Value at Risk Models for a Nonlinear Hedged Portfolio

A thesis

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by

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Abstract:

This thesis addresses some practical issues that are similar to what a risk manager would be facing. To protect portfolio against unexpected turbulent drop, risk managers might use options to hedge the portfolio. Since the price of an option is not a linear function of the price of the underlying security or index, consequently option hedged portfolio's value is a not linear combination of the market prices of the underlying securities,

Three Value-at-Risk (VaR) models, traditional estimate based Monte Carlo model, GARCH based Monte Carlo model, and resampling model, are developed to estimate risk of non-linear portfolios. The results from the models by setting different levels of hedging strategies are useful to evaluate and compare these strategies, and therefore may assist risk managers in making practical decisions in risk management.

Acknowledgements

I am very thankful to Professor Domokos Vermes for his broad knowledge, kind help, and generous guidance. Without his support, my thesis project would not go far as one tenth as it is now. Numerous improvements have been made during the ongoing journey of my thesis project. Fortunately I finished this journey with his encouragement.

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Chapter 1

1. Introduction

1.1 Risk Management Based on Value-at-Risk

The objective of this project is to compare the cost and effectiveness of various hedging strategies from the perspective of the value-at-risk (VaR) risk measure. Three different quantitative computational methods are used to calculate VaR.

The portfolio to be considered is a diversified portfolio consisting of 20 securities with fixed allocations. The components of the portfolio are either mutual funds or equities of large companies. The portfolio has significant exposure to all major risk factors, including domestic U.S. stock market risk, interest rate risk, foreign exchange risk, and commodities risk.

To hedge against U.S. stock market risk, the manager of the portfolio is allowed to spend a small percentage of the total portfolio value every month to buy put option on the S&P500 index. In the model constructed, the strike price can be chosen, the fraction of the total portfolios value to be spent on options can also be chosen. Time assumptions of options are that the options are bought on the first day of every day at the market price, and the options expire on the last day of the same month.

An unhedged portfolio's value is a weighted linear combination of the market prices of the underlying securities. But the price of an option is not a linear function of the price of the underlying security or index. Consequently the value of a protected, option hedged portfolio is a nonlinear function of the prices of the underlying securities and index.

The quality of the risk protection offered by the hedging strategy will be measured by the 95% value-at-risk (VaR) metrics one month out in the future. This VaR is the dollar amount of the largest loss that can occur by the end of the month with a probability not greater than 5%. In other words, there is 95% probability that at the end of the month the portfolio value will be above the 95% VaR threshold. Put it in another way, in long-term average only once in every twenty months will the loss exceed the 95% VaR.

A successful manager shall produce high return while keeping tight control over the risks. For instance, he shall not incur loss exceeding the VaR limit more frequently than once in every 20 months. As an example to the contrary, a manager who turns in a large return at the end of the month 24 but who incurs loss exceeded the VaR limit four times during that period will not be considered reliable.

The value at risk needs to be assessed and communicated to the client at the beginning of every investment period. In the institutional world regulators require banks and insurance companies keep reserves proportional to their portfolio's value-at-risk.

As a forward looking risk measure, VaR has to be calculated from observed past prices and applied to the future. In this project, three different methods are used to calculate VaR:

1. Traditional estimate based Monte Carlo Simulation
2. GARCH based Monte Carlo Simulation
3. VaR estimation Based on resampling of historical data

Each of the three methods is implemented in Excel Visual Basic for applications (VBA). Then the VaR of various portfolios with different hedging strategies will be evaluated on a monthly basis over a two year sample interval between Jan 2000 and Dec

2001. This will enable us to compare the hedging strategies and also the VaR evaluation methods.

1.2 Overview of VaR Modeling Approaches

In this section we give a concise overview of the three methods used to evaluate VaR. More details will follow in subsequent chapters where we describe the implementation of each method.

In the language of probability theory, the 95% VaR is the difference between the mean of the portfolio value and the lower 5% quantile of the probability distribution of the future portfolio value. Hence VaR evaluation is the estimation of the 5% quantile.

If daily security returns are assumed to be independent, normally distributed random variable and the portfolio value is a linear combination of the underlying security prices, then the portfolio value has a log-normal distribution, such that the VaR can be obtained from the known formula for the quantile of the log-normal distribution. We are not following this approach since it can’t be generalized to fat-tailed (non-normal) return distributions or to hedged portfolios containing options or other non-linear derivatives.

In all three of our approaches we simulate the future portfolio value distribution and estimate its 5% quantile by the value that separates the 5% smallest of the simulated values from the largest 95%. More precisely, we simulate the daily returns on the underlying securities (and index) and combine the returns into the portfolios value at the end of the month. The three methods differ in the way how the simulated daily returns are generated.

In the two Monte Carlo approaches we model the daily returns on the 21 securities as a 21 dimensional joint-normally distributed random vector. Then we use a random number generator to draw pseudo-random deviates from this distribution. By simply changing the function in the random number generator, we would be able to generate other, e.g., fat-tailed return distribution.

To specify a multidimensional normal distribution we have to give its mean vector and its covariance matrix. In risk management the mean is usually assumed to be zero. We estimate the covariance matrix from a historical database of daily returns for the securities. In the Traditional estimate based Monte Carlo Simulation method we estimate the covariance matrix by the traditional sample covariance. In the GARCH based approach we use the recursive GARCH (1, 1) estimator. The parameters of the GARCH process are chosen to maximize the likelihood function.

The third VaR modeling approach considered in this project uses resampling instead of the Monte Carlo method. In this case, we are not making any assumption about the distribution of the daily returns and hence we don't need to estimate its covariance matrix. Instead, we simulate the distribution of the daily returns by randomly resampling from the historical database of past daily returns.

In all three modeling approaches we base our simulated future values on the past three months of observations. We repeat this procedure on the first trading day of every month. The ultimate result is the VaR applicable to the last day of that month. We then compare the loss "predicted" by the VaR to the actual loss or gain realized over the month in question.

Chapter 2

2. Background

2.1 Risk Management and Risk Measures

Most investors wish to get the most returns at minimum risk. Risk is defined to be uncertainty of the future returns. There are five main categories of financial risks:

- Market risk
- Credit risk
- Liquidity risk
- Operational risk
- Legal risk

Market risk is exposure to the uncertain market value of portfolio. Credit risk originates from the fact that counterparties may be unwilling or unable to fulfill their contractual obligations. Liquidity risk compounds both market risk and credit risk. Operational risk generally can be defined as arising from human and technical errors or accidents. Legal risk arises when a transaction proves unenforceable in law [1].

2.2 What is Value at Risk

Value at Risk (VaR) is a category of risk measures, unlike market risk metrics such as the Greeks, duration and convexity, or beta, which are applicable to only certain asset categories or certain sources of market risk, VaR can be applied to all asset categories and can cover all sources of market risk. Therefore VaR is very attractive to senior managers and thus is widely used by banks, security firms, commodity and energy merchants, and other trading organizations.

What is VaR? Value at Risk (VaR) summarizes the worst loss over a target horizon with a given level of confidence [1]. The most advantage of VaR is that it can summarize the maximum loss in a single dollar value.

For example, a risk manager may give the following statement: "We are 90% certain that we will not lose more than $5000 next day." This statement means that the loss level next day will not exceed $5000 with 90% probability under normal conditions. [7]

The following figure gives a visual illustration of the above statement.

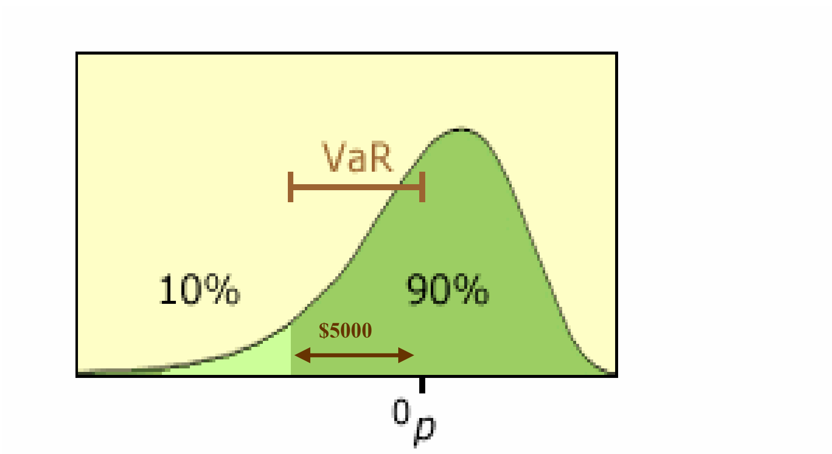


Figure 2.1: VaR description

2.3 How to Calculate VaR

To calculate Value at Risk measure we need two inputs, the portfolio's holdings and historical market data. A transformation method will then be used to combine market data and portfolio holdings, and to get the daily returns on the portfolios. From these we can calculate the probability distribution of the portfolio value N days out in the future. The VaR is the x% quantile of this probability distribution. Steps of calculating VaR are:

1. Calculate the current value of the portfolio by using the current assets prices and holdings

2. Set the time horizon
3. Measure the variability of the risk factors
4. Calculate the future value of the portfolio by using future assets prices and holdings
5. Set the confidence interval
6. Calculate the VaR value

The flow chart below gives visual interpretation of the procedures to calculate VaR.

