^{\kappa,\nu}$ in $W_0^{1,2}(\Omega)$ satisfying the equilibrium equation,

$$-\operatorname{div} (\varepsilon(\chi^{\kappa,\nu}) \nabla \varphi^{\kappa,\nu}) = f, \quad (3.21)$$

converges weakly in $W_0^{1,2}(\Omega)$ to the state variable φ^κ associated with $\varepsilon(\theta^\kappa, \gamma^\kappa)$ through the homogenized equilibrium equation,

$$-\operatorname{div} (\varepsilon(\theta^\kappa, \gamma^\kappa) \nabla \varphi^{\kappa,\nu}) = f. \quad (3.22)$$

With the construction (3.20) in mind, we state the following theorem that guarantees the existence of a recovery sequence of configurations.

Theorem 5. *For a given design $(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa))$ in $D_{\mathfrak{D}}^\kappa$, the sequence of configurations given by $\{\chi^{\kappa,\nu}\}_{\nu=1}^\infty$ in (3.20), is a recovery sequence, i.e.,*

$$\lim_{\nu \rightarrow \infty} F(\chi^{\kappa,\nu}, \varepsilon(\chi^{\kappa,\nu}), \nabla \hat{\varphi}) = RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}). \quad (3.23)$$

Proof. Indeed, in general one can write,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} F(\chi^{\kappa, \nu}, \varepsilon(\chi^{\kappa, \nu}), \nabla \hat{\varphi}) &= \lim_{\nu \rightarrow \infty} |\nabla \varphi^{\kappa, \nu} - \nabla \hat{\varphi}|^2 dx = \\ &= \int_{\Omega} |\nabla \varphi^{\kappa} - \nabla \hat{\varphi}|^2 dx + \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla \varphi^{\kappa, \nu} - \nabla \varphi^{\kappa}|^2 dx. \end{aligned} \quad (3.24)$$

Taking similar steps as in the proof of Theorem 2, we further apply the corrector theory of homogenization given in [8], and derive that,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \int_{\Omega} |\nabla \varphi^{\kappa, \nu} - \nabla \varphi^{\kappa}|^2 dx &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |(P^{\kappa, \nu} - I) \nabla \varphi^{\kappa} + z^{\kappa, \nu}|^2 dx = \\ &= \lim_{\nu \rightarrow \infty} \sum_{i=1}^{N(\kappa)} \int_{\Omega_i} |(P_i^{\kappa, \nu} - I) \nabla \varphi^{\kappa} + z_i^{\kappa, \nu}|^2 dx. \end{aligned} \quad (3.25)$$

Here $P^{\kappa, \nu}$ is the corrector matrix associated with $\chi^{\kappa, \nu}$, while $P_i^{\kappa, \nu}$ is the corrector matrix related to the layered material in the subdomain Ω_i , given by,

$$P_i^{\kappa, \nu} = R(\gamma_i^{\kappa}) \begin{pmatrix} \frac{h_{\theta_i^{\kappa}}}{[\alpha(1-\chi^{\kappa, \nu}) + \beta\chi^{\kappa, \nu}]} & 0 \\ 0 & 1 \end{pmatrix} R^T(\gamma_i^{\kappa}).$$

Moreover $P_i^{\kappa, \nu} \rightharpoonup I$ weakly in $L^2(\Omega_i)^{2 \times 2}$ as $\nu \rightarrow \infty$, and since $P_i^{\kappa, \nu} \in L^\infty(\Omega_i)^{2 \times 2}$, it follows from the corrector theorem given in [8] that $z_i^{\kappa, \nu} \rightarrow 0$ strongly in $L^2(\Omega_i)^2$.

As a consequence, the sequence of equalities in (3.25) continues as follows,

$$\lim_{\nu \rightarrow \infty} \sum_{i=1}^{N(\kappa)} \int_{\Omega_i} |(P_i^{\kappa, \nu} - I) \nabla \varphi^{\kappa} + z_i^{\kappa, \nu}|^2 dx =$$

$$= \lim_{\nu \rightarrow \infty} \sum_{i=1}^{N(\kappa)} \left(\int_{\Omega_i} R(\gamma_i^\kappa) \begin{pmatrix} \left(\frac{h_{\theta_i^\kappa}}{[\alpha(1-\chi^{\kappa,\nu}) + \beta\chi^{\kappa,\nu}]} - 1 \right)^2 & 0 \\ 0 & 0 \end{pmatrix} R^T(\gamma_i^\kappa) \nabla \varphi^\kappa \cdot \nabla \varphi^\kappa dx \right) =$$

$$= \sum_{i=1}^{N(\kappa)} \int_{\Omega_i} R(\gamma_i^\kappa) H(\theta_i^\kappa) R^T(\gamma_i^\kappa) \nabla \varphi^\kappa \cdot \nabla \varphi^\kappa dx = \int_{\Omega} R(\gamma^\kappa) H(\theta^\kappa) R^T(\gamma^\kappa) \nabla \varphi^\kappa \cdot \nabla \varphi^\kappa dx,$$

where the matrices $R(\gamma)$ and $H(\theta)$, are given as before. By substituting the latest result back into (3.24), we obtain that,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} F(\chi^{\kappa,\nu}, \varepsilon(\chi^{\kappa,\nu}), \nabla \hat{\varphi}) &= \int_{\Omega} |\nabla \varphi^\kappa - \nabla \hat{\varphi}|^2 dx + \int_{\Omega} R(\gamma^\kappa) H(\theta^\kappa) R^T(\gamma^\kappa) \nabla \varphi^\kappa \cdot \nabla \varphi^\kappa dx \\ &= RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}), \end{aligned}$$

and the theorem follows.

Chapter 4

Optimal Designs

For an arbitrary initial partition $T_{\bar{\kappa}}$ we consider its further refinements, i.e. the nested family of partitions $\{T^\kappa\}_{\kappa \leq \bar{\kappa}}$ that $T_{\bar{\kappa}}$ includes. In this chapter we show that the optimal designs associated with these refinements represent a minimizing sequence of designs for the RP problem and then by approaching each optimal design on the sequence by fine layers of α and β materials, we construct a minimizing sequence of configurations for the P problem. A discussion on optimal designs of the RP -problem attainable by minimizing sequences of configurations for a special class of target potentials, is given in Section 4.2.

4.1 Minimizing Sequences of Configurations

We call a nested family of partitions $\{T^\kappa\}_{\kappa \leq \epsilon}$ of Ω the one satisfying,

$$\kappa_1 < \kappa_2 \leq \epsilon \Rightarrow \forall \Omega_i^{\kappa_1} \in T_{\kappa_1} \quad \exists \Omega_j^{\kappa_2} \in T_{\kappa_2} : \Omega_i^{\kappa_1} \subset \Omega_j^{\kappa_2}. \quad (4.1)$$

For any given partition $T_{\bar{\kappa}}$ the sequence of refinements of this partition is denoted by $\{T_\kappa\}_{\kappa \leq \bar{\kappa}}$ and is a nested family of partitions as described by (4.1).

