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A Novel Approach to Electromagnetic Simulations of Waveguides Using the Finite Element Method

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Abstract

We propose the use of the scalar, fifth-order Hermite interpolation polynomial in the Finite Element Method as an alternative to the vector finite element approach to solutions of Maxwell's equations in two dimensions. We analyzed the behavior of electromagnetic waves in homogeneous waveguides and inhomogeneous waveguides. As with vector finite elements, we are able to suppress the spurious solutions by appropriately setting the boundary conditions at interfaces. However with the derivative continuity provided by the quintic Hermite shape functions , we are able to provide greater accuracy than the vector finite element of equal polynomial order. Our scheme therefore has proven successful in calculations involving electromagnetic fields while at the same time providing better results than standard methods.

Contents

List of Figures

Chapter 1

Introduction

The idea of guiding electromagnetic waves along conducting rods have been of interest for a long time. As early as 1899, Sommerfeld thought of guiding magnetic waves along a circular conducting wire [1]. With the advent of digital computation it became possible to consider problems with waveguide geometries and characteristics that had no closed-form analytical solutions. In particular, the numerical simulation of electromagnetic waves in the microwave frequencies attracted much interest from their use in transmitters and receivers of radio waves.

The Theory of Electromagnetic theory was developed by Faraday and Maxwell as early as the 19^{th} century. Shortly afterwards the theory was verified experimentally by Hertz and the study of electromagnetic waves or the study of light became of increasing interest among physicists. Although the earlier works were concentrated on transmission of electrical signals of low frequency, many breakthroughs were made: of which, works from the like of Oliver Heaviside have contributed to the development of modern transmission-line theory [2]. In 1897, Lord Rayleigh presented an extensive study of the propagation of electromagnetic waves and in particular of propagation in cylindrical waveguides which highlighted the limitations of electromagnetic fields in propagation [3]. Lord Rayleigh realized that propagation only occurred above a cutoff frequency of propagation which was dependent on the cross-sectional dimension. Such allowed wave propagation was therefore of relatively high frequencies and was hard to reproduce on a practical scale, hindering much of the research in waveguides. Rayleigh's work was then revived by the likes of G. Southworth [4] and W. Barrow [5] more than 40 years later with advances in emission of waves. The development of the magnetron during World War II provided the technology required for a reliable source of emission of radio waves. Subsequently, works by Schwinger and Marcuvitz laid down the foundation for numerical simulation of waveguides with an intergral equation formulation of the problem [6].

However as early as the 1970s, it was known that solving Maxwell equations numerically gave rise to solutions with no physical counterparts [7]. In particular these solutions seemed to not obey the divergence conditions of the electric and magnetic fields. These erratic solutions were thus called spurious solutions. Extensive research to eliminate those non-physical solutions was then carried out [8] [9]. One such attempt was carried out using the "penalty method" [10] [11]. The penalty method introduces a term in the calculation that force the solutions to obey the divergence condition of the field. However this method require also the use of penalty factor together with the above term. However the penalty method required a proper choice of the penalty factor to be first identified to obtain correct solutions. Too small and too large choices for the penalty are not able to completely eliminate spurious solutions.

After numerous unsatisfactory results, the vector finite element method was proposed [12] [13]. The vector finite element method make use of the so-called edge elements. The edge elements are able to ensure the continuity of the tangential components of the field across interfaces. By satisfying the tangential boundary conditions, the spurious solutions are effectively eliminated [14]. However while the edge elements are able to satisfy tangential continuity across interfaces, they do not ensure the continuity of the normal components of the fields. This lack of constraint on the normal fields leads to "pixelated" solutions at the edges of the elements.

We propose the use of an alternative set of polynomial basis functions, the scalar fifth-order Hermite interpolation polynomials for the numerical calculations of electromagnetic fields. We solve the Maxwell's equations in two dimensions using the finite element method. In our approach, each of the component of the field is represented by the scalar Hermite shape functions. The Hermite shape functions allow us to ensure C_2-continuity along the boundaries as well as C_1-continuity across the boundaries: thus eliminating spurious solutions and allowing for more accurate solutions across elements.

Furthermore the scalar formulation of our approach offers more flexibility in our calculations as compared with vector finite elements. For instance, we are, in principle, able to consider coupled Schrödinger-Maxwell problems using a multi-physics formulation.

In the next chapter, we first go through the derivation of Hermite Interpolation Polynomials which we chose as our set of basis function for our FEM approach. In the following chapters, we give an account of the physics of homogeneous waveguides and inhomogeneous waveguides. We then present solutions obtained from our FEM formulation and contrast with these standard results in literature.

Chapter 2

Fifth-order Hermite Interpolation Polynomials

2.1 Derivation of the Hermite Interpolation Polynomials on the Standard Triangle

In our Finite Element Method, we seperate the domain in which we are interested into smaller elements. In 2-d, we choose triangles as our smaller elements as it is possible to replicate most shapes that the domain takes accurately as long as the elements are small enough i.e. with sufficiently dense meshes.

We therefore derive the Hermite shape functions for the standard triangle, shown in Fig.2.1. In the standard triangle, the nodal coordinates are specified as $(\xi_0, \eta_0), (\xi_1, \eta_1), (\xi_2, \eta_2)$.

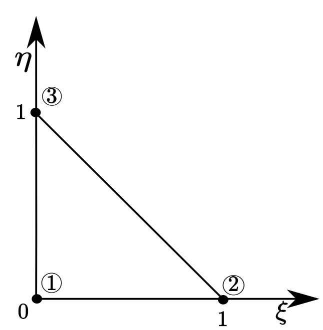


Figure 2.1: Standard Triangle

Our aim is to derive a set of 18 polynomials corresponding to 3 sets of 6 functions satisfying the following conditions

$$\begin{aligned}
N_i^{(0)}(\xi_j,\eta_j) &= \delta_{ij}\delta_{k,0}\ , \\
\left.\frac{\partial N_i^{(\xi)}(\xi,\eta)}{\partial \xi}\right|_{(\xi_j,\eta_j)} &= \delta_{ij}\delta_{k,\xi}\ , \\
\left.\frac{\partial N_i^{(\eta)}(\xi,\eta)}{\partial \eta}\right|_{(\xi_j,\eta_j)} &= \delta_{ij}\delta_{k,\eta}\ , \\
\left.\frac{\partial^2 N_i^{(\xi\xi)}(\xi,\eta)}{\partial \xi^2}\right|_{(\xi_j,\eta_j)} &= \delta_{ij}\delta_{k,\xi\xi}\ , \\
\left.\frac{\partial^2 N_i^{(\xi\eta)}(\xi,\eta)}{\partial \eta\partial \xi}\right|_{(\xi_j,\eta_j)} &= \delta_{ij}\delta_{k,\xi\eta}\ , \\
\left.\frac{\partial^2 N_i^{(\eta\eta)}(\xi,\eta)}{\partial \eta^2}\right|_{(\xi_j,\eta_j)} &= \delta_{ij}\delta_{k,\eta\eta}\ .
\end{aligned} \tag{2.1}$$

So, the values and first and second derivatives of shape function N_0 are 0 at all nodes except node 0, where its value is 1, and the values and derivatives of shape function $N_0^{(\xi)}$ is 0 at all nodes except node 0 where its ξ derivative is 1.

We now develop a set of polynomials with 18 terms in which the first 15 basis term comes from the Pascal triangle

$$\begin{aligned}
< 1;\quad \xi\quad \eta;\quad \xi^2\quad \xi\eta\quad \eta^2;\quad \xi^3\quad \xi^2\eta\quad \xi\eta^2\quad \eta^3; \\
\xi^4\quad \xi^3\eta\quad \xi^2\eta^2\quad \xi\eta^3\quad \eta^4 >
\end{aligned} \tag{2.2}$$

and the last three are chosen to be $\xi^5 - 5\xi^3\eta^2, \xi^2\eta^3 - \xi^3\eta^2$, and $\eta^5 - 5\xi^3\eta^2$, such that the normal derivative along the edges will be cubic in ξ and η. The shape functions are then just linear combinations of these basis functions. Then from the set of restrictions on the shape functions defined in Eq. (2.1) , we have 18^2 equations which we can put into the following 18×18 matrix form

$$\begin{bmatrix} a_0 & b_0 & \dots & r_0 \\ a_0^{(\xi)} & b_0^{(\xi)} & \dots & r_0^{(\xi)} \\ a_0^{(\eta)} & b_0^{(\eta)} & \dots & r_0^{(\eta)} \\ a_0^{(\xi\xi)} & b_0^{(\xi\xi)} & \dots & r_0^{(\xi\xi)} \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{(\eta\eta)} & b_2^{(\eta\eta)} & \dots & r_2^{(\eta\eta)} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ \xi_0 & 1 & \dots & 0 \\ \eta_0 & 0 & \dots & 0 \\ \xi_0^2 & 2\xi_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_0^5 - 5\xi_0^3\eta_0^2 & -15\xi_0^2\eta_0^2 & \dots & 20\eta_2^3 - 10\xi_2^3 \end{bmatrix} = \mathbb{I}. \tag{2.3}$$

It follows directly from the above matrix equation that the coefficient matrix is just the inverse of the matrix formed by explicitly calculating the values of the sets of the basis functions and their derivatives at each node. With the coefficient matrix we then obtain the complete set of shape functions by multiplying them into the basis functions.

2.2 Transformation of the Standard Polynomials to an arbitrary triangle

In meshing the region concerned in FEM calculations, the triangles used are rarely isosceles right-angled triangles but can take arbitrary forms. We therefore require shape functions for interpolation over arbitraty triangles compared to our previously derived shape functions for standard right angle triangles. In order to generalize the shape functions for the arbitrary triangle, we look at the transformation from the standard triangle to the arbitrary one.

Let ξ and η be the variables of the local coordinate system of the standard triangle and let x and y be the global coordinate variables of the arbitrary triangle. Also let x_i, y_i for $i = 0, 1, 2$ denote the vertex coordinates for the arbitrary triangle.

Using the set of linear shape functions for triangles

$$\begin{aligned} N_0(\xi,\eta) &= 1 - \xi - \eta \ , \\ N_1(\xi,\eta) &= \xi \ , \\ N_2(\xi,\eta) &= \eta \ . \end{aligned} \tag{2.4}$$

we can write the global coordinates in terms of the local coordinates

$$\begin{aligned} x &= x_0 N_0 + x_1 N_2 + x_2 N_2 \\ y &= y_0 N_0 + y_1 N_2 + y_2 N_2 \end{aligned} \tag{2.5}$$

Written in matrix form, we then obtain the transformation matrix from the local coordinate system to the global one

$$\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = \begin{bmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{2.6}$$

It then follows that the inverse transformation from the arbitrary to the standard triangle is obtained by pre-multiplying both side with the inverse of the linear transformation matrix. By then observing that the transformation matrix is none other than the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{2.7}$$

We can obtain equations for the partial derivatives of the local variables with respect to the global ones in terms of the partial derivatives of the global coordinates with respect to the local ones. Taking the determninant of the $\mathbf{J}$, we have

$$|\mathbf{J}| = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \xi} \tag{2.8}$$

we then have

$$\begin{aligned} \frac{\partial \xi}{\partial x} &= \frac{1}{|\mathbf{J}|} \cdot \frac{\partial y}{\partial \eta}\,; & \frac{\partial \xi}{\partial y} &= -\frac{1}{|\mathbf{J}|} \cdot \frac{\partial x}{\partial \eta}\,; \\ \frac{\partial \eta}{\partial x} &= -\frac{1}{|\mathbf{J}|} \cdot \frac{\partial y}{\partial \xi}\,; & \frac{\partial \eta}{\partial y} &= \frac{1}{|\mathbf{J}|} \cdot \frac{\partial x}{\partial \eta}\,. \end{aligned} \tag{2.9}$$

This set of equalities will prove useful when trying to obtain the shape functions for the global variables.

2.3 Evaluation of the Polynomial Values and Derivatives

In the above sections we obtained the shape functions for the standard triangle and also went over the transformation from any arbitrary triangle to the standard triangle. In this section we will go over the steps to obtain the proper shape functions in the global coordinates system from the local coordinates system. First of all we observe that for the interpolation of a function in the local coordinates system we have the following equation

$$\begin{aligned} f(\xi,\eta) &= \sum_i \big(f_i N_i + f'_{i,\xi} N_i^{(\xi)} + f'_{i,\eta} N_i^{(\eta)} + \\ &\quad f''_{i,\xi\xi} N_i^{(\xi\xi)} + f''_{i,\xi\eta} N_i^{(\xi\eta)} + f''_{i,\eta\eta} N_i^{(\eta\eta)} \big) \end{aligned} \tag{2.10}$$

However in most cases, we are actually working with arbitrary triangles in a global coordinates system (x, y) . Therefore we are interested in $f(x, y)$ rather than $f(\xi, \eta)$. Analogous to the previous equation we expect something like

$$\begin{aligned} f(x,y) &= \sum_i \big(f_i M_i + f'_{i,x} M_i^{(x)} + f'_{i,y} M_i^{(y)} + \\ &\quad f''_{i,xx} M_i^{(xx)} + f''_{i,xy} M_i^{(xy)} + f''_{i,yy} M_i^{(yy)} \big) \end{aligned} \tag{2.11}$$

where the set of shape functions $M_i^(j)$ are polynomials in x and y.In order to obtain these shape functions we look at the effect of differentiating the functions with respect to the local coordinates. Using the chain rule the first derivatives would yield

$$\begin{aligned} \frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} \end{aligned} \tag{2.12}$$

Similarly for the 2nd derivatives we have

$$\begin{aligned} \frac{\partial^2 f}{\partial \xi^2} &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2 + \cdot 2\frac{\partial^2 f}{\partial x \partial y}\left(\frac{\partial y}{\partial \xi}\cdot\frac{\partial y}{\partial \xi}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial \xi}\right)^2 \\ \frac{\partial^2 f}{\partial \xi \partial \eta} &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial x}{\partial \eta}\right) + \frac{\partial^2 f}{\partial x \partial y}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial \xi}\cdot\frac{\partial y}{\partial \eta}\right) \\ \frac{\partial^2 f}{\partial \eta^2} &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial \eta}\right)^2 + 2\cdot\frac{\partial^2 f}{\partial x \partial y}\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial y}{\partial \eta}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2 \end{aligned} \tag{2.13}$$

From the previous sets of equation and together with the fact that the function value should be the same in both coordinate systems, we obtain the following matrix equation

$$\begin{bmatrix} f \\ f'_\xi \\ f'_\eta \\ f''_{\xi\xi} \\ f''_{\xi\eta} \\ f''_{\eta\eta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & 0 & 0 & 0 \\ 0 & \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & (\frac{\partial x}{\partial \xi})^2 & 2\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi} & (\frac{\partial y}{\partial \xi})^2 \\ 0 & 0 & 0 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \\ 0 & 0 & 0 & (\frac{\partial x}{\partial \eta})^2 & 2\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta} & (\frac{\partial y}{\partial \eta})^2 \end{bmatrix} \begin{bmatrix} f \\ f'_x \\ f'_y \\ f''_{xx} \\ f''_{xy} \\ f''_y \end{bmatrix}. \tag{2.14}$$

By then substituting these sets of equations in Eq. (2.10) we obtain the following relation

$$\{M(x, y)\}_i = \{N(\xi, \eta)\}_i \cdot \mathbf{T} \tag{2.15}$$

where $\mathbf{T}$ is the transformation matrix obtained above and M_i is the set of shape functions associated with the i^{th} node in the global coordinates.

