Integrating Behavioral Economics into System Dynamics

Souleymane Bah

A Dissertation
Submitted to the Faculty
of the
WORCESTER POLYTECHNIC INSTITUTE
in partial fulfillment of the requirements for the
Degree of Doctor of Philosophy
in
Economics and System Dynamics

May 2023

APPROVED:

____

Professor Michael J. Radzicki, Primary Advisor, WPI

____

Professor Sharon Johnson, Committee Member, WPI

____

Professor Alexander D. Smith, Committee Member, WPI

Abstract

Dynamic models driven by judgment and decision-making from behavioral and experimental economics better capture how people make decisions compared to classical models based on delay time constants and multiplier effects or table functions. The underlying assumptions on table functions are sometimes challenging to justify.

Behavioral economists elect to use discrete-time modeling with difference equations.

In system dynamics, we build models with the concept of continuous-time and do it on a discrete machine, i.e., a digital computer, and this requires approximating the time unfolding continuously with discrete time steps. This leads to confusion in the system dynamics community on how to properly model discrete-time features in continuous-time models. Even though system dynamicists commonly use continuous-time models with differential equations, discrete-time models are consistent with system dynamics. This issue is important in system dynamics in general and particularly in bringing behavioral economics models with richer, more realistic human decision-making structures and empirically verified with human decision subjects into system dynamics modeling. We first clarify this issue to the system dynamic community. Also, we do this to make sure by replicating correctly behavioral economics models and structures into system dynamics modeling and looking at the issue of discrete-time and continuous-time in a manner that satisfies both behavioral economists and system dynamicists, and we preserve the results published in the behavioral economics literature.

We developed a formal, teachable method of bringing behavioral economics into system dynamics. We applied and validated this method using a macro model, the Samuelson multiplier-accelerator model, a micro model, the Cobweb model, and the behavioral economics model, the quasi-hyperbolic discounting model. We also applied this method in two chapters of this dissertation. First, to improve an existing model and paper published in 2019 in the system dynamics review journal and second, to revisit and explore two well-known lifecycle models. We replicated Brumberg's and Modigliani's lifecycle hypothesis in system dynamics and built Shefrin's and Thaler's behavioral lifecycle hypothesis from the description given in their paper. We also tested a couple of predictions provided by Shefrin and Thaler in their behavioral lifecycle hypothesis paper.

Acknowledgment

I am humbled and immensely grateful to all the individuals whose unwavering support, guidance, and encouragement have contributed to the completion of this dissertation. Their invaluable contributions have enriched my academic journey; this accomplishment would not have been possible without their assistance.

First and foremost, I express my deepest gratitude to my supervisor, Professor Michael Radzicki. His exceptional mentorship, expertise, and dedication have been instrumental in shaping this research. In addition, I am grateful for his insightful feedback, which has significantly enhanced the quality of this work.

I extend my sincere appreciation to the members of my dissertation committee, Professor Sharon Johnson and Professor Alexander Smith. Their insightful suggestions, constructive critiques, and rigorous evaluation have helped refine this research. I am grateful for their time, expertise, and commitment to scholarly excellence.

I am indebted to the WPI faculty, whose unwavering support and resources have played a pivotal role in my academic development. Their commitment to fostering an environment of intellectual growth and innovation has shaped my research pursuits. I express my heartfelt thanks to Professor Robert Krueger, Professor James Lyneis, Professor Oleg Pavlov, Professor Khalid Saeed, Professor Lamine Sagna, and Professor Winston Soboyejo for their support and feedback, which have laid a strong foundation for this research. Their intellectual stimulation and collaborative spirit have made this journey more meaningful and enjoyable.

My gratitude extends to my family and friends, whose unconditional love, encouragement, and patience have been a constant strength throughout this arduous journey. Their unwavering belief in my abilities and the sacrifices made on my behalf have been an immeasurable inspiration. I am deeply thankful for their support and understanding during the highs and lows of this endeavor.

I would like to acknowledge my colleagues and classmates, Christine Bolton, James Hacunda, Anne Johnson, Frederick Kautz, Professor Saeed Langarudi, Christine Tang, and Professor Raafat Zaini, who have walked alongside me during this academic pursuit. Their camaraderie, thought-provoking discussions, and shared experiences have enriched my understanding and expanded my research horizons.

Furthermore, I would like to express my appreciation to the numerous scholars, researchers, and authors whose groundbreaking works have paved the way for this dissertation. Their contributions to economics and system dynamics have been a constant source of inspiration and guided my research endeavors. I am grateful for their commitment to advancing knowledge and laying the foundation for this work.

I am immensely thankful to all those who have supported and encouraged me throughout this dissertation journey.

To my beloved parents, wife, and children

Contents

Chapter 1 Introduction

In today's interconnected and dynamic world, understanding and managing complex systems is crucial for tackling a wide range of challenges. Traditional economic models have long relied on the assumption of rationality, assuming that individuals make decisions based on logical reasoning, self-interest, and utility maximization. However, extensive research has shown that human behavior is far more intricate and nuanced than this simplified view suggests.

Behavioral economics, a field that combines psychology and economics, offers valuable insights into how individuals actually make decisions, considering cognitive biases, social influences, and emotional factors.

Simultaneously, system dynamics provides a powerful framework for analyzing the behavior of complex systems over time. By considering the interconnections, feedback loops, and dynamics among various components, system dynamics allows us to comprehend the behavior of these systems in a holistic and comprehensive manner. It enables us to examine how changes in one part of the system can ripple through via a network of interacting feedback loops and impact the entire system. It also allow us to identify unintended consequences of the system.

Dynamic models driven by judgment and decision-making from behavioral and experimental economics capture better how people make decisions than classical models based on delay time constants and multiplier effects or table functions. The underlying assumptions on table functions in system dynamics are challenging to justify from a behavioral standpoint. Behavioral economists elect to use discrete-time modeling with difference equations. In system dynamics, we build models with the concept of continuous-time. We do it on a discrete machine, i.e., a digital computer, and this requires approximating the time unfolding continuously with discrete-time steps. This leads to confusion in the system dynamics community on how to model discrete-time features in continuous-time models properly. Even though system dynamicists commonly use continuous-time models with differential equations, discrete-time models are consistent with system dynamics. This issue is important in system dynamics in general and particularly in bringing behavioral economics models with richer, more realistic human decision-making structures and empirically verified with human decision subjects into system dynamics modeling. We do this to ensure that by replicating behavioral economics models and structures correctly into system dynamics modeling and looking at the issue of discrete-time and continuous-time in a manner that satisfies both behavioral economists and system dynamicists, we preserve the results published in the behavioral economics literature.

This integration combines the rich insights of behavioral economics and the analytical tools of system dynamics, fostering a more accurate and realistic understanding of the behavior of complex systems. Moreover, it offers a multidimensional perspective on decision-making and system behavior.

This dissertation is organized into three interrelated chapters in sequence.

The first chapter presents a formal, repeatable, and teachable method to bring behavioral economics into system dynamics. We will first clarify the issue of discrete-time versus continuous-time modeling and how it matters for policy change. We will also present a discrete-time molecule with stock and flow structure to build discrete-time models in system dynamics without using a dt (i.e., solution interval) of 1. We called this molecule cornerstone for building discrete-time models in System Dynamics. It can be applied beyond the economics domain. The discrete-time molecule can be used to build, for example, healthcare, manufacturing, finance, business strategy, environmental models, etc. The second molecule or generic structure we developed replicates the quasi-hyperbolic discounting or present-bias in System Dynamics. We successfully applied these molecules in this dissertation's second and third chapters.

The second chapter explores and analyzes some modeling problems in the paper published in the System Dynamics Review 2019 titled "Building a Bridge to behavioral economics: countervailing cognitive biases in lifetime saving decisions." (Bahaddin et al. 2019). Each problem will be examined, and a proposed solution with a new model based on best practices in system dynamics modeling will be given. Finally, we will explore from the corrected model the three cognitive biases in lifetime savings and how two of them interact and eventually cancel out, as described in Bahaddin et al.'s paper. The experiments will be run first on the corrected feedback-poor model, and in the next step, the fixed model will be extended to a feedback-rich version.

The third chapter is an application of the results previous first two chapters. This chapter will compare and contrast two lifetime saving models developed by Nobel laureates Franco Modigliani and Richard Thaler. Modigliani's Standard Lifecycle Hypothesis (LCH) and Thaler's Behavioral Lifecycle Hypothesis (BLC) are two influential theories in the field of economics that attempt to explain how individuals manage their consumption and savings over their lifetime. We will test a couple of predictions made by Shefrin and Thaler in their BLC paper using the system dynamics methodology and tools.

Chapter 2 Behavioral Economics and System Dynamics: Toward an Integrative Approach

2.1 Introduction

This chapter aims to present a formal, repeatable, and teachable method to bring behavioral economics into system dynamics. We will first explore, analyze, understand, and as such, be conversant with the proper way of using discrete or continuous-time, judgment, and decision-making in dynamic modeling. Then, we will clarify their differences, under which circumstances to apply one instead of the other or a combination of both, and the fact that choosing one approach instead of the other impacts policy change. We argue that dynamic models driven by guiding principles on when and how to use discrete-time (Pidd, 1992) or continuous-time (Forrester, 1961) or a mix of both (Kalecki, 1935) and judgment and decision-making, from experimental and behavioral economics, capture better how people make decisions than classical models based on delay time constants and multiplier effects or table functions. According to Veblen, every economic analysis should include human behavior. To illustrate our purpose, we will explore how to incorporate discrete-time structures into continuous-time models and standalone discrete-time models in system dynamics in a manner that satisfies both behavioral economists and system dynamicists and preserve the results from the behavioral economic literature.

The plan of this chapter is as follows. The second section defines the two modeling techniques. The third section presents a historical background of discrete-time and continuous-time modeling. The fourth section presents the issue and debate about these two modeling techniques. It also gives an overview of how discrete-time and continuous-time modeling are used and what their differences are in the field of economics, system dynamics, engineering, and computer science. The fifth section provides a sweeping overview of different discretization techniques used in continuous-time modeling to approximate the time unfolding continuously in discrete-time steps. In the sixth section, we present a formal, teachable, and repeatable method based on the mean value theorem on how to convert a discrete-time into continuous-time. We will apply this technique, along with a generic structure presented in this chapter to build a discrete-time model in system dynamics with a solution interval of less than 1 to build three well-known models: the Samuelson-Hicks' multiplier-accelerator macro model, the Cobweb micro model, and the present-bias behavioral economics model in system dynamics. We do this to make sure, first, we can faithfully reproduce discrete-time economic models, particularly behavioral economics models in system dynamics, in a manner that satisfies both economics and system dynamicists and, second, compare and contrast the converted continuous-time model with its original discrete-time version and whether or not this has an impact for policy changes. The seventh section presents when to use discrete-time modeling, continuous-time modeling, or a mix of both. The final section concludes and lays out further research and directions.

2.2 Discrete-time and continuous-time modeling

There are two types of people in the modeling world: those who prefer continuous-time models and those who prefer discrete-time models. But in reality, both are necessary to understand complex systems fully.

A discrete-time system is a system for which the state, input, and output variables are defined only for discrete moments in time.

Discrete time models are convenient for matching a model to time series data recorded at regular intervals. For example, they are also very useful in running computerized lab experiments in the economics field. Moreover, they are usually the most intuitive to understand of all forms of dynamic models. Historically, economists usually built models with a two-stage approach (e.g., now and later).

These models assume a stepwise mode of execution. At any particular time instant, the model is in a specific state and defines how this state changes and what the state will be at the next time. The time unit between states is chosen and takes an integer value, e.g., one day, one month, or one year. The system is assumed to change only at each discrete time tick.

One of the main characteristics of discrete-time models is that the system's behavior between the time intervals is not captured and has no influence or causal relation with the system. So, to represent the system's behavior over time between two discrete-times t and t+1, modelers usually take the value of t as constant and move it to t+1, and at t+1, they replace it with the value of t+1. In other words, the system is assumed to be in a steady state or in equilibrium between every discrete time interval.

A continuous system is one in which the state variables change continuously with respect to time. With the continuous-time model, we do not specify the next state directly but use a derivative function to specify the rate of change of the state variables. At any particular time instant on the time axis, given a state and an input value, we only know the rate of change of that state. From this information, the state at any point in the future has to be computed at each solution interval time.

2.3 Discrete-time or continuous-time modeling: a historical perspective

Traditionally, in dynamic modeling, particularly in economic modeling, we have discrete-time and continuous-time modeling, which is either linear or nonlinear. These models can either be simulated and run on an analog or digital computer and can be either linear or nonlinear. Linear systems are simple and have an analytical solution. However, they are not robust, and their behaviors are sensitive to their parameter values, whereas nonlinear systems are more robust, and their behavior are less sensitive to their parameter values. So, in dynamic modeling, it is important to understand these three concepts and how they can adequately be used together: 1. discrete-time vs. continuous-time, 2. linear vs. nonlinear, and 3. analog computer vs digital computer.

When early dynamic economic modeling was developed, there were no computers. For example, Harrod was writing his “An Essay in dynamic economics” in 1936 before the digital computers. The first commercial computer was developed by the German Konrad Zuse in 1942 and MIT introduced the first digital computer with magnetic-core memory and real-time graphics with the Whirlwind project in 1955 under the direction of Jay Forrester. Forrester, the system dynamics inventor, also patented the magnetic core memory used in digital computers. The goal of the Whirlwind project, funded by the US Navy, was to create a computer to drive a high-quality real-time aircraft flight simulator to train their pilots. In the early stage of the project, the Whirlwind team started by building a large analog computer to continuously update the aircraft dashboard simulator from the pilot’s control inputs. However, they soon realized the limitations of using the analog computer to run continuous-time simulations. They found out the analog computer was inflexible, slow to solve their problems. Also, they required more capacity by adding more parts to the machine while increasing the costs to operate the computer. So, they decided to switch to a more flexible digital computer solution that allows more simulation accuracy. This project demonstrated that continuous-time modeling can be done on a discrete-time machine. The step time of the model is discrete but the model is continuous. The stepping of the model approximates the underlying continuous model's flow of something, e.g., people, products, etc. The limitation of modeling continuous-time on a digital, discrete machine is on the ceiling between integration error when the step time or solution interval is big or round-off error when the solution interval is small, as we cannot make the solution interval infinitesimally small.

Continuous-time models and discrete-time models are different mathematical models with other mathematical properties. For example, discrete-time models use difference equations, whereas continuous-time models use differential equations.

Discrete time models are usually the most intuitive to understand of all forms of dynamic models. These models assume a stepwise mode of execution. At any particular time instant, the model is in a particular state and defines how it changes and what the state will be at the next time. The behavior mode between two states is assumed to be in equilibrium or a resting state. The time unit between states is chosen and takes an integer value, e.g., one day, one month or one year. Discrete-time notation is convenient for matching a model to time series data recorded at regular intervals. It is also very useful in running computerized lab experiments in economics for example.

With continuous-time model, we do not specify the next state directly but use a derivative function to specify the rate of change of the state variables. At any particular time instant on the time axis, given a state and an input value, we only know the state's rate of change. From this information, the state at any point in the future has to be computed. Hence, to build a discrete-time model using system dynamics method based on a continuous-time approach, it is important to understand how to properly use the solution interval in the model. The solution interval is the length of time interval at which the simulation software solves the model equations and calculates the values of all variables. Forrester presented in great detail how to properly select the solution interval with a dynamic system model in appendix D of his seminal Industrial Dynamics book (Forrester 1961). In our models, we will follow the Forrester approach and the rule of thumb he proposed in this book.

The second important point is building confidence in these models, particularly discrete-time models. Contrary to discrete-time models, methods and guidelines for testing continuous-time models have been extensively studied in the system dynamics field. One common mistake inexperienced modelers make in building discrete-time models using system dynamics is setting the solution interval and fixing it to one. Testing a model's robustness by halving the solution interval in discrete-time models is not easy, and by in large important time delays are buried in the solution interval (Low, 1976, Keen, 2021). Nevertheless, discrete-time models are very useful to test and build confidence in a model. To reject a model using calibration techniques should be based first and foremost on robust models.

Laibson, in his paper titled: “Hyperbolic discount functions, undersaving, and savings policy, 1996”, p.8, described how the discrete quasi-hyperbolic function graph is created. He stated that: “ The points of the discrete-time quasi-hyperbolic function have been connected to generate the smooth curve in Figure 1.1”. Based on these findings, we will attempt to show how to properly model each of these modeling concepts in system dynamics. Laibson, later, in this seminal paper titled” “Golden Eggs and Hyperbolic Discounting” felt that a continuous-time version could solve one of the limitations of his golden eggs model. He concluded his paper by stating: “A second problem associated with the model is the anomalous prediction that consumers will always face a binding self-imposed liquidity constraint. For example, the golden eggs model predicts that after making their consumption choice, consumers should have no liquid funds left in their bank accounts. This prediction contradicts many consumers’ experiences. However, this problem can be readily addressed by introducing a precautionary savings motive for holding liquidity. For example, consider a continuous-time analog of the golden eggs model and assume that instantaneous liquidity needs arrive with some hazard rate. Then in equilibrium the consumer will only rarely completely exhaust her liquidity”. This shows some interest from behavioral economists on continuous-time modeling.

2.4 Discrete time versus Continuous time in different fields

2.4.1. Discrete time versus continuous-time in Economics

Traditionally, in dynamic modeling and particularly in economic modeling (Hicks, Laibson, Samuelson, etc.), we use discrete-time modeling, why? What are the advantages of using discrete-time modeling? Romanchuk (Romanchuk, 2017) argued in favor of modeling in discrete-time not only because economic data are available in discrete-time but also because of its simplicity. On the other hand, system dynamicists and other economists (Allen, Bergstrom, Gandolfo, Goodwin, Harrod, Forrester, Keen, Koopmans, Phillips, Radzicki, Tustin, etc.) use continuous-time modeling. What are the motivations of this group to elect modeling in continuous-time over discrete-time? Forrester, the founder of system dynamics, argued that feedback systems should be formulated continuously since real systems are continuous. Other scholars such as R. G. D. Allen and Baumol focused on the solutions of differential and difference equations. They also compared and contrasted discrete-time and continuous-time techniques.

In his book Mathematical Economics (Allen, 1966), Allen started by analyzing, comparing, and contrasting the Cobweb model, which relates to a market of a single good, using both the discrete-time and the continuous-time techniques. One of the main differences of these two techniques using the Cobweb model, for example, is the interpretation of the sellers' and buyers' decisions. In analyzing the discrete-time, difference equations models, the concepts of *ex-ante* and the *ex-post* are fundamental. The *ext ante* is the expected or planned output, while the *ex-post* is the actual or realized output. In the Cobweb model, for example, the sellers, based on the *ex-post* of the previous period, decide how much to sell and expect the price to be the previous price, P_{t-1}, which they bring forward the supply S_t, while the buyers, when they discover the price, P_t, in the current period, decide how much to buy. The transition from ex-ante to ex post and the relationship between the variables must be specified in the discrete-time model. In the continuous-time however, the *ex-ante* and *ex-post* feature becomes irrelevant as the decisions and price adjustments occur continuously. Another difference is in the solution of the two techniques. For example, the discrete-time version of the Cobweb model involves a power function $c^{t,}$ whereas its continuous-time version involves an exponential function e^{ct}. In solving the differential equations, the approach is first to find the general solution and then derive the specific solution which corresponds to the initial conditions. With the difference equations, the process is reversed. In this later case, we first find a particular discrete-time solution and then obtain the general solution, which is more difficult to derive than the differential equations with the continuous-time model. Moreover, Allan showed that the difference equations can have discrete or continuous solutions. As opposed to Forrester, who recommended starting with the continuous solution, Allen recommended starting with the discrete solution not because they can be easily solved but also because they are sufficient for a wide variety of economic problems. The intuition behind continuous solution of a difference equation is that the series of the discrete-time delay also called the "series of the equally-spaced values" by Allen, can start from any value, for example, ½ instead of 1, and varied continuously. In general, both the difference and the differential equations can be solved using the Laplace transform.

Like Allan, Baumol also looked at the differences between discrete-time and continuous-time, focusing on the methods of solving the difference and differential equations. With the first-order difference equations, Baumol proceeds with a trial and error method by trying different formulas to see if they satisfy the difference equation and the initial conditions. This method, similar to an algorithm, runs step by step to find the solution of the difference equation. Regarding the higher order difference equations, he uses the classical method of the characteristic equation, also called the "reduced equation" by Samuelson.

Baumol approach to analyzing and comparing the discrete-time and continuous-time is similar to Allen. When describing the oscillatory movements for both discrete-time and continuous-time systems, he showed that the real numbers system is insufficient. In many cases, the solution has complex roots. The sign of real part of the complex number and the sign of the real number of the roots can be used to determine the oscillatory (or not) behavior of both the discrete-time and continuous-time systems. Baumol provides a qualitative analysis with a summary of the behavior oscillatory modes of both the difference and differential equations from the cases of nature of the roots (Baumol, 1970). For example, in the difference equation case, the system generates oscillations when the roots are real and negative. However, in the differential equation case, the system generates no oscillations. Baumol compared these two techniques on Harrod's model. In the difference equation case, Harrod's model is in the form of $y_t = \frac{v}{v-s} * y_{t-1}$, where v is the capital-output ratio, and *s is* the marginal propensity to save. Both *s* and *v* are positive. When s is greater than v, we can see that the output will oscillate from a negative to a positive value in one year and then from a positive to a negative value in the following year. This is a peculiar or "queer" case, as the output should always be positive. In the differential equation case, Harrod's model is in the form of $\frac{\partial y}{\partial t} = \left(\frac{s}{v}\right) * y$, which has an analytical solution that is $y_t = y_0 e^{\left(\frac{s}{v}\right)t}$. However, we can see that as both s and v are positive, the output will generate an exponential growth behavior and the peculiar case in the difference equation disappears.

We will compare these two modeling approaches using the well-known Samuelson's linear multiplier-accelerator model. This model will be built both in discrete-time and continuous-time using system dynamics methodology, tools and techniques. When comparing these models, we will extend our analysis and explore the implications of these two techniques regarding policy change. Next, we will shed some light on when it is the most appropriate to use discrete-time and when to use continuous-time or both with the mixed differential-difference equations modeling (Kalecki, 1935). For example, Forrester in very early in system dynamics development in Industrial Dynamics, and later, Sterman in Business Dynamics in 2000 described when to use continuous-time modeling. Also, Gandolfo in (Gandolfo, 1997) provided a list of arguments grouped in eight categories in favor of continuous-time modeling. We will look at the view of these authors and others who favor using the discrete-time technique, such as (Romanchuk, 2017).

In behavioral economics, where discrete-time is commonly used, we see some interests in modeling in continuous-time (Laibson, 1997).

2.4.2. Discrete time versus continuous-time in system dynamics

In system dynamics, we commonly model in continuous-time, and differential equation is the basic mathematical operator. Many authors in the system dynamics field have looked at the issue of discrete-time vs. continuous-time modeling. Forrester, in the very early in system dynamics development in Industrial Dynamics and, later, Sterman in Business Dynamics in 2000, described when to use continuous-time modeling. Forrester suggested always starting modeling in continuous-time and testing the model; when there is a solid reason to add discreteness, a discrete-time structure can be added. Regarding the choice of the solution interval, Forrester suggested that in any model, the solution interval should be less than half of the shortest first-order delay in the system (Forrester, 1971). Low, in 1976, criticized and identified a couple of issues from Samuelson's discrete-time linear multiplier-accelerator and proposed a solution from a system dynamics perspective. The first issue, from a system dynamics point of view, is the assumption that there is only one delay time which is the model time. Any other delay is embedded in the model time. In the Samuelson's multiplier-accelerator model, the model time is in years but the consumption time and the investment time should have different time delay. Consumption should be specified in weeks or months, and investment in years. The second issue highlighted by Low is that there is no distinction between stocks and flows in a discrete-time model's variables. Discrete-time models generally do not specify stock and flow structures (Sterman, 2000, p. 798). Low reconstructed Samuelson-Hicks' model in system dynamics with DYNAMO and showed that the original multiplier-accelerator model fails to conserve important physical flows such as investment.

According to Low, the non-conservation of stuff in Samuelson-Hicks' discrete-time model reveals other problems. More specifically, the material and behavioral time constants and information smoothing delays are buried in the solution interval of the original model. In such situations, testing the model for robustness by halving the solution interval may generate inconsistent behaviors. However, we believe Low's solution to fix Samuelson-Hicks' model has some issues. Low's Samuelson-Hicks version in System Dynamics has been published in two books (Legasto, Forrester, and Lyneis, 1980, p. 107) and (Randers, 1980, p.76). He used the "information smoothing delays" formulation to model the discrete-time structure in Samuelson-Hicks' model in system dynamics. Unfortunately, low didn't provide the resulting behavior of his discrete-time model. In our view, Low's Samuelson-Hicks' discrete-time model does not faithfully represent the original model with the correct time shape behavior. We have a better solution that we will present in this chapter.