Theorem 6. *The system of the discrete design spaces $\{D_{\Theta}^{\kappa}\}_{\kappa \leq \bar{\kappa}}$ associated with the refinements of $T_{\bar{\kappa}}$, is dense in D_{Θ} , i.e., for every $(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_{\Theta}$, there exists a sequence of $\{(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa}))\}_{\kappa \leq \bar{\kappa}} \in \{D_{\Theta}^{\kappa}\}_{\kappa \leq \bar{\kappa}}$, for which,*

$$\lim_{\kappa \rightarrow 0^+} (\theta^{\kappa}, \gamma^{\kappa}) = (\theta, \gamma) \text{ a.e. in } \Omega, \quad (4.2)$$

$$\lim_{\kappa \rightarrow 0^+} \varepsilon(\theta^{\kappa}, \gamma^{\kappa}) = \varepsilon(\theta, \gamma) \text{ a.e. in } \Omega, \quad (4.3)$$

and,

$$\lim_{\kappa \rightarrow 0^+} RF(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa}), \nabla \hat{\varphi}) = RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}). \quad (4.4)$$

Proof. For a given design $(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_{\Theta}$, we choose any partition $T_{\bar{\kappa}}$ of Ω and consider its refinements $\{T^{\kappa}\}_{\kappa \leq \bar{\kappa}}$. For every refinement T^{κ} , we construct $(\theta^{\kappa}, \gamma^{\kappa}, \varepsilon(\theta^{\kappa}, \gamma^{\kappa})) \in D_{\Theta}^{\kappa}$, described for each subdomain Ω_i^{κ} of Ω as follows,

$$\begin{aligned} \theta_{|\Omega_i^{\kappa}}^{\kappa} &= \frac{1}{\text{meas } \Omega_i^{\kappa}} \int_{\Omega_i^{\kappa}} \theta(x) \, dx = \theta_i^{\kappa}, \\ \gamma_{|\Omega_i^{\kappa}}^{\kappa} &= \frac{1}{\text{meas } \Omega_i^{\kappa}} \int_{\Omega_i^{\kappa}} \gamma(x) \, dx = \gamma_i^{\kappa}, \\ \varepsilon(\theta^{\kappa}, \gamma^{\kappa})_{|\Omega_i^{\kappa}} &= \varepsilon(\theta_i^{\kappa}, \gamma_i^{\kappa}) = R(\gamma_i^{\kappa}) \Lambda(\theta_i^{\kappa}) R^T(\gamma_i^{\kappa}). \end{aligned}$$

On the set of the intersection of the Lebesgue points of the functions θ and γ , we have that $(\theta^{\kappa}, \gamma^{\kappa}) \rightarrow (\theta, \gamma)$ almost everywhere in Ω , as $\kappa \rightarrow 0^+$. This delivers the convergence in (4.2), and the followings,

$$\varepsilon(\theta^{\kappa}, \gamma^{\kappa}) \rightarrow \varepsilon(\theta, \gamma) \text{ a.e. in } \Omega \text{ as } \kappa \rightarrow 0^+,$$

$$R(\gamma^{\kappa}) H(\theta^{\kappa}) R^T(\gamma^{\kappa}) \rightarrow R(\gamma) H(\theta) R^T(\gamma) \text{ a.e. in } \Omega \text{ as } \kappa \rightarrow 0^+.$$

From the theory of G -convergence, $\varepsilon(\theta^\kappa, \gamma^\kappa)$ G -converges to $\varepsilon(\theta, \gamma)$, which further implies that the sequence of state variables $\varphi^\kappa \in W_0^{1,2}(\Omega)$, satisfying the equilibrium equation,

$$-\operatorname{div} (\varepsilon(\theta^\kappa, \gamma^\kappa) \nabla \varphi^\kappa) = f, \quad (4.5)$$

converges weakly in $W_0^{1,2}(\Omega)$ to the $W_0^{1,2}(\Omega)$ solution φ of the homogenized equilibrium equation,

$$-\operatorname{div} (\varepsilon(\theta, \gamma) \nabla \varphi) = f. \quad (4.6)$$

Following the same arguments given in the proof of Theorem 3, we find that the sequence $\{\varphi^\kappa\}_{\kappa \leq \bar{\kappa}}$ converges strongly in $W_0^{1,2}(\Omega)$ to φ . Proceeding along the same lines as in the proof of Theorem 3 and using the Lebesgue convergence theorem and the estimate (3.15), one can easily show that (4.4) holds.

We now identify minimizing sequences of designs for the RP problem. We consider any nested family of partitions denoted by $\{T_\kappa\}_{\kappa > 0}$. For each value of κ we consider the optimal design for the discrete RP^κ problem denoted by $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$.

Theorem 7. *The sequence $\{(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))\}_{\kappa > 0}$, satisfies the non-increasing monotonicity condition,*

$$\text{for } \kappa < \kappa', \quad RP^\kappa = RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}) \leq RP^{\kappa'} = RF(\bar{\theta}^{\kappa'}, \bar{\gamma}^{\kappa'}, \varepsilon(\bar{\theta}^{\kappa'}, \bar{\gamma}^{\kappa'}), \nabla \hat{\varphi}),$$

and is a minimizing sequence for the RP problem, i.e.

$$\lim_{\kappa \rightarrow 0^+} RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}) = RP,$$

or equivalently

$$\{RP^\kappa\}_{\kappa > 0} \searrow RP \text{ as } \kappa \rightarrow 0^+.$$

Proof. The monotonicity follows immediately from the fact that $\kappa < \kappa'$ implies that $D_{\Theta}^{\kappa'} \subset D_{\Theta}^{\kappa}$. We note that the monotonicity property and the zero lower bound, imply the existence of the limit,

$$\lim_{\kappa \rightarrow 0^+} RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}).$$

Since for every $\kappa > 0$, $D_{\Theta}^{\kappa} \subset D_{\Theta}$, RP is a lower bound for the monotonically decreasing sequence of $RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi})$, and as a result,

$$RP \leq \lim_{\kappa \rightarrow 0^+} RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}). \quad (4.7)$$

On the other hand, for the nested family of partitions $\{T^\kappa\}_{\kappa > 0}$ and for any given $(\theta, \gamma, \varepsilon(\theta, \gamma))$ in D_{Θ} , it follows from Theorem 6 that there exists a sequence $\{(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa))\}_{\kappa > 0}$ for which the followings hold,

$$RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}) \leq RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}), \quad (4.8)$$

and

$$\begin{aligned} \lim_{\kappa \rightarrow 0^+} RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}) &\leq \lim_{\kappa \rightarrow 0^+} RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}) = \\ &= RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}). \end{aligned} \quad (4.9)$$

It is now evident that,

$$\lim_{\kappa \rightarrow 0^+} RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla \hat{\varphi}) \leq \inf_{(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_\Theta} RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}) = RP \quad (4.10)$$

and from (4.7) and (4.10), the theorem follows .

With Theorems 5 and 6 in hand, it is possible to identify a sequence of configurations specified by χ^j for which,

$$RP = \lim_{j \rightarrow \infty} F(\chi^j, \varepsilon(\chi^j), \nabla \hat{\varphi}). \quad (4.11)$$

Indeed we consider a minimizing sequence for RP as given by Theorem 6. To each element $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$ of the sequence we apply Theorem 5 to find a recovery sequence of configurations $\{\chi^{\kappa, \nu}\}_{\nu=1}^\infty$. In this way we see that,

$$\begin{aligned} RP = \lim_{\kappa \rightarrow 0^+} RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)) &= \lim_{\kappa \rightarrow 0^+} \lim_{\nu \rightarrow \infty} F(\chi^{\kappa, \nu}, \varepsilon(\chi^{\kappa, \nu}), \nabla \hat{\varphi}) = \\ &= \lim_{\kappa \rightarrow 0^+} F(\chi^{\kappa, \nu(\kappa)}, \varepsilon(\chi^{\kappa, \nu(\kappa)}), \nabla \hat{\varphi}), \end{aligned} \quad (4.12)$$

from where it follows that we can extract a sequence of configurations $\{\chi^{\kappa_j, \nu_j}\}_{j=1}^\infty$ for which,

$$RP = \lim_{j \rightarrow \infty} F(\chi^{\kappa_j, \nu_j}, \varepsilon(\chi^{\kappa_j, \nu_j}), \nabla \hat{\varphi}), \quad (4.13)$$

which proves (4.11). We now establish the following result.

Theorem 8.

$$P = RP$$

i.e.,

$$\inf_{\chi \in ad_{\Theta}} F(\chi, \varepsilon(\chi), \nabla \hat{\varphi}) = \inf_{(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_{\Theta}} RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}). \quad (4.14)$$

Proof. From Remark 2 in Chapter 2, we have that $P \geq RP$. On the other hand (4.11) and the fact that,

$$\forall j = 1 \dots \infty, \quad F(\chi^{\kappa_j, \nu_j}, \varepsilon(\chi^{\kappa_j, \nu_j}), \nabla \hat{\varphi}) \geq P, \quad (4.15)$$

imply that $RP \geq P$, and we conclude that $RP = P$.