Now that we have showed how to carry out the interpolation of a function over any triangle, we now turn to the interpolations of the derivatives of that same function. In order to do so, we write the different differential operators with respect to x and y in terms of the local coordinates ξ and η. As an example we look at the differential operator for x-derivatives. Using the chain rule we obtain

$$D_x = \frac{\partial \xi}{\partial x} D_\xi + \frac{\partial \eta}{\partial x} D_\eta \tag{2.16}$$

where for instance D_x is the operator corresponding to taking the derivative with respect to x. In a similar way for the differential operator of double derivative with respect to x first and then y, we have

$$\begin{aligned} D_{xy} \;=\; & \left(\frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi}{\partial y}\right) D_{\xi\xi} + \left[\left(\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x}\right)\right] D_{\xi\eta} \\ & + \left(\frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial y}\right) D_{\eta\eta} \end{aligned} \tag{2.17}$$

Following the same procedure, we can find the different differential operators with respect to the global coordinates in terms of their local counterparts and by then applying them to the 'scaled' interpolation polynomials for the functions, we obtain the proper interpolation polynomials for the derivatives in the global coordinates.

In the next chapter, we look at the homogoneous waveguides and using the shape functions derived in this chapter, we solve for the allowed modes of propagations in an FEM scheme.

Chapter 3

Fields in a Conducting Homogeneous Waveguides

3.1 The Equations of Motion for E and H Fields

The electromagnetic fields in a waveguide are evaluated with the finite element method using Hermite interpolation polynomials. While the FEM allows us to consider a waveguide of arbitrary cross-section, we consider the prototypical rectangular waveguide as an example to investigate the performance of Hermite FEM. We shall assume that the waves propagate in the longitudinal direction, labeled the z-direction, as shown in Fig. (3.1).

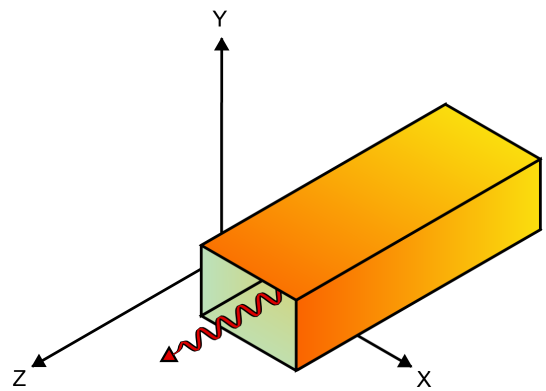


Figure 3.1: A rectangular waveguide, with waves propagating in the z-direction. The walls are assumed to be perfect conductors.

To calculate the electric and magnetic fields propagating through a waveguide, we begin with Maxwell's equations, in MKS units,

$$\nabla \cdot \mathbf{D} = \rho, \tag{3.1}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{3.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{3.4}$$

In the above, we express $\mathbf{D}$ and $\mathbf{B}$ in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$,

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{3.5}$$

If the medium is isotropic, the quantities ϵ and μ will be scalar quantities, rather than second-rank tensors. Let us define the dimensionless quantities ϵ_r and μ_r so that

$$\epsilon = \epsilon_r \epsilon_0, \qquad \mu = \mu_r \mu_0, \tag{3.6}$$

where ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively.

In waveguide problems, we are usually concerned with time harmonic electromagnetic waves propagating through the waveguide with a known wave number. Therefore we expect the electric and magnetic fields to have the forms

$$\mathbf{E} = \mathbf{E}_o(x, y)e^{i(k_z z - wt)}, \tag{3.7}$$

$$\mathbf{H} = \mathbf{H}_o(x, y)e^{i(k_z z - wt)}. \tag{3.8}$$

Note that, for the perfect conductor at the periphery of the waveguide, we set the current $\mathbf{J}$ equal to zero. Taking the time derivatives of Eqs. (3.7-3.8) and substituting the results into Eqs. (3.2-3.3), we obtain

$$\mathbf{H} = -\frac{i}{\mu\omega}\nabla \times \mathbf{E}, \tag{3.9a}$$

$$\mathbf{E} = \frac{i}{\epsilon\omega}\nabla \times \mathbf{H}. \tag{3.9b}$$

To perform finite element analysis on waves propagating through the waveguide, it is convenient to use Maxwell's equations to define an action integral. Such an integral can be discretized within the framework of the FEM. The fields are represented by Hermite interpolation polynomials on triangles, multiplied by the values of the fields and their derivatives at the vertices (nodes) of the triangle. The integration of the action can now be performed over spatial variables to obtain the discretized action in terms of the nodal parameters. Then a variation with respect to the nodal parameters, the principle of stationary action, yields a matrix equation of motion that is solved numerically to obtain solutions to the waveguide problem.

The Lagrangian density can be obtained either by substituting Eq. (3.9a) into Eq. (3.9b) or vice versa. Substitution of Eq. (3.9a) into Eq. (3.9b) yields

$$\mathbf{E} = \frac{i}{\epsilon\omega}\nabla \times \left(-\frac{i}{\mu\omega}\nabla \times \mathbf{E}\right), \tag{3.10}$$

which simplifies to

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) - \omega^2\epsilon\mathbf{E} = 0. \tag{3.11}$$

We can relate the frequency to the wave vector k_0 by the relationships

$$\epsilon_0\mu_0 = \frac{1}{c^2}, \tag{3.12}$$

$$\frac{\omega}{c} = k_0, \tag{3.13}$$

so that Eq. (3.11) becomes

$$\nabla \times \left(\frac{1}{\mu_r \mu_0} \nabla \times \mathbf{E} \right) - \left(\frac{k_0^2}{\epsilon_0 \mu_0} \right) \epsilon_r \epsilon_0 \mathbf{E} = 0. \tag{3.14}$$

Factoring out μ_0, we obtain

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \mathbf{E} \right) - k_0^2 \epsilon_r \mathbf{E} = 0. \tag{3.15}$$

Instead of the electric field formulation, we may express the governing equation exclusively in terms of $\mathbf{H}$ by substituting Eq. (3.9b) into Eq. (3.9a), yielding

$$\mathbf{H} = -\frac{i}{\mu\omega} \nabla \times \left(\frac{i}{\epsilon\omega} \nabla \times \mathbf{H} \right), \tag{3.16}$$

which simplifies to

$$\nabla \times \left(\frac{1}{\epsilon} \nabla \times \mathbf{H} \right) - \omega^2 \mu \mathbf{H} = 0. \tag{3.17}$$

Again invoking Eqs. (3.12-3.13) we obtain

$$\nabla \times \left(\frac{1}{\epsilon_r \epsilon_0} \nabla \times \mathbf{H} \right) - \left(\frac{k_0^2}{\epsilon_0 \mu_0} \right) \mu_r \mu_0 \mathbf{H} = 0. \tag{3.18}$$

Now factoring out ϵ_0, we obtain

$$\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \mathbf{H} \right) - k_0^2 \mu_r \mathbf{H} = 0. \tag{3.19}$$

Next we investigate Eq. (3.15) and Eq. (3.19), observing that either equation may be used to set up the action integral.

In order to define the Lagrangian density used in the action integral, we use the vector identity

$$\begin{aligned} \nabla \cdot (\mathbf{P} \times \mathbf{R}) &= \epsilon_{ijk} \left(\partial_i P_j \right) R_k - P_j \epsilon_{jik} \partial_i R_k \\ &= (\nabla \times \mathbf{P}) \cdot \mathbf{R} - \mathbf{P} \cdot (\nabla \times \mathbf{R}) . \end{aligned} \tag{3.20}$$

Now let $\mathbf{R} = \alpha \nabla \times \mathbf{Q}$. Then from Eq. (3.20),

$$\nabla \cdot (\mathbf{P} \times (\alpha \nabla \times \mathbf{Q})) = (\nabla \times \mathbf{P}) \cdot (\alpha \nabla \times \mathbf{Q}) - \mathbf{P} \cdot (\nabla \times (\alpha \nabla \times \mathbf{Q})) \,. \tag{3.21}$$

This leads to the integrals

$$\begin{aligned} \int_V d^3r \nabla \cdot [\mathbf{P} \times (\alpha \nabla \times \mathbf{Q})] = \int_V d^3r \, (\nabla \times \mathbf{P}) \cdot (\alpha \nabla \times \mathbf{Q}) \\ - \int_V d^3r \, \mathbf{P} \cdot [\nabla \times (\alpha \nabla \times \mathbf{Q})] \,. \end{aligned} \tag{3.22}$$

The left side can be reduced to a surface integral by Gauss's Theorem, so we have

$$\begin{aligned} \int_V d^3r \mathbf{P} \cdot [\nabla \times (\alpha \nabla \times \mathbf{Q})] = \int_V d^3r \, (\nabla \times \mathbf{P}) \cdot (\alpha \nabla \times \mathbf{Q}) \\ - \oint_S ds \hat{\mathbf{n}} \cdot [\mathbf{P} \times (\alpha \nabla \times \mathbf{Q})] \,. \end{aligned} \tag{3.23}$$

We can now set $\mathbf{Q} = \mathbf{H}$, $\mathbf{P} = \mathbf{H}^*$, and $\alpha = \epsilon_r^{-1}$ to obtain

$$\begin{aligned} \int_V d^3r \mathbf{H}^* \cdot \left[\nabla \times \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \right] = \int_V d^3r \, (\nabla \times \mathbf{H}^*) \cdot \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \\ - \oint_S ds \hat{\mathbf{n}} \cdot \left[\mathbf{H}^* \times \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \right] . \end{aligned} \tag{3.24}$$

Note that, since

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \mathbf{R} \cdot (\mathbf{P} \times \mathbf{Q}) \,, \tag{3.25}$$

the surface term may be rewritten as

$$\hat{\mathbf{n}} \cdot \left[\mathbf{H}^* \times \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \right] = -\mathbf{H}^* \cdot \left[\hat{\mathbf{n}} \times \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \right] . \tag{3.26}$$

For time-harmonic fields, it has been shown in Eq. (3.9b) that $\mathbf{E}$ is proportional to $\epsilon_r^{-1} (\nabla \times \mathbf{H})$. Assuming the waveguide to be constructed of a perfectly conducting material, one of the boundary conditions is that $\hat{n} \times \mathbf{E} = 0$. Thus, the surface term in Eq. (3.24) is exactly zero.

Suppose, instead of a perfect electric conductor, that we have a magnetic wall at the waveguide boundary. In this case the relevant boundary condition is $\hat{n} \times \mathbf{H} = 0$. In this case we may write the surface term as

$$\hat{\mathbf{n}} \cdot \left[\mathbf{H}^* \times \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \right] = \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) \cdot [\hat{\mathbf{n}} \times \mathbf{H}^*] . \tag{3.27}$$

Since $\hat{\mathbf{n}} \times \mathbf{H}^* = 0$, the surface term once again vanishes.

Note that it is possible to work instead with the electric field formulation of Eq. (3.15). In this case, the surface term takes the form

$$\oint_S ds \hat{\mathbf{n}} \cdot \left[\mathbf{E}^* \times \frac{1}{\mu_r} (\nabla \times \mathbf{E}) \right] . \tag{3.28}$$

Note that, regardless of whether the perfect conductor ($\hat{\mathbf{n}} \times \mathbf{E} = 0$) or the magnetic wall ($\hat{\mathbf{n}} \times \mathbf{H} = 0$) case is used, Eq. (3.25) may be used to set the surface term equal to zero.

The result is that, for a perfect electric or magnetic conductor, there will not be a nonzero surface term in the action integral, regardless of whether Eq. (3.15) or Eq. (3.19) is used.

Continuing from Eq. (3.19), we can now readily construct the action integral. Generally, it is preferable to work with the magnetic field formulation because magnetic fields are continuous throughout most materials found in nature. Suppose we define the operator $\mathcal{L}$ as

$$\mathcal{D} = \nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \right) - k_0^2 \mu_r . \tag{3.29}$$

Then, following Jin [17, p.212], the operator can be used to construct an action integral of the form

$$\mathcal{A} = \frac{1}{2} \langle \mathcal{D}\mathbf{H}, \mathbf{H} \rangle = \frac{1}{2} \int_V d^3 r \mathbf{H}^* \cdot (\mathcal{D}\mathbf{H}) , \tag{3.30}$$

provided that the operator is self-adjoint; that is,

$$\langle \mathcal{D}\mathbf{H}, \mathbf{H} \rangle = \langle \mathbf{H}, \mathcal{D}\mathbf{H} \rangle . \tag{3.31}$$

The self-adjointness of $\mathcal{D}$ is verified in Jin's book [17, p.215-p.216]. To eliminate the double-curl from the action, we substitute Eq. (3.29) into Eq. (3.30).

Then we reduce the overall derivative order of the action integral by invoking Eq. (3.24), keeping in mind that the surface term vanishes for a perfect electric or magnetic conductor. The drawback is that the differentiation of the inverse of the dielectric may, in some cases, have an adverse effect on the convergence rate of solutions for inhomogeneous waveguides. The resulting action is

$$\mathcal{A} = \frac{1}{2}\int_V d^3r \left[(\nabla \times \mathbf{H}^*) \cdot \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) - k_0^2 \mu_r \mathbf{H}^* \cdot \mathbf{H} \right]. \tag{3.32}$$

Similarly, using the electric field formulation of Maxwell's equations yields

$$\mathcal{A} = \frac{1}{2}\int_V d^3r \left[(\nabla \times \mathbf{E}^*) \cdot \frac{1}{\mu_r} (\nabla \times \mathbf{E}) - k_0^2 \epsilon_r \mathbf{E}^* \cdot \mathbf{E} \right]. \tag{3.33}$$

The following section describes the discretization of the vector operators involved in the action, allowing the minimization of the action to be solved numerically. The magnetic field formulation, Eq. (3.32), will be used for most of the subsequent calculations.

To formulate the Lagrangian density for our calcuation, we can also start from the usual definition of the Lagrangian density for electromagnetic fields, Eq (3.35).

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}\big(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*\big) \qquad (3.34) \\ &= \frac{1}{4}\Big(\epsilon\mathbf{E}\cdot\mathbf{E}^* - \mu\mathbf{H}\cdot\mathbf{H}^*\Big) \end{aligned}$$

From Eq. (3.9a) and Eq. (3.9b), we can easily get back Eq. (3.32) and Eq. (3.33) since

$$\mathcal{A} = \int \mathcal{L} d^3r \tag{3.35}$$

For example on subsitution of **H** from Eq. (3.9a), we get

$$\begin{aligned} \mathcal{A} &= \frac{1}{4}\int \left\{ \frac{1}{\omega^2}(\nabla\times\mathbf{H})\frac{1}{\epsilon}(\nabla\times\mathbf{H}^*) - \mu\mathbf{H}\cdot\mathbf{H}^* \right\} d^3r \\ &= \frac{1}{4}\frac{c^2\mu_0}{\omega^2}\int \left\{ (\nabla\times\mathbf{H})\frac{1}{\epsilon_r}(\nabla\times\mathbf{H}^*) - \mu_r k^2 \mathbf{H}\cdot\mathbf{H}^* \right\} d^3r \end{aligned}$$

where we used

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

and

$$c = \frac{\omega}{k}$$

The extra factor in front of the action integral is irrevelant in eigenvalue problems and therefore, we are left with similar forms of the action integral in both derivation.

