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Discussions of No Arbitrage in Financial Markets

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Abstract

We consider a financial market with one continuous time risky price process and one continuous time risk-free price process. We assume all the trading takes place at finitely many time points in this market. We provide necessary and sufficient conditions on the discounted price process so that the market does not admit arbitrage possibilities.

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1 Introduction and Background

One of the most fundamental and important concepts in the theory of mathematical finance is the notion of arbitrage. Stated informally, an arbitrage opportunity is the possibility for an investor to gain money without any initial capital investment and without the risk of losing money. Naturally, it is reasonable to expect that arbitrage should not be allowed in financial models. Indeed, a great deal of the theory of mathematical finance rests on the assumption that arbitrage opportunities should not exist. In practice, this is a reasonable assumption as well, since if such an opportunity were to present itself, it would be exploited very quickly, and the market would shift such that the opportunity for arbitrage would disappear. A descriptive analogy is that arbitrage opportunities are like empty parking spaces in a crowded lot: It is reasonable to assume that no empty spaces exist at any given time, since as soon as a space becomes free, it will immediately be taken by someone.

In this paper we will present the background necessary for an understanding of a simple model of a financial market. We will discuss a general discrete-time multiperiod market model with a risk-free asset and a single risky asset. In addition to giving definitions of certain fundamental ideas in mathematical finance (such as martingales and self-financing trading strategies), we will also require the use of a change of numéraire result and the Dalang-Morton-Willinger Theorem, and we will give statements of these results as they relate to our model. Using these tools, we will be able to present our main result, which states that the no-arbitrage property of the market is preserved when the discounted price process of the risky security is composed with a strictly monotone function.

1.1 The Model Setup

The model that we will be using is a basic multiperiod model with a single risky security. We will assume that all of the uncertainty in the market is captured by a finite probability space $(\Omega, \mathcal{F}, P)$. The elements of the model are specified as follows:

- $n+1$ trading dates: $t = t_0, t_1, \ldots, t_n$ chosen from the set $[0, \infty]$. These represent the times at which transactions are possible.
- Finite sample space Ω with K elements; $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$. The elements represent "states of the world," i.e. possible outcomes for the market.
- A probability measure P on Ω such that $P(\omega) > 0$ for all $\omega \in \Omega$. This condition allows us to avoid being overly technical, as is often necessary in discussions in probability; when we refer to a condition being true "almost surely," we mean that the condition holds for every $\omega \in \Omega$. Similarly, when we refer to an event having positive probability, we simply mean that there exists $\omega \in \Omega$ such that the condition holds.
- A filtration $\{\mathcal{F}_{t_0}, \mathcal{F}_{t_1}, \ldots, \mathcal{F}_{t_n}\}$ representing the information available to investors at each trading date; $\mathcal{F}_t$ represents information available

at time t.

- A single risk-free asset with price process B_t, having risk-free interest rate r. This process can be interpreted as representing the value of a bank account; specifically, B_t represents the amount of money in the account at time t when \$1 was initially deposited at time 0. Naturally, the process should have $B_0 = 1$ and should be strictly positive for every ω. Typically B is an increasing process. The interest rate r can itself be a random variable, but for the purposes of this discussion we will only take r to be a positive constant. We will assume that interest is compounded continuously, so that the value of the bank account at time t is $B_t = e^{rt}$.

- A single risky security with price process X_t, where X_t is nonnegative. This process should be thought of as representing the price of a risky asset (e.g. a single share of common stock of a company) at time t.

In other words, we are considering a model in which we allow investors to trade at certain fixed dates (for example, once per day for some stretch of time). Investors have information available to them: Namely, that at the last date the market will be in one of K possible states (represented by the finite sample space), and each of these states has some nonzero probability of occurring. As dates pass, investors gain more information about the future state of the market by observing the information that the market presents at each trading date; that is, at each trading date, investors know more about the possible market outcomes than they did at the previous date. The filtration involved in the model setup describes how this information can become available; the structure of information will be elaborated in the next section.

