

Graph Theory with Applications to Statistical Mechanics

Eric Reich, WPI

September 11, 2014

Abstract

This work will have two parts. The first will be related to various types of graph connectivity, and will consist of some exposition on the work of Andreas Holtkamp on local variants of vertex connectivity and edge connectivity in graphs. The second part will consist of an introduction to the field of physics known as *percolation theory*, which has to do with infinite connected components in certain types of graphs, which has numerous physical applications, especially in the field of statistical mechanics.

Contents

1	Introduction	4
1.1	Basic terminology and notation	4
1.2	Types of Connectivity	7
1.2.1	Vertex-connectivity in graphs	8
1.2.2	Edge-Connectivity in graphs	9
1.2.3	Restricted edge-connectivity	9
1.2.4	Local restricted edge-connectivity	10
2	Vertex Connectivity	11
2.1	Results on maximum local connectivity in graphs with bounded clique number	11
2.2	Maximum local connectivity in diamond-free and p -diamond-free graphs	13
2.3	Maximum local connectivity in $K_{2,p}$ -free graphs	15
3	Edge Connectivity	16
3.1	Maximum (local) edge connectivity in diamond-free graphs	16
3.2	Restricted edge-connectivity	18
3.3	λ_3 -optimality in triangle-free graphs	18
3.4	Local k -restricted edge-connectivity	20
4	Percolation	24
4.1	Introduction	24
4.2	Percolation in the linear lattice	26
4.3	Percolation in the Bethe Lattice	28
4.4	Application - Magnetism	31

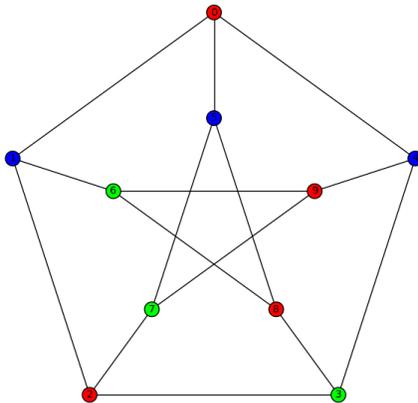
1 Introduction

The first part of this work will contain some exposition on many theorems on various types graph connectivity. We follow the thesis of Holtkamp [1].

1.1 Basic terminology and notation

In the first part of this work, we shall give an overview of some of the terminology of graph connectivity. As an illustrative example for these concepts, we will make use of the *Petersen Graph*, pictured below:

Figure 1: The Petersen graph.



We begin by establishing some notation. Let G be a graph with n vertices. For an unordered pair of vertices u, v of G , recall that $e = uv$ is the *edge* connecting u and v . We first define the *vertex* and *edge* sets of G :

Definition 1.1.1. The *vertex set* $V(G)$ of a graph G is the collection of all vertices of G .

Definition 1.1.2. The *edge set* $E(G)$ of a graph G is the collection of all edges of G .

We may also define notions of *open and closed neighborhoods* for a vertex v of a graph.

Definition 1.1.3. For a vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ is the set of all vertices adjacent to v . The *closed neighborhood* is defined to be $N_G(v) \cup \{v\}$.

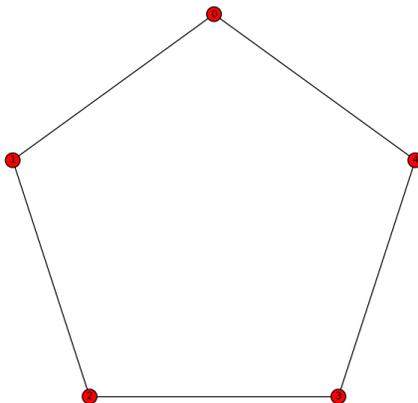
For $X, Y \subseteq V(G)$, we define $(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$ and $|X, Y| = |(X, Y)|$. Similarly, the neighborhood of a collection of vertices $N[X] = \bigcup_{v \in X} N[v]$.

Next, we introduce the notion of an induced subgraph.

Definition 1.1.4. For a graph G and a set X of vertices of G , the *induced subgraph* $G[X]$ is the graph composed of the vertices in X along with those edges whose endpoints are both in X .

For a set of vertices X of G , the notation $G - X$ will denote the graph induced by $V(G) - X$, and $G - S$ denotes a subgraph in which a collection of edges $S \subset E(G)$ is removed from G .

Figure 2: The subgraph induced by the vertices v_0, \dots, v_4 of the Petersen Graph.



For a graph G and vertex v , we will denote by $n(G) = |V(G)|$, $m(G) = |E(G)|$, $d(v) = |N(v)|$ the *order* of G , *size* of G , and *degree* of v respectively. The minimum and maximum degrees of a graph G , denoted by $\delta(G)$ and $\Delta(G)$ respectively, are defined as $\min \{d(v) : v \in G\}$ and $\max \{d(v) : v \in G\}$ respectively. Going back to our example of the Petersen graph, we see that each vertex has $d(v) = 3$, so $\delta = \Delta = 3$, $n = 10$ and $m = 15$.

Now, we recall the notions of a *cycle* and a *path* in a graph G :

Definition 1.1.5. Let $\{v_1, \dots, v_p\} \subset V(G)$ with $\{v_1v_2, \dots, v_pv_1\} \subset E(G)$. Then we say that $C_p = v_1v_2 \dots v_p$ forms a cycle of length p provided that $p \geq 3$.

Our induced subgraph above is an example of a 5-cycle.

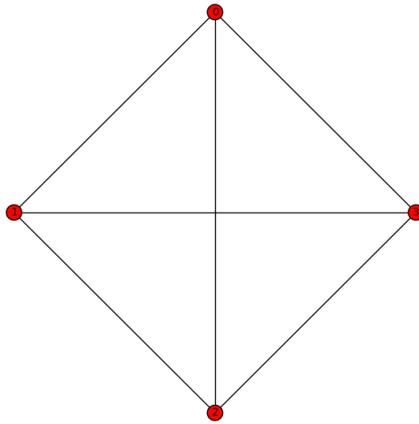
Definition 1.1.6. Let u, w be two vertices of G . Then we say that $P = u_0u_1 \dots u_m$ is a *path* of length $m \geq 1$ from u to w if $u_0 = u$, $u_m = w$ and $u_i \neq u_j$ for all $i, j \in \{1, 2, \dots, m\}$. The length of the shortest path between two vertices u, w is called the *distance* between them.

In the Petersen graph, for instance, there is a path between v_0 and v_3 : $\{v_0v_5, v_5v_8, v_8v_3\}$.

Finally, we review some notation for a few special types of graphs that will be of significance in this work:

Definition 1.1.7. A *complete graph* K_n is a graph with n vertices and all possible edges.

Figure 3: The complete graph on 4 vertices, K_4 .



Definition 1.1.8. A *clique* is an induced complete subgraph of a graph G . The *clique number* $\omega(G)$ is the maximum order over all cliques of G .

In our Petersen graph example, the clique number ω is 2.

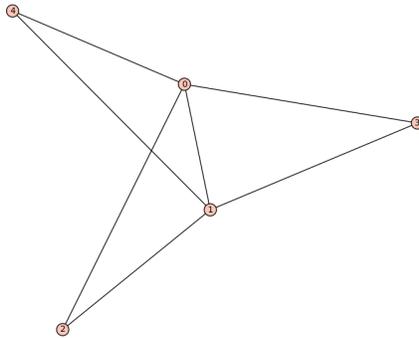
Definition 1.1.9. A graph G is said to be *p-partite* (or *p-colorable*) if its vertex set $V(G)$ may be partitioned into p independent sets, where an *independent* set of vertices is one whose induced subgraph contains no edges. If G is q -colorable but not $q-1$ -colorable for some q , we say that $q = \chi(G)$ is the *chromatic number* of G .