3.1.1 Array Representation of Fields and their Derivatives

From Maxwell's equations, we have the divergence condition,

$$\nabla \cdot (\mu_r \mathbf{H}) = 0, \tag{3.36}$$

which makes it possible to remove one of the three vector components from the action integral by expressing it in terms of the remaining two. Since the fields are known to have a sinusoidal variation in the z-direction, it is natural to eliminate the z-component of the field from the action. First we define the transverse magnetic field, $\mathbf{H}_t$, as $H_x\hat{\imath}+H_y\hat{\jmath}+0\hat{k}$. We also introduce the transverse gradient operator ∇_t, as $\partial_x\hat{\imath}+\partial_y\hat{\jmath}+0\hat{k}$. In the following we express the curl of the magnetic field in an array form,

$$\nabla \times \mathbf{H} = (\nabla \times \mathbf{H})_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \hat{\mathbf{i}} + (\nabla \times \mathbf{H})_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \hat{\mathbf{j}} + (\nabla \times \mathbf{H})_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \hat{\mathbf{k}} \tag{3.37}$$

In this format we express the curl as

$$\nabla \times \mathbf{H} = \begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} \tag{3.38}$$

The Transverse Curl Operator

Similarly, the matrix form of the transverse curl operator is

$$\nabla_t \times \mathbf{H}_t = \begin{pmatrix} 0 \\ 0 \\ \partial_x H_y - \partial_y H_x \end{pmatrix} \tag{3.39}$$

It is also helpful to write out the divergence condition in terms of the transverse gradient operator,

$$\nabla_t \cdot \mu_r \mathbf{H}_t + \partial_z H_z = 0, \tag{3.40}$$

which can be rewritten for time-harmonic waves propagating in the z-direction as

$$\nabla_t \cdot \mu_r \mathbf{H}_t = -ik_z H_z. \tag{3.41}$$

We seek a means of expressing $\nabla \times \mathbf{H}$ in terms of $\nabla_t \times \mathbf{H}_t$, eliminating the axial component of the magnetic field from the action integral prior to discretization. Such a method is described in detail by Jin [17, p.257-259]. First, we observe that

$$k_0^2 \mu_r \mathbf{H}^* \cdot \mathbf{H} = k_0^2 \mu_r \mathbf{H}_t^* \cdot \mathbf{H}_t + k_0^2 \mu_r \left| H_z \right|^2 . \tag{3.42}$$

The Lagrangian Density

We verify below the statement of Eq. (7.72) from Jin [17, p.257], that the Lagrangian density of Eq. (3.32) may be rewritten as

$$\begin{aligned}\mathcal{A} = \frac{1}{2} \int_V d^3 r \Bigg[& (\nabla_t \times \mathbf{H}_t^*) \cdot \frac{1}{\epsilon_r} (\nabla_t \times \mathbf{H}_t) - \mathbf{H}_t^* \cdot k_0^2 \mu_r \mathbf{H}_t - k_0^2 \mu_r \left| H_z \right|^2 \\ & + (\nabla_t H_z - \partial_z \mathbf{H}_t)^* \cdot \frac{1}{\epsilon_r} (\nabla_t H_z - \partial_z \mathbf{H}_t) \Bigg] .\end{aligned} \tag{3.43}$$

First we prove that the sum of the first and fourth terms of Eq. (3.43) yields the same result as the first term of Eq. (3.32). To prove this, rewrite Eq. (3.38) as

$$\nabla \times \mathbf{H} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} , \tag{3.44}$$

where

$$\begin{aligned} P &= \partial_y H_z - \partial_z H_y, \\ Q &= \partial_z H_x - \partial_x H_z, \\ R &= \partial_x H_y - \partial_y H_x. \end{aligned} \tag{3.45}$$

It follows that

$$(\nabla \times \mathbf{H}^*) \cdot \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) = \frac{1}{\epsilon} \left(|P|^2 + |Q|^2 + |R|^2 \right) . \tag{3.46}$$

We rewrite some of the terms of Eq. (3.43) as

$$\nabla_t \times \mathbf{H}_t = \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix} \tag{3.47}$$

and

$$\nabla_t H_z - \partial_z \mathbf{H}_t = \begin{pmatrix} \partial_x H_z - \partial_z H_x \\ \partial_y H_z - \partial_z H_y \\ 0 \end{pmatrix} = \begin{pmatrix} -Q \\ P \\ 0 \end{pmatrix} \tag{3.48}$$

It follows from Eq. (3.47) and Eq. (3.48) that

$$\begin{aligned}
(\nabla_t \times \mathbf{H}_t^*) \cdot & \frac{1}{\epsilon_r} (\nabla_t \times \mathbf{H}_t) + (\nabla_t H_z - \partial_z \mathbf{H}_t)^* \cdot \frac{1}{\epsilon_r} (\nabla_t H_z - \partial_z \mathbf{H}_t) \\
&= \frac{1}{\epsilon_r} \left(R^* R + (-Q)^*(-Q) + P^* P \right) \\
&= \frac{1}{\epsilon_r} \left(|P|^2 + |Q|^2 + |R|^2 \right) \\
&= (\nabla \times \mathbf{H}^*) \cdot \frac{1}{\epsilon_r} (\nabla \times \mathbf{H}) .
\end{aligned} \tag{3.49}$$

Thus, the action of Eq. (3.32) is equivalent to the action of Eq. (3.43). The equality of the terms containing μ_r has already been verified via Eq. (3.42).

Elimination of H_z

Continuing from Eq. (3.43), we now seek to eliminate terms involving H_z from the action. Jin asserts that

$$\frac{1}{\epsilon_r} \nabla_t H_z \cdot \nabla_t H_z^* = \nabla_t \cdot \left(H_z^* \frac{1}{\epsilon_r} \nabla_t H_z \right) - H_z^* \nabla_t \cdot \left(\frac{1}{\epsilon_r} \nabla_t H_z \right) . \tag{3.50}$$

We can verify Eq. (3.50) immediately by applying the product rule to the middle term. Next, we verify Jin's result that

$$\nabla_t \cdot \left(\frac{1}{\epsilon_r} \nabla_t H_z \right) = \partial_z \nabla_t \cdot \left(\frac{1}{\epsilon_r} \mathbf{H}_t \right) - k_0^2 \mu_r H_z . \tag{3.51}$$

Beginning with the double-curl formulation of Maxwell's equations from Eq. (3.19), we take the z-component of each side,

$$\left[\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \mathbf{H} \right) \right]_z = k_0^2 \mu_r H_z . \tag{3.52}$$

We write Eq. (3.52) out term-by-term by multiplying Eq. (3.38) by ϵ_r^{-1} and then taking the curl of the product, yielding

$$\begin{aligned}\left[\nabla\times\left(\frac{1}{\epsilon_r}\nabla\times\mathbf{H}\right)\right]_z &= \partial_x\left[\frac{1}{\epsilon_r}\left(\partial_z H_x - \partial_x H_z\right)\right] - \partial_y\left[\frac{1}{\epsilon_r}\left(\partial_y H_z - \partial_z H_y\right)\right]\\ &= \partial_z\left[\partial_x\left(\frac{1}{\epsilon_r}H_x\right) + \partial_y\left(\frac{1}{\epsilon_r}H_y\right)\right] - \partial_x\left(\frac{1}{\epsilon_r}\partial_x H_z\right) - \partial_y\left(\frac{1}{\epsilon_r}\partial_y H_z\right)\\ &= \partial_z\nabla_t\cdot\left(\frac{1}{\epsilon_r}\mathbf{H}_t\right) - \nabla_t\cdot\left(\frac{1}{\epsilon_r}\nabla_t H_z\right).\end{aligned} \tag{3.53}$$

Substituting Eq. (3.52) into Eq. (3.53), we find that Eq. (3.51) is indeed true.

The Action Integral in Terms of Transverse Fields

Finally, we invoke the two-dimensional divergence theorem,

$$\iint_\Omega \left[\nabla_t\cdot\left(\frac{1}{\epsilon_r}H_z^*\nabla_t H_z\right)\right] d\Omega = \oint_\Gamma \left[\frac{1}{\epsilon_r}H_z^*\nabla_t H_z\cdot\hat{n}\right] d\Gamma, \tag{3.54}$$

Which can be rewritten as

$$\iint_\Omega \left[\nabla_t\cdot\left(\frac{1}{\epsilon_r}H_z^*\nabla_t H_z\right)\right] d\Omega = \oint_\Gamma \left[\frac{1}{\epsilon_r}H_z^*\frac{\partial H_z}{\partial n}\right] d\Gamma. \tag{3.55}$$

Expanding the last dot product of Eq. (3.43) and substituting Eqs. (3.50), (3.51), and (3.55) we obtain

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int_V d^3r \Bigg[&(\nabla_t\times\mathbf{H}_t^*)\cdot\frac{1}{\epsilon_r}(\nabla_t\times\mathbf{H}_t) - \mathbf{H}_t^*\cdot\left(k_0^2\mu_r - \frac{k_z^2}{\epsilon_r}\right)\mathbf{H}_t\\ &+jk_zH_z^*\nabla_t\cdot\left(\frac{1}{\epsilon_r}\mathbf{H}_t\right) + \frac{jk_z}{\epsilon_r}\nabla_t H_z^*\cdot\mathbf{H}_t - \frac{jk_z}{\epsilon_r}\mathbf{H}_t^*\cdot\nabla_t H_z\Bigg]\\ &+\oint_\Gamma\left(\frac{1}{\epsilon_r}H_z^*\frac{\partial H_z}{\partial n}\right)d\Gamma,\end{aligned} \tag{3.56}$$

where Γ denotes the outer edge of the waveguide cross-section. We can then use the product rule to verify that

$$H_z^*\nabla_t\cdot\left(\frac{1}{\epsilon_r}\mathbf{H}_t\right) + \frac{1}{\epsilon_r}\nabla_t H_z^*\cdot\mathbf{H}_t = \nabla_t\cdot\left(\frac{1}{\epsilon_r}H_z^*\mathbf{H}_t\right), \tag{3.57}$$

$$\frac{1}{\epsilon_r}\mathbf{H}_t^* \cdot \nabla_t H_z = \nabla_t \cdot \left(\frac{1}{\epsilon_r} H_z \mathbf{H}_t^*\right) - H_z \nabla_t \cdot \left(\frac{1}{\epsilon_r}\mathbf{H}_t^*\right). \tag{3.58}$$

Substituting Eq. (3.57) and Eq. (3.58) into Eq. (3.56), and again using the two-dimensional divergence theorem, now yields

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int_V d^3r \bigg[&(\nabla_t \times \mathbf{H}_t^*) \cdot \frac{1}{\epsilon_r}(\nabla_t \times \mathbf{H}_t) - \mathbf{H}_t^* \cdot \left(k_0^2 \mu_r - \frac{k_z^2}{\epsilon_r}\right)\mathbf{H}_t \\ &-ik_z H_z^* \nabla_t \cdot \left(\frac{1}{\epsilon_r}\mathbf{H}_t\right)\bigg] d\Omega \\ &+ \oint_\Gamma \left(\frac{1}{\epsilon_r} H_z^* \frac{\partial H_z}{\partial n} - \frac{ik_z}{\epsilon_r} H_z^* H_n + \frac{ik_z}{\epsilon_r} H_z H_n^*\right) d\Gamma.\end{aligned} \tag{3.59}$$

We now seek to simplify Eq. (3.59) by proving that

$$j\omega\epsilon_0 H_z^* E_{\hat{\tau}} = \frac{1}{\epsilon_r}\left(H_z^* \frac{\partial H_z}{\partial n} - ik_z H_z^* H_n\right), \tag{3.60}$$

where $\hat{\tau}$ denotes the direction tangent to the edge of the waveguide cross-section. To verify Eq. (3.60), we begin by moving the ϵ_r term to the left side, then invoke Eq. (3.9b) and cancel out the H_z^* term to obtain

$$-(\nabla \times \mathbf{H})_{\hat{\tau}} = \frac{\partial H_z}{\partial n} - ik_z H_n \tag{3.61}$$

We now rewrite Eq. (3.61) as

$$-\hat{\tau} \cdot (\nabla \times \mathbf{H}) = \hat{n} \cdot \nabla H_z - \partial_z (H \cdot \hat{n}) \tag{3.62}$$

Note that the tangential unit vector $\hat{\tau}$ may be represented as

$$\hat{\tau} = n_y \hat{i} - n_x \hat{j} \tag{3.63}$$

where n_x and n_y are components of the normal unit vector. It is obvious, then, that $\hat{n}$, $\hat{\tau}$, and $\hat{z}$ form an orthonormal basis. Expanding Eq. (3.62) in terms of components, using Eq. (3.38), we get

$$\begin{pmatrix} n_y \\ -n_x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix} \cdot \left[\begin{pmatrix} \partial_x H_z \\ \partial_y H_z \\ \partial_z H_z \end{pmatrix} - \begin{pmatrix} \partial_z H_x \\ \partial_z H_y \\ \partial_z H_z \end{pmatrix}\right] \tag{3.64}$$

It is clear that both sides of Eq. (3.64) are equal. Thus Eq. (3.60) is true. Furthermore, for a perfect electric conductor, there will be no tangential electric field component. For a perfect magnetic conductor, there is no axial magnetic field component at the waveguide boundary. Thus the surface integral of Eq. (3.60) vanishes both for electric and magnetic conductors. With this in mind, Eq. (3.59) now simplifies to

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int_V d^3r \Bigg[&(\nabla_t \times \mathbf{H}_t^*) \cdot \frac{1}{\epsilon_r}(\nabla_t \times \mathbf{H}_t) - \mathbf{H}_t^* \cdot \left(k_0^2\mu_r - \frac{k_z^2}{\epsilon_r}\right)\mathbf{H}_t \\ &-ik_z H_z^* \nabla_t \cdot \left(\frac{1}{\epsilon_r}\mathbf{H}_t\right)\Bigg] d\Omega + \oint_\Gamma \left(\frac{ik_z}{\epsilon_r} H_z H_n^*\right) d\Gamma.\end{aligned} \tag{3.65}$$

Finally, we eliminate the z-component entirely, reducing the integral to two dimensions, by using the divergence condition, Eq. (3.41), to obtain

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int_\Omega d^2r \Bigg[&(\nabla_t \times \mathbf{H}_t^*) \cdot \frac{1}{\epsilon_r}(\nabla_t \times \mathbf{H}_t) - \mathbf{H}_t^* \cdot \left(k_0^2\mu_r - \frac{k_z^2}{\epsilon_r}\right)\mathbf{H}_t \\ &+\frac{1}{\mu_r}\nabla_t \cdot (\mu_r \mathbf{H}_t)\, \nabla_t \cdot \left(\frac{1}{\epsilon_r}\mathbf{H}_t^*\right)\Bigg] d\Omega + \oint_\Gamma \left(\frac{ik_z}{\epsilon_r \mu_r}\nabla_t \cdot (\mu_r \mathbf{H}_t)\, H_n^*\right) d\Gamma.\end{aligned} \tag{3.66}$$

Eq. (3.66) is the integral which we choose to discretize when performing the finite element calculations. The advantage over previous expressions for the action integral is that the overall derivative order in the action is reduced from 2 to 1.