4.2 Optimal Designs of the Relaxed Problem

For a special set of target potentials we characterize in this section optimal designs of the relaxed RP problem attainable by minimizing sequences of configurations. To accomplish this we recall Theorem 1 and the related discussion in Section 1.2, as well as the representation of the relaxed functional given earlier in (2.12) as,

$$RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}) = \int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^2 dx + \int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla \varphi \cdot \nabla \varphi dx.$$

It is instructive to write the second term of the above representation, in a form where $\varepsilon(\theta, \gamma)$ appears explicitly. Manipulation gives,

$$R(\gamma)H(\theta)R^T(\gamma) = \frac{(m_{\theta}I - \varepsilon(\theta, \gamma))^2}{(1 - \theta)\beta(\beta - \alpha)} + \frac{(m_{\theta}I - \varepsilon(\theta, \gamma))}{\beta}. \quad (4.16)$$

It now becomes clear that,

$$\int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla\varphi \cdot \nabla\varphi dx = 0 \quad \text{iff} \quad \varepsilon(\theta, \gamma)\nabla\varphi = m_{\theta}\nabla\varphi. \quad (4.17)$$

With this fact in mind and the results of [15] given in Section 2.1, we state the following theorem which accounts for oscillations appearing in minimizing sequences of configurations.

Theorem 9. *There exists a dense G_{δ} subset K of $W_0^{1,2}(\Omega)$ such that for $\hat{\varphi}$ in K :*

(1) *There exists a minimizer in D_{Θ} for the RP problem.*

(2) *At a minimizer $(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}))$ of the RP problem,*

$$RF(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}), \nabla\hat{\varphi}) = \int_{\Omega} |\nabla\bar{\varphi} - \nabla\hat{\varphi}|^2 dx \quad \text{and} \quad \varepsilon(\bar{\theta}, \bar{\gamma})\nabla\bar{\varphi} = m_{\bar{\theta}}\nabla\bar{\varphi},$$

where $\bar{\varphi}$ is the potential associated with the design.

(3) *$P = RP$.*

(4) *Any cluster point of any minimizing sequence in ad_{Θ} of the P problem is a minimizer of the RP problem and any minimizer of the RP problem in D_{Θ} is a limit of a minimizing sequence for the P problem.*

Here the convergence of sequences of designs is with respect to the G -convergence .

Proof. From the results [15] given in Section 2.1, for a target potential $\hat{\varphi} \in K$, and for any minimizing sequence $\{(\chi^{\nu}, \varepsilon(\chi^{\nu}))\}_{\nu=1}^{\infty}$ of the P problem, its associated sequence of state variables $\{\varphi^{\nu}\}_{\nu=1}^{\infty}$ solving the equilibrium equation,

$$-\text{div}(\varepsilon(\chi^{\nu})\nabla\varphi^{\nu}) = f, \quad (4.18)$$

is Cauchy in the $W_0^{1,2}(\Omega)$ norm given by $\|u\|^2 = \int_{\Omega} |\nabla u|^2 dx$. From the completeness of $W_0^{1,2}(\Omega)$, there exists a potential $\bar{\varphi} \in W_0^{1,2}(\Omega)$ such that $\varphi^\nu \rightarrow \bar{\varphi}$ strongly in $W_0^{1,2}(\Omega)$. Passing to subsequences if necessary, the sequence $\{(\chi^\nu, \varepsilon(\chi^\nu))\}_{\nu=1}^\infty$ weak $L^\infty(\Omega)$ star converges to $(\bar{\theta}, m_{\bar{\theta}})$, while the compactness property of G -convergence implies that the sequence $\{\varepsilon(\chi^\nu)\}_{\nu=1}^\infty$ G -converges to an effective tensor $\bar{\varepsilon}^e$ where,

$$-\operatorname{div}(\bar{\varepsilon}^e \nabla \bar{\varphi}) = f. \quad (4.19)$$

In this context we mention that the work in [15] and [3], show that the condition,

$$\bar{\varepsilon}^e \nabla \bar{\varphi} = m_{\bar{\theta}} \nabla \bar{\varphi}, \text{ a.e. in } \Omega, \quad (4.20)$$

is necessary and sufficient for the strong convergence of gradients associated with sequences $\{\varepsilon^e(\chi^\nu)\}_{\nu=0}^\infty$ G -converging to $\bar{\varepsilon}^e$ and weak L^∞ star converging to $m_{\bar{\theta}}$. This implies that $m_{\bar{\theta}}$ is an eigenvalue of $\bar{\varepsilon}^e$.

To establish Theorem 9, we take advantage of the geometry of the set of effective tensors for two dimensional problems. As given in [5] and [13], the effective tensors associated with the density $\bar{\theta}(x)$ are all 2-by-2 symmetric matrices with eigenvalues λ_1, λ_2 , lying for almost all x in Ω , in the set given by the inequalities,

$$\begin{aligned} \sum_{k=1}^2 \frac{1}{\lambda_j - \alpha} &\leq \frac{1}{h_{\bar{\theta}} - \alpha} + \frac{1}{m_{\bar{\theta}} - \alpha}, \\ \sum_{k=1}^2 \frac{1}{\beta - \lambda_j} &\leq \frac{1}{\beta - h_{\bar{\theta}}} + \frac{1}{\beta - m_{\bar{\theta}}}. \end{aligned} \quad (4.21)$$

The constraints on the eigenvalues of $\bar{\varepsilon}^e$ given by (4.21) together with (4.20), allow us to uniquely identify $\bar{\varepsilon}^e$ as the effective tensor given by,

$$\bar{\varepsilon}^e = \varepsilon(\bar{\theta}, \bar{\gamma}) = R(\bar{\gamma})\Lambda(\bar{\theta})R^T(\bar{\gamma}), \quad (4.22)$$

where the angle $\bar{\gamma}$ is chosen according to the requirement given by (4.20). For this choice of the angle, from (4.17), we also have the local relation,

$$R(\bar{\gamma})H(\bar{\theta})R^T(\bar{\gamma})\nabla\bar{\varphi} = 0, \quad a.e. \text{ in } \Omega, \quad (4.23)$$

and we conclude that

$$P = \int_{\Omega} |\nabla\bar{\varphi} - \nabla\hat{\varphi}|^2 dx = RF(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}), \nabla\hat{\varphi}). \quad (4.24)$$

In view of Theorem 8 we deduce that the design $(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}))$ is the optimal design for the RP problem. This establishes parts (1), (2), and (3) of Theorem 9.

To proceed with part (4) of the theorem, we recall the notion of a cluster point (θ, ε^e) for a sequence of configurations $\{(\chi^\nu, \varepsilon(\chi^\nu))\}_{\nu=1}^\infty$. The definition of a cluster point (θ, ε^e) implies the existence of a subsequence such that $\{\chi^\nu\}_{\nu=1}^\infty$ weak $L^\infty(\Omega)$ star converges to θ and $\{\varepsilon(\chi^\nu)\}_{\nu=1}^\infty$ G -converges to ε^e . Arguments identical to those given above show that any cluster point of any minimizing sequence for the P problem is a minimizing design for the RP problem. This establishes one side of the implication in (4). The other side of the implication in (4) follows immediately from the construction of a recovery sequence of configurations based upon Theorems 5 and 6, and equation (4.13).

Remark 3. *When the minimizing sequences of configurations of the P problem with target potentials in the set K , oscillate locally in the form of layers of the two*

dielectrics, the layers are asymptotically parallel to the optimal (limit) gradient $\nabla\bar{\varphi}$.

Such configurations allow for the best effective conductivity properties to be aligned with the direction of the gradient which is consistent with the physical intuition.

Chapter 5

Numerical Approach

In Section 5.1 of this chapter, we provide an outline of the method used for the numerical solution of the discrete design problem. The numerical examples included in Section 5.2 provide optimal designs for several settings and illustrate how electrostatics fields can be controlled by using functionally graded materials.