In our model we consider a risk-free asset (B), representing a money market account (or bank account), which we assume to have a constant rate of interest. The importance of the risk-free asset is that it may be invested in without risk (as the name suggests); investors are guaranteed a certain return when investing into the money market account. We also consider a single risky asset (X), whose value at each trading date is random. There is risk involved for investors since there is a chance they can either gain or lose money from an investment in the risky asset. Again, the value of X can be interpreted as the price per share of a company's common stock.

These are the elements of the model. In the next few sections, we will present a bit more additional background information on general tools and how they will apply to our model.

1.2 Information Structures

One of the aspects of conventional financial market models is the filtration which represents how information about asset prices is revealed to investors. The revealing of information is modeled in terms of subsets of the sample space Ω. At time $t = t_0$ we assume that every state ω in the sample space is a possibility, and when we reach time $t = t_n$, we assume that the "true state" of the world ω becomes apparent to every investor. In the intermediate time, the information that gradually becomes available to investors allows them to anticipate the eventual true state of the world, and rule out states that become impossible.

Pliska [8] gives a description of the structure of information involving partitions of the sample space Ω, in which there exists a sequence $\{A_t\}$ of subsets of Ω, such that $A_{t_0} = 0$, $A_{t_n} = \{\omega\}$ for some $\omega \in \Omega$, and $A_{t_n} \subset A_{t_{n-1}} \subset \ldots \subset A_{t_1} \subset A_{t_0}$. In this way, investors know at time t_i that the true state is some element of A_{t_i}. At each date t_i, the partition is then defined as the collection of all possible time $t = t_i$ subsets.

A more common way of specifying the structure of information is by the use of σ-algebras, which correspond directly with the partition model.

Definition 1.1. *Let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a σ-algebra on Ω if it has the following properties:*

- $\Omega \in \mathcal{F}$
- *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$*
- *If $F, G \in \mathcal{F}$, then $F \cup G \in \mathcal{F}$*

Every σ-algebra on Ω corresponds to a unique partition of Ω. This means that the information structure can be organized as a sequence of σ-algebras, $\{\mathcal{F}; t = t_0, t_1, \ldots, t_n\}$, and we call the sequence a *filtration*. Note that $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_{t_n}$ contains all subsets of Ω.

At time t_i, the information available to investors is contained in an element of the partition of Ω corresponding to $\mathcal{F}_{t_i}$; the information is described by an event A that we refer to as an *atom* of $\mathcal{F}_{t_i}$. The investors therefore know that the information available at the date t_{i+1} will be a subset of A (and thereby be contained in the partition of Ω corresponding to $\mathcal{F}_{t_{i+1}}$). In other words, if investors have some atom A of information available today, then their information tomorrow will be a subset of A.

1.3 Stochastic Processes

Definition 1.2. *A* stochastic process *is a function $S(t, \omega)$ of both t and ω (we will typically omit the ω and only write $S(t)$).*

For fixed ω, $S(t, \omega)$ is a deterministic function called the *sample path.* For fixed t, $S(t, \omega)$ is a random variable.

Definition 1.3. *A random variable X is called* measurable *with respect to a σ-algebra $\mathcal{F}$ if the function $\omega \to X(\omega)$ is constant on any subset in the partition corresponding to $\mathcal{F}$. An equivalent definition is the following: For any real number x, the set $\{\omega \in \Omega : X(\omega) = x\}$ is an element of $\mathcal{F}$.*

The concept of measurability is intuitive, but not always clear directly from the definition. Loosely speaking, we say a random variable is measurable with respect to a σ-algebra if the information contained in that σ-algebra is sufficient to determine the value of the random variable.

Definition 1.4. *A stochastic process is called* adapted *to the filtration* $\{\mathcal{F}; t = t_0, t_1, \ldots, t_n\}$ *if for every* $t \in \{t_0, t_1, \ldots, t_n\}$, *the random variable* $S\,(t)$ *is* $\mathcal{F}_t$*-measurable.*

This follows naturally from the previous definitions; we want to consider stochastic processes whose value at each time point can be determined by the information that the filtration reveals at that point. The next definition gives another extension of this idea.