3.2 Classification of Propagating Waves

The waves propagating in a waveguide may be broken down into three main categories, based on the directions of the electric and magnetic fields. In a transverse electromagnetic (**TEM**) wave, both the electric field and the magnetic field are confined to the the transverse direction. If the wave is propagating in the z-direction, the fields will only have components in the x- and y-directions.

TEM waves can propagate at any frequency, but they do not appear in simple waveguides consisting of a single enclosing surface. Since the TEM wave is the solution to an electrostatic problem in two dimensions, it will not appear if the entire enclosing boundary is an equipotential surface. However, TEM waves are the dominant mode in structures with multiple surfaces, such as coaxial cables, and may appear in waveguides with multiple dielectrics.

When the waveguide consists of a single hollow conductor, with a single isotropic dielectric material, the only propagating modes will be the transverse electric (**TE**) and transverse magnetic (**TM**) modes. In a TE mode, the axial component of the electric field must equal zero everywhere. However, the magnetic field is permitted to have components in all three directions. This gives the boundary conditions

$$E_z = 0, \qquad \left[\frac{\partial B_z}{\partial n}\right]_S = 0, \tag{3.67}$$

where S denotes the surface of the waveguide. In a TM mode, only the electric field is permitted to have a component in the axial direction. Thus the boundary conditions are

$$B_z = 0, \qquad [E_z]_S = 0. \tag{3.68}$$

Some examples of the different types of modes, reproduced from [19] are given on the following two pages. Note a slightly counterintuitive choice of nomenclature: transverse electric modes are also called H-modes, and transverse magnetic modes are also called E-modes.

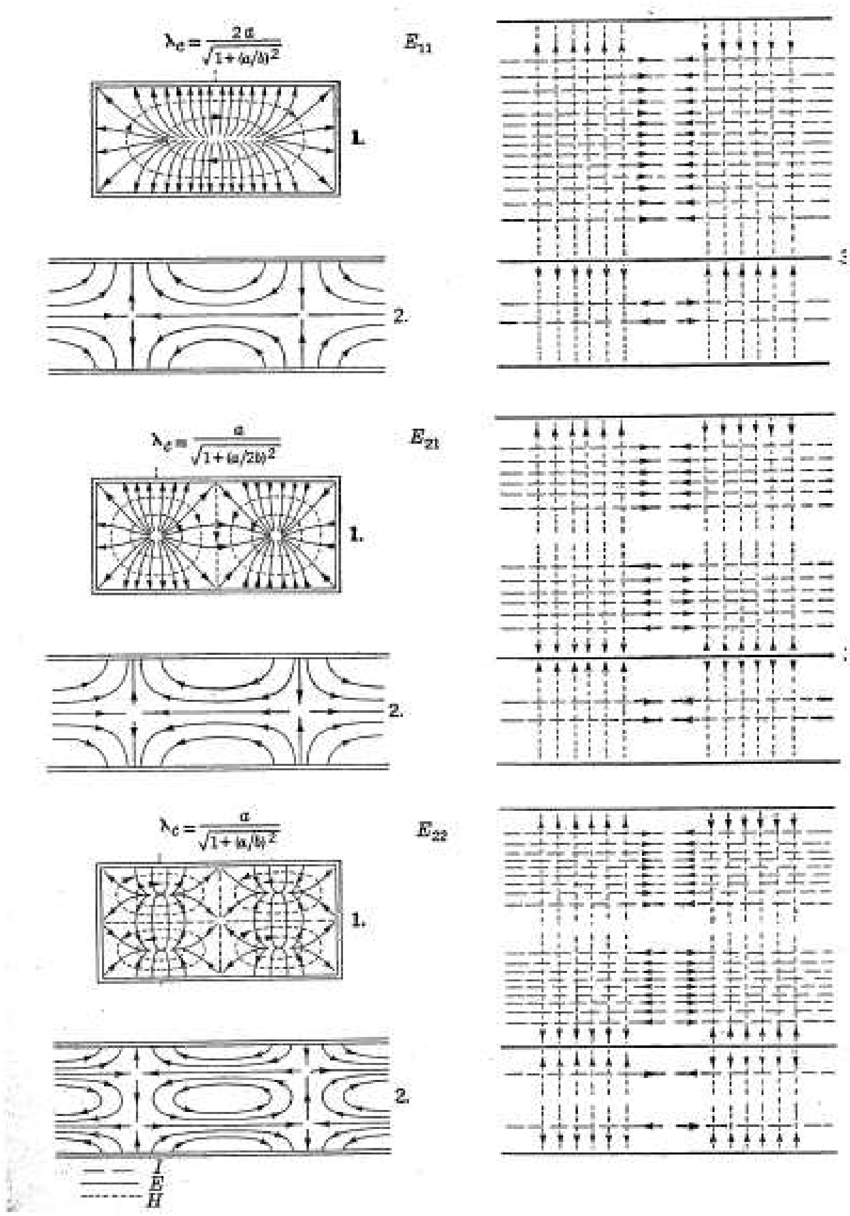


Figure 3.2: Some examples of propagating E-modes in a rectangular waveguide, reproduced from [19, p.59].

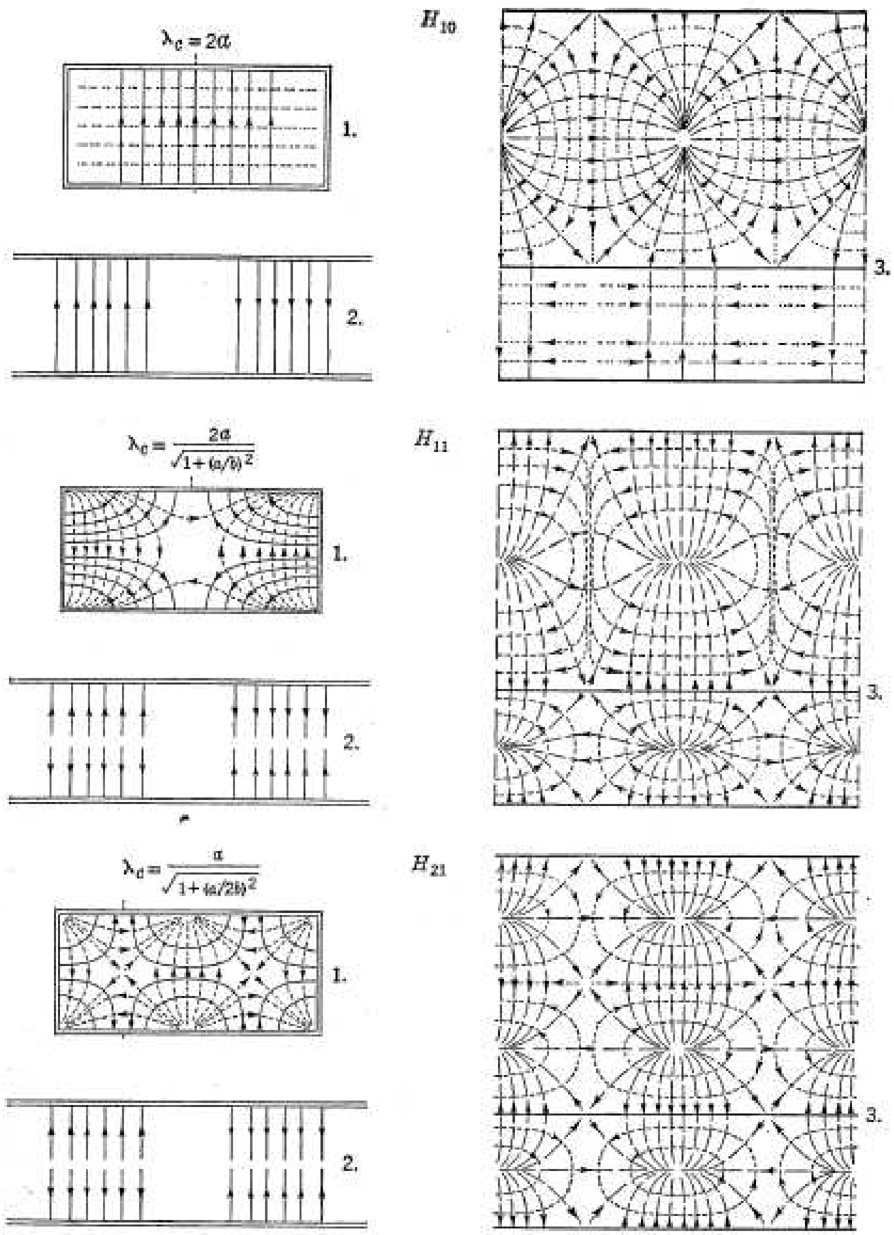


Figure 3.3: Some examples of propagating H-modes in a rectangular waveguide, reproduced from [19, p.63].

3.3 Boundary Conditions

Due to the nature of the electric and magnetic field, at every interface, the fields have to obey the boundary conditions given in [18, p. 18],

$$\begin{aligned} (\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} &= \sigma, \\ (\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} &= 0, \\ \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) &= 0, \\ \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{K}, \end{aligned} \tag{3.69}$$

where the subscripts refer to the electric and magnetic field in two distinct dielectric materials, σ is the idealized surface charge density, and $\mathbf{K}$ is the idealized surface current.

In other words, the normal component of the magnetic field and the tangential component of the electric field are continuous across any interface. Any discontinuity in the normal component of $\mathbf{D}$ is equal to the charge density at the interface, and any discontinuity in the magnetic field equals the current at the interface.

Suppose we are given a waveguide with a single dielectric material, enclosed by a perfect conductor. We then have

$$\begin{aligned} \hat{\mathbf{n}} \times \mathbf{E_1} &= 0, \\ \hat{\mathbf{n}} \cdot \mathbf{H_1} &= 0, \end{aligned} \tag{3.70}$$

while for perfect magnetic conductor we have

$$\begin{aligned} \hat{\mathbf{n}} \times \mathbf{H_1} &= 0, \\ \hat{\mathbf{n}} \cdot \mathbf{E_1} &= 0. \end{aligned} \tag{3.71}$$

We then express the normal unit vector in terms of its components

$$\hat{n} = n_x \hat{x} + n_y \hat{y}. \tag{3.72}$$

Correspondingly the tangent unit vector can be written as

$$\hat{t} = n_y \hat{x} - n_x \hat{y}, \tag{3.73}$$

and the directional derivative along the tangent as

$$\hat{t} \cdot \nabla = n_y \partial_x - n_x \partial_y. \tag{3.74}$$

3.3.1 Perfect Conductor

We now return to the case of the perfect conductor,

$$\hat{\mathbf{n}} \times \mathbf{E_1} = 0, \tag{3.75}$$

$$\hat{\mathbf{n}} \cdot \mathbf{H_1} = 0. \tag{3.76}$$

Then Eq. (3.75) yields

$$n_x H_x = n_y H_y, \tag{3.77}$$

while Eq. (3.76) yields

$$\partial_y H_x = \partial_x H_y. \tag{3.78}$$

Suppose that we have a rectangular waveguide with walls parallel to the xz- and yz-planes, as in Fig. (3.1). Then we can differentiate Eq. (3.77) and Eq. (3.78) with respect to x and y to obtain a complete set of boundary conditions for the waveguide.

For a boundary edge along $\hat{x}$ ($n_x = 0$), we obtain

$$\begin{aligned} H_y &= 0, \\ \partial_x H_y &= 0, \\ \partial_y H_x &= 0, \\ \partial_x^2 H_y &= 0, \\ \partial_y^2 H_x &= 0, \\ \partial_x \partial_y H_x &= 0. \end{aligned} \tag{3.79}$$

Likewise, for a boundary edge along $\hat{y}$ ($n_y = 0$), we obtain

$$\begin{aligned} H_x &= 0, \\ \partial_y H_x &= 0, \\ \partial_x H_y &= 0, \\ \partial_y^2 H_x &= 0, \\ \partial_x^2 H_x &= 0, \\ \partial_x \partial_y H_y &= 0. \end{aligned} \tag{3.80}$$

3.3.2 Magnetic Wall

In the case of the Magnetic Wall we have the boundary conditions

$$\hat{\mathbf{n}} \times \mathbf{H_1} = 0, \tag{3.81}$$

$$\hat{\mathbf{n}} \cdot \mathbf{E_1} = 0. \tag{3.82}$$

Using Eqs.(3.81-3.82), we obtain a set of boundary conditions for the magnetic wall case. For a boundary edge along $\hat{x}$ ($n_x = 0$), we obtain

$$\begin{aligned} H_x &= 0, \\ \partial_x H_x &= -\partial_y H_y, \\ \partial_y H_y &= 0, \\ \partial_y^2 H_y &= 0, \\ \partial_x \partial_y H_x &= \partial_x \partial_y H_y. \end{aligned} \tag{3.83}$$

For a boundary edge along $\hat{y}$ ($n_y = 0$), we obtain

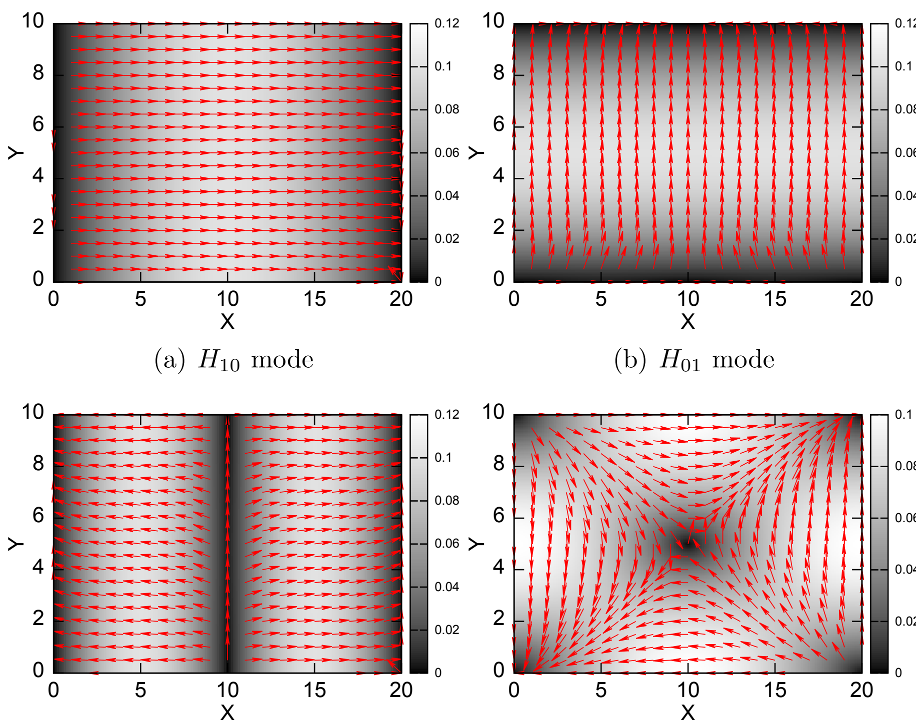
$$\begin{aligned} H_y &= 0 \\ \partial_x H_x &= -\partial_y H_y, \\ \partial_x H_y &= 0, \\ \partial_x^2 H_x &= -\partial_x \partial_y H_y, \\ \partial_x^2 H_y &= 0. \end{aligned} \tag{3.84}$$

3.4 Results of E&M Calculations

After discretizing the action integral, FEM was performed on the two-dimensional waveguide cross-section using a set of Hermite interpolation elements. The Hermite elements exhibit C_1 continuity throughout the finite element mesh. These elements were obtained from [15, p.227].Since they also possess second derivative degrees of freedom, is is possible to include second derivative boundary conditions. However, it is not guaranteed that the second derivative terms will be continuous across interelement boundaries.

