On the Constructions of Certain Fractal Mixtures

by

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Abstract

The purpose of this paper is to construct sets, measures and energy forms of certain mixed nested fractals which are spatially homogeneous but not strictly self-similar. We start with the constructions of regular nested fractals, such as Sierpiński gaskets in $\mathbb{R}^n$ and Koch curves in $\mathbb{R}^2$, by employing the iterated map system. Then we show that under the open set condition, the unique invariant (self-similar) measure consists with the normalized Hausdorff measure ristricted on the invariant set. The energy forms construced on regular Sierpinski gaskets and Koch curves is also proved to be a closed form. Next, we use the similar idea, by extending the iterated maps system into a general case, to construct the mixture sets, as well as measures and energy forms. It can be seen that the elements so constructed will not have any strict self-similarity, but them indeed satisfy some weak self-similar properties.

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Chapter 1

Introduction

A fractal is by definition a set for which the Hausdorff dimension strictly exceeds the topological dimension, i.e., a set with non-integral Hausdorff dimension, given by Benoit Mandelbrot in his book [17]. Such sets, when they have the additional property of being strictly self-similar, have been used to to model various physical phenomena. Meanwhile, in [16], Lindstrom was able to describe a family of fractals, called by him *nested fractals*, to be a good mathematical model for what physicists call *finitely ramified fractals*, which are self-similar bodies that can be disconnected by a finite number of cuts. For example, the Sierpiński gasket and the Koch curve are two particular nested fractals that will be mainly dicussed in this paper. For very regular self-simillar fractals, it is possible to construct the unique invariant set K and invariant Hausdorff measure μ on K based on the contraction principle in complete metric spaces. Those notions have been studied in a general framework by Hutchinson [11]. Moreover, the Dirichlet form for the regular Sierpiński gasket has been introduced in Fukushima-Shima [9] as a basis to formulate the spectral analysis for the gasket.

However, in the mathematical physics literature, the main interest is not in regular fractals, but in irregular objects which are believed to exhibit “fractal” properties. We call this kind of structures by “irregular fratals” or “fractal mixtures”. Sets of this type, and their diffusions, have been studied recently by Barlow-Hambly [1]. The main focus of this thesis will be on constructing the sets, measures and energy forms that are not strictly self-similar. Results obtained in this paper are used to prepare for the future study and research. We will not consider the case of non nested fractals, such as the Sierpiński carpet, because it asks for employing quite different techniques. The paper is organized as follows: In the next chapter, I will begin by recalling the contraction principle in a metric space. After introducing contractive maps and the completeness of Hausdorff metric space of compact sets, the proof of the existence and uniqueness of invariant sets is given, based on which certain fractal sets will be constructed in following chapters. I use Chapter 3 to describe the properties of contractive similitudes in Euclidean space, as well as those of invariant sets. In addition, it is necessary to talk about the Hausdorff dimension of such invariant sets under given contractive similitudes satisfying the open set condition. In order to help with understanding, basic concepts of Hausdorff measure are also given. Chapter 4 is devoted to developing theories of invariant measures, which are proved by the contraction principle. Some properties of such measure will be shown. In particular, the invariant measure consists with the Hausdorff measure under the open set condition. Examples in fractals, such as the Koch curve, Sierpiński gasket and carpet, are shown including pictures in Chapter 5. Then, in Chapter 6, the reader is first introduced to the iterated map system. Energy forms on certain regular fractals are constructed later. Furthermore, we can show that such energy form is bilinear, closed, and also satisfies the Markov property. That is to say this energy form is a Dirichlet form. It doesn’t enter the scene of any fractal mixtures until Chapter 7. I extend the iterated map system to a general case which depends on a given positive integer sequence. Once the new system has been explained, we will use the similar idea that was developed in the previous chapters to construct the sets, measures and energy forms on irregular Sierpiński gaskets. Finally, we will list some future works in the last chapter.

Complete proofs of the main results will be presented. For some of the more difficult results, only the easiest non-trivial case of the proof (such as the case of two dimensions) is included here, with a reference to the complete proof in a more advanced text.

Chapter 2

Contractions

2.1 Contraction Principle

Let (X, d) be a complete metric space. We say $\lim_{n\to\infty} x_n = x$ for $x, x_n \in X$, if $d(x_n, x) \to 0$ in $\mathbb{R}$ as $n \to \infty$.

A map $f : X \to X$ is said to be a *contraction*, if there exists $0 < r < 1$ such that

$$d(f(x), f(y)) \leq r d(x, y)$$

for every $x, y \in X$. The smallest one of such constant r is given by

$$r = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)},$$

and is called the *Lipschitz constant* of f, denoted by $Lip(f)$.

Notice a contraction map is continuous. For notational purposes we define $f^n(x)$, $x \in X$ for $n \geq 0$ inductively by $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$.

One important result is known as Banach's contraction principle followed.

Theorem 2.1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a con-*

traction. Then f has a unique fixed point $p \in X$ such that $f(p) = p$. Furthermore, for any $x \in X$ we have

$$\lim_{n\to\infty} f^n(x) = p$$

with

$$d(f^n(x), p) \leq \frac{r^n}{1-r} d(x, f(x)).$$

Proof. We first show uniqueness. Suppose there exist $x, y \in X$ with $f(x) = x$, $f(y) = y$. Then

$$rd(x,y) \leq d(x,y) = d(f(x), f(y)) \leq rd(x,y).$$

Therefore $d(x,y) = 0$, which implies $x = y$.

To show existence, we first show that $\{f^n(x)\}$ is a Cauchy sequence. Since

$$d(f^n(x), f^{n+1}(x)) \leq rd(f^{n-1}(x), f^n(x)) \leq \cdots \leq r^n d(x, f(x)),$$

thus for every $\epsilon > 0$, we can find an $N \in \mathbb{R}$ large enough such that for all $m > n > N$, we have

$$\begin{aligned} d(f^n(x), f^m(x)) \leq & d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^{n+2}(x)) \\ & + \cdots + d(f^{m-1}(x), f^m(x)) \\ \leq & r^n d(x, f(x)) + \cdots + r^{m-1} d(x, f(x)) \\ \leq & r^n d(x, f(x))(1 + r + r^2 + \cdots) \\ = & \frac{r^n}{1-r} d(x, f(x)) \leq \frac{r^N}{1-r} d(x, f(x)) < \epsilon. \end{aligned}$$

This shows that $\{f^n(x)\}$ is a Cauchy sequence. Since X is complete, there exists a $p \in X$ such that $\lim_{n\to\infty} f^n(x) = p$. Moreover the continuity of f yields

$$p = \lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)) = f(p).$$

Thus p is a fixed point of f. Finally we have

$$\lim_{m\to\infty} d(f^n(x), f^m(x)) = d(f^n(x), p) \leq \frac{r^n}{1-r} d(x, f(x)).$$

□

2.2 Metric Space of Compact Sets

Let (X, d) be a complete metric space. If $x \in X, K \subset X$, then define the *distance* between x and K by

$$d(x, K) = \inf \{d(x, y) : y \in K\}. \tag{2.1}$$

For $\epsilon > 0$, define the *ϵ-neighbourhood* of K by

$$K_\epsilon = \{x \in X : d(x, K) < \epsilon\}. \tag{2.2}$$

Let $\mathcal{B}$ be the class of non-empty closed bounded subsets of X, $\mathcal{C}$ be the class of non-empty compact subsets of X.

Definition 2.2.1. *Hausdorff metric δ on $\mathcal{C}$ is defined by*

$$\delta(A, B) = \sup \{d(x, B), d(y, A) : x \in A, y \in B\}. \tag{2.3}$$

Theorem 2.2.1. *$(\mathcal{C}, \delta)$ is a complete metric space under Hausdorff metric.*

The proof of this theorem is not trivil, for details, see reference [2] or [14].

We list some elementary properties to be used in the following sections. Let $f : X \to X$, and $f_i : X \to X$ for $i = 1, \ldots, N$. Denote $A_i = f_i(A)$ for $A \subset X$. Then for $A \subset X, B \subset X$

(i) $\delta(f(A), f(B)) \leq Lip(f)\delta(A, B)$,

(ii) $\delta(\bigcup_{i=1}^N A_i, \bigcup_{i=1}^N B_i) \leq \sup_{i=1,\ldots,N} \delta(A_i, B_i)$.

2.3 Invariant Sets

Let (X, d) be a complete metric space. $\psi_i : X \to X$ for $i = 1, \ldots, N$ are contraction maps with

$$d(\psi_i(x), \psi_i(y)) \leq r_i d(x, y)$$

where $0 < r_i < 1$ for $i = 1, \ldots, N$. We assume that $r_i = Lip(\psi_i)$.

Define a set-to-set map Ψ by

$$\Psi(A) := \bigcup_{i=1}^{N} \psi_i(A), \quad A \subset X \tag{2.4}$$

where $\psi_i(A) = \{\psi_i(a) : a \in A\}$. Denote n-time iterated map $\Psi \circ \cdots \circ \Psi$ by Ψ^n.

Notice that each ψ_i is considered as a set-to-set map, and Ψ is also a set-to-set map imaging the subset $A \subset X$ into the subset $\Psi(A) \subset X$. We now study the map Ψ on the space $(\mathcal{C}, \delta)$. We first show some properties of the set $\Psi(B)$ and the map $B \longmapsto \Psi(B)$ when $B \in \mathcal{C}$.

Lemma 2.3.1. Ψ *is a contraction map on* $\mathcal{C}$ *in the Hausdorff metric.*

Proof. By the properties listed in Section 2.2, we have

$$\delta\left(\Psi(A), \Psi(B)\right) = \delta\left(\bigcup_{i=1}^{N} \psi_i(A), \bigcup_{i=1}^{N} \psi_i(B)\right)$$

$$\leq \max_{1\leq i\leq N} \delta\left(\psi_i(A), \psi_i(B)\right) \leq \max_{1\leq i\leq N} \{r_i\}\, \delta(A, B).$$

Let $r = \max_{1\leq i\leq N}\{r_i\}$. Then $0 < r < 1$ and $\delta\left(\Psi(A), \Psi(B)\right) \leq r\delta(A, B)$. $\square$

Lemma 2.3.2. *Let $B \in \mathcal{C}$. Then $\Psi(B) \in \mathcal{C}$.*

Proof. Since we have proven Ψ is a contration map on $(\mathcal{C}, \delta)$, that Ψ is a continuous map. Moreover, a continuous image of a compact set is compact. Review that $\mathcal{C}$ is a class of non-empty compact subsets of X. Therefore, $\Psi(B) \in \mathcal{C}$ when $B \in \mathcal{C}$. $\square$

Similar to the definition of a *fixed point* in Section 2.1, we give a definition of an *invariant set* under a set-to-set contraction map.

Definition 2.3.1. *The set $K \subset X$ is invariant with respect to Ψ, if*

$$K = \Psi(K) = \bigcup_{i=1}^{N} \psi_i(K). \tag{2.5}$$

Furthermore, a theorem showing the existence and uniqueness of an *invariant set* is given.

Theorem 2.3.1. *There is a unique non-empty compact set $K \in \mathcal{C}$ which is invariant with respect to Ψ. Moreover, for an arbitrary non-empty compact set $A \in \mathcal{C}$, $\Psi^p(A) \to K$ as $p \to \infty$ in the Hausdorff metric.*

Proof. Since $(\mathcal{C}, \delta)$ is a complete space in Hausdorff metric, from Lemma 2.3.1 we know $\Psi : \mathcal{C} \to \mathcal{C}$ is contraction. Then by the contraction principle, there exists a unique fixed point $K \in \mathcal{C}$ such that $\Psi(K) = \bigcup_{i=1}^{N} \psi_i(K) = K$, i.e. K is invariant with respect to Ψ. In addition, for any $A \in \mathcal{C}$, we have $\lim_{p\to\infty} \Psi^p(A) = K$. $\square$

2.4 Properties of Invariant Sets

Continue the notations in Section 2.3. Denote $\psi_{i_1,\ldots,i_p} = \psi_{i_1} \circ \cdots \circ \psi_{i_p}$, and by $s_{i_1,\ldots,i_p}$, the fixed points of $\psi_{i_1,\ldots,i_p}$. For arbitrary $A \subset X$, denote $\psi_{i_1,\ldots,i_p}(A) = A_{i_1,\ldots,i_p}$.

Notice that $\Psi^p(A) = \bigcup_{i_1,\ldots,i_p} A_{i_1,\ldots,i_p}$ where for every set of indeces $i_1,\ldots,i_p \in \{1,\ldots,N\}$. If A is bounded, then $diam(A_{i_1,\ldots,i_p}) \leq r_{i_1} \cdot \cdots \cdot r_{i_p} diam(A) \to 0$ as $p \to \infty$.

By $\hat{i}_1,\ldots,\hat{i}_p$, we mean the infinite sequence $i_1,\ldots,i_p,i_1,\ldots,i_p \ldots i_1,\ldots,i_p \ldots$.

Property 2.4.1. *Let K be the compact invariant set of Ψ. Then*

1. $K_{i_1\ldots i_p} = \bigcup_{i_{p+1}=1}^{N} K_{i_1\ldots i_p,i_{p+1}}$.

2. $K \supset K_{i_1} \supset \cdots \supset K_{i_1\ldots i_p} \supset \cdots$, *and* $\bigcap_{p=1}^{\infty} K_{i_1\ldots i_p}$ *is a singleton whose member is denoted as* $k_{i_1\ldots i_p\ldots}$. *K is the union of these singletons.*

3. $\psi_{j_1\ldots j_q}(k_{i_1\ldots i_p\ldots}) = k_{j_1\ldots j_q i_1\ldots i_p\ldots}$.

4. $k_{\hat{i}_1\ldots\hat{i}_p} = s_{i_1\ldots i_p}$,*and in particular* $s_{i_1\ldots i_p} \in K$.

   Also $k_{i_1\ldots i_p\ldots} = \lim_{p\to\infty} s_{i_1\ldots i_p}$, *and in particular, this limit exists.*

5. *K is the closure of the set of fixed points of* $\psi_{i_1\ldots i_p}$.

6. *The coordinate map* $\pi : \boldsymbol{C}(N) \to K$ *given by* $\pi(\alpha) = k_\alpha$ *is a continuous map onto K.*

7. *If A is a non-empty bounded set, then* $d(A_{i_1\ldots i_p}, k_{i_1\ldots i_p\ldots}) \to 0$ *uniformly as* $p \to \infty$.

Proof. 1. Since

$$K = \bigcup_{i=1}^{N} \psi_i(K) = \bigcup_{i,j} \psi_i(\psi_j(K)) = \bigcup_{i,j} \psi_{ij}(K) = \bigcup_{i,j} K_{ij},$$

then

$$K = \bigcup_{i_1\dots,i_p} K_{i_1\dots,i_p}.$$

Similarly,

$$K_{i_1\dots,i_p} = \psi_{i_1\dots,i_p}(K) = \psi_{i_1\dots,i_p}\left(\bigcup_{i_{p+1}=1}^{N} \psi_{i_{p+1}}(K)\right)$$

$$= \bigcup_{i_{p+1}=1}^{N} \psi_{i_1,\dots,i_{p+1}}(K) = \bigcup_{i_{p+1}=1}^{N} K_{i_1,\dots,i_p i_{p+1}}.$$

2. From 1, we have $K \supset K_{i_1} \supset \cdots \supset K_{i_1\dots i_p} \supset \cdots$. Since $diam(K_{i_1,\dots,i_p}) \to 0$ as $p \to \infty$, that $\bigcap_{p=1}^{\infty} K_{i_1,\dots,i_p}$ is a singelton, whose unique member is denoted by $k_{i_1,\dots,i_p\dots}$. Since $K = \bigcup_{i_1\dots,i_p} K_{i_1\dots,i_p}$, that K is the union of $k_{i_1,\dots,i_p\dots}$.

3. Since $\psi_{j_1,\dots,j_q}(K_{i_1,\dots,i_p}) = K_{j_1,\dots,j_q i_1,\dots,i_p}$, then we have

$$\psi_{j_1,\dots,j_q}(k_{i_1,\dots,i_p\dots}) = \psi_{j_1,\dots,j_q}\bigcap_{p=1}^{\infty} K_{i_1,\dots,i_p}$$

$$= \bigcap_{p=1}^{\infty} K_{j_1,\dots,j_q i_1,\dots,i_p} = k_{j_1,\dots,j_q i_1,\dots,i_p\dots}.$$

4. By the above $\psi_{i_1,\dots,i_p}(k_{\hat{i}_1,\dots,\hat{i}_p}) = k_{\hat{i}_1,\dots,\hat{i}_p}$, it follows $k_{\hat{i}_1,\dots,\hat{i}_p}$ is the unique fixed point $s_{i_1,\dots,i_p}$ of $\psi_{i_1,\dots,i_p}$, which implies both $s_{i_1,\dots,i_p}, k_{i_1,\dots,i_p\dots} \in K_{i_1,\dots,i_p}$. Since

$$\lim_{p\to\infty} diam(K_{i_1,\dots,i_p}) = 0,$$

thus $\lim_{p\to\infty} s_{i_1,\dots,i_p} = k_{i_1,\dots,i_p\dots}$.

5. From 2 and 4, we get 5 immediately.

6. Suppose $\alpha =< \alpha_1 \dots \alpha_p \dots > \in \mathbf{C}(N)$ and $\epsilon > 0$. Then $\pi(\alpha) = k_{\alpha_1\dots\alpha_p\dots}$,and so there is a q such that $K_{\alpha_1\dots\alpha_q} \subset \{x \in K : d(x, \psi(\alpha)) < \epsilon\}$. Since $K_{\alpha_1\dots\alpha_q}$ is

the image of the open set $\{\beta : \beta_i = \alpha_i, \text{ if } i \leq q\}$, it follows π is continuous.