Definition 1.5. *A stochastic process is said to be* predictable *with respect to the filtration* $\{\mathcal{F}; t = t_0, t_1, \ldots, t_n\}$ *if for every* $t \in \{t_0, t_1, \ldots, t_n\}$, *the random variable* $S\,(t)$ *is* $\mathcal{F}_{t-1}$*-measurable.*

Predictability of a stochastic process is more powerful than adaptedness: It means that the value of the stochastic process at each point in time can be determined *one period in advance.*

These definitions are standard and common among researchers in mathematical finance. Nevertheless, we feel that it is a good idea to review them for the purposes of this report, to make sure that the reader is sufficiently well-informed for the sections that follow, particularly in regards to how these definitions should relate to our model.

1.4 The Class of Trading Strategies

We will be interested in trading strategies $H = \left\{H_{t_0}, H_{t_1}, \ldots, H_{t_{n-1}}\right\}$. Each H_{t_i} is a stochastic process $H_{t_i} = \left(H_{t_i}^B, H_{t_i}^X\right)$, where $H_{t_i}^B$ represents the amount of shares of the risk-free asset that are bought (or sold) at time t_i and held until time t_{i+1}. Likewise, $H_{t_i}^X$ represents the amount of shares of the risky security. A positive value corresponds to a long position in the asset, and a negative value represents a short position. In this way, at each trading date t_i, H_{t_i} specifies an investor's portfolio.

The value process of a trading strategy H is defined as

$$V_{t_i} = H_{t_i}^B B_{t_i} + H_{t_i}^X X_{t_i}$$

This represents, as the name suggests, the value of the portfolio that the investor is holding at time t_i; that is, how much money the investor would earn (or lose) by closing his positions in each asset.

Specifically, the initial investment is $V_{t_0} = H_{t_0}^B B_{t_0} + H_{t_0}^X X_{t_0}$.

We also define the *gains process* of a trading strategy at time t_i as

$$G_{t_i}^H = V_{t_i}^H - V_{t_0}^H$$

The gains process represents the value that the portfolio has gained or lost since the initial investment. This process plays an important role in defining self-financing trading strategies and arbitrage; those definitions will be discussed in the following sections.

1.5 The Self-Financing Constraint

The notion of a *self-financing* trading strategy is fairly intuitive, but at the same time, an important tool in our market model. The formulation is very straightforward: Suppose we create a portfolio H whose time $t = t_0$ positions are $\left(H_{t_0}^B, H_{t_0}^X\right)$. The value of the portfolio at time t_0 is then $V_{t_0} = H_{t_0}^B B_{t_0} + H_{t_0}^X X_{t_0}$. We hold our positions in the two assets until time $t = t_1$, at which point the values of the assets have changed. Before making any changes to the portfolio, we note that its value has changed to $H_{t_0}^B B_{t_1} + H_{t_0}^X X_{t_1}$.

At time t_1 we are again allowed to alter our positions in each of the assets; suppose we now choose positions $\left(H_{t_1}^B, H_{t_1}^X\right)$. The constraint we are introducing here is the *self-financing* constraint: Naturally, we mean that we wish for the trading strategy to finance itself. That is, we will not invest more money from our pockets, nor will we pocket any money from our portfolio. We want the value of the portfolio to remain the same when we change our positions in each asset. In other words, any change in the value of the portfolio is a result of changes in the value of the investments. Formally, this means that we want to impose the following condition:

$$H_{t_0}^B B_{t_1} + H_{t_0}^X X_{t_1} = H_{t_1}^B B_{t_1} + H_{t_1}^X X_{t_1}$$

Or equivalently,

$$B_{t_1}\left(H_{t_1}^B - H_{t_0}^B\right) + X_{t_1}\left(H_{t_1}^X - H_{t_0}^X\right) = 0.$$

We therefore define the class of self-financing trading strategies as the set of all trading strategies which satisfy this condition:

$$\mathcal{H}^{SF} = \left\{ H = \left(H_{t_i}^B, H_{t_i}^X\right)_{i\geq 0} : B_{t_i}\left(H_{t_i}^B - H_{t_{i-1}}^B\right) + X_{t_i}\left(H_{t_i}^X - H_{t_{i-1}}^X\right) = 0 \right\}$$

The self-financing constraint is important in defining the notion of an arbitrage opportunity, which we will introduce later.