5.1 Numerical Procedure

As described in Chapter 3, for a given partition T^κ of the design domain Ω , the number of subdomains is $N(\kappa)$ and the design variable $(\theta, \gamma, \varepsilon(\theta)) \in D_{\Theta}^\kappa$ can be represented by a vector in $R^{2N(\kappa)}$, with components $(\theta_1, \gamma_1, \dots, \theta_i, \gamma_i, \dots, \theta_{N(\kappa)}, \gamma_{N(\kappa)})$. These components are the constant values (θ_i, γ_i) that (θ, γ) takes in each subdomain Ω_i , and are subject to the box constraints,

$$\begin{aligned} 0 \leq \theta_i \leq 1, \quad i = 1 \dots, N(\kappa), \\ 0 \leq \gamma_i \leq 2\pi, \quad i = 1 \dots, N(\kappa). \end{aligned} \tag{5.1}$$

Recall the relaxed objective functional,

$$\begin{aligned} RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}) &= \int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^2 dx + \\ &+ \int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla \varphi \cdot \nabla \varphi dx, \end{aligned}$$

where the state variable φ solves the equilibrium equation (2.13).

We include the resource constraint $\sum_{i=1}^{N(\kappa)} \theta_i \text{meas}(\Omega_i) \leq \Theta \text{meas}(\Omega)$, by adding to the relaxed objective functional the penalty term,

$$\ell \times \left(\int_{\Omega} \theta dx - \Theta \text{meas}(\Omega) \right), \quad \text{for } \ell > 0.$$

The discrete design problem becomes,

$$\min_{(\theta, \gamma)} L(\theta, \gamma) = \min_{(\theta, \gamma)} [RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}) + \ell \times \left(\int_{\Omega} \theta dx - \Theta \text{meas}(\Omega) \right)], \quad (5.2)$$

where (θ, γ) are subject to the constraints given by (5.1). The numerical procedure is a straight forward application of the steepest decent method , given in [11], and carried out in what follows. We consider changes in (θ, γ) and correspondingly in φ and ε , as

$$\begin{aligned} (\theta, \gamma) &\rightarrow (\theta + \varepsilon \tilde{\theta}, \gamma + \varepsilon \tilde{\gamma}) \\ \varphi(\theta, \gamma) &\rightarrow \varphi(\theta + \varepsilon \tilde{\theta}, \gamma + \varepsilon \tilde{\gamma}) = \varphi + \varepsilon \tilde{\varphi} + O(\varepsilon^2) \\ \varepsilon(\theta, \gamma) &\rightarrow \varepsilon(\theta + \varepsilon \tilde{\theta}, \gamma + \varepsilon \tilde{\gamma}) = \varepsilon(\theta, \gamma) + \underbrace{\varepsilon (\varepsilon'_\theta \tilde{\theta} + \varepsilon'_\gamma \tilde{\gamma})}_{\tilde{\varepsilon}} + O(\varepsilon^2). \end{aligned}$$

Here $\tilde{\varepsilon}$ and $\tilde{\varphi}$ are related through the equation,

$$\nabla \cdot (\tilde{\varepsilon} \nabla \varphi + \varepsilon \nabla \tilde{\varphi}) = 0, \quad (5.3)$$

which is a linearization of the equilibrium equation (2.13), and therefore one can write,

$$\int_{\Omega} (\tilde{\varepsilon} \nabla \varphi + \varepsilon \nabla \tilde{\varphi}) \cdot \nabla \lambda \, dx = 0, \quad \forall \lambda \in W_0^{1,2}. \quad (5.4)$$

We calculate next the change in the functional L . By $T(\theta, \gamma)$ we will denote $T(\theta, \gamma) = R(\gamma)H(\theta)R^T(\gamma)$, and we use the prime superscript signs to indicate (partial) differentiation with respect to the subscript variable.

$$\begin{aligned} \delta L &= L(\theta + \varepsilon \tilde{\theta}, \gamma + \varepsilon \tilde{\gamma}) - L(\theta, \gamma) = \\ &= \int_{\Omega} (|\nabla \varphi - \nabla \hat{\varphi}|_{(\theta + \varepsilon \tilde{\theta}, \gamma + \varepsilon \tilde{\gamma})}^2 - |\nabla \varphi - \nabla \hat{\varphi}|_{(\theta, \gamma)}^2) \, dx + \varepsilon l^* \int_{\Omega} \tilde{\theta} \, dx + \\ &+ \int_{\Omega} (T \nabla \varphi \cdot \nabla \varphi |_{(\theta + \varepsilon \tilde{\theta}, \gamma + \varepsilon \tilde{\gamma})} - T \nabla \varphi \cdot \nabla \varphi |_{(\theta, \gamma)}) \, dx = \\ &= \varepsilon [2 \int_{\Omega} (\nabla \varphi - \nabla \hat{\varphi}) \cdot \nabla \tilde{\varphi} \, dx + l \int_{\Omega} \tilde{\theta} \, dx] + \\ &+ \int_{\Omega} [T(\theta, \gamma) + \underbrace{\varepsilon (T'_\theta \tilde{\theta} + T'_\gamma \tilde{\gamma})}_{\tilde{T}}] (\nabla \varphi + \varepsilon \nabla \tilde{\varphi}) \cdot (\nabla \varphi + \varepsilon \nabla \tilde{\varphi}) \, dx - \\ &- \int_{\Omega} T(\theta, \gamma) \nabla \varphi \cdot \nabla \varphi \, dx + O(\varepsilon^2) = \\ &= \varepsilon [2 \int_{\Omega} (\nabla \varphi - \nabla \hat{\varphi}) \cdot \nabla \tilde{\varphi} \, dx + l \int_{\Omega} \tilde{\theta} \, dx + \\ &+ \int_{\Omega} ((T'_\theta \tilde{\theta} + T'_\gamma \tilde{\gamma}) \nabla \varphi \cdot \nabla \varphi + 2T(\theta, \gamma) \nabla \varphi \cdot \nabla \tilde{\varphi}) \, dx] + O(\varepsilon^2). \end{aligned}$$

Reorganizing the terms in the last equality gives,

$$\begin{aligned} \delta L &= \varepsilon [\int_{\Omega} 2(\nabla \varphi - \nabla \hat{\varphi} + T(\theta, \gamma) \nabla \varphi) \cdot \nabla \tilde{\varphi} \, dx + \int_{\Omega} (T'_\theta \nabla \varphi \cdot \nabla \varphi + l) \tilde{\theta} \, dx + \\ &+ \int_{\Omega} T'_\gamma \nabla \varphi \cdot \nabla \varphi \tilde{\gamma} \, dx] + O(\varepsilon^2). \end{aligned}$$

According to the adjoint method, our plan is to add the zero-term in (5.4) to our expression for the variation of the functional δL and by using a proper choice for λ , simplifying δL so that it does not depend on $\tilde{\varphi}$. Clearly,

$$\begin{aligned} \delta L = & \epsilon \left[\int_{\Omega} (\nabla \varphi - \nabla \hat{\varphi} + 2T(\theta, \gamma) \nabla \varphi) \cdot \nabla \tilde{\varphi} \, dx + \int_{\Omega} (T'_{\theta} + l) \nabla \varphi \cdot \nabla \varphi \, \tilde{\theta} \, dx + \right. \\ & \left. + \int_{\Omega} T'_{\gamma} \nabla \varphi \cdot \nabla \varphi \, \tilde{\gamma} \, dx \right] + \epsilon \left[\int_{\Omega} \underbrace{ (\varepsilon'_{\theta} \tilde{\theta} + \varepsilon'_{\gamma} \tilde{\gamma}) }_{\tilde{\varepsilon}} \nabla \varphi + \varepsilon \nabla \tilde{\varphi} \right] \cdot \nabla \lambda \, dx \left. \right] + O(\epsilon^2). \end{aligned}$$

and after reorganizing terms,

$$\begin{aligned} \delta L = & \epsilon \int_{\Omega} [2(\nabla \varphi - \nabla \hat{\varphi}) + 2T(\theta, \gamma) \nabla \varphi + \varepsilon \nabla \lambda] \cdot \nabla \tilde{\varphi} \, dx + \\ & + \epsilon \int_{\Omega} [\varepsilon'_{\theta} \nabla \varphi \cdot \nabla \lambda + T'_{\theta} \nabla \varphi \cdot \nabla \varphi + l] \tilde{\theta} \, dx + \\ & + \epsilon \int_{\Omega} [\varepsilon'_{\gamma} \nabla \varphi \cdot \nabla \lambda + T'_{\gamma} \nabla \varphi \cdot \nabla \varphi] \tilde{\gamma} \, dx + O(\epsilon^2). \end{aligned}$$