The Petersen graph, for instance, has clique number 2, and is 3-partite, so it has chromatic number $\chi = 3$.

Definition 1.1.10. A *triangle-free* graph is one that has no C_3 as a subgraph.

Definition 1.1.11. A *diamond* is the graph obtained by removing a single edge from K_4 . A *p-diamond* is a graph consisting of $p+2$ vertices, with a pair of connected vertices that have p common neighbors and no other edges.

Figure 4: A 3-diamond.



1.2 Types of Connectivity

In this section, we will give an overview of the various types of connectivity commonly used in graph theory.

Definition 1.2.1. A graph G is said to be connected if, given any two vertices u, v of G , there exists a path between u and v .

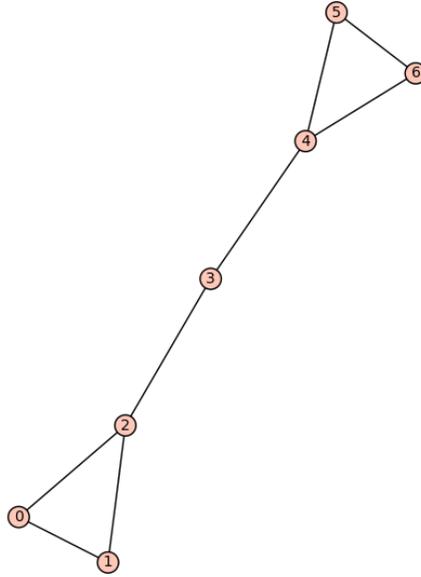
Definition 1.2.2. A *component* is a maximal connected induced subgraph of a graph G .

Definition 1.2.3. A vertex v of a graph G is said to be a *cut-vertex* if its removal divides G into at least two components. If a graph has no cut-vertices, it is said to be *2-connected*.

Definition 1.2.4. A subset $X \subset V(G)$ of a connected graph G is said to be a separating set if $G - X$ consists of at least two components. A *minimal* separating set is one that is minimal with respect to inclusion, and a *minimum* separating set is one of minimal cardinality.

Example

Figure 5: A connected graph. Deleting the vertex v_3 separates G into two components, so that v_3 is a cut-vertex of G and forms a separating set.



1.2.1 Vertex-connectivity in graphs

Definition 1.2.5. The connectivity number $\kappa(G)$ of a graph G is the smallest number of vertices whose deletion disconnects the graph.

It is also possible to define a local notion of connectivity, as follows:

Definition 1.2.6. The *local connectivity* $\kappa(u, v)$ between two vertices u, v of a graph G is defined to be the maximum number of internally disjoint $u-v$ paths in G . It is a consequence of Menger's theorem[17] that $\kappa(G) = \min \{\kappa(u, v) : u, v \in V(G)\}$.

In addition, the maximum number of internally disjoint $u-v$ paths is equal to the minimum cardinality of the separating set S separating u and v in the event that $uv \notin E(G)$. For $uv \in E(G)$ there exists a vertex subset S separating u and v in $G - uv$ with $S = \kappa_{G-uv}(u, v) = \kappa_G(u, v) - 1$. We also have $\kappa(G) \leq \delta(G)$ and $\kappa(u, v) \leq \min \{d(u), d(v)\}$.

Definition 1.2.7. A graph is said to be *maximally connected* when $\kappa(G) = \delta(G)$ and $\kappa(u, v) \leq \min \{d(u), d(v)\}$. A graph is said to be *maximally local connected* when $\kappa(u, v) = \min \{d(u), d(v)\}$.

1.2.2 Edge-Connectivity in graphs

In order to talk about edge-connectivity, we first introduce the definition of an edge-cut in a graph G :

Definition 1.2.8. An *edge-cut* is a subset $S \subseteq E(G)$ such that $G - S$ is disconnected.

Definition 1.2.9. The *edge connectivity* $\lambda(G)$ is the smallest number of edges whose deletion disconnects the graph.

Now, we define the local edge-connectivity number:

Definition 1.2.10. The *local edge-connectivity number* $\lambda(u, v)$ between two vertices u, v of G is the maximum number of edge-disjoint u, v paths in G .

Menger's theorem once again tells us that:

$$\lambda(G) = \min \{ \lambda_G(u, v) : u, v \in V(G) \}$$

It is also known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, which is known as *Whitney's inequality*. Similarly to vertex connectivity, we have $\lambda(G) \leq \delta(G)$ and $\lambda_G(u, v) \leq \min \{ d(u), d(v) \}$.

Definition 1.2.11. A graph G is said to be *maximally edge-connected* when $\lambda(G) = \delta(G)$ and *maximally local edge connected* when $\lambda(u, v) = \min \{ d(u), d(v) \}$ for all pairs of vertices u, v in G .

1.2.3 Restricted edge-connectivity

The definitions in this section are due to Fabrega and Fiol [18]

Definition 1.2.12. An edge-cut S is said to be k -restricted if every component of $G - S$ has at least k vertices. The k -restricted edge-connectivity number $\lambda_k(G)$ is then defined to be the minimum cardinality over all k -restricted edge-cuts of G .

Definition 1.2.13. A graph is said to be k -restricted edge-connected if $\lambda_k(G)$ exists.

Definition 1.2.14. A k -restricted edge cut (X, \overline{X}) is said to be a *minimum k -restricted edge-cut* if $[X, \overline{X}] = \lambda_k(G)$.

Note that for a minimum k -restricted edge cut (X, \overline{X}) , the graph $G - (X, \overline{X})$ has exactly two connected components. For such a cut, the set X is called a *fragment* of G . Let:

$$r_k(G) = \min \{ |X| : X \text{ is a } k \text{ fragment of } G \}$$

Definition 1.2.15. A k -fragment for which $|X| = r_k(G)$ is said to be a *k -atom* of G .

Definition 1.2.16. The *minimum k -edge-degree* $\xi_k(G)$ is defined to be:

$$\xi_k(G) = \min \{ [X, \overline{X}] : |X| = k \text{ and } G[X] \text{ is connected.} \}$$

A k restricted edge-connected graph G for which $\lambda_k(G) \leq \xi_k(G)$ is said to be *optimally- k -restricted edge-connected* (or λ_k -optimal). A graph for which every minimum k -restricted edge-cut isolates a connected subgraph of order k is said to be *super- λ_k* .

1.2.4 Local restricted edge-connectivity

Definition 1.2.17. A graph is said to be *local k -restricted edge-connected* if for each pair of vertices x, y of G , there exists an edge-cut such S such that each component of G has order at least k , and x and y are in different components of $G - S$.

The size of such a cut is denoted by $\lambda_k(x, y)$, which is referred to as the *local k -restricted edge-connectivity* number of x and y .

The quantity

$$\xi_k(x, y) = \min \{ [X, \overline{X}] : |X| = k, G[X] \text{ is connected, } |\{x, y\} \cap X| = 1 \}$$

denotes the number of edges between a connected subgraph of order k that contains one of x, y , and the remaining graph. Analogous to λ_k -optimality, we define the notion of *local k -restricted edge-connectivity* as follows:

Definition 1.2.18. A graph is said to be *local k -restricted edge-connected* if $\lambda_k(x, y) = \xi_k(x, y)$ for all pairs of vertices x, y in G .

In the coming sections, we will see how these definitions are applied to graphs of various types.

2 Vertex Connectivity

In this section, we present some illustrative examples from the work of Andreas Holtkamp on vertex connectivity - specifically, results on graphs with bounded clique number, p -diamond-free graphs, and $K_{2,p}$ -free graphs. Theorems will be listed, followed by their corresponding examples.