For self-financing trading strategies, we will show that an intuitive property holds in our model: The total gain of the trading strategy is equal to the sum of the individual gains of the investments in the bank account and risky security between consecutive trading dates.

Proposition 1.1. *The following identity holds for trading strategies* $H = (H_{t_0}, H_{t_1}, \ldots, H_{t_n})$ *belonging to* $\mathcal{H}^{SF}$*:*

$$V_{t_k} - V_{t_0} = \sum_{i=0}^{k-1} H_{t_i}^B\left(B_{t_{i+1}} - B_{t_i}\right) + \sum_{i=0}^{k-1} H_{t_i}^X\left(X_{t_{i+1}} - X_{t_i}\right)$$

Proof. Recall that the self-financing constraint means the following:

$$H_{t_{i-1}}^B B_{t_i} + H_{t_{i-1}}^X X_{t_i} = H_{t_i}^B B_{t_i} + H_{t_i}^X X_{t_i}$$

Consider the sum

$$\sum_{i=0}^{k-1} H_{t_i}^B \left(B_{t_{i+1}} - B_{t_i}\right) + \sum_{i=0}^{k-1} H_{t_i}^X \left(X_{t_{i+1}} - X_{t_i}\right)$$

$$= \sum_{i=0}^{k-1} \left[H_{t_i}^B \left(B_{t_{i+1}} - B_{t_i}\right) + H_{t_i}^X \left(X_{t_{i+1}} - X_{t_i}\right)\right]$$

The $i = 0$ term of this sum is

$$\begin{aligned} & H_{t_0}^B \left(B_{t_1} - B_{t_0}\right) + H_{t_0}^X \left(X_{t_1} - X_{t_0}\right) \\ = & \left(H_{t_0}^B B_{t_1} + H_{t_0}^X X_{t_1}\right) - \left(H_{t_0}^B B_{t_0} + H_{t_0}^X X_{t_0}\right) \end{aligned} \tag{1}$$

The $i = 1$ term is

$$\begin{aligned} & H_{t_1}^B \left(B_{t_2} - B_{t_1}\right) + H_{t_1}^X \left(X_{t_2} - X_{t_1}\right) \\ = & \left(H_{t_1}^B B_{t_2} + H_{t_1}^X X_{t_2}\right) - \left(H_{t_1}^B B_{t_1} + H_{t_1}^X X_{t_1}\right) \end{aligned} \tag{2}$$

Note that, due to the self-financing constraint, the first term of (1) and the second term of (2) will cancel. Similarly, the first term of the $i = 1$ case will cancel the second term of the $i = 2$ case, and so on. After these cancellations, we will be left with the first term of the $i = k - 1$ case and the second term of the $i = 0$ case, so

$$\begin{aligned} & \sum_{i=0}^{k-1} \left[H_{t_i}^B \left(B_{t_{i+1}} - B_{t_i}\right) + H_{t_i}^X \left(X_{t_{i+1}} - X_{t_i}\right)\right] \\ = & \left(H_{t_{k-1}}^B B_{t_k} + H_{t_{k-1}}^X X_{t_k}\right) - \left(H_{t_0}^B B_{t_0} + H_{t_0}^X X_{t_0}\right) \end{aligned} \tag{3}$$

Finally, note that the self-financing constraint implies that

$$\left(H_{t_{k-1}}^B B_{t_k} + H_{t_{k-1}}^X X_{t_k}\right) = \left(H_{t_k}^B B_{t_k} + H_{t_k}^X X_{t_k}\right)$$

so we are left with

$$\begin{aligned}
&\sum_{i=0}^{k-1} \left[H_{t_i}^B \left(B_{t_{i+1}} - B_{t_i}\right) + H_{t_i}^X \left(X_{t_{i+1}} - X_{t_i}\right)\right] \\
&= \left(H_{t_k}^B B_{t_k} + H_{t_k}^X X_{t_k}\right) - \left(H_{t_0}^B B_{t_0} + H_{t_0}^X X_{t_0}\right) \\
&= V_{t_k} - V_{t_0}
\end{aligned} \tag{4}$$