Before considering the effects of different dielectric materials, a homogeneous waveguide was simulated with $\epsilon_r = 1, \mu_r = 1, k_z = 1.00$. The resulting functions were classified according to their z-components. Modes which exhibited a substantial z-component of the magnetic field were classified as H-modes, while those with a substantial z-component of the electric field were classified as E-modes. For the homogeneous waveguide with a single conducting boundary, no mixing of the different modes is expected to occur.

The resulting eigenfunctions are presented in Figs.(3.4-3.6). The numerically calculated modes bear a close resemblance to the analytically determined modes shown in Figs.(3.2-3.3).



(a) H_{10} mode (b) H_{01} mode (c) H_{20} mode (d) H_{11} mode

Figure 3.4: Modes $1 - 4$ for a perfectly conducting waveguide, $k_z = 1.0$ with waveguide dimensions in a 2:1 ratio.

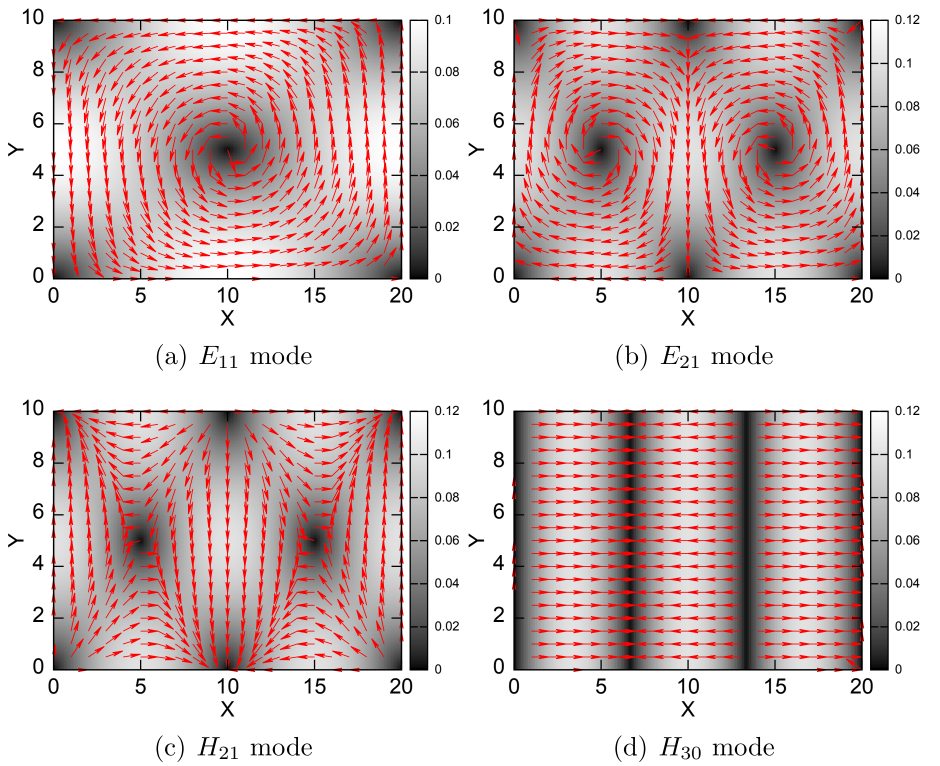
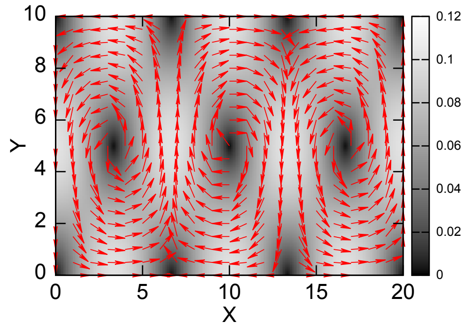
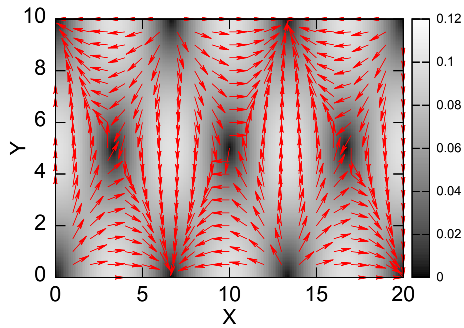


Figure 3.5: Modes $5-8$ for a perfectly conducting waveguide, $k_z = 1.0$ with waveguide dimensions in a 2:1 ratio.



(a) E_{31} mode



(b) H_{31} mode

Figure 3.6: Modes $9-10$ for a perfectly conducting waveguide, $k_z = 1.0$ with waveguide dimensions in a 2:1 ratio.

In addition, the eigenvalues corresponding to four propagating modes are plotted against the value of k_z in Fig.(3.7). Errors in the eigenvalue calculations for the homogeneous waveguide varied from $1E-15$ to $1E-8$, as shown in Table 1. This highlights the high accuracy obtained with our scheme especially for the first mode.

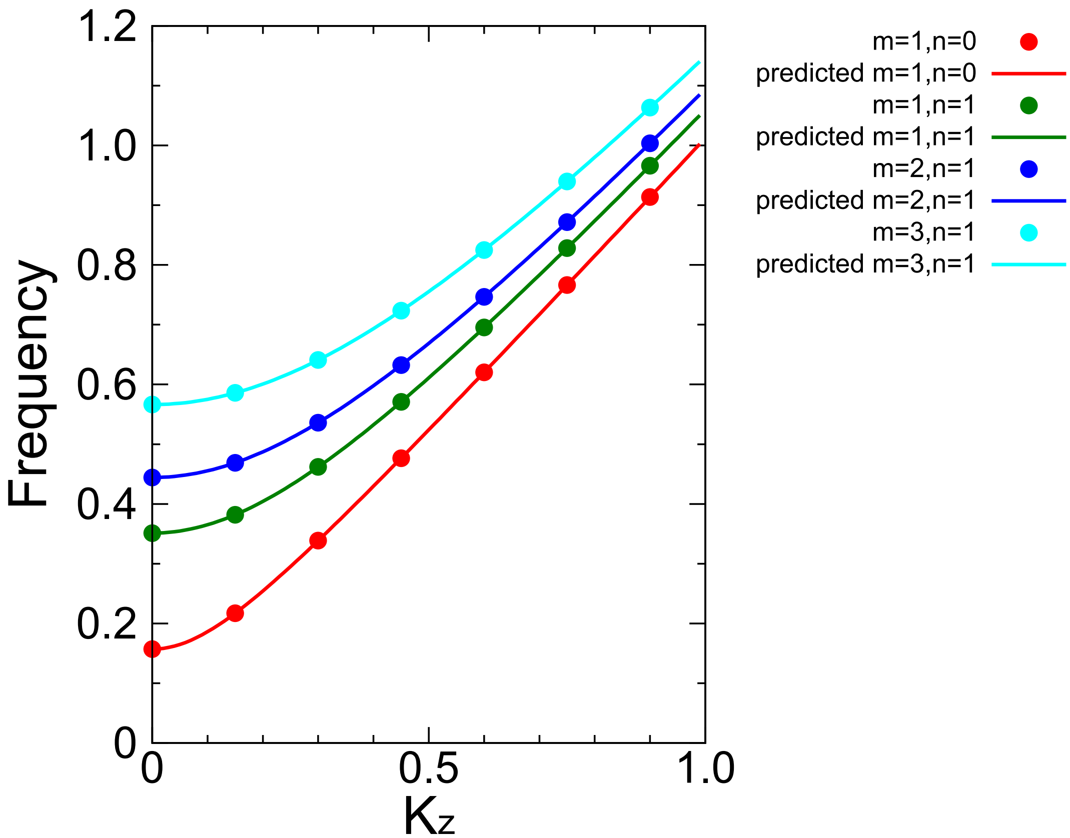


Figure 3.7: Eigenvalues of several propagating modes, plotted along with their predicted values. The predicted and actual eigenvalues are nearly indistinguishable on the plot.

Table 3.1: Numerically calculated eigenvalues versus their predicted values for the ten lowest-energy modes of the homogeneous rectangular waveguide.

Predicted	Actual	Error
1.024674011	1.024674011	1.00E-15
1.098696044	1.098696042	2.20E-09
1.098696044	1.098696046	2.30E-09
1.123370055	1.123370055	3.40E-10
1.123370055	1.123370056	5.90E-10
1.197392088	1.197392092	3.70E-09
1.197392088	1.197392093	4.90E-09
1.222066099	1.222066102	3.10E-09
1.320762143	1.320762178	3.50E-08
1.320762143	1.320762182	3.90E-08

Chapter 4

Inhomogeneous Waveguides

4.1 Implication of Imhomogeneity of Waveguides

Inhomogeneous, or composite, waveguides consist of multiple dielectric materials in the same cross-section. Generally, the addition of new dielectrics to a waveguide tends to complicate calculations, because the permeability and permittivity are discontinuous at the interface between dielectrics. However, it is possible to avoid the discontinuity problems by approximating the interface between dielectrics with a continuous function, roughly resembling a step function.

Suppose, for example, we are given a rectangular waveguide with relative permittivities ϵ_1 and ϵ_2 on the left and right of an interface, respectively, as shown in Fig. (4.1). Instead of forcing the dielectric to be discontinuous at

the interface, it is possible to simplify calculations by expressing the dielectric as a smooth function closely resembling a step function,

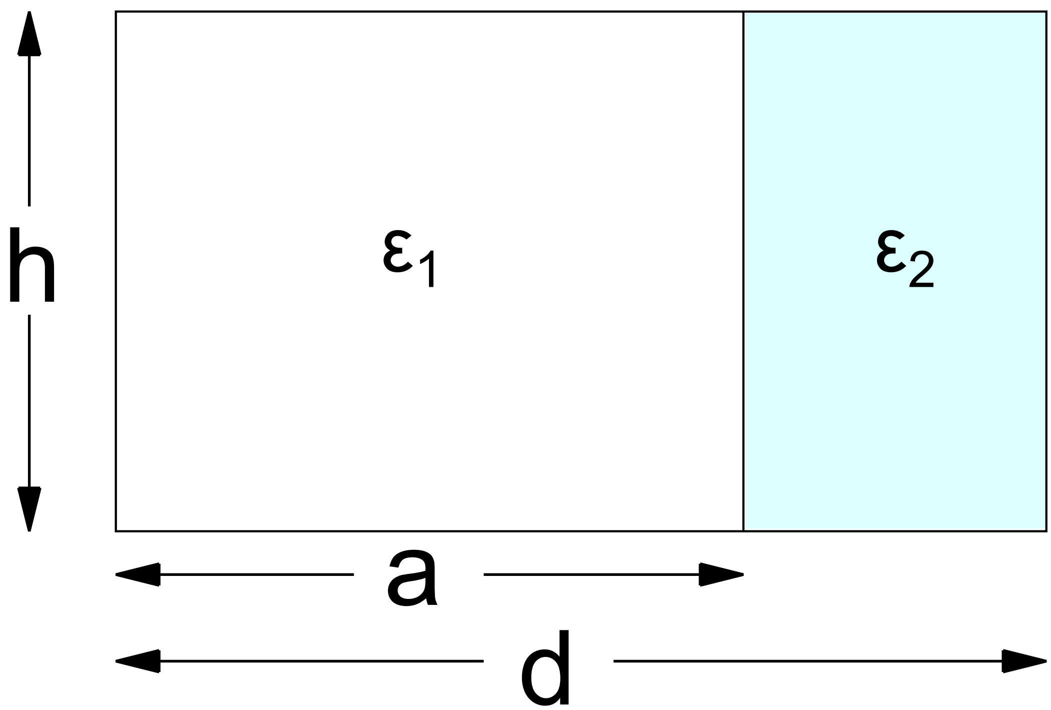


Figure 4.1: The dimensions of a partially filled waveguide are shown. These dimensions will be used when calculating the solutions to Maxwell's equations within the waveguide.

$$\epsilon_r(x) = \left[\frac{\tanh\left(s\left(x-a\right)\right)+1}{2}\right]\left(\epsilon_2-\epsilon_1\right)+\epsilon_1. \tag{4.1}$$

The parameter s determines how sharp the transition between dielectric properties is. In the limit as s becomes arbitrarily large, the permittivity function becomes a step function. For $\epsilon_1 = 1$ and $\epsilon_2 = 2$, the effects of various values of s are illustrated in Fig. (4.2). When utilizing the form of the permittivity or permeability given by Eq. (4.1), it is vital to consider the partial derivatives of the dielectric properties when discretizing the action integral of Eq. (3.66). If the reciprocal of the dielectric function is represented by a smooth function which changes only in a small region surrounding the interface, then the mesh should be refined close to the interface so that the finite elements can accurately represent the derivative of the smoothed function.

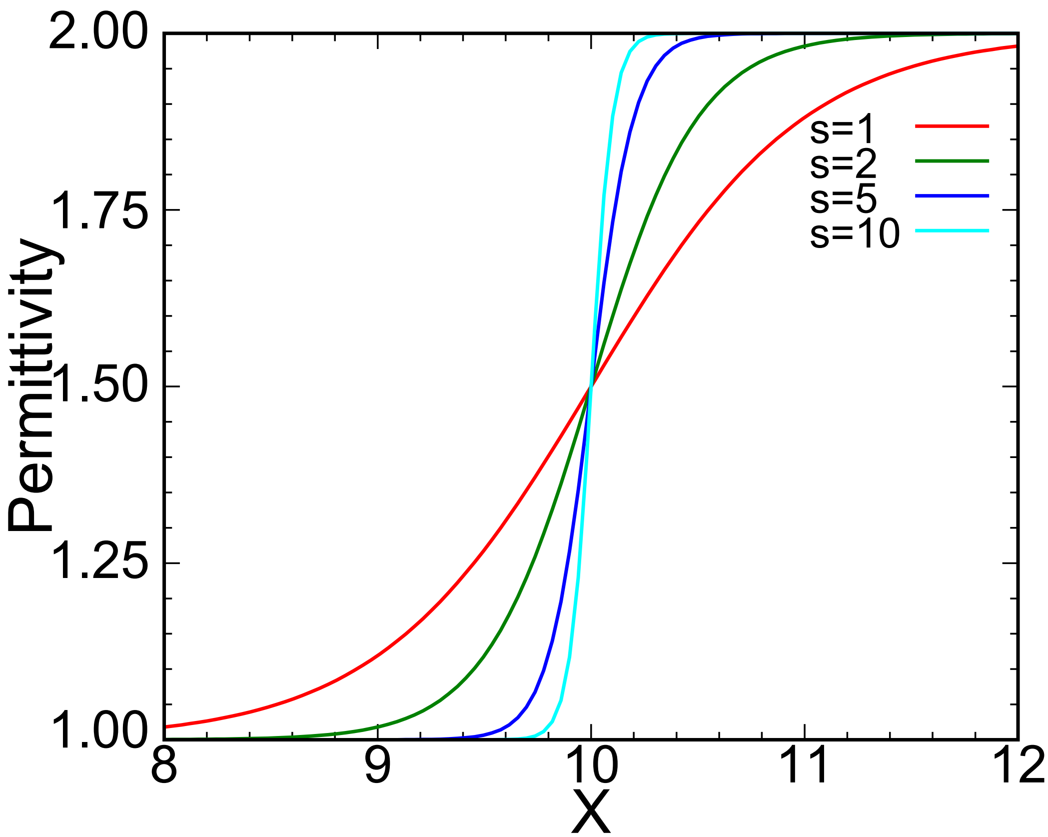


Figure 4.2: Typical plots of the permittivity as defined in Eq. (4.1), for various values of the sharpness parameter s. Here $\epsilon_1 = 1$, $\epsilon_2 = 2$, and $a = 14.5$.