7. Suppose $A \subset X$ is non-empty bounded set. Then

$$\begin{aligned} d(A_{i_1,\dots,i_p}, k_{i_1,\dots,i_p\dots}) =& d(\psi_{i_1,\dots,i_p}(A), \psi_{i_1,\dots,i_p}(k_{i_{p+1}\dots})) \\ \leq& r_{i_1} \cdot \dots \cdot r_{i_p} d(A, k_{i_{p+1}}) \\ \leq& r_{i_1} \cdot \dots \cdot r_{i_p} \sup \{d(a,b) : a \in A, b \in K\} \\ \leq& constant \left(\max_{1 \leq i \leq N} r_i \right)^p \\ \rightarrow& 0 \end{aligned}$$

as $p \rightarrow \infty$.

□

2.5 Similitudes in Metric Space

Let (X, d) be a complete metric space.

Definition 2.5.1. *A map $f : X \rightarrow X$ is called a similitude if $d(f(x), f(y)) = rd(x,y)$, $\forall x, y \in X$ and some fixed $r \in \mathbb{R}$. Moreover, $f : X \rightarrow X$ is said to be a contractive similitude if $r \in (0,1)$.*

Notice that from the definition we know that a contractive similitude f is also a contraction map with $Lip(f) = r$. Therefore, there exists a fixed point p in X such that $f(p) = p$.

The notion of similitudes (contractive similitudes) can be given in any arbitrary metric space. However, we are interested in a particular case where the metric space is $\mathbb{R}^n$ with Euclidean distance d. Relative properties of invariant sets in Euclidean space will be given in the following chapter.

Chapter 3

Similarities

3.1 Similitudes in Euclidean Space

Let (X, d) be a complete metric space. In this section, we only consider the case that $X = \mathbb{R}^n$ and the Euclidean distance d.

Denote

$$\mu_r : \mathbb{R}^n \to \mathbb{R}^n \text{ be the homothety } \mu_r(x) = rx, r \geq 0,$$

$$\tau_b : \mathbb{R}^n \to \mathbb{R}^n \text{ be the translation } \tau_b(x) = x - b.$$

Proposition 3.1.1. *$f : \mathbb{R}^n \to \mathbb{R}^n$ is a similitude iff $f = \mu_r \circ \tau_b \circ O$ for some homothety μ_r, translation τ_b and orthonormal transformation O.*

Proof. $(\Leftarrow)$ is obvious.

$(\Rightarrow)$ Let f be a similitude with $Lip(f) = r$. Set $g(x) = r^{-1}(f(x) - f(0))$, then $f(x) = \mu_r \circ \tau_{-r^{-1}S(0)} \circ g$. Need to prove g is orthonormal transformation, i.e. preserve the inner product and linear. Since

$$
\begin{aligned}
(g(x), g(y)) =&(r^{-1}(f(x)-f(0)), r^{-1}(f(y)-f(0))) \\
=&r^{-2}(f(x)-f(0), f(y)-f(0)) \\
=&\frac{r^{-2}}{2}\left[||f(x)-f(0)||^2+||f(y)-f(0)||^2-||f(x)-f(y)||^2\right] \\
=&\frac{r^{-2}}{2}\left[(d(f(x), f(0)))^2+(d(f(y), f(0)))^2-(d(f(x), f(y)))^2\right] \\
=&\frac{r^{-2}}{2}\left[r^2(d(x,0))^2+r^2(d(y,0))^2-r^2(d(x,y))^2\right] \\
=&\frac{1}{2}\left[(d(x,0))^2+(d(y,0))^2-(d(x,y))^2\right] \\
=&\frac{1}{2}\left[||x||^2+||y||^2-||x-y||^2\right] \\
=&(x,y),
\end{aligned}
$$

it follows g preserves inner products.

Let $\{e_i : 1 < i < N\}$ be an orthonormal basis for $\mathbb{R}^n$. Then $\{g(e_i) : 1 < i < N\}$ is also an orthonormal basis. Hence

$$
g(x) = \sum_{i=1}^{N}(g(x), g(e_i))g(e_i) = \sum_{i=1}^{N}(x, e_i)g(e_i).
$$

It follows g is linear. Therefore g is an orthonormal transformation. $\square$

Remark 3.1.1. *If $\psi_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1, \ldots, N$ are contractive similitudes with Lipschitz constants r_i. Then for $A \subset \mathbb{R}^n$, $\Psi(A) := \bigcup_{i=1}^{N} \psi_i(A)$ is a contractive similitude in $(\mathcal{C}, \delta)$, where $\mathcal{C}$ is the class of non-empty compact subsets of $\mathbb{R}^n$ and δ is the Hausdorff metric on $\mathcal{C}$. Moreover, there exists a unique compact invariant set $K \in \mathcal{C}$ such that $\Psi(K) = K$.*

Now we are interested in the dimension of the invariant set K of Ψ. Before showing the Euclidean properties of K, we give some notions of Hausdorff dimension and Hausdorf measures in the following sections.

3.2 Hausdorff Measures

Now we introduce certain "lower dimensional" measures on $\mathbb{R}^n$, which allow us to measure certain "very small" subsets of $\mathbb{R}^n$. These are the Hausdorff measures $\mathscr{H}^k$, defined in terms of the diameters of various efficient coverings. The idea is that A is an "k-dimensional subset" of $\mathbb{R}^n$ if $0 < \mathscr{H}^k(A) < \infty$, even if A is very complicated geometrically, such as in the case of fractals.

Definition 3.2.1. *Let $A \subset \mathbb{R}^n, 0 \leq k < \infty, 0 < \epsilon \leq \infty$. Set*

$$\mathscr{H}^k_\epsilon(A) = \inf \left\{ \sum_{i=1}^{\infty} \alpha(k) 2^{-k} (diam C_i)^k : A \subset \bigcup_{i=1}^{\infty} C_i, diam C_i \leq \epsilon \right\} \tag{3.1}$$

where

$$\alpha(k) = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)},$$

with $\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx$, $(0 < t < \infty)$ be the gamma function.

Define

$$\mathscr{H}^k(A) = \lim_{\epsilon \to 0} \mathscr{H}^k_\epsilon(A) = \sup_{\epsilon > 0} \mathscr{H}^k_\epsilon(A). \tag{3.2}$$

We call $\mathscr{H}^k$ the k-dimensional Hausdorff measure on $\mathbb{R}^n$, for $A \subset \mathbb{R}^n$.

Remark 3.2.1. *$\mathscr{H}^k$ will not always be finite on bounded sets. In fact, we have $\mathscr{H}^k(A) \in [0, \infty]$.*

By the definition of Hausdorff measure, we can easily prove that: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschiz, i.e., $Lip(f) < \infty$, then $\mathscr{H}^k(f(A)) \leq (Lip(f))^k \mathscr{H}^k(A)$. If f is a similitude, then $f_\# \mathscr{H}^k := \mathscr{H}^k \circ f^{-1} = (Lip(f))^{-k} \mathscr{H}^k$.

Let $\mathscr{L}^n$ be the n-dimensional Lebesgue measure on $\mathbb{R}^n$. Observe that

$$\mathscr{L}^n(B(x,r)) = \alpha(n)r^n$$

for all balls $B(x,r) \subset \mathbb{R}^n$. We will see later that if k is an integer, $\mathscr{H}^k$ agrees with ordinary "k-dimensional surface area" on nice sets.

We now show some results of the Hausdorff measure without proof. Although these results will not be used in this paper, they play an important role in the research of Hausdorff measures. Moreover, they will be helpful for us to understand the relative theory of Hausdorff measures.

- $\mathscr{H}^k$ is a Borel regular measure $(0 \leq k < \infty)$.
- n-dimensional Lebesgue measure and n-dimensional Hausdorff measure agree on $\mathbb{R}^n$, i.e. $\mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$.
- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and one-to-one, $n \leq m$. Then for each $\mathscr{L}^n$-measurable subset $A \subset \mathbb{R}^n$,

  $$\mathscr{H}^n(f(A)) = \int_A J(f)d\mathscr{L}^n,$$

  where $J(f)$ is the Jacobian of f.

For more details and proof, see reference [4].

Example 3.2.1 (Surface area of a graph)**.** *Assume $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ by*

$$f(x) = (x, g(x)).$$

For each open set $U \subset \mathbb{R}^n$, define the graph of g over U by

$$G = G(g, U) = \{(x, g(x)) : x \in U\} \subset \mathbb{R}^{n+1}.$$

Then

$$\mathscr{H}^n(G) = \text{“surface area” of } G = \int_U J(f)dx.$$

3.3 Hausdorff Dimension

Before defining the Hausdorff dimension of a subset of $\mathbb{R}^n$, we first show a lemma to help with understanding the following concepts.

Lemma 3.3.1. *Let $A \subset \mathbb{R}^n$ and $0 \leq k < t < \infty$.*

(i) If $\mathscr{H}^k(A) < \infty$, then $\mathscr{H}^t(A) = 0$,

(ii) If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^k(A) = +\infty$.

Proof. Suppose $\mathscr{H}^k(A) < \infty$ and $\epsilon > 0$. Then there exist sets $\{C_i\}_{i=1}^{\infty}$ such that $diamC_i \leq \epsilon, A \subset \bigcup_{i=1}^{\infty} C_i$ and

$$\sum_{i=1}^{\infty} \alpha(k)2^{-k} \left(diamC_i\right)^k \leq \mathscr{H}_\epsilon^k(A) + 1 \leq \mathscr{H}^k(A) + 1.$$

Then

$$\begin{aligned}
\mathscr{H}_\epsilon^t(A) \leq & \sum_{i=1}^{\infty} \alpha(t)2^{-t} \left(diamC_i\right)^t \\
= & \frac{\alpha(t)}{\alpha(k)} 2^{k-t} \sum_{i=1}^{\infty} \alpha(k)2^{-k} \left(diamC_i\right)^k \left(diamC_i\right)^{t-k} \\
\leq & \frac{\alpha(t)}{\alpha(k)} 2^{k-t} \epsilon^{t-k} (\mathscr{H}^k(A) + 1).
\end{aligned}$$

Send $\epsilon \to 0$ to conclude $\mathscr{H}^t(A) = 0$. We proved assertion (i). Assertion (ii) follows from (i) at once. $\square$

Definition 3.3.1. *The Hausdorff dimension of a subset $A \subset \mathbb{R}^n$ is defined to be*

$$d_{\mathscr{H}} = d_{\mathscr{H}}(A) = \inf\left\{0 \le k < \infty : \mathscr{H}^k(A) = 0\right\}. \tag{3.3}$$

Notice that, by Lemma 3.3.1, $\mathscr{H}^t(A) = 0$ for all $t > d_{\mathscr{H}}$ and $\mathscr{H}^t(A) = +\infty$ for all $t < d_{\mathscr{H}}$.

3.4 Euclidean Properties of Invariant Sets

Continue the notations in Section 2.4 and 3.1. Let (X, d) be $\mathbb{R}^n$ with Euclidean metric. Denote by K the unique compact invariant set of Ψ. For convenience, we set $d_{\mathscr{H}} = d_{\mathscr{H}}(K)$.

Let $\gamma(t) = \sum_{i=1}^N r_i^t$. Then $\gamma(0) = N$ and $\gamma(t) \searrow 0$ as $t \to \infty$. Hence there is a unique $d_{\mathscr{S}} \in \mathbb{R}$ such that $\sum_{i=1}^N r_i^{d_{\mathscr{S}}} = 1$.

Definition 3.4.1. *$d_{\mathscr{S}}$ is said to be the similarity dimension of $\{\psi_1, \ldots, \psi_N\}$, if $\sum_{i=1}^N r_i^{d_{\mathscr{S}}} = 1$.*

Now our main objective is to prove that the similarity dimension $d_{\mathscr{S}}$ equals to the Hausdorff dimension $d_{\mathscr{H}}$ of K under certain condition.

Proposition 3.4.1. *Let K be the unique compact invariant set of Ψ, then we have $\mathscr{H}^{d_{\mathscr{S}}}(K) < +\infty$ and so $d_{\mathscr{H}} \le d_{\mathscr{S}}$.*

Proof. By Property 2.4.1 1, we know $K = \bigcup_{i_1,\ldots,i_p} K_{i_1\ldots i_p}$ and

$$\sum_{i_1,\ldots,i_p} (diam K_{i_1\ldots i_p})^{d_{\mathscr{S}}} = \sum_{i_1,\ldots,i_p} r_{i_1}^{d_{\mathscr{S}}} \cdot \cdots \cdot r_{i_p}^{d_{\mathscr{S}}} (diam K)^{d_{\mathscr{S}}} = (diam K)^{d_{\mathscr{S}}}.$$

Since

$$diam K_{i_1,\dots,i_p} \leq \left(\max_{1\leq i\leq N}\{r_i\}\right)^p diam K \to 0$$

as $p \to \infty$. By the definition of Hausdorff measure, we have

$$\mathscr{H}^{d_{\mathscr{S}}}(K) \leq \alpha(d_{\mathscr{S}})2^{-d_{\mathscr{S}}}(diam K)^{d_{\mathscr{S}}} < \infty.$$

It follows that $d_{\mathscr{H}} \leq d_{\mathscr{S}}$.

□

We next prove $d_{\mathscr{H}} \geq d_{\mathscr{S}}$. Before showing that, we define an important conception called *open set condition.*

Definition 3.4.2 (Open Set Condition)**.** $\{\psi_1,\dots,\psi_N\}$ *satisfies the open set condition (o.s.c.) if there exists a non-empty open set O such that*

(i) $\bigcup_{i=1}^N \psi_i O \subset O$,

(ii) $\psi_i O \cap \psi_j O = \emptyset$ *if* $i \neq j$.

Definition 3.4.3. *The lower (upper) k-dimensional density of $A \subset X$ at points $x \in X$ is defined respectively by*

$$\theta^k_*(A,x) = \liminf_{r\to 0} \frac{\mathscr{H}^k(A\cap B(x,r))}{\alpha(k)r^k} \tag{3.4}$$

$$\theta^{*k}(A,x) = \limsup_{r\to 0} \frac{\mathscr{H}^k(A\cap B(x,r))}{\alpha(k)r^k} \tag{3.5}$$

Likewise, for a measure μ on X, we define

$$\theta^k_*(\mu,x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\alpha(k)r^k} \tag{3.6}$$

$$\theta^{*k}(\mu,x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\alpha(k)r^k} \tag{3.7}$$

Thus we get $\theta^k_*(A,x) = \theta^k_*(\mathscr{H}^k \lfloor A, x)$.

The upper density turns out to be more important than the lower density. The main results we will use are

(i) $\theta^{*k}(\mu,x) \geq \lambda, \forall x \in A \Rightarrow \mathscr{H}^k(A) \leq \lambda^{-1}\mu(A)$,

(ii) $\theta^{*k}(\mu,x) \leq \lambda, \forall x \in A \Rightarrow \mathscr{H}^k(A) \geq 2^{-k}\lambda^{-1}\mu(A)$.

for $\mu \in \mathcal{M}$. $\mathcal{M}$ is the set of Borel regular measures having bounded support and finite mass, i.e. $\mathbf{M}(\mu) = \mu(X) < \infty$. For a reference see [6].

If $0 < \mu(A) < \infty$ and $0 < \theta^{*k}(\mu,x) < \infty$, then we have $0 < \mathscr{H}^k(K) < \infty$.

Lemma 3.4.1. *Suppose $0 < c_1 < c_2 < \infty$ and $0 < \rho < \infty$. Let $\{U_i\}$ be a family of disjoint open sets in $\mathbb{R}^n$. Suppose each U_i contains a ball of radius ρc_1 and is contained in a ball of ρc_2. Then at most $(1+2c_2)^n c_1^{-n}$ of the $\bar{U}_i$ meet $B(0,\rho)$.*

Proof. Suppose $\bar{U}_1, \ldots, \bar{U}_k$ meet $B(0,\rho)$. Then each of $\bar{U}_1, \ldots, \bar{U}_k$ is a subset of $B(0,(1+2c_2)\rho)$. Summing the volumes of the k corresponding disjoint spheres of radius ρc_1, we have

$$k\alpha_n \rho^n c_1^n \leq \alpha_n (1+2c_2)^n \rho^n,$$

and hence $k \leq (1+2c_2)^n c_1^{-n}$. □

Now we show an important theorem which gives us the value of the Hausdorff dimension of K.

Theorem 3.4.1. *Suppose $\{\psi_1, \ldots, \psi_N\}$ satisfies the o.s.c., then $0 < \mathscr{H}^{d_{\mathscr{S}}}(K) < \infty$. In particular $d_{\mathscr{H}} = d_{\mathscr{S}}$.*

Proof. Let μ be the invariant measure of $\mathcal{T}$ in Section 4.2. Denote O the open set asserted to exist by the o.s.c.. First prove that there exists constants κ_1, κ_2 such that

$$0 < \kappa_1 \leq \theta^{d_{\mathscr{S}}}_*(\mu,k) \leq \theta^{*d_{\mathscr{S}}}(\mu,k) \leq \kappa_2 < \infty$$

for all $k \in K$.