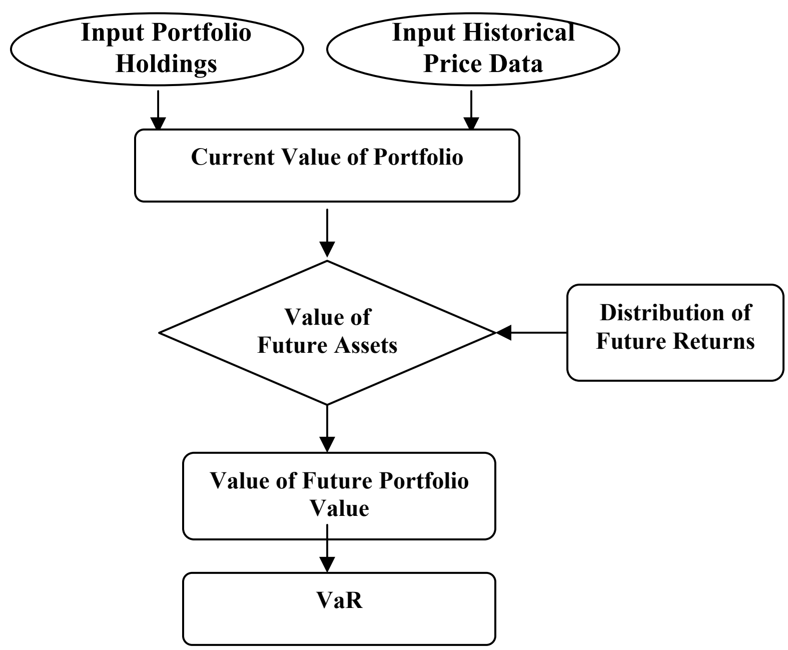


Figure 2.2 Simple Overview of Calculation of VaR

The crucial step is to obtain the probability distribution of the future portfolio value. There are mainly two methods to do this:

- Historical Simulation Method
- Model Based Method

In the Historical Simulation Method one observes how the portfolio would have performed in various past x-day periods in the market. Two important Model Based Methods are Monte Carlo simulation and resampling (Bootstrap).

Inputs of Historical Data and Holdings

Two inputs for calculating VaR are original portfolio holdings ${}^{0}\omega$ and historical prices P for market variables of the portfolio. The first input ${}^{0}\omega$ indicates the number of initial shares of each asset held in the portfolio at the beginning, the second input P is a vector of the historical prices of each asset. ${}^{0}P$ is price at time zero, ${}^{-t}P$ is price at t time periods back. From historical the prices P, we can calculate the historical returns vector R for the assets in the portfolio.

Current portfolio Value

Current portfolio value is easily to be got by

$$ {}^{0}V = \sum_{i=1}^{n} \omega_i \, {}^{0}P_i $$

Approaches to Get the Future Value of Portfolio

If changes of the market variable were assumed multivariate normality, then it is easy to estimate future prices of the market variables. Based on future prices of the market variables, we can estimate the volatility of the portfolio by a linear approximation from changes in the market variables.

However, if the portfolio includes options or mortgage based bonds, the portfolio changes are not linear from the changes of the market variables. For nonlinear portfolio, we often assume the joint normal distribution of risk factors and the returns. Denote the current prices of assets ${}^{0}P$, and the future returns of the market variable ${}^{1}R$. Future returns of the assets can be given by ${}^{1}R = \frac{{}^{1}P - {}^{0}P}{{}^{0}P}$. Then the future prices of assets can be got by

$$
\begin{aligned}
{}^{1}P &= {}^{0}P + {}^{0}P \frac{{}^{1}P - {}^{0}P}{{}^{0}P} \\
&= {}^{0}P + {}^{0}P * {}^{1}R \\
&= {}^{0}P(1 + {}^{1}R)
\end{aligned}
$$

Then to estimate the futures price of market variables is equivalent to estimate the future returns.

To estimate the future returns, four wildly used approaches are as follows:

- Monte Carlo Simulation Method
- Resampling Method
- Quadratic Method
- Historical Simulation

The first two methods are used in our project. The basic concept behind Monte Carlo approach is to simulate repeatedly a random process for the financial variables, covering a wide range of possible situations. In resampling method, the variables are randomly drawn from the historical sample pool with specified distribution.

The Future Value of the Portfolio

Denote the future prices of the market variables ${}^{1}P$, the future value of the portfolio ${}^{1}V$, then

$$
{}^{1}V = \sum_{i=1}^{n} {}^{1}\omega_i \, {}^{1}P_i
$$

Calculate VaR

VaR has two parameters, one is the time horizon *N*, which is measured in days, the other is *X*, the confidence interval. In VaR simulations, usually time period is set to the same as the time horizon, *N* days. The future value of the portfolio, ${}^{1}V$, is the portfolio value after *N* days. Large amount of simulations is drawn, say, 2000 times. The difference between the mean value of 2000 simulations and the quantile value corresponding to the confidence interval is VaR.

Chapter 3

3. Evaluation of VaR

This chapter illustrates in detail three models used to evaluate VaR.

3.1 Variables Used in VaR Models

Notations and variables used in our VaR models are:

${}^{-t}P_i$ represents historical price of the i^{th} security on t time periods back in the past

${}^{t}P_i$ represents future price of the i^{th} security on t time periods forward in the future

${}^{-t}R_i$ represents historical return of the i^{th} security on t time periods back in the past

${}^{0}\omega_i$ represents original holdings of the i^{th} security in the portfolio

${}^{0}n_i$ represents original shares of the i^{th} security in the portfolio

${}^{1}n_i$ represents future shares of the i^{th} security in the portfolio

${}^{0}V$ represents current value of the portfolio

${}^{1}V$ represents future value of the portfolio

3.2 Overview of the Evaluation Procedure

This project uses a scenario that is similar to what a risk manager would be facing. A portfolio was set up for a client beginning in January 2000 and ending in December 2001. The portfolio consists of twenty securities, most of which are mutual funds and S&P500 companies. It will also consist of one put option of S&P500 index to protect against the risk of an unexpected market drop.

The risk manager will be supposed to implement VaR with a confidence level of 95% using one of three model-building methods. His choices are to leave the portfolio unprotected, i.e. not to include the put option or to buy different levels of protection through the put option. He can choose the strike price of the option and the percentage of the total portfolio value that is spent on buying the option. The client wants the biggest profit at acceptable risk for the two-year investment. The proper risk management is considered to be that the VaR threshold is not breached more frequently than 5% of the times.

VaR measurement is calculated to tell the risk manager what loss level of the portfolio will be with a confidence degree of probability. Chapter 2 has stated procedures of calculating VaR in general. A slightly different practical procedure used in our project is given in the following flow chart.