Which is what we wanted to prove. □

1.6 The Definition of Arbitrage

Definition 1.6. *We define an arbitrage opportunity as any self-financing trading strategy $H \in \mathcal{H}^{SF}$ which satisfies the conditions*

$$G_{t_n}^H = V_{t_n}^H - V_{t_0}^H \geq 0 \textit{ for all } \omega \in \Omega \textit{ and } G_{t_n}^H > 0 \textit{ for some } \omega \in \Omega$$

This is the formal definition of an arbitrage opportunity. Intuitively, these conditions mean that the trading strategy is guaranteed not to lose value, and has some possibility of gaining value; they represent the ability to earn a risk-free profit.

Definition 1.7. *We say that a market does not admit arbitrage (or, the market is arbitrage-free) if there exists no self-financing arbitrage opportunity in the market.*

As we stated before, it is our goal to study market models which do not admit arbitrage. This is, intuitively speaking, a condition on the ”fairness” of a market; we do not want there to be opportunities for investors to make riskless profits. These are models in which we should be able to fairly price derivative securities.

2 Change of Numéraire Result

In general terms, a numéraire is a standard unit of an asset against which the values of other assets are measured. In common practice, a country's currency acts as a numéraire, determining the relative value of goods under a common measure. For certain situations, applying a change of numéraire can be helpful or necessary; one natural example of a numéraire change is the change between the currencies of two countries. In other situations it may be appropriate to change the numéraire to a zero-coupon bond, or make a change to compensate for inflation, or in the case of this model, to adjust asset prices by discounting at the risk-free interest rate.

It has been shown (by Delbaen and Schachermayer [4]) that the No-Arbitrage property of a market may depend on the choice of numéraire, so it is important to make sure that the No-Arbitrage property remains intact when changing the numéraire.

For the purposes of this model, it will be necessary to treat the value of the bank account process as numéraire, and express the value of the risky security in terms of the bank account value. In order to simplify the model, we will discount the bank account process and the risky security's price process using the risk-free interest rate. We will use the fact that the market given by (B_t, X_t) admits no arbitrage if and only if the market given by $\left(1, \tilde{X}_t\right)$ admits no arbitrage, where $\tilde{X}_t = X_t/B_t$ denotes the discounted price process of the risky security. A more generalized version of this result has been shown [4], and so we will present our version without proof.

In the market $(1, \tilde{X})$, a self-financing trading strategy $\tilde{H}$ is given by:

$$\tilde{H} = (\tilde{H}_{t_0}, \tilde{H}_{t_1}, \ldots, \tilde{H}_{t_{n-1}})$$

where $\tilde{H}_{t_i}$ is a random variable for each i (i.e. H is a discrete stochastic process).

These are the amounts invested in the risky asset; these values are enough to determine the trading strategy since we can always choose a unique set of weights for the risk-free asset such that the trading strategy is self-financing.

The corresponding gains process is given by

$$G_{t_i}^{\tilde{H}} = \tilde{H}_{t_0}(\tilde{X}_{t_1} - \tilde{X}_{t_0}) + \ldots + \tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}})$$

Definition 2.1. *We say that the market* $(1, \tilde{X})$ *does not admit arbitrage if there exists no* $\tilde{H}$ *such that*

$$G_{t_n}^{\tilde{H}} \geq 0 \text{ for all } \omega \in \Omega \text{ and } G_{t_n}^{\tilde{H}} > 0 \text{ for some } \omega \in \Omega$$

Theorem 2.1. *The following are equivalent:*

- *The no-arbitrage property holds in the market* (B_t, X_t).
- *The no-arbitrage property holds in the market* $\left(1, \tilde{X}_t\right)$.