Note that

$$\mu(K_{i_1,\ldots,i_p}) \geq (\mathscr{H}^{\mathscr{S}}(K))^{-1}\mathscr{H}^{\mathscr{S}}\lfloor K_{i_1,\ldots,i_p}(K_{i_1,\ldots,i_p})$$

$$= r_{i_1}^{d_{\mathscr{S}}} \cdot \cdots \cdot r_{i_p}^{d_{\mathscr{S}}} \mu(\psi_{i_1,\ldots,i_p}^{-1} K_{i_1,\ldots,i_p}) = r_{i_1}^{d_{\mathscr{S}}} \cdot \cdots \cdot r_{i_p}^{d_{\mathscr{S}}} \mu(K) = r_{i_1}^{d_{\mathscr{S}}} \cdot \cdots \cdot r_{i_p}^{d_{\mathscr{S}}}.$$

Let $k = k_{i_1,\ldots,i_p\ldots}$ and consider the ball $B(k,\rho)$. Choose the least ρ such that $K_{i_1,\ldots,i_p} \subset B(k,\rho)$. Then we have $r_{i_1} \cdot \cdots \cdot r_{i_p}(diamK) \geq \rho r_1$ (recall $r_1 \leq \cdots \leq r_N$). Thus

$$\frac{\mu B(k,\rho)}{\alpha(d_{\mathscr{S}})\rho^{d_{\mathscr{S}}}} \geq \frac{\mu(K_{i_1,\ldots,i_p})}{\alpha(d_{\mathscr{S}})\rho^{d_{\mathscr{S}}}} \geq \frac{r_{i_1}^{d_{\mathscr{S}}} \cdot \cdots \cdot r_{i_p}^{d_{\mathscr{S}}}}{\alpha(d_{\mathscr{S}})\rho^{d_{\mathscr{S}}}} \geq \frac{r_1^{d_{\mathscr{S}}}}{\alpha(d_{\mathscr{S}})(diamK)^{d_{\mathscr{S}}}}$$

Hence $\theta_*^{d_{\mathscr{S}}}(\mu,k) \geq r_1^{d_{\mathscr{S}}}\alpha^{-1}(d_{\mathscr{S}})(diamK)^{-d_{\mathscr{S}}}$ for $k \in K$.

Suppose O contains a ball of radius c_1 and is contained in a ball of radius c_2. For each sequence $j_1 \ldots j_q \ldots$ select the least q such that $r_1\rho \leq r_{j_1} \cdot \cdots \cdot r_{j_q} \leq \rho$. Let I be the set of $< j_1 \ldots j_q >$ thus selected. Thus $\left\{O_{j_1\ldots j_q} :< j_1 \ldots j_q >\in I\right\}$ is a collection of disjoint open sets. Moreover, each such $O_{j_1\ldots j_q}$ contains a ball of radius $r_{j_1} \cdot \cdots \cdot r_{j_q}c_1$ and hence of radius $r_1c_1\rho$, and is contained in a ball of radius $r_{j_1} \cdot \cdots \cdot r_{j_q}c_2$ and hence of radius ρc_2. It follows from Lemma 3.4.1 that at most $(1+2c_2)^n(r_1c_1)^{-n}$ of the $\bar{O}_{j_1\ldots j_q}$ meet $B(k,\rho)$. Hence at most $(1+2c_2)^n(r_1c_1)^{-n}$ of the $K_{j_1\ldots j_q}$ meet $B(k,\rho)$. Then

$$\frac{\mu(B(k,\rho))}{\alpha(d_{\mathscr{S}})\rho^{d_{\mathscr{S}}}} \leq \frac{(1+2c_2)^n}{r_1^n c_1^n} \cdot \frac{\rho^{d_{\mathscr{S}}}}{\alpha(d_{\mathscr{S}})\rho^{d_{\mathscr{S}}}} = \frac{(1+2c_2)^n}{\alpha(d_{\mathscr{S}})r_1^n c_1^n}$$

It follows $\theta^{*d_{\mathscr{S}}}(\mu,k) \leq (1+2c_2)^n(\alpha(d_{\mathscr{S}})r_1^n c_1^n)^{-1}$.

If we let $\kappa_1 = r_1^{d_{\mathscr{S}}}\alpha^{-1}(d_{\mathscr{S}})(diamK)^{-d_{\mathscr{S}}}, \kappa_2 = (1+2c_2)^n(\alpha(d_{\mathscr{S}})r_1^n c_1^n)^{-1}$, then we have

$$0 < \kappa_1 \leq \theta_*^{d_{\mathscr{S}}}(\mu,x) \leq \theta^{*d_{\mathscr{S}}}(\mu,k) \leq \kappa_2 < \infty.$$

Now use the results of k-dimentional density of μ at point k, we have

$$0 < \mathscr{H}^{d_{\mathscr{S}}}(K) < \infty$$

which implies $d_{\mathscr{S}} = d_{\mathscr{H}}$. $\square$

Corollary 3.4.1. *Suppose $\{\psi_1, \ldots, \psi_N\}$ satisfies the o.s.c.. If $r_i = r = \frac{1}{\alpha}$ for $i = 1, \ldots, N$, then $d_{\mathscr{H}}(K) = \frac{\log N}{\log \alpha}$.*

Now we know the Hausdorff dimension of the invariant set with respect to $\{\psi_1, \ldots, \psi_N\}$ under the open set condition. Does there exist a measure so-called an "invariant measure" with respect to $\{\psi_1, \ldots, \psi_N\}$? What are the similarity properties of this measure? In the following chapter, we will show the existence of this special invariant measure which equals the normalized Hausdorff measure $\mathscr{H}^{d_{\mathscr{S}}}$ restricted on K.

Chapter 4

Invariant Measures

In this chapter, similar to the theory of the invariant set, we will show relative definitions and properties of the invariant measure with respect to a set of contractive similitudes in a complete metric space. The main tool we are using is the contraction principle which has already been shown in Section 2.1. Before giving out the definition of invariant measures, we first aim to show the completeness of the metric space of Borel regular measures.

4.1 Metric Space of Borel Regular Measures

Let (X, d) be a complete metric space.

Definition 4.1.1. *A measure μ on X is said to be* Borel regular *iff all Borel sets are measurable and for each $A \subset X$ there exists a Borel set $B \supset A$ with $\mu(A) = \mu(B)$.*

We define the *support* of μ to be the closed set

$$spt\mu = X \backslash \bigcup \{A : A \text{ open}, \mu(A) = 0\}.$$

For $A \subset X, E \subset X$, $\mu \lfloor A(E) = \mu(A \cap E)$.

Define *mass* of μ by $\mathbf{M}(\mu) = \mu(X)$. $\mathcal{M}$ is the set of Borel regular measures having bounded support and finite mass.

Set

$$\mathcal{M}^1 = \{\mu \in \mathcal{M} : \mathbf{M}(\mu) = 1\},$$

$$\mathcal{BC}(X) = \{\phi : X \to \mathbb{R} : \phi \text{ is continuous and bounded on bounded subset}\}.$$

For $\mu \in \mathcal{M}, \phi \in \mathcal{BC}(X)$, define $\mu(\phi) = \int \phi d\mu$. Then we say $\mu_n \rightharpoonup \mu$ as $n \to \infty$ iff $\mu_n(\phi) \to \mu(\phi)$ for all $\phi \in \mathcal{BC}(X)$.

We introduce a metric L on $\mathcal{M}^1$ to enable a following theorem to hold.

Definition 4.1.2. *For $\mu, \nu \in \mathcal{M}^1$, the L metric is defined by*

$$L(\mu, \nu) = \sup\{|\mu(\phi) - \nu(\phi)| : \phi : X \to \mathbb{R}, Lip\phi \leq 1\}. \tag{4.1}$$

Notice that $\phi \in \mathcal{BC}$ in the definition. We can check L is a metric by verifying $L(\mu, \nu) < +\infty$, the only part that is not straightforward. Suppose $spt\mu \cup spt\nu \subset B(a, r) = \{x \in X : d(x, a) < r\}$, then for $Lip\phi \leq 1$

$$|\mu(\phi) - \nu(\phi)| = |\mu(\phi - \phi(a) + \phi(a)) - \nu(\phi - \phi(a) + \phi(a))|$$

$$= |\mu(\phi - \phi(a)) - \nu(\phi - \phi(a))| \leq \mu(r) + \nu(r) = 2r < +\infty.$$

Theorem 4.1.1. *$\mathcal{M}^1$ is a complete space under the L metric.*

Proof. Let E be a bounded subset of X. $\{\mu_1, \mu_2, \ldots, \mu_n, \ldots\}$ is a sequence of elements in $\mathcal{M}^1$ with $spt\mu_n \subset E$ for every n such that $L(\mu_n, \mu_m) \to 0$ as $n, m \to \infty$. We will construct a measure $\mu \in \mathcal{M}^1$ such that $L(\mu_n, \mu) \to 0$ as $n \to \infty$.

Let $\phi \in \mathcal{BC}(X)$ and ϕ is not a constant on E. Then for every $\epsilon < 0$, we have

$$|\int \phi d\mu_m - \int \phi d\mu_n| = |\int_E \phi d\mu_m - \int_E \phi d\mu_n| \leq \epsilon$$

Therefore $\int \phi d\mu_n$ converges to some $f(\phi) \in \mathbb{R}$ as $n \to \infty$. Notice that if $\phi = c$ on E with c a constant, then $f(\phi) = c$. $f(\phi)$ is a linear functional of $\phi \in \mathcal{BC}$. Since $|\int \phi d\mu_n| \leq ||\phi||_\infty$ for every n, that $|f(\phi)| \leq ||\phi||_\infty$ for every $\phi \in \mathcal{BC}(X)$. By Riesz's theorem, there exists a μ on X, such that

$$f(\phi) = \int \phi d\mu$$

for every $\phi \in \mathcal{BC}(X)$. Moreover,

$$|\int \phi d\mu_n - \int \phi d\mu| \to 0$$

as $n \to \infty$. Since $\int \phi d\mu_n = 0$ whenever $\phi \not\equiv 0$ on X/E for every n, that $spt\mu \subset E$, which means $spt\mu$ is bounded. By choosing $\phi = 1$ on E, we have

$$\mu(X) = \int \phi d\mu = \lim_{n\to\infty} \int \phi d\mu_n = \lim_{n\to\infty} \int_E \phi d\mu_n = \lim_{n\to\infty} \mu_n(X) = 1.$$

Thus $\mu \in \mathcal{M}^1$ and $L(\mu_n, \mu) \to 0$ as $n \to \infty$. $\square$

4.2 Invariant Measures

Let (X, d) be a complete metric space. $\{\psi_1, \ldots, \psi_N\}$ is a set of contractive similitudes in X with $Lip(\psi_i) = r_i$ for $i = 1 \ldots N$.

Let $m = \{m_1, \ldots, m_N\}$ be a family of positive constants with $m_i \in (0, 1)$ for $i = 1, \ldots, N$ such that $\sum_{i=1}^N m_i = 1$.

If $f : X \to X$ is continuous and sends bounded sets to bounded sets, in particular f is a contraction map, then for every $\mu \in \mathcal{M}^1$, we have $f_{\#}\mu = \mu \circ f^{-1} \in \mathcal{M}^1$. We also define $f_{\#}\mu(\phi) = \mu(\phi \circ f)$ for $\phi \in \mathcal{BC}(X)$.

For $\mu \in \mathcal{M}^1$, define $\mathcal{T}(\mu) = \sum_{i=1}^{N} m_i \psi_{i\#}\mu = \sum_{i=1}^{N} m_i \mu \circ \psi_i^{-1}$. Then we can see that $\mathcal{T} = (\mathcal{T}; m_1, \ldots, m_N)$ is a map of space $\mathcal{M}^1$ into itself. Denote n-time iterated map $\mathcal{T} \circ \cdots \circ \mathcal{T}$ by $\mathcal{T}^n$.

Definition 4.2.1. *μ is an invariant measure of $\mathcal{T}$, if*

$$\mu = \mathcal{T}(\mu) = \sum_{i=1}^{N} m_i \mu \circ \psi_i^{-1}. \tag{4.2}$$

Notice that for every $\phi \in \mathcal{BC}(X)$, if μ is an invariant measure of $\mathcal{T}$, then $\mu(\phi) = \int \phi d\mu = \sum_{i=1}^{N} m_i \int \phi \circ \psi_i d\mu$.

Lemma 4.2.1. *For any $m = \{m_1, \ldots, m_N\}$, $\mathcal{T} : \mathcal{M}^1 \to \mathcal{M}^1$ is a contraction map in the L metric.*

Proof. To establish the contraction of $\mathcal{T}$, suppose $Lip\phi \leq 1$ and $r = \max_{1 \leq i \leq N} \{r_i\}$. Then for $\mu, \nu \in \mathcal{M}^1$,

$$\begin{aligned}
\mathcal{T}(\mu)(\phi) - \mathcal{T}(\nu)(\phi) &= \sum_{i=1}^{N} (m_i \psi_{i\#}\mu)(\phi) - \sum_{i=1}^{N} (m_i \psi_{i\#}\nu)(\phi) \\
&= \sum_{i=1}^{N} (m_i(\mu(\phi \circ \psi_i) - \nu(\phi \circ \psi_i)) \\
&= \sum_{i=1}^{N} m_i r(\mu(r^{-1}\phi \circ \psi_i) - \nu(r^{-1}\phi \circ \psi_i)) \\
&\leq \sum_{i=1}^{N} m_i r L(\mu, \nu) = rL(\mu, \nu)
\end{aligned}$$

So $L(\mathcal{T}(\mu), \mathcal{T}(\nu)) \leq rL(\mu, \nu)$ with $r < 1$. □

Theorem 4.2.1. *For every $m = \{m_1, \ldots, m_N\}$, there exists a unique $\mu \in \mathcal{M}^1$ such that $\mathcal{T}(\mu) = \mu$. For any $\nu \in \mathcal{M}^1$, $\mathcal{T}^p(\nu) \to \mu$ as $p \to \infty$ in the L metric.*

Proof. Since $(\mathcal{M}^1, L)$ is a complete metric space. From Lemma 4.2.1, we know $\mathcal{T} : \mathcal{M}^1 \to \mathcal{M}^1$ is contraction. Therefore, by the conraction principle in Section 2.1, there exists a unique fixed point $\mu \in \mathcal{M}^1$ such that $\mathcal{T}(\mu) = \mu$, which means μ is an invariant measure of $\mathcal{T}$ for a certain $m = \{m_1, \ldots, m_N\}$. Furthermore, for any $\nu \in \mathcal{M}^1$, $\mathcal{T}^p(\nu) \to \mu$ as $p \to \infty$ in the L metric, which means $L\left(\mathcal{T}^p(\nu), \mu\right) \to 0$ in $\mathbb{R}$ as $p \to \infty$. □

Now our objective is to prove that by choosing a special $m = \{m_1, \ldots, m_N\}$, the invariant measure μ of $\mathcal{T}$ equals a Hausdorff measure.

4.3 Invariant Measures as Hausdoff Measures

Continue notations in Section 4.2. K denotes the invariant set of Ψ

Recall now $\sum_{i=1}^{N} r_i^{d_{\mathscr{S}}} = 1$. Let $m_i = r_i^{d_{\mathscr{S}}}$, then $\sum_{i=1}^{N} m_i = 1$ and $m_i \in (0, 1)$ for $i = 1, \ldots, N$.

Now we present an important theorem of invariant measures under the o.s.c.. Notice that we can apply the properties in Section 3.4.

Theorem 4.3.1. *Suppose $\{\psi_1, \ldots, \psi_N\}$ satisfies the o.s.c.. If we choose $m = \{m_1, \ldots, m_N\}$ by setting $m_i = r_i^{d_{\mathscr{S}}}$, then the unique invariant measure of $\mathcal{T}$ is $\mu_0 = (\mathscr{H}^{d_{\mathscr{S}}}(K))^{-1} \mathscr{H}^{d_{\mathscr{S}}} \lfloor K$.*

Proof. Denote O the open set asserted to exist by the o.s.c.. By Property 2.4.1 7, we have $K_i \subset \bar{O}_i$. Since $O_i \cap O_j = \emptyset$ if $i \neq j$, that $K_i \cap K_j = \emptyset$ for $i \neq j$. Therefore

$\mathscr{H}^{d_{\mathscr{S}}}(K_i \cap K_j) = 0$ for $i \neq j$ and so

$$\mathscr{H}^{d_{\mathscr{S}}} \lfloor K = \sum_{i=1}^{N} \mathscr{H}^{d_{\mathscr{S}}} \lfloor K_i = \sum_{i=1}^{N} \mathscr{H}^{d_{\mathscr{S}}} \lfloor \psi_i(K),$$

Notice that for $E \subset X$,

$$\begin{aligned}(\mathscr{H}^{d_{\mathscr{S}}} \lfloor \psi_i(K))(E) =& \mathscr{H}^{d_{\mathscr{S}}}(\psi_i(K) \cap E) = \mathscr{H}^{d_{\mathscr{S}}}(\psi_i(K \cap \psi_i^{-1}(E))) \\ =& r_i^{d_{\mathscr{S}}} \mathscr{H}^{d_{\mathscr{S}}}(K \cap \psi_i^{-1}(E)) = r_i^{d_{\mathscr{S}}}(\mathscr{H}^{d_{\mathscr{S}}} \lfloor K)(\psi_i^{-1}(E)) \\ =& r_i^{d_{\mathscr{S}}} \psi_{i\#}(\mathscr{H}^{d_{\mathscr{S}}} \lfloor K)(E)\end{aligned}$$

Hence

$$\mathscr{H}^{d_{\mathscr{S}}} \lfloor K = \sum_{i=1}^{N} r_i^{d_{\mathscr{S}}} \psi_{i\#}(\mathscr{H}^{d_{\mathscr{S}}} \lfloor K),$$

Let $\mu_0 = (\mathscr{H}^{d_{\mathscr{S}}}(K))^{-1} \mathscr{H}^{d_{\mathscr{S}}} \lfloor K$, it follows that $\mu_0 = \sum_{i=1}^{N} r_i^{d_{\mathscr{S}}} \psi_{i\#}(\mu_0)$, and $\mathbf{M}(\mu_0) = 1$. Therefore $\mu_0 = \mathcal{T}(\mu_0)$. By uniquesness, we have μ_0 is the invariant measure of $\mathcal{T}$. $\square$

Chapter 5

Examples in Fractals

In this chapter, we will show three particular fractal examples, which are the *Koch curve*, the *Sierpiński gasket* and the *Sierpiński carpet*. Recall some notations:

Let $(\mathbb{R}^D, d)$ be the D-dimensional Euclidean space with Euclidean distance d, where $D \geq 1$ is an integer.