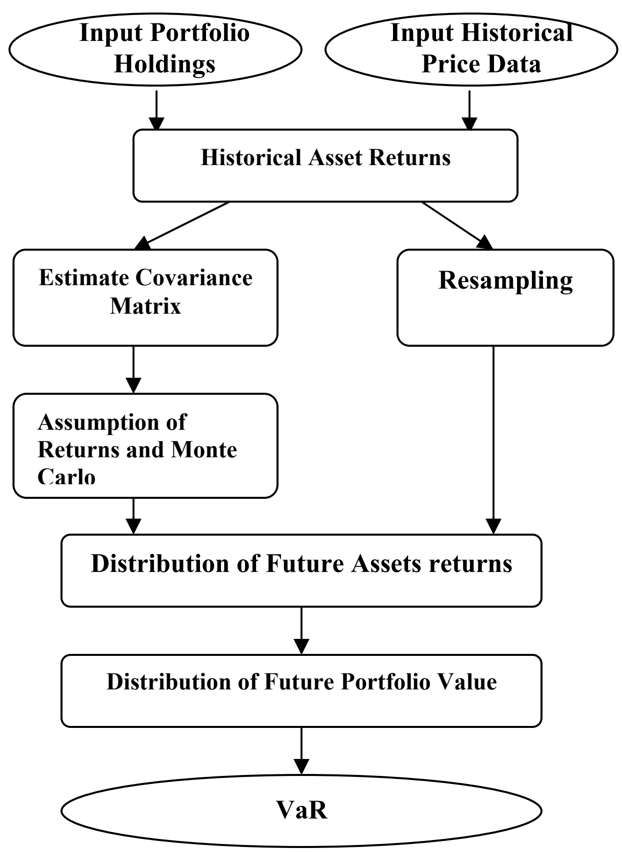


Figure 3.1 Simple Overview of Evaluation of VaR

3.3 Historical Returns and Historical Volatilities

We collected historical price data of 20 securities from Mar 1990 to Nov 2003. The original portfolio holdings are known to be ${}^{0}\omega_1, {}^{0}\omega_2, \ldots, {}^{0}\omega_{20}$.

Denote the historical prices of the market variables at t time periods back in the past ${}^{-t}P$. Denote ${}^{-t}R$ as the percentage price change, the returns, of the market variables between *t* and *t-1* time periods back in the part.

$${}^{-t}R = \frac{{}^{-t}P - {}^{-(t-1)}P}{{}^{-(t-1)}P}$$

In VaR calculation, the returns are assumed to be statistically independent and normally distributed, and the mean of the returns is zero.

Denote the historical volatility during *m* time periods in the past σ.

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} {}^{-i}R^2$$

In calculating volatility, time period is usually set to be one day, σ calculated from the above equation is daily volatility. Assuming 252 trading days in a year, the annualized volatility will be $\sqrt{252}\sigma \approx 16\sigma$.

3.4 Methods to Estimate Covariance Matrix

3.4.1 Traditional Method

One approach to estimate the covariance matrix is to put equal weights on all historical returns. Assume the mean returns are zero, then the variance is given by

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} {}^{-i}R^2$$

The covariance of the historical returns of stock *l* and stock *k* is given by

$$\text{cov}_{lk} = \frac{\sum_{i=1}^{m} {}^{-i}r_l \, {}^{-i}r_k}{m-1}$$

The covariance matrix of *n* stocks is given by

$$COV = \begin{pmatrix} \text{var}\, R_1 & \text{cov}(R_1, R_2) & . & . & \text{cov}(R_1, R_n) \\ \text{cov}(R_2, R_1) & \text{var}\, R_2 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \text{cov}(R_n, R_1) & . & . & . & \text{var}\, R_n \end{pmatrix}$$

3.4.2 GARCH Method

A well-known method to estimate the volatilities is Generalized AutoRegressive Heteroskedastic (GARCH) Model. The GARCH model assumes that the variance of returns follows a predictable process. It assigns more weights on recent variance. The simplest model is GARCH (1, 1) model. The formula is as following:

$${}^{-(t-1)}\sigma^2 = \gamma V_L + \alpha({}^{-(t-1)}R)^2 + \beta({}^{-t}\sigma)^2$$

Where γ, α, β are positive constant weights that sum to one, $\gamma + \alpha + \beta = 1$, and V_L is long-run average variance rate.

Estimating GARCH (1, 1) Parameters

We can estimate the γ, α, β by using maximum likelihood method from return data. Assume that the returns are R_1, R_2, R_3, …. R_m that are normally distributed, and the mean of the returns are zero. Denote the variance by v. The likelihood of R_i being observed is the value of the probability density function, given by

$$\frac{1}{\sqrt{2\pi\nu}}\exp(\frac{-R_i^{\ 2}}{2\nu})$$

The likelihood of the m observations occurring in the order in which they are observed is

$$\prod_{i=1}^{m}\frac{1}{\sqrt{2\pi\nu}}\exp(\frac{-R_i^{\ 2}}{2\nu})$$

By maximum likelihood method, the best estimate of ν is the value that maximizes the value of this expression. Since logarithm function is increasing function, maximizing a function is equivalent to maximizing the logarithm of it. Thus we want to maximize

$$\sum_{i=1}^{m}(-\ln(\nu)-\frac{R_i^{\ 2}}{\nu}) \quad \text{i.e.} \quad -m\ln(\nu)-\sum_{i=1}^{m}\frac{R_i^{\ 2}}{\nu}$$

Optimization functions, such as Solver in Excel VBA, or MatLab toolbox, can be used to get optimal values of α, β . Then $\gamma = 1-\alpha-\beta$.

Use GARCH (1, 1) to Get Historical Volatilities

Setting V_L to be the starting volatility at time zero, we can use

$${}^{-(t-1)}\sigma^2 = \gamma V_L + \alpha({}^{-(t-1)}R)^2 + \beta({}^{-t}\sigma)^2$$

to get the historical volatilities.

Use GARCH (1, 1) to forecast Future Volatilities

Denote the long-term correlations of the historical returns as $Correl$. Suppose the correlation between the returns will not be changed while time changes. Then the estimated covariance matrix can be got by

$$COV = \begin{pmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & . & & \\ & . & . & . & \\ & & . & \sigma_{n-1} & \\ 0 & & & & \sigma_n \end{pmatrix} Correl \begin{pmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & . & & \\ & . & . & . & \\ & & . & \sigma_{n-1} & \\ 0 & & & & \sigma_n \end{pmatrix}$$

Where COV is 20×20 matrix

Similarly the variance of the portfolio is given by

$$\sigma_p{}^2 = \sum_{i=1}^{20} \omega_i COV(i,i)\omega_i + \sum_{i=1}^{20}\sum_{j<i} \omega_i COV(i,j)\omega_j$$

3.5 Monte Carlo Simulation

Based on two covariance estimations, Monte Carlo simulations will be used to simulate the distribution of future returns. The basic concept behind Monte Carlo approach is to simulate repeatedly a random process for variables based on specified distribution. The chart below is a flow chart of using Monte Carlo Simulation.

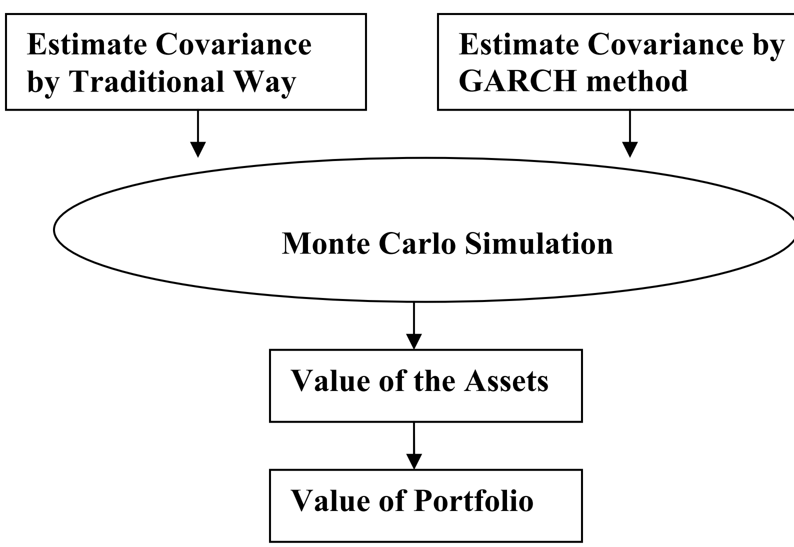


Figure 3.2 Estimation Based Monte Carlo Simulation

The simulation can be carried out by the following steps:

1. Assume the distribution of the returns
2. Generate a pseudo random numbers, $\varepsilon_1, \varepsilon_2, \varepsilon_3 ..., \varepsilon_n$ from which the future prices of the assets are simulated
3. Calculate the value of the assets(or portfolio) from data in step 2
4. Repeat steps 2, 3 as many times as necessary, normally, more than 1000 times.

This process creates a sequence of the future value of the portfolio ${}^1V_1, {}^1V_2, ..., {}^1V_{1000}$. We can sort the future values of the portfolio, and get the expected value and the 5% quantile of the smallest sorted values. The difference of the two numbers is VaR of the portfolio.