3 The Dalang-Morton-Willinger Theorem

To introduce the Dalang-Morton-Willinger Theorem, a few other preliminary definitions are needed; we need to discuss the concept of equivalent probability measures, and define conditional expectation and martingales. Then we will be ready to state the Dalang-Morton-Willinger Theorem.

3.1 Equivalent Probability Measure

Definition 3.1. *Two probability measures* P *and* Q *are called* equivalent *if* $P(\omega) = 0 \Leftrightarrow Q(\omega) = 0$ *for every event* ω *in the sample space* Ω.

Essentially, the definition means that the two probability measures P and Q agree on which events have measure zero. For the purposes of our model, since there is no event $\omega \in \Omega$ such that $P(\omega) = 0$, there can also be no event such that $Q(\omega) = 0$, if Q is a measure equivalent to P. In short, since all of our outcomes ω occur with positive probability under P, they must also all be assigned positive probability by Q.

3.2 Conditional Expectation

Definition 3.2. *If X is a random variable, and $\mathcal{G}$ is a σ-algebra, then the conditional expectation of X with respect to $\mathcal{G}$ (denoted $E[X|\mathcal{G}]$) is a random variable with the following properties:*

- *$E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.*
- *$\int_A X dP = \int_A E[X|\mathcal{G}]\, dP$ where $A \in \mathcal{G}$.*

Conditional expectation has the following properties:

- $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$ (conditional expectation is linear)
- $E[E[X|\mathcal{G}]] = E[X]$
- $E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_2]$ if $\mathcal{G}_2 \subset \mathcal{G}_1$
- $E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_1]$ if $\mathcal{G}_1 \subset \mathcal{G}_2$
- $E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$ if X is $\mathcal{G}$-measurable.

3.3 Martingales

Definition 3.3. *An adapted stochastic process is called a* martingale *if* $E[X_t|\mathcal{F}_s] = X_s$.

For the purposes of our model, we will say that a process is a martingale if $E[X_{t_m}|\mathcal{F}_{t_n}] = X_{t_n}$. Intuitively, this means that if a process is a martingale, we do not expect its value to have any tendency to rise or fall.

3.4 Dalang-Morton-Willinger Theorem

Theorem 3.1. *The market* $\left(1, \tilde{X}\right)$ *satisfies the no-arbitrage condition if and only if there is a probability measure* Q *equivalent to* P *such that the process* $\tilde{X}$ *is a martingale under* Q.

The theorem has been widely applied and proven under many different conditions. Essentially, it asserts that if there are no arbitrage opportunities in the market model, then there necessarily exists a martingale measure equivalent to the real-world measure. Similarly, if the existence of an equivalent martingale measure can be shown, then it must follow that the market does not admit arbitrage. Naturally, the existence of a martingale measure is typically important since it allows for risk-neutral pricing of assets.

Even though this result provides a necessary and sufficient condition for no-arbitrage, it is not trivial to check if a process admits an equivalent martingale measure or not. As an example, consider a martingale M. It does not immediately follow that the process e^M admits an equivalent martingale measure. However, our result in the next section will show that e^M does in fact admit an equivalent martingale measure.

4 No-Arbitrage Conditions with Simple Integrands

4.1 A Lemma

Before proving the main result, it will be necessary to present a lemma. The proof of the main result will follow almost immediately from the statement of the lemma.

Lemma 4.1. *The market* $(1, \tilde{X})$ *does not admit arbitrage if and only if the discounted price process* $\tilde{X}$ *satisfies the condition* $(\star)$ *below,*

Condition $(\star)$**:** *For any two trading dates* $t_i < t_j$ *and any atom* A *in the partition of* Ω *corresponding to* $\mathcal{F}_{t_i}$*, either*

$$\tilde{X}_{t_i} = \tilde{X}_{t_j} \text{ for all } \omega \text{ in } A \text{ , or}$$

$$\tilde{X}_{t_i} > \tilde{X}_{t_j} \text{ for some } \omega \text{ in } A \text{ , and } \tilde{X}_{t_i} < \tilde{X}_{t_j} \text{ for some } \omega \text{ in } A.$$

This statement means that either $\tilde{X}_{t_i} = \tilde{X}_{t_j}$ identically on the event A, or $\tilde{X}_{t_i} > \tilde{X}_{t_j}$ and $\tilde{X}_{t_i} < \tilde{X}_{t_j}$ both occur with some probability on the event A.