Barlas et al. also looked at this issue and built a supply chain model in discrete-time using System Dynamics. In (Barlas, 2011) on p. 2, they point out that by choosing a solution interval, dt, of 1, we can build a discrete-time version of the model:

> "For consistency with models and policies in the inventory literature, time is modeled discrete (DT=1 time period). This makes the model time-discrete that is necessary to represent standard ordering policies like Order-up-to-S and (s, S), as will be described below. Another policy analyzed, the anchor-and-adjust ordering rule typically used in system dynamics models, is normally time-continuous. As will be explained below, in the assumed parameter settings, it was possible to represent the anchor-and-adjust ordering rule with D=1 as well, without causing any erroneous dynamics."

This is similar to the Low approach of modeling discrete-time in system dynamics. Bahaddin et al. (Bahaddin, 2019) also build a discrete-time using a solution interval of 1 time period. In the second chapter of this dissertation, we will analyze and fix this issue and other issues we found in Bahaddin's model. Other authors in the system dynamics field (Ossimitz et al., 2008), also compared the two approaches. Ossimitz et al. argued that system dynamics is compatible with both discrete-time and continuous-time. Even though continuous-time is at the core of system dynamics modeling, they pointed out some disadvantages of continuous-time. One of them, they argued, is that with continuous-time, in some situations, it is possible to eliminate the structural difference between points in time related to stocks and time-interval related to flows by making the time-interval infinitesimally short. They concluded their paper:

> “We see the conflict between a continuous and a discrete concept of time right in the core of SD methodology. By addressing it explicitly and making both concepts of time clear to SD learners might help them to understand better both system dynamics and the world. “

Sterman addressed the issue raised above by Ossimitz et al. in their concluding remarks in the article published in the system dynamics Review journal, marking the 60th anniversary of the field (Sterman, 2018).

> “System dynamics models can be implemented using a variety of different simulation architectures. These vary in their representation of time (continuous or discrete), state variables (continuous or discrete), and uncertainty (stochastic or deterministic). Ordinary differential equations, stochastic differential equations, discrete event simulations, agent-based models and dynamic network models are common computational architectures offering different choices on these dimensions.”

He reminded us of what Jay wrote about his view on the continuous and discrete concept of time.

> “… in formulating a model of an industrial operation, we suggest that the system be treated, at least initially, on the basis of continuous flows and interactions of the variables. Discreteness of events is entirely compatible with the concept of information-feedback systems … As a starting point, the dynamics of the continuous-flow model are usually easier to understand and should be explored before complications of discontinuities and noise are added” (Forrester, 1961, pp. 64-65).

Sterman also emphasizes the importance of the quantitative approach and the use of analytical methods in building empirically grounded models.

We do not see a conflict, per se, between the continuous and discrete concepts of time. Knowing when and how to properly build empirically grounded models in line with what the scholars in the system dynamics field laid out as best practices and recommendations is very important.

2.4.3. Discrete time versus continuous-time in Engineering and Computer Science

Engineers also use both discrete-time and continuous-time. They often convert signals, not the model itself, from discrete-time to continuous-time and vice-versa. And the general approach to continuous-time signal processing takes the form of cascading operations (Mayhan, 1984). The signal is converted into discrete-time, processed on a digital computer, and then reconstructed back into continuous-time from the processed digital representation. The theoretical basis of this approach lies in the Sampling Theorem (Oppenheim and Willsky, 1997). The Sampling Theorem, a fundamental concept in signal processing and communications, states that for a given analog continuous-time signal, there is a minimum rate at which the signal must be sampled to avoid information loss when it is reconstructed from its digital samples. It is a mathematical framework for converting an analog continuous-time signal into a digital discrete-time signal. The signal conversion between these two representations involves using mathematical techniques such as the Z-transform, Laplace transform or the Fourier Transform. The Z-transform is used for discrete-time signals, the Laplace transform for continuous-time and discrete-time signals, and the Fourier transform for continuous-time signals. For example, the Fourier transform is widely used in radio frequency communication systems, wireless communication with autonomous vehicles, image processing, and control systems. The choice of the technique depends on the specific application and the desired outcome. Each representation has its advantages and disadvantages. Discrete-time signals are more suitable for representing digital signals which can be sampled and processed on a computer. Since they can have a finite number of values, they can be processed more precisely than continuous-time signals.

Furthermore, as they can be sampled at regular intervals, the data rate can be reduced and the signal processing can be simplified. However, sampling a continuous-time can result in loss of information as a discrete-time signal approximates the continuous-time signal. Continuous time signals, on the other hand, capture more precisely the dynamic of the underlying physical systems. However, compared to discrete-time signals, continuous-time signals are more susceptible to noise and interferences.

To enable and implement these signal processing techniques and conversions from one representation to the other, MATLAB, a software package widely used in engineering and computer science, provides a method via the built-in function “d2c” for converting a discrete-time system model to continuous-time, using either zero-order hold (zoh), first-order hold (foh) or bilinear (Tustin) approximation methods. This is useful for performing stability or frequency response analyses on a continuous-time model. MATLAB also offers a convenient way to convert a continuous-time model to a discrete-time model via the built-in function “c2d”. This is necessary for simulating and analyzing digital control systems using digital technology. In addition to MATLAB, Python, one of the most popular programming languages in data science and machine learning, offers a built-in function similar to the MATLAB “c2d” called “cont2discrete”. It is possible to call and use MATLAB, Python, or other programming language functions from system dynamics (Dungan, 2016), (Hesan et al, 2014), PySD. However, this later approach might be cumbersome for non-programmers building system dynamics models.

Since we focus more on the models and model conversions than the signals and signal conversions, we will not discuss the mathematical and computer science techniques applied to signals analysis and conversions in this dissertation.

2.5 Discretization of continuous-time models

Discretization is a numerical method commonly used by a digital machine or computer to approximate solutions to differential equations by dividing the continuous domain into a finite number of discrete points. This technique is used in system dynamics. It is a process used in numerical analysis of converting continuous values from a model or a function into a discrete, finite set of values to approximate the behavior of a system overtime.

There are several types of discretization methods for differential equations. Some of the most commonly used are:

Forward difference: this method approximates the derivative at a point using the function values at the current point and the next point in the discretized domain. The Euler method is a forward difference discretization method for approximating the solution of a first-order ordinary differential equation (ODE).

Backward difference: this method approximates the derivative at a point using the function values at the current point and the previous point in the discretized domain.

Central difference: this method approximates the derivative at a point using the function values at the current point and both the previous and next points in the discretized domain.

Upwind difference: this method approximates the derivative at a point using the function value at the current point and the value at the nearest upwind (i.e., in the flow direction).

Downwind difference: this method approximates the derivative at a point using the function value at the current point and the value at the nearest point downwind (i.e., opposite to the direction of the flow).

Crank-Nicolson method: this method combines the forward and backward difference methods based on the trapezoidal rule and gives a second-order approximation of the derivative. It provides a good balance between accuracy and computational efficiency.

Tarasov exact discretization method: this method is a numerical technique that allows for the accurate and efficient solution of differential equations by discretizing the differential operators to preserve the exactness of the underlying equations. The main feature of this method is that it uses the power-law memory technique. This technique is called power-law memory because the method describes the relationship between the solution interval and the memory of the system. This method ensures that the discrete solution is exact and avoids errors arising from other discretization methods, such as Euler's method. Tarasov exact discretization method has been applied in the study of nonlinear economic models and has shown how it can be used to obtain exact solutions and numerical approximations of these models. In (Tarasova, Tarasov, 2016, 2017), the authors applied the exact discretization method with power-law memory to discretize the continuous-time Harrod-Domar's model, allowing for more accurate simulations of the dynamics model. Their approach highlights the importance of precise discretization techniques for obtaining reliable simulation results. Therefore, we recommend implementing this discretization method in system dynamics software packages.

There are also many other variations and extensions of the basic methods described above, such as the Runge-Kutta and Adams-Bashforth methods, which can provide higher order accuracy and produce more accurate approximations than Euler's. The Euler discretization error depends on the size of the solution interval. The smaller the solution interval in the Euler method, the smaller the error; the larger the solution interval, the more computing power is required. The choice of which method to use depends on the problem, the type of models being analyzed, the specific properties of the differential equation, and the desired balance between accuracy and computational efficiency. Different discretization methods are available in most system dynamics software packages. Euler's and the Runge-Kutta methods are the most used techniques in system dynamics modeling.

2.6 Comparing discrete-time and continuous-time

In this section, we will compare and contrast the two modeling techniques using a macro model, Samuelson’s multiplier-accelerator model, a micro model, the cobweb model, and a behavioral economic model, the quasi-hyperbolic discounting.

Before reproducing, comparing, and contrasting the three models in the two techniques, we will first build and describe the discrete-time molecule that will use in building the three models in discrete-time.

2.6.1. Discrete time molecule

The discrete-time molecule can be used to build any discrete-time model in system dynamics. The molecule must be used with the Euler integration method. A molecule like this is very important in bringing behavioral economics structures into system dynamics. An example of this is demonstrated in the section below with the quasi-hyperbolic discount model.

Below are the five equations of the molecule:

1. Initial_Lagged_var1_t-1 = 0
2. Lagged var1 t–1 = Initial_Lagged_var1_t-1
3. Chge in Lagged var1 t -1 = (IF TIME = INT(TIME) THEN (var_1 - Lagged_var1_t-1)/DT ELSE 0)
4. Lagged var2 t –1 = DELAY(Lagged_var1_t-1, 1 , Lagged_var1_t-1)
5. var2 t = (IF TIME = INT(TIME) THEN Lagged_var1_t-1 ELSE Lagged_var2_t-1)

The INT(input) returns the integer value of the input. This is similar to Sterman and Oliva’s Pick function (Sterman, 1984), (Oliva, 1995) or the SAMPLE function in DYNAMO system dynamics programming language. The stock "Lagged var1 t–1" and flow "Chge in Lagged var1 t -1" create the lag of the variable var1 in discrete-time which is different than the information smoothing formulation Low used in (Legasto, Forrester, and Lyneis, 1980, p. 107) and (Randers, 1980, p.76).

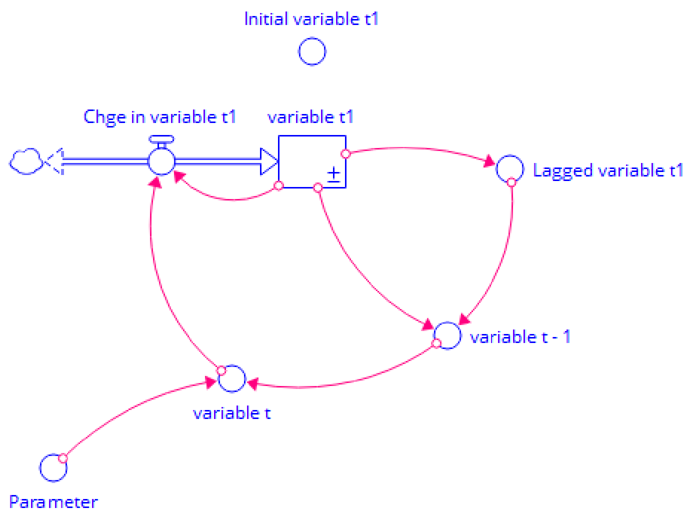


Fig.2.1. Discrete-time molecule

2.6.2. Converting a discrete-time model into a continuous-time model

In (Goodman, 1989, p. 21), a conversion example from discrete-time to continuous-time is given using the DYNAMO, the first system dynamics simulation programming language. To convert a discrete-time model to its continuous-time version, Goodman replaces DT or Δt, the time interval in the discrete-time to dt, i.e., the solution interval in the continuous-time version. This solution is not generalizable and challenging, particularly on second-order difference equations. Our solution is similar but different. We offer a generic and repeatable solution to convert a discrete-time model into continuous-time. This solution is based on the discrete-time mean value theorem. The discrete-time mean value theorem extends the mean value theorem for continuous functions to the discrete case, where the function is defined only at discrete points in time. It is used to derive formulas for approximating the derivative of a function using its values at discrete-time points. The discrete-time mean value theorem states that for any function f(t) that is defined and continuous over the interval [t- Δt, t] and differentiable over the open interval (t- Δt, t), there exists a point c within (t- Δt, t) such that:

$f(t) - f(t-\Delta t) = f'(c)^{*} \Delta t$

where f'(c) is the average rate of change of the function over the interval [t- Δt, t].

$$\frac{f(t) - f(t-\Delta t)}{\Delta t} = f'(c)$$

Taking the limit as $\Delta t \rightarrow 0$, and keeping in mind that $c \in [t - \Delta t, t]$, one gets

$$\lim_{\Delta t \rightarrow 0} \frac{f(t)-f(t-\Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} f'(c)$$

$$\lim_{\Delta t \rightarrow 0} f'(c) = f'(t)$$

$$f'(t) = \frac{df(t)}{dt}$$

In the conversion process, we will replace to the limit $\frac{f(t)-f(t-\Delta t)}{\Delta t}$ by $\frac{df(t)}{dt}$ to create the continuous-time equation. For example, when $\Delta t = 1$, we will replace the expression

$f(t) - f(t-1) = f'_t$ by $\frac{df(t)}{dt}$ and the expression $[f(t) - f(t-1)] - [f(t-1) - f(t-2)] = f'_t - f'_{t-1} = f''_t$ in the discrete-time equation will be replaced by $\frac{d^2 f(t)}{dt^2}$

It is convenient to write the equation with the highest derivative on the right-hand side to build the system dynamics from the continuous-time equation converted from the discrete-time equations.

2.6.3. Samuelson Multiplier-Accelerator Model in Discrete-time

In this section, we will reproduce in system dynamics the Samuelson linear Multiplier-Accelerator model initially developed in discrete-time by Paul Samuelson in the mid-30s. One of the discrete-time versions will be built using an infinite order delay (IOD) (Radzicki, 2019), and the other version will be based on the new discrete-time generic structure or molecule presented in this chapter. Using the previous method, we will also convert the discrete-time model into a continuous-time model.

The Samuelson Multiplier-Accelerator model explains how changes in investment can affect the level of output and income in an economy over time. The model is based on two key ideas: the multiplier effect and the accelerator effect. The multiplier effect, on the one hand, refers to the notion that an initial increase in investment can lead to a much more significant increase in output and income in the economy. However, the multiplier alone cannot adequately explain the cyclical and cumulative nature of the economic fluctuations. The accelerator effect, on the other hand, refers to the idea that changes in the level of investment can lead to changes in the output and income growth rate over time. The interaction between the multiplier and accelerator gives rise to cyclical movements in economic activity. The equations of Samuelson's multiplier-accelerator model are defined as follows:

$$Y_t = g_t + C_t + I_t \quad (1)$$

$$C_t = \alpha\ Y_{t-1} \quad (2)$$

$0 < \alpha < 1$; α is the average propensity to consume.

$$I_t = \beta\ (C_t - C_{t-1}) \quad (3)$$

β is the acceleration coefficient which is the same as the "relation" in Hansen terminology.

$$g_t = 1 \quad (4)$$

Substituting C_t in I_t

$$I_t = \alpha\ \beta\ (Y_{t-1} - Y_{t-2}) \quad (5)$$

Rewriting equation (1) with (2) and (5), we have

$$Y_t = 1 + \alpha\ Y_{t-1} + \alpha\ \beta\ Y_{t-1} - \alpha\ \beta\ Y_{t-2} \quad (6)$$

$$Y_t = 1 + \alpha\ (1 + \beta)\ Y_{t-1} - \alpha\ \beta\ Y_{t-2} \quad (7)$$

The model below has three sub-models, one continuous-time, and two discrete-time sub-models.

Model 1 (the discrete-time molecule presented in this chapter) and model 2 (Radzicki, 2019) generate precisely the same behavior. The main difference between these two models is that model 1 presents a stock-flow structure explicitly showing major and minor feedback loops. Model 1 uses a similar but different approach than the Sterman and Oliva's Pick function (Sterman, 1984), (Oliva, 1995) to create the discrete-time behavior. This Pick function extracts every data point available from time series data to calculate summary statistics for historical fit. In model 1, we use the built-in function INTEGER in Vensim and INTEG in Stella, available in system dynamics software packages and programming languages. Model 1 must run with the Euler method. With the INTEG or INTEGER function, we extract discrete-time values from the system dynamics model. Model 2 (Radzicki, 2019), much more straightforward, has implicit stocks, the infinite order delay. Model 2 works both with Euler and Runge Kutta methods. With model2, we do not see the stock flow structure which is important in system dynamics modeling. With model1, we use the molecule to see the stock flow structure, the lags, the minor and major loops.

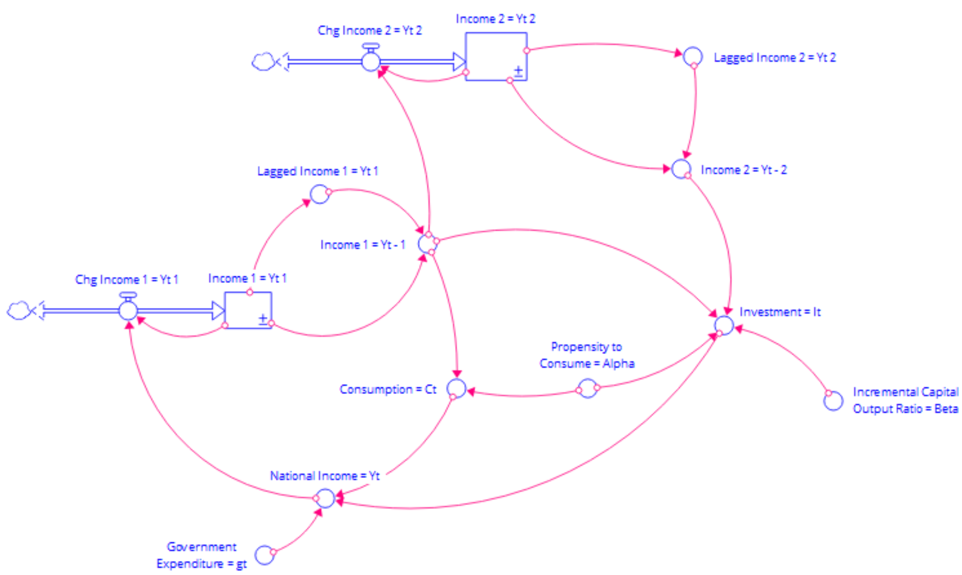


Fig. 2.2. Discrete-time model 1

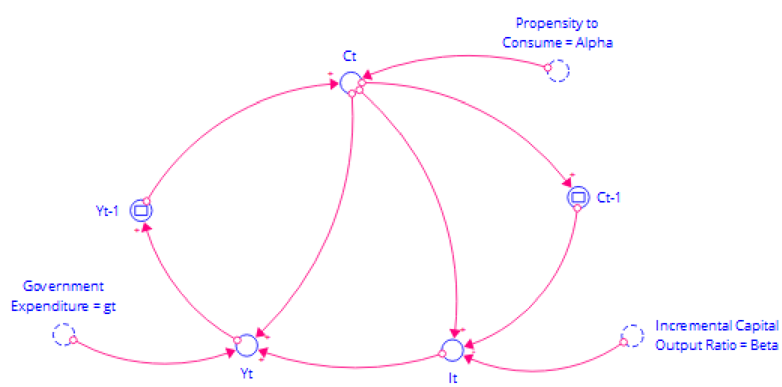


Fig. 2.3. Discrete-time model 2

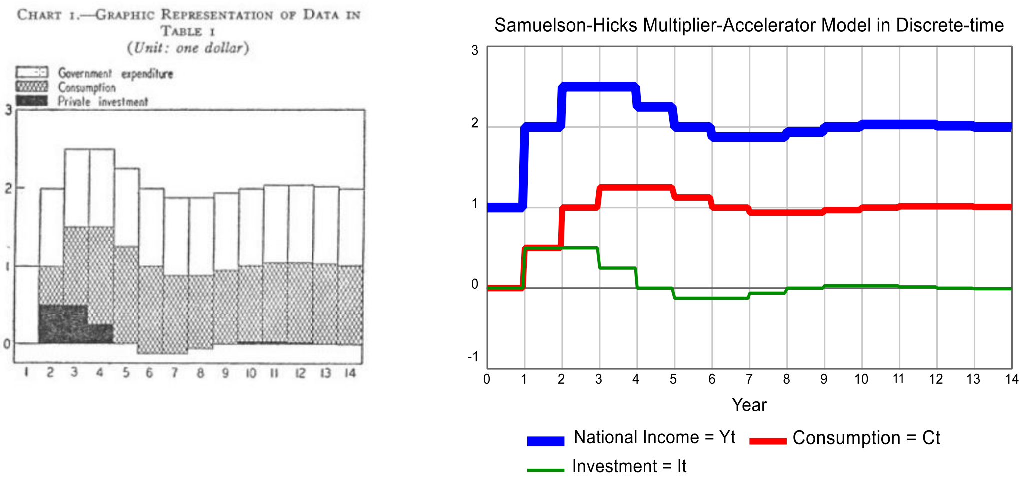


Fig. 2.4. Samuelson's MA (Samuelson, 1939) and its discrete-time rendition in system dynamics

2.6.4. Samuelson Multiplier-Accelerator Model in Continuous-time

We will start with the discrete-time model and convert it following the conversion process described in the previous section on converting a discrete-time model into a continuous-time model.

The solution of the discrete-time Samuelson Multiplier-Accelerator model (Samuelson, 1936, p76) is:

$$\mathbf{Y_t = 1 + \alpha(1+\beta)\ Y_{t-1} - \alpha\beta\, Y_{t-2}} \qquad (8)$$

$$Y_t = 1 + \alpha\,[(1+\beta)\ Y_{t-1} - \beta\ Y_{t-2}]$$

$$Y_t = 1 + \alpha\,[\,Y_{t-1} + \beta\, Y_{t-1} - \beta\, Y_{t-2}\,]$$

$$Y_t = 1 + \alpha\,[\,Y_{t-1} + \beta\,(Y_{t-1} - Y_{t-2})]$$

$$Y_t = 1 + \alpha\,[\,Y_{t-1} + \beta\,(Y_t - Y_{t-1}) - \beta\,(Y_t - Y_{t-1}) + \beta\,(Y_{t-1} - Y_{t-2})]$$

$$Y_t = 1 + \alpha\,[Y_{t-1} + \beta\,(Y_t - Y_{t-1}) - \beta\,((Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}))];\ \ Y'_t = Y_t - Y_{t-1}$$

$$Y_t = 1 + \alpha\,[Y_{t-1} + \beta\, Y'_t - \beta\,((Y'_t - Y'_{t-1})]$$

$$Y_t = 1 + \alpha\, Y_{t-1} + \alpha\beta\, Y'_t - \alpha\,\beta\,((Y'_t - Y'_{t-1})]$$

$$Y_t = 1 + \alpha\, Y_t - \alpha\, Y_t + \alpha\, Y_{t-1} + \alpha\beta\, Y'_t - \alpha\,\beta\,(Y'_t - Y'_{t-1})];\ \ Y''_t = Y'_t - Y'_{t-1}$$

$$Y_t = 1 + \alpha\, Y_t - \alpha(Y_t - Y_{t-1}) + \alpha\beta\, Y'_t - \alpha\,\beta\, Y''_t$$

$$Y_t - \alpha\, Y_t = 1 - \alpha Y'_t + \alpha\beta\, Y'_t - \alpha\,\beta\, Y''_t$$

$$(1-\alpha)\, Y_t = 1 - \alpha Y'_t + \alpha\beta\, Y'_t - \alpha\,\beta\, Y''_t$$

$(1 - \alpha)\, Y_t = 1 + \alpha(\beta - 1)\, Y'_t - \alpha\, \beta\, Y''_t$

$\alpha\, \beta\, Y''_t = 1 + (\alpha - 1)\, Y_t + \alpha(\beta - 1)\, Y'_t$

$$Y''_t = \frac{1}{\alpha\, \beta} + \frac{(\alpha - 1)}{\alpha\, \beta}\, Y_t + \frac{(\beta - 1)}{\beta}\, Y'_t \qquad (9)$$

Applying the results from section the conversion procedure and mean value theorem, and taking the limit as the solution interval approaches zero, we have:

$$\frac{d^2Y(t)}{dt^2} = \frac{1}{\alpha\, \beta} + \frac{(\alpha - 1)}{\alpha\, \beta}\, Y(t) + \frac{(\beta - 1)}{\beta}\, \frac{dY(t)}{dt} \qquad (10)$$

$$\frac{dY(t)}{dt} = X(t)$$

$$\frac{d^2Y(t)}{dt^2} = \frac{dX(t)}{dt}$$

From equation (2), we have

$C_t = \alpha\, Y_{t-1}$

$C_t = \alpha\, (Y_t - Y_t + Y_{t-1})$

$C_t = \alpha\, (Y_t - (Y_t - Y_{t-1}))$; $\qquad Y'_t = Y_t - Y_{t-1}$

$C_t = \alpha\, (Y_t - Y'_t)$

$$C(t) = \alpha\, (Y(t) - \frac{dY(t)}{dt}) \qquad (11)$$

From equation (5), we have :

$I_t = \alpha\, \beta\, (Y_{t-1} - Y_{t-2})$

$I_t = \alpha\, \beta\, (\, (Y_t - Y_{t-1}) - (\, (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})\,)\,)$

$I_t = \alpha\, \beta\, (\, Y'_t - (\, Y'_t - Y'_{t-1}\,)\,)$, taking the limit as the solution interval approaches to zero,

$$I(t) = \alpha\, \beta\, (\, \frac{dY(t)}{dt} - \frac{d^2Y(t)}{dt^2}\,) \qquad (12)$$

The system dynamics translation of equation (10) is shown in Fig. 2.6.