Before we present any theorems, note the following observation due to Holtkamp [1]:

Observation 2.0.1. *Every maximally local connected graph is maximally connected.*

To see this, recall that for a maximally local connected graph G , $\kappa(u, v) = \min \{d(u), d(v)\}$ for all pairs of vertices u, v of G . This implies that:

$$\kappa(G) = \min_{u, v \in V(G)} \{\kappa(u, v)\} = \min_{u, v \in V(G)} \{d(u), d(v)\} = \delta(G)$$

2.1 Results on maximum local connectivity in graphs with bounded clique number

We begin with a few theorems on sufficient conditions for p -partite graphs to be maximally (local) connected, due to Topp and Volkmann[2]:

Theorem 2.1.1. *Let $p \geq 2$ be an integer. If G is a p -partite graph such that*

$$n(G) \leq \delta(G) \left(\frac{2p-1}{2p-3} \right)$$

then $\kappa(G) = \delta(G)$.

Furthermore, as a consequence of our observation above, we can extend this result to the following theorem[5] on maximum local connectivity:

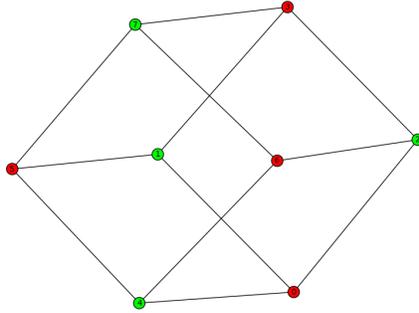
Theorem 2.1.2. *Let $p \geq 2$ be an integer, and let G be a p -partite graph satisfying*

$$n(G) \leq \delta(G) \left(\frac{2p-1}{2p-3} \right)$$

then G is maximally local connected.

It is clear from the picture of the cube graph, below, that $\kappa(G) = \delta(G)$, as 3 vertices must be deleted in order to disconnect the cubic graph. To see that the cubic graph is maximally local connected, note that for any pair u, v of vertices, there are 3 internally disjoint $u - v$ paths in G .

Figure 6: The cube graph. Here we have a 2-partite graph with $\delta = 3$, so the conditions of the inequality are satisfied.



Now, we also have the following theorems [4][6] on graphs with a bounded clique number, as an application of Turan's theorem[3] to the previous theorems:

Theorem 2.1.3. *Let $p \geq 2$ be an integer, and let G be a connected graph with clique number $\omega(G) \leq p$. If $n(G)$ satisfies*

$$n(G) \leq \delta(G) \left(\frac{2p-1}{2p-3} \right)$$

then $\kappa(G) = \delta(G)$.

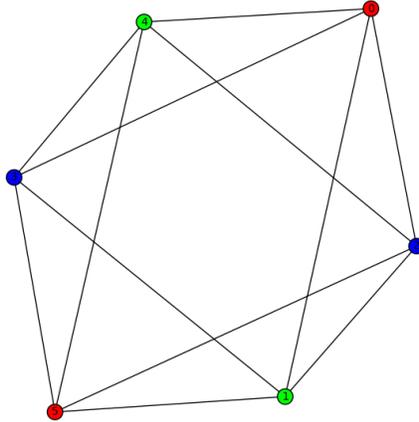
Theorem 2.1.4. *Let $p \geq 2$ be an integer, and let G be a graph with clique number $\omega(G) \leq p$. If $n(G)$ satisfies*

$$n(G) \leq \delta(G) \left(\frac{2p-1}{2p-3} \right)$$

then G is maximally local connected.

An example of this is the octahedral graph. To see that the octahedral graph is in fact maximally (local) connected, note that it has $\delta = 4$. It is easy to see from the picture that at least 4 vertices must be deleted from the graph to disconnect it. For any two vertices u, v , we can find at most four paths (which happen to be of length two) in G that are internally disjoint, so we see that $\kappa(u, v) = \min_{u,v} \{d(u), d(v)\}$.

Figure 7: The octahedral graph. Here we have a graph with $\delta = 4$ and $\omega \leq 3$ so the conditions of the inequality are satisfied.



2.2 Maximum local connectivity in diamond-free and p -diamond-free graphs

Note that bipartite graphs are diamond-free. We begin with the following result due to Volkmann [7]:

Theorem 2.2.1. *Let G be a connected diamond-free graph of order n and minimum degree $\delta \geq 3$. If $n \leq 3\delta$, then $\kappa(G) = \delta(G)$.*

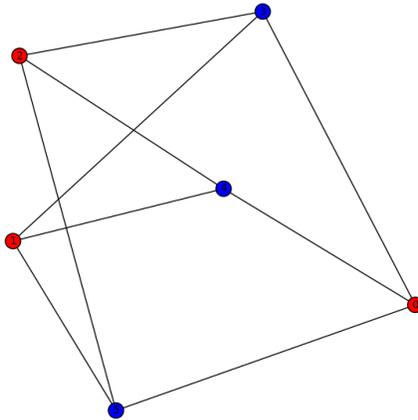
This bound is not tight, and can be improved, as we see in the following theorems due to Holtkamp and Volkmann [8]:

Theorem 2.2.2. *Let G be a connected diamond-free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G) - 1$, then G is maximally local connected.*

Theorem 2.2.3. *Let G be a connected diamond-free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G)$ and $d_G(x) \notin \{\delta(G) + 1, \delta(G) + 2\}$ for each vertex $x \in V(G)$, then G is maximally local connected.*

To see that this graph is maximally connected, note that since it is a complete bipartite graph on 6 vertices, at least 3 vertices must be removed in order to disconnect it. Furthermore, since each vertex has degree 3, we need only observe that for any pair of vertices $\kappa(u, v)$, there are 3 vertex-disjoint paths from one vertex to another, which is clear from the picture.

Figure 8: A 2-diamond free graph with $\delta = 3$, so it satisfies the given inequality.



Now, we have the following results on p -diamond free graphs, due to Holtkamp and Volkmann [8]:

Theorem 2.2.4. *Let $p \geq 2$ be an integer, and let G be a connected p -diamond free graph. In addition, let u, v , be two vertices of G , and let $r = \min \{d_G(u), d_G(v)\} - \delta(G)$. Then,*

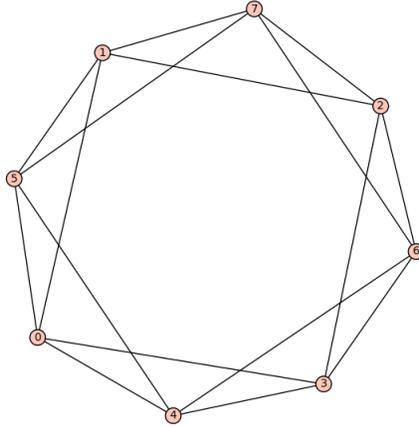
1. *If $uv \notin E(G)$, and $n(G) \leq 3\delta(G) + r - 2p + 2$, then $\kappa_G(u, v) = \delta(G) + r$.*
2. *If $uv \in E(G)$ and $n(G) \leq 3\delta(G) + r - 2p + 1$, then $\kappa_G(u, v) = \delta(G) + r$*

Theorem 2.2.5. *Let $p \geq 3$ be an integer, and let G be a connected, p -diamond free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally local connected.*

Theorem 2.2.6. *Let $p \geq 3$ be an integer, and let G be a connected, p -diamond free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally connected.*

Because of the symmetry of this graph, we can see easily that at least 4 vertices must be deleted from the graph for it to be deleted. Furthermore, if we choose two vertices u, v in G , and note that every pair of vertices is at distance one or two from another, we can see that there are 4 paths that are internally disjoint from u to v , and each vertex has degree 4, so G is maximally local edge-connected.