For the choice of $\lambda \in W_0^{1,2}$, satisfying the so-called adjoint equation,

$$-\nabla \cdot (\varepsilon(\theta, \gamma) \nabla \lambda) = 2 \nabla \cdot ((\nabla \varphi - \nabla \hat{\varphi}) + T(\theta, \gamma) \nabla \varphi), \quad (5.5)$$

the choice $(\tilde{\theta}, \tilde{\gamma})$ given by,

$$\begin{cases} \tilde{\theta} &= - (\varepsilon'_{\theta} \nabla \varphi \cdot \nabla \lambda + T'_{\theta} \nabla \varphi \cdot \nabla \varphi + l) \\ \tilde{\gamma} &= - (\varepsilon'_{\gamma} \nabla \varphi \cdot \nabla \lambda + T'_{\gamma} \nabla \varphi \cdot \nabla \varphi), \end{cases} \quad (5.6)$$

is a direction of descent for the functional L , since,

$$\delta L = -\epsilon \int_{\Omega} |(\tilde{\theta}, \tilde{\gamma})|^2 \, dx + O(\epsilon^2).$$

This direction is called the steepest descent direction, and the method based on it is called the steepest descent method. At a minimum $(\tilde{\theta}, \tilde{\gamma}) = 0$, gives us the optimality necessary conditions,

$$\begin{cases} \varepsilon'_\theta \nabla \varphi \cdot \nabla \lambda + T'_\theta \nabla \varphi \cdot \nabla \varphi + l = 0 \\ \varepsilon'_\gamma \nabla \varphi \cdot \nabla \lambda + T'_\gamma \nabla \varphi \cdot \nabla \varphi = 0. \end{cases} \quad (5.7)$$

From the second equation above, one can easily derive the optimality condition for the rotation angle γ ,

$$\tan(2\gamma) = \frac{V \sin(2\gamma_1) + \sin(\gamma_1 + \gamma_2)}{V \cos(2\gamma_1) + \cos(\gamma_1 + \gamma_2)}, \quad (5.8)$$

where $\nabla \varphi = |\nabla \varphi|(\cos \gamma_1, \sin \gamma_1)$, $\nabla \lambda = |\nabla \lambda|(\cos \gamma_2, \sin \gamma_2)$ and $V = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 \frac{\theta(1-\theta)h_\theta^2 |\nabla \varphi|}{(h_\theta - m_\theta) |\nabla \lambda|}$.

The Adjoint Method Algorithm. The algorithm is a repeated application until convergence of the following steps,

- Given (θ, γ) solve for φ and λ in $W_0^{1,2}$,

$$-\nabla \cdot \left(\begin{bmatrix} \varepsilon(\theta, \gamma) & 0 \\ 2(I + T(\theta, \gamma)) & \varepsilon(\theta, \gamma) \end{bmatrix} \begin{pmatrix} \nabla \varphi \\ \nabla \lambda \end{pmatrix} \right) = \begin{pmatrix} f \\ -\Delta \hat{\varphi} \end{pmatrix} \quad (5.9)$$

- Evaluate the functional,

$$L(\theta, \gamma) = \| \nabla \varphi - \nabla \hat{\varphi} \|_{L^2}^2 + \int_{\Omega} T(\theta, \gamma) \nabla \varphi \cdot \nabla \varphi \, dx + l \int_{\Omega} (\theta - \Theta_0) \, dx.$$

- Update (θ, γ) as,

$$\begin{cases} \theta \leftarrow \theta - \underbrace{(\varepsilon'_\theta \nabla \varphi \cdot \nabla \lambda + T'_\theta \nabla \varphi \cdot \nabla \varphi + l)}_{\tilde{\theta}} \epsilon \\ \gamma \leftarrow \gamma - \underbrace{(\varepsilon'_\gamma \nabla \varphi \cdot \nabla \lambda + T'_\gamma \nabla \varphi \cdot \nabla \varphi)}_{\tilde{\gamma}} \epsilon. \end{cases} \quad (5.10)$$

unless $0 \leq -\Delta L \leq Tolerance$.

Descendence of the functional L assures convergence of the algorithm.

5.2 Numerical Examples

We provide numerical examples that illustrate how electrostatic fields can be controlled using functionally graded materials. For all examples the design domain is chosen to be the square centered at the origin given by $\Omega = (-1, 1) \times (-1, 1)$ and we choose the target field to be zero, i.e., $\nabla \hat{\varphi} = (0, 0)$. The discrete design is associated with a partition of Ω into 20,000 subdomains of diameter on the order of 10^{-2} .

For the first two examples the charge distribution is taken to be uniform in Ω and given by $f = 1$. We choose $\alpha = 1$ and $\beta = 2$ and constrain the amount of good dielectric to be 40% of the design domain. The density distribution θ of the better dielectric material in the optimized discrete design is given in Figure 5.1: a. Here the darkest regions consist of pure β dielectric, the white regions are occupied by pure α dielectric and the regions of graded conductivity properties are given by the intermediate shades. The layer normals in the graded parts of the design are given by the arrows in Figure 5.1: a. The contours are the level lines of the electric potential. Note that the layer normals are tangential to the level lines, hence perpendicular to the electric field. We emphasize that Figure 5.1: a, gives the necessary geometric information for manufacturing graded materials. Furthermore,

the continuity property expressed in Theorem 5, guarantees that we can construct a two phase configuration that is nearly optimal.

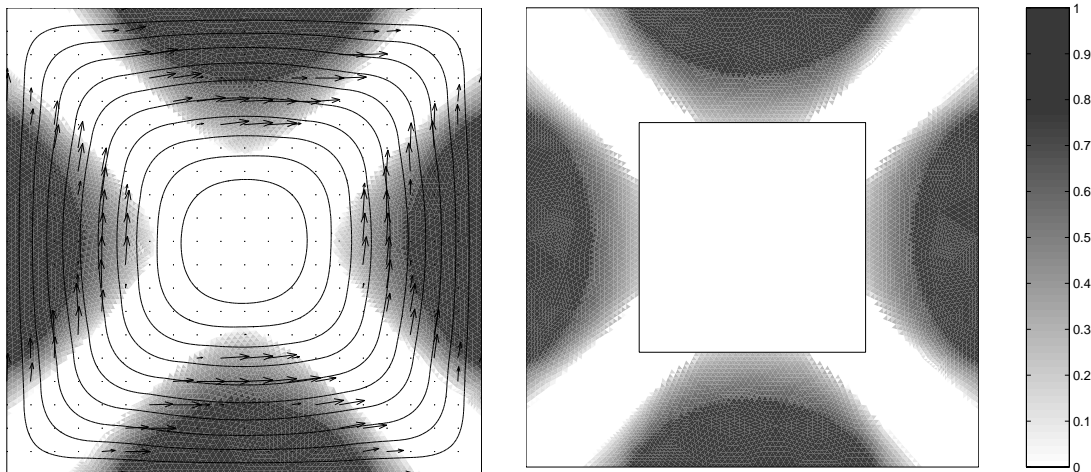


Figure 5.1: a

Figure 5.1: b

For the second example we consider a subdomain W of the design domain Ω . Here we take $W = \Omega \setminus \{(-1/2, 1/2) \times (-1/2, 1/2)\}$. We consider the problem,

$$P = \inf_{\chi \in \text{ad}_{\Theta}} \int |\nabla \varphi|^2 dx. \quad (5.11)$$

The theory presented in this paper easily generalizes to this case and the relaxed problem is,

$$RP = \inf_{(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_{\Theta}} \left\{ \int_W |\nabla \varphi|^2 dx + \int_W R(\gamma) H(\theta) R^T(\gamma) \nabla \varphi \cdot \nabla \varphi dx \right\}, \quad (5.12)$$

and $P = RP$.