$\psi_i : \mathbb{R}^D \to \mathbb{R}^D$ for $i = 1, \ldots, N$ are contractive similitudes with $Lip(\psi_i) = \frac{1}{\alpha}$ where $\alpha > 1$.

For any $A \subset \mathbb{R}^D$, define $\Psi(A) := \bigcup_{i=1}^{N} \psi_i(A)$. For more details of iteration of maps, see Section 6.1.

5.1 The Koch Curve

Consider $D = 2, N = 4$. For arbitrary $\alpha \in (2, 4]$, the *Koch curve* in $\mathbb{R}^2$ is defined in the following manner:

Let $z_0, z_1 \in \mathbb{R}^2$ and I be the unit segement joining z_0 and z_1. Let I_i for $i = 1, \ldots, 4$ be the segments of length $1/3$ joining: z_0 to z_2; z_2 to z_3; z_3 to z_4; z_4 to z_1, respectively. See Fig. 5.1.

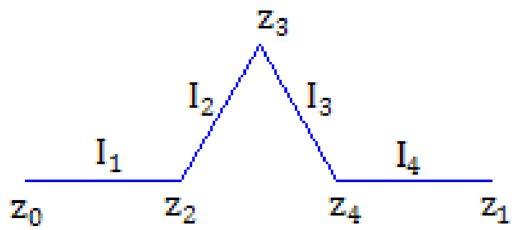


Figure 5.1: Koch graph

For instance, if $z_0 = (0,0)$ and $z_1 = (1,0)$, then

$$z_2 = (1/3, 0), \quad z_3 = (1/2, \sqrt{3}/6), \quad z_4 = (2/3, 0).$$

Consider 4 contractive similitudes $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ in $\mathbb{R}^2$:

$$\psi_1(z) = \frac{z}{\alpha}, \qquad \psi_2(z) = \frac{z}{\alpha}e^{i\theta} + \frac{1}{\alpha}$$

$$\psi_3(z) = \frac{z}{\alpha}e^{-i\theta} + \frac{1}{2} + \frac{i\sin(\theta)}{\alpha}, \qquad \psi_4(z) = \frac{z+\alpha-1}{\alpha}$$

where $\theta = \cos^{-1}\left(\frac{\alpha}{2} - 1\right)$ and $z \in \mathbb{C}$. They map I onto I_i preserving orientation. We can easily see that $Lip(\psi_i) = \frac{1}{\alpha}$ for $i = 1, \ldots, 4$.

We put $\Gamma = \{z_0, z_1\}$ and $V_0 = \Gamma$,

$$V_n = \Psi^n(\Gamma), \quad n \geq 0.$$

Then the *Koch curve* K is the compact set

$$K = cl\left(\bigcup_{n=0}^{\infty} V_n\right).$$

In the case that $z_0 = (0,0), z_1 = (1,0)$ and $z_3 = (1/2, \sqrt{3}/6)$, let O be the open triangle with vertices z_0, z_1 and z_3. Then we can check that $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ satisfies the o.s.c. such that

$$\bigcup_{i=1}^{4} \psi_i(O) \subset O$$

and

$$\psi_i(O) \cap \psi_j(O) = \emptyset, \quad \text{if } i \neq j.$$

Therefore, we can apply Corollary 3.4.1 to get the Hausdorff dimension of the *Koch curve* is $d_{\mathscr{H}}(K) = \frac{\log N}{\log \alpha}$.

In the following, we will show the constructions of the *Koch curve* under different values of α.

(i) $\alpha = 2.01, N = 4$. $d_{\mathscr{H}}(K) = \frac{\log 4}{\log 2.01} \approx 1.98$.

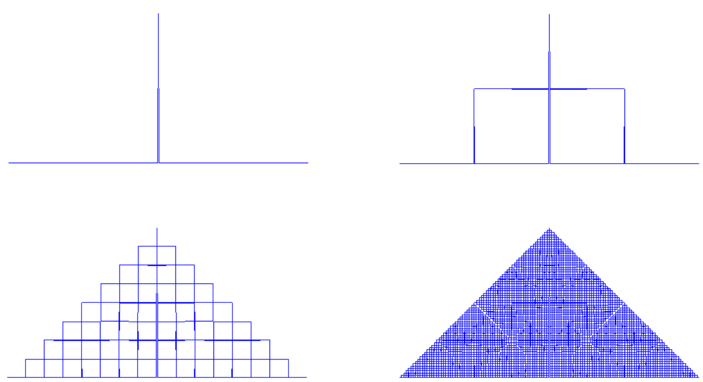


Figure 5.2: Koch iterations $\alpha = 2.01$

(ii) $\alpha = 3, N = 4$. $d_{\mathscr{H}}(K) = \frac{\log 4}{\log 3} \approx 1.26$.

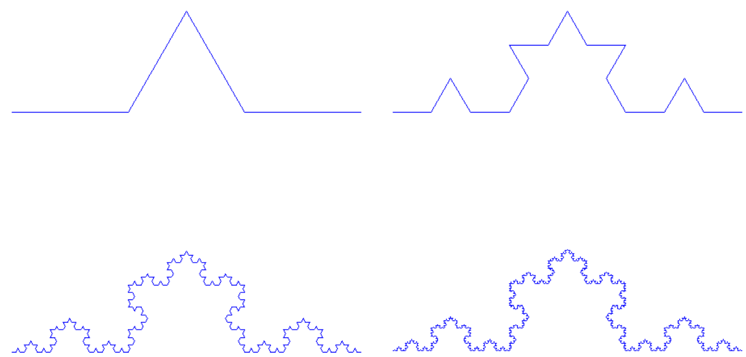


Figure 5.3: Koch iterations $\alpha = 3$

(iii) $\alpha = 3.9, N = 4$. $d_{\mathscr{H}}(K) = \frac{\log 4}{\log 3.9} \approx 1.02$.

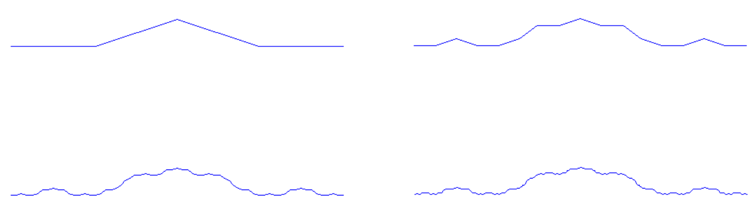


Figure 5.4: Koch iterations $\alpha = 3.9$

5.2 The Sierpiński Gasket

Consider $D \geq 2, \alpha = 2$ and $N = D + 1$. Let $z_1, \ldots, z_N \in \mathbb{R}^D$ and $|z_i - z_j| = 1$ for $i \neq j$. $\{\psi_1, \ldots, \psi_N\}$ is a family of contractive similitudes

$$\psi_i(z) = z_i + \frac{1}{\alpha}(z - z_i), \quad i = 1, \ldots, N$$

with $Lip(\psi_i) = \frac{1}{\alpha}$.

We put $\Gamma = \{z_1, \ldots, z_N\}$ and $V_0 = \Gamma$,

$$V_n = \Psi^n(\Gamma), \quad n \geq 0.$$

Then the *Sierpiński gasket* of $\mathbb{R}^D$ is

$$K = cl\left(\bigcup_{n=0}^{\infty} V_n\right).$$

Note that each V_n is obtained from V_{n-1} by adding the midpoints to every pair of vertices belonging to the same triangle $\psi_{i|(n-1)}(\Gamma)$ of size $2^{-(n-1)}$ in V_{n-1}. Moreover, $\Gamma \subset \Psi(\Gamma)$. So the sequence $V_0, V_1, \ldots, V_n, \ldots$ is monotone increasing. See Fig.5.5.

Since when $D = 2, \alpha = 2$ and $N = 3$, thus $\Gamma = \{z_1, z_2, z_3\}$. Let O be the open triangle with vertices z_1, z_2 and z_3. Then we can check that $\{\psi_1, \psi_2, \psi_3\}$ satisfies the o.s.c.. Hence, by applying Corollary 3.4.1, we have $d_{\mathscr{H}}(K) = \frac{\log 3}{\log 2} \approx 1.59$.

Now we can perform a similar construction. Let $D = 2, \alpha = 3$ and $N = 6$. The 6 contractive similitudes carry the unit triangle of vertices Γ into each one of the 6 "upward facing" triangles obtained by deleting the 3 "downward facing" triangles. See Fig.5.6. Constructing the increasing sequence $V_0, V_1, \ldots, V_n, \ldots$ as in the dyadic case leads to $K = cl\left(\bigcup_{n=0}^{\infty} V_n\right)$. Such a K is also a Sierpiński gasket in $\mathbb{R}^D, D = 2$. By choosing the same open set O as in dyadic case, the Hausdorff dimension of the triadic Sierpiński gasket is $d_{\mathscr{H}}(K) = \frac{\log 6}{\log 3} \approx 1.63$.

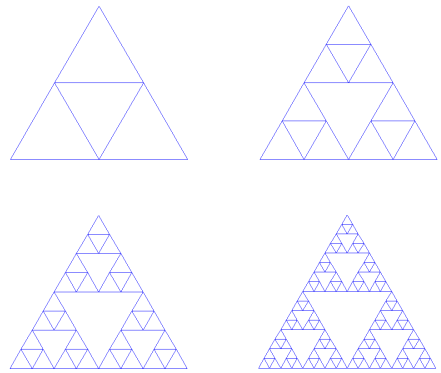


Figure 5.5: Sierpiński gasket $\alpha = 2$

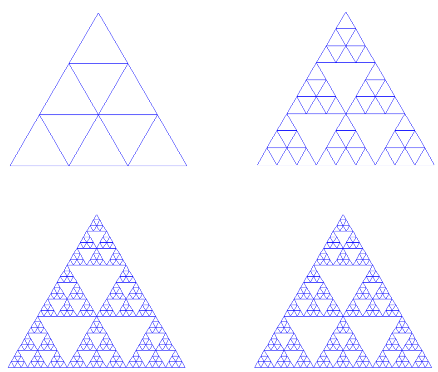


Figure 5.6: Sierpiński gasket $\alpha = 3$

In fact, we can construct a whole family of *Sierpiński curves* for integers $\alpha \geq 2$ in $\mathbb{R}^2$, by choosing $N = \alpha(\alpha+1)/2$ contractive similitudes which map the unit triangle into N "upward facing" triangles of side α^{-1}. Similar constructions can be proceeded in $\mathbb{R}^D$ for $D \geq 2$.

5.3 The Sierpiński Carpet

Consider $D = 2, N = 8$ and $\alpha = 3$. Let $\Gamma = \{z_1, z_2, z_3, z_4\}$ be a set of 4 vertices of a square in $\mathbb{R}^D$. $\{\psi_1, \ldots, \psi_8\}$ is a family of contractive similitudes with $Lip(\psi_i) = \frac{1}{\alpha}$ which carry the square of vertices Γ into each one of the N smaller subsquares obtainded by deleting the central subsquare. Note that $V_0, V_1, \ldots, V_n, \ldots$ is monotone increasing. See Fig. 5.7.

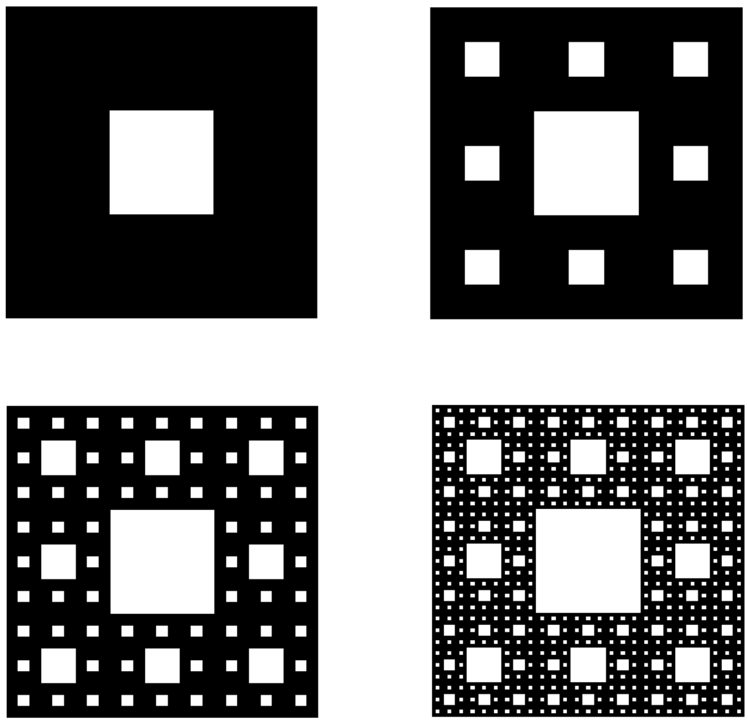


Figure 5.7: Sierpiński carpet

We put $V_0 = \Gamma$ and

$$V_n = \Psi^n(\Gamma), \quad n \geq 0.$$

Then we obtain the *Sierpiński carpet*

$$K = cl\left(\bigcup_{n=0}^{\infty} V_n\right).$$

Let O be the open square of vertices z_1, z_2, z_3, z_4. We can check that $\{\psi_1, \ldots, \psi_8\}$ satisfies the o.s.c.. Hence, by Corrollary 3.4.1, we get the Hausdorff dimension of the *Sierpiński carpet* is $d_{\mathscr{H}}(K) = \frac{\log 8}{\log 3} \approx 1.89$.

Similar constructions can be carried out in $\mathbb{R}^D, D \geq 2$.

Chapter 6

Energy Forms on Self-similar Fractals

In this chapter, our objective is to construct an energy form $E[u]$ on some fractals K, such as the Koch curve and the Sierpiński gasket, which will take the place of the classical Dirichlet integral

$$E[u] = \int_K |\nabla u|^2 dx$$

without making use of the notion of ∇u.

We will only show the construction of energy forms on so-called *nested fractals* (cf. [16]), which is also called the by the physicists *finitely ramified fractals*: that is, it can be disconnected by removing finitely many points. The proofs in this chapter relied very heavily on the fact that the Sierpiński gasket and Koch curve are nested fractals. By contrast, the Sierpiński carpet is not a nested fractal. Thus it is required to employ quietly different techniques.

6.1 Iteration of Maps

Before constructing the energy form, we first give some general notations that will be used.

Let $\psi = \{\psi_1, \ldots, \psi_N\}, \quad N \geq 1$ be a family of N maps $\psi_i : \mathbb{R}^D \to \mathbb{R}^D$. By Ψ we denote the set-to-set mapping

$$\Psi(E) = \bigcup_{i=1}^{N} \psi_i(E), \quad E \subset \mathbb{R}^D, \tag{6.1}$$

and by φ_n for $n \in \mathbb{N}$, the composed set-to-set mapping in $\mathbb{R}^D$

$$\varphi_n = \underbrace{\Psi \circ \cdots \circ \Psi}_{n} \tag{6.2}$$

with $\varphi_0 = Id$.

Let Γ be a non-empty compact subset of $\mathbb{R}^D$ such that

$$\Gamma \subset \Psi(\Gamma). \tag{6.3}$$

Then define the invariant fractal as

$$K = cl\left(\bigcup_{n=0}^{\infty} \varphi_n(\Gamma)\right). \tag{6.4}$$

Now Set

$$W = \otimes_{i=1}^{\infty} \{1, \ldots, N\}$$

to be the set of all sequences of integers $w = (w_1, w_2, \ldots)$ with $1 \leq w_i \leq N$.

$$W_n = \otimes_{i=1}^{n} \{1, \ldots, N\}$$

to be the set of all finite sequences of integers $w|n = (w_1, w_2, \ldots, w_n)$ with $1 \le w_i \le N$, $1 \le i \le n$. For $w \in W$ and $n \in \mathbb{N}$, we set

$$\psi_{w|n} = \psi_{w_1} \circ \cdots \circ \psi_{w_n}$$

The subsets

$$K_{w|n} = \psi_{w|n}(K)$$

of K are called *n-complexes* and the sets

$$\Gamma_{w|n} = \psi_{w|n}(\Gamma)$$

are called *n-cells.*

For $E \subset \mathbb{R}^D$, we have

$$\varphi_n(E) = \bigcup_{w \in W_n} \psi_{w|n}(E).$$

Therefore, if we set $V_0 = \Gamma$ and

$$V_n = \varphi_n(V_0), \quad n \ge 1,$$

then

$$K = cl\left(\bigcup_{n=0}^{\infty} V_n\right).$$

For $n \ge 1$, we have the decompositions of V_n into n-cells

$$V_n = \bigcup_{w|n \in W_n} \Gamma_{w|n}$$

and of K into n-complexes

$$K = \bigcup_{w|n \in W_n} K_{w|n}.$$

Remark 6.1.1. *If Γ is chosen to be a subset of the set of all fixed points of the maps ψ_i, then the sets $V_n = \varphi_n(\Gamma), n \geq 0$ form a monotone increasing sequence of subsets of $\mathbb{R}^D$.*

Now we give the definition of *essential fixed points*. Let $\{z_1, \ldots, z_N\}$ be the set of fixed points of $\psi = \{\psi_1, \ldots, \psi_N\}$. If $p \in \{z_1, \ldots, z_N\}$, there exists $q \in \{z_1, \ldots, z_N\}$, $q \neq p$, and $\psi_i(p) = \psi_j(q)$, $i \neq j$, then p is called an *essential fixed point* of ψ. Essential fixed points are important because they tell us how the different parts of the fractal are put together; inessential fixed points serve no such purpose.