Generating Normal Random Number

A critical part of Monte Carlo simulation is the generation of the standard random variables. Firstly random numbers are generated from a uniform distribution over the interval [0, 1]. More properly speaking, these numbers are "pseudo" random because they are generated from an algorithm using a predefined rule. Then these random numbers are used to generate random returns from the normal distribution of returns through the inverse cumulative probability distribution function.

Multidimensional Normal Random Deviates with Given Covariance Matrix:

If returns are normally distribution vectors $N_n(\mu, \Sigma)$, we can transform standard normal distribution to multidimensional normal random deviates.

If X is *n* dimensional standard normal random vector with independent corresponds, the covariance matrix Σ , $\Sigma = I = \begin{pmatrix} 1 & 0 & . & 0 & 0 \\ 0 & 1 & . & . & 0 \\ 0 & . & . & . & . \\ . & . & . & . & 0 \\ 0 & 0 & . & 0 & 1 \end{pmatrix}$, which is n independent standard normal deviates.

We can construct $N_n(\mu, \Sigma)$ pseudorandom vector $y^{[k]}$ from previous $N_n(0,1)$ pseudorandom vector $x^{[k]}$ by setting $y^{[k]} = kx^{[k]} + \mu$, where k is the Cholesky matrix of Σ.

3.6 Resampling Model

In resampling model, random numbers are generated from uniform distribution, and then they are used to get sample with replacement from a pool of past historical returns. In this way, the distribution of historical returns is not necessary to be assumed as in the Monte Carlo simulation.

The steps to perform resampling are as followings:

- Generate a random number N. N is between 1 to size the past three month's returns. We generate the future returns by randomly selecting one return at a time from the sample over the past three months with replacement. Define the index choice as N_1, a number between 1 and 60, the selected return is R_{N_1}.
- Repeating the operation for total of 20 replications yields a total of 20 pseudo numbers for future returns. From these future returns, we can calculate the values of assets in the future.

The most important advantage of bootstrap method is that it doesn't require the distribution of the selected variable. It can include fat tails for the returns. But bootstrap method has some limitations. For small sample sizes, the bootstrapped distribution is a poor approximation to the actual one. Thus when I implemented bootstrap method, about 180 days data were used.

3.7 Calculate Protected Portfolio Value

In this section, we explain the method to calculate the value of the portfolio, protected and unprotected. If the total value of the protected portfolio value is lower than the unprotected portfolio, then the hedging is ineffective to protect the portfolio.

3.7.1 Calculate the Value of Protected Portfolio

The protected portfolio is composed of 20 stocks and one put option on S&P 500 index. The protected portfolio has nonlinear relationship with the changes of the returns. The left graph below shows relationship between the value of the protected portfolio and the returns, the right graph shows distribution of portfolio value.

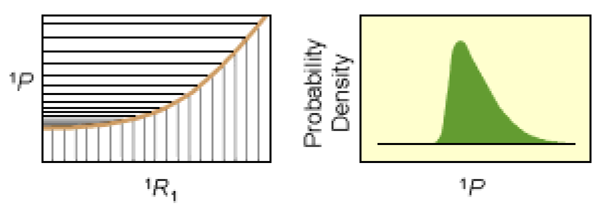


Figure 3.3 Nonlinear Portfolio

We divide calculation of nonlinear portfolio into two parts. First we calculate the linear part of the portfolio, and then we calculate the option value of the portfolio. The summation of the two parts is the total value of the portfolio.

The steps to get the value of the portfolio are as following:

- Calculate the covariance matrix based on past three month returns

- Use this to get the portfolio value on the day at the end of the month. Nonlinear portfolio value equals the value of linear part of the portfolio and the value of the option.

3.7.2 Apply Covariance Matrix to Get Values of Assets

Denote the shares of i^{th} stock as n_i. Then a security price *m* days out in the future is given by

$$^{m}P = {}^{0}P\prod_{i=1}^{m}(1+r_i)$$

Then we can get

$$^{m}P = {}^{0}P\frac{^{2}P}{^{1}P}\frac{^{3}P}{^{2}P}......\frac{^{m}P}{^{m-1}P} = {}^{m}P$$

For $^{m}P = {}^{0}P\prod_{i=1}^{m}(1+r_i)$, we can use Taylor expansion to approximate the equation. Then

$$^{m}P = {}^{0}P\exp\left(\sum_{i=1}^{m}\ln(1+r_i)\right) = {}^{0}P\exp\left(\sum_{i=1}^{m}r_i\right)$$

The value of the security is $n_i * {}^{m}P$.

3.7.3 Calculate Option Value

On the first day of each month, the risk manager can buy a put option on S&P 500 index, which expires at closing on the last day of the same month. We do the following steps to calculate the price of the option on the first day of each month:

1. Simulation S&P 500 index as the 21^{st} securities of the portfolio
2. Use covariance estimate to get the volatility of the S&P 500 index
3. Set the proportion of money put into buying option
4. Set the striking price of the option

5. Calculate price of the option at the beginning of the month, accordingly shares of the option and value of the option

When we calculate the beginning price, we use Black-Scholes formula

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where T is the expiration period for the put option on S&P500 index,

K is the strike price

S_0 is the beginning price of the underlying stock at time 0

r is risk free interest rate

σ is the volatility of the underlying stock

$$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Suppose at the end of the month the S&P500 index price is S_T, we can calculate the option price by

$$Max(K - S_T, 0)$$

3.7.4 Method to Calculate the Value of Unprotected Portfolio Value

Since unprotected portfolio only contains stocks, it is a linear portfolio. We can use covariance matrix which is introduced in 3.8.2 to get the value of unprotected portfolio.

The procedure to get the value of the portfolio is as following:

- Calculate the covariance matrix based on past three month returns

- Use this to simulate the portfolio value on the day at the end of the month. Its value equals the summation of value of the securities.

Chapter 4

4. Using VaR for Risk Management

In previous chapter, we built models to calculate the VaR of the portfolio. In this chapter, we show results from our models, we also compare the results from different hedging strategies.

4.1 Two Year Return and Volatility

Returns are always the most important thing in investment. For the two years investment, we must calculate the two years returns. Suppose for the first year, the value of the portfolio is V_0 , at the end of the two years the value of the portfolio is V_{24} .

Then the two-year returns are given by $R_{24} = \dfrac{V_{24} - V_0}{V_0}$

Two years volatility are given by $\sigma_{24} = \dfrac{1}{24}\sum_{i=1}^{24} R_i$

Both parameters are calculated for the unprotected and protected portfolio.

4.2 Application of VaR for Risk Management

Firstly, we put a small percent of money on option, 0.1% of the value of the portfolio in put option, and set the strike price to be 90% of the closing price of S&P500 index. This hedging strategy gives little effect of hedging even in the turbulent stock market from Jan 2000 to Dec 2001.

Then, we put 1% of the value of the portfolio in option, and set the strike price to be 95% or 99% of the closing price S&P500 index. One thousand simulations are drawn.