Here the goal is to screen as much electric field away from the domain W as possible. The good dielectric is constrained to occupy 40% of Ω . The density distribution of the good dielectric in the optimal design is given in Figure 5.1: b. We point out that we allow the two dielectrics to be placed anywhere in Ω , however the algorithm automatically uses the good dielectric only in W . This is consistent

with intuition.

For the next example we take the charge distribution to be 1 everywhere outside of W and zero inside W . As before we take $\alpha = 1$ and $\beta = 2$.

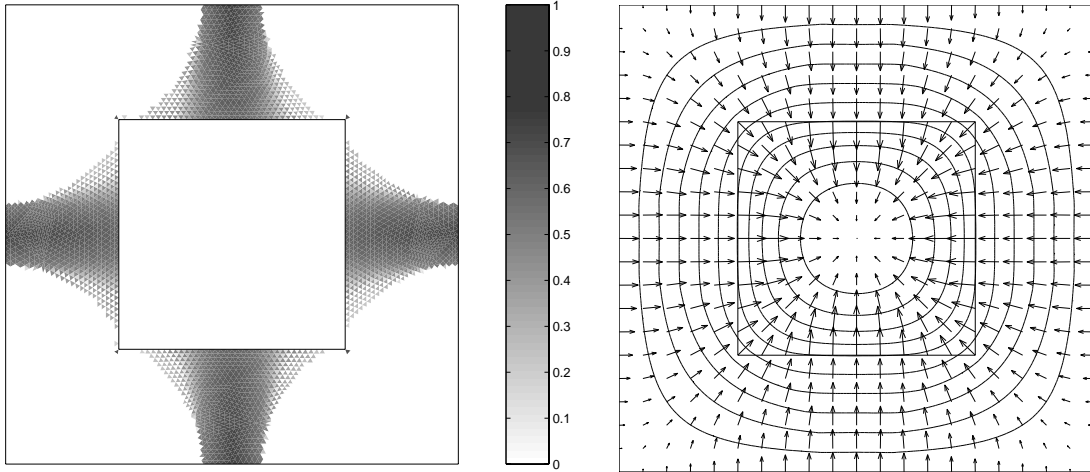


Figure 5.2: a

Figure 5.2: b

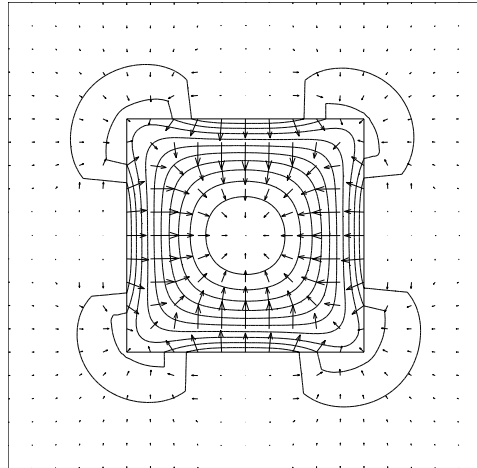


Figure 5.3

The good dielectric is constrained to occupy 15% of the design domain. The density distribution for the optimal design is given in Figure 5.2: a. In Figure 5.2: b, we plot the level lines of the potential and the electric field associated with the design. Last we consider the same layout as in Figure 5.2: a, but with $\alpha = 1$ and

$\beta = 1000$ and we plot the electric field for this case in Figure 5.3. For this layout and choice of β we see that the electric field has been screened away from W .

Chapter 6

Abstract Results on Related Problems and Future Work

In the first two sections of this chapter we give abstract results on problems which can be solved by the same procedures used to solve the P problem and the relaxed RP problem earlier in Chapters 2-5. The formulation of an open problem to be treated in the future is given in the last section of the chapter.

6.1 Optimal Design Problem on Flux Fields

A problem analogous to the P problem (1.6) can be formulated in the same space ad_{Θ} of admissible designs of pure materials, for the flux field $D = \epsilon \nabla \varphi$, as shown below,

$$P_1 = \inf_{\chi \in ad_{\Theta}} F_1(\chi, \varepsilon(\chi), \hat{D}), \quad (6.1)$$

where the objective functional denoted by $F_1(\chi, \varepsilon(\chi), \hat{D})$ is given by,

$$F_1(\chi, \varepsilon(\chi), \hat{D}) = \int_{\Omega} |D - \hat{D}|^2 dx. \quad (6.2)$$

Here $D = \varepsilon(\chi)\nabla\varphi$ is a solution of,

$$-\operatorname{div} D = f, \quad D = \varepsilon(\chi)\nabla\varphi, \quad (6.3)$$

for $\varphi \in W_0^{1,2}(\Omega)$. In the context of dielectric materials, $D = \varepsilon\nabla\varphi$ represents the polarization field, $E = \nabla\varphi$ represents the electric field, while φ represents the electric potential.

The procedure to solve the P_1 problem (6.1) is identical to that of solving the P problem (1.6) described in Chapters 2-4, while the numerical experiments for the P_1 problem are to be completed in the near future.

We highlight here only some adjustments to be made on some of the statements related to the P problem (1.6), in Chapters 2-4 as we go along and solve the P_1 problem.

On the way to relaxing the P_1 problem, we investigate a weakly convergent sequence $\{D^\nu\}_{\nu=1}^\infty$, with the weak limit \tilde{D} , related to a piecewise oscillating sequence of designs given by $\chi^\nu(x) = \chi(x, \nu x \cdot n)$, $\nu = 1, 2, \dots, \infty$. In general one writes,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \int_{\Omega} F_1(\chi^\nu, \varepsilon(\chi^\nu), \hat{D}) dx &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |D^\nu - \hat{D}|^2 dx = \\ &= \lim_{\nu \rightarrow \infty} \int_{\Omega} |D^\nu - \tilde{D}|^2 dx + \int_{\Omega} |\tilde{D} - \hat{D}|^2 dx. \end{aligned} \quad (6.4)$$

However, for this particular case, one can find the closed form expression for (6.4), given by,

$$\begin{aligned}
\lim_{\nu \rightarrow \infty} \int_{\Omega} |D^\nu - \hat{D}|^2 dx &= \int_{\Omega} \varepsilon^{-1}(\theta, \gamma) R(\gamma) H_1(\theta) R^T(\gamma) \varepsilon^{-1}(\theta, \gamma) \tilde{D} \cdot \tilde{D} dx + \\
&+ \int_{\Omega} |\tilde{D} - \hat{D}|^2 dx.
\end{aligned} \tag{6.5}$$

Here as before θ shows the relative thickness of the β layer, γ shows the direction of the normal to the layers, $R(\gamma)$ is the rotation matrix with angle γ while the matrix $H_1(\theta)$ is a function of the density θ given by,

$$H_1(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & (\beta - \alpha)^2 \theta (1 - \theta) \end{pmatrix}. \tag{6.6}$$

The following equilibrium conditions hold,

$$-\operatorname{div} D^\nu = f, \quad D^\nu = \varepsilon(\chi^\nu) \nabla \varphi^\nu, \tag{6.7}$$

$$-\operatorname{div} \tilde{D} = f, \quad \tilde{D} = \varepsilon(\theta, \gamma) \nabla \tilde{\varphi}, \tag{6.8}$$

where as before, the effective tensor $\varepsilon(\theta, \gamma)$ given in (2.7), is the G -limit of the sequence of dielectric tensors $\{\varepsilon(\chi^\nu)\}_{\nu=1}^\infty$.