Figure 9: A 3-diamond free graph that satisfies with $\delta = 4$. so satisfies the given inequality.



2.3 Maximum local connectivity in $K_{2,p}$ -free graphs

Note that a $K_{2,p}$ -free graph is also p -diamond free, so the following two results are easy corollaries of the previous result on p -diamond free graphs, due to Holtkamp and Volkmann [9]:

Corollary 2.3.1. *Let $p \geq 3$, and let G be a connected $K_{2,p}$ -free graph. If $n(G) \leq 3\delta(G) - 2p + 2$ then G is maximally locally connected.*

Corollary 2.3.2. *Let $p \geq 3$, and let G be a connected $K_{2,p}$ -free graph. If $n(G) \leq 3\delta(G) - 2p + 2$ then G is maximally connected.*

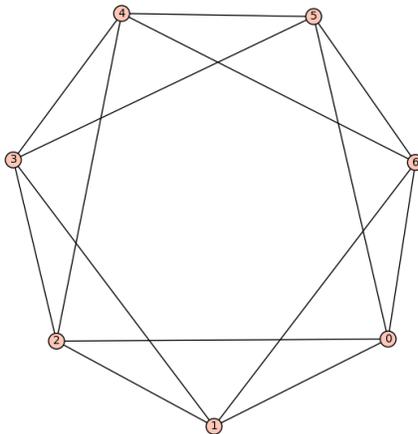
Now, we move on to some results on $K_{2,p}$ -free graphs in the special case of $p = 4$. The results are due to Holtkamp and Volkmann [8]

Theorem 2.3.1. *Let G be a connected $K_{2,4}$ -free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G) - 5$, then G is maximally local connected.*

Corollary 2.3.3. *Let G be a connected $K_{2,4}$ -free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G) - 5$, then G is maximally connected.*

It's easy to see that the graph satisfies the given inequality and that the deletion of at least 4 vertices is required to disconnect the graphs. It can be seen that G is maximally local connected by similar reasoning to the example of the p -diamond free graph.

Figure 10: A $K_{2,4}$ -free graph on 7 vertices with minimum degree $\delta = 4$.



3 Edge Connectivity

3.1 Maximum (local) edge connectivity in diamond-free graphs

Similar to the case of local vertex connectivity, we may make the following observation due to Holtkamp [1]:

Observation 3.1.1. *Every maximally local edge-connected graph is maximally edge-connected.*

To see this, note that for a maximally local edge-connected graph, we have $\lambda(u, v) = \min_{u, v \in V(G)} \{d(u), d(v)\}$. Thus, we have:

$$\lambda(G) = \min_{u, v \in V(G)} \lambda(u, v) = \min_{u, v \in V(G)} \{d(u), d(v)\} = \delta(G)$$

We have several results on local edge-connectivity in diamond-free graphs: A theorem due to Holtkamp[10], and several corollaries due to Holtkamp[10], Volkmann [11], and and Fricke, Oellerman, and Swart [12], respectively:

Theorem 3.1.1. *Let G be a diamond-free graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally local edge-connected.*

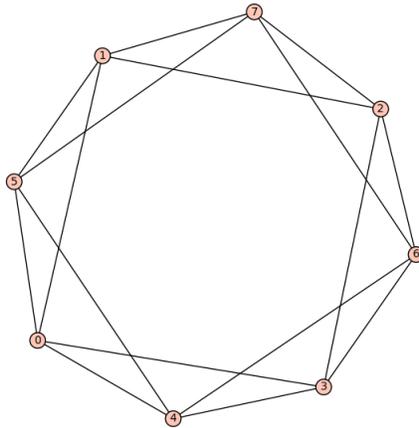
Corollary 3.1.1. *Let G be a diamond-free graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally edge-connected.*

Corollary 3.1.2. *Let G be a bipartite graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally local edge-connected.*

Corollary 3.1.3. *Let G be a bipartite graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally edge-connected.*

For the purposes of comparison, let us consider again the 2-diamond free graph we already used:

Figure 11: A 3-diamond free graph that also has $\delta = 4$, so satisfies the given inequality.



We can easily see from the picture that the deletion of at least 4 vertices is necessary in order to disconnect the graph. By a similar argument to the vertex-connectivity case, we can see that for any two vertices u, v , there are at least 4 edge-disjoint paths between any two vertices u, v .

3.2 Restricted edge-connectivity

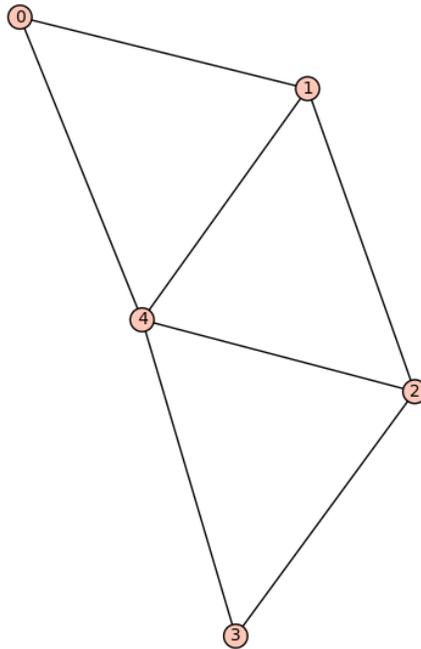
We have the following results categorizing 2-restricted edge-connected graphs, due to Esfahanian and Hakimi [19]:

Theorem 3.2.1. *Every connected graph of order $n \geq 4$ except a star $K_{1,n-1}$ is 2-restricted edge-connected and satisfies $\lambda(G) \leq \lambda_2(G) \leq \xi(G)$*

We also have the following result due to Yuan and Liu [13], that gives a sufficient condition for a graph to be λ_2 -optimal:

Theorem 3.2.2. *Let G be a connected triangle-free graph of order $n \geq 4$. If $d(u) + d(v) \geq 2\lfloor \frac{n+2}{4} \rfloor + 1$ for each pair of vertices u, v at distance 2, then G is λ_2 -optimal.*

Figure 12: A λ_2 -optimal graph.



3.3 λ_3 -optimality in triangle-free graphs

Bonsma, Ueffing, and Volkmann [15] discovered the following characterization of graphs that are not 3-restricted edge-connected:

Theorem 3.3.1. *A connected graph G is 3-restricted edge-connected iff $n \geq 6$ and it is not isomorphic to the net N or any graph in the family F depicted in the Figure below:*

Figure 13: The net N .

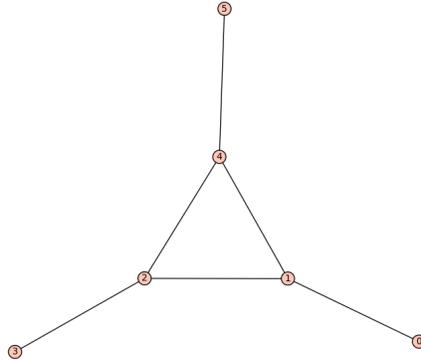
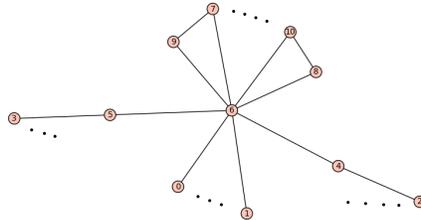


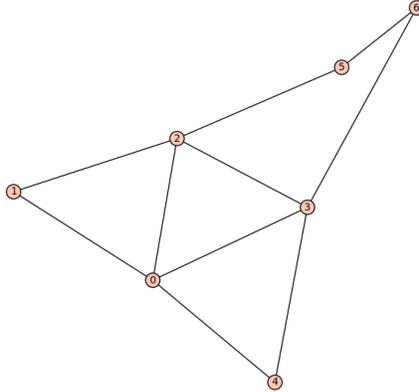
Figure 14: The family F that is not local 3-restricted edge-connected. Note that the black dots are connecting the vertices v_2 and v_3 .