6.2 Energy Forms on Sierpiński Gasket

We consider the "dyadic" Sierpiński gasket K in $\mathbb{R}^D, D \geq 2$, with $\alpha = 2$ and $N = D + 1$. Recall notations in Section 5.2:

$\psi = \{\psi_1, \ldots, \psi_N\}$ is a family of similitudes of K. Let $\Gamma = \{z_0, \ldots, z_D\}$ be the set of vertices of an equilateral unit simplex in $\mathbb{R}^D$, where Γ is a subset of the set of all fixed points of maps ψ_i, for $i = 1 \ldots N$. Then

$$V_0 = \Gamma \subset V_1 = \Psi(\Gamma) \subset \cdots \subset V_n = \Psi^n(\Gamma) \subset \cdots$$

$$V^\infty = \bigcup_{n=0}^{\infty} V_n, \qquad K = cl(V^\infty).$$

For arbitrary $u : V^\infty \to \mathbb{R}$, we define

$$E_0[u] = \frac{1}{2} \sum_{\xi, \eta \in \Gamma} |u(\xi) - u(\eta)|^2, \tag{6.5}$$

and

$$E_1[u] = \rho \sum_{i=1}^{N} E_0[u \circ \psi_i], \tag{6.6}$$

where ρ is a renormalization factor of the energy form to be determined later. Then we have

$$\begin{aligned} E_2[u] =& \rho \sum_{i=1}^{N} E_1[u \circ \psi_i] \\ =& \rho^2 \sum_{w_1=1}^{N} \sum_{w_2=1}^{N} E_0[u \circ \psi_{w_1} \circ \psi_{w_2}] \\ =& \rho^2 \sum_{w|2 \in W_2} E_0[u \circ \psi_{w|2}], \end{aligned}$$

so for $n \geq 1$

$$E_n[u] = \rho^n \sum_{w|n \in W_n} E_0[u \circ \psi_{w|n}], \tag{6.7}$$

or more explicitly,

$$E_n[u] = \rho^n \sum_{w|n \in W_n} \frac{1}{2} \sum_{\xi, \eta \in \Gamma} |u(\psi_{w|n}(\xi)) - u(\psi_{w|n}(\eta))|^2. \tag{6.8}$$

Now come back to $\rho > 0$, which is chosen according to the *Gauss variational principle* stated as

$$\min_{u|(V_1 - V_0)} E_1[u] = E_0[u]. \tag{6.9}$$

For instance, when $D=2$, we denote the values of u on Γ by

$$u(z_0)=A, \qquad u(z_1)=B, \qquad u(z_2)=C,$$

and on $V_1 - V_0$ by

$$u\left(\frac{z_0+z_1}{2}\right)=c, \quad u\left(\frac{z_1+z_2}{2}\right)=a, \quad u\left(\frac{z_2+z_0}{2}\right)=b.$$

Lemma 6.2.1. *Let A, B, C be real constants. Then*

$$\begin{aligned}
&\min_{a,b,c}(|A-c|^2+|c-b|^2+|b-A|^2 \\
&+|c-B|^2+|B-a|^2+|a-c|^2 \\
&+|b-a|^2+|a-C|^2+|C-b|^2) \\
&=\frac{3}{5}\left\{|A-B|^2+|B-C|^2+|C-A|^2\right\}.
\end{aligned}$$

The minimizing $\bar{a}, \bar{b}, \bar{c}$ are

$$\bar{a}=\frac{A+2B+2C}{5}, \quad \bar{b}=\frac{2A+B+2C}{5}, \quad \bar{c}=\frac{2A+2B+C}{5}. \tag{6.10}$$

By Lemma 6.2.1, we have

$$\rho=\frac{5}{3}.$$

It can be seen that, in order to calculate ρ, it is sufficient to apply this principle only between $E_0[u]$ and $E_1[u]$, which requies solving a quadratic minimization problem. In the general case $D \geq 1$, by solving a linear systerm of equations, the value of ρ determined by the Gauss variational principle is

$$\rho = \frac{N+2}{N} = \frac{D+3}{D+1}. \tag{6.11}$$

For details, see Rammal [24], Fukushima-Shima [9]. In fact, there is another way to determine the value of ρ, which is based on *decimation*(cf. [20]).

Note that only the restrictions $u = u|V_n$ of u to V_n enters the expression $E_n[u]$ and

$$E_0[u|V_0] \leq E_1[u|V_1] \leq \cdots \leq E_n[u|V_n] \leq \cdots . \tag{6.12}$$

We now define the form

$$E[u] = \sup_{n \geq 0} E_n[u|V_n] \tag{6.13}$$

on the domain

$$D_E^\infty = \left\{ u : V^\infty \to \mathbb{R} : \sup_{n \geq 0} E_n[u|V_n] < +\infty \right\} \tag{6.14}$$

Note that the equality of 6.12 holds everywhere if $\bar{u}$ is the function obtained by starting with $\bar{u}|V_0 = \{A, B, C\}$ and extending $\bar{u}$ from V_0 to V_1, by defining $\bar{u}(p)$ at each dyadic $p \in V_1 - V_0$ to be the "average values"

$$\left\{ \frac{A+2B+2C}{5}, \frac{2A+B+2C}{5}, \frac{2A+2B+C}{5} \right\}.$$

Do the same extension from V_{n-1} to V_n, by defining $\bar{u}$ at each new dyadic point, which belongs to the same triangle with vertices $\Gamma_{w|n-1}$, to be the "average values" of $\bar{u}$ at $\Gamma_{w|n-1}$(cf. [28]). We say that such a $\bar{u}$ on V^∞ is the *harmonic extension* of

$u|V_0$, which keeps energy stationary. Hence, $D_E^\infty \neq \emptyset$, as it contains the harmonic extension of $u|V_0$.

The following estimate shows that each $u \in D_E^\infty$ admits a unique continuous extension to $K = cl(V^\infty)$.

Lemma 6.2.2. *There exists a constant c such that for every $u : V^\infty \to \mathbb{R}$ and for arbitrary p and q in V^∞, the following estimate holds:*

$$|u(p) - u(q)| \leq c\sqrt{\sup_{n\geq 0} E_n[u|V_n]}|p-q|^{\beta_{Eucl}} \tag{6.15}$$

where

$$\beta_{Eucl} = \frac{1}{2}\frac{\log\rho}{\log\alpha} = \frac{1}{2}\frac{\log((D+3)/(D+1))}{\log 2}. \tag{6.16}$$

We will use the following properties of the Sierpiński gasket to prove the lemma. For the proof of these properties, see reference [21].

Property 6.2.1. *(1) There exists a $\gamma > 0$ such that $K_{i|m} \cap K_{j|m} = \emptyset$ implies $dist(K_{i|m}, K_{j|m}) \geq \gamma\alpha^{-m}$ for every m, (2) If $i|m \neq j|m$, then $K_{i|m} \cap K_{j|m} = \Gamma_{i|m} \cap \Gamma_{j|m}$.*

Proof. (Lemma 6.2.2)

Let $p, q \in V^\infty \subset K$. Since $K = \bigcup_{i|m \in W_m} K_{i|m}$, thus $p \in K_{i|m}$ and $q \in K_{j|m}$ for some $i|m, j|m \in W_m$.

Assume that $|p-q| < \gamma \leq 1$. Then $\exists m \geq 0$ such that

$$\gamma\alpha^{-(m+1)} \leq |p-q| \leq \gamma\alpha^{-m} \tag{6.17}$$

So $dist(K_{i|m}, K_{j|m}) \leq |p-q| < \gamma\alpha^{-m}$, which implies $K_{i|m} \cap K_{j|m} \neq \emptyset$ by property (1). Then, by property (2), we have $\Gamma_{i|m} \cap \Gamma_{j|m} \neq \emptyset$. Thus $\exists a \in \Gamma_{i|m} \cap \Gamma_{j|m}$ such that

$$a = \psi_{i|m}(\xi) = \psi_{j|m}(\eta) \tag{6.18}$$

where $\xi, \eta \in \Gamma$.

Consider $n \geq m$. There exists the smallest $n \geq m$ such that $p, q \in V_n$. Then $p = \psi_{i|n}(\bar{\xi})$ and $q = \psi_{j|n}(\bar{\eta})$ where $\bar{\xi}, \bar{\eta} \in \Gamma$.

Now we need to construct a chain of points connecting p to q "from two sides". Start with

$$p = \psi_{i|n}(\bar{\xi}) = \psi_{i_1 \ldots i_m i_{m+1} \ldots i_n}(\bar{\xi}) =: x_n$$

Let

$$x_{n-1} = \psi_{i|n-1}(\bar{\xi}) = \psi_{i_1 \ldots i_m i_{m+1} \ldots i_{n-1}}(\bar{\xi})$$

$$x_{n-k} = \psi_{i|n-k}(\bar{\xi})$$

where $0 \leq k \leq n - m$. Now we have points $x_n, x_{n-1}, ..., x_m$. Then insert point a by defining $x_{m-1} := a = \psi_{i|m}(\xi)$.

Doing the same starting with $y_n = q$. Let $y_{n-k} = \psi_{j|n-k}(\bar{\eta})$ where $0 \leq k \leq n-m$. Insert $y_{m-1} = a = \psi_{j|m}(\eta)$.

We have constructed a chain:

$$p = x_n, x_{n-1}, ..., x_m, x_{m-1} = a = y_{m-1}, y_m, ..., y_n = q.$$

with a property that two consecutive points in the chain belong to the same cell.

Check for $k = 0$. Let $\bar{\xi}$ be the fixed point of ψ_{i_0} , so $x_{n-1} = \psi_{i_1 \ldots i_{n-1} i_0}(\bar{\xi})$. If $i_0 = i_n$, then $x_n = x_{n-1}$. If $i_0 \neq i_n$, then $\psi_{i_n}(\bar{\xi}) = \psi_{i_0}(\bar{\bar{\xi}})$ for some $\bar{\bar{\xi}} \in \Gamma$. So $x_n = \psi_{i_1 \ldots i_n}(\bar{\xi}) = \psi_{i_1 \ldots i_{n-1} i_0}(\bar{\bar{\xi}})$. Therefore $x_n, x_{n-1} \in \Gamma_{i_1 \ldots i_{n-1} i_0}$.

Now we start to estimate $|u(p)-u(q)|$. By the chain constructed above, we have

$$|u(p)-u(q)|^2 \leq \sum_{k=0}^{n-m} 2^{n-m+1}\left[|u(x_{n-k})-u(x_{n-k-1})|^2 + |u(y_{n-k})-u(y_{n-k-1})|^2\right].$$

Since $\bar{\xi} = \psi_{i_0}(\bar{\xi})$ with $\psi_{i_{n-k}}(\bar{\xi}) = \psi_{i_0}(\bar{\bar{\xi}})$, that

$$|u(x_{n-k})-u(x_{n-k-1})|^2 = |u\left(\psi_{i|n-k-1}\psi_{i_{n-k}}(\bar{\xi})\right) - u\left(\psi_{i|n-k-1}\psi_{i_0}(\bar{\xi})\right)|^2$$

$$= |u\left(\psi_{i|n-k-1}\psi_{i_0}(\bar{\bar{\xi}})\right) - u\left(\psi_{i|n-k-1}\psi_{i_0}(\bar{\xi})\right)|^2$$

$$\leq \sum_{i|n-k} |u\left(\psi_{i|n-k}(\bar{\bar{\xi}})\right) - u\left(\psi_{i|n-k}(\bar{\xi})\right)|^2$$

$$\leq \sum_{i|n-k}\left\{\frac{1}{2}\sum_{\xi',\eta'}|u\left(\psi_{i|n-k}(\xi')\right) - u\left(\psi_{i|n-k}(\eta')\right)|^2\right\}$$

Multiply both sides by ρ^{n-k} to obtain

$$\rho^{n-k}|u(x_{n-k})-u(x_{n-k-1})|^2 \leq E_{n-k}[u].$$

Clearly, the same result holds for terms with y. So we get

$$\begin{aligned}
|u(p)-u(q)|^2 &\leq 2^{n-m+2}\sum_{k=0}^{n-m}\rho^{k-n}E_{n-k}[u] \\
&\leq 2^{n-m+2}\rho^{-n}E_n[u]\sum_{k=0}^{n-m}\rho^k \\
&= 2^{n-m+2}\rho^{-n}E_n[u]\frac{\rho^{n-m+1}-1}{\rho-1} \\
&\leq \frac{4\cdot 2^{n-m}}{\rho-1}E_n[u]\rho^{1-m}
\end{aligned}$$

Since $\rho^{1-m} = \alpha^{(1-m)(\log_\alpha \rho)}$. Let $\beta = \frac{\log \rho}{2 \log \alpha}$, and by equation 6.17, we have

$$|u(p) - u(q)|^2 \leq \frac{4\alpha^{4\beta}}{\gamma^{2\beta}(\rho - 1)} 2^{n-m} E_n[u] |p - q|^{2\beta}$$

Finally we have

$$|u(p) - u(q)| \leq c \sqrt{\sup_{n \geq 0} E_n[u]} |p - q|^\beta.$$

$\square$

From the estimate in Lemma 6.2.2, we know that u is uniformly continuous on V^∞. As $K = cl(V^\infty)$, we have the following corollary.

Corollary 6.2.1. *Every function $u \in D_E^\infty$ can be uniquely extended to a continuous function on K.*

We continue to denote the extension by u and define the energy form

$$E[u] = \lim_{n \to \infty} E_n[u] \tag{6.19}$$

on the domain

$$D_E = \left\{ u \in C(K) : \sup_{n \geq 0} E_n[u|V_n] < +\infty \right\}. \tag{6.20}$$

Moreover, for every $u \in D_E$, the estimate in Lemma 6.2.2 will hold, by which we find that $D_E \subset C^{0,\beta_{Eucl}}(K)$.

Lemma 6.2.3. *D_E is complete under the norm*

$$||u||_{D_E} = (||u||^2_{L^2(K,\mu)} + E[u])^{1/2} \tag{6.21}$$

Proof. Choose a Cauchy sequence $\{u_n\}$ in D_E such that

$$||u_n - u_m||_{D_E} = \Big(||u_n - u_m||^2_{L^2(K,\mu)} + E[u_n - u_m]\Big)^{1/2} \to 0$$

for $n, m \to \infty$. Then we have

$$||u_n - u_m||^2_{L^2(K,\mu)} \to 0$$

$$E[u_n - u_m] \to 0.$$

Thus we have $||u_n||_{L^2(K,\mu)} \leq C_1$ and $E[u_n] \leq C_2$, because Cauchy sequences are bounded.

First we show that $u_n(x)$ is uniformly bounded on K.

For any $x, y \in K$, we have

$$\begin{aligned} |u_n(x)| &\leq |u_n(x) - u_n(y)| + |u_n(y)| \\ &\leq c\sqrt{E[u_n]}|x-y|^\beta + |u_n(y)| \\ &\leq cC_2 diam(K)^\beta + |u_n(y)| \\ &\leq cC_2 + |u_n(y)| \end{aligned}$$

where c, C_2 are constant. As $\mu(K) = \int_K d\mu = 1$, integrating on both sides in $\mu(dy)$ gives

$$|u_n(x)| \leq cC_2 + \int_K |u_n(y)| d\mu(y)$$

By Schwarz inequality,

$$\begin{aligned}|u_n(x)| &\leq cC_2 + \mu(K)^{1/2}\left(\int_K |u_n(y)|^2 d\mu(y)\right)^{1/2} \\ &\leq cC_2 + C_1^{1/2}\end{aligned}$$

where C_1 is constant.

Additionally, it can be proved that the functions $u_n(x)$ are equicontinuous, since for any $x, y \in K$, we have

$$|u_n(x) - u_n(y)| \leq c\sqrt{E[u_n]}|x-y|^\beta \leq cC_2 diam(K)^\beta \leq cC_2.$$

Hence, $\{u_n(x)\}$ is uniformly bounded and equicontinuous on K. By Ascoli-Arzelá theorem, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in C(K)$ such that

$$||u_{n_k} - u||_\infty \to 0$$

for $k \to \infty$. It follows that $u \in L^2(K,\mu)$ as $C(K) \subset L^2(K,\mu)$, and

$$||u_n - u||_{L^2(K,\mu)} \to 0 \tag{6.22}$$

for $n \to \infty$.

Now we want to prove that $u \in D_E$, and $E[u_n - u] \to 0$ as $n \to \infty$.

Since $E_k[u_n - u]$ is a finite sum, that for a fixed n, we have

$$E_k[u_n - u] = \lim_{m\to\infty} E_k[u_n - u_m] \leq \lim_{m\to\infty} E[u_n - u_m].$$

Let $k \to \infty$, then

$$E[u_n - u] \leq \lim_{m\to\infty} E[u_n - u_m]$$

$$\limsup_{n\to\infty} E[u_n - u] \leq \lim_{n,m\to\infty} E[u_n - u_m] = 0$$

which implies

$$\lim_{n\to\infty} E[u_n - u] = 0.$$

Therefore, we proved that there is a $u \in D_E$ such that

$$||u_n - u||_{D_E} = \Big(||u_n - u||^2_{L^2(K,\mu)} + E[u_n - u]\Big)^{1/2} \to 0$$

for $n \to \infty$, i.e., the completeness of D_E. $\square$

Lemma 6.2.4. *D_E is dense in $C(K)$.*

For the proof, see reference [22].

Now we define the space $H^1(K)$ to be the completion of D_E in the norm

$$||u||_{H^1} = (||u||^2_{L^2(K,\mu)} + E[u])^{1/2}$$

and extend $E[u]$ to the completed space $H^1(K)$.

We obtain the bilinear form $E(u, v)$ with domain $H^1(K)$ by

$$E(u,v) = \frac{1}{2}\left\{E[u+v] - E[u] - E[v]\right\} = \frac{1}{4}\left\{E[u+v] - E[u-v]\right\}, u, v \in H^1(K)$$

i.e., replace the quadratic term $|u(\psi_{w|n}(\xi)) - u(\psi_{w|n}(\eta))|^2$ by the bilinear term $(u(\psi_{w|n}(\xi)) - u(\psi_{w|n}(\eta)))(v(\psi_{w|n}(\xi)) - v(\psi_{w|n}(\eta)))$ in the definition of $E_n[u]$ and

$$E(u,v) = \sup_{n\geq 0} E_n(u|V_n, v|V_n) = \lim_{n\to\infty} E_n(u|V_n, v|V_n). \tag{6.23}$$

so that $E(u, v)$ is a closed, symmetric bilinear form with dense domain $H^1(K)$ in $L^2(K, \mu)$.