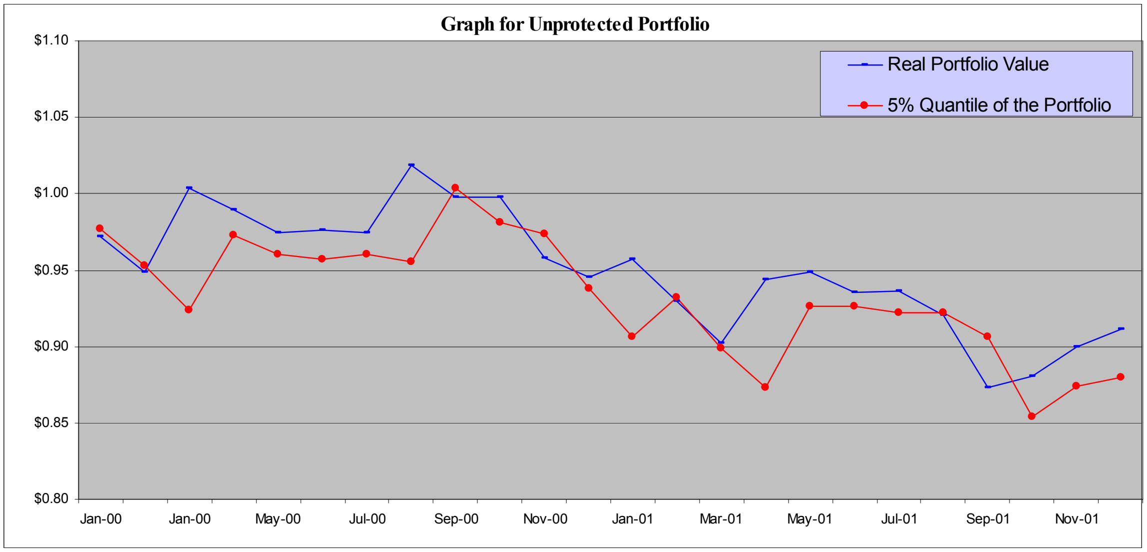


Figure 4.1 Unprotected Portfolio

with 1% invest on options, with option strike price is 95% of closing price

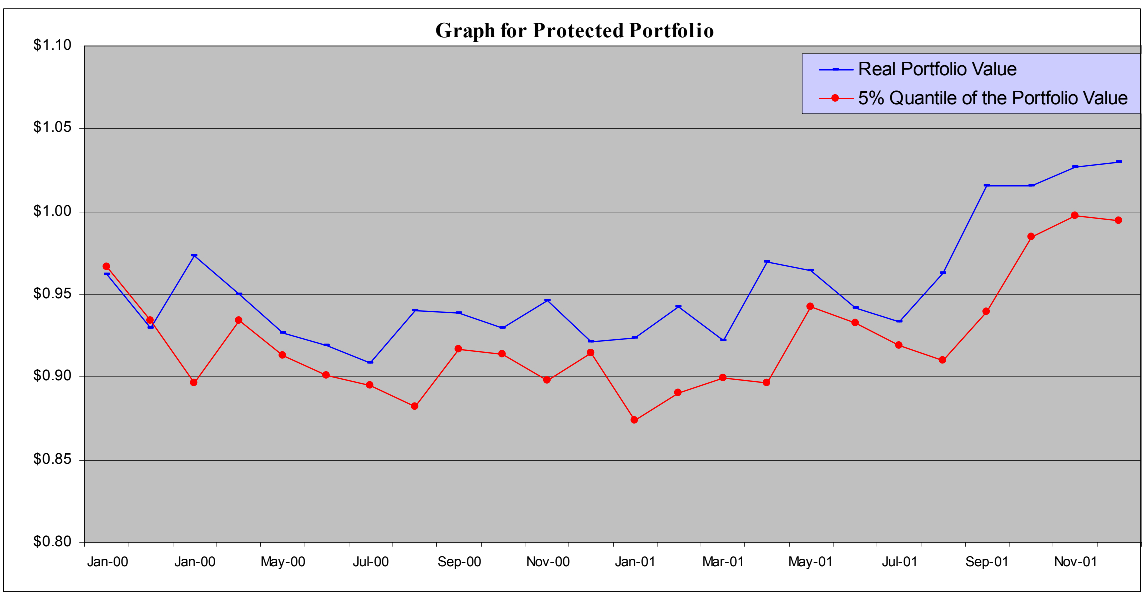


Figure 4.2 Protected Portfolio

with 1% invest on options, with option strike price is 95% of closing price

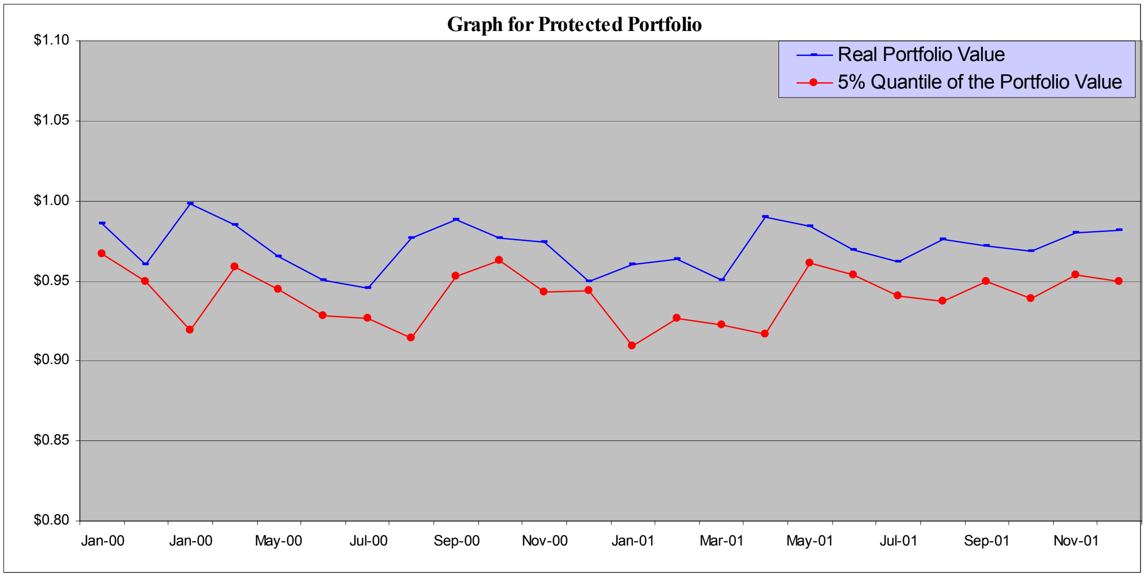


Figure 4.3 Protected Portfolio

with 1% invest on options, with option strike price is 99% of closing price

Two strike prices, at 95%, or at 99% of the closing price of S&P500 index, are both effective for hedging purpose. Considering the cost of option, 95% strike price is more desirable, since the option price with 99% strike price is more expensive than the option price with 95% strike price.

Further, we investigated whether the hedging will perform better if we put more money on option. The answer is yes. Results of putting 2% and 1% of total portfolio value on options with 99% strike price are shown below.