Holtkamp, Meierling, and Montejao [14] found the following result on triangle-free graphs with sufficiently high degree:

Theorem 3.3.2. *Let G be a connected triangle-free graph of order $n \geq 6$. If $d(u) + d(v) \geq 2\lfloor \frac{n}{4} \rfloor + 3$ for each pair u, v of non-adjacent vertices, then G is λ_3 -optimal.*

Figure 15: A 3-restricted edge-connected graph that is not triangle-free.



3.4 Local k -restricted edge-connectivity

In this section, I will present a few results on local k -restricted edge-connectivity. The results are all due to Holtkamp and Meierling [20]

We begin with a series of observations due to Holtkamp [1]:

Observation 3.4.1. *Every local k -restricted edge-connected graph is k -restricted edge connected.*

Observation 3.4.2. *Every local $k + 1$ -restricted edge-connected graph is k -restricted edge connected, and satisfies $\lambda_k(G) \leq \lambda_{k+1}(G)$.*

Observation 3.4.3. *Every local λ_k -optimal graph with $\lambda_k(G) \leq \zeta_k(G)$ is λ_k -optimal.*

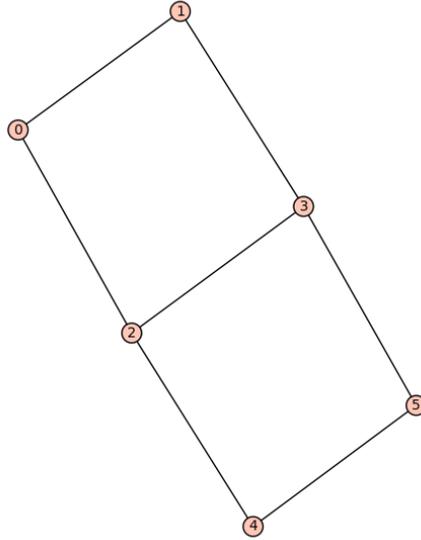
One of the main results used for showing the local k -restricted edge-connectivity of a graph is the following:

Lemma 3.4.1. *A connected graph G is local k -restricted edge-connected iff for every pair of vertices x, y of G , there exist disjoint sets $\{x, x_1, \dots, x_{k-1}\}$ and $\{y, y_1, \dots, y_{k-1}\}$ such that the induced subgraphs $G[X]$ and $G[Y]$ are connected.*

Theorem 3.4.1. *Let G be a connected graph of order at least $2k$. If G has a cut-vertex that isolates a component of order at most k , then G is not local k -restricted edge connected.*

See figure 17 for an example of this graph.

Figure 16: A local 2-restricted edge-connected graph that satisfies the conditions of the above Lemma.



We also have the following characterization of local k -restricted edge-connected graphs:

Theorem 3.4.2. *Let G be a connected graph of order $n \geq 2k$ and let x and y be two vertices of G .*

If $\kappa(x, y) \geq 2$ and no cut vertex w that leaves x and y in a common component of order at most $2k - 2$ (if $w \neq x, y$) or leaves x and y in a component of order at most $k - 1$ (if $w = x$ or $w = y$), respectively, there exists a k -restricted edge-cut separating x and y .

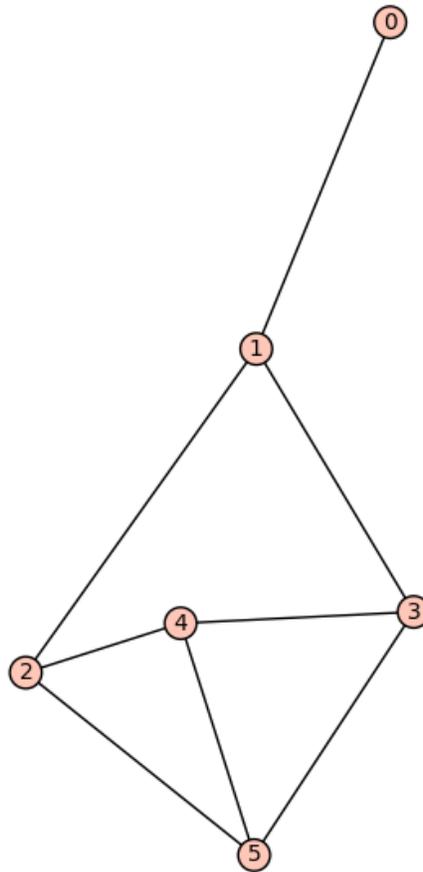
If $\kappa(x, y) = 1$ and there exists a cut-vertex that leaves one of x and y in a component of order s with $k \leq s \leq n - k$, then there exists a k -restricted edge-cut separating x and y .

Corollary 3.4.1. *Let G be a connected graph of order at least $2k$. If G has no cut-vertex that isolates a component of order at most $2k - 2$, then G is local k -restricted edge-connected.*

The following corollary is useful in determining whether a graph of sufficiently large order is local 2-restricted edge-connected:

Corollary 3.4.2. *A connected graph of order at least 4 is not local 2-restricted edge connected iff it contains a vertex of degree 1 or it contains two adjacent vertices of degree 2 that have a common neighbor.*

Figure 17: An example of a graph that is not local 2-restricted edge-connected. v_1 is the cut-vertex described in the theorem, as it isolates a component of order $1 \leq 2$ when removed.



Note that all of the above examples that were local 2-restricted edge-connected necessarily satisfy these criteria.

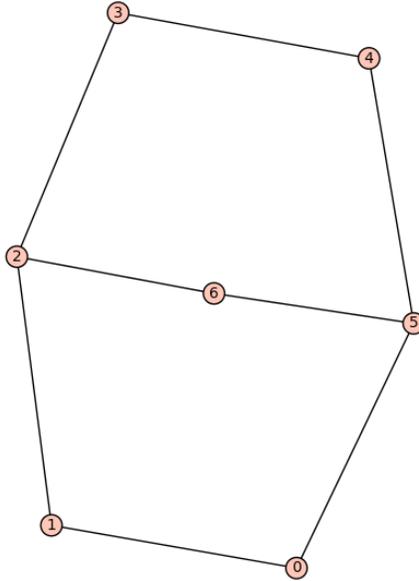
Corollary 3.4.3. *Every graph of minimum degree at least 3 is local 2-restricted edge-connected.*

This prevents the existence of any prohibited structures described in the previous corollary in the graph.

Now, we have a similar result on the types of graphs which cannot be local 3-restricted edge-connected:

Theorem 3.4.3. *A connected graph of order at least 6 is not local 3-restricted*

Figure 18: An example of a 2-connected graph. Based on a previous result, it is obviously local 2-restricted edge-connected. However, it's easy to see that it meets the criteria of the above theorem, as there is no one cut-vertex that isolates a component of order at most 2.



edge-connected iff it satisfies at least one of the following four conditions:

It contains a cut-vertex v that isolates a component of order at most 3

It contains a cut-vertex v that isolates a component of order 4 such that at least two of its vertices are not adjacent to v .

It contains a cut-vertex v that isolates a paw such that one of its vertices of degree 2 is not adjacent to v .

It contains a cut-vertex that isolates a path of order 4.