The space $H_0^1(K)$ is the space of all functions $u \in H^1(K)$ such that $u|\Gamma = 0$. By the representation theory of closed symmetric billinear forms (see F.2), there exists a self-adjoint operator Δ, defined with domain D_Δ dense in $H_0^1(K)$, such that

$$E(u, v) = -\int_K (\Delta u) v d\mu \tag{6.24}$$

for every $u \in D_\Delta$ and $v \in H_0^1(K)$.

6.3 Energy Forms on Koch Curve

We first show a lemma in the following elementary minimization problems, which will play an important role in the construction of energy form on Koch curve.

Lemma 6.3.1. *Let A, B be real constants. Then*

$$\min_{a,b,c} \left\{ |A - a|^2 + |a - c|^2 + |c - b|^2 + |b - B|^2 \right\} = \frac{1}{4} |A - B|^2.$$

The minimizing $\bar{a}, \bar{b}, \bar{c}$ are given by

$$\bar{a} = \frac{3A + B}{4}, \quad \bar{b} = \frac{A + 3B}{4}, \quad \bar{c} = \frac{A + B}{2}.$$

Let $D = 2, \alpha = 3, N = 4$. $\{\psi_1, \ldots, \psi_4\}$ is a family of contractive similitudes. In complex notation, $z = x_1 + ix_2$:

$$\psi_1(z) = \frac{z}{3}, \qquad \psi_2(z) = \frac{z}{3} e^{i\frac{\pi}{3}} + \frac{1}{3}$$

$$\psi_3(z) = \frac{z}{3}e^{-i\frac{\pi}{3}} + \frac{1}{2} + i\frac{\sqrt{3}}{6}, \qquad \psi_4(z) = \frac{z}{3} + \frac{2}{3}.$$

Let $z_0 = (0,0), z_1 = (1,0)$. Put $\Gamma = \{z_0, z_1\}$ and

$$V_n = \Psi^n(\Gamma), \quad n \geq 0.$$

with $V_0 = \Gamma$ and $V^\infty = \bigcup_{n=0}^\infty V_n$. Then the *Koch curve* is the compact set

$$K = cl\,(V^\infty)\,.$$

For arbitrary $u : V^\infty \to \mathbb{R}$, we define

$$E_0[u] = \frac{1}{2}\sum_{\xi,\eta\in\Gamma} |u(\xi) - u(\eta)|^2,$$

and for $n \geq 1$

$$E_n[u] = \rho^n \sum_{w|n\in W_n} E_0[u \circ \psi_{w|n}],$$

where $\rho > 0$ is chosen according to the *Gauss variational principle*:

$$\min_{u|(V_1 - V_0)} E_1[u] = E_0[u].$$

If we denote the values of u on $V_0 = \Gamma$ by

$$u(z_0) = A, \qquad u(z_1) = B$$

and the values of u on $V_1 - V_0$ by

$$u(z_2) = a, \qquad u(z_3) = c, \qquad u(z_4) = b,$$

then by Lemma 6.3.1, we find that

$$\min_{u|(V_1 - V_0)} E_1[u] = \rho \frac{1}{4} E_0[u]$$

Therefore the variational principle uniquely determines the value

$$\rho = 4.$$

Similar to the construction on Sierpiński gaskets, we define the form

$$E[u] = \sup_{n \geq 0} E_n[u|V_n]$$

on the domain

$$D_E^\infty = \left\{ u : V^\infty \to \mathbb{R} : \sup_{n \geq 0} E_n[u|V_n] < +\infty \right\}$$

where $D_E^\infty \neq \emptyset$. We can also get a similar estimate as been shown in Lemma 6.2.2.

Lemma 6.3.2. *There exists a constant c such that for every $u : V^\infty \to \mathbb{R}$ and for arbitrary p and q in V^∞, the following estimate holds:*

$$|u(p) - u(q)| \leq c \sqrt{\sup_{n \geq 0} E_n[u|V_n]} |p - q|^\beta$$

where

$$\beta = \frac{1}{2} \frac{\log \rho}{\log \alpha} = \frac{1}{2} \frac{\log 4}{\log 3}.$$

Now extend the energy form $E[u]$ onto the domain

$$D_E = \left\{ u \in C(K) : E[u] = \sup_{n \geq 0} E_n[u|V_n] < +\infty \right\}.$$

Furthermore, define the space $H^1(K)$ to be the completion of D_E in the norm

$||u||_{H^1}$ and extend $E[u]$ to the completed space $H^1(K)$.

Then we obtain a closed symmetric bilinear form $E(u,v)$ with dense domain $H^1(K)$ in $L^2(K,\mu)$. By the representation theory, there exists a self-adjoint operator Δ, defined with domain D_Δ dense in $H_0^1(K)$, such that

$$E(u,v) = -\int_K (\Delta u)v d\mu$$

for every $u \in D_\Delta$ and $v \in H_0^1(K)$.

Remark 6.3.1. *Notice a special case that, when we choose $\alpha = 4$, the Koch curve becomes a segment, such as the interval $[0,1]$. The relative energy form becomes a "dyadic" energy.*

Chapter 7

Fractal Mixtures

In this chapter, unlike those described so far, we will investigate more general models which can be seen as mixtures of self-similar fractals. They are constructed by the general iterated maps system. Furthermore, after showing some asymptotic properties, we will look at how to construct the volume measures and energy forms on certain fractal mixtures, such as irregular Sierpiński gaskets.

7.1 General Iteration of Maps

Let A be a finite set of integers $a \geq 2$. For $a \in A$, let

$$\psi^{(a)} = \left\{ \psi_1^{(a)}, \ldots, \psi_{N_a}^{(a)} \right\}$$

be a family of $N_a \geq 2$ contractive similitudes in $\mathbb{R}^D$. Denote $\Psi^{(a)}$ as a set-to-set mapping in $\mathbb{R}^D$ such that

$$\Psi^{(a)}(E) = \bigcup_{i=1}^{N_a} \psi_i^{(a)}(E), \quad E \subset \mathbb{R}^D$$

Let $\Xi = A^{\mathbb{N}}$ be the set of sequence $\xi = (\xi_1, \xi_2, \ldots)$ in A. For $n \in \mathbb{N}$, denote $\varphi_n^{(\xi)}$ as a set-to-set mapping in $\mathbb{R}^D$ such that

$$\varphi_n^{(\xi)} = \Psi^{(\xi_1)} \circ \cdots \circ \Psi^{(\xi_n)}$$

with $\varphi_0^{(\xi)} = Id$.

Let Γ be a nonempty compact subset of $\mathbb{R}^D$, $\Gamma \subset \Psi^{(a)}(\Gamma)$, then the fractal $K^{(\xi)}$ associated with ξ is defined by

$$K^{(\xi)} = cl\left(\bigcup_{n=0}^{\infty} \varphi_n^{(\xi)}(\Gamma)\right)$$

Define the left shift operator θ on Ξ: $\theta\xi = (\xi_2, \xi_3, \ldots)$ for $\xi = (\xi_1, \xi_2, \ldots)$. The family $\left\{\varphi_n^{(\xi)}\right\}_{\xi \in \Xi}$ has the property

$$\varphi_n^{(\xi)} = \varphi_m^{(\xi)} \circ \varphi_{n-m}^{(\theta^m \xi)}$$

for $n \geq m \geq 1$.

Note that the set $K^{(\xi)}$ is not in general invariant, but the family $\left\{K^{(\xi)}\right\}_{\xi \in \Xi}$ does satisfy the property

$$K^{(\xi)} = \varphi_n^{(\xi)}\left(K^{(\theta^n \xi)}\right), \quad \xi \in \Xi, n \in \mathbb{N}.$$

For $\xi \in \Xi$, let

$$W^{(\xi)} = \otimes_{i=1}^{\infty} \{1, \ldots, N_{\xi_i}\}$$

be the set of all sequences of integers $w = (w_1, w_2, \ldots)$ with $1 \leq w_i \leq N_{\xi_i}$

$$W_n^{(\xi)} = \otimes_{i=1}^{n} \{1, \ldots, N_{\xi_i}\}$$

be the set of all finite sequences of integers $w|n = (w_1, w_2, \ldots, w_n)$ with $1 \le w_i \le N_{\xi_i}$, $1 \le i \le n$.

For $w \in W^{(\xi)}$ and $n \in \mathbb{N}$, we set

$$\psi_{w|n}^{(\xi)} = \psi_{w_1}^{(\xi_1)} \circ \cdots \circ \psi_{w_n}^{(\xi_n)}$$

The sets

$$K_{w|n}^{(\xi)} = \psi_{w|n}^{(\xi)} \left(K^{(\theta^n \xi)} \right)$$

are called *n-complexes*, and the sets

$$\Gamma_{w|n}^{(\xi)} = \psi_{w|n}^{(\xi)}(\Gamma)$$

are called *n-cells*.

Then for $E \subset \mathbb{R}^D$,

$$\varphi_n^{(\xi)}(E) = \bigcup_{w|n \in W_n} \psi_{w|n}^{(\xi)}(E).$$

Therefore, if we set $V_0 = \Gamma$ and

$$V_n^{(\xi)} = \varphi_n^{(\xi)}(V_0), \quad n \ge 1,$$

then

$$K^{(\xi)} = cl \left(\bigcup_{n=0}^{\infty} V_n^{(\xi)} \right).$$

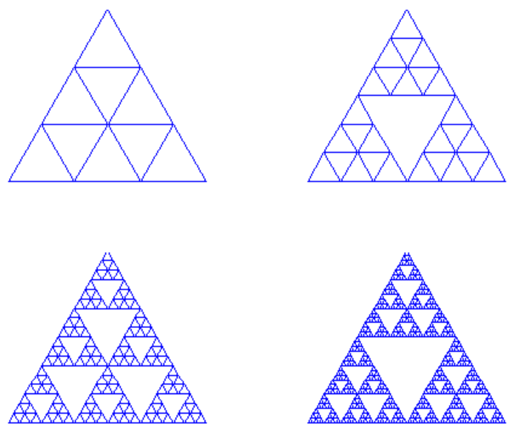
For $n \ge 1$, we have the decompositions of $V_n^{(\xi)}$ into n-cells

$$V_n^{(\xi)} = \bigcup_{w|n \in W_n^{(\xi)}} \Gamma_{w|n}^{(\xi)}$$

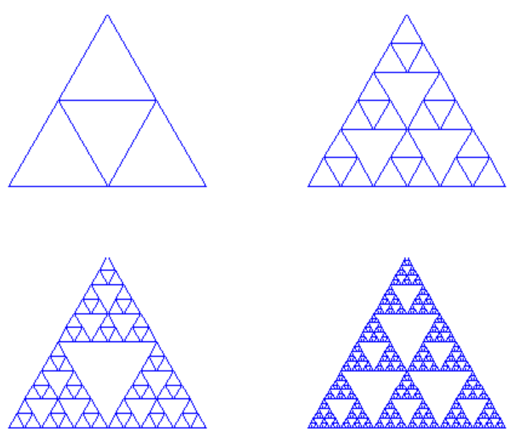
and of $K^{(\xi)}$ into n-complexes

$$K^{(\xi)} = \bigcup_{w|n \in W_n^{(\xi)}} K_{w|n}^{(\xi)}.$$

Example 7.1.1 (Irregular Sierpiński gasket)**.** *Consider $D = 2$ and $A = \{2, 3\}$. Then we have $N_a = 3$ if $a = 2$, while $N_a = 6$ if $a = 3$. For a fixed finite sequence $\xi = (2, 3, 2, 3)$, we have*



For $\xi = (3, 2, 3, 2)$, we have



From the example above, we can see that the set $K^{(\xi)}$ obviously depends on the specific sequence ξ.

7.2 Construction of Irregular Sierpiński Gaskets

In this section, we simply show how to construct the set of irregular Sierpiński gaskets (i.e. mixtures of Sierpiński gaskets) based on the general iteration of maps.

Let $\Gamma = \{z_0, z_1, z_2\}$ be the set of an equilateral unit simplex in $\mathbb{R}^D$. Let A be a finite set of integers $a \geq 2$.

For example, when $D = 2$, $\Gamma = \left\{(0,0), (1,0), (1/2, \sqrt{3}/2)\right\}$ and $A = 2, 3$.

For $a \in A$, we set $\alpha_a = a$. Consider contractive similitudes

$$\psi^{(a)} = \left\{\psi_1^{(a)}, \ldots, \psi_{N_a}^{(a)}\right\}$$

where

$$\psi^{(a)}(x) = b_i^{(a)} + \alpha_{(a)}^{-1}(x - b_i^{(a)}), \quad x \in \mathbb{R}^D,$$

for $i = 1, \ldots, N_a$, which carry the simplex into each one of the N_a "upward facing" smaller simplices obtained by decomposing the simplex into α_a^D equilateral simplices of side α_a^{-1}. In fact, for every $a \in A$, Γ is the set of the essential fixed points of the family $\psi^{(a)}$. Also note that every family $\psi^{(a)}$, $a \in A$, satisfies the open set condition.

For $\xi \in \Xi = A^{\mathbb{N}}$, let $V_0^{(\xi)} = \Gamma$, then

$$V_0^{(\xi)} = \Gamma \subset V_1^{(\xi)} = \varphi_1^{(\xi)}(\Gamma) \subset \cdots \subset V_n^{(\xi)} = \varphi_n^{(\xi)}(\Gamma) \subset \cdots,$$

Denote $V^{(\xi)} = \bigcup_{n=0}^{\infty} V_n^{(\xi)}$. Finally we get the irregular Sierpiński gasket $K^{(\xi)}$ as

$$K^{(\xi)} = cl\left(V^{(\xi)}\right).$$

7.3 Asymptotic Properties

Consider the mixtures of Sierpiński gasket. Given a family of contractive similitudes $\psi^{(a)} = \left\{\psi_1^{(a)}, \psi_2^{(a)}, \dots \psi_{N_a}^{(a)}\right\}$ in $\mathbb{R}^D$, there exists a constant $\alpha_a \in (1, \infty)$ such that

$$|\psi_i^{(a)}(x) - \psi_i^{(a)}(y)| = \alpha_a^{-1}|x-y|, \quad x, y \in \mathbb{R}^D,$$

for every $i = 1, \dots, N_a$ Assume that they satisfy the so-called *open set condition.* Then for $a \in A$ there exists a unique compact invariant set $K_a = \Psi^{(a)}(K_a)$, and an invariant Hausdorff measure

$$\mu_a(\cdot) = \sum_{i=1}^{N_a} N_a^{-1}\mu_a((\psi^{(a)})^{-1}(\cdot)),$$

and an invariant energy form

$$E_a(u, v) = \sum_{i=1}^{N_a} \rho_a E_a(u \circ \psi^{(a)}, v \circ \psi^{(a)}), \quad u, v \in D_{E_a}.$$

The constants

$$\alpha_a, N_a, \rho_a, \quad a \in A,$$

are the basic scaling factors for length, volume, and energy on the fractal K_a.

For a fixed sequence $\xi \in \Xi = A^{\mathbb{N}}$, the mixtures of Sierpiński gasket $K^{(\xi)}$ is now constructed by the maps $\Psi^{(a)}$ associated with $\psi^{(a)}, a \in A$, as described in the first section.

We set $\alpha^{(\xi)}(0) = N^{(\xi)}(0) = \rho^{(\xi)}(0) = 1$ and for $n \geq 1$,

$$\alpha^{(\xi)}(n) = \prod_{n}^{i=1} \alpha_{\xi_i}, \quad N^{(\xi)}(n) = \prod_{i=1}^{n} N_{\xi_i}, \quad \rho^{(\xi)}(n) = \prod_{i=1}^{n} \rho_{\xi_i}; \tag{7.1}$$

moreover,

$$\delta^{(\xi)}(n) = \frac{1}{2}\frac{\log(N^{(\xi)}(n)\rho^{(\xi)}(n))}{\log \alpha^{(\xi)}(n)} \tag{7.2}$$

and

$$\nu^{(\xi)}(n) = 2\frac{\log N^{(\xi)}(n)}{\log(N^{(\xi)}(n)\rho^{(\xi)}(n))}. \tag{7.3}$$

The parameter $\delta^{(\xi)}(n)$ is the one that restores the "Einstein ralation"

$$N^{(\xi)}(n)\rho^{(\xi)}(n) = \alpha^{(\xi)}(n)^{2\delta^{(\xi)}(n)}. \tag{7.4}$$

Remark 7.3.1. *Two quantities above: $\delta^{(\xi)}(n)$ and $\nu^{(\xi)}(n)$, will play the role of an effective index of the ramification existing in our fractal at the* n*th length scale and the intrinsic homogeneous dimension of $K^{(\xi)}$ respectively.*

Definition 7.3.1. *For $\xi \in \Xi$ and $n \geq 1$, we define the frequency of each $a \in A$ in ξ by*

$$h_a^{(\xi)}(n) = \frac{1}{n}\sum_{h=1}^{n} 1_{\{\xi_i = a\}}. \tag{7.5}$$

In addition, $h_a^{(\xi)}(n)$ also gives the frequency with which the family $\psi^{(a)}$ occurs up to step n of the iteration.

Assume that for $\xi \in \Xi$, there exists constants $p_a \geq 0, a \in A$, with $\sum_{a\in A} p_a$, such that

$$h_a^{(\xi)}(n) \to p_a \quad \text{ as } n \to \infty \text{ for each } a \in A, \tag{7.6}$$

$$|h_a^{(\xi)}(n) - p_a| \leq \frac{c}{n}, \quad n \geq 1, a \in A, \tag{7.7}$$

where c is a constant.