For example, if the atom $A = \{\omega_1, \omega_2\}$, then the statement would be satisfied if $\tilde{X}_{t_i} = \tilde{X}_{t_j}$ in both outcomes ω_1 and ω_2. The statement would also be satisfied if $\tilde{X}_{t_i} > \tilde{X}_{t_j}$ in ω_1 and $\tilde{X}_{t_i} < \tilde{X}_{t_j}$ in ω_2.

To imply that the no-arbitrage condition holds, the statement must be true for every such atom A.

Proof. (**Proof of Lemma 4.1**) ($\Rightarrow$) By way of contradiction, assume the market $(1, \tilde{X})$ does not admit arbitrage, but the condition ($\star\star$) does not hold, then there are two trading dates $t_i < t_j$ and an atom $A \in \mathcal{F}_{t_i}$, such that either

$$(i) \quad \tilde{X}_{t_j} \geq \tilde{X}_{t_i} \quad \text{for all } \omega \in A \text{ and } \tilde{X}_{t_j} > \tilde{X}_{t_i} \text{ for some } \omega \text{ in } A$$

or

$$(ii) \quad \tilde{X}_{t_j} \leq \tilde{X}_{t_i} \quad \text{for all } \omega \in A \text{ and } \tilde{X}_{t_j} < \tilde{X}_{t_i} \text{ for some } \omega \text{ in } A$$

In the case of (i) let $\tilde{H} = (\tilde{H}_{t_1}, \tilde{H}_{t_2}, \dots, \tilde{H}_{t_{n-1}})$ be such that $\tilde{H}_{t_k} = 0$ for $k = 1, 2, \dots, i-1, j, j+1, \dots, n-1$ and $\tilde{H}_{t_l} = 1_A$ for $l = i, i+1, \dots, j-1$. Then $\tilde{G}^{\tilde{H}}_{t_n} = 1_A(\tilde{X}_{t_j} - \tilde{X}_{t_i})$ and

$$\tilde{G}^{\tilde{H}}_{t_n} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}^{\tilde{H}}_{t_n} > 0 \text{ for some } \omega.$$

So $\tilde{H}$ is an arbitrage opportunity. In the case of (ii), let $\tilde{\tilde{H}} = (\tilde{\tilde{H}}_{t_1}, \tilde{\tilde{H}}_{t_2}, \dots, \tilde{\tilde{H}}_{t_{n-1}})$ be such that $\tilde{\tilde{H}}_{t_k} = 0$ for $k = 1, 2, \dots, i-1, j, j+1, \dots, n-1$ and $\tilde{\tilde{H}}_{t_l} = -1_A$ for $l = i, i+1, \dots, j-1$. Then $\tilde{G}^{\tilde{\tilde{H}}}_{t_n} = -1_A(\tilde{X}_{t_j} - \tilde{X}_{t_i})$ and so $\tilde{\tilde{H}}$ is an arbitrage opportunity. Contradiction.

($\Leftarrow$) Assume the condition ($\star$) does hold. By way of contradiction, assume $(1, \tilde{X})$ admits arbitrage. Then there is a self-financing trading strategy $\tilde{H} = (\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_{t_{n-1}})$, such that the corresponding gains process satisfies

$$\tilde{G}^{\tilde{H}}_{t_n} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}^{\tilde{H}}_{t_n} > 0 \text{ for some } \omega.$$

Let i be the minimum of all k that satisfy

$$\tilde{G}^{\tilde{H}}_{t_k} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}^{\tilde{H}}_{t_k} > 0 \text{ for some } \omega.$$

Case I: $i = 1$

Then

$$\tilde{H}_{t_0}(\tilde{X}_{t_1} - \tilde{X}_{t_0}) \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{H}_{t_0}(\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0 \text{ for some } \omega.$$