A simple corollary of this is:

Corollary 3.4.4. *Every graph with at least 6 vertices and minimum degree at least 4 is local 3-restricted edge-connected.*

This again prevents any of the above prohibited structures from being present in the graph.

For $k \geq 4$, graphs that are locally k -restricted edge-connected have not been entirely characterized. However, we have the following result:

Theorem 3.4.4. *Every connected graph G of order at least $2k$ and minimum degree $k + 1$ is local k -restricted edge-connected.*

Figure 19: Forbidden structures in local 2-restricted edge-connected graphs. The vertex colored differently from the rest of the graph represents the remainder of an arbitrary graph that the structure is connected to.

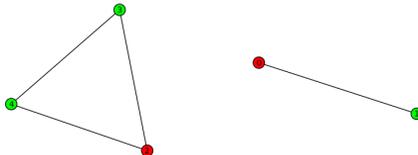
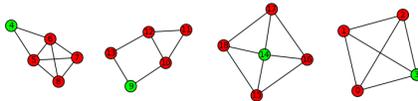


Figure 20: Forbidden structures in local 3-restricted edge-connected graphs. The vertex colored differently from the rest of the graph represents the remainder of an arbitrary graph that the structure is connected to.



4 Percolation

4.1 Introduction

Percolation theory is an area of physics that has a great deal to do with graph theory.

We are concerned with the following problem: Suppose we have a very large *lattice* L , with a large number of *sites*. Each site may be occupied with probability p , or unoccupied (with probability $1 - p$). What can we say about the *clusters* in the lattice, or, groups of neighboring occupied sites?

This is one type of percolation, known as *site percolation*. There is another type of percolation, known as *bond percolation*, in which we imagine every site to be occupied, and consider bonds between sites. Each bond may be *open*, with probability p , or *closed*, with probability $1 - p$. We can think of this like a graph with the vertices being the sites and open bonds being the edges connecting the vertices. We are then interested in describing clusters, which are in this case defined to be collections of sites connected by bonds, or, in graph-theoretic terms, connected components of the lattice.

We will be concerned only with site percolation in this work. We will discuss the two best-known and simplest exactly solved models in percolation theory - the Bethe lattice and the one-dimensional lattice. The key features of the theory appear in these simplified models and generalize to more complex models that are not exactly solvable (and typically require computer simulation). Our discussion will follow that of Stauffer and Aharony [21].

Figure 21: A visualization of site percolation, from [22]

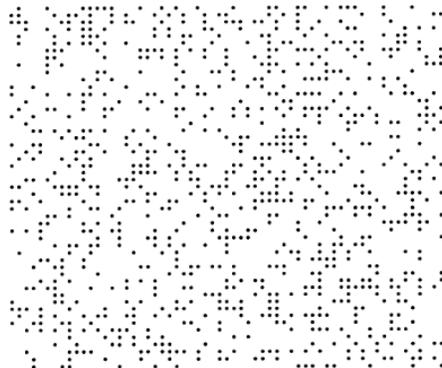
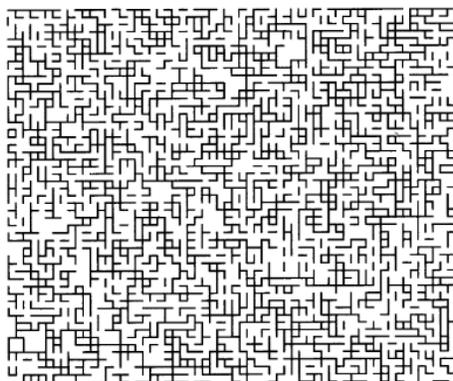


Figure 22: A visualization of bond percolation, from [22]



For each model, there are a few quantities that we are interested in determining:

The *percolation threshold* p_c is the value of the site occupation probability p for which an infinite cluster appears in an infinite lattice,

The *cluster number* n_s , or the number of s -clusters per lattice site,

The *average cluster size* S .

The *correlation function* $g(r)$, which gives the probability that a site at a distance r from an occupied site belongs to the same cluster as that occupied site.

The *correlation length* ξ .

The *network strength* P .

4.2 Percolation in the linear lattice

The percolation problem can be solved exactly in $d = 1$ dimensions. Although simple, many important features of the one-dimensional percolation model generalize to higher dimensions.

Consider an infinitely long linear "lattice" in one dimension, with several sites placed at fixed distances from one another. Each site may be occupied with probability p .

In order for two clusters to be separated from one another, the sites adjacent to the far-left and far-right ends of the cluster must be empty. Suppose we want to find the number of clusters of size s on a lattice of length L , where we will eventually take $L \rightarrow \infty$. To find out, consider solving the problem of determining whether a given occupied site is the left end of an s -cluster. The probability of s arbitrary sites on the lattice being occupied is p^s . There are two unoccupied sites neighboring each end of the cluster, which are empty with probability $(1 - p)^2$. Thus the probability of a given occupied site being the left end of an s -cluster is $p^s(1 - p)^2$. Now, on a linear lattice of length L , there are L sites, so that the number of s -clusters on a lattice of length L is then $L(1 - p)^2 p^s$. As L goes to infinity, it is preferable to talk about the number of *s-clusters per lattice site* obtained by dividing the number of clusters of size s by the length L . Clearly the number of s -clusters per lattice site is $p^s(1 - p)^2$.

Then, the probability of an arbitrary given site being part of an s -cluster (not necessarily the left end, as in the derivation above) is just sn_s .

In order to determine the percolation threshold, consider what happens when $p = 1$. All sites are occupied, and the entire chain forms one cluster. For $p < 1$, there will exist unoccupied sites, and for a chain of length L , there will be on average $(1 - p)L$ such sites. For fixed p , this quantity goes to infinity as L goes to infinity, so that there will always be at least one unoccupied site on the lattice, so that there will never be a cluster connecting both ends of the chain. Thus we see that there is no percolating cluster for $p < 1$, so that $p_c = 1$, so that the region $p > p_c$ is not observable on this lattice.

Now, suppose we choose a random point on the lattice that we know is part of finite a cluster. We want to determine the average size of such a cluster. We know that the probability of a given occupied site being part of an s -cluster is sn_s , and the probability of a random occupied site being part of a cluster of any size is $\sum_s sn_s$. Then, $w_s = sn_s / \sum_s sn_s$ is the probability that a cluster to which an arbitrary site belongs contains exactly s sites. The *average cluster*

size S is then given by:

$$S = \sum_s s w_s$$

This is equivalent to:

$$\sum_s \frac{n_s s^2}{\sum_s s n_s}$$

To evaluate this sum, recall that the probability that an occupied site belongs to a cluster of size s is $s n_s$, as mentioned above. Then, the probability of a site belonging to a cluster of any size is simply the probability that it is occupied, so that $p = \sum_s s n_s$. To calculate the numerator, we write:

$$(1-p)^2 \sum_s s^2 p^s = (1-p)^2 \left(p \frac{d}{dp} \right)^2 \sum_s p^s$$

from which we see that

$$S = \frac{1+p}{1-p}$$

Recall that the *correlation function* $g(r)$ gives the probability that a site at a distance r from an occupied site belongs to the same cluster as that occupied site. In general, for a pair of sites at distance r to be members of the same cluster, all r sites in between them must be occupied, from which we see $g(r) = p^r$ for all p and r . For $p < 1$ the correlation function goes to zero as r goes to infinity, indeed we can write:

$$g(r) = \exp\left(\frac{-r}{\xi}\right)$$

where ξ is the correlation length, and is defined as follows:

$$\xi = -\frac{1}{\ln(p)} = \frac{1}{p_c - p}$$

The last equality only holds for p close to $p_c = 1$.