We set

$$\alpha^{(\xi)} = \prod_{a\in A} \alpha_a^{p_a}, \quad N^{(\xi)} = \prod_{a\in A} N_a^{p_a}, \quad \rho^{(\xi)} = \prod_{a\in A} \rho_a^{p_a}. \tag{7.8}$$

By the assumption of asymptotic condition above, we have

$$\left(\alpha^{(\xi)}(n)\right)^{1/n} = \prod_{a\in A} \alpha_a^{h_a^{(\xi)}(n)} \to \alpha^{(\xi)},$$

$$\left(N^{(\xi)}(n)\right)^{1/n} = \prod_{a\in A} N_a^{h_a^{(\xi)}(n)} \to N^{(\xi)},$$

$$\left(\rho^{(\xi)}(n)\right)^{1/n} = \prod_{a\in A} \rho_a^{h_a^{(\xi)}(n)} \to \rho^{(\xi)},$$

as $n \to \infty$. Moreover, set

$$\delta = \frac{1}{2}\frac{\sum_a p_a \log(N_a\rho_a)}{\sum_a p_a \log \alpha_a} \tag{7.9}$$

$$\nu = 2\frac{\sum_a p_a \log \alpha_a}{\sum_a p_a \log(N_a\rho_a)} \tag{7.10}$$

Then as $n \to \infty$

$$\delta^{(\xi)}(n) \to \delta, \quad \nu^{(\xi)}(n) \to \nu.$$

7.4 Construction of Measures

We will freely use the notations given in the previous sections. Following Chapter 4, we proceed by describing the volume measure $\mu^{(\xi)}$ on $K^{(\xi)}$. Although $\mu^{(\xi)}$ here is not strictly invariant, we are able to show some similar properties, like those given in section 4.2.

Consider the complete metric space $(\mathcal{M}^1, L)$, the definition of which is first given in section 4.1. Let $\xi = (\xi_1, \xi_2, \ldots) \in \Xi = A^{\mathbb{N}}$. For $\mu \in \mathcal{M}^1$, we set

$$\mathcal{T}^{(\xi_j)}(\mu) = \sum_{i=1}^{N_{\xi_j}} \frac{1}{N_{\xi_j}} \mu \circ (\psi_i^{(\xi_j)})^{-1}, \quad \text{for } j \geq 1 \tag{7.11}$$

$$\mathcal{T}_n^{(\xi)}(\mu) = \mathcal{T}^{(\xi_1)} \circ \mathcal{T}^{(\xi_2)} \circ \cdots \circ \mathcal{T}^{(\xi_n)}(\mu) \tag{7.12}$$

for $n \geq 1$, with $\mathcal{T}_0^{(\xi)}(\mu) = \mu$. By Lemma 4.2.1, we have

$$L(\mathcal{T}^{(\xi_j)}(\mu), \mathcal{T}^{(\xi_j)}(\nu)) \leq N_{\xi_j}^{-1} L(\mu, \nu), \quad \mu, \nu \in \mathcal{M}^1, \tag{7.13}$$

which implies $\mathcal{T}^{(\xi_j)}$ is a contraction map on $\mathcal{M}^1$. Hence, $\mathcal{T}_n^{(\xi)}$ is also a contraction map. Now we denote

$$\mathcal{T}^{(\xi)}(\mu) = \lim_{n \to \infty} \mathcal{T}_n^{(\xi)}(\mu) \tag{7.14}$$

for $\mu \in \mathcal{M}^1$.

Theorem 7.4.1. *Fix $\xi \in \Xi$, for any $\mu \in \mathcal{M}^1$, there exists a unique measure $\mu^{(\xi)} \in \mathcal{M}^1$ such that $\mathcal{T}^{(\xi)}(\mu) = \mu^{(\xi)}$.*

Proof. We first show that $\left\{\mathcal{T}_n^{(\xi)}(\mu)\right\}$ is a Cauchy sequence for a fixed ξ. Since

$$\begin{aligned}
L(\mathcal{T}_n^{(\xi)}(\mu), \mathcal{T}_{n+1}^{(\xi)}(\mu)) &\leq N_{\xi_1}^{-1} L(\mathcal{T}_{n-1}^{(\theta\xi)}(\mu), \mathcal{T}_n^{(\theta\xi)}(\mu)) \\
&\leq N_{\xi_1}^{-1} N_{\xi_2}^{-1} L(\mathcal{T}_{n-2}^{(\theta^2\xi)}(\mu), \mathcal{T}_{n-1}^{(\theta^2\xi)}(\mu)) \\
&\leq \cdots \\
&\leq (N^{(\xi)}(n))^{-1} L(\mathcal{T}_0^{(\theta^n\xi)}(\mu), \mathcal{T}_1^{(\theta^n\xi)}(\mu)) \\
&= (N^{(\xi)}(n))^{-1} L(\mu, \mathcal{T}_1^{(\theta^n\xi)}(\mu)).
\end{aligned}$$

Suppose $m > n$. Then we have

$$\begin{aligned}
L(\mathcal{T}_n^{(\xi)}(\mu), \mathcal{T}_m^{(\xi)}(\mu)) \leq & L(\mathcal{T}_n^{(\xi)}(\mu), \mathcal{T}_{n+1}^{(\xi)}(\mu)) + L(\mathcal{T}_{n+1}^{(\xi)}(\mu), \mathcal{T}_{n+2}^{(\xi)}(\mu)) \\
& + \cdots + L(\mathcal{T}_{m-1}^{(\xi)}(\mu), \mathcal{T}_m^{(\xi)}(\mu)) \\
\leq & (N^{(\xi)}(n))^{-1} L(\mu, \mathcal{T}_1^{(\theta^n\xi)}(\mu)) + (N^{(\xi)}(n+1))^{-1} L(\mu, \mathcal{T}_1^{(\theta^{n+1}\xi)}(\mu)) \\
& + \cdots + (N^{(\xi)}(m-1))^{-1} L(\mu, \mathcal{T}_1^{(\theta^{m-1}\xi)}(\mu))
\end{aligned}$$

Let

$$M = \max_{a \in A} L(\mu, \mathcal{T}^{(a)}(\mu))$$

then

$$L(\mathcal{T}_n^{(\xi)}(\mu), \mathcal{T}_m^{(\xi)}(\mu)) \leq \left[(N^{(\xi)}(n))^{-1} + (N^{(\xi)}(n+1))^{-1} + \cdots + (N^{(\xi)}(m-1))^{-1}\right] M.$$

Let $m, n \to \infty$, then we get

$$L(\mathcal{T}_n^{(\xi)}(\mu), \mathcal{T}_m^{(\xi)}(\mu)) \to 0.$$

Therefore, $\left\{\mathcal{T}_n^{(\xi)}(\mu)\right\}$ is Cauchy. On the other hand, we know the space $(\mathcal{M}^1, L)$ is complete. Thus there exists a unique $\mu^{(\xi)} \in \mathcal{M}^1$ such that

$$\lim_{n\to\infty} \mathcal{T}_n^{(\xi)}(\mu) = \mu^{(\xi)},$$

i.e.,

$$\mathcal{T}^{(\xi)}(\mu) = \mu^{(\xi)}.$$

$\square$

For all $w \in W$, $n \geq 0$, the measure $\mu^{(\xi)}$ is defined to be the unique Radon measure on $K^{(\xi)}$ such that

$$\mu^{(\xi)}(K_{w|n}^{(\xi)}) = N^{(\xi)}(n)^{-1}. \tag{7.15}$$

Obviously, we have

$$\mu^{(\xi)}(K^{(\xi)}) = N^{(\xi)}(0)^{-1} = 1. \tag{7.16}$$

Furthermore, it is not difficult to see that the family of measures $\{\mu^{(\xi)}\}_{\xi\in\Xi}$ satisfies the relation

$$\mu^{(\xi)}(K_{w|n}^{(\xi)}) = \sum_{w|n\in W_n} N^{(\xi)}(n)^{-1}\mu^{(\theta^n\xi)}(\psi_{w|n}^{-1}(K_{w|n}^{(\xi)})), \tag{7.17}$$

for $n \geq 1$. In fact, for $n = 0$, we have

$$\begin{aligned}\sum_{w|n \in W_n} N^{(\xi)}(n)^{-1} \mu^{(\theta^n \xi)}(\psi_{w|n}^{-1}(K^{(\xi)})) &= \sum_{w|n \in W_n} N^{(\xi)}(n)^{-1} \mu^{(\theta^n \xi)}(K^{(\theta^n \xi)}) \\ &= \sum_{w|n \in W_n} N^{(\xi)}(n)^{-1} \\ &= N^{(\xi)}(n) N^{(\xi)}(n)^{-1} \\ &= 1 = \mu^{(\xi)}(K^{(\xi)}).\end{aligned}$$

For $n > 0$, let $n = n_0$ be fixed, then we have

$$\begin{aligned}\sum_{w|n \in W_n} N^{(\xi)}(n)^{-1} \mu^{(\theta^n \xi)}(\psi_{w|n}^{-1}(K_{n_0}^{(\xi)})) &= N^{(\xi)}(n_0)^{-1} \mu^{(\theta^{n_0} \xi)}(\psi_{w|n_0}^{-1}(K_{n_0}^{(\xi)})) \\ &= N^{(\xi)}(n_0)^{-1} \mu^{(\theta^{n_0} \xi)}(K_{n_0}^{(\theta^{n_0} \xi)}) \\ &= N^{(\xi)}(n_0)^{-1} \\ &= \mu^{(\xi)}(K_{n_0}^{(\xi)}).\end{aligned}$$

By the properties shown above, we can also write

$$\int_{K^{(\xi)}} f d\mu^{(\xi)} = \sum_{w|n \in W_n} N^{(\xi)}(n)^{-1} \int_{K^{(\theta^n \xi)}} f \circ \psi_{w|n} d\mu^{(\theta^n \xi)}. \tag{7.18}$$

for every function $f \in L^1(K^{(\xi)}, \mu^{(\xi)})$.

7.5 Energy Forms on Irregular Sierpiński Gaskets

In presenting the construction of the energy form $E^{(\xi)}$, we proceed in the same way based on the *Gauss principle*, as in Section 6.2 (See also [8]).

Continue the notations in Section 7.2 and 7.3. To simplify the notation, we omit reference to ξ in quantities depending on ξ.

Take a function $u : V^{(\xi)} \to \mathbb{R}$. Recall that we should write $u|V_0$, $u|V_1 \ldots$ for the restriction of u to V_0, $V_1 \ldots$. However, we simply write u in all cases for convenience.

Now we proceed the similar process in the case of regular Sierpiński gasket. Define

$$E_0(u,u) = \frac{1}{2} \sum_{x,y\in\Gamma} |u(x) - u(y)|^2. \tag{7.19}$$

Set

$$E_n(u,u) = \rho(n) \sum_{w|n\in W_n} E_0(u \circ \psi_{w|n}, u \circ \psi_{w|n}). \tag{7.20}$$

Then we can write

$$E_n(u,u) = \rho(n) \sum_{w|n\in W_n} \frac{1}{2} \sum_{x,y\in\Gamma} |u(\psi_{w|n}(x)) - u(\psi_{w|n}(y))|^2 \tag{7.21}$$

The choice of $\rho(n)$ above ensures that E_n satisfies the Gauss principle

$$\min_{u|(V_n - V_{n-1})} E_n(u,u) = E_{n-1}(u,u).$$

Recall the process in the case of regular Sierpiński gaskets. We only need to apply the principle between $E_0[u]$ and $E_1[u]$ to find ρ, which is then used in each construction step from $E_{n-1}[u]$ to $E_n[u]$. Now we use the same idea. Note that ρ_a only depends on $a \in A$. Therefore, we can apply different ρ_{ξ_n} in each step from $E_{n-1}[u]$ to $E_n[u]$. Hence we have

$$E_0(u,u) \leq E_1(u,u) \leq \cdots \leq E_n(u,u).$$

Now define the form

$$E(u,u) = \sup_{n\geq 0} E_n(u,u) \tag{7.22}$$

with domain

$$D_E^{(\xi)} = \left\{ u : V^{(\xi)} \to \mathbb{R} : \sup_{n\geq 0} E_n(u,u) < +\infty \right\}. \tag{7.23}$$

Similar to Lemma 6.2.2, the following estimate allows extending each $u \in D_E^{(\xi)}$ to $K = cl\left(V^{(\xi)}\right)$.

Lemma 7.5.1. *There exists a constant c such that for every $u : V^{(\xi)} \to \mathbb{R}$ and for arbitrary p and q in $V^{(\xi)}$, the following estimate holds:*

$$|u(p) - u(q)| \leq c\sqrt{\sup_{n\geq 0} E_n[u|V_n]}|p-q|^\beta \tag{7.24}$$

where $\beta = \frac{\log \rho}{2\log \alpha}$ with $\rho = \min_{a\in A}\{\rho_a\}$ and $\alpha = \max_{a\in A}\{\alpha_a\}$.

Notice that if there is only one ρ, which means we go back to the regular Sierpiński gasket case, then the estimate above will reduce to the one shown in Section 6.2.

We will use the following properties of the irregular Sierpiński gasket.

Property 7.5.1. *(1) There exists a $\gamma > 0$ such that $K_{i|m} \cap K_{j|m} = \emptyset$ implies $dist(K_{i|m}, K_{j|m}) \geq \gamma\alpha^{-1}(m)$ for every m, (2) If $i|m \neq j|m$, then $K_{i|m} \cap K_{j|m} = \Gamma_{i|m} \cap \Gamma_{j|m}$.*

Proof. (Lemma 7.5.1) Let $p, q \in V^{(\xi)} \subset K$. Since $K = \bigcup_{w|m\in W_m} K_{w|m}$, thus $p \in K_{i|m}$ and $q \in K_{j|m}$ for some $i|m, j|m \in W_m$.

Assume that $|p-q| < \gamma \leq 1$. Denote $\alpha = \max_{a\in A} \alpha_a$. Then $\exists m \geq 0$ such that

$$\gamma\alpha^{-(m+1)} \leq \gamma\alpha^{-1}(m+1) \leq |p-q| \leq \gamma\alpha^{-1}(m) \tag{7.25}$$

So $dist(K_{i|m}, K_{j|m}) \leq |p-q| < \gamma\alpha^{-1}(m)$, which implies $K_{i|m} \cap K_{j|m} \neq \emptyset$ by property (1). Then, by property (2), we have $\Gamma_{i|m} \cap \Gamma_{j|m} \neq \emptyset$. Thus $\exists a \in \Gamma_{i|m} \cap \Gamma_{j|m}$ such that

$$a = \psi_{i|m}(x) = \psi_{j|m}(y) \tag{7.26}$$

where $x, y \in \Gamma$.

Consider $n \geq m$. There exists the smallest $n \geq m$ such that $p, q \in V_n$. Then $p = \psi_{i|n}(\bar{x})$ and $q = \psi_{j|n}(\bar{y})$ where $\bar{x}, \bar{y} \in \Gamma$.

Now we need to construct a chain of points connecting p to q "from two sides". Start with

$$p = \psi_{i|n}(\bar{x}) = \psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_m}^{(\xi_m)} \circ \psi_{i_{m+1}}^{(\xi_{m+1})} \circ \cdots \circ \psi_{i_n}^{(\xi_n)}(\bar{\xi}) =: x_n$$

Let

$$x_{n-1} = \psi_{i|n-1}(\bar{x}) = \psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_m}^{(\xi_m)} \circ \psi_{i_{m+1}}^{(\xi_{m+1})} \circ \cdots \circ \psi_{i_{n-1}}^{(\xi_{n-1})}(\bar{x})$$

$$x_{n-k} = \psi_{i|n-k}(\bar{x}) = \psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_{n-k}}^{(\xi_{n-k})}(\bar{x})$$

where $0 \leq k \leq n-m$. Now we have points $x_n, x_{n-1}, ..., x_m$. Then insert point a by defining $x_{m-1} := a = \psi_{i|m}(x)$. For convenience, we denote

$$\psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_m}^{(\xi_m)} \circ \psi_{i_{m+1}}^{(\xi_{m+1})} \circ \cdots \circ \psi_{i_n}^{(\xi_n)} = \psi_{i_1 \ldots i_m i_{m+1} \ldots i_n}.$$

Doing the same starting with $y_n = q$. Let $y_{n-k} = \psi_{j|n-k}(\bar{y})$ where $0 \leq k \leq n-m$. Insert $y_{m-1} = a = \psi_{j|m}(y)$.

We have constructed a chain:

$$p = x_n, x_{n-1}, ..., x_m, x_{m-1} = a = y_{m-1}, y_m, ..., y_n = q.$$

with a property that two consecutive points in the chain belong to the same cell.

Check for $k = 0$. Let $\bar{x}$ be the fixed point of $\psi_{i_0}^{(\xi_n)}$, so $x_{n-1} = \psi_{i_1 \dots i_{n-1} i_0}(\bar{x})$. If $i_0 = i_n$, then $x_n = x_{n-1}$. If $i_0 \neq i_n$, then $\psi_{i_n}^{(\xi_n)}(\bar{x}) = \psi_{i_0}^{(\xi_n)}(\bar{\bar{x}})$ for some $\bar{\bar{x}} \in \Gamma$. So $x_n = \psi_{i_1 \dots i_n}(\bar{\xi}) = \psi_{i_1 \dots i_{n-1} i_0}(\bar{\bar{\xi}})$. Therefore $x_n, x_{n-1} \in \Gamma_{i_1 \dots i_{n-1} i_0}$.