4.2 Exact Solutions for the Inhomogeneous Waveguide

We calculate the exact eigenvalues and eigenfunctions of an inhomogeneous waveguide with cross-section dimensions $d \times h$, having relative permittivity $\epsilon_r = \epsilon_1$ in the region $0 \leq x < a$ and $\epsilon_r = \epsilon_2$ in the region $a < x \leq d$, as shown in Fig. (4.1). During the finite element calculations, the dielectric in the left region was assumed to be air, so that $\epsilon_1 = 1$ and $\epsilon_2 \geq \epsilon_1$.

Recall that Maxwell's equations may be expressed as

$$\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \mathbf{H} \right) - k_0^2 \mu_r \mathbf{H} = 0. \tag{3.19}$$

By assuming that the change from ϵ_1 to ϵ_2 occurs over an infinitesimally thin region centered at $x = a$, one may express the inverse of the dielectric function as a step function,

$$\frac{1}{\epsilon_r} = \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right) \theta\left(x - a\right) + \frac{1}{\epsilon_1}, \tag{4.2}$$

Assume that $\mu_r = 1$ everywhere. Note that we can rewrite the double curl as

$$\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \mathbf{H} \right) = \nabla \left(\frac{1}{\epsilon_r} \right) \times \left(\nabla \times \mathbf{H} \right) + \frac{1}{\epsilon_r} \nabla \times \left(\nabla \times \mathbf{H} \right). \tag{4.3}$$

The second term in Eq. (4.3) can be simplified by noting that

$$\nabla \times \left(\nabla \times \mathbf{H} \right) = \nabla \left(\nabla \cdot \mathbf{H} \right) - \nabla^2 \mathbf{H}, \tag{4.4}$$

and also by noting that the divergence of $\mathbf{H}$ is zero for uniform permeability. By including the type of permittivity expressed in Eq. (4.2), the gradient in Eq. (4.3) can be rewritten as

$$\nabla \left(\frac{1}{\epsilon_r} \right) = \begin{pmatrix} \delta\left(x - a\right) \Lambda \\ 0 \\ 0 \end{pmatrix}, \text{ where } \Lambda = \frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}. \tag{4.5}$$

so that Eq. (3.19) can be rewritten as

$$\begin{pmatrix} \delta\left(x - a\right) \Lambda \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} - \frac{1}{\epsilon_r} \nabla^2 \mathbf{H} - k_0^2 \mathbf{H} = 0. \tag{4.6}$$

The cross product in Eq. (4.6) can be expressed as

$$\begin{pmatrix} \delta\left(x-a\right)\Lambda \\ 0 \\ 0 \end{pmatrix} \times \left(\nabla \times \mathbf{H}\right) = \begin{pmatrix} 0 \\ -\delta\left(x-a\right)\Lambda\left(\partial_x H_y - \partial_y H_x\right) \\ \delta\left(x-a\right)\Lambda\left(\partial_z H_x - \partial_x H_z\right) \end{pmatrix}. \tag{4.7}$$

Finally, we have three equations for the three vector components:

$$\frac{1}{\epsilon_r}\nabla^2 H_x + k_0^2 H_x = 0, \tag{4.8a}$$

$$\delta\left(x-a\right)\Lambda\left(\partial_x H_y - \partial_y H_x\right) + \frac{1}{\epsilon_r}\nabla^2 H_y - k_0^2 H_y = 0, \tag{4.8b}$$

$$\delta\left(x-a\right)\Lambda\left(\partial_z H_x - \partial_x H_z\right) - \frac{1}{\epsilon_r}\nabla^2 H_z - k_0^2 H_z = 0. \tag{4.8c}$$

In the following, we begin considering these vector components in order to determine the eigenvalues and eigenfunctions for the inhomogeneous waveguide. The solutions to Eqs. (4.8a) and (4.8b) will be considered. Note that it is only necessary to solve the set of differential equations for H_x and H_y, not H_z. Having assumed that the magnetic field has a sinusoidal variation in the z-direction, the divergence condition may be expressed as

$$\partial_x H_x + \partial_y H_y + ik_z H_z = 0. \tag{4.9}$$

Once H_x and H_y are determined, the divergence condition may be used to calculate H_z.

4.2.1 Boundary Conditions

Before discussing the solution to the differential equation governing each component, it is prudent to discuss the boundary conditions governing a rectangular conducting waveguide with sides parallel to the x- and y-axes. We know in this case that the electric field can only have a normal component and the magnetic field can only have a tangential component at the boundary, so that

$$\hat{\mathbf{n}} \cdot \mathbf{H} = 0, \tag{4.10}$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0. \tag{4.11}$$

Expanding Eq. (4.10) we have

$$n_x H_x + n_y H_y = 0. \tag{4.12}$$

For the left and right boundaries, which are parallel to the y-axis, Eq. (4.12) simplifies to

$$H_x = 0 \text{ at } x = 0, d. \tag{4.13}$$

For the top and bottom boundaries, we get

$$H_y = 0 \text{ at } y = 0, h. \tag{4.14}$$

In addition, Eq. (4.11) may be used to generate derivative boundary conditions at the edges. Recall that one of Maxwell's equations reads

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}. \tag{4.15}$$

Assuming that dielectric properties do not vary over time and there is no current density at the boundary, we have

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{4.16}$$

Assuming that the electric and magnetic fields have time-harmonic forms, Eq. (4.16) simplifies to

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E}. \tag{4.17}$$

Solving Eq. (4.17) for $\mathbf{E}$ and substituting into Eq. (4.11) yields

$$\hat{n} \times \left(-\frac{1}{i\omega\epsilon} \nabla \times \mathbf{H} \right) = 0. \tag{4.18}$$

Expand the cross product to obtain

$$\begin{pmatrix} n_y \left(\partial_x H_y - \partial_y H_x\right) \\ -n_x \left(\partial_x H_y - \partial_y H_x\right) \\ n_x \left(\partial_z H_x - \partial_x H_z\right) - n_y \left(\partial_y H_z - \partial_z H_y\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.19}$$

So the first and second vector components of the cross product each yield

$$\partial_x H_y = \partial_y H_x. \tag{4.20}$$

On the left and right boundaries, we have already shown that $H_x = 0$. Since these edges are parallel to the y-axis and $H_x = 0$ along these edges, it follows that $\partial_y H_x = 0$. From Eq. (4.20) we then obtain a derivative condition on H_y:

$$\partial_x H_y = 0, \text{ at } x = 0, d. \tag{4.21}$$

Similarly, on the top and bottom edges, the derivative condition is

$$\partial_y H_x = 0, \text{ at } y = 0, h. \tag{4.22}$$

Together, Eqs. (4.13), (4.14), (4.21), and (4.22) constitute all the boundary conditions for a conducting waveguide of width d and height h.

4.2.2 The x-Component of the Magnetic Field

Before considering the y- or z- components of the field, we separate and solve Eq. (4.8a),

$$\frac{1}{\epsilon_r}\nabla^2 H_x + k_0^2 H_x = 0. \tag{4.8a}$$

We assume time-harmonic variation in the z-direction,

$$H_x(x, y, z) = H_x(x, y)\, e^{i(k_z z - \omega t)}, \tag{4.23}$$

and separate the solution further by writing

$$H_x(x, y, z) = X_x(x)\, Y_x(y)\, e^{i(k_z z - \omega t)}. \tag{4.24}$$

The subscript x is used as a reminder that Eq. (4.24) is the separated form of the x-component of the field. Factoring out the exponential term incorporating the dependence on z and t, we reduce Eq. (4.8a) to the separated form

$$-\frac{1}{\epsilon_r}\frac{\partial^2 X_x}{\partial x^2} Y_x - \frac{1}{\epsilon_r}\frac{\partial^2 Y_x}{\partial y^2} X_x + \frac{1}{\epsilon_r} k_z^2 X_x Y_x - k_0^2 X_x Y_x = 0. \tag{4.25}$$

Finally, divide Eq. (4.25) by $X_x Y_x$ and multiply by $-\epsilon_r$ to obtain

$$\frac{X_x''}{X_x} + \frac{Y_x''}{Y_x} + \left(k_0^2 \epsilon_r - k_z^2\right) = 0. \tag{4.26}$$

Next we proceed to investigate the different possible solutions to the separated equations. Note that every term in Eq. (4.26) can be considered a constant, because the first is only dependent on x, the second is only dependent on y, and the rest are all constants. We will begin by investigating the y-dependence of H_x.

y-Dependence: $Y_x''/Y_x > 0$

Suppose that the y-dependent term of Eq. (4.26) is positive, i.e.

$$\frac{Y_x''}{Y_x} = \kappa_y^2. \tag{4.27}$$

The general solution to this equation is

$$Y_x(y) = C_1 \cosh(\kappa_y y) + C_2 \sinh(\kappa_y y). \tag{4.28}$$

Recalling the boundary conditions from Eq. (4.22), we determine that the only allowable solution is $C_1 = 0$, $C_2 = 0$. Thus there are no non-trivial solutions of this type for Y_x.

y-Dependence: $Y_x''/Y_x = 0$

Suppose that the y-dependent term of Eq. (4.26) is exactly zero,

$$\frac{Y_x''}{Y_x} = 0. \tag{4.29}$$

Integration yields the general form of the solution,

$$Y_x(y) = C_1 y + C_2. \tag{4.30}$$

Recalling the boundary conditions from Eq. (4.22), we determine that the only allowable solution is $C_1 = 0$, but the derivative boundary conditions are satisfied for any value of C_2. Thus the non-trivial solution for $Y_x(y)$ is a constant.

y-Dependence: $Y_x''/Y_x < 0$

Suppose that the y-dependent term of Eq. (4.26) is negative, i.e.

$$\frac{Y_x''}{Y_x} = -k_y^2. \tag{4.31}$$

The general solution to this equation is

$$Y_x(y) = C_1 \cos(k_y y) + C_2 \sin(k_y y). \tag{4.32}$$

The boundary conditions from Eq. (4.22) require that $C_2 = 0$. However, a non-trivial solution does exist of the form

$$Y_x(y) = C_1 \cos(k_{n,y} y), \quad k_{n,y} = \frac{n\pi}{h}, \quad n = 0, 1, 2 \cdots, \tag{4.33}$$

where the solution $n = 0$ is the constant solution which was identified in the previous case, $Y_x''/Y = 0$. Now that all possible solutions for Y_x have been determined, we seek solutions for X_x. The equation of X_x can be set up by substituting Eq. (4.33) into Eq. (4.26):

$$\frac{X_x''}{X_x} = \left(\frac{n\pi}{h}\right)^2 + k_z^2 - k_0^2\epsilon_1, \quad 0 \le x \le a, \tag{4.34a}$$

$$\frac{X_x''}{X_x} = \left(\frac{n\pi}{h}\right)^2 + k_z^2 - k_0^2\epsilon_2, \quad a \le x \le d. \tag{4.34b}$$

In the following we solve these equations for all possible functional forms of X_x.

x-Dependence: Fully Exponential Solution

Assume that the right-hand sides of Eqs. (4.34a) and (4.34b) are both positive, yielding equations of the forms

$$\frac{X_x''}{X_x} = \kappa_{1,x}^2, \quad 0 \le x \le a. \tag{4.35a}$$

$$\frac{X_x''}{X_x} = \kappa_{2,x}^2, \quad a \le x \le d. \tag{4.35b}$$

In order to satisfy Dirichlet boundary conditions at $x = 0$ and $x = d$ the solutions must be hyperbolic sine functions,

$$X_x(x) = C_1 \sinh(\kappa_{1,x} x), \quad 0 \le x \le a. \tag{4.36a}$$

$$X_x(x) = C_2 \sinh(\kappa_{2,x}(x - d)), \quad a \le x \le d. \tag{4.36b}$$

The derivatives may be expressed as

$$X_x(x) = C_1 \kappa_{1,x} \cosh(\kappa_{1,x} x), \quad 0 \le x \le a. \tag{4.37a}$$

$$X_x(x) = C_2 \kappa_{2,x} \cosh(\kappa_{2,x}(x - d)), \quad a \le x \le d. \tag{4.37b}$$

We require H_x and its derivative to be continuous. By substituting $x = a$ into the expressions for H_x and its derivative, we obtain

$$C_1 \sinh(\kappa_{1,x} a) = C_2 \sinh(\kappa_{2,x}(a-d)). \tag{4.38a}$$

$$C_1 \kappa_{1,x} \cosh(\kappa_{1,x} a) = C_2 \kappa_{2,x} \cosh(\kappa_{2,x}(a-d)). \tag{4.38b}$$

Divide Eq. (4.38a) by Eq. (4.38b) to obtain a dispersion relation for $\kappa_{1,x}$ and $\kappa_{2,x}$.

$$\frac{1}{\kappa_{1,x}} \tanh(\kappa_{1,x} a) = \frac{1}{\kappa_{2,x}} \tanh(\kappa_{2,x}(a-d)). \tag{4.39}$$

Since $a < d$, the right side of this equation is less than or equal to zero for any value of $\kappa_{2,x}$. The left side is greater than or equal to zero for any value of $\kappa_{1,x}$. Therefore, the dispersion relation is only satisfied when $\kappa_{1,x} = \kappa_{2,x} = 0$. This is a trivial solution which equals zero everywhere.

x-Dependence: Fully Sinusoidal Solution

Assume that the right-hand sides of Eqs. (4.34a) and (4.34b) are both negative, yielding equations of the forms

$$\frac{X_x''}{X_x} = -k_{1,x}^2, \quad 0 \le x \le a, \tag{4.40a}$$

$$\frac{X_x''}{X_x} = -k_{2,x}^2, \quad a \le x \le d. \tag{4.40b}$$

In order to satisfy Dirichlet boundary conditions at $x = 0$ and $x = d$ the solutions must be sine functions,

$$X_x(x) = C_1 \sin(k_{1,x} x), \quad 0 \le x \le a, \tag{4.41a}$$

$$X_x(x) = C_2 \sin(k_{2,x}(x-d)), \quad a \le x \le d. \tag{4.41b}$$

The derivatives may be expressed as

$$X_x(x) = C_1 k_{1,x} \cos(k_{1,x} x), \quad 0 \le x \le a, \tag{4.42a}$$

$$X_x(x) = C_2 k_{2,x} \cos(k_{2,x}(x-d)), \quad a \le x \le d. \tag{4.42b}$$

We require H_x and its derivative to be continuous. By substituting $x = a$ into the expressions for H_x and its derivative, we obtain

$$C_1 \sin(k_{1,x} a) = C_2 \sin(k_{2,x}(a-d)), \tag{4.43a}$$

$$C_1 k_{1,x} \cos\left(k_{1,x} a\right) = C_2 k_{2,x} \cos\left(k_{2,x}\left(a-d\right)\right). \tag{4.43b}$$