We see that since the length of a cluster with s sites is $s - 1$, the average cluster size S is roughly proportional to the correlation length ξ .

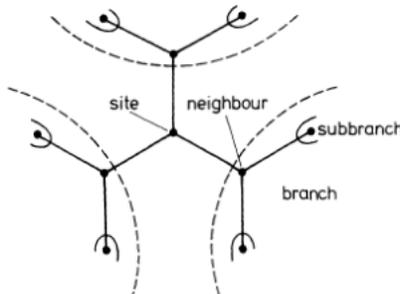
In general, though certain quantities diverge at the percolation threshold, their divergence behavior can be described by *power laws*, which remains generally true even in higher dimensions where models are not exactly solvable.

4.3 Percolation in the Bethe Lattice

The other simple exactly solvable percolation model is the Bethe Lattice.

In the Bethe lattice, one begins at a central site from which z bonds emanate. Each bond connects to z other sites which also have z other bonds emanating: one connecting to the previous site, and $z - 1$ others connecting to other sites.

Figure 23: A visualization of the Bethe lattice, with $z = 3$.



To calculate the percolation threshold p_c of the Bethe lattice, we begin at the origin and attempt to find an infinite path of occupied sites starting from that origin. Traversing such a path, we see that at each new site there are $z - 1$ bonds emanating in directions other than the one we came. Each one leads to a new neighbor, which is occupied with probability p , from which it follows that there are on average $(z - 1)p$ new neighbors to choose from to continue the path. If $(z - 1)p < 1$, the average number of paths leading to infinity decreases by this factor at each new site visited along the path. Even if z is very large and all the sites adjacent to the occupied origin are also occupied, the probability of finding a continuous path of occupied neighbors goes to zero exponentially with path length provided that $p \leq \frac{1}{z - 1}$. As a result, we obtain:

$$p_c = \frac{1}{z - 1}$$

For p larger than the percolation threshold, note that there is still not always a path from the origin going to infinity. For instance, all the neighbors of the origin could be empty.

Definition 4.3.1. The *percolation probability* P is the probability that an arbitrarily selected site belongs to an infinite cluster.

This quantity is clearly zero for $p < p_c$, so we are interested only in the region $p > p_c$. P is sometimes referred to as the *strength* of the network, and p as the concentration. We want to determine the value of P .

Now, let Q be the probability that an arbitrary site is not connected to infinity through one fixed branch originating at that site. From now on, we will take $z = 3$ for simplicity. For a given neighbor of our starting site, the probability that the two sub-branches beginning at that neighbor do not go to infinity is Q^2 . Thus pQ^2 is the probability that a given neighbor is occupied but none of its sub-branches lead to infinity. The probability that this neighbor is not occupied is $1 - p$, so that $Q = (1 - p) + pQ^2$. This quadratic has solutions $Q = 1$ and $Q = \frac{1 - p}{p}$. Now, the probability $P - p$ that the origin is occupied but is connected to infinity by none of its three branches is pQ^3 , so that $P = p(1 - Q^3)$. This gives zero for $Q = 1$ and

$$\frac{P}{p} = 1 - \left(\frac{1 - p}{p}\right)^3$$

The first solution $Q = 1$ corresponds to $p > p_c$ and the second $Q = \frac{1 - p}{p}$ corresponds to $p < p_c$.

We may also calculate the mean cluster size S , which in the case of the Bethe lattice is the average number of sites of the cluster to which the origin belongs. Let T be the average cluster size for one branch, that is, the average number of sites to which the origin is connected and belongs to one branch. Sub-branches T have the same cluster size as the origin itself. If a neighbor to the origin is empty (probability $1 - p$), the cluster size for this branch is zero. If not, the sub branch contributes its own mass and the mass T of its two sub-branches emanating, so that we obtain:

$$T = (1 - p)0 + p(1 + 2T)$$

Which has solution $T = \frac{p}{1 - 2p}$ for $p < p_c = 1/2$. The total cluster size is zero if the origin is empty, and $1 + 3T$ if occupied. Therefore, the mean size S is given by:

$$S = 1 + 3T = \frac{1 + p}{1 - 2p}$$

This is the mean cluster size below the percolation threshold p_c . Near $p_c = 1/2 = p$, we see that S is proportional to $\frac{1}{1 - p_c}$. If there is no infinite network below p_c , it is possible that there does exist a very weak one, i.e. P is very small, above p_c . For instance, at $p = 1/2$, we see that $P = 0$, and when p approaches p_c from above, P is proportional to $p - p_c$.

Both of these are further examples of *critical phenomena*, where quantities go to zero or infinity following power laws. This is one way in which the percolation problem is similar to a phase transition.

Now, let us calculate n_s , the average number per site of clusters containing s sites. The size s is, similar to the one-dimensional case, related to the perimeter t , which is the number of empty neighbors of occupied cluster sites. An isolated site has 3 empty neighbors, and an isolated pair of occupied sites have 4 empty neighbors. In the case of a general z , an isolated site has z unoccupied neighbors, and an isolated pair has $2z-2$ unoccupied neighbors. Each additional site added to this pair gives a further $z-2$ unoccupied neighbors, so that $t = (z-2)s + 2$. For large s , the perimeter t is proportional to s , and the ratio t/s is:

$$\frac{t}{s} = \frac{1-p_c}{p_c}$$

As $t/s = z-2$ and we know that $p_c = \frac{1}{z-1}$.

From the our equation derived earlier for the cluster number, we see that:

$$g_s p^s (1-p)^{2+(z-2)s}$$

Let us again set $z=3$. Instead of calculating the cluster number n_s , instead consider the ration $\frac{n_s(p)}{n_s(p_c)}$. Substituting, we see that this is equal to:

$$\left[\frac{(1-p)}{(1-p_c)} \right]^2 [1 - a(p-p_c)^2]$$

This quantity is proportional to $\exp(-cs)$, where $a=4$

$$c = -\ln(1 - a(p-p_c)^2)$$

We see that c is proportional to $(p-p_c)^2$.

Now, we want to find the behavior of the cluster number near the critical point p_c . Recall from our calculation in one dimension that S is proportional to $\sum_s s^2 n_s$ because the denominator remains finite near p_c . Let us assume that the decay of $n_s(p_c)$ is proportional to $s^{-\tau}$. To calculate the value of this sum for the Bethe lattice, we assume that p is only slightly smaller than p_c , and use the trick of converting a sum to an integral to see that S is proportional to the integral:

$$\int s^{2-\tau} \exp(-cs) ds$$

Which we see is proportional to $c^{\tau-3}$ after changing variables. Then, from the definition of c , we see that the quantity S is in fact proportional to $(p-p_c)^{2\tau-6}$. Since we have already shown that S is proportional to $\frac{1}{p-p_c}$, it follows that $2\tau-6 = -1$ from which we see that $\tau = 5/2$. This exponent is sometimes referred to as the *Fisher exponent*. Thus, we see that n_s is proportional to $s^{-\frac{5}{2}} \exp(-cs)$ and c is proportional to $(p-p_c)^2$, where the first proportionality holds for p and large s , and the second only for p near the threshold p_c .

4.4 Application - Magnetism

In our treatment of the one-dimensional lattice, we wrote that there was somewhat of an analogy between the percolation problem and phase transitions. We will explore this in more detail in this section. In particular, we will show how identifying some quantities from theories of ferromagnetism such as the spontaneous magnetization, susceptibility, etc. with quantities from percolation theory like the percolation threshold, mean cluster size, and cluster strength, can generate a new percolation model equivalent to a statistical-mechanical one.