Now we start to estimate $|u(p) - u(q)|$. By the chain constructed above, we have

$$|u(p) - u(q)|^2 \leq \sum_{k=0}^{n-m} 2^{n-m+1} \left[|u(x_{n-k}) - u(x_{n-k-1})|^2 + |u(y_{n-k}) - u(y_{n-k-1})|^2 \right].$$

Since $\bar{x} = \psi_{i_0}^{(\xi_{n-k})}(\bar{x})$ with $\psi_{i_{n-k}}(\bar{x}) = \psi_{i_0}^{(\xi_{n-k})}(\bar{\bar{x}})$, that

$$|u(x_{n-k}) - u(x_{n-k-1})|^2 = |u\left(\psi_{i|n-k-1}\psi_{i_{n-k}}(\bar{x})\right) - u\left(\psi_{i|n-k-1}\psi_{i_0}^{(\xi_{n-k})}(\bar{x})\right)|^2$$

$$= |u\left(\psi_{i|n-k-1}\psi_{i_0}^{(\xi_{n-k})}(\bar{\bar{x}})\right) - u\left(\psi_{i|n-k-1}\psi_{i_0}^{(\xi_{n-k})}(\bar{x})\right)|^2$$

$$\leq \sum_{i|n-k} |u\left(\psi_{i|n-k}(\bar{\bar{x}})\right) - u\left(\psi_{i|n-k}(\bar{x})\right)|^2$$

$$\leq \sum_{i|n-k} \left\{ \frac{1}{2} \sum_{x',y'} |u\left(\psi_{i|n-k}(x')\right) - u\left(\psi_{i|n-k}(y')\right)|^2 \right\}$$

Multiply both sides by $\rho(n-k)$ to obtain

$$\rho(n-k)|u(x_{n-k}) - u(x_{n-k-1})|^2 \leq E_{n-k}[u].$$

Clearly, the same result holds for terms with y. So we get

$$|u(p) - u(q)|^2 \leq 2^{n-m+2} \sum_{k=0}^{n-m} \rho^{-1}(n-k) E_{n-k}[u].$$

Now let

$$\rho = \min_{a \in A} \rho_a.$$

Then we have

$$\begin{aligned}
|u(p) - u(q)|^2 &\leq 2^{n-m+2} E_n[u] \sum_{k=0}^{n-m} \rho^{k-n} \\
&= 2^{n-m+2} \rho^{-n} E_n[u] \frac{\rho^{n-m+1} - 1}{\rho - 1} \\
&\leq \frac{4 \cdot 2^{n-m}}{\rho - 1} E_n[u] \rho^{1-m}
\end{aligned}$$

Since $\rho^{1-m} = \alpha^{(1-m)\log_\alpha \rho}$. Let $\beta = \frac{\log \rho}{2 \log \alpha}$, and by equation 7.25, we have

$$|u(p) - u(q)|^2 \leq \frac{4\alpha^{4\beta}}{\gamma^{2\beta}(\rho - 1)} 2^{n-m} E_n[u] |p - q|^{2\beta}$$

Finally we have

$$|u(p) - u(q)| \leq c \sqrt{\sup_{n \geq 0} E_n[u]} |p - q|^\beta.$$

$\square$

Corollary 7.5.1. *Every function $u \in D_E^{(\xi)}$ can be uniquely extended to a continuous function on K.*

We continue to denote the extension by u and define the energy form

$$E[u] = \lim_{n \to \infty} E_n[u] \tag{7.27}$$

on the domain

$$D_E = \left\{ u \in C(K) : \sup_{n \geq 0} E_n[u|V_n] < +\infty \right\}. \tag{7.28}$$

Moreover, for every $u \in D_E$, the estimate in Lemma 7.5.1 will hold, by which we find that $D_E \subset C^{0,\beta}(K)$.

Chapter 8

Future Work

Following the investigations described in this thesis, a number of future works could be taken up:

- We have constructed the irregular Sierpiński gasket by general iteration of contractive similitudes. Then we want to reverse this process through the deconstruction by some proper metric, which will lead to relative inequlity theory, such as Poincaré inequalities, capacity inequalities and Harnack inequalities.

- Spectral analysis on certain fractal mixtures. For instance, describe the eigenvalues of the Laplacian on the irregular Sierpiński gasket, which will be proceeded by constructing the discrete Laplacian on pre-gasket and studying the limit of their eigenvalues. Moreover, discuss the relationship between the Laplacian and the self-adjoint operator associated with the energy form.

- Optimal control problem on fractal mixtures. One direction of this research is to find the optimal sequence ξ, based on which the cost function will obtain its extreme value over mixed fractal-type domains. Another interesting direction is said to be optimal fractal-type domains in an environment, in which some financial principles will also be employed.

Appendix A

Metric Spaces

Definition A.0.1 (Metric Space)**.** *A metric space is a pair* (X, d)*, where* X *is a set and* $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ *is a metric (distance function) on* X *such that for all* $x, y, z \in X$ *we have:*

1. $d(x, y) = 0$ *iff* $x = y$.

2. $d(x, y) = d(y, x)$.

3. $d(x, y) \leq d(x, z) + d(z, y)$.

Definition A.0.2 (Convergence of a sequence)**.** *A sequence* (x_n) *in a metric space* (X, d) *is said to be convergent if there is an* $x \in X$ *such that*

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

i.e., for every $\epsilon > 0$*, there exists an* $N \in \mathbb{Z}^+$ *such that*

$$d(x, x_n) < \epsilon$$

for all $n > N$.

Definition A.0.3 (Cauchy sequence)**.** *A sequence* (x_n) *in a metric space* (X, d) *is said to be Cauchy if for every* $\epsilon > 0$*, there exists an* $N \in \mathbb{Z}^+$ *such that*

$$d(x_m, x_n) < \epsilon$$

for every $m, n > N$*.*

Definition A.0.4 (Completeness)**.** *A metric space* $X = (X, d)$ *is said to be complete if every Cauchy sequence in* X *converges.*

Notice that every convergent sequence is Cauchy.

Definition A.0.5 (Isometric spaces)**.** *Let* $X = (X, d)$ *and* $\tilde{X} = (\tilde{X}, \tilde{d})$ *be metric spaces. Then:*

(i) A mapping $f : X \to \tilde{X}$ *is said to be isometric or an isometry if* f *preserves distances, that is, for all* $x, y \in X$*, we have*

$$d(x, y) = \tilde{d}(f(x), f(y)).$$

(ii) The space X *is said to be isometric with the space* $\tilde{X}$ *if there exists a bijective isometry of* X *onto* $\tilde{X}$*. The spaces* X *and* $\tilde{X}$ *are called isometric spaces.*

Theorem A.0.1 (Completion)**.** *For a metric space* $X = (X, d)$*, there exists a complete metric space* $\hat{X} = (\hat{X}, \hat{d})$ *which has a subspace that is isometric with* X *and is dense in* $\hat{X}$*. This space* $\hat{X}$ *is unique up to isometry, and is called the completion of* X*.*

For more information, see reference [13].

Appendix B

Compactness

Definition B.0.6 (Compact). *A metrix space X is said to be compact if every open covering of X has a finite subcollection which also covers X.*

Definition B.0.7 (Sequentially compact). *A space X is said to be sequentially compact if every sequence from X contains a convergent subsequence.*

Definition B.0.8 (Bolzano-Weierstrass property). *A space X is said to have the Bolzano-Weierstrass property if every infinite sequence in X has at least on cluster point.*

Theorem B.0.2 (Borel-Lebesgue). *Let X be a metric space. Then the following are equivalent:*
(i) X is compact.
(ii) X has the Bolzano-Weierstrass property.
(iii) X is sequentially compact.

Theorem B.0.3 (Heine-Borel Theorem). *Every closed and bounded subset of real numbers is compact.*

Proposition B.0.1. *A closed subset of a compact space is compact. A compact subset of a metric space is closed and bounded.*

Proposition B.0.2. *The continuous image of a compact set is compact.*

Remark B.0.1. *Notice that if a metric space* (X, d) *is not* $\boldsymbol{R}^n$*, a bounded closed subset of* X *may be not compact. One example is* L^2 *space.* $\{\sin(nx)\}$ *is a set of functions with* $n \in \boldsymbol{N}$ *and* $x \in [-\pi, \pi]$*. Then* $\{\sin(nx)\}$ *is bounded closed subset of* L^2*, but it is not compact. Since* $||\sin(nx) - \sin(mx)||_p = \sqrt{2\pi}$ *for* $n \neq m$*, that nothing other than constant sequence from* $\{\sin(nx)\}$ *will be Cauchy and convergent. Hence, not compact.*

For more information, see reference [13], [26].

Appendix C

Borel Measures

Definition C.0.9 (Hausdorff space). *X is a Hausdorff space if the follwing is true: If $p, q \in X$ and $p \neq q$, then p has a neighborhood U and q has a neighborhood V such that $U \cap V = \emptyset$.*

Definition C.0.10 (Locally compact). *X is locally compact if every point in X has a neighborhood whose closure is compact.*

Theorem C.0.4 (Riesz representation theorem). *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that*

1. *$\Lambda f = \int_X f d\mu$ for every $f \in C_c(X)$,*

2. *$\mu(K) < \infty$ for every compact set $K \subset X$,*

3. *For every $E \in \mathfrak{M}$, we have*

$$\mu(E) = \inf \{\mu(V) : E \subset V, V \text{ open}\}.$$

4. *The relation*

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$

holds for every open set E*, and for every* $E \in \mathfrak{M}$ *with* $\mu(E) < \infty$*.*

5. *If* $E \in \mathfrak{M}, A \subset E$*, and* $\mu(E) = 0$*, then* $A \in \mathfrak{M}$*.*

Definition C.0.11 (Borel measure)**.** *A measure* μ *defined on the* σ*-algebra of all Borel sets in a locally compact Hausdorff space* X *is called a Borel measure on* X*.*

Definition C.0.12 (Regular)**.** *A Borel set* $E \subset X$ *is outer regular or inner regular, respectively, if* E *has property 3 and 4 of Theorem C.0.4. If every Borel set in* X *is both outer and inner regular,* μ *is called regular.*

For more informations, see reference [27].

Appendix D

Hilbert Space

D.1 Properties of Hilbert Space

Hilbert Space H has the following five properties:

1. H is linear
2. Scalar Products $(u, v), \forall u, v, w \in H, a \in \mathbf{R}$
   - $(au, v) = a(u, v)$
   - $(u + v, w) = (u, w) + (v, w)$
   - $(u, v) = (v, u)$
   - $(u, u) > 0$ if $u \neq 0$
   - $(u, u)^{1/2} = ||u||$
3. H is infinite dimensional
4. H is complete
5. H is separable

D.2 Convergence in Hilbert Space

Definition D.2.1 (Strong Convergence). *the sequence $\{u_n\} \subset H$ converges to u if*

$$\lim ||u_n - u|| = 0$$

Definition D.2.2 (Weak Convergence). *If given $\{u_n\}$, these exists a fixed element u s.t. $(u_n, v) \to (u, v), \forall v \in H$, then $\{u_n\}$ is weakly convergent.*

Definition D.2.3 (Weak Cauchy). *A sequence $\{u_n\}$ of elements of H with the property that $\forall \rho \in H$, the sequence of real numbers $\{(\rho, u_n)\}$ is a Cauchy sequence.*

Definition D.2.4 (Weak Compact). *A subset A of H s.t. every infinite sequence of elements of A contains a sub-sequence that is weakly convergent to an element in A.*

Theorem D.2.1. *Strong Convergence implies Weak Convergence*

Theorem D.2.2. *In finite dimensional spaces, there is no distinction between strong and weak convergence.*

D.3 Completely Continuous Operators

Definition D.3.1. *An operator F in H is call continuous if $Fu_n \to u$, whenever $u_n \to u$*

Definition D.3.2. *The operator F is completely continuous if every weakly convergent sequence is transformed into a strongly convergent sequence.*

D.4 Eigenvalues

Let F be any self-adjoint, positive-definite, completely continuous operator.

Definition D.4.1. *A real number λ, for which the equation $Fu - \lambda u = 0$ has a nontrivial solution u, is called an eigenvalue of F with corresponding eigenvector u.*

Theorem D.4.1. *IF F is a self-adjoint, positive-definite, completely continuous operator with domain H, then the set of all eigenvalues λ_i of F, arranged in non-increasing order, is an infinite sequence of positive numbers converging to zero,*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \to 0$$

For more informations, see reference [10], [13].

Appendix E

Sobolev Space

E.1 Weak Derivatives

Definition E.1.1 (Weak Derivatives). *Suppose $u, v \in L^1_{loc}(U)$, and $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all test functions $\phi \in C_c^\infty(U)$, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Lemma E.1.1 (Uniqueness of weak derivatives). *A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set a measure zero.*

E.2 Sobolev Space $W^{k,p}$

Definition E.2.1. *The Sobolev space*

$$W^{k,p}(U)$$

consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex $\alpha \leq k, D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Definition E.2.2. *If $u \in W^{k,p}(U)$, we define its norm to be*

$$||u||_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx\right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} ess\sup_U |D^\alpha u| & (p = \infty). \end{cases}$$

Theorem E.2.1. *For each $k = 1, 2, ...$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Remark E.2.1. *If $p = 2$, we usually write*

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, ...).$$

E.3 Sobolev Space $H^1(\Omega)$

Definition E.3.1. *Let Ω be a nonempty, open subset of $\mathbb{R}^n$. $H^1(\Omega)$ consists of functions $f \in L^2(\Omega)$ such that there exists a sequence $\{f_n\} \subset C^1(\bar{\Omega})$ with $\{\nabla f_n\}$ Cauchy in $L^2(\Omega)$, and f_n converging to f in $L^2(\Omega)$.*

Lemma E.3.1. *If $f \in H^1(\Omega)$, then f has a weak derivative $\nabla \in L^2(\Omega)$.*

Lemma E.3.2. *$H^1(\Omega)$ is a Hilbert space when equipped with the inner product*

$$(f,g)_{H^1(\Omega)} := \int_\Omega fg dx + \int_\Omega \nabla f \cdot \nabla g dx.$$

Let $I := (a,b) \subset \mathbb{R}$ and consider an important theorem of $H^1(I)$ which does not always hold for general domain $\Omega \subset \mathbb{R}^n$.

Theorem E.3.1. *$H^1(I) \subset C(I)$, i.e., for every $f \in H^1(I)$, there exists $g \in C(I)$ with $f = g$ a.e..*

For more, see reference [3],[23].

Appendix F

Dirichlet Forms

F.1 Self-adjoint Operator

Definition F.1.1 (Hilbert adjoint operator)**.** *Let $T : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then $T^* : H_2 \to H_1$ is an adjoint operator of T if for every $x \in H_1$ and $y \in H_2$*

$$(Tx, y) = (x, T^*y).$$

Theorem F.1.1. *T^* of T exists, is unique, and is a bounded linear operator with norm $||T^*|| = ||T||$.*

Definition F.1.2 (Self-adjoint operator)**.** *A bounded linear operator $T : H \to H$ on a Hilbert space H is said to be self-adjoint if $T^* = T$.*

F.2 Closed Forms

Let H be a Hilbert space with inner product $(\cdot, \cdot)$.

Definition F.2.1 (Symmetric form)**.** *A non-negative definite symmetric billinear form densely defined on H is called a symmetric form on H.*

Definition F.2.2 (Closed form)**.** *A symmetric form f is closed in H if its domain $D[f]$ is complete under the inner product $f(u,v) + (u,v)$ for $u, v \in H$.*

Theorem F.2.1. *A symmetric form f is closed if and only if there exists a non-negative self-adjoint operator Λ in the closure $\overline{D[f]}$ in H, with domain $D[\Lambda] \subset D[\sqrt{\Lambda}] = D[f]$ such that $f(u,v) = (\sqrt{\Lambda}u, \sqrt{\Lambda}v)$ for every $u, v \in D[f]$. Moreover, $f(u,v) = (\Lambda u, v)$ for every $u \in D[\Lambda], v \in D[f]$.*

F.3 Markovian Forms

Let X be a locally compact separable Hausdorff space, and m be a positive Radon measure on X such that $supp[m] = X$.

Definition F.3.1. *A form f on $L^2(X,m)$ is called Markovian if it satisfies the following conditions*

(i)For each $\epsilon > 0$, there exists a $\eta_\epsilon : \mathbb{R} \to [-\epsilon, 1+\epsilon]$, with $\eta_\epsilon(t) = t$ for $t \in [0,1]$ and $0 \leq \eta_\epsilon(t') - \eta_\epsilon(t) \leq t' - t$ for every $t' < t$.

(ii)If $u \in D[f]$, then $\eta_\epsilon \circ u \in D[f]$ and $f(\eta_\epsilon \circ u, \eta_\epsilon \circ u) \leq f(u,u)$.

Proposition F.3.1. *A closed form f on $L^2(X,m)$ is Markovian if and only if the following condition is satisfied:*

If $u \in D[f]$, $v = (0 \vee u) \wedge 1$, then $v \in D[f]$ and $f(v,v) \leq f(u,u)$ where $(0 \vee u) \wedge 1 = \inf\{\sup\{u, 0\}, 1\}$.

F.4 Dirichlet Forms

Definition F.4.1. *A Dirichlet form is by definition a symmetric form on $L^2(X,m)$ which is not only Markovian but also closed.*

Theorem F.4.1 (Beurling-Deny representation formular)**.** *Any regular Dirichlet form f on $L^2(X,m)$ can be expressed for $u, v \in D[f] \cap C_0(X)$ as*

$$f(u,v) = f^c(u,v) + \int_{X\times X-d} (u(x)-u(y))(v(x)-v(y))J(dx,dy) + \int_X u(x)v(x)k(dx).$$

Here f^c is a symmetric form with domain $D[f^c] = D[f] \cap C_0(X)$ and satisfies the following condition:

$$f^c(u,v) = 0$$

for $u \in D[f^c]$ and $v \in \vartheta(u)$, where

$$\vartheta(u) = \{v \in D[f^c] : v \text{ is constant on a neighborhood of } supp[u]\}.$$

For more informations, see reference [7], [18].

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