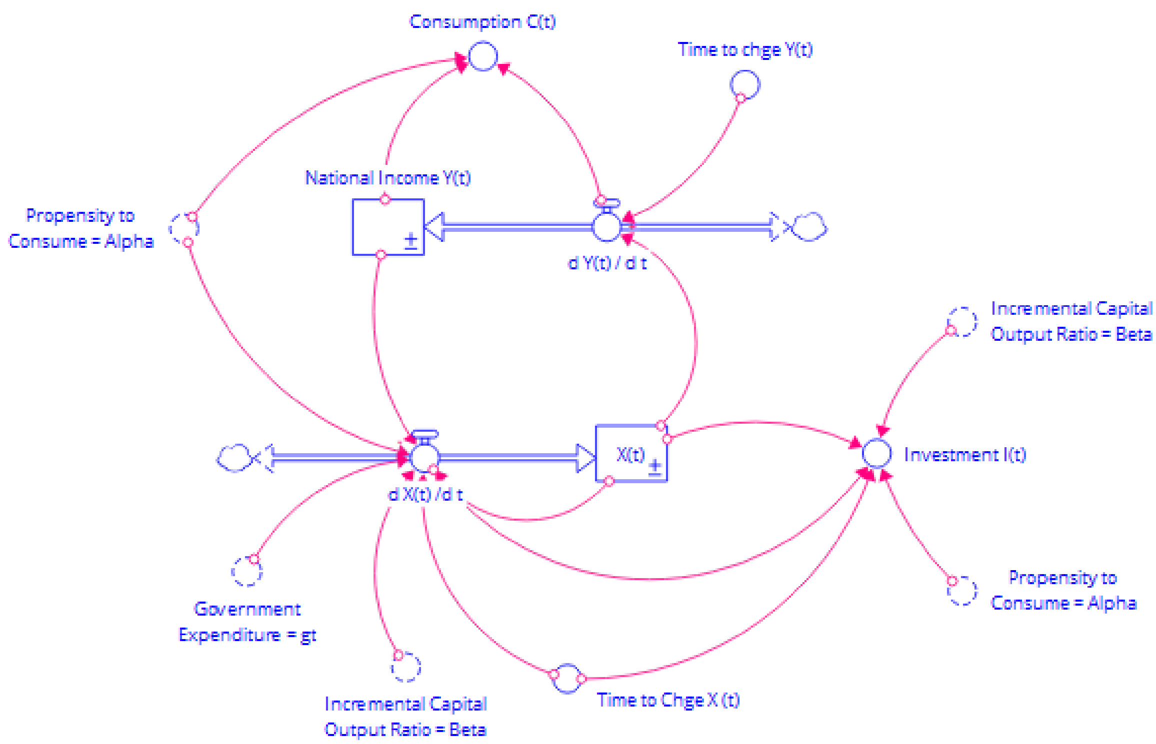


Fig. 2.6. Samuelson's MA model converted in continuous-time

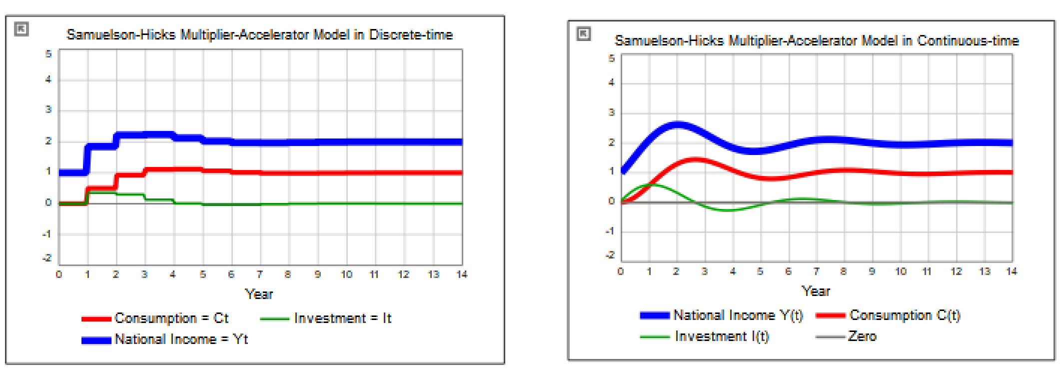


Fig. 2.7. Discrete-time vs. continuous-time, $\alpha = 0.5$, $\beta = 0.8$

Depending on the parameter's values, the continuous-time and the discrete-time Samuelson's multiplier-accelerator model can generate different behaviors. For example, Samuelson's multiplier-accelerator continuous-time model generates a new region when the incremental Capital Output Ration beta =1 as compared to the discrete-time version. For example, the graphs below in Fig. 2.8 are the results with alpha = 0.5

and beta = 1 in discrete and continuous-time. In this region, the continuous-time model generates sustained oscillations whatever the policy change, for example, by increasing government spending. In contrast, with the discrete-time model, the system will reach equilibrium from year 13.

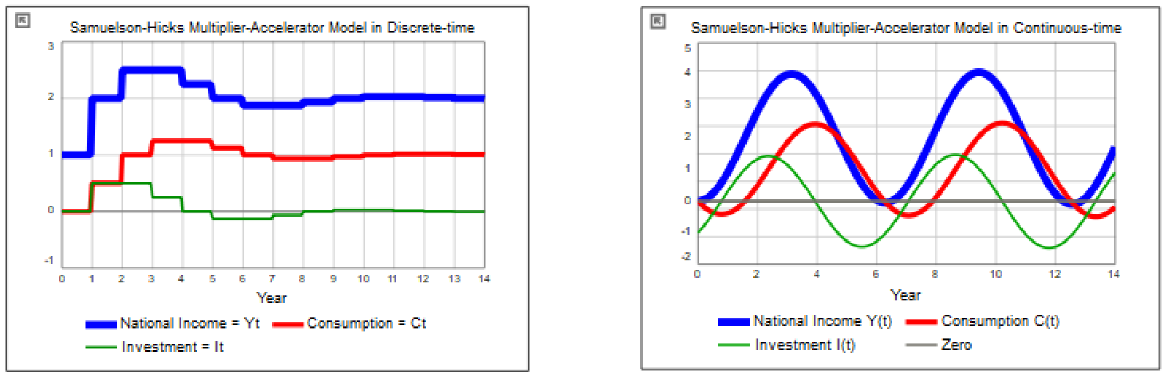


Fig. 2.8. Discrete-time vs. continuous-time, α = 0.5, β = 1

2.6.5. Cobweb Model in Discrete-time

The Cobweb model is the second model to compare the two modeling methods. The cobweb model is a dynamic microeconomic model that describes the behavior of prices and quantities in a market where there is a time delay between the adjustment of supply and demand. The suppliers decide how much to sell at ex-post based on the previous price and bring it forward to the market in the next period. The buyers, in ex-post, in the current period, when they discover the price, decide how much to buy. The transition between the two periods has to be specified, generally one time lag.

The equations of the classical cobweb model in the discrete-time model are as follows (R.G. Allen, 1965, p.4)

$D_t = a + b * P_t$

$S_t = c + d * P_{t-1}$

$p_t = g * p_{t-1}$

$p_t = P_t - P_e$; P_e is the equilibrium price

$$g = \frac{d}{c}$$

The solution of this model is:

$p_t = p_0 * g^t$ or

$P_t = P_e + (P_0 - P_e) * g^t$

We used the extended Cobweb model described in (R.G.D. Allen, 1965, pp. 13-14) and built it into system dynamics. This model introduces a price expectation on the supply side as

$P_{t-1} - \rho \Delta P_{t-2}$; where $0 < \rho \leq 1$, ΔP_{t-2} is the price increase from period t-2,

$\Delta P_{t-2} = P_{t-1} - P_{t-2}$

The equations of the extended model are as follows (R.G. Allen, 1965, p. 14, equation 4):

$$p_t = c\,(1 - \rho)\, p_{t-1} + c\rho\, p_{t-2} \qquad (13)$$

$p_t = P_t - P_e$

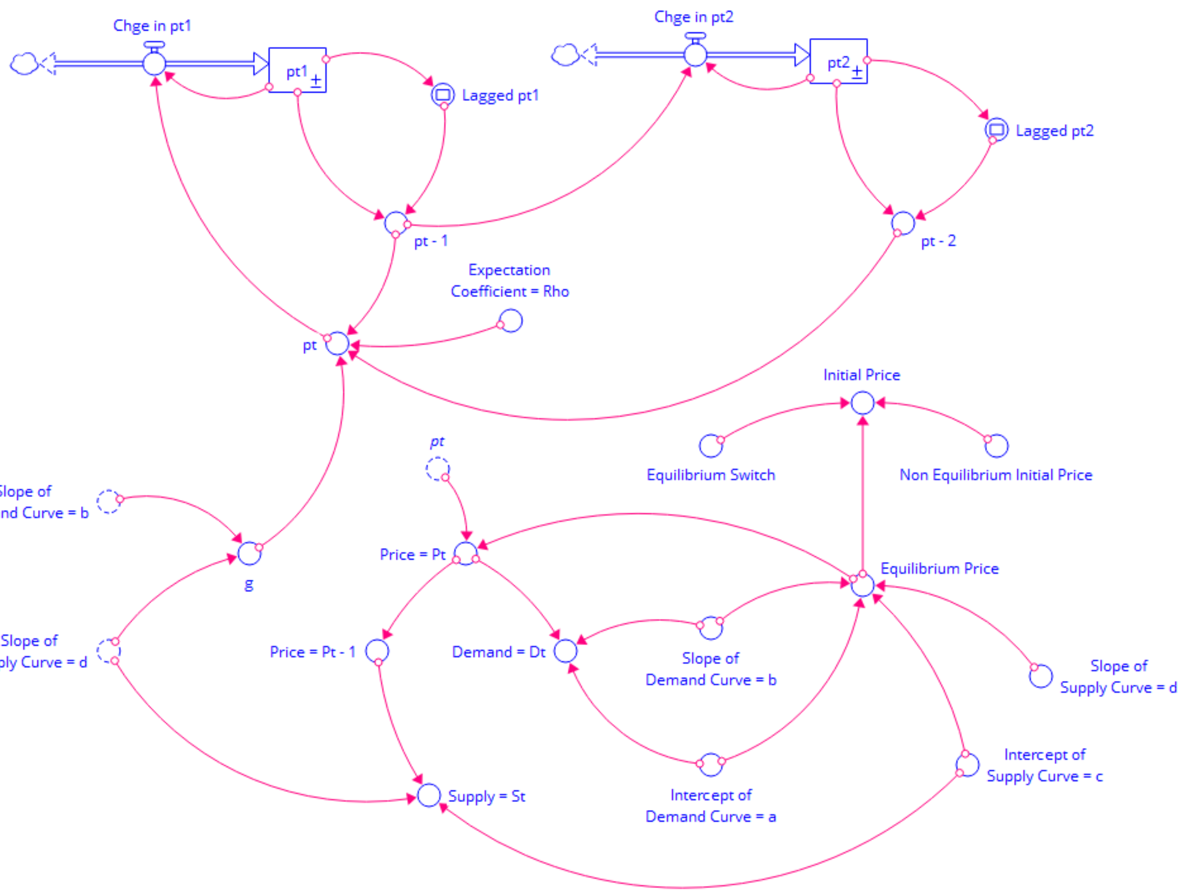
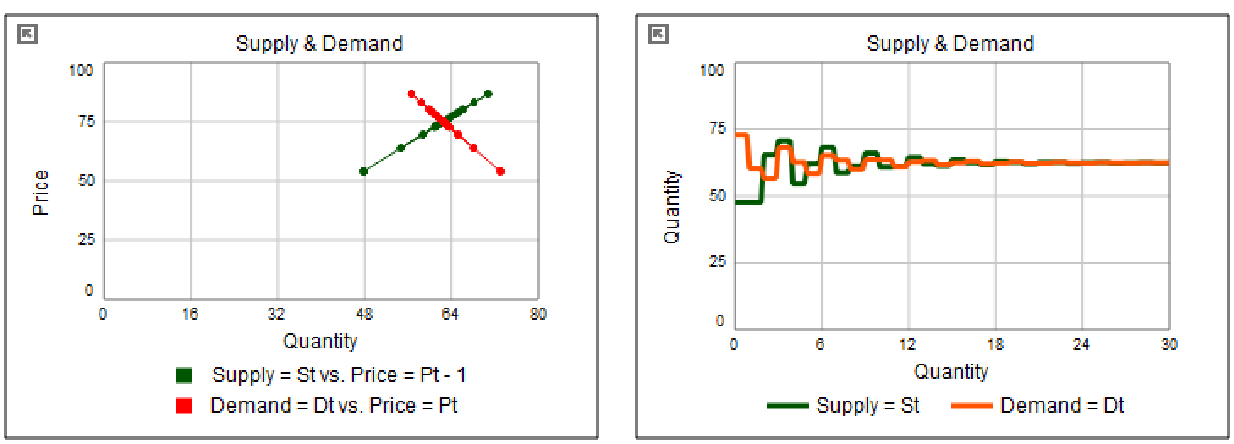


Fig. 2.9. Cobweb model in discrete-time



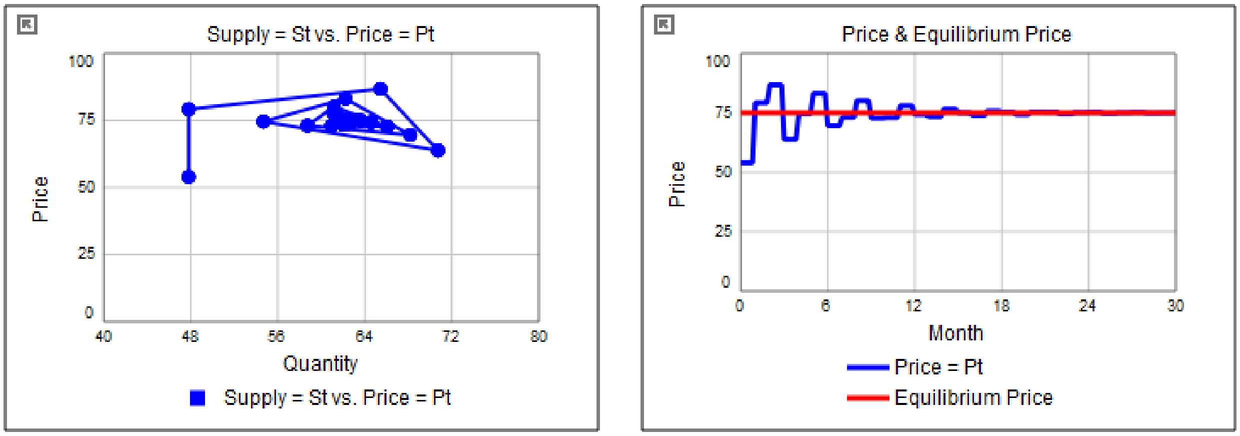


Fig. 2.10. Cobweb model outputs in discrete-time

2.6.6. Cobweb Model in Continuous-time

Interestingly, equation (13) is similar to Samuelson's multiplier-accelerator discrete-time second-order linear equation (8), $Y_t = 1 + \alpha\,(1 + \beta)\;\; Y_{t-1} - \alpha\beta\, Y_{t-2}$ where $\beta = -\rho\,, \alpha = g \;=\; \frac{d}{c}$ (the supply curve and the demand curve ratio) and excluding government spending from the equation. The continuous-time solution from the discrete-time version of Samuelson's multiplier-accelerator is:

$$\frac{d^2Y(t)}{dt^2} = \frac{1}{\alpha\,\beta} + \frac{(\alpha - 1)}{\alpha\,\beta}\,Y(t) + \frac{(\beta - 1)}{\beta}\,\frac{dY(t)}{dt}$$

Substituting β by $-\rho$ and α by g, we have:

$$\frac{d^2p(t)}{dt^2} = \frac{(1-g)}{g\,\rho}\,p(t) + \frac{(\rho+1)}{\rho}\,\frac{dp(t)}{dt} \qquad (14)$$

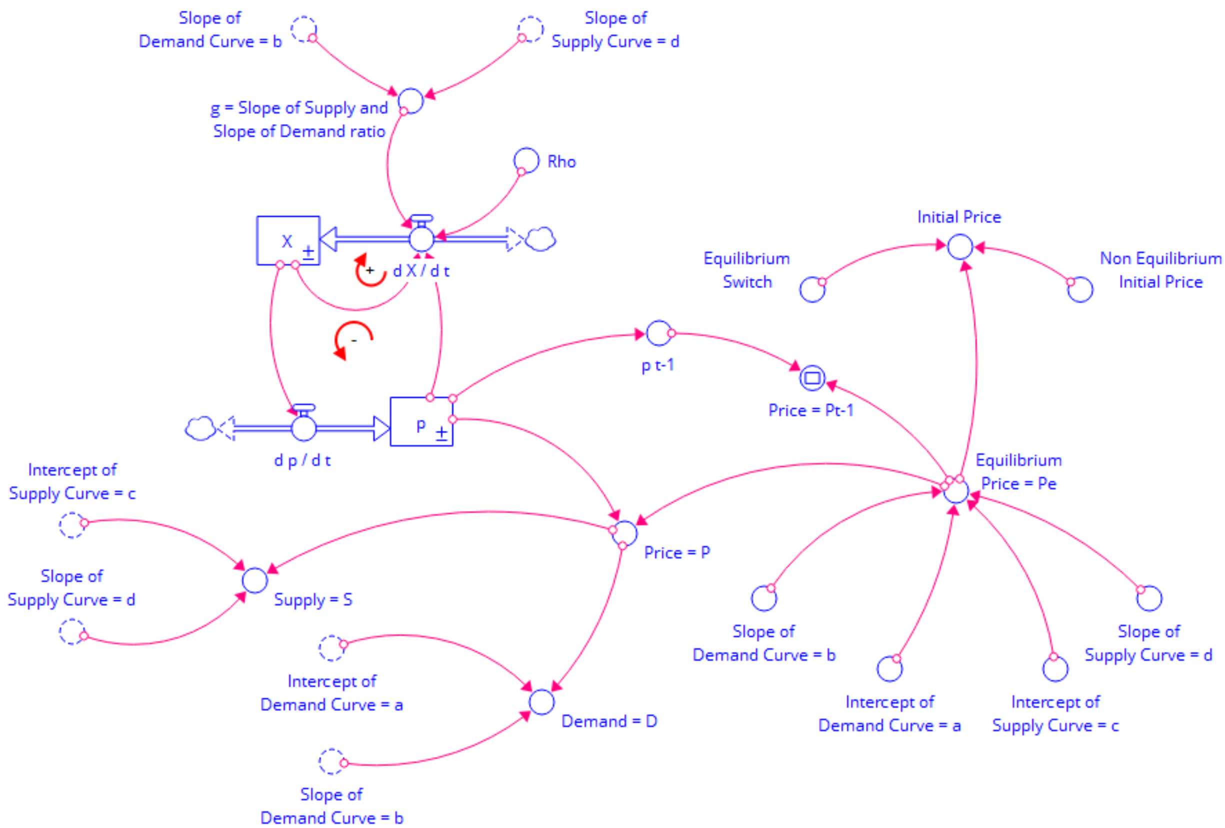


Fig. 2.11. Cobweb model in continuous-time

The continuous-time version of this Cobweb model converted from the discrete-time version generates a different and inconsistent behavior from the same input parameters as in its original discrete-time version. The continuous-time version will have an impact on policy change. In this case, we will reject this continuous version and build a new one, empirically grounded, from scratch following the system dynamics standard method.

One of the limitations of both discrete-time models, Samuelson's multiplier-accelerator and the Cobweb, but discrete-time in general, is they do not have explicit stock, e.g., inventory of goods in the case of the Cobweb model. And in reality, for the Cobweb model, for example, it takes time to produce the goods. As the goods are made, they enter the stock and backlog before being shipped and sold to the market.

Other variations of the Cobweb model incorporate stock of goods (R.G.D Allen, 1965, pp. 15-19). However, even though the discrete-time version and the converted versions of these models specify the stock of goods (R.G.D. Allen, 1965, pp. 15-19), they do not explicitly model the stock or inventory of goods as done in system dynamics.

2.6.7. Quasi-hyperbolic discount model

This section will describe and build the third model based on the behavioral economics literature related to intertemporal choice. We will look at the present-bias model, the quasi-hyperbolic discount model. The quasi-hyperbolic discount model is a concept in behavioral economics that seeks to explain how people make decisions about rewards available at different points in time in the future. Unlike the standard exponential discounting model that assumes people are rational and discount future rewards at a constant rate, the quasi-hyperbolic discount model proposes that people use two different discount rates: a "present-biased" rate for short-term rewards and a "future-biased" rate for long-term rewards. The present-biased rate reflects the higher value people place on instant gratification. The future-biased rate reflects the greater willingness to wait for delayed gratification and to receive larger rewards later. This means that people tend to prefer smaller rewards that are available immediately over larger rewards that will be available later. However, as the delay between the decision and the reward increases, people become more patient, and their discount rate shifts to the future-biased rate. The quasi-hyperbolic discount model has been used to explain a wide range of phenomena, such as procrastination, addiction, and savings behavior

This model can be used as a molecule in a larger system dynamics structure to model intertemporal problems such as retirement lifecycle with a utility function and budget constraints specified. We will show how to faithfully reproduce the present-bias in system dynamics both in discrete-time and continuous-time.

The quasi-hyperbolic discount model (Laibson, 1997) is described as follows:

$$\text{Quasi} - \text{hyperbolic discount (t)} = \begin{cases} 1 \text{ , t} = 0 \\ \beta\delta^{t}\text{, t} > 0 \end{cases} \text{, } 0 < \beta, \delta <= 1 \qquad (15)$$

"Note that the quasi-hyperbolic discount function is a discrete-time function with values {1, $\beta\delta$, $\beta\delta^2$, $\beta\delta^3$, …}", (Laibson, 1997, p. 450).

The parameter δ, which is the same as in the exponential discounting model, called the standard discount factor captures the long-run and time-consistent discounting, whereas the parameter β, called the present bias factor captures the short-term impatience and time inconsistency discounting. When the beta factor is equal to 1, the present bias model is the same as the exponential discounting model.

The system dynamics representation of equation (15) is shown in Fig. 2.13. To test and validate our model, we compare it with the exact analytical solution of the quasi-hyperbolic function and ensure it replicates the behavioral economics literature results.

Fig.1.12. below shows the behavior modes of the three discounting functions from the behavioral economics literature (Laibson, 1997, p.450).

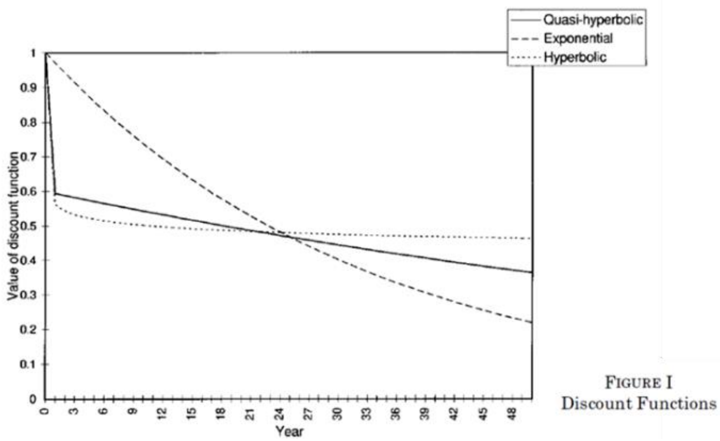


Fig. 2.12. Discount Functions, (Laibson, 1997, p. 450)

Notice the function $\text{discount}_t = a_0 * \delta^t$ where $a_0 = 1$ is the solution of the function $\text{discount}_t = \delta * \text{discount}_{t-1}$

These discounting functions can be reproduced with system dynamics tools using a solution interval of less than 1 as shown in Fig. 1.16.