Motivated by this case, we pose in the relaxed space D_Θ given in (2.11), the relaxed design problem formulated as,

$$RP_1 = \inf_{(\theta, \gamma, \varepsilon(\theta, \gamma)) \in D_\Theta} RF_1(\theta, \gamma, \varepsilon(\theta, \gamma), \hat{D}), \tag{6.9}$$

where the new objective functional denoted by $RF_1(\theta, \gamma, \varepsilon(\theta, \gamma), \hat{D})$ is given by,

$$\begin{aligned}
RF_1(\theta, \gamma, \varepsilon(\theta, \gamma), \hat{D}) &= \int_{\Omega} |D - \hat{D}|^2 dx + \\
&+ \int_{\Omega} \varepsilon^{-1}(\theta, \gamma) R(\gamma) H_1(\theta) R^T(\gamma) \varepsilon^{-1}(\theta, \gamma) D \cdot D dx, \quad (6.10)
\end{aligned}$$

and for $\varphi \in W_0^{1,2}(\Omega)$,

$$-\operatorname{div} D = f, \quad D = \varepsilon(\theta, \gamma) \nabla \varphi. \quad (6.11)$$

We recall that the extended space of designs D_{Θ} contains the original space of designs ad_{Θ} , and as we choose $\theta = \chi$ we have $\varepsilon(\theta, \gamma) = \varepsilon(\chi)$, $H_1(\theta) = 0$ and,

$$F_1(\chi, \varepsilon(\chi), \hat{D}) = RF_1(\theta, \gamma, \varepsilon(\theta, \gamma), \hat{D}). \quad (6.12)$$

The statements of Section 4.2, related to the characterization of the minimizers of the RP problem for a special class of target potentials are equivalently given below for the RP_1 problem. The special class of target fluxes for the RP_1 problem, follows from Theorem 1 applied for the strong $L^2(\Omega)$ closure of the set SD_{Θ} , defined as,

$$\mathcal{SD}_{\Theta} = \begin{cases} D \mid -\operatorname{div} D = f, & D = \varepsilon(\chi) \nabla \varphi, & \varphi \in W_0^{1,2}(\Omega), \\ \chi \in ad_{\Theta}. \end{cases} \quad (6.13)$$

The strong $L^2(\Omega)$ closure of the set SD_{Θ} , denoted by $\bar{\mathcal{SD}}_{\Theta}$, is established in [3], and is given by,

$$\bar{\mathcal{SD}}_{\Theta} = \begin{cases} D \mid -\operatorname{div} D = f, & D = h_{\theta} \nabla \varphi, & \varphi \in W_0^{1,2}(\Omega), \\ \text{for some } \theta \in L^{\infty}(\Omega, [0, 1]), \text{ such that } h_{\theta} = ((1 - \theta)\alpha^{-1} + \theta\beta^{-1})^{-1}. \end{cases}$$

From the definition of $\bar{S}D_\Theta$ and the application of Theorem 1 it follows that there exists a dense G_δ subset K_1 of target flux \hat{D} in $L^2(\Omega)$, for which,

$$\begin{aligned} P_1 &= \inf_{\chi \in ad_\Theta} \int_{\Omega} |D - \hat{D}|^2 dx = \inf_{D \in \bar{S}D_\Theta} \int_{\Omega} |D - \hat{D}|^2 dx = \\ &= \min_{D \in \bar{S}D_\Theta} \int_{\Omega} |D - \hat{D}|^2 dx. \end{aligned} \quad (6.14)$$

We can now characterize the minimizers of the RP_1 problem, the same way as we did in Theorem 9 for the RP problem. We close our discussion with the equivalent statement of Theorem 9, for the flux problem. Its proof goes along the same lines as that of Theorem 9, having in mind the following,

$$\int_{\Omega} \varepsilon^{-1}(\theta, \gamma) R(\gamma) H_1(\theta) R^T(\gamma) \varepsilon^{-1}(\theta, \gamma) D \cdot D dx = 0 \quad \text{iff} \quad \varepsilon(\theta, \gamma) D = h_\theta D, \quad (6.15)$$

which can be easily derived from the fact that,

$$R(\gamma) H_1(\theta) R^T(\gamma) = (\beta - (\beta - \alpha)\theta) \cdot [\varepsilon(\theta, \gamma) - h_\theta I]. \quad (6.16)$$

Theorem 10. *There exists a dense G_δ subset K_1 of $L^2(\Omega)$ such that for \hat{D} in K_1 ,*

- (1) *There exists a minimizer in D_Θ for the RP_1 problem.*
- (2) *At a minimizer $(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}))$ of the RP_1 problem,*

$$RF_1(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}), \nabla \bar{\varphi}) = \int_{\Omega} |\bar{D} - \hat{D}|^2 dx \quad \text{and} \quad \varepsilon(\bar{\theta}, \bar{\gamma}) \bar{D} = h_{\bar{\theta}} \bar{D},$$

where \bar{D} and $\bar{\varphi}$ are respectively the flux (polarization field) and the potential associated with the design.

- (3) $P_1 = RP_1$.

(4) Any cluster point of any minimizing sequence in ad_Θ of the P_1 problem is a minimizer of the RP_1 problem and any minimizer of the RP_1 problem in D_Θ is a limit of a minimizing sequence for the P_1 problem.

Here the convergence of sequences of designs is with respect to the G -convergence.

Remark 4. When the minimizing sequences of configurations of the P_1 problem with target fluxes in the set K_1 , oscillate locally in the form of layers of the two dielectrics, the normal to the layers is asymptotically parallel to the optimal (limit) polarization field $\bar{D} = \varepsilon(\bar{\theta}, \bar{\gamma})\nabla\bar{\varphi}$ and the optimal (limit) electric field $\bar{E} = \nabla\bar{\varphi}$.

6.2 Optimal Design Problem on Gradient Fields- General Design Space Formulation

In this section we pose the P problem (1.6) on a larger design space containing ad_Θ , which includes all effective dielectric tensors representing composite materials of the α and β dielectrics, under the constraint in the amount of the β material. This extended space of designs can be parametrically characterized by the density function θ of the β material in Ω , and the effective permittivity tensor ε , belonging respectively to the sets Ad_Θ and G_θ described below.

$$Ad_\Theta = \left\{ \theta \in L^\infty(\Omega, [0, 1]) : \int_\Omega \theta \, dx \leq \Theta \right\} \quad (6.17)$$

$$G_\theta = \left\{ \varepsilon \in L^\infty(\Omega, S_\theta) : \theta \in Ad_\Theta \right\}, \quad (6.18)$$

where $S_{\theta(x)}$ represents the set of effective tensors for a density $\theta(x)$ for x a.e. in Ω , and it is given explicitly in [5] and [9].

We formulate our optimal design problem as,

$$GP = \inf_{\theta \in Ad_{\Theta}} \inf_{\varepsilon_{\theta} \in G_{\theta}} GF(\theta, \varepsilon, \nabla \hat{\varphi}), \quad (6.19)$$

with the objective functional given as before by,

$$GF(\theta, \varepsilon, \nabla \hat{\varphi}) = \int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^2 dx, \quad (6.20)$$

and with $\varphi \in W_0^{1,2}(\Omega)$ satisfying the equilibrium equation,

$$-\operatorname{div}(\varepsilon \nabla \varphi) = f. \quad (6.21)$$

One can easily see the relation $GP \leq P = RP$ among the three introduced problems. Our goal for the GP problem (6.19), is to identify an optimal design or a minimizing sequence of designs, and since we can characterize an optimal design of the GP problem as a rank one laminate design, we can next reformulate the problem in the relevant space of rank one laminate designs D_{Θ} only. This fact and the nature of objective functional (6.20), enable us to identify minimizing sequences of designs of the GP problem through refinements, in the same way we did for the RP problem previously posed on the design space D_{Θ} with objective functional given by (2.12). This approach is summarized in the following discussion.