Recall that in statistical physics, the probability of a system being in an energy state E_i is given by p_i , where:

$$\frac{\exp(E_i)}{\sum_j \exp\left(\frac{E_j}{kT}\right)}$$

and the expected value of a quantity A that takes the value A_i in energy state i is then $\sum_i A_i p_i$.

One well known model of magnetism from statistical physics is the *Ising Model*. In this model, we have a collection of magnetic dipoles with energy levels $E_1 = -H$, $E_2 = H$. The magnetic dipole is assumed to be able to point only up or down, i.e. parallel or antiparallel to the magnetic field H , and this dipole associated with an atom will be referred to as the *spin*.

In ferromagnetic materials, neighboring spins have an "exchange" interaction with energy $-J$ if the spins are parallel, and J if the spins are antiparallel. Thus, the total energy for the Ising model is:

$$E = -J \sum_{ik} S_i S_k - H \sum_i S_i$$

Where the S_i are spins that may take the values ± 1 .

In a ferromagnet, recall that the *Curie point* is the temperature below which spontaneous magnetization m_0 occurs in zero external field, and the susceptibility χ is the zero-field derivative $\frac{dm}{dH}$. Near the critical point, we know that:

$$\begin{aligned} C_v &\propto |T - T_c|^{-\alpha} \\ m_0 &\propto (T - T_c)^{-\beta} \\ \chi &\propto |T - T_c|^{-\gamma} \\ \xi &\propto |T - T_c|^{-\nu} \end{aligned}$$

Here the correlation length ξ is the range over which one spin nontrivially affects the orientation of other spins. In the two-dimensional theory, it is known

that the values of the critical exponents are $\alpha = 0, \beta = 1/8, \gamma = 7/4, \nu = 1$.

To form an analogy with percolation models, identify the spontaneous magnetization with cluster strength, susceptibility with mean cluster size, and temperatures $T > T_c$ with concentrations $p < p_c$.

Now, imagine that only some fraction p of lattice sites are occupied by spins, and the remaining fraction $1 - p$ remains unoccupied. As in percolation, the spins are distributed randomly. This is known as the *site-diluted quenched Ising model*.

Suppose that we are at very low temperatures, so that $H \propto T$ and $kT \ll J$. Spins within a single percolation cluster will be parallel to each other in equilibrium. The probability to flip spins in a cluster involves powers of $\exp\left(\frac{-2J}{kT}\right)$, since an energy of $2J$ is required to break a bond between two spins. In addition, different clusters have different spins and do not influence one another, so that each finite cluster with s can be thought of as a 'super-spin' with total energy $\pm sH$. Therefore, the probability for a spin to point parallel to H is:

$$\frac{\exp\left(\frac{-sH}{kT}\right)}{\exp\left(\frac{-sH}{kT}\right) + \exp\left(\frac{sH}{kT}\right)}$$

and the probability for a spin to point in the opposite direction is:

$$\frac{\exp\left(\frac{sH}{kT}\right)}{\exp\left(\frac{-sH}{kT}\right) + \exp\left(\frac{sH}{kT}\right)}$$

The difference between these two probabilities multiplied by the cluster size s is then the magnetization per cluster:

$$m_{cluster} = s \tanh\left(\frac{sH}{kT}\right)$$

Then, if an infinite cluster is present, its contribution to the total magnetization is $\pm P$, depending on the orientation of the cluster. The total magnetization per lattice site is then:

$$m = \pm P + \sum_s s n_s \tanh\left(\frac{sH}{kT}\right)$$

As H goes to zero, only the infinite cluster remains, so that $m_0 = \pm P$. At small values of H , a Taylor expansion of \tanh tells us that the zero-field

derivative $\frac{dm}{dH}$, which gives the susceptibility, is proportional to the mean cluster size S :

$$\chi = \sum_s \frac{s^2 n_s}{kT} \propto S$$

Now, if at low temperatures, we want the spin concentration p to approach the percolation threshold p_c , we must have $m_0 \propto (p - p_c)^\beta$, and $\chi \propto |p - p_c|^\gamma$. Note that these are *not* the undiluted Ising exponents, but percolation exponents like were discussed in earlier sections. Thus we have a correspondence between a percolation model and an Ising model, where the spontaneous magnetization is exactly the infinite cluster size, the susceptibility is exactly the mean cluster size, and the percolation threshold is exactly the transition for ferromagnetism.

References

- [1] A. Holtkamp, *Connectivity in Graphs and Digraphs*, PhD Thesis, RWTH Aachen University, 2013.
- [2] J. Topp and L. Volkmann, *Sufficient conditions for equality of connectivity and minimum degree of a graph*, J. Graph Theory **17** (1993), 695700.
- [3] P. Turán, *An extremal problem in graph theory*, Mat. Fiz. Lapok **48** (1941), 436452.
- [4] A. Hellwig and L. Volkmann, *On connectivity in graphs with given clique number*, J. Graph Theory **52** (2006), 714.
- [5] L. Volkmann, *Restricted arc-connectivity of digraphs*, Inform. Process. Lett. **103** (2007), 234239
- [6] A. Holtkamp and L. Volkmann, *On local connectivity of graphs with given clique number*, J. Graph Theory **63** (2010), 192197
- [7] P. Dankelmann, A. Hellwig, and L. Volkmann, *On the connectivity of diamond-free graphs*, Discrete Appl. Math. **155** (2007), 21112117.
- [8] A. Holtkamp and L. Volkmann, *On the connectivity of p -diamond-free graphs*, Discrete Math. **309** (2009), 60656069.
- [9] A. Holtkamp and L. Volkmann, *On local connectivity of $K_{2,p}$ -free graphs*, Australas. J. Combin. **51** (2011), 153158
- [10] A. Holtkamp, *On local edge-connectivity of diamond-free graphs*, Australas. J. Combin. **49** (2011), 153158.
- [11] L. Volkmann, *Bemerkungen zum p -fachen Zusammenhang von Graphen*, An. Univ. Bucuresti Mat. **37** (1988), no. 1, 7579.

- [12] G. Fricke, O.R. Oellermann, and H.C. Swart, *The edge-connectivity, average edge-connectivity and degree conditions*, unpublished manuscript (2000).
- [13] J. Yuan and A. Liu, *Sufficient conditions for λ_k -optimality in triangle-free graphs*, Discrete Math. **310** (2010), 981987.
- [14] A. Holtkamp, D. Meierling, and L.P. Montejano, *k -restricted edge- connectivity in triangle-free graphs*, Discrete Appl. Math. **160** (2012), 1345 1355.
- [15] P. Bonsma, N. Ueffing, and L. Volkmann, *Edge-cuts leaving components of order at least three*, Discrete Math. **256** (2002), 431439.
- [16] J. Yuan, A. Liu, and S. Wang, *Sufficient conditions for bipartite graphs to be super- k -restricted edge-connected*, Discrete Math. **309** (2009), 28862896
- [17] K. Menger, *Zur allgemeinen Kurventheorie*, Fund. Math. **10** (1927), 96115.
- [18] J. Fàbrega and M.A. Fiol, *On the extraconnectivity of graphs* , Discrete Math. **155** (1996), 4957.
- [19] A.H. Esfahanian and S.L. Hakimi, *On computing a conditional edge- connectivity of a graph*, Inform. Process. Lett. **27** (1988), 195199.
- [20] A. Holtkamp and D. Meierling, *Local restricted edge-connectivity*, submitted for publication.
- [21] Stauffer, D. and Aharony, A. (2003). *Introduction to Percolation Theory*. Great Britain: Taylor and Francis.
- [22] Grimmett, G.R. (1999). *Percolation*. Germany: Springer.