Divide Eq. (4.43a) by Eq. (4.43b) to obtain a dispersion relation for $k_{1,x}$ and $k_{2,x}$.

$$\frac{1}{k_{1,x}} \tan\left(k_{1,x} a\right)) = \frac{1}{k_{2,x}} \tan\left(k_{2,x}\left(a-d\right)\right). \tag{4.44}$$

Since both $k_{1,x}$ and $k_{2,x}$ may be defined in terms of the parameters k_0, n, and k_z, the dispersion relation may be solved to determine the allowed values of k_0 for this type of mode. The dispersion relation for a completely sinusoidal field is well-documented. [19] [20]

x-Dependence: Sinudoidal and Exponential Solutions

Suppose that the right-hand side of Eq. (4.34a) is positive, and the right-hand side of Eq. (4.34b) is negative. In this case, the equations take the forms

$$\frac{X_x''}{X_x} = \kappa_{1,x}^2, \quad 0 \le x \le a, \tag{4.45a}$$

$$\frac{X_x''}{X_x} = -k_{2,x}^2, \quad a \le x \le d. \tag{4.45b}$$

In order to satisfy Dirichlet boundary conditions at $x = 0$ and $x = d$ the solutions must be hyperbolic sine and sine functions, respectively:

$$X_x\left(x\right) = C_1 \sinh\left(\kappa_{1,x} x\right), \quad 0 \le x \le a, \tag{4.46a}$$

$$X_x\left(x\right) = C_2 \sin\left(k_{2,x}\left(x-d\right)\right), \quad a \le x \le d. \tag{4.46b}$$

The derivatives may be expressed as

$$X_x\left(x\right) = C_1 \kappa_{1,x} \cosh\left(\kappa_{1,x} x\right), \quad 0 \le x \le a, \tag{4.47a}$$

$$X_x\left(x\right) = C_2 k_{2,x} \cos\left(k_{2,x}\left(x-d\right)\right), \quad a \le x \le d. \tag{4.47b}$$

We require H_x and its derivative to be continuous. By substituting $x = a$ into the expressions for H_x and its derivative, we obtain

$$C_1 \sinh\left(\kappa_{1,x} a\right) = C_2 \sin\left(k_{2,x}\left(a-d\right)\right), \tag{4.48a}$$

$$C_1 \kappa_{1,x} \cosh\left(\kappa_{1,x} a\right) = C_2 k_{2,x} \cos\left(k_{2,x}\left(a-d\right)\right). \tag{4.48b}$$

Divide Eq. (4.48a) by Eq. (4.48b) to obtain a dispersion relation for $\kappa_{1,x}$ and $k_{2,x}$.

$$\frac{1}{\kappa_{1,x}} \tanh\left(\kappa_{1,x} a\right)) = \frac{1}{k_{2,x}} \tanh\left(k_{2,x}\left(a-d\right)\right). \tag{4.49}$$

This dispersion relation may be used to determine the allowed values of k_0 for this type of mode.

Note that the opposite case, in which the field is sinusoidal in the left region and exponential in the right region, is not possible if $\epsilon_2 \geq \epsilon_1$. Based on Eqs. (4.34a) and (4.34b), the quantity X_x''/X_x must be greater in the left region than in the right region. Having a sinusoidal solution in the left region and an exponential solution in the right, however, requires that X_x''/X_x is negative in the left region and positive in the right region, leading to a contradiction.

The combination of a sine and a hyperbolic sine solution is analogous to an asymmetric quantum well. The sinusoidal solutions in the region with ϵ_1, which may be considered the shallower quantum well, are similar to below-barrier states which will tunnel through the shallower region while exhibiting exponential decay.

In conclusion, two types of non-trivial solution are possible for H_x. One exhibits a sine-like behavior throughout the waveguide cross-section. The other exhibits a sine-like behavior in the region of higher dielectric constant and a hyperbolic sine-like behavior in the region of lower dielectric constant. Both types of solutions are either constant or exhibit a cosine-like dependence in the y-direction.

4.2.3 The y-Component of the Magnetic Field

Now we consider the functional form of solutions to Eq. (4.8b),

$$\delta\left(x-a\right)\Lambda\left(\partial_x H_y - \partial_y H_x\right) + \frac{1}{\epsilon_r}\nabla^2 H_y - k_0^2 H_y = 0, \tag{4.8b}$$

These solutions will be similar to those for the H_x component, except for discontinuities arising from the Dirac δ function term. First we integrate Eq. (4.8b) across the dielectric interface:

$$\int_{a-\Delta}^{a+\Delta}\left[\delta\left(x-a\right)\Lambda\left(\partial_x H_y-\partial_y H_x\right)+\frac{1}{\epsilon_r}\nabla^2 H_y+k_0^2 H_y\right]dx=0. \tag{4.50}$$

The integration of every term will be treated separately, and we will take the limit as Δ approaches zero to determine the behavior of the magnetic field and its derivatives across the discontinuity in ϵ_r. First, consider the last term in the integral, $k_0^2 H_y$. Because the magnetic field must be continuous, integration over an infinitesimally small region leads to

$$\lim_{\Delta\to 0}\int_{a-\Delta}^{a+\Delta}\left[k_0^2 H_y\right]dx=0. \tag{4.51}$$

Next, consider the Laplacian operator term, recalling that

$$\frac{1}{\epsilon_r}=\Lambda\theta\left(x-a\right)+\frac{1}{\epsilon_1}. \tag{4.52}$$

Expanding the Laplacian operator, we have

$$\frac{1}{\epsilon_r}\nabla^2 H_y=\frac{1}{\epsilon_r}\left(\partial_x^2 H_y+\partial_y^2 H_y-k_z^2 H_y\right). \tag{4.53}$$

During integration, the k_z^2 term will vanish in the limit as Δ approaches zero due to H_y being continuous. As we will see later, the Dirichlet boundary conditions on H_y at the top and bottom edges require that the y-dependence must take the form of a sine function of frequency $m\pi/h$ for some positive integer m. Therefore we can express the second derivative with respect to y as

$$\partial_y^2 H_y=-\left(\frac{m\pi}{h}\right)^2 H_y, \tag{4.54}$$

and this term will also cancel as Δ approaches zero due to the continuity of H_y.

We perform integration by parts on the term carrying two derivatives with respect to x:

$$\begin{aligned}\lim_{\Delta\to 0}\int_{a-\Delta}^{a+\Delta}\left[\frac{1}{\epsilon_r}\partial_x^2 H_y\right]dx&=\lim_{\Delta\to 0}\left[\frac{1}{\epsilon_r}\partial_x H_y\right]_{a-\Delta}^{a+\Delta}\\&\quad-\lim_{\Delta\to 0}\int_{a-\Delta}^{a+\Delta}\left[\Lambda\delta\left(x-a\right)\partial_x H_y\right]dx.\end{aligned} \tag{4.55}$$

Note that the integral term on the right-hand side of Eq. (4.55) cancels the $\partial_x H_y$ term in Eq. (4.50), simplifying that integral to

$$\lim_{\Delta\to 0}\int_{a-\Delta}^{a+\Delta}\left[\Lambda\delta\left(x-a\right)\left(\partial_y H_x\right)\right]dx-\lim_{\Delta\to 0}\left[\frac{1}{\epsilon_r}\partial_x H_y\right]_{a-\Delta}^{a+\Delta}=0. \tag{4.56}$$

Finally, since all of the allowed solutions for H_x are continuous, the term containing the Dirac delta function in Eq. (4.56) can be integrated, leaving

$$\Lambda\partial_y H_x\left(x=a\right)+\frac{\partial_x H_{y,L}}{\epsilon_1}-\frac{\partial_x H_{y,R}}{\epsilon_2}=0, \tag{4.57}$$

$$\frac{1}{\epsilon_2}\left(\partial_x H_{y,R}-\partial_y H_x\left(x=a\right)\right)=\frac{1}{\epsilon_1}\left(\partial_x H_{y,L}-\partial_y H_x\left(x=a\right)\right) \tag{4.58}$$

where $\partial_x H_{y,L}$ and $\partial_x H_{y,R}$ represent the x-derivatives as x approaches a from the left and right, respectively. Note that this may be rewritten as

$$\lim_{x\to a^-}\frac{1}{\epsilon_1}\left(\nabla\times H_t\right)=\lim_{x\to a^+}\frac{1}{\epsilon_2}\left(\nabla\times H_t\right), \tag{4.59}$$

which imposes an internal boundary condition on the axial electric field.

Continuing from Eq. (4.58), substitute the solutions for H_x,

$$H_x=\begin{Bmatrix} C_{1n}\sin(k_{1n}a)\\ C_{1n}\sinh(\kappa_{1n}a)\end{Bmatrix}\cos(\frac{n\pi y}{h}), x=a^-, \tag{4.60a}$$

$$H_x=C_{2n}\sin(k_{2n}(a-d))\cos(\frac{n\pi y}{h}), x=a^+, \tag{4.60b}$$

Where the curly braces may be interpreted as meaning that the solution can take one form or the other for a given value of n. By enforcing continuity of H_x at $x=a$, the two expressions can be used to express $C_{1,n}$ as a function of $C_{2,n}$ or vice-versa,

$$C_{1,n}=\frac{C_{2,n}\sin(k_2(d-a))}{\begin{Bmatrix}\sin(k_{1n}a)\\ \sinh(\kappa_{1n}a)\end{Bmatrix}}. \tag{4.61}$$

Differentiating the expressions for H_x and substituting into Eq. (4.58), we get

$$\frac{1}{\epsilon_2}\left(\partial_x X_{y,R}(x)\sin(\frac{m\pi y}{h}) - \frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)\sin(\frac{n\pi y}{h})\right)$$
$$= \frac{1}{\epsilon_1}\left(\partial_x X_{y,L}(x)\sin(\frac{m\pi y}{h}) - \frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)\sin(\frac{n\pi y}{h})\right), \quad (4.62)$$

noting that satisfaction of the boundary conditions at $y = 0$ and $y = h$ requires that H_y has a sine function variation in the y-direction. To ensure continuity across the boundary, we require either $n = 0$, $m = 0$, or $n = m$. The case investigated here, $n = m$ with both n and m nonzero, can be used to couple the solutions for H_x and H_y for a certain frequency and permittivity.

In addition, if both field components are nonzero, they must propagate at the same frequency. It follows that k_2 has the same value for both components. Also, either k_1 or κ_1 will be the same for both fields, depending on whether the fields assume trigonometric or hyperbolic forms, respectively. As will be shown in the following sections, the y-component of the magnetic field will be of the functional form

$$H_y = \begin{Bmatrix} \mathcal{D}_{1n}\sin(k_{1n}a) \\ \mathcal{D}_{1n}\sinh(\kappa_{1n}a) \end{Bmatrix}\cos(\frac{n\pi y}{h}), x = a^-, \quad (4.63a)$$

$$H_y = \mathcal{D}_{2n}\sin(k_{2n}(a-d))\sin(\frac{n\pi y}{h}), x = a^+. \quad (4.63b)$$

Again, we use the continuity of the field at $x = a$ to express one coefficient in terms of the other,

$$\mathcal{D}_{1,n} = \frac{\mathcal{D}_{2,n}\sin(k_2(d-a))}{\begin{Bmatrix} \cos(k_{1n}a) \\ \cosh(\kappa_{1n}a) \end{Bmatrix}}. \quad (4.64)$$

Substitute this result into Eq. (4.62) and cancel the common factor of $\sin(n\pi y/h)$ to obtain

$$\frac{1}{\epsilon_2}\left(\mathcal{D}_{2n}k_2n\sin\left(k_2n(d-a)\right) - \frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)\right)$$
$$= \frac{1}{\epsilon_1}\left(\mathcal{D}_{2n}k_2n\begin{Bmatrix} -k_1\sin\left(k_1a\right) \\ -\kappa_1\sinh\left(\kappa_1a\right) \end{Bmatrix} - \frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)\right). \quad (4.65)$$

This equation can be rearranged to provide an expression for $\mathcal{D}_{2n}$ in terms of C_{2n}:

$$\frac{1}{\epsilon_2}\left(\mathcal{D}_{2n}\sin\left(k_{2n}(d-a)\right)k_{2n}-\frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)\right)=$$
$$\frac{1}{\epsilon_1}\left(\mathcal{D}_{2n}\cos\left(k_2(d-a)\right)\begin{Bmatrix}-k_{1n}\tan\left(k_{1n}a\right)\\ \kappa_{1n}\tanh\left(\kappa_{1n}a\right)\end{Bmatrix}-\frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)\right) \tag{4.66}$$

$$\mathcal{D}_{2n}=\frac{\left(\frac{1}{\epsilon_2}-\frac{1}{\epsilon_1}\right)\frac{n\pi}{h}C_{2n}\sin\left(k_{2n}(d-a)\right)}{\frac{1}{\epsilon_2}k_{2n}\sin\left(k_{2n}(d-a)\right)-\frac{1}{\epsilon_1}\cos\left(k_{2n}(d-a)\right)\begin{Bmatrix}-k_{1n}\tan\left(k_{1n}a\right)\\ \kappa_{1n}\tanh\left(\kappa_{1n}a\right)\end{Bmatrix}}. \tag{4.67}$$

Since the values of k_{1n}, κ_{1n}, and k_{2n} are all found by solving the differential equation for H_x, Eq. (4.67) provides a means of coupling H_x and H_y and relating the values of the amplitudes of these field components.