One can easily establish the existence of an optimal design for the GP problem by using the direct method of the calculus of variations and the fact that our admissible space of designs is G -closed. We are then able to characterize optimal designs for the GP problem, by first writing the problem in a variational form and eliminating the equilibrium equation (6.21) through the introduction of the Lagrange multiplier

$\lambda \in W_0^{1,2}(\Omega)$, as shown below,

$$GP = \min_{\theta \in Ad_\Theta} \min_{\varepsilon \in G_\theta} \min_{\varphi \in W_0^{1,2}} \sup_{\lambda \in W_0^{1,2}} \left[\int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^2 dx + \int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \lambda dx - \int_{\Omega} f \lambda dx \right].$$

We then switch some of the above operations based on saddle point arguments, and obtain,

$$GP = \min_{\theta \in Ad_\Theta} \min_{\varphi \in W_0^{1,2}} \sup_{\lambda \in W_0^{1,2}} \inf_{\varepsilon \in G_\theta} \left[\int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^2 dx + \int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \lambda dx - \int_{\Omega} f \lambda dx \right].$$

Finally for fixed θ in Ad_Θ , φ and λ in $W_0^{1,2}$, we find the necessary optimality condition for the effective tensor ε , expressed as,

$$\varepsilon^{opt} \nabla \varphi \cdot \nabla \lambda = \min_{\varepsilon \in G_\theta} (\varepsilon \nabla \varphi \cdot \nabla \lambda), \quad \text{a.e. in } \Omega.$$

From the extremal property of the effective tensors for rank one laminates and Mirsky's Lemma, the existence of an optimal design with effective tensor $\varepsilon(\theta, \gamma) \in D_\Theta$ now follows. In order to further identify a minimizing sequence of designs, we use the same procedure as for the RP problem, explained in detail in Chapters 3 and 4 and briefly described in the next two steps.

I) We consider a partition of Ω into disjoint subdomains of diameter less than κ and introduce a discrete approximation of the design space D_Θ , denoted by D_Θ^κ . The design space D_Θ^κ consists of designs of rank one laminates with constant effective properties in each subdomain of the partition. We show that the design problem given by,

$$GP^\kappa = \inf_{(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa)) \in D_\Theta^\kappa} GF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla \hat{\varphi}),$$

has an optimal design denoted by $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))$.

II) As we further refine the partitions of Ω , the sequence of optimal designs $\{(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa))\}_{\kappa>0}$ for the discrete problems $\{GP^\kappa\}_{\kappa>0}$, is a minimizing sequence for the GP problem.

We emphasize here that the continuity property of the GF functional, equivalently stated in Theorems 3 and 6 for the RF functional, follows from the strong $L^2(\Omega)$ convergence of the sequence of gradients associated with the sequence of designs.

The numerical procedure when solving the discrete problem GP^κ , provides us with a design of piece-wise constant effective permittivities corresponding to rank one laminate materials. Notice that the functionals of the RP and GP problems change only by the term,

$$\int_{\Omega} R(\gamma)H(\theta)R^T(\gamma)\nabla\varphi \cdot \nabla\varphi dx,$$

which for the parameters of the numerical examples given in Section 5.2, is a high order term and can be ignored. Therefore the pictures and the numerical results shown in Section 5.2 are valid for the GP problem as well. At this point the question if $GP = RP$ or if there is a strict inequality, even for the special class K of target potentials, remains unanswered and needs to be explored.

6.3 Future Work

An interesting problem to be considered in the future is the P problem (1.6) formulated in the $L^p(\Omega)$ space of gradient fields. The goal of the problem still remains to find configurations of the two α and β dielectric materials that support an electric field which is as close as possible to a target field, but in terms of the L^p norm,

where $2 < p \leq \infty$. The problem is formulated as follows,

$$P_p = \inf_{\chi \in ad_{\Theta}} F_p(\chi, \varepsilon(\chi), \nabla \hat{\varphi}), \quad (6.22)$$

with an objective functional given by,

$$F_p(\chi, \varepsilon(\chi), \nabla \hat{\varphi}) = \int_{\Omega} |\nabla \varphi - \nabla \hat{\varphi}|^p dx. \quad (6.23)$$

The variable $\varphi \in W_0^{1,p}$ is a solution of the equilibrium equation (1.1), the target potential $\hat{\varphi} \in W_0^{1,p}$, while the admissible space of designs ad_{Θ} is described in (1.4).

Appendix A

Weak Convergence and G -Convergence

In this Appendix we provide the definitions of the weak convergence, G -convergence and its more generalized concept of H -convergence.

A.1 Weak Convergence

Assume that $1 \leq p < \infty$, and $p' = \frac{p}{p-1}$.

Definition 1. A sequence $\{f^\nu\}_{\nu=1}^\infty \subset L^p(\Omega)$ converges weakly to $f \in L^p(\Omega)$, provided that for each $g \in L^{p'}(\Omega)$,

$$\int_{\Omega} f^\nu g dx \rightarrow \int_{\Omega} f g dx \quad \text{as } \nu \rightarrow \infty.$$

Remark 5. Extending this terminology to the Sobolev space $W^{1,p}(\Omega)$, we say that $\{f^\nu\}_{\nu=1}^\infty \subset W^{1,p}(\Omega)$ converges weakly to f in $W^{1,p}(\Omega)$, provided that $\{f^\nu\}_{\nu=1}^\infty$ converges weakly to f in $L^p(\Omega)$, and $\{Df^\nu\}_{\nu=1}^\infty$ converges weakly to Df in $L^p(\Omega)$, on each component.

Remark 6. Weak compactness property. *For any bounded sequence in $L^p(\Omega)$, $1 < p < \infty$, there exists a subsequence which weakly converges to an element of that space.*

Definition 2. A sequence $\{f^\nu\}_{\nu=1}^\infty \subset L^\infty(\Omega)$ converges weakly star to $f \in L^\infty(\Omega)$, provided that for each $g \in L^1(\Omega)$,

$$\int_{\Omega} f^\nu g dx \rightarrow \int_{\Omega} f g dx \quad \text{as } \nu \rightarrow \infty.$$

Remark 7. Weak star compactness property. *For any bounded sequence in $L^\infty(\Omega)$, there exists a subsequence which weakly star converges to an element of that space.*

A.2 G-Convergence

Let $M(\alpha, \beta, \Omega)$, be the set of tensors described as,

$$M(\alpha, \beta, \Omega) = \{ A \in L^\infty(\Omega)^4 : \alpha|\lambda|^2 \leq A\lambda \cdot \lambda \leq \beta|\lambda|^2, \quad \forall \lambda \in R^2 \},$$

where $0 < \alpha < \beta$.

Definition 3. A sequence $\{A^\nu\}_{\nu=1}^\infty$ of elements of $M(\alpha, \beta, \Omega)$, G-converges to an element A of $M(\alpha', \beta', \Omega)$, if and only if, for any $f \in W^{-1,2}(\Omega)$, the solution φ^ν of,

$$\begin{cases} -\operatorname{div} (A^\nu \nabla \varphi^\nu) = f & \text{in } \Omega, \\ \varphi^\nu \in W_0^{1,2}(\Omega), \end{cases} \quad (\text{A.1})$$

is such that,

$$\begin{cases} \varphi^\nu \rightarrow \varphi & \text{weakly in } W_0^{1,2}(\Omega), \\ A^\nu \nabla \varphi^\nu \rightarrow A \nabla \varphi & \text{weakly in } L^2(\Omega), \end{cases} \quad (\text{A.2})$$

where φ is the solution of,

$$\begin{cases} -\operatorname{div}(A \nabla \varphi) = f & \text{in } \Omega, \\ \varphi \in W_0^{1,2}(\Omega). \end{cases} \quad (\text{A.3})$$

Remark 8. For any sequence A^ν , $\nu = 1 \dots \infty$ of symmetric matrices in $M(\alpha, \beta, \Omega)$, there is always a G -convergent subsequence converging to a matrix A , which is also symmetric and is an element of $M(\alpha, \beta, \Omega)$.

Remark 9. If equation (A.1) is interpreted as the equation for the electrostatic potential φ^ν , A^ν as the tensor of dielectric permittivity, $E^\nu = \nabla \varphi$ as the electric field, and $D^\nu = A^\nu \nabla \varphi^\nu$ as the polarization field, then convergence (A.2) is a statement about the weak convergence of the fields E^ν and D^ν . Moreover the electrostatic energy $e^\nu = (D^\nu, E^\nu) = \int_\Omega A^\nu \nabla \varphi^\nu \cdot \nabla \varphi^\nu dx$, weakly converges to $(D, E) = \int_\Omega A \nabla \varphi \cdot \nabla \varphi dx$.

Remark 10. The generalized concept of G -convergence is the concept of H -convergence, for which conditions (A.1)-(A.3), apply locally for any $\omega \subset\subset \Omega$.

For more information on G -convergence and H -convergence, see [12] and [8].

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