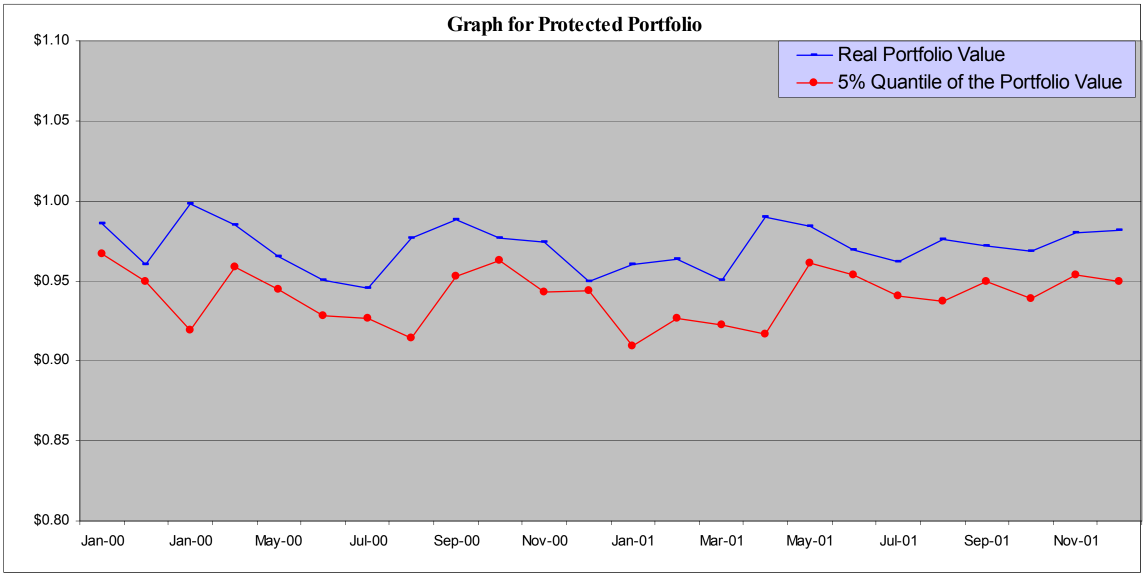


Figure 4.4 Protected Portfolio

1% invest on options, with option strike price is 99% of closing price

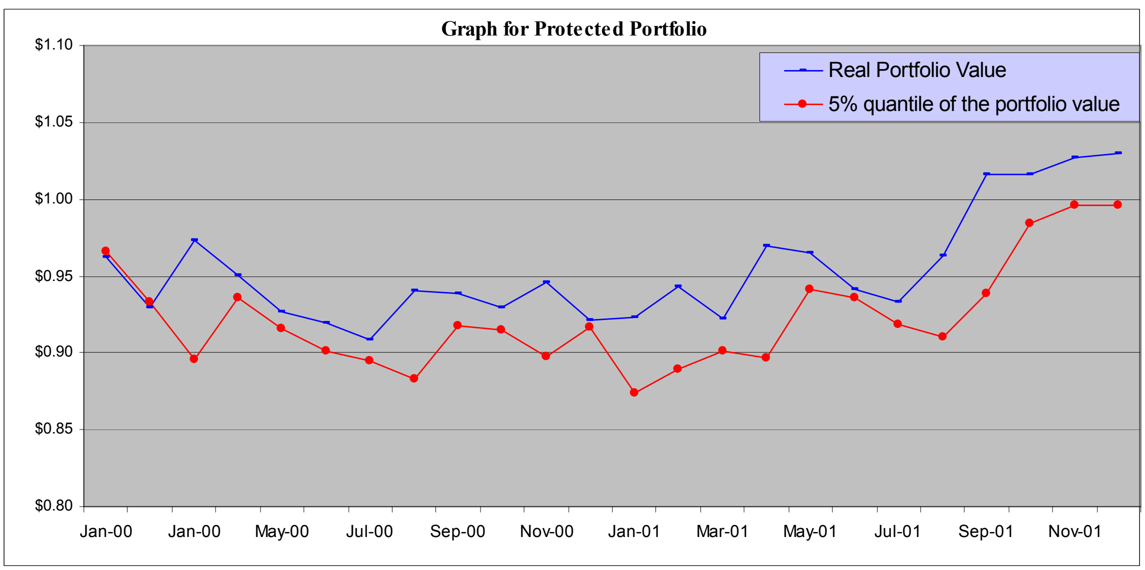


Figure 4.5 Protected Portfolio

2% invest on options, with option strike price is 99% of closing price

It shows that if we invest more money on put option, we get better hedging effect when the stock market drops violently.

Are these models work well in bull stock market? We tried the model from Jan 1993 to Dec 1994. It shows that put option hedging strategy is not good in those years, since the hedging is not working.

4.3 Comparison of VaR for Three Models

We also found that three different models give similar results.

The first chart is for protected portfolio using traditional estimate based Monte Carlo simulation with 1% investment on option with strike price at 99% of closing price of S&P500 index (1000 simulations).

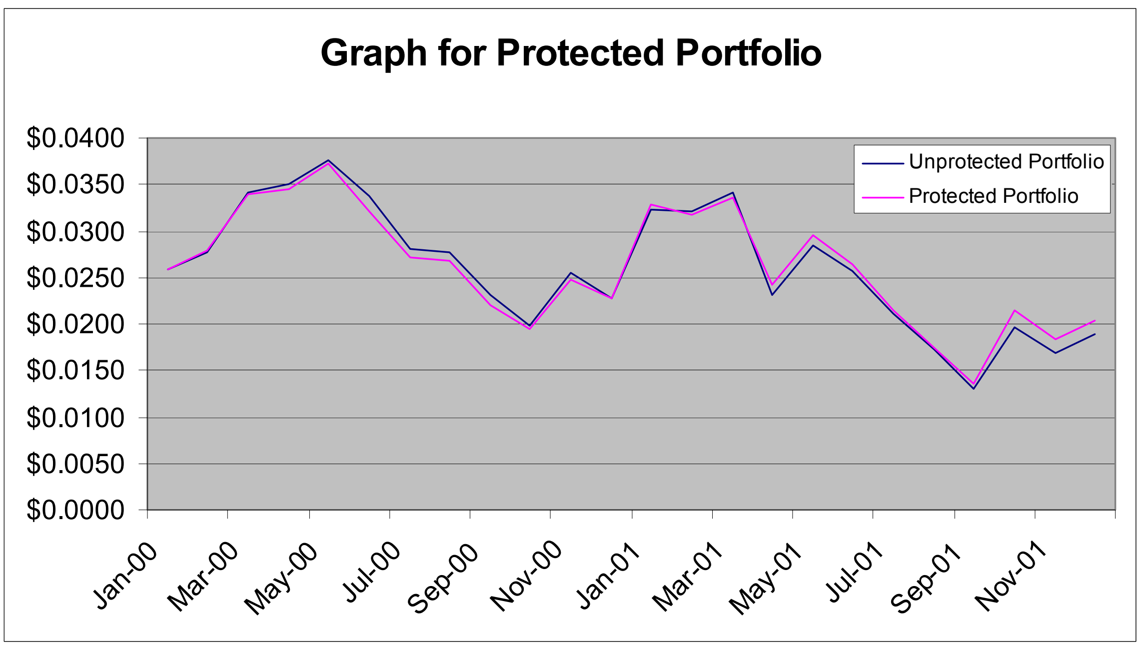


Figure 4.6 VaR chart of protected portfolio

Using traditional estimate based Monte Carlo simulation

The second chart is results from using GARCH estimate based Monte Carlo simulation.

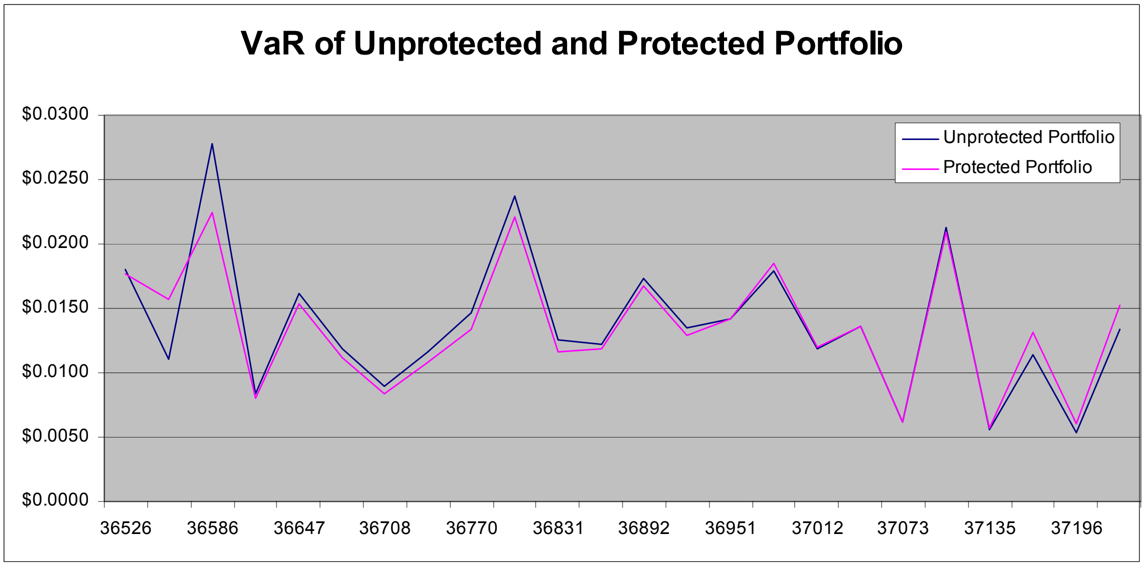
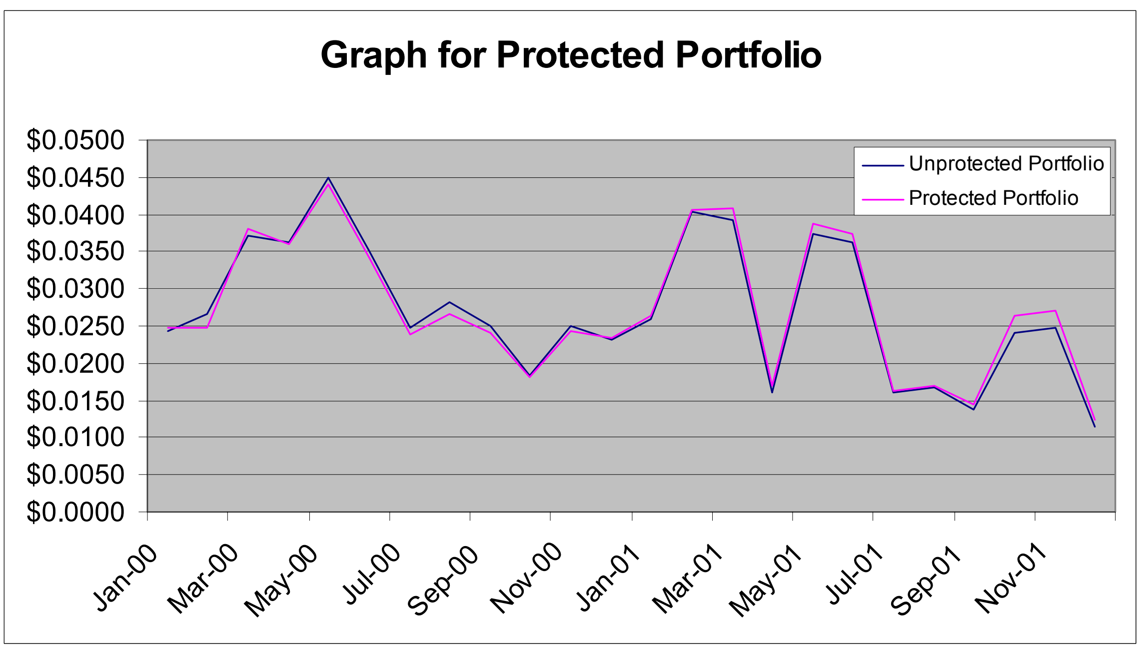


Figure 4.7 VaR chart of protected portfolio

Using GARCH based Monte Carlo simulation

The third chart is the results from using GARCH resampling method.