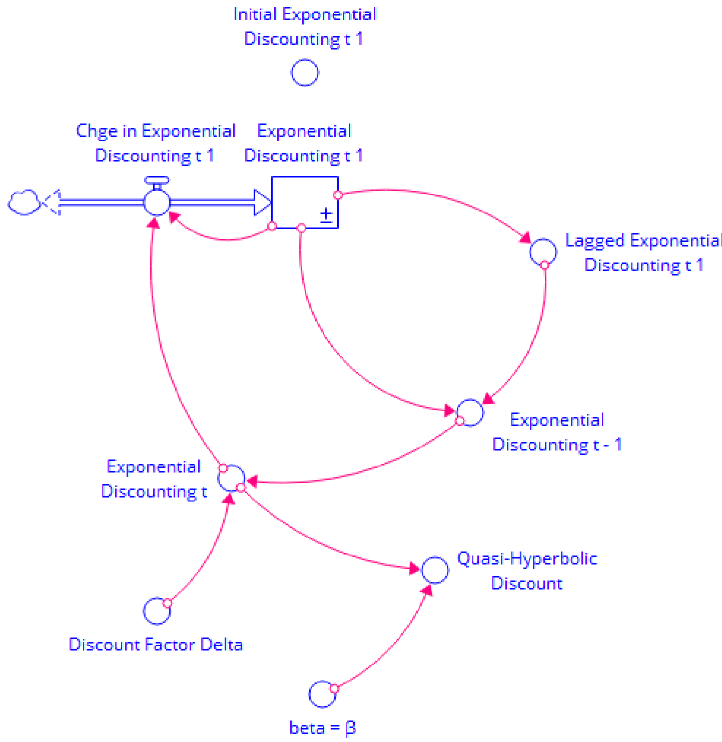


Fig. 2.13. Quasi-hyperbolic discount model in discrete-time

Here is an example of how to use the discrete-time version of the quasi-hyperbolic discount model can be used:

The agent has an exogenous stream of income in N+1 periods Y_0, Y_1, , ... , Y_N

She can consume in each period C_t , save $S_t = Y_t - C_t$, and earn a fixed interest r on her savings.

Her instantaneous utility is:

$u(C_t) = \ln(C_t)$

$\max_{c_0, c_1, \ldots, c_N} U = \ln(C_0) + \sum_{t=0}^{N} \beta * \delta^t * \ln(C_t)$

s.t. budget constraint:

$\sum_{t=0}^{N} \frac{C_t}{(1+r)^t} = \sum_{t=0}^{N} \frac{Y_t}{(1+r)^t}$

2.6.8. Converting the discrete-time present-bias model into continuous-time

In this section, we will convert the expression $\delta^t, t > 0$ in equation (15) from discrete-time to continuous-time. Let $f(t) = \delta^t$.

$f(t-1) = \delta^{t-1}$

$f(t-1) = \delta^t\delta^{-1}$

$f(t-1) = f(t)\delta^{-1}$

$f(t) = \delta f(t-1)$

$f(t) = \delta f(t) - \delta f(t) + \delta f(t-1)$

$f(t) = \delta f(t) - \delta\,(f(t) - f(t-1))$

$f(t) - \delta f(t) = \delta f(t) - \delta\,(f(t) - f(t-1))$

$-\delta\left(f(t) - f(t-1)\right) = (1-\delta)\,f(t)$

$\delta\left(f(t) - f(t-1)\right) = (\delta - 1)\,f(t)$

$f(t) - f(t-1) = -\frac{(1-\delta)}{\delta} f(t)$; replacing $f(t) - f(t-1)$ by $\frac{df(t)}{dt}$, we have

$$\frac{df(t)}{dt} = -\frac{(1-\delta)}{\delta} f(t) \qquad (16)$$

$$\frac{df(t)}{dt} = -\rho\, f(t) \text{ ; where } \rho = \frac{(1-\delta)}{\delta} \text{ is the discount rate} \qquad (17)$$

$$f(t) = e^{-\rho t} \qquad (18)$$

The continuous-time version of the converted discrete-time quasi-hyperbolic is:

$$\text{Converted Quasi} - \text{Hyperbolic Discount} = \begin{cases} 1, t = 0 \\ \beta\, f(t), t > 0 \end{cases}, \quad 0 < \beta, \delta <= 1 \qquad (19)$$

The system dynamics representation of equation (18) is shown in Fig. 1.14. below. The variable *Initial Converted Exponential Discount f (t)* is equal to 1.

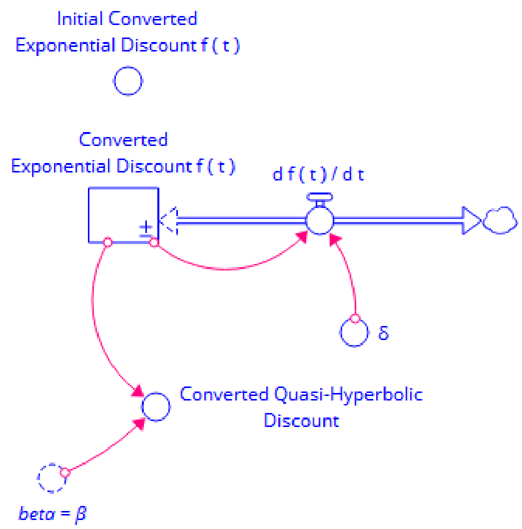


Fig. 2.14. Quasi-hyperbolic discounting model in continuous-time
converted from a discrete-time version

We can see in Table. 1.1., that the behavior of this formulation is very close to the discrete-time version. However, at some points in time, the values of the quasi-hyperbolic discount functions of the discrete-time and converted continuous-time are slightly different. Therefore, a better formulation will be to first rewrite the delta discount function in continuous-time with the best match to its original discrete-time version.

$\ln(\delta^{t}) = \ln(\delta\, t)$

$e^{\ln(\delta^{t})} = e^{\ln(\delta\, t)} = 1 * e^{\ln(\delta\, t)}$

Let $g\,(\,t\,) = e^{\ln(\delta\,t)}$ (20)

Equation (20) is the exact analytical solution of $\frac{dg(t)}{dt} = \ln(\delta\, t)\, g(t)$ (21)

The new continuous-time version of the discrete-time quasi-hyperbolic model is as follows:

$$\text{Quasi} - \text{Hyperbolic Discount } = \begin{cases} 1\,, t = 0 \\ \beta\, g(t), t > 0 \end{cases},\ 0 < \beta, \delta <= 1,\ g(t) = \ln(\delta\, t)\, g(t) \quad (22)$$

The system dynamics representation of equation (22) is shown in Fig. 1.15 below, and the behavior mode is shown in Fig. 1.16.

The individual has an exogenous stream of income in N+1 periods Y_0, Y_1, , ... , Y_N

She can consume in each period C_t , save $S_t = Y_t - C_t$, and earn a fixed interest r on her savings in each period.

Her instantaneous utility is:

$u(C_t) = \ln(C_t)$

$$\max_{c_0,c_1,\ldots,c_N} U = \ln(C0) + \int_{t=0}^{N} \beta * e^{\ln(\delta)t} * \ln(C_t) \text{ or}$$

$$\max_{c_0,c_1,\ldots,c_N} U = \ln(C0) + \int_{t=0}^{N} \beta * e^{-\rho t} * \ln(C_t)$$

s.t. budget constraint:

$$\int_{t=0}^{N} \frac{C_t}{(1+r)^t} = \int_{t=0}^{N} \frac{Y_t}{(1+r)^t}$$

We correctly reproduced the quasi-hyperbolic model (Laibson, 1997, p. 450) in system dynamics continuous-time and discrete-time with a solution interval of less than 1.

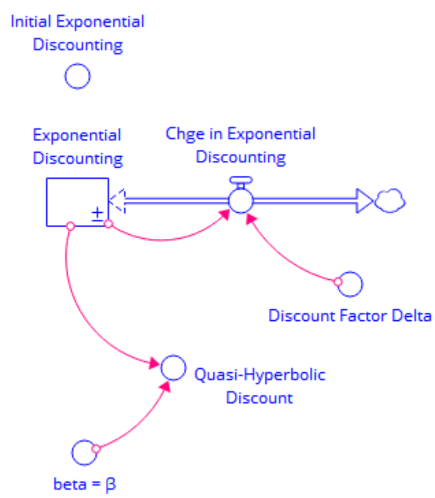


Fig. 2.15. Quasi-hyperbolic model in continuous-time

Time	Quasi-Hyperbolic Discount Discrete-time	Quasi-Hyperbolic Discount Continuous-time	Converted Quasi-Hyperbolic Discount
0	1.000	1.000	1.000
1	1.000	1.000	1.000
2	0.588	0.588	0.564
3	0.582	0.582	0.547
4	0.576	0.576	0.530
5	0.571	0.571	0.514
6	0.565	0.565	0.498
7	0.559	0.559	0.483
8	0.554	0.554	0.468
9	0.548	0.548	0.454
10	0.543	0.543	0.440

Table. 2.1. Exponential discount functions comparison

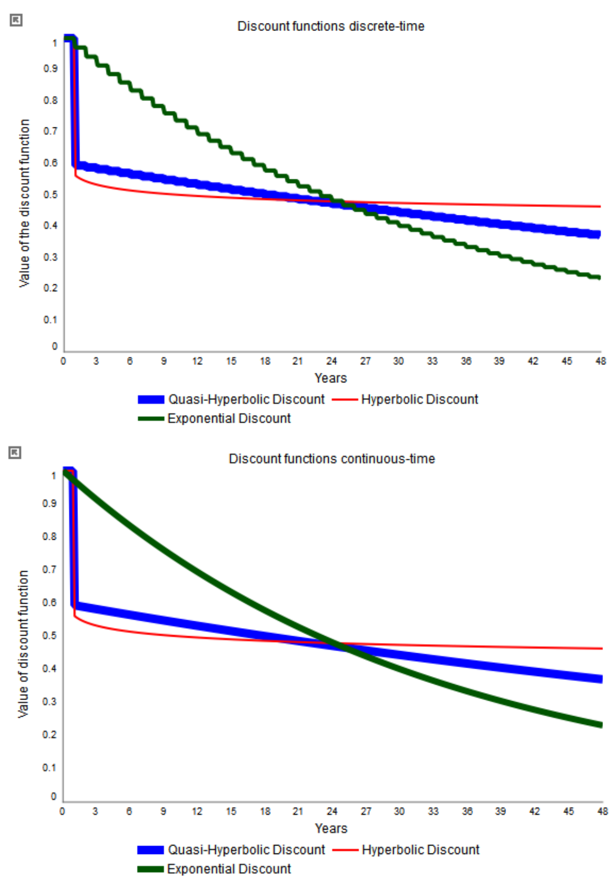


Fig. 2.16. Replication of discounting functions
in discrete-time and continuous-time with a solution interval less than 1

The quasi-hyperbolic system dynamics molecule, either discrete or continuous-time, can be used in a larger system dynamics model to model intertemporal problems.

2.7 When to use discrete-time and when to use continuous-time modeling

Discrete-time is more appropriate for solving operational and tactical issues and where events occur in discrete-time (Lane and Morecroft). Discrete-time is also very useful in gaming, where the decisions are taken at discrete-time intervals and synchronously in the case of a multiplayer game. In reality and quite often, individuals make decisions at different points in time based on multiple sources of information at hand, for example, when comparing the current state and the desired state of the system to take corrective actions, and those decisions can be influenced by each individual's perception, expectations and biases. A good example is the financial market, where decisions at the aggregate level are made in continuous-time. Therefore, continuous-time modeling is more appropriate, particularly in strategic decision situations where judgment, perception, and expectation play a major role. Nature integrates. Modeling problems from a system perspective using the continuous-time approach with stocks and flows, feedback loops, and time delay, enriched with behavioral economics structures, better captures how individuals make decisions and provide more insights to make better recommendations. Discreteness exists in systems. When building differential-difference models or a mixed of continuous-time and discrete-time models, always start with the continuous-time approach and then add discreteness when justified (Forrester, 1961, p.66):

> "When a model has progressed to a point where such refinements are justified, and there is reason to believe that discreteness are has significant influence on system behavior discontinuous variables should be explored to determine their effect on the model".

The choice of the modeling approach should always come after the problem definition and the specification of the underlying assumptions.

2.8 Discussion and future research

In this chapter, we presented a formal, teachable, and repeatable method of bringing decision-making structures from behavioral economics into system dynamics in a manner that is acceptable to both system dynamicists and economists and preserves the results from the behavioral economics literature.

When building models using system dynamics and deciding which tools to use, whether discrete-time or continuous-time, we recommend starting with the problem definition and specifying the underlying assumptions, then, choose the modeling technique, discrete-time or continuous-time, which better matches the problem's underlying assumptions. The resulting model can be either fully discrete or fully continuous or mixed of discrete and continuous, i.e., with differential-difference equations. For a mix of discrete-time and continuous-time models, we suggest following Forrester's recommendation to always start with continuous-time, test the model and when the continuous-time model is robust enough, then add the discrete-time structures. It is crucial to remember that during the conversion process from discrete-time to continuous-time, the model's assumptions can implicitly change (not always) due to the differences in nature of the two types of equations, i.e., the difference and differential equations. There are situations where the converted continuous-time model matches its discrete-time original version without impacting policy change. However, these are exceptional situations.

Some future areas of research about this topic would be (1) to define a method how for selecting a discretization technique for a specific model, (2) to analyze in detail and study the possibility of implementing Tarasov's discretization with power-law memory technique into system dynamics software packages, (3) to compare and contrast the two continuous-time quasi-hyperbolic discount models presented in this chapter in terms of best fit with real data, (4) to develop more behavioral economics structures such as prospect theory, social preferences models in system dynamics using the method presented in this chapter or an extended version of this method and (5) to build an automated tool integrated into system dynamics software packages which convert discrete-time models into continuous-time using the method and the process described in this chapter.

“The two most powerful warriors are patience and time.” - Leo Tolstoy

"Don't save what is left after spending; spend what is left after saving." - Warren Buffett

Chapter 3 Improving a bridge: Dynamic Inconsistency and Self-Control in Lifetime Saving

3.1 Introduction

The goal of this chapter is to explore and analyze a couple of modeling problems found in the paper published in the SDR (System Dynamics Review) 2019, titled “Building a bridge to behavioral economics: countervailing cognitive biases in lifetime saving decisions” and propose a solution based on best practices in system dynamics.

First, we applaud the Bahaddin et al. attempt and contribution to bringing behavioral economics into system dynamics. The behavioral economics literature presents a feedback-poor view of lifetime savings, whereas system dynamics offers a feedback-rich view in dynamic modeling. Developing a catalog of empirically grounded behavioral economics molecules and instructions for their proper use is very important. Most important economic choices are made over time and affect later decisions, most often, inheritably made in the face of fundamental uncertainty. Classical economic theory assumes that individuals are rational and their choices are driven by utility maximization. When individuals are faced with decisions that involve uncertainty, experiments have shown that people have difficulties and often seem to deviate systematically from making rational choices to maximize their utility. Their economic choices and decisions are usually ill-informed and influenced by cognitive biases and non-economics factors such as emotions. The emerging field of behavioral economics, which combines psychology and economics, has yielded important insights into people's choices. Behavioral economics studies the effects of psychological, cognitive, emotional, cultural, and social factors on the economic decisions of individuals. In other words, it uses psychology to understand theories of production, theories of the firm, household behavior, and institutions.

The Bahaddin and al. research areas of interest are to understand better the effects of interacting biased and how they cancel out. “The theorical problem of interest in our research centers on the countervailing cognitive biases, a less understood area of theory which hypothesizes that two competing cognitive biases can interact or even cancel out each other.” (Bahaddin et al. 2019). Their paper analyzes the optimal trade-offs between spending and saving for retirement using system dynamics modeling. The first model (feedback-poor) reproduces the theoretical lifetime utility results from the behavioral economics literature. The second one (feedback-poor) adds two cognitive biases that partially cancel each other out in a counterintuitive manner. The third model (feedback-rich) modifies the first to show how to replace an economic optimization calculation with a decision-making heuristic based on information feedback.

We found a couple of modeling problems in their paper. Firstly, the formulation of the Last consumption variable in the model used as first-order control and to drain out everything left in the Wealth stock is sensitive to the solution interval. Secondly, Labor Income is modeled as a stock. In our view, it should be flow. Thirdly, the models presented by the authors are overall sensitive to the solution interval, and they choose a solution interval of 1. This is probably due to the authors’ perceived need for discrete-time modeling. But there is a better way to create discrete-time molecules in system dynamics with proper time shapes and insensitive to the solution interval. Fourthly, their models' formulation of the exponential discounting function is sensitive to the solution interval. We have a correction to this problem. Lastly, the authors present an outdated discounting model and do not explain why they chose it. Moreover, they cite but do not use a more recent discounting model, the quasi-hyperbolic, that is standard in the behavioral economics literature. We will develop a system dynamics molecule or generic structure for this model.

The plan of this chapter is as follows. The first section presents an improvement to the model presented in the SDR paper by describing the problems. Then, we offer a solution to each of them with a proper way to build the behavioral economics model using the system dynamics approach with best practices (Forrester, 1971). The second section presents the life-cycle model from their paper with an introduction to the present-bias hyperbolic discounting model. In the third section, we will run some experiments and explore countervailing biases on the feedback-poor version of the model. In the fourth section, we will follow the authors’ approach and extend our model to the feedback-rich version. The final section lays out further research and directions.

3.2 Improving Bahaddin et al.'s model

In Fig. 3.1, we highlighted areas in the authors' model where we found problems. Each of them will be looked at in detail, and a solution will be given.

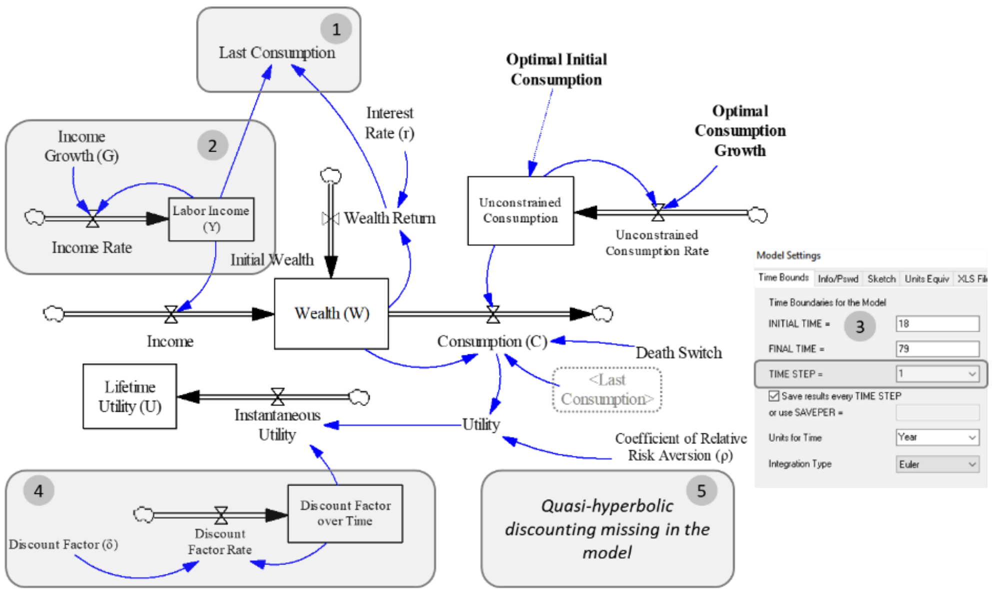


Fig. 3.1. Mapping identified problems in the authors' model

Last Consumption and first-order control

The first problem is related to the first-order control in the *Wealth* stock. The *Last consumption* auxiliary variable in their model captures the fact that the individual will not save for the time after T+1. In the last period T, she will consume everything that is left. The equations in their models are as follows:

Last consumption = IF THEN ELSE(Time > (FINAL TIME - TIME STEP / 2) - 1, "Labor Income (Y)" + Wealth Return + "Wealth (W')" / TIME STEP , 0) ~$/year

Current Consumption = (min (Unconstrained Consumption, "Wealth (W')" / TIME STEP)) * (Death Switch) + Last Consumption ~$/year

Death Switch = 1+STEP(-1, FINAL TIME-1) ~dimentionless

TIME STEP = 1 ~year

Consumption increases when the solution interval (i.e., the TIME STEP in the equation) is smaller than 1. How the solution interval is used in Consumption and *Last*

Consumption shows that these equations are sensitive to the solution interval. Therefore, the individuals will have different utility depending on the value of the solution interval. Therefore, we conclude that the model is not stable.

To fix this problem, we propose replacing *Last consumption* with an outflow and renaming *Consumption* as *Current consumption*. The *Current consumption* flow and the new *Last Consumption* flow are exclusive and separated by the variable Death Time. *Current Consumption* equals zero from a year before *Death Time*. *Last Consumption* flow turns on and drains out everything left in the *Wealth* stock from the last year at time T.

The new equations are as follows:

Current Consumption = IF THEN ELSE(Time <= Death Time - 1, ("Discrete Current Consumption (DCC)") , 0) ~$/year

Last Consumption = IF THEN ELSE(Time = Death Time - TIME STEP, "Wealth (W)" / TIME STEP, 0) ~$/year

Death time = FINAL TIME ~year

TIME STEP = 0.0078125 ~year

Labor Income equation

The second problem we identified in the authors' model is how income is modeled. The authors' model has two income-related equations: Labor Income Y modeled as stock and Income modeled as flow. One of the model assumptions is that individuals have a fixed income till the age of 58, the retirement age. As the amount is constant, the growth rate G should be 0 instead of 1 defined in the authors' model. Even though in the authors' paper, Labor Income (Y) modeled as stock captures that behavior, we recommend modeling income as flow to follow a stock-flow consistency approach (Godley, Lavoie 2012) with a STEP function to switch Labor Income (Y) to 0 from the retirement age at 58. The new Labor Income (Y) flow equation is as follows:

Labor Income (Y) = Normal Labor Income (Y) * (1 + Income Growth Rate (G)) * (1 - Retirement Switch) / Time to Chg Labor Income (Y) ~$/year

Normal Labor Income (Y) = 1000 ~$

Income Growth Rate (G) = 0 ~fraction

Retirement Switch = STEP (1, Retirement Time + TIME STEP) ~dimentionless

Retirement Time = 58 ~years

Time to Chg Labor Income (Y) = 1 ~year

Solution interval

The third problem we identified in the authors' model is related to the value of the solution interval used in the model. The solution interval, also called time step is the length of time interval at which the simulation software solves the model equations and calculates the values of all variables. Any change in the model takes at least a time period equal to the solution interval to occur.

The value of the model solution interval in the authors' paper is set to 1. In system dynamics modeling, we generally chose the value of solution interval smaller than 1. A solution interval of 1 deviates from the traditional system dynamics paradigm. The first test we did was to modify the solution interval's value to check the model's robustness. This shows that model in the authors' paper is sensitive to solution interval, indicating a possible integration error. We conclude that the model is not robust enough.

Understanding the concept and the mechanics of the solution interval used in system dynamics modeling is important in building robust system dynamics models. Having a solution interval too large leads to an integration error; therefore, we suggest choosing a small solution interval. On the other hand, having a solution interval too small generates a round-off error (Radzicki, 2019). It is important to find a tradeoff between these two extremes. How, then, smaller the solution interval should be? The smaller the solution interval, the more calculations the computer needs to do, the more time it takes to do the computations, and the larger the results files generated by the simulation will be. With modern-day computers and computing power, this is no longer an issue.

Nevertheless, modelers should be aware of the tradeoff between the size of the solution interval and the processing time/storage requirements. Therefore, the second question we might ask is if there is any rule of thumb for choosing the solution interval. The answer to this question depends on the time constant in the model.

If the model contains higher order delays, for example, third order delay (resp. n-th order delay), the smallest time constant is one-third (resp. 1/n) of the total delay time.

A simulation model where the solution interval exceeds the time constant produces incorrect results. The solution interval should be a fraction less than half of the shortest delay existing in the system (Forrester, 1971).

A robust model should have a solution interval that is insensitive to the behavior generated by the model. Therefore, when simulating a model and ensuring it is not sensitive to solution interval, it is always recommended to repeatedly cut the solution interval in half and check that the behavior generated by the model is caused by its structure rather than an error from the change in the solution interval.