Next we consider the case $H_x = 0$, for which we can revise Eq. (4.57) to define the interface boundary condition as

$$\frac{\partial_x H_{y,L}}{\epsilon_1}=\frac{\partial_x H_{y,R}}{\epsilon_2}. \tag{4.68}$$

Assuming that H_y is separable, we write it as $H_y = X_y(x)\,Y_y(y)\,e^{i(k_z z-\omega t)}$, yielding the separated form of the differential equation,

$$\frac{X_y''}{X_y}+\frac{Y_y''}{Y_y}+\left(k_0^2\epsilon_r-k_z^2\right)=0. \tag{4.69}$$

We now consider the special case $H_x = 0$, for which the field H_y is continuous across the interface at $x = a$ and the derivative $\partial_x H_y$ obeys Eq. (4.68). At the edges of the waveguide cross section, the field is governed by the boundary conditions given in Eqs. (4.14) and (4.21). We seek to determine all valid expressions for H_y for this case.

y-Dependence: $Y_y''/Y_y > 0$

Suppose that the y-dependent term of Eq. (4.69) is positive, i.e.

$$\frac{Y_y''}{Y_y}=\kappa_y^2. \tag{4.70}$$

The general solution to this equation is

$$Y_y(y) = C_1 \cosh(\kappa_y y) + C_2 \sinh(\kappa_y y). \tag{4.71}$$

Recalling the boundary conditions from Eq. (4.22), we determine that the only allowable solution is $C_1 = 0$, $C_2 = 0$. Thus there are no non-trivial solutions of this type for Y_y.

y-Dependence: $Y_y''/Y_y = 0$

Suppose that the y-dependent term of Eq. (4.69) is exactly zero,

$$\frac{Y_y''}{Y_y} = 0. \tag{4.72}$$

Integration yields the general form of the solution,

$$Y_y(y) = C_1 y + C_2. \tag{4.73}$$

Recalling the boundary conditions from Eq. (4.22), we determine that the only allowable solution is $C_1 = 0$ and $C_2 = 0$. Unlike Y_x, Y_y is not satisfied by a constant solution because the boundary conditions at $y = 0$ and $y = h$ act on the function instead of its derivative.

y-Dependence: $Y_y''/Y_y < 0$

Suppose that the y-dependent term of Eq. (4.69) is negative, i.e.

$$\frac{Y_y''}{Y_y} = -k_y^2. \tag{4.74}$$

The general solution to this equation is

$$Y_y(y) = C_1 \cos(k_y y) + C_2 \sin(k_y y). \tag{4.75}$$

The boundary conditions from Eq. (4.14) require that $C_1 = 0$. However, a non-trivial solution does exist of the form

$$Y_y(y) = C_2 \sin(k_{m,y} y), \quad k_{m,y} = \frac{m\pi}{h}, \quad m = 1, 2, 3 \cdots, \tag{4.76}$$

where the solution $m = 0$ is omitted for H_y because it leads to H_y equalling zero throughout the entire waveguide cross-section. Now that all possible solutions for Y_y have been determined, we seek solutions for X_y. The equation of X_y can be set up by substituting Eq. (4.76) into Eq. (4.69):

$$\frac{X_y''}{X_y} = \left(\frac{m\pi}{h}\right)^2 + k_z^2 - k_0^2\epsilon_1, \quad 0 \le x \le a, \tag{4.77a}$$

$$\frac{X_y''}{X_y} = \left(\frac{m\pi}{h}\right)^2 + k_z^2 - k_0^2\epsilon_2, \quad a \le x \le d. \tag{4.77b}$$

In the following we solve these equations for all possible functional forms of X_y.

x-Dependence: Fully Exponential Solution

Assume that the right-hand sides of Eqs. (4.77a) and (4.77b) are both positive, yielding equations of the forms

$$\frac{X_y''}{X_y} = \kappa_{1,x}^2, \quad 0 \le x \le a, \tag{4.78a}$$

$$\frac{X_y''}{X_y} = \kappa_{2,x}^2, \quad a \le x \le d. \tag{4.78b}$$

In order to satisfy derivative boundary conditions at $x = 0$ and $x = d$ the solutions must be hyperbolic cosine functions,

$$X_y(x) = C_1 \cosh(\kappa_{1,x} x), \quad 0 \le x \le a, \tag{4.79a}$$

$$X_y(x) = C_2 \cosh(\kappa_{2,x}(x - d)), \quad a \le x \le d. \tag{4.79b}$$

The derivatives may be expressed as

$$X_y(x) = C_1\kappa_{1,x} \sinh(\kappa_{1,x} x), \quad 0 \le x \le a, \tag{4.80a}$$

$$X_y(x) = C_2\kappa_{2,x} \sinh(\kappa_{2,x}(x - d)), \quad a \le x \le d. \tag{4.80b}$$

We require H_y to be continuous, and require its derivative to obey Eq. (4.68). By substituting $x = a$ into the expressions for H_y and its derivative, we obtain

$$C_1 \cosh(\kappa_{1,x} a) = C_2 \cosh(\kappa_{2,x}(a - d)), \tag{4.81a}$$

$$\frac{C_1\kappa_{1,x}}{\epsilon_1} \sinh(\kappa_{1,x} a) = \frac{C_2\kappa_{2,x}}{\epsilon_2} \sinh(\kappa_{2,x}(a - d)). \tag{4.81b}$$

Divide Eq. (4.81b) by Eq. (4.81a) to obtain a dispersion relation for $\kappa_{1,x}$ and $\kappa_{2,x}$.

$$\frac{\kappa_{1,x}}{\epsilon_1} \tanh\left(\kappa_{1,x} a\right) = \frac{\kappa_{2,x}}{\epsilon_2} \tanh\left(\kappa_{2,x}\left(a-d\right)\right). \tag{4.82}$$

Since $a < d$, the right side of this equation is less than or equal to zero for any value of $\kappa_{2,x}$. The left side is greater than or equal to zero for any value of $\kappa_{1,x}$. Therefore, the dispersion relation is only satisfied when $\kappa_{1,x} = \kappa_{2,x} = 0$. This is a trivial solution which equals zero everywhere.

x-Dependence: Fully Sinusoidal Solution

Assume that the right-hand sides of Eqs. (4.77a) and (4.77b) are both negative, yielding equations of the form

$$\frac{X_y''}{X_y} = -k_{1,x}^2, \quad 0 \le x \le a, \tag{4.83a}$$

$$\frac{X_y''}{X_y} = -k_{2,x}^2, \quad a \le x \le d. \tag{4.83b}$$

In order to satisfy derivative boundary conditions at $x = 0$ and $x = d$ the solutions must be cosine functions,

$$X_y\left(x\right) = C_1 \cos\left(k_{1,x} x\right), \quad 0 \le x \le a, \tag{4.84a}$$

$$X_y\left(x\right) = C_2 \cos\left(k_{2,x}\left(x-d\right)\right), \quad a \le x \le d. \tag{4.84b}$$

The derivatives may be expressed as

$$X_y\left(x\right) = -C_1 k_{1,x} \sin\left(k_{1,x} x\right), \quad 0 \le x \le a, \tag{4.85a}$$

$$X_y\left(x\right) = -C_2 k_{2,x} \sin\left(k_{2,x}\left(x-d\right)\right), \quad a \le x \le d. \tag{4.85b}$$

We require H_y and to be continuous, and require its derivative to obey Eq. (4.68). By substituting $x = a$ into the expressions for H_y and its derivative, we obtain

$$C_1 \cos\left(k_{1,x} a\right) = C_2 \cos\left(k_{2,x}\left(a-d\right)\right), \tag{4.86a}$$

$$\frac{C_1 k_{1,x}}{\epsilon_1} \sin\left(k_{1,x} a\right) = \frac{C_2 k_{2,x}}{\epsilon_2} \sin\left(k_{2,x}\left(a-d\right)\right). \tag{4.86b}$$

Divide Eq. (4.86b) by Eq. (4.86a) to obtain a dispersion relation for $k_{1,x}$ and $k_{2,x}$.

$$\frac{k_{1,x}}{\epsilon_1}\tan\left(k_{1,x}a\right)) = \frac{k_{2,x}}{\epsilon_2}\tanh\left(k_{2,x}\left(a-d\right)\right). \tag{4.87}$$

Since both $k_{1,x}$ and $k_{2,x}$ may be defined in terms of the parameters k_0, m, and k_z, the dispersion relation may be solved to determine the allowed values of k_0 for this type of mode.

x-Dependence: Sinudoidal and Exponential Solutions

Suppose that the right-hand side of Eq. (4.77a) is positive, and the right-hand side of Eq. (4.77b) is negative. In this case, the equations take the forms

$$\frac{X_y''}{X_y} = \kappa_{1,x}^2, \quad 0 \le x \le a, \tag{4.88a}$$

$$\frac{X_y''}{X_y} = k_{2,x}^2, \quad a \le x \le d. \tag{4.88b}$$

In order to satisfy derivative boundary conditions at $x = 0$ and $x = d$ the solutions must be cosine and hyperbolic cosine functions,

$$X_y\left(x\right) = C_1\cosh\left(\kappa_{1,x}x\right), \quad 0 \le x \le a, \tag{4.89a}$$

$$X_y\left(x\right) = C_2\cos\left(k_{2,x}\left(x-d\right)\right), \quad a \le x \le d. \tag{4.89b}$$

The derivatives may be expressed as

$$X_y\left(x\right) = C_1\kappa_{1,x}\sinh\left(\kappa_{1,x}x\right), \quad 0 \le x \le a, \tag{4.90a}$$

$$X_y\left(x\right) = -C_2k_{2,x}\sin\left(k_{2,x}\left(x-d\right)\right), \quad a \le x \le d. \tag{4.90b}$$

We require H_y and to be continuous, and its derivative to obey Eq. (4.68). By substituting $x = a$ into the expressions for H_y and its derivative, we obtain

$$C_1\cosh\left(\kappa_{1,x}a\right) = C_2\cos\left(k_{2,x}\left(a-d\right)\right). \tag{4.91a}$$

$$C_1\kappa_{1,x}\sinh\left(\kappa_{1,x}a\right) = -C_2k_{2,x}\sin\left(k_{2,x}\left(a-d\right)\right). \tag{4.91b}$$

Divide Eq. (4.91b) by Eq. (4.48a) to obtain a dispersion relation for $\kappa_{1,x}$ and $k_{2,x}$.

$$\frac{\kappa_{1,x}}{\epsilon_1}\tanh\left(\kappa_{1,x}a\right)) = -\frac{\kappa_{2,x}}{\epsilon_2}\tanh\left(k_{2,x}\left(a-d\right)\right). \tag{4.92}$$

This dispersion relation may be used to determine the allowed values of k_0 for this type of mode.

As explained in the section concerning H_x, the opposite case, in which the field is sinusoidal in the left region and exponential in the right region, is not possible if $\epsilon_2 \geq \epsilon_1$. Based on Eqs. (4.77a) and (4.77b), the quantity X_y''/X_y must be greater in the left region than in the right region. Having a sinusoidal solution in the left region and an exponential solution in the right, however, requires that X_y''/X_y is negative in the left region and positive in the right region, leading to a contradiction.

Therefore all solutions for H_y will fall into one of two groups. The first type of solution has a cosine-like behavior in both regions, with a sine-like dependence on y. The second type has a cosine-like behavior in the region of higher dielectric constant and a hyperbolic cosine-like behavior in the region of lower dielectric constant, again with a sine-like dependence on y. Unlike the case of H_x, a non-trivial solution which is independent of y cannot be achieved. This is because a nonzero constant does not satisfy the y-dependent part of the separated equation.

4.3 Results for Inhomogeneous Waveguide

In Fig.(4.3), we display our numerical results obtained using FEM.

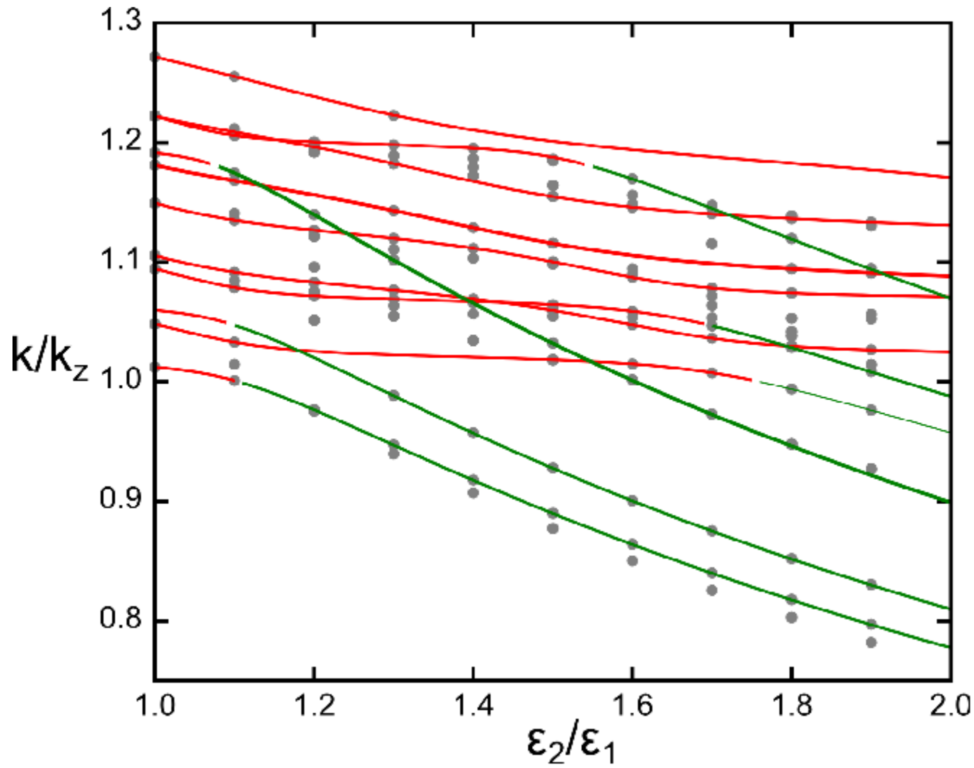


Figure 4.3: Numerical result for the dispersion relation using FEM with Hermite interpolation polynomials contrasted with the analytical dispersion relation

We observe that we obtain a few extra modes in our numerical calculations compared to what we would expect from our analytical consideration of the inhomogeneous waveguide in the earlier sections. These solutions do not appear to be spurious since they obey the boundary conditions imposed on them as well as obeying the divergence condition of the fields. We are currently looking at any other possible combination of solution in our theoretical consideration of the expected spectrum. On the numerical side, we are looking at further insight into the extra modes by considering TM and TE modes separately.

Chapter 5

Conclusion

Early attempts at using the discretization method represented by the Finite Element Method in solving the Maxwell's equations in waveguides have led to the occurrence of spurious solutions as with the Lagrange interpolation polynomials do not implicitly satisfy the proper inter-element boundary conditions. As from 1985, with the advent of vector finite element method and edge elements, the spurious solutions were eliminated from the EM fields calculations. The edge elements are able to provide tangential continuity along the edges which in effect eliminates spurious solutions. However the edge elements do not impose any continuity condition on the normal component of the field to the edge. This introduces inaccuracies in the calculation of the fields.

We developed an independent alternative through scalar, fifth-order Hermite interpolation polynomials that allow us to represents vector fields in an FEM scheme. The Hermite interpolation polynomials ensures 2^{nd}-derivative continuity within elements and 1^{st} derivative continuity from one element to another. We investigated our approach over several standard test cases. In each case, *we observe superior level of accuracy.* We are able to obtain eigenvalues up to 3 orders of magnitude higher than calculations done with vector finite elements of similar polynomial order. We are also able generate electromagnetic field distribution in exquisite details in both homogeneous waveguides and inhomogeneous waveguides.

We have shown throughout this project that via scalar quintic Hermite interpolation polynomials, we obtain higher flexibility in regards to specifying boundary conditions allowing us to eliminate spurious solutions in waveguide problems and at the same time yielding solutions with higher accuracy.

Moreover our calculations using Hermite FEM provide great prospects in calculations other than waveguides. For instance, using a similar approach, it is possible to reproduce through Hermite FEM, the band spectrum of photonic crystals. In addition our scalar FEM approach allows for the possibility of multiphysics simulation combining the Maxwell's equation with Shrödinger equation.

Chapter 6

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