Chapter 5

5. Findings

From the perspective of the value-at-risk (VaR) risk measures, we can compare the effectiveness of various hedging strategies. Our findings are generalized as follows:

- Option based hedging strategy shows good hedging results from Jan 2000 to Dec 2001 when the stock market has extreme drop.
- In volatile market like 2000-2001 buying put option at strike price 95% of current closing price protects portfolio better.
- Spending 2% of portfolio value on options given better returns than spending 1% in turbulent years like 2000-2001.
- In bull market, for instance 1992-1994, put option hedging strategy is not working.

Appendix A

A.1 Traditional Estimate Based and Monte Carlo Simulation

Unprotected Portfolio (1000simulations)

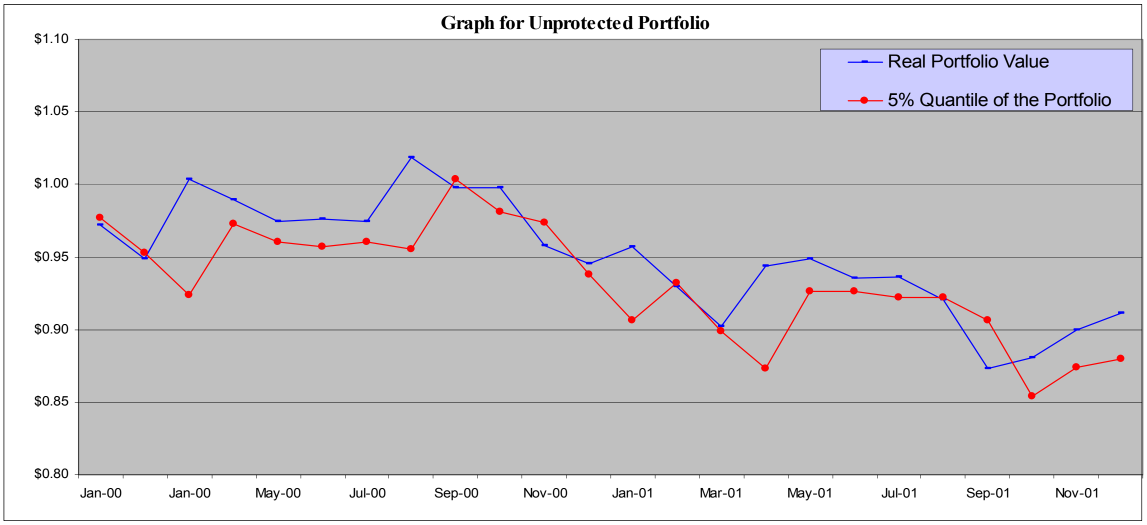


Figure A.1 Unprotected Portfolio

with 1% invest on options, with option strike price is 99% of closing price of S&P500

Protected portfolio using Monte Carlo 1% investment on option with 99% of current closing price of S&P500 (1000 simulations)

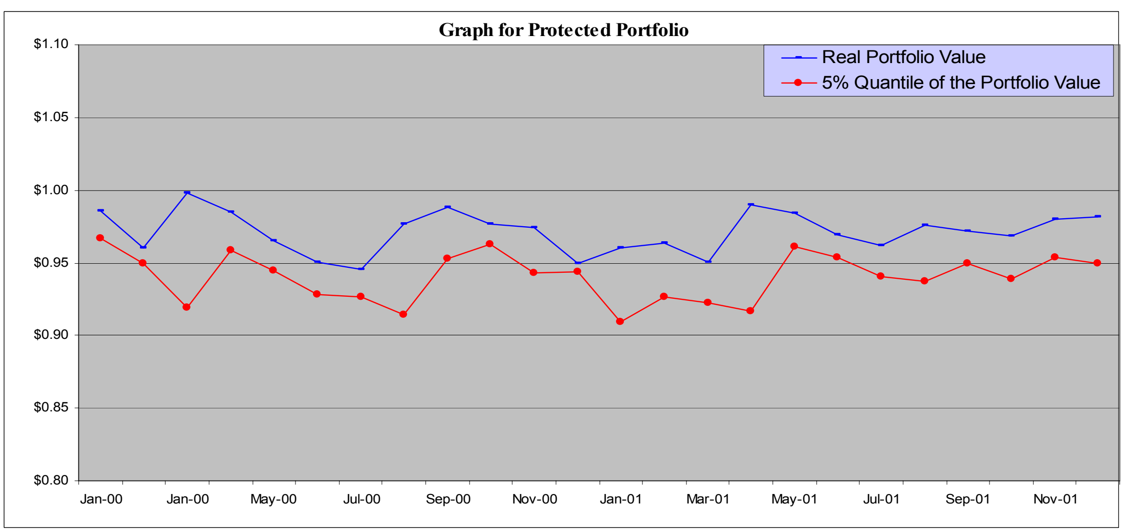


Figure A.2 Protected Portfolio

with 1% invest on options, with option strike price is 99% of closing price of S&P500

A.2 GARCH Based Monte Carlo Simulation

Unprotected portfolio (1000 simulations)

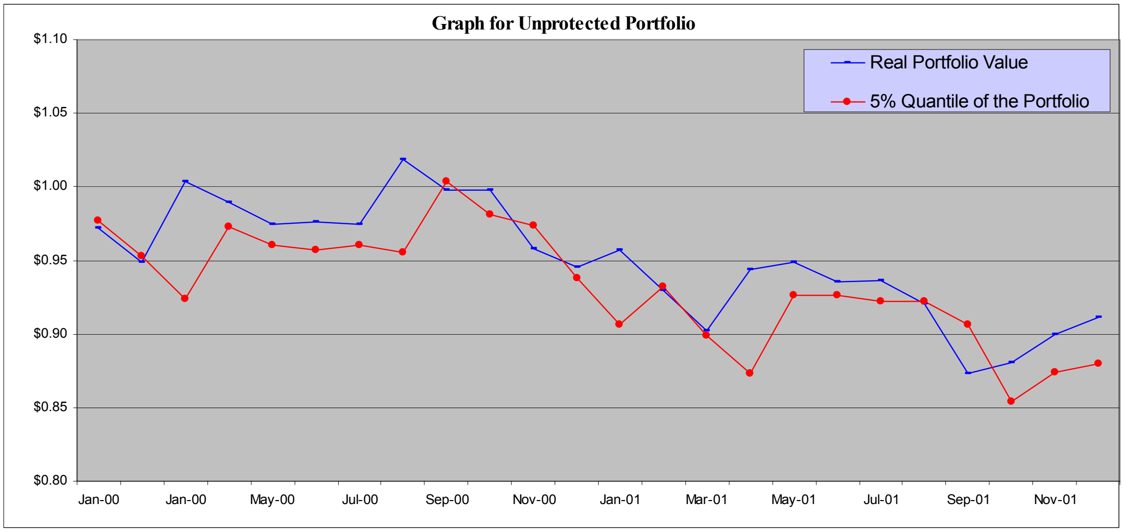


Figure A.3 Unprotected Portfolio

with 1% invest on options, with option strike price is 99% of closing price of S&P500

Protected portfolio using Monte Carlo method 1% investment on option with 99% of current closing price of S&P500 (1000 simulations)