Exponential discounting function formulation

The fourth problem we found in the authors' paper is related to the formulation of the exponential discounting or discounted utility function formulation used in the model. The exponential discounting function (Samuelson, 1936), is defined as follows:

$$Exponential\ Discounting\ (t) = \begin{cases} 1\ , t = 0 \\ \delta^t, t > 0 \end{cases} \qquad (1)$$

0<= δ<=1, where δ is the discount factor

The equations in the authors' paper related to the exponential discount function are defined as follows:

Discount Factor over Time = INTEG (Discount Factor Rate, 1), ~ dimentionless

Discount Factor Rate = (("Discount Factor (δ)"-1) * Discount Factor over Time)/TIME STEP, ~ dimentionless/year

"Discount Factor (δ)" = 0.99, ~fraction

Initial Discount Factor over Time = 1, ~dimentionless

Since the *Discount Factor over Time* equation depends on the solution interval (i.e. TIME STEP), this formulation generates a different behavior when the solution interval is less than 1. When the solution interval is smaller than 1, the *Discount Factor over Time*, δ, becomes steeper. To illustrate that, we did a test to compare and contrast the exponential discounting model in their paper with the exact analytical solution of δ^t

represented in equation (1) with a solution interval equal to 0.0625. As illustrated in Fig. 2.2, the simulation results show a significant discrepancy between the two models.

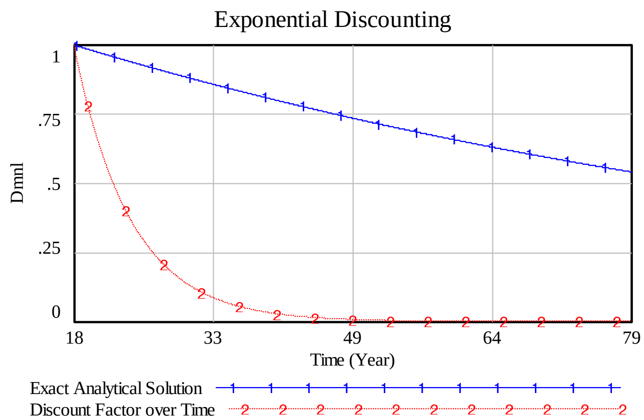


Fig.3.2. Exponential discounting discrepancy between the authors' model and the exact analytical solution when the solution interval is changed from 1 to 0.0625

To fix this problem, we will use the discrete-time molecule developed in chapter one

Since $\delta \leq 1$, a better and "natural" formulation would be to model "Chg in Exponential Discounting" as an outflow since Exponential Discounting generates an exponential decay behavior.

As illustrated in Fig. 3.3., the system dynamics model formulation matches exactly the exact analytical solution, δ^t, whatever the value of the solution interval is smaller than 1. This new discounting model is insensitive to the solution interval.

Any subsequent analysis and policy recommendations are questionable if there is a discrepancy between a model formulation and its mathematical representation.

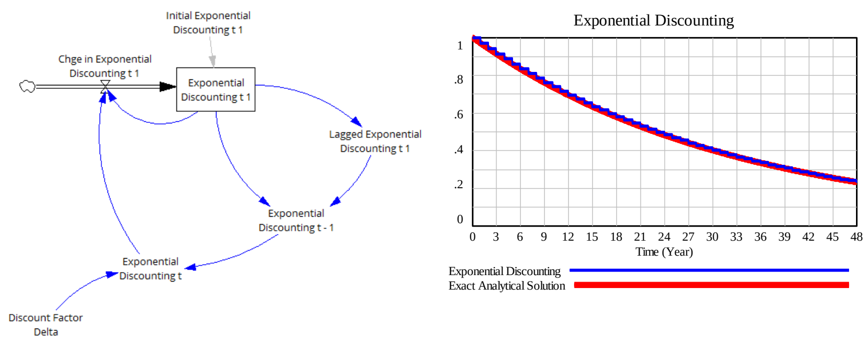


Fig. 3.3. Exponential discounting model and exact analytical solution

Quasi-hyperbolic discounting

The fifth and last problem is related to the type of discounting model used, the exponential discounting or discounted utility (DU) model. Their paper doesn't explain the choice of this model as opposed to the other and more recent forms of discounting models, such as present-bias (Laibson, 1997) even though it is referenced in the same paper (Bahaddin et al., 2019, pp.190, 191), mentioned in (Bahaddin et al., 2017) and largely documented in the behavioral economics literature.

Almost all current choices have future consequences which involve a certain degree of uncertainty and an intertemporal tradeoff and are influenced by cognitive biases. There are generally two types of intertemporal choices:

- Choices involving investment, such as consumption versus saving and its impact on retirement, building a new plan, and equipment, have immediate costs and delayed benefits characterize these choices.
- Immediate benefits and delayed costs characterize choices involving temptation, such as addiction, and procrastination.

The exponential discounting model has a constant discounting factor, meaning that the preference of today's self versus future self is discounted at the same rate and constant over time. The main feature of this model is that it preserves time or dynamic consistency. This means that the trade-offs between receiving rewards today and receiving them with a delay are independent of when that delay occurs. In other words, an agent with a time consistency preference does not change her mind in the future since her perception of the future is precisely the same as she planned it would be. Unfortunately, experiments widely documented in the behavioral economics literature have shown that agents often seem to deviate systematically from the implications of the exponential, time-consistent discounting model.

Evidence from psychology has shown that agents tend to put more psychological weight on short-term horizon tradeoffs at the expense of their long-term interests. For example, a person would prefer a candy today to two candies tomorrow, but she would prefer two candies in 101 days to one in 100 days. But 100 days from now, she would ostensibly again choose one candy immediately instead of two the next day. Her oneself will change again, and she will act impatiently. The tradeoffs between today and tomorrow are treated more impatiently than between 100 days and 101 days. For the long horizon, agents tend to be more patient. The rate at which agents discount future rewards declines as the length of the delay increases. That is, they have a preference for smaller sooner rewards and larger later rewards (Loewenstein et al. Prelec, 1992, Frederick et al., 2002). This model is called time inconsistent instead of the time consistent exponential discounting model.

First used by Phelps and al. 1968 and later by Laibson 1997, the quasi-hyperbolic discount function is a discrete-time function expressed as a set of values $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots\}$. It is a time-inconsistent discounting model, also called the present-bias model.

Laibson noted that "the discrete-time quasi-hyperbolic function has been connected to generate the smooth curve" [32, p. 8]. In the first chapter of this dissertation, we will develop the quasi-hyperbolic model in discrete and continuous-time in system dynamics. In this chapter, we will use the discrete.

$$Quasi - hyperbolic\ discount\ (t) = \begin{cases} 1\ , t = 0 \\ \beta\delta^t, t > 0 \end{cases}, 0 < \beta, \delta <= 1 \qquad (2)$$

The parameter δ, which is the same as in the exponential discounting model, called the standard discount factor captures the long-run and time-consistent discounting. In contrast, the parameter β, called the present bias factor, captures the short-term impatience and time inconsistency discounting. When the beta factor equals 1, the present-bias model is the same as the exponential discounting model.

The system dynamics representation of equation (2) is shown in Fig. 3.6. To test and validate our model, we compare it with the exact analytical solution of the quasi-hyperbolic function and ensure it replicates the behavioral economics literature results.

Fig. 3.4. below shows the behavior of the three discounting functions from the behavioral

economics literature (Laibson, 1997).

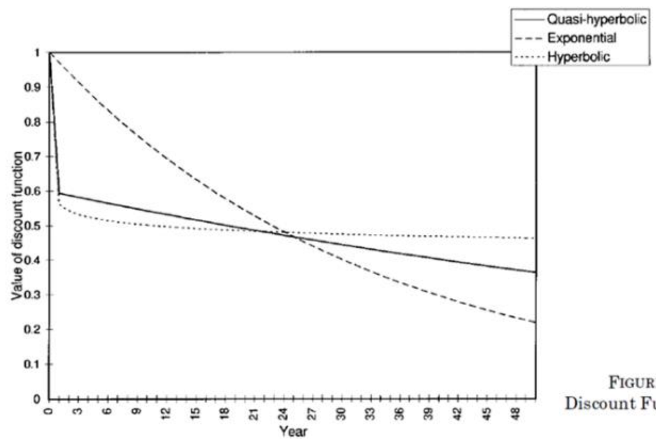
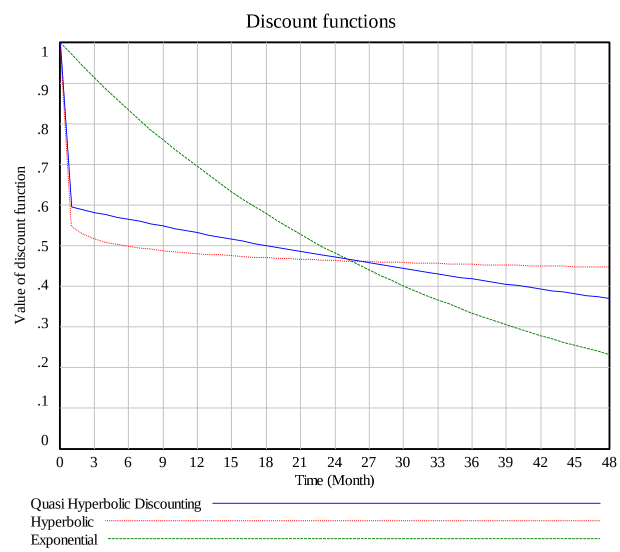


FIGURE I
Discount Functions

Fig. 3. 4. Discount Functions

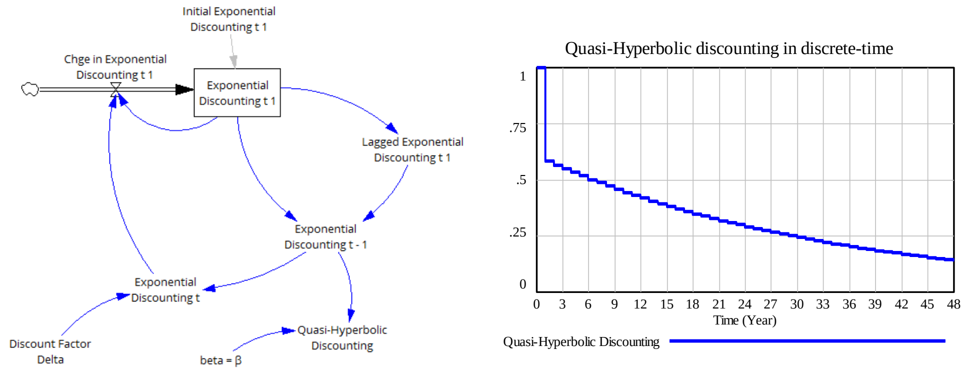
These discounting functions can be reproduced with system dynamics tools using a solution interval of 1, as shown in Fig. 3.5. But as discussed earlier, the proper way to represent these functions in system dynamics is to use a solution interval of less than 1.

Fig. 3.5. Replication of discounting functions with a solution interval equal to 1



We will reuse the system dynamics quasi-hyperbolic molecule developed in chapter one. This chapter will use the discrete-time version since Bahaddin's model is in discrete-time.

Fig. 3.6. Discrete time quasi-hyperbolic discounting molecule with solution interval less than 1, β = 0.6, δ = 0.97



3.3 Lifetime savings and utility model

In this section, we will use the system dynamics tool to replicate the lifetime savings or life-cycle model described in the authors' paper but apply the solutions to the problems discussed in the previous sections. We will start with the feedback-poor view model and portray the lifetime utility model.

Irvin Fisher introduced the lifetime savings model, which Franco Modigliani later proposed. It is an economic concept analyzing individual consumption and saving behavior. The economic agents aim to maximize their consumption and savings from the time they enter the labor market. The model assumes that individuals have a constant income throughout their lifetime. Bahaddin et al. used Modigliani's life-cycle model, also called the standard life-cycle hypothesis, LCH, with the standard Constant Relative Risk Aversion (CRRA) utility function, u(C_t).

$$u\,(C_t) = \begin{cases} \frac{C_t^{1-\rho}}{1-\rho}\,, & \text{if } \rho < 1 \\ \ln\,(C_t)\,, & \text{if } \rho = 1 \end{cases}$$

The individual will not save at time T+1 and will consume everything left before she dies. Therefore, we will use the quasi-hyperbolic function (Laibson, 1997) instead of the exponential discounting function described in the previous sections.

In this model, the individual starts with an initial wealth (W_0). She receives at each period the same amount of labor income (Y_t) and interest on income at interest (r). She chooses to consume an amount of (Ct) at each period to maximize their utility subject to their income and wealth. A key assumption in the model is the following:

$$Y_t = Y_{t+1} \quad (3)$$

$$W_{t+1} = (1+r) * (W_t + Y_t - C_t) \quad (4)$$

$$\max_{c_t, c_{t+1}, \ldots, c_T} U = u(c_1) + \beta \sum_{t=1}^{T} \delta^t u(c_t) \quad (5)$$

s.t. budget constraint:

$$\sum_{t=0}^{N} C_t = \sum_{t=0}^{N} W_t$$

$\beta = 0.6, \delta = 0.97, r = 0.05$

$0 \leq \beta \leq 1, 0 \leq \delta \leq 1$

The lifetime saving model in system dynamics is shown in Fig. 3.7.

We will use the Powell optimization algorithm in the Vensim software package to optimize the lifetime utility U. The variable being optimized in the model is the *Optimal Lifetime Utility*, and the optimization parameters are *Optimal Growth Rate* and *Initial Optimal Consumption Growth*.

We run optimization runs with the discrete-time quasi-hyperbolic discounting function. As a base case, we will use $\beta = 1$ for the optimization to have a baseline to compare our model with Bahaddin's model.

The optimal values found with $\beta = 1$, $r = 0.05$ and $\rho = 0.67$ are:

Optimization with the discrete-time quasi-hyperbolic function:

Initial Optimal Consumption = 235.543

Optimal Consumption Growth Rate = 0.0595243

Our *Optimal Consumption Growth Rate* is the same as in Bahaddin's model. However, their *Initial Optimal Consumption* is different than ours. Their *Initial Optimal Consumption* is 259.1 versus 235.54.

The lifetime utility can be optimized using the Euler equation in the behavioral economic literature. This is similar to the approach used in Bahaddin's paper (Bahaddin, 2019)

$$\left(\frac{C_{t+1}}{C_t}\right)^{\rho} = (1 + r) * \delta * [C'_{t+1} * (\beta - 1) + 1] \text{ (Laibson, 1996)} \qquad (7)$$

where C'_{t+1} is the marginal propensity to consume

When $\beta = 1$, the Euler equation becomes:

$$\left(\frac{C_{t+1}}{C_t}\right)^{\rho} = (1 + r) * \delta \qquad (8)$$

To gain confidence in our model, we will compare the simulation results with the outcome from the Euler equation using Eq. 9. We follow the same approach as in Bahaddin's paper.

$$\left(\frac{C_{t+1}}{C_t}\right)^{\rho} = (1 + r) * \delta = (1 + 0.05) * 0.99 = 1.03950 \qquad (9)$$

The consumption ratio from the simulation is given by:

$$\left(\frac{C_{t+1}}{C_t}\right)^{\rho}_{simulation} = (1 + \Psi)^{\rho} \qquad (10)$$

Ψ is the optimal consumption growth rate in the model

$$\left(\frac{C_{t+1}}{C_t}\right)^{\rho}_{simulation} = (1 + 0.0595248)^{0.67} = 1.0394999 \qquad (11)$$

Eq. 9 and 11 show the consumption ratio from the Euler equation matches its corresponding value from the simulation. This result gives more confidence in our model. This result is in line with Bahaddin's optimal model. The present-bias do not influence these parameters and optimization since β = 1 in this case.

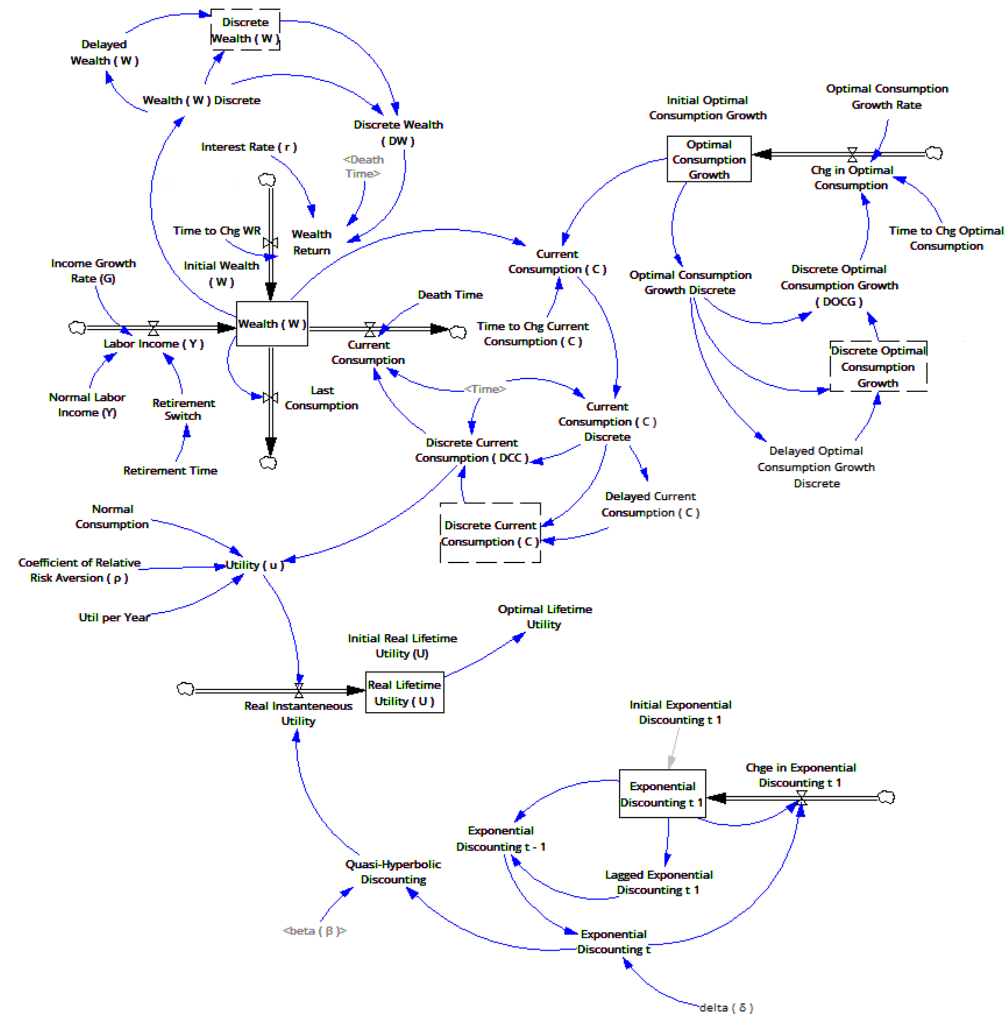
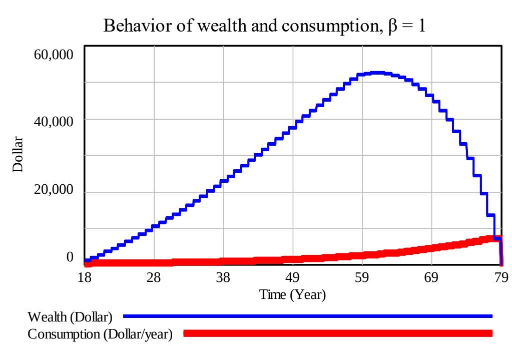
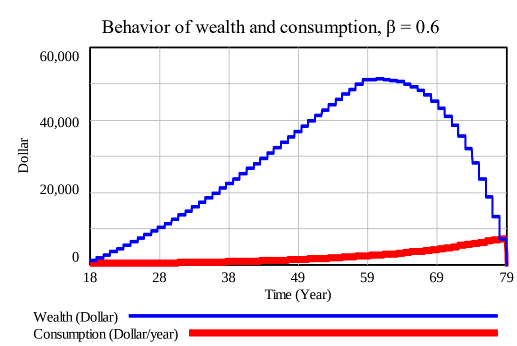


Fig. 3.7. The lifetime savings with quasi-hyperbolic discounting model





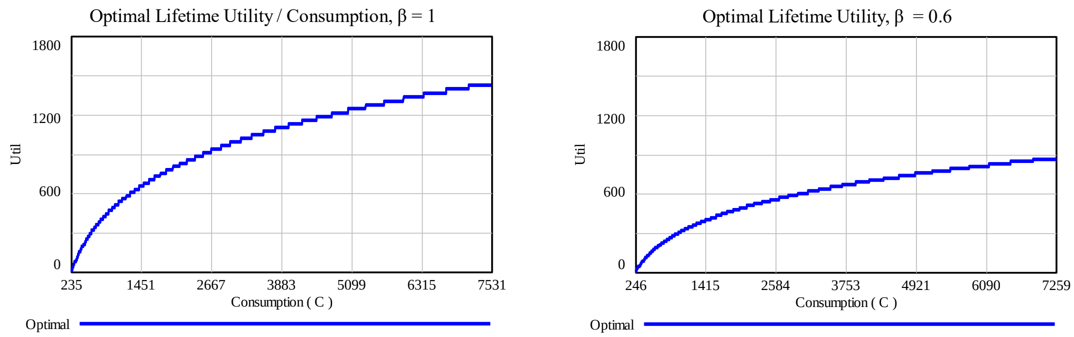


Fig. 3.8. Optimal behavior of wealth, consumption, and real lifetime utility,
$\delta = 0.99$, β = 1 (left graphs), β = 0.6 (right graphs)

As shown in Fig.3.8, The optimal lifetime utility of present-biased or time inconsistency individuals (β = 0.6), is 39.4% lower than time consistent individuals (β = 1). In addition, consumption (resp. wealth) of time-inconsistent individuals is 2.6% (resp. 3.6%) lower than of time-consistent, unbiased individuals.

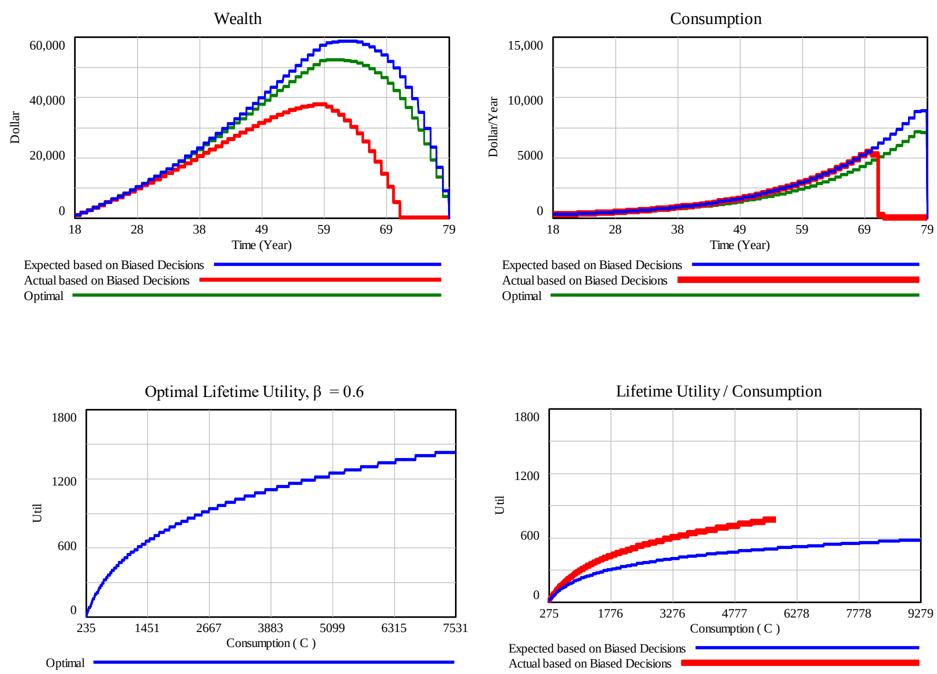
3.4 Feedback-poor countervailing biases model

In this section, we use the feedback-poor model introduced in Fig. 3.7 to explore the implications and effects of three biases on individuals' utility: the misperception of interest rate (r), the coefficient of relative risk aversion (ρ), and the present-bias (β). First, we run a few experiments to look at the utility of an induvial who believes that the interest rate (r) and the coefficient of relative risk aversion (ρ) are 20 percent higher than their actual values and impatient with a present bias (β) equal to 0.6.

We ran a few simulations from the actual values to better understand how these biases interact and compared them with the optimal behavior discussed in previous sections. The first simulation, optimization scenario #2 from Table 2.1, captures the overconfidence bias of individuals who believe the interest rate (r) is higher than 20% of their actual values. The second simulation, optimization scenario #3, captures the misperception of risk of individuals with a coefficient of relative risk aversion (ρ) of 20 percent higher than their actual values. Finally, the third simulation, optimization scenario #3, shows the countervailing biases between the interest rate (r) and misperception of risk of individuals with and the coefficient of relative risk aversion (ρ). We also did sensitivity analysis to find the combination of the parameter range of the interest rate (r) and the coefficient of relative risk aversion (ρ) where the optimal lifetime can be found.

The introduction of the present-bias in the model has s significant impact on the lifetime utility. In the optimized scenario #4 in Table. 2.1, the Bahaddin' s model with the classical exponential discounting produced a normalized lifetime utility of 0.97 versus 0.54 from our model with the present-bias, quasi-hyperbolic discounting.

Fig. 3.9. Actual, optimal and biased behavior of a time inconsistent individuals who misperceive r and ρ to be both 20% higher than their actual values and a present-biased β = 0.6.



Optimized Scenario	Bahaddin et al. (β = 1) Normalized Consumption Growth	Bahaddin et al. (β = 1) Normalized Lifetime Utility	Bah et al. (β = 1) Normalized Consumption Growth	Bah et al. (β = 1) Normalized Lifetime Utility	Bah et al. (β = 0.6) Normalized Consumption Growth	Bah et al. (β = 0.6) Normalized Lifetime Utility
1: The individual is unbiased	1.0000	1.0000	1.0000	1.0000	-	-
2: The individual misperceives r to be 20% higher than its real value	1.2530	0.9698	1.2537	0.9090	1.2311	0.5505
3: The individual misperceives ρ to be 20% higher than its real value	0.8292	0.9968	0.8293	0.9969	0.8050	0.6034
4: The individual misperceives ρ to be 24.3% higher and r to be 20% higher than their real values	**1.0013** (*)	0.9702	**1.0013** (*)	0.8980	0.9778	0.5441
5: The individual misperceives ρ to be 21.7% higher and r to be 20% higher than their real values	1.0233	0.9703	1.0234	0.8985	**0.9999** (*)	0.5444
6: The individual misperceives ρ to be 10% higher and r to be 20% higher than their real values	1.1353	0.9720	1.1360	0.9084	1.1125	0.5798

(*) Consumption growth rate where the two biases called out

Table 3.1. Normalized final lifetime utility for an unbiased individual compared to ten optimized scenarios with varying perceptual biases

3.5 Feedback-rich lifetime saving model

The feedback-rich model is an example where the two fields of behavioral economics and system dynamics are bridged together. The difference between the feedback-poor optimal model and the feedback-rich heuristic model is how consumption is derived. In the heuristic model, the individuals consider savings after a minimum consumption level from different spending categories, such as the cost of rent, car loan, etc., is met (Stone, 1954, Achury et al., 2012, Chetty and Szeidl, 2016). Bahaddin et al. incorporated this concept as a Minimum Consumption exogenous variable in their heuristic, feedback-rich model. The authors' heuristic model is derived from their feedback-poor optimal model. Therefore, the problems identified in this chapter's previous sections are also present in their heuristic model, such as the use of a solution internal of 1, the formulation of the income equation and discount function, etc. Our model is based on Bahaddin's work but integrates the solutions discussed in the previous sections in this chapter.

The heuristic model captures three loops. The new Minimum Consumption variable is the source of the two new loops. The first loop R1, also in the optimal feedback-poor model, is a first-order positive or reinforcing feedback loop that generates new wealth with an annual interest rate of 5%. The second loop B1 is a negative or balancing loop. As income rises, the Risk-Adjusted Consumption, the total income minus the minimum income, also increases. As Risk-Adjusted Consumption rises, the Current Consumption rises, and the stock of Wealth depletes, closing the negative feedback loop. The third loop R2 or the second positive feedback loop, is driven by the table function, which captures the theory of savings in the variable Effect of Coefficient of Relative Risk Aversion. When the total income to Minimum Consumption ratio is less than 1, the individual will not save. As the ratio of savings fraction increases above 1, the individual will save a larger and larger fraction of Risk-Adjusted Savings. The new model and the simulation results are shown in Fig. 3.10.

Consumption in the heuristic model is higher in the beginning than in the feedback-poor optimal model and is nearly constant until retirement time, when it starts to decline. Moreover, as illustrated in Fig. 3.10, the heuristic endogenous model does not produce an optimal solution like the feedback-poor model. By comparison, consumption in the optimal model rises exponentially till death time. The heuristic model seems more realistic than the feedback-poor optimal model.

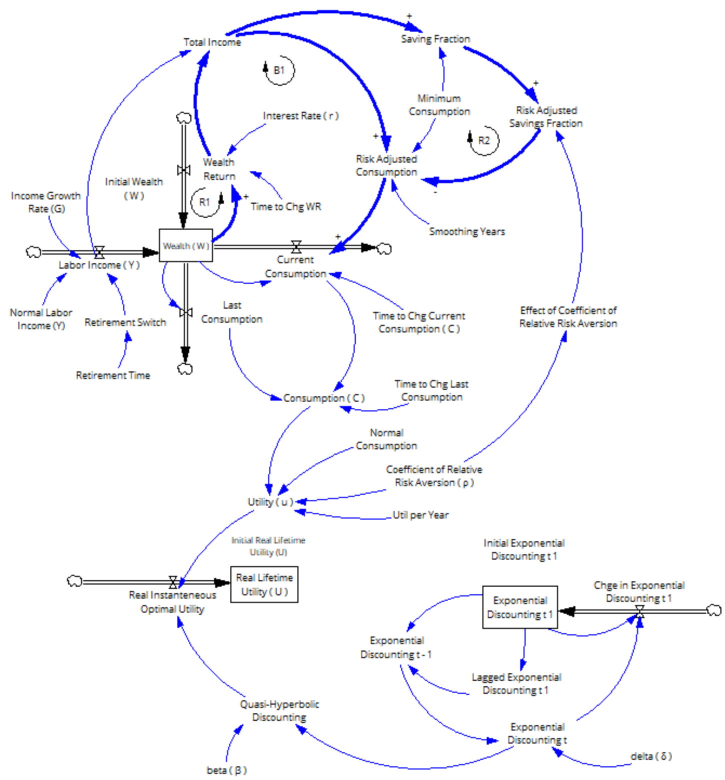


Fig. 3.10. Feedback-rich version of a simple heuristic consumption model

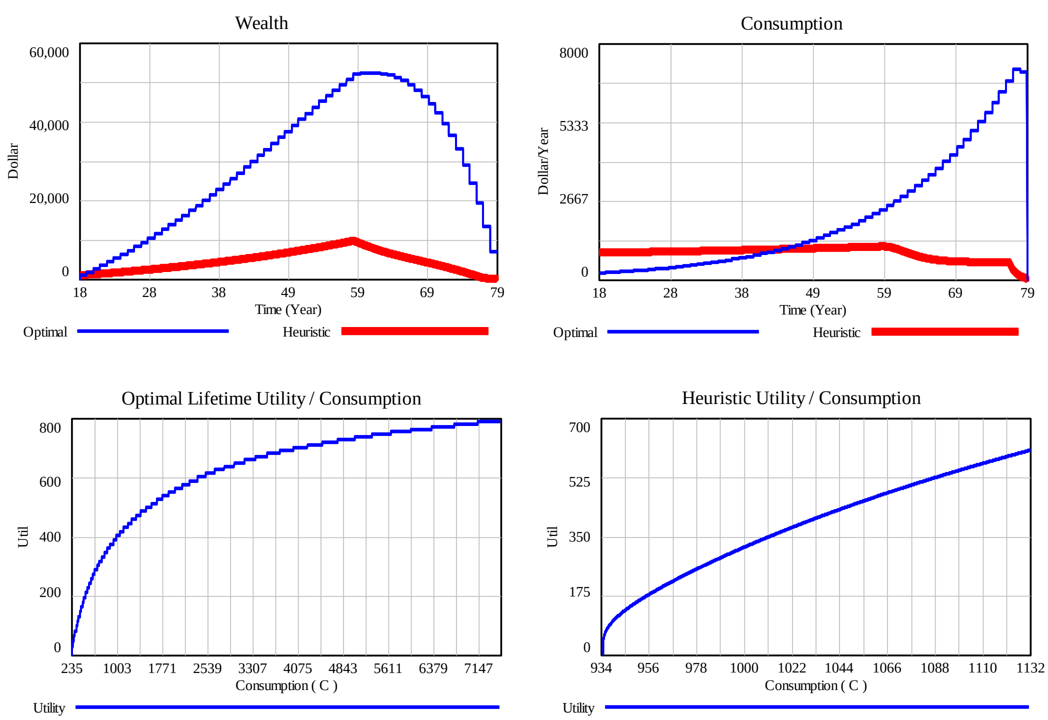


Fig. 3.11. Feedback-poor optimal model and feedback-rich heuristic model behavior modes

3.6 Discussion and future research

In this chapter, we identified some modeling problems found in Bahaddin et al. paper published in the System Dynamics Review 2019 titled "Building a bridge to behavioral economics: countervailing cognitive biases in lifetime saving decisions". We offered solutions to these problems using best practices in system dynamics. We have also demonstrated that our models are consistent with those developed in the behavioral literature. Moreover, this chapter shows the importance of model testing to build confidence in system dynamics models and ensure the models respond plausibly under extreme conditions set on the parameter values and policies (Homer, 1993). One of the first tests we recommend is to ensure the model is not sensitive to the solution interval. We also extended the model incorporating a quasi-hyperbolic discounting or present-bias model and developed this model as a generic structure or molecule.

As other areas of research, we suggest (1) investigating the use of cumulate prospect theory (Kahneman et al. 1979), which overcome the Allais paradox and replace the expected utility formulation in both models, the countervailing bias and the heuristic model (2) analyze present-bias individuals separately as sophisticated and naïve and (3) run surveys or use statistical techniques to estimate the biased parameters in both models. We also suggest (4) exploring and analyzing the heuristic model replacing the standard lifecycle model based on Modigliani's theory with Shefrin and Thaler's behavioral lifecycle model (BLC. Shefrin and Thaler have criticized Modigliani's life-cycle hypothesis model (LCH) (Shefrin, Thaler, 1988). They extended and enriched the LCH model with behavioral components. The main difference between the two theories is that the BLC considers the psychological and social factors influencing people's spending and saving behavior.

In contrast, Modigliani's LCH model assumes that people are rational and make decisions in their best interest based on their lifetime income and consumption goals. Thaler and Shefrin added three main components to the BLC model: self-control, mental accounting, and framing. BLC assumes that households treat components of their wealth as nonfungible.

Chapter 4 Revisiting Modigliani’s and Thaler s Life-Cycle Hypothesis Models

4.1 Introduction

This third chapter is an application of the first two chapters. We will revisit, compare and contrast two life time saving models developed by two Nobel laureates, Franco Modigliani, and Richard Thaler. Modigliani's Standard Lifecycle Hypothesis (LCH) and Thaler's Behavioral Lifecycle Hypothesis (BLC) are two influential theories in the field of economics that attempt to explain how individuals manage their consumption and savings throughout their lifetime. While both views share some assumptions about individuals’ consumption and saving over time, they differ in treating several key factors, such as the role of psychological biases and the importance of income fungibility. In recent years, there has been a growing interest in revisiting these models and exploring their assumptions and limitations. By reviewing these models, we can better understand the complex interplay between economic factors and psychological biases that shape individual consumption behavior. In this exploration, we will reproduce the LCH model and build a simple BLC model from (Shefrin and Thaler, 1981 and 1988) in system dynamics. We will discuss their similarities and differences and how they have influenced our understanding of individual consumption and saving lifetime behavior in economics.

System Dynamics is an interdisciplinary methodology and field of study that focuses on understanding and managing the behavior of complex systems over time. Jay W. Forrester developed it at MIT in the early 50s, and has since been widely applied in various domains, including business, engineering, public policy, environmental management, and social sciences. System dynamics allows for the representation of feedback loops, time delays, nonlinearities, soft variables, etc., which are critical in understanding how individual behaviors and decisions interact over time. It is a method, technique, and tool which helps identify unintended consequences and leverage points in complex systems. System dynamics can help decision-makers define better policies and redesign better and more robust systems.

The standard lifecycle hypothesis model developed in the early 1950s is one of the most used models to explain an individual's consumption behavior over their lifetime. This model considers a set of factors such the age, income, consumption, and budget constraints. This model's primary assumption is that consumers seeking to maximize their utility throughout their lifetime are rational and forward planning. Modigliani's life-cycle theory has been criticized by behavioral economists (Shefrin, Thaler, 1988). Thaler and Shefrin suggest that many people may lack the self-control to reduce spending now and save more for the future. Therefore, they extended Modigliani's model with behavioral components. One of the main differences between the two theories is that the BLC considers the psychological and social factors that influence people's spending and saving behavior.

In contrast, Modigliani's LCH model assumes that people are rational and make decisions in their best interest based on their lifetime income and consumption goals. Shefrin and Thaler added three main components to the BLC model: self-control, mental accounting, and framing. According to the BLC model, self-control is a key determinant of an individual's financial success over their lifetime. It refers to an individual's ability to regulate behavior, resist instant gratification, and make choices that align with long-term goals. Mental accounting refers to the way individuals categorize and allocate their financial resources. The mental accounting component suggests that individuals do not necessarily treat all money as equivalent and may assign different values and priorities to various sources or uses of money. The third component, framing, refers to how individuals perceive and interpret information based on the context in which it is presented, and how the information is presented can influence an individual's decision-making.

The plan of this chapter is as follows. In the second section, we will reproduce Modigliani's Standard LCH model in system dynamics. We will first start by building a simple model of the lifecycle to give an intuition of the key concepts of individuals' consumption and saving behavior over their lifetime. We will then extend this model, adding the key assumptions of Modigliani's and Brumberg's Standard Lifecycle Hypothesis model. Finally, we will run the model to ensure we reproduce the same results as in the economics literature. In the third section, we will build a simple BLC model based on Shefrin's and Thaler's description (Shefrin and Thaler, 1988) in system dynamics. Before building the BLC model in System Dynamics, we will explore and explain the key behavioral components Thaler and Shefrin incorporated into their model. We will then present the model's assumptions and explain how we will build each behavioral component in system dynamics. In the fourth section, we will run the model to test two of Shefrin's and Thaler's predictions from their paper. The final section lays out further research and directions.

4.2 Modigliani's life-cycle hypothesis model

Modigliani's and Brumberg's lifecycle hypothesis paper was first published (Kurihara, 1952) and then in Modigliani's collected papers book in 2005. This chapter presents a theory of lifetime saving and consumption. The theory of consumption is not new. Many other influential authors have studied and published about consumption in economics. Some of the most notable include:

John Maynard Keynes is widely regarded as one of the most influential economists of the 20th century, and his work on consumption and saving is particularly influential. In his 1936 book "The General Theory of Employment, Interest and Money", Keynes argued that consumption and saving decisions are driven by a combination of current income, expectations of future income, psychology and uncertainty. In this book, he stated:

> "The fundamental psychological law, upon which we are entitled to depend with great confidence both a priori from our knowledge of human nature and from the detailed facts of experience, is that men are disposed, as a rule and on the average, to increase their consumption as their income increases, but not by as much as the increase in their income."

Milton Friedman (Friedman, 1957), another Nobel laureate in 1985, is best known for his work on the permanent income hypothesis of consumption, which argues that individuals aim to maintain a stable level of consumption over time based on their expected long-term income.

James Tobin (Tobin, 1958), another Nobel laureate in 1981, developed a portfolio theory of investment, which has important implications for consumption decisions. Tobin argued that individuals balance expected returns and risks of different investments when making consumption and saving decisions. His theory emphasizes the role of financial market conditions and investment behavior in influencing consumption decisions.

Recently, two contemporary economists, Robert Hall, and Gregory Mankiw, made important contributions to the study of consumption in macroeconomics. They have developed models of consumption that incorporate features such as liquidity constraints, habit formation, and the role of expectations in driving consumption decisions.

The life-cycle consumption theory states that individuals save during their working time, when they receive income to fund their retirement and dis-save later during their retirement time. The model assumes that individuals may be more likely to treat their income or assets as a single, interchangeable pool of resources. In other words, they treat their income and wealth as fungible.

More specifically, the model assumes that individuals

- plan their consumption and saving decisions according to a fixed lifespan
- expect to live another T years when they enter the job market
- maintain the same level of consumption and wish to have the smoothest possible path of consumption throughout their lifetime
- retire R years from the time they enter the job market
- expect to earn a fixed income Y per year
- have a wealth, W, coming from their income and from which they use for their consumption
- expect neither to receive nor a desire to leave an inheritance
- have rational expectations about their future income and consumption, and they use this information to make consumption and saving decisions

The lifecycle hypothesis predicts wealth accumulation follows a "hump-shaped" behavior that is low at the beginning of adulthood called the dependency phase, peaks at the retirement age, called the maturity phase, and then declines in old age during retirement.

The model below in Fig. 4.1. is a simple model of lifecycle, which gives an intuition of the key concepts of consumption, saving, and wealth of individuals' lifetime behavior. This model assumes a fixed amount of income accumulating to generate wealth, which the individual consumes. The consumption C, depends on wealth W, income Y, retirement year R, and lifespan T.

$$C = \frac{W+RY}{T} \qquad (1)$$

$$C = \frac{1}{T}\,W + \frac{R}{T}\,Y \qquad (2)$$

Subject to budget constraints:

Total Income = Total Consumption + Total Saving.

To model individuals' rational behavior, we use the exponential discounting function. However, psychology, empirical, and behavioral economics studies (Anslie, 1975), (Thaler, 1981), (Laibson, 1997) and others have rejected the exponential discounting formulation and replaced it with the hyperbolic and quasi-hyperbolic discounting function.

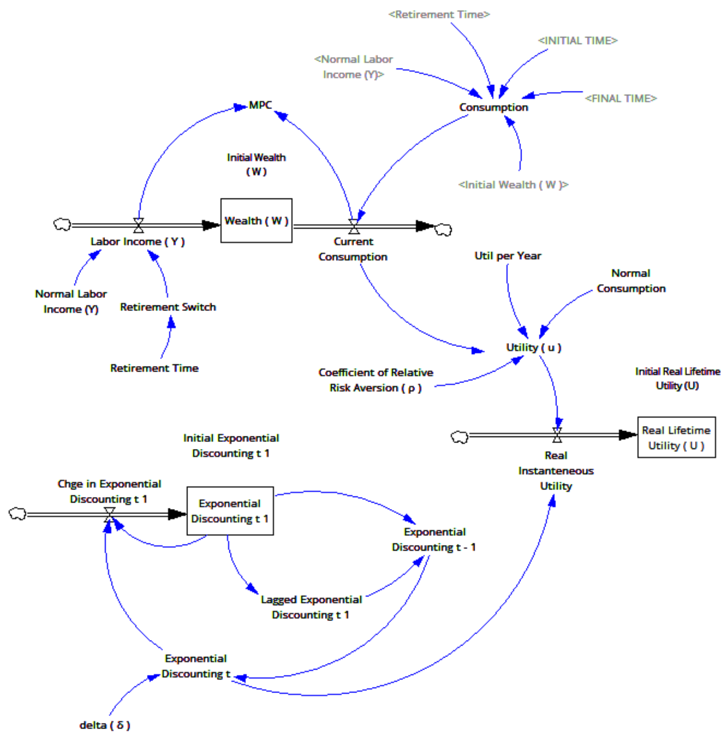


Fig. 4.1. Simple Life-cycle hypothesis model

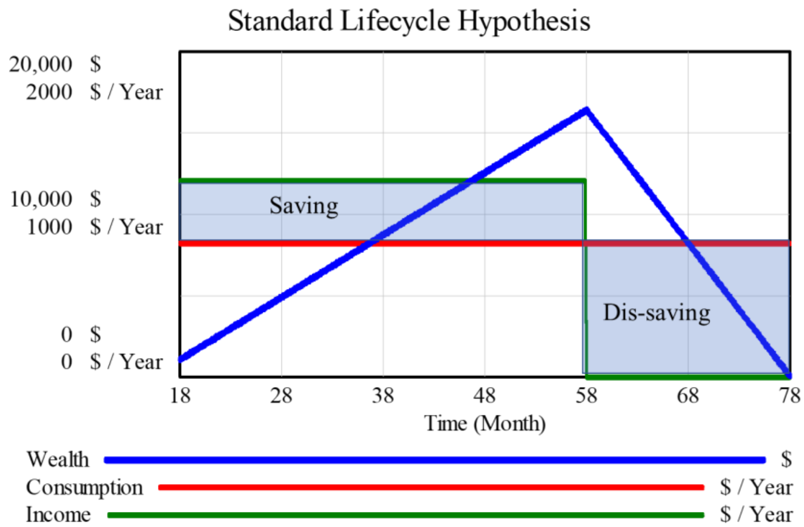


Fig. 4.2. Wealth and Saving and Dis-Saving

We build a new version of the standard lifecycle hypothesis by changing the consumption function from the previous model. We ran the model and optimized the lifetime utility from the two marginal propensities, *alpha,* and *beta,* using the Powell built-in function in Vensim system dynamics software. The marginal propensity to consume out of income, *alpha*, out of income represents the fraction of an additional unit of income that individuals choose to consume and measures the responsiveness of consumption to changes in income. The second parameter, *beta*, is the marginal propensity to consume out of wealth, representing the fraction of additional wealth that individuals choose to consume. Individuals also take into account their accumulated wealth when making consumption decisions.

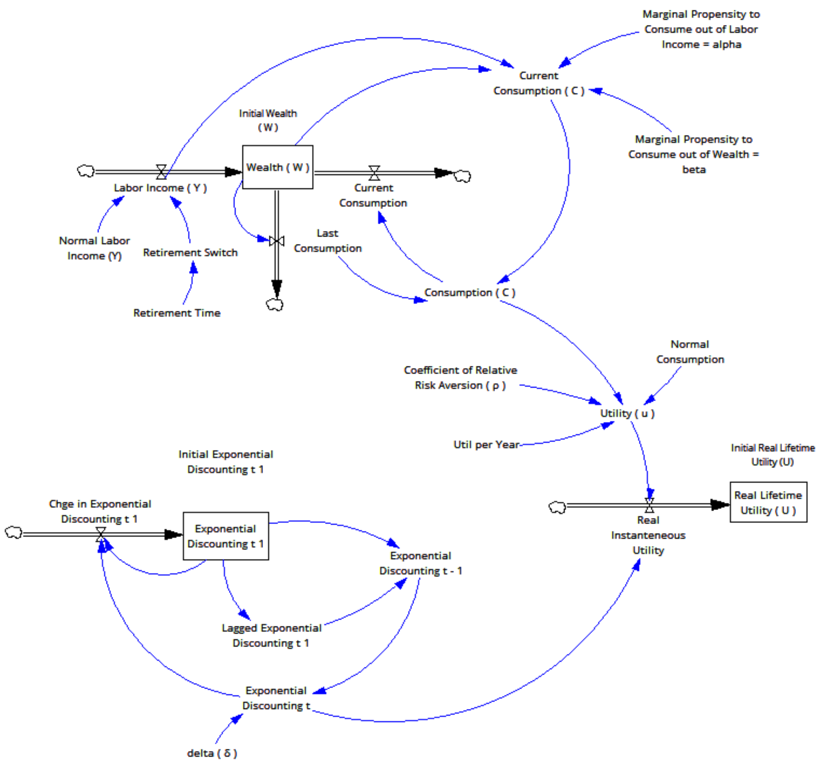


Fig. 4.3. Standard Life-cycle hypothesis model

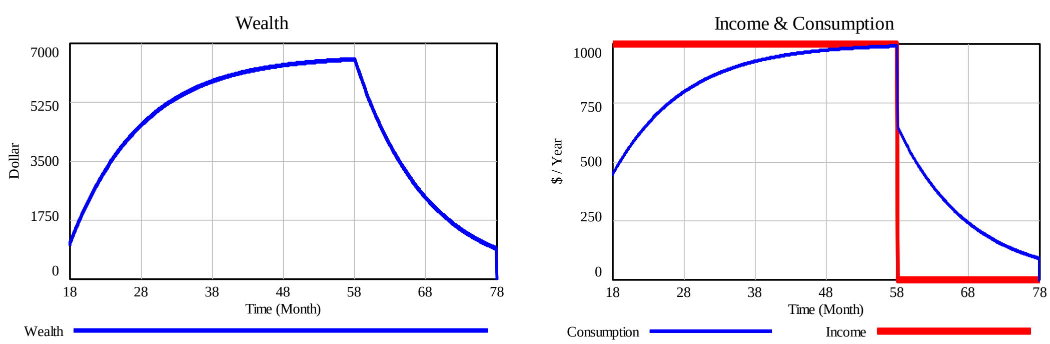


Fig. 4.4. Wealth and Saving and Dis-Saving

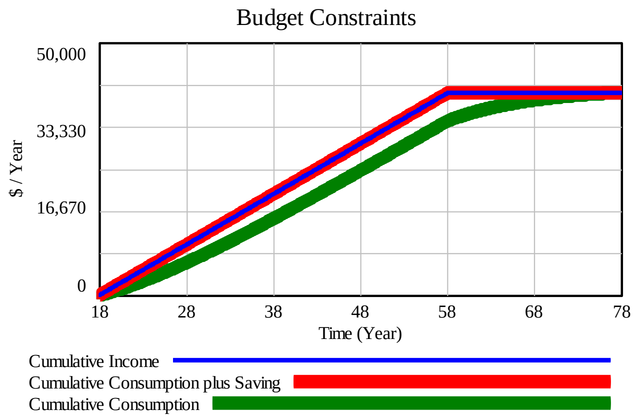


Fig. 4.5. Budget constraints

In this model, wealth and consumption follow a similar behavior pattern as described in Fig. 4.4. The consumption increases at a decreasing rate during the dependence stage and peaks at retirement age. At retirement age, as income drops to zero, consumption plummets and decreases at an increasing rate during retirement.