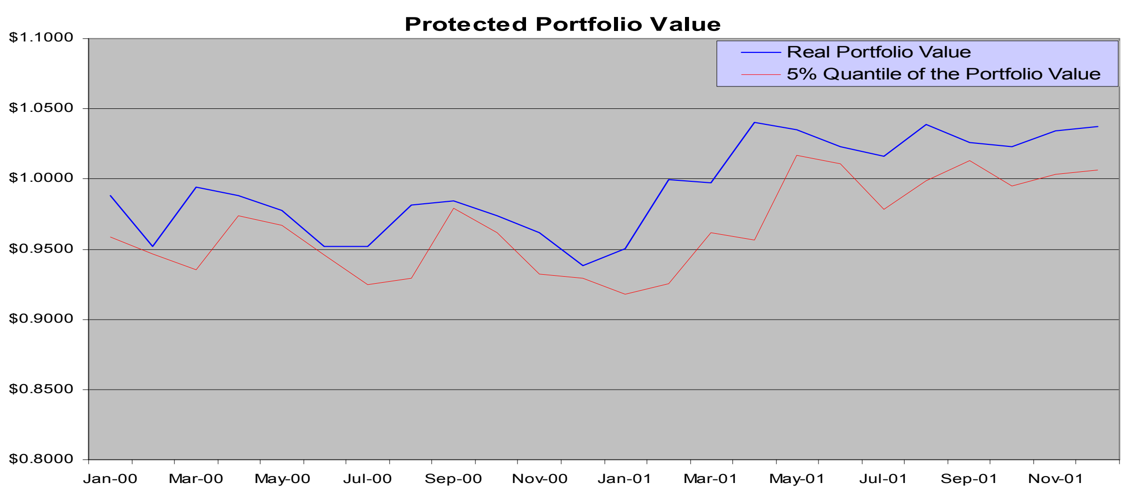


Figure A.4 Protected Portfolio

with 1% invest on options, with option strike price is 99% of closing price of S&P500

A.3 Resampling Results

Unprotected portfolio figure (1000 replications)

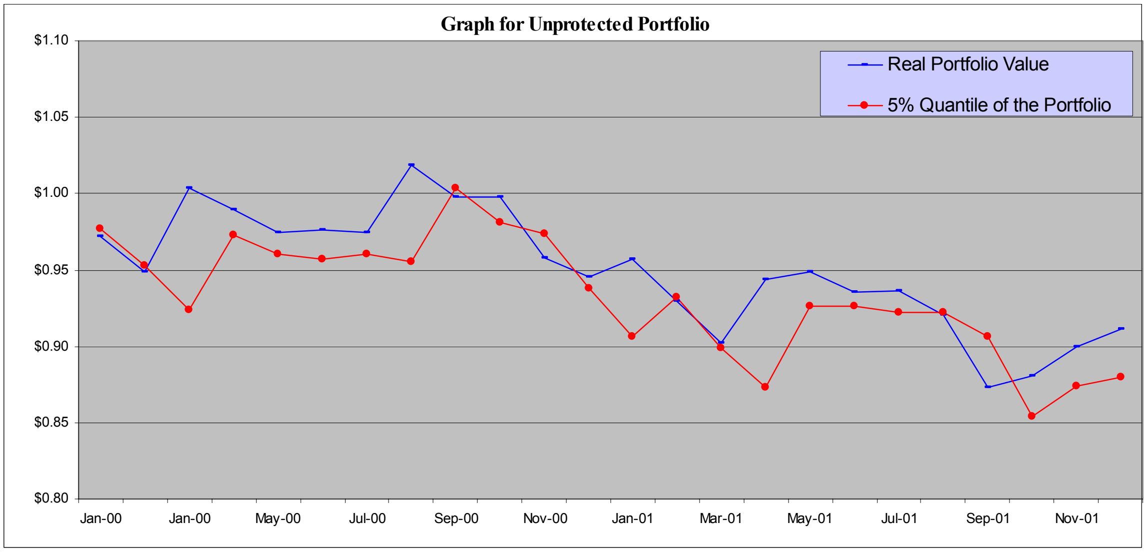


Figure A.5 Unprotected Portfolio

with 1% invest on options, with option strike price is 99% of closing price of S&P500

Protected portfolio using resampling 1% investment on option with 99% of closing price of S&P500 (1000 replications)

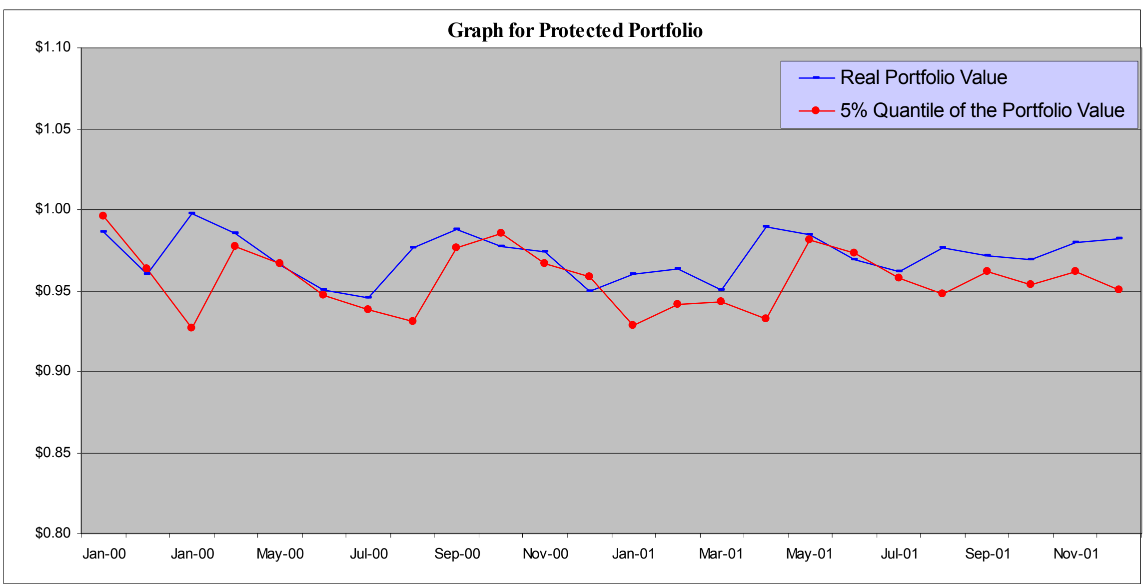


Figure A.6 Protected Portfolio

with 1% invest on options, with option strike price is 99% of closing price of S&P500

References

[1] Philippe Jorion, Value at risk: the Benchmark for Controlling Market risk, McGraw-Hill Professional Book Group

[2] Mark Britten-Jones and Stephen M. Schaefer, Non-Linear Value-at-Risk, European Finance Review

[3] Simon Benninga and Zvi Wiener (1998), Value-at-Risk, Mathematicain Education Research, Vol. 7 No. 4

[4] John Hull and Alan White, Value at Risk When Daily Changes in Market Variables are not Normally Distributed, Journal of Derivatives, Vol 5, No. 3, spring 1998, PP 9-19

[5] Dowd, K. (1998): Beyond Value at Risk, Wiley, Chichester.

[6] Darrell and Jun Pan, An Overview of Value at Risk

[7] John C Hull. Options futures, and other Derivatives (5th edition), Pearson education

[8] Wendy L. Martinez, Angel R. Martinez, Computational Statistics handbook with MATLAB, Champan&Hall /CRC

[9] Les Clewlow and Chris Strickland, Implementing derivatives models, Wiley

[10] http://www.quantnotes.com/publications/var.htm

[11] http://www.cooper.edu/engineering/chemechem/monte.html

[12] Istvan Manno, Introduction to the Monte-Carlo Method, Akademiai Kiado, Budapest

[13] George S. Fishman, Monte Carlo Concepts, Algorithms and applications, Springer

[14] Reuven Y. Rubinstein, Simulation and the Monte Carlo Method, John Wiley& Sons

[15] Clifford E. Lunneborg, Data Analysis by resampling: Concepts and Applications, Duxbury Press

[16] S.N. Lahiri, Resampling Methods for dependent Data, Springer Series in Statistics

[17] Robert Litterman, Kurt Winkelmann, Risk Management series-estimating covariance matrices

[18] Levine Stephan Krehbel Berenson, Statistics for Managers, 3rd edition

[19] Peter Albrecht (2003): Risk Measures, John Wiley&Sons

[20] http://www.riskglossary.com/articles/