4.3 Thaler's behavioral life-cycle hypothesis model

Thaler's BLC (Behavioral Life-Cycle) model, like the LCH model, also explains how individuals save and invest over their lifetime. The BLC model critiques and extends Modigliani's LCH model, considering individuals may not always make rational decisions due to behavioral biases and heuristics. The model includes three key concepts: self-control, mental accounting, and framing. This section will describe these three components and how we will model them in system dynamics.

Self-control (Thaler, 1981) (Laibson et al., 1998) refers to an individual's ability to resist immediate gratification in favor of long-term goals. In the BLC model, individuals with high self-control are more likely to save and invest for their future, while those with low self-control may struggle to resist temptation and prioritize long-term goals. Shefrin and Thaler cited Ainslie's contributions to self-control. Ainslie (Anslie, 1975) introduced the concept of hyperbolic discounting and argued that this could explain the tendency for individuals to value immediate rewards more than future rewards. Psychology has shown that individuals tend to value immediate rewards more than future rewards at the expense of their long-term interests. Lowenstein, Laibson, Thaler, and others in behavioral economics have built on these early researchers' work and developed new insights and approaches to understanding self-control and its implications for decision-making. One of the most well-known and common models of this cognitive bias in the behavioral economics literature is present-bias or quasi-hyperbolic discounting (Laibson, 1997). In the first chapter of this dissertation, we developed a quasi-hyperbolic discounting molecule in system dynamics based on Laibson's work (Laibson, 1997). We applied this molecule to the lifecycle model in the second chapter of this dissertation. In this third chapter, we will reuse this molecule to implement the self-control component in the BLC model.

Mental Accounting (Thaler, 1985): the standard LCH model assumes fungibility. This means all money is treated the same, regardless of source. In the LCH theory, individuals treat their assets as a single resource used for consumption. However, empirical results from behavioral economics have shown that individuals process various components of wealth and assign specific labels or mental accounts to different parts of their wealth where certain accounts are more tempting than others. Mental accounting is how individuals categorize and allocate their financial resources into different "accounts" based on subjective criteria such as income source or the funds' intended use. The BLC model suggests that individuals may be more likely to spend money from one account (such as a discretionary spending account) than from another (such as a retirement account), even if the money is functionally equivalent. To illustrate this, Shefrin and Thaler conducted a small survey of MBA students at Santa Clara University. The survey consists of three scenarios, where they were asked how much they expect their consumption to rise.

Scenario 1: you’ve been given a special bonus at work, for which you will receive $200 a month for twelve months.

Scenario 2: you’ve been given a special bonus at work for which you will receive a lump sum of $2400.

Scenario 3: you’ve been told that a distant relative has left you a small inheritance which has been an after-tax value of $2400, but you will not receive the money for five years.

In these three scenarios, the individual wealth will increase by the same amount of $2400. According to the fungibility principle, the use of this extra money should not be influenced by where it comes from. It turns out that the result of this survey rejects the fungibility principle and supports the differential MPC hypothesis. The survey results show that the annual median MPC for scenario1, 2, and 3 are resp. $1200, $785, and $0. So, we can think of individuals keep in mind three mental accounts of a current income account, a discretionary or asset account, and a future income account. The current income account is for day-to-day spending such as groceries, rent, etc. The discretionary or asset account is for saving and investing that can be used for college for fees or for a down payment to buy a house. The future income account is for future spending after retirement, for example.

We extended and added two account stocks in the LCH model described in the previous section to model mental accounting in system dynamics. We follow the same approach as Shefrin and Thaler. We added two new accounts in our model, a *Discretionary Saving* account and the *Pension Saving* account. Shefrin and Thaler used a 10% MPC out of *Income* for these two accounts, with 4% MPC out of Income allocated to the *Discretionary Saving* account and 6% MPC to the *Pension Saving* account. The remaining 10% of the *Income* will go to the current account, *Wealth*, which will be used for everyday consumption. We will use the same parameter estimates as Shefrin and Thaler for these accounts. The individuals cannot consume from *Pension Saving* during the dependency phase. At retirement age, the full amount accumulated on the *Pension Saving* account will be transferred to the *Wealth* account, and the *Pension Contributions* will be shut off. Unlike Pension Saving, individuals can use the Discretionary Saving during the dependency phase, for example, a down payment to buy a house.

Framing (Kahneman and Tversky, 1984), the third component of the BLC, refers to how information is presented, and can influence an individual's perception and decision-making. In the BLC model, individuals may be more likely to save and invest when information is framed positively (such as emphasizing the potential gains) rather than negatively (such as highlighting the potential losses). In the context of the BLC, framing plays an important role in shaping individual financial behavior and can have important implications for financial decision-making over an individual's lifetime. For example, how retirement savings options are presented to individuals can influence how much they choose to save for retirement. By framing retirement savings as a percentage of income rather than a fixed dollar amount, individuals may be more likely to save more for retirement, as the percentage frame highlights the importance of saving a consistent proportion of income over time. Mental accounting is another example of framing. Shefrin and Thaler stated that" "*The decomposition of wealth into mental accounts constitutes an example of framing*" (Shefrin, 1988, p.615).

In our model, the pension account builds up during the dependency phase, and no fund is transferred to the Wealth account till the retirement year at time 58. At year 58, the full amount in the Pension account is transferred to the Wealth account. Regarding the Discretionary account, we transferred 70% to the Wealth account at year 45, and at retirement age at time 58, the full remaining amount in the Discretionary Saving account is transferred to the Wealth account.

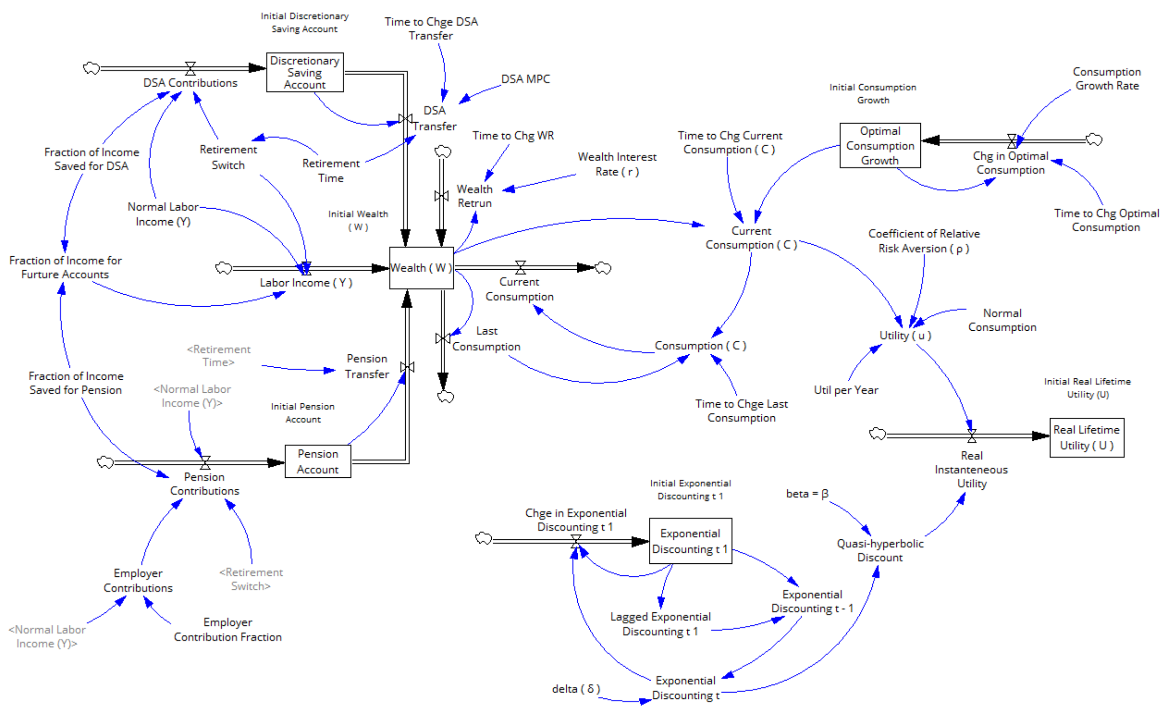


Fig. 4.6. A Simple Behavioral lifecycle model

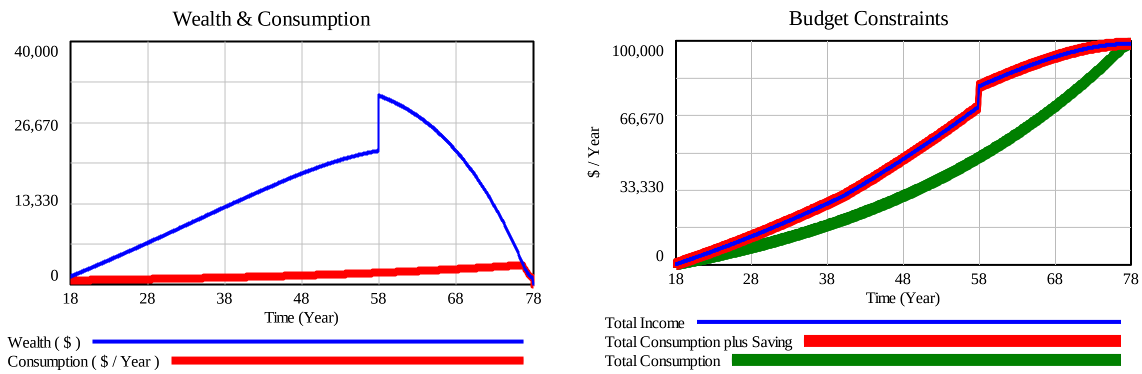


Fig. 4.7. Wealth, consumption, and budget constraints

4.4 Thaler's predictions on the behavioral life-cycle hypothesis model

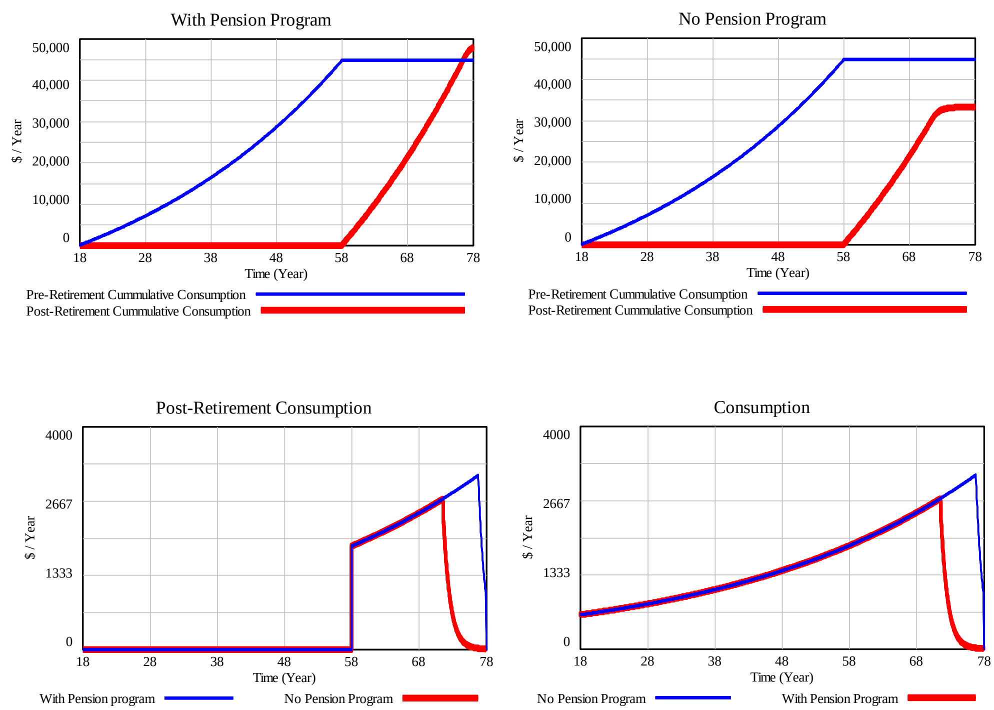
This section will discuss and test two of Shefrin's and Thaler's predictions using the system dynamics version of the BLC model we developed in this chapter.

Prediction 3: "In the absence of sufficiently large Social Security and pension programs, retirement consumption will be less than preretirement consumption."

This prediction suggests that in the absence of adequate Social Security and pension programs, people will consume less in retirement than before retirement. In other words, in the absence of Social Security and private pensions, consumption in old age relative to lifetime consumption would be significantly lower. Retired households often face the challenge of sustainable consumption levels experienced early in retirement due to inadequate savings. Older people respond to this shortfall by gradually reducing real consumption over time. This is because they will have to rely on their own savings and investments, which may not be sufficient to maintain their pre-retirement standard of living. As a result, they will need to reduce their consumption. This prediction is based on the idea that people tend to save less than they should for retirement and, therefore, may not have enough resources to support their desired consumption levels in retirement. It highlights the importance of having adequate retirement savings programs to ensure that individuals can maintain their standard of living in retirement.

We didn't incorporate a social security structure in our model. This is beyond the scope of this chapter. However, we did include a simplified version of a pension program within the mental accounting structure as described in (Shefrin and Thaler, 1988). We did two runs with the model to test this prediction. One run with a pension program, we called optimal, and another run without a pension program. The model's results in Fig. 4.8 show that Shefrin and Thaler prediction three is verified.

Fig. 4.8. Prediction 3 simulation results



Prediction 4. "The saving rate increases with permanent income."

The idea is that as people earn more, they can meet their current needs and wants while saving for the future. Shefrin and Thaler argued that the marginal cost of exercising willpower decreases as permanent income increases. In other words, saving becomes relatively easier for individuals with higher income levels. They argue that saving is a luxury for people with low incomes, indicating that the ability to save is influenced by income and consumption levels. As a person's income increases over the long run, they tend to save a more significant proportion of their income. Therefore, as permanent income increases, the target level of consumption also increases, leading to a rise in saving to achieve the desired level of consumption in the future. This prediction implies that individuals with higher permanent income will have a higher saving rate than those with lower permanent income, ceteris paribus.

The authors also discussed the proportionality principle, which refers to the idea that consumption is proportional to permanent income, regardless of the income level. According to this principle, individuals would maintain a consistent saving rate irrespective of income. However, the authors rejected the proportionality principle and proposed an alternative based on this prediction. The rejection of the proportionality principle mentioned by Shefrin and Thaler is supported by Ervin Fischer (Fischer, 1930) and later with empirical evidence presented by Thomas Mayer (Mayer, 1972) and further studies conducted by Diamond and Hausman in 1984. The conclusion is that the proportionality hypothesis, which suggests a direct proportionality between income and saving, is invalidated.

Prediction 5. "Holding wealth constant, consumption tracks income."

This prediction suggests that when for a given level of wealth, changes in consumption will closely follow changes in income. Individuals tend to spend more when they earn more, even if their overall level of wealth remains the same. This prediction is based on the idea that individuals are motivated by current income rather than total wealth when making consumption decisions. Therefore, as their income increases, they tend to spend more on consumption, even if overall wealth remains constant. This prediction highlights the importance of income fluctuations in affecting consumption decisions. It implies that individuals will likely adjust their consumption patterns based on income changes rather than overall wealth. This is consistent with the idea that consumption is driven by the relative income of the individual, which is a key assumption in the standard economic models of consumption.

We run two simulations, one with the base case and the second one where we increase the Labor Income (Y) by 30% from year 40. The results of the simulations are shown in Fig. 4.9. Our model confirms Shefrin's and Thaler's prediction 5.

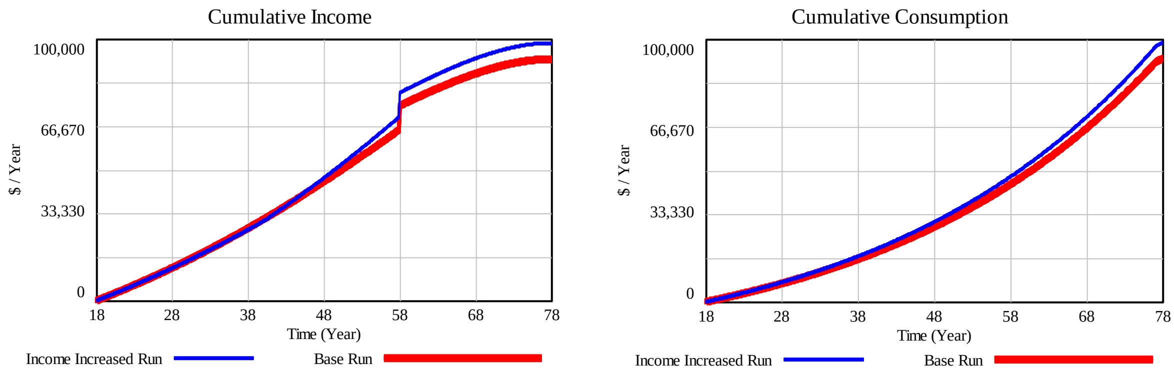


Fig. 4.9. Prediction 5 simulation results

4.5 Discussion and future research

In this chapter, we reproduced and presented Brumberg's and Modigliani's Life Cycle Hypothesis (LCH) and built a simple BLC model from Shefrin's and Thaler's paper (Shefrin and Thaler, 1988) in system dynamics. Then, we focused more on the BCL model and tested two of Shefrin's and Thaler's predictions with our version of the BLC model. Our BLC model validates the two predictions, 3 and 5, we tested. These two models share some common assumptions.

Firstly, both models assume that individuals plan for their consumption and savings over their lifetime. Modigliani's LCH model assumes that individuals plan their consumption and savings over their lifetime to maintain a constant living level. On the other hand, Shefrin and Thaler's BLC model assumes that individuals plan their consumption and savings over their lifetime to achieve specific financial goals.

Secondly, both models assume that individuals face uncertainty in their income and investment returns over their lifetime. Modigliani's LCH model assumes that individuals face uncertain income streams and investment returns, which they try to smooth out over their lifetime. Shefrin and Thaler's BLC model assumes that individuals face uncertain investment returns, which they try to mitigate by using mental accounting and other behavioral strategies.

However, they also have some key differences. For example, Modigliani's LCH model assumes that individuals have rational expectations and behave optimally given their information and preferences. On the other hand, Shefrin's and Thaler's BLC model incorporates behavioral factors such as self-control, mental accounting, and framing, which can lead to deviations from optimal behavior and the classical lifecycle model.

In addition, Modigliani's LCH model assumes that all individuals have the same time preferences, income profiles, and investment opportunities. Shefrin and Thaler's BLC model recognizes that individuals prefer immediate gratification and have self-control problems and other behavioral biases when saving for the future. They may also have different financial goals and risk attitudes, which can influence their saving and investment decisions.

The value of building such models, particularly the BLC model in system dynamics, is many folds. Firstly, this model can be incorporated into a larger system dynamics model to explore the impact of different interventions or management strategies and facilitates the evaluation of policy options on the behavior of the lifecycle system over time with what-if scenarios analysis. For example, a policy might involve increasing funding for social security or new pension programs. Secondly, we can run Monte Carlo simulations on this model to test for the robustness of Shefrin's and Thaler's predictions. This can help study the model's resilience in the face of behavioral biases and shocks, which is very difficult to do without a simulation model like system dynamics. Thirdly, this model can be embedded into a game that can be used in lab experiments in experimental economics settings. Finally, this approach will help policymakers assess the effectiveness and unintended consequences of interventions designed to address behavioral biases and promote desired economic outcomes. By combining the insights of behavioral economics with the system-level perspective of system dynamics, researchers and policymakers can gain a deeper understanding of the complexities of economic behavior and develop more robust and effective interventions to promote desirable economic outcomes.

Some future areas of research to investigate this topic would be (1) testing the other Shefrin's and Thaler's predictions, (2) Incorporate the BLC model and analyzing the effect of habit formation (Caroll, Overland, and Weil, 2000) with a path dependence where the utility of today's consumption depends on past consumptions and (3) revisit the model presented in chapter 2 of this dissertation and look at under which circumstances the cognitive biases of risk aversion and overconfidence cancel out under BLC model assumptions and lastly (4), use prospect theory (Kahneman and Tversky, 1979) in the BLC model in combination with the above-proposed areas or research.

Chapter 5 Conclusion

In this research, we developed, tested, and applied a formal, repeatable, and teachable method to bring behavioral economics into system dynamics that satisfies both behavioral economics and system dynamics.

On the one hand, behavioral economists elect to model in discrete-time. On the other hand, system dynamicists commonly model in continuous-time. Discrete time models are consistent with system dynamics. However, there's still confusion in the system dynamics community on how to correctly model discrete-time features in system dynamics. We clarified this issue of discrete-time versus continuous-time modeling and how it matters for policy change. We showed that the choice between a discrete-time model and its converted version in continuous-time matters under the same assumptions for policy change. We tested this using a concrete example with the Samuelson multiplier-accelerator model. We also developed a new discrete-time molecule with stock and flow structure to build discrete-time models in system dynamics without using a dt (i.e., solution interval) of 1. This molecule is the cornerstone for building discrete-time models in system dynamics. This molecule can be applied beyond the economics domain. The discrete-time molecule can be used to build, for example, healthcare, manufacturing, finance, environmental, and organizational change discrete-time models or embedded in a mix and larger differential-difference equation models. The second molecule or generic structure we developed replicates the quasi-hyperbolic discounting or present-bias in system dynamics. We successfully applied these molecules in this dissertation's second and third chapters. We also developed a formal method based on Goodman's approach (Goodman, 1980), which we generalized, to convert discrete-time models into continuous-time. In chapter one, we used this method to compare and contrast the three models we replicated and converted them from discrete-time to continuous-time. The last two chapters apply the technique and results described in the first chapter.

One of the key benefits of integrating behavioral economics into system dynamics is the recognition that human decision-making is not solely driven by rationality (Sterman, 2000), (Simon, 1972), self-interest, and utility maximization. Behavioral economics brings to light the cognitive biases, social influences, and emotional factors that impact decision-making processes. By incorporating these behavioral aspects into system dynamics models, we can create more accurate representations of real-world systems where human behavior plays a crucial role. Furthermore, this integration enables us to explore the feedback loops between individual behavior and the dynamical system. It illuminates how the decisions and actions of individuals within a system can reverberate through feedback loops, influencing the system's overall behavior. By understanding these interactions, we can identify potential unintended consequences and leverage points where interventions and policy changes can significantly impact system outcomes. By incorporating behavioral factors, we can better anticipate and simulate the effects of individual choices on the system as a whole, providing more robust insights into long-term system behavior.

Integrating behavioral economics into system dynamics has broad applicability in various fields. Some notable areas of application include:

Experimental Economics: Experimental Economics labs provide controlled environments where researchers can design and conduct experiments to observe and analyze human behavior in economic decision-making scenarios. For example, a study to conduct an Experimental Economics lab that aims to understand the dynamics of market behavior and price formation. To study these dynamics, researchers can design an experiment where participants engage in a simulated market environment and are given a set of assets to trade, and their task is to determine the value and trading price of these assets. By incorporating behavioral economics principles, researchers can introduce realistic factors such as risk aversion and decision-making under uncertainty into the experiment. Additionally, system dynamics modeling can be employed to capture the interactions and feedback loops between individual behavior and market outcomes. Therefore, researchers can develop system dynamics models that simulate the dynamics of price formation, market supply and demand, market pressures, exogenous shocks, and the impact of individual trading decisions on market stability. During the experiment, participants' behaviors and trading decisions are recorded, and the data is analyzed to understand how individual behavior and market dynamics are interconnected. The data is then used to ground and refine the system dynamics models, incorporating insights from behavioral economics to improve the accuracy and realism of the model.

Gaming: integrating behavioral economics into system dynamics can have direct implications not only for the gaming industry itself but also on individual and group learning experiences. By considering player decision-making, social influences, and the complex dynamical system within games, this integrated approach enhances the game design, player engagement, and the overall gaming experience and learning. It helps understand player behavior, identify new biases, and create personalized gaming experiences.

Environmental sustainability: behavioral economics integrated into system dynamics can contribute to addressing environmental challenges and develop interventions that promote sustainable behaviors and mitigate environmental degradation. It helps understand human behavior and decision-making about environmental issues like resource consumption, waste management, and climate change.

Integrating behavioral economics into system dynamics open up exciting avenues for further research and exploration. Some areas of additional investigation include:

Decision-making under uncertainty: investigating the interplay between behavioral biases and decision-making processes in the face of fundamental uncertainty with prospect theory within system dynamics models can enhance our understanding of how individuals make choices in complex and uncertain environments. In addition to previous studies in this area, this research can shed light on risk perceptions, information processing, and decision-making strategies under varying levels of uncertainty.

Multi-agent systems and multi-method simulation: exploring the dynamics of multi-agent systems, where individuals or entities are embedded with cognitive biases, interact, and influence each other's behaviors, is an area of growing interest. This can be modeled by integrating behavioral economics into a multi-method approach to gain insights into emergent behaviors, cooperation, competition, and the formation of complex social structures.

Dynamics of social norms: understanding the dynamics of social norms and their influence on individual behavior using insights from the combination of behavioral economics and system dynamics is an area that can benefit from further research. Examining how social norms evolve, spread, and interact with behavioral factors can provide insights into collective behaviors, cultural shifts, and the emergence of new norms. This area of research can be combined with multi-agent systems using a multi-method simulation approach.

Technological applications: exploring the application of behavioral economics and system dynamics in emerging technological fields, such as artificial intelligence, the internet of things, smart systems, and autonomous decision-making, for example, autonomous driving cars, can provide insights into human-machine interactions and the ethical implications of these technologies.

Supplementary materials: the models and source codes presented in this dissertation are available for download from the WPI eProjects website.

References

Achury C, Hubar S, Koulovatianos C. 2012. Saving rates and portfolio choice with subsistence consumption. Review of Economic Dynamics 15(1): 108–126.

Ainslie, G. 1975. Specious reward: A behavioral theory of impulsiveness and impulse control. Psychological Bulletin Journal. American Psychological Association. Vol. 82-4, pp 463-496.

Ainslie, G., & Haslam, N. (1992). Hyperbolic discounting. In G. Loewenstein & J. Elster (Eds.), Choice over time (pp. 57–92). Russell Sage Foundation.

Allen, R.G.D. (1965) Mathematical Economics. 2nd Edition. London / New York: Macmillan / St. Martin's Press.

Angeletos GM, Laibson D, Repetto A, Tobacman J, Weinberg S. 2001. The hyperbolic consumption model: calibration, simulation, and empirical evaluation. Journal of Economic Perspectives 15(3): 47–68.

Angner E., 2016. A Course in behavioral economics 2nd ed. Macmillan International Higher Education

Ashraf N, Camerer CF, Loewenstein G. 2005. Adam Smith, behavioral economist. Journal of Economic Perspectives 19(3): 131–145.

Bahaddin B, Weinberg S, Luna-Reyes L, Andersen D. 2017. Simulating lifetime saving decisions: the behavioral economics of countervailing cognitive biases. In 35th International System Dynamics Conference. System Dynamics Society: Cambridge, MA.

Bahaddin B, Weinberg S, Luna-Reyes L, Andersen D. 2019. Building a bridge to behavioral economics: countervailing cognitive biases in lifetime savings decisions. System Dynamics Review. John Wiley and Sons

Barlas, Y. (1989). Multiple tests for validations of system dynamics type of simulation models. European Journal of Operational Research, 42(1), 59–87.

Barlas, Y., & Carpenter, S. (1990). Philosophical roots of model validation: Two paradigms. System Dynamics Review, 6(2), 148–166.

Barlas, Y., and Carpenter, S. (1990). Philosophical roots of model validation: Two paradigms. System Dynamics Review, 6(2), 148–166.

Baumol, W.J., 1970. Economic Dynamics: An Introduction. 3rd edition. Macmillan Bellman R. 1957. Dynamic programming. Princeton University Press: Princeton, NJ.

Bellman R. 1957. Dynamic programming. Princeton University Press: Princeton, NJ.

Benartzi S, Thaler RH. 1995. Myopic loss aversion and the equity premium puzzle. Quarterly Journal of Economics 110(1): 73–92.

Benartzi, S., and Thaler, R. H. 2007. Heuristics and biases in retirement savings behavior. Journal of Economic Perspectives, 21: 81–104.

Benzion, U., Rapoport, A., Yagil, J., 1989. Discount rates inferred from decisions: an experimental study Journal: Management Science. Volume: 35: 270-284. Issue: 3. Pages: 270—284

Berns, G.S., Laibson, D. and Loewenstein, G., 2007. Intertemporal choice–toward an integrative framework. Trends in cognitive sciences, 11(11), pp.482-488.

Beshears J, Choi JJ, Hurwitz J, Laibson D, Madrian BC. 2015. Liquidity in retirement savings systems: an international comparison. American Economic Review 105(5): 420–425.

Breierova B, 1998. Mistakes and misunderstandings: DT error. MIT Literature Collection. D-4695

Brekke KA, Moxnes E. 2003. Do numerical simulation and optimization results improve management?

Experimental evidence. Journal of Economic Behavior and Organization 50(1): 117–131.
Cagan, P. 1965. The Eflect of Pension Plans on Aggregate Savings. New Yorlc: National Bureau of Economic Research
Caliendo F, Huang KX. 2008. Overconfidence and consumption over the life cycle. Journal of Macroeconomics 30(4): 1347–1369.
Camerer C, Issacharoff S, Loewenstein G, O'Donoghue T, Rabin M. 2003. Regulation for conservatives: behavioral economics and the case for "Asymmetric Paternalism". University of Pennsylvania Law Review 151(3): 1211–1254.
Carroll, C., Overland, J., Weil, D. 2000. Saving and Growth with Habit Formation. American Economic Review Journal. Vol. 90. Issue 3.
Cavana, R.Y., Dangerfield, B., Pavlov, O.V., Radzicki, M.J., Wheat, I.D. Feedback Economics: Economic Modeling with System Dynamics. Springer 2021
Chetty R, Szeidl A. 2016. Consumption commitments and habit formation. Econometrica 84(2): 855–890.
Dangerfield B, Roberts C. 1996. An overview of strategy and tactics in system dynamics optimization. Journal of the Operational Research Society 47(3): 405–423.
Deaton, A. 1987. Life-cycle models of consumption: Is the evidence consistent with the theory? In T.F. Bewley (ed.), Advances in Econometrics—Fifth World Congress, Vol. II, pp. 121–148. Cambridge, England: Cambridge University Press.
Deaton, A. 1992. Understanding Consumption. Oxford: Oxford University Press.
DellaVigna S. 2009. Psychology and economics: evidence from the field. Journal of Economic Literature 47(2): 315–372.
Diamond, P., Jerry, H. 1984. Individual Retirement and Saving Behavior. Journal of Public Economics, 81-114.
Diamond, P., Koszegi, B. 1999. Quasi-hyperbolic discounting and retirement. Journal of Public Economics 87 (2003) 1839–1872
Diamond, P., Koszegi, B. 1999. Quasi-hyperbolic discounting and retirement. Journal of Public Economics 87 (2003) 1839–1872
Doyle, J.K., Ford, D.N., Radzicki, M.J. and Trees, W.S., 2001. Mental models of dynamic systems. Encyclopedia of Life Support Systems.
Ericson, K.M. and Laibson, D., 2019. Intertemporal choice. In Handbook of Behavioral Economics: Applications and Foundations 1 (Vol. 2, pp. 1-67). North-Holland.
Federal Reserve. 2018. Report on the economic well-being of U.S. households in 2017. Board of Governors of the Federal Reserve System: Washington, DC.
Fisher, I., 1930. The Theory of Interest. London: MacMillan.
Ford, D.N. and Sterman, J.D., 1998. Dynamic modeling of product development processes. System Dynamics Review: The Journal of the System Dynamics Society, 14(1), pp.31-68.
Forrester, J.W., 1961. Industrial dynamics. Cambridge MIT press. Available from Pegasus communications: Waltham MA.
Forrester, J.W., 1968. Market growth as influenced by capital investment. Industrial Management Review.
Forrester JW. 1971. Principles of Systems. Pegasus Communications.
Forrester JW. 2003. Economic theory for the new millennium. System Dynamics Review 29(1): 26–41.

Forrester, JW., Senge PM., 1980. Tests for building confidence in system dynamics models. In System Dynamics ed. by A. A. Legasto Jr. et al. New York: North Holland.

Frederick S., Loewenstein G., and O'Donoghue T. 2002. Time discounting and time preference: A Critical review. Journal of Economic Literature Vol. 40, No. 2, pp. 351-401.

Friedman, M. 1957. A theory of the consumption function. Princeton, NJ: Princeton University Press.

Gino, F. and Pisano, G., 2008. Toward a theory of behavioral operations. Manufacturing & Service Operations Management, 10(4), pp.676-691.

Goodman, M. 1989. Study Notes in System Dynamics. System Dynamics Series. Pegasus Communications. ISBN: 1-883823-40-4

Gul, F., Pesendorfer, W. 2004. Self-control and the theory of consumption. Econometrica, Vol. 72, No. 1, pp. 119-158. The Econometric Society

Hall, R. E. 1978. Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence. Journal of Political Economy, 86(6), 971-987.

Harris, C. and Laibson, D., 2013. Instantaneous gratification. The Quarterly Journal of Economics, 128(1), pp.205-248.

Homer, JB. 1983. Partial-model testing as a validation tool for system dynamics. In International System Dynamics Conference Proceedings. Chestnut Hill. MA.

Kahneman D, Tversky A. 2002. Maps of bounded rationality: a perspective on intuitive judgment and choice. Prize lecture. Economic Sciences 2002: 449–489.

Kahneman, D., Tversky, A. 1979. Prospect theory: an analysis of decision under risk. Econometrica 47:263–91.

Kahneman, D., Tversky, A. 1984. Choices, values, and frames. American Psychological Association. Vol. 39. Issue 4, pp. 341-350

Katona, G. 1965. Private Pensions and Individual Saving. Ann Arbor. University of Michigan Press

Kraft H, Munk C, Seifried FT, Wagner S. 2017. Consumption habits and humps. Economic Theory 64(2): 305–330.

Laibson D. 1997. Golden eggs and hyperbolic discounting. Quarterly Journal of Economics 112(2): 443–478.

Laibson D., Harris C. 2001. Dynamic choices of hyperbolic consumers. Econometrica, Vol 69, No 4

Laibson, D. 1996. Hyperbolic discount functions, undersaving, and savings policy. Working Paper 5635. National Bureau of Economic Research.

Laibson, D. 1996. Hyperbolic discount functions, undersaving, and savings policy". Working Paper 5635. National Bureau of Economic Research. 1996

Laibson, D., Repetto, A., Tobacman J., Hall, R., Gale W. and Akerlof, G., 1998. Self-control and saving for retirement. Brookings Papers on Economic Activity, , Vol. 1998, No. 1 (1998), pp. 91-196.

Laibson, D., Repetto, A., Tobacman J., Hall, R., Gale W. and Akerlof, G., 1998. Self-control and saving for retirement. Brookings Papers on Economic Activity, , Vol. 1998, No. 1 (1998), pp. 91-196.

Langarudi S, Bar-On I. 2018. Utility perception in system dynamics models. Systems 6(4): 37.

Legasto A.A., Forrester, J. W., Lyneis, J. M. 1980. System dynamics, Studies in the Management Sciences, Vol. 14, Amsterdam: North Holland

Loewenstein G., Prelec D., 1992. Anomalies in intertemporal choice: evidence and an interpretation. The Quarterly Journal of Economics 107, 573-597.

Loewenstein, G., Elster, J., 1992. Choice Over Time. Russell Sage Foundation.

Low, G.W. 1980. The Principle of Conservation and the Multiplier-Accelerator Theory of Business Cycles. Proceedings of the 1976 System Dynamics Conference.

Lyneis, J.M., 1973. Working capital management: a systems dynamic approach.

Lyneis, J.M., 1975. Designing financial policies to deal with limited financial resources. Financial Management, pp.13-24.

Lyneis, J.M., 1980. Corporate planning and policy design : a system dynamics approach. Cambridge, Mass. MIT Press/Wright-Allen series in system dynamics.

Lyneis, J.M., 1993. A dynamic model of technology diffusion. Pugh-Roberts Associates.

Lyneis, J.M., 1999. System dynamics for business strategy: a phased approach. System Dynamics Review: The Journal of the System Dynamics Society, 15(1), pp.37-70.

Lyneis, J.M., 2000. System dynamics for market forecasting and structural analysis. System Dynamics Review: The Journal of the System Dynamics Society, 16(1), pp.3-25.

Martinez, I., Richardson, G. 2001. Best practice in system dynamics modeling. The 19th International Conference of the System Dynamics Society, Atlanta, GA

Mass, NJ. 1991. Diagnosing Surprise Model Behavior: A tool for evolving behavioral and policy insights. System Dynamics Review. 7(1): 68-86.

Mayer, T., 1972. Permanent Income, Wealth and Consumption. Berkeley: University of California Press

McFadden, D., 1991. Advances in computation, statistical methods, and testing of discrete choice models. Marketing Letters, 2(3), pp.215-229.

Meadows, D. 1972. Dynamics of Commodity Production Cycles. Cambridge, Massachusetts: Wright-Allen Press, Inc.

Mickens, R. E., 1988. Difference equation models of differential equations. Journal of Mathematical and Computer Modelling. Vol. 11. pp. 528-530.

Mickens, R. E., 2005. Dynamic consistency: a fundamental principle for constructing nonstandard finite difference schemes for differential equations. Journal of Difference Equations and Applications. Vol. 11, Issue 7. Publisher: Taylor and Francis Group

Mickens, R.E., 1999. Discretizations of nonlinear differential equations using explicit nonstandard methods. Journal of Computational and Applied Mathematics

Modigliani, F. 1986. 'Life Cycle, Individual Thrift, and the Wealth of Nations', The American Economic Review, 76(3), pp. 297–313.

Modigliani, F. and Brumberg, R. 1954. Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data. In: Kurihara, K., Ed., Post Keynesian Economics, New Brunswick: Rutgers University Press, 388-436.

Modigliani, F., 2005. The Collected Papers of Franco Modigliani. The MIT Press. Volume 6. DOI: https://doi.org/10.7551/mitpress/1923.001.0001. ISBN: 97802622800516

Morrison BJ, Rudolph JW, Carroll JS. 2013. Dynamic modeling as a multidiscipline collaborative journey. System Dynamics Review 29(1): 4–25.

Moxnes E. 2015. An Introduction to deterministic and stochastic optimization. In Analytical methods for dynamic modelers, Rahmandad H, Oliva R, Osgood N (eds). MIT Press: Cambridge, MA; 243.

Nerlove, M. 1958. Adaptive Expectations and Cobweb Phenomena. Quarterly Journal of Economics. MIT Press. Vol. 72-2. pp. 227—240.

Nichols, M., Pavlov, O. and Radzicki, M.J., 2006. The circular and cumulative structure of administered pricing. Journal of Economic Issues, 40(2), pp.517-526.

O'Donoghue T, Rabin M, 1999. Doing it now or later. American Economic Review, 89(1), pp. 103–24.

O'Donoghue T, Rabin M. 2015. Present bias: lessons learned and to be learned. American Economic Review 105(5): 273–279.

Oakley D, Kenneally K. 2015. Retirement security 2015: roadmap for policy makers— Americans' views of the retirement crisis. National Institute for Retirement Security: Washington: DC.

Oliva, R. 1995. A Vensim® Module to calculate summary statistics for historical fit. System Dynamics Group, MIT. Memo D-4584. Cambridge, MA

Paich, M. 1985. Generic Structures. System Dynamics Review. 1(1): 126-132

Pan, J., Webb, C.S. and Zank, H., 2015. An extension of quasi-hyperbolic discounting to continuous time. Games and Economic Behavior, 89, pp.43-55.

Payne JW, Bettman JR, Schkade DA, Schwarz N, Gregory R. 1999. Measuring constructed preferences: towards a building code. In Elicitation of preferences, Fischhoff B, Manski CF (eds). Springer: Berlin; 243–275.

Phelps, E. S., Pollak, R. A. 1968. On Second-Best National Saving and Game Equilibrium Growth. The Review of Economic Studies, Vol. 35, Issue 2, Pages 185–199, https://doi.org/10.2307/2296547

Phelps, E. S., Pollak, R. A. 1968. On Second-Best National Saving and Game Equilibrium Growth. The Review of Economic Studies, Volume 35, Issue 2, Pages 185–199, https://doi.org/10.2307/2296547

Pidd, M. 2002. Computer simulation in management 4th edition. John Wiley & Sons, Inc. ISBN: 0-471-97931-7

Qudrat-Ullah H. On the validation of system dynamics type simulation models. Telecommun Syst (2012) 51:159–166. DOI 10.1007/s11235-011-9425-4

Radzicki M. 1988. Institutional dynamics: an extension of the institutionalist approach to socioeconomic analysis. J Econ Issues 22(3):633–665

Radzicki, M.J., 1989. Reference modes and the optimal shape parameter. System Dynamics Review, 5(2), pp.192-198.

Radzicki, M.J., 1990. Methodologia oeconomiae et systematis dynamis. System Dynamics Review, 6(2), pp.123-147.

Radzicki M., Sterman J. 1993. Evolutionary economic and system dynamics. System Dynamics Review

Radzicki, M.J. and Sterman, J.D., 1994. Evolutionary economics and system dynamics. Evolutionary concepts in contemporary economics, pp.61-89.

Radzicki, M. and Taylor, R.A., 1997. Introduction to system dynamics. A Systems Approach to Understanding Complex Policy Issues, US Department of Energy's.

Radzicki, M.J., 2004, July. Expectation formation and parameter estimation in uncertain dynamical systems: the system dynamics approach to post Keynesian-institutional economics. In Proceedings of the Int System Dynamics Conference, Oxford, UK.

Radzicki, M.J., 2005. Institutional economics, post Keynesian economics, and system dynamics: Three strands of a heterodox economics braid. Economics (ICAPE), 5(7).

Radzicki M., 2019. Lecture note: SD590 system dynamics modeling WPI 2019.

Radzicki M., 2019. System dynamics and its contribution to economics and economic modeling. Springer Science Business Media LLC R. A. Meyers (ed.), Encyclopedia of Complexity and Systems Science

Rahmandad H. 2015. Connecting strategy and system dynamics: an example and lessons learned. System Dynamics

Review 31(3): 149–172.
Ramsey FP. 1928. A mathematical theory of saving. Economic Journal 38(152): 543–559.
Randers, J. 1980. Elements of the Systems Dynamics Method. The MIT Press Wright-Allen Series in System Dynamics
Rasmusen, E., 2008. Some common confusions about hyperbolic discounting. SSRN Electronic Journal.
Repenning NP. 2003. Selling system dynamics to (other) social scientists. System Dynamics Review 19(4): 303–327.
Samuelson P. 1937. A Note on measurement of utility. The Review of Economic Studies. Vol. 4, No. 2, pp. 155-161
Samuelson P. 1939. Interactions between the Multiplier Analysis and the Principle of Acceleration. The Review of Economics and Statistics. Vol. 21, No. 2, pp. 75-78 (4 pages). Published By: The MIT Press
Sayama Hiroki. Introduction to the modeling and analysis of complex systems. Open SUNY Textbooks, Year: 2015 ISBN: 978-1-942341-09-3
Schlager, K.J., 1964. How managers use industrial dynamics. IMR; Industrial Management Review (pre-1986), 6(1), p.21.
Shefrin, H., Thaler, R. 1988. The behavioral life-cycle hypothesis. Economic Inquiry vol. 6. pp. 609-643
Shefrin, H.M. and Thaler, R. 1977. An Economic Theory of Self-Control. National Bureau of Economic Research (Working Paper Series). Available at: https://doi.org/10.3386/w0208
Shefrin, H.M. and Thaler, R.H. 1988. 1992. Mental Accounting, Saving, Self-Control. in G. Loewenstein & J. Elster (Eds.), Choice over time (pp. 287–330). Russell Sage Foundation
Simon H. 1978. Rational decision-making in business organizations. Nobel memorial lecture. Economic Sciences 1978: 343–371.
Simon, H.A., 1972. Theories of bounded rationality. Decision and organization, 1(1), pp.161-176.
Sterman, J.D. 1984. Appropriate Summary Statistics for Evaluating the Historical Fit of System Dynamics Models. Dynamica, 10 (Winter), 51-66.
Sterman, J.D. 1987. Systems Simulation. Expectation formation in behavioral simulation models. Behavioral Science 32, 190-211. https://doi.org/10.1002/bs.3830320304
Sterman, J.D. 1988. Modeling the formation of expectations : the history of energy demand forecasts. International Journal of Forecasting 4, 243-259.
Sterman, J.D. 1995. A behavioral analysis of learning curve strategy. Cambridge, Mass. : Sloan School of Management, Massachusetts Institute of Technology.
Sterman, J.D. 2000. Business dynamics: Systems thinking and modeling for a complex world. Irwin/McGraw-Hill.
Sterman, J.D. 2018. System dynamics at sixty: the path forward. DOI: 10.1002/sdr.1601. System Dynamics Review. John Wiley
Stokey NL. 1989. Recursive methods in economic dynamics. Harvard University Press: Cambridge, MA.
Strotz, Robert. 1955. “Myopia and inconsistency in dynamic utility maximization.” Review of Economic Studies, 23(3): 165–180.
Tarasov, V.E. 2009. Map of Discrete System into Continuous. Journal of Mathematical Physics.Vol.47. No.9. (2006) 092901
Tarasov, V.E. 2015. Exact Discrete Analogs of Derivatives of Integer Orders: Differences as Infinite Series. Journal: Journal of Mathematics. Vol. 2015. DOI: 10.1155/2015/134842

Tarasov, V.E. 2017. Exact Discretization of an Economic Accelerator and Multiplier with Memory. Journal of Fractal and Fractional. Vol. 1, No. 1. DOI: 10.3390/fractalfract1010006

Tarasov, V.E., Tarasova, V.V. 2019. Harrod–Domar Growth Model with Memory and Distributed Lag. Journal: Axioms. Vol. 8. No. 1. DOI: 10.3390/axioms8010009

Tarasova, V. 2017. Exact Discretization of an Economic Accelerator and Multiplier with Memory. Journal: Fractal and Fractional. Vol. 1, Issue: 1. DOI: 10.3390/fractalfract1010006

Tarasova, V.V, Tarasov V. E. 2017. Accelerators in macroeconomics: Comparison of discrete and continuous approaches. Journal: American Journal of Economics and Business Administration. Vol.9. No.3. P.47-55

Thaler RH. 2016. Behavioral economics: past, present and future. American Economic Review 106(7): 1577–1600.

Thaler, R., 1985. Mental Accounting and Consumer Choice." Marketing Science, Summer 1985, 199-214

Thaler, R., Shefrin H., 1981. An Economic Theory of Self-Control. Journal of Political Economy 89, 392-406.

Thaler, R.H., H.M. Shefrin., 1981. An Economic Theory of Self-Control. Journal of Political Economy 89, 392-406.

Thaler, R.., and Benartzi, S. 2004. Save more tomorrow: Using behavioral economics to increase employee saving. Journal of Political Economy, 112: S164–S187

Thaler, R. and Sunstein, C.R. 2021. Nudge. The Final edition. New Haven: Yale University Press.

Tobin, J. 1958. Liquidity preference as behavior towards risk. Review of Economic Studies, 25(2), 65-86.

Wang, M. and Jie, F., 2020. Managing supply chain uncertainty and risk in the pharmaceutical industry. Health services management research, 33(3), pp.156-164.

Webb, C.S., 2016. Continuous quasi-hyperbolic discounting. Journal of Mathematical Economics, 64, pp.99-106.

Wynne Godley, Marc Lavoie: Monetary Economics. An Integrated Approach to Credit, Money, Income, Production and Wealth. Palgrave Macmillan, New York 2012, ISBN 978-0-230-30184-9.

Zhang , L., 2013. Saving and retirement behavior under quasi-hyperbolic discounting. Journal of Economics. Springer Vienna. Vol. 109 Issue: 1.

Zhang , L., 2013. Saving and retirement behavior under quasi-hyperbolic discounting. Journal of Economics. Springer Vienna. Volume: 109 Issue: 1.

Zhao, X., Zhou, Y. and Xie, J., 2017. An inventory system with quasi-hyperbolic discounting rate. IISE Transactions, 49(6), pp.593-602.