Let $F = \{\tilde{H}_{t_0} > 0\}$ and $E = \{\tilde{H}_{t_0} < 0\}$, then either $F \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0\} \neq \emptyset$ or $E \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) < 0\} \neq \emptyset$. Otherwise, we would have $\tilde{G}^{\tilde{H}}_{t_1} \equiv 0$ which contradicts the minimality of $i = 1$. Note that since $\tilde{H}_{t_0}$ is $\mathcal{F}_{t_0}$-measurable, we have $F, E \in \mathcal{F}_{t_0}$. Now, if $F \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0\} \neq \emptyset$ then there is an atom A in F such that $A \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0\} \neq \emptyset$. So we have $\tilde{X}_{t_1} \geq \tilde{X}_{t_0}$ for all ω in A and $\tilde{X}_{t_1} > \tilde{X}_{t_0}$ for some ω in A. This contradicts the condition ($\star$). If $E \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) < 0\} \neq \emptyset$, we obtain a contradiction similarly.

Case II: $i > 1$

Then, either $\tilde{G}_{t_{i-1}} \equiv 0$, or $\tilde{G}_{t_{i-1}} < 0$ for some ω.

(i) If $\tilde{G}_{t_{i-1}} \equiv 0$, then $\tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}) \geq 0$ for all ω and $\tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}) > 0$ for some ω. So we have the same situation as in **Case I**. Therefore we can obtain a contradiction in a similar way as in Case I.

(ii) If $\tilde{G}_{t_{i-1}} < 0$ for some ω,

Let $F = \{\tilde{G}_{t_{i-1}} < 0\}$, then since $\tilde{G}_{t_i} \geq 0$ for all ω in F we have $\tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}) > 0$ for all ω in F. Let $B = \{\tilde{H}_{t_{i-1}} > 0\}$ and $D = \{\tilde{H}_{t_{i-1}} < 0\}$, then either $F \cap B \neq \emptyset$ or $F \cap D \neq \emptyset$. So assume $F \cap B \neq \emptyset$. Note that $\tilde{X}_{t_i} > \tilde{X}_{t_{i-1}}$ for all ω in $F \cap B$. Also note that $F \cap B \in \mathcal{F}_{t_{i-1}}$. So there is an atom A of Ω corresponding to $\mathcal{F}_{t_{i-1}}$, such that $\tilde{X}_{t_i} > \tilde{X}_{t_{i-1}}$ for all ω in A. This contradicts condition $(\star)$. If $F \cap D \neq \emptyset$ we again obtain a contradiction similary.

□

4.2 Main Result

The main result is a theorem showing how the no-arbitrage property of a market holds under composition with a strictly monotone function.

Theorem 4.1. *Let f be a strictly monotone function. The following are equivalent:*

- *The no-arbitrage property holds in* $\left(1, \tilde{X}\right)$.
- *The no-arbitrage property holds in* $\left(1, f(\tilde{X})\right)$.

Proof. The proof follows almost immediately from the lemma we presented previously.

($\Rightarrow$) First, suppose that the no-arbitrage property holds in $\left(1, \tilde{X}\right)$. Now let $t_i < t_j$ and an atom A of $\mathcal{F}_{t_i}$ be given. Additionally, let f be a strictly monotone function. There are two cases:

Case I: $P(A \cap \{\tilde{X}_{t_i} = \tilde{X}_{t_j}\}) = P(A)$

Certainly, $\tilde{X}_{t_i} = \tilde{X}_{t_j} \Rightarrow f(\tilde{X}_{t_i}) = f(\tilde{X}_{t_j})$,
or equivalently, the set $\left\{\tilde{X}_{t_i} = \tilde{X}_{t_j}\right\}$ is a subset of $\left\{f(\tilde{X}_{t_i}) = f(\tilde{X}_{t_j})\right\}$.

So $P\left(A \cap \left\{f(\tilde{X}_{t_i}) = f(\tilde{X}_{t_j})\right\}\right) = P(A)$ as well.

Case II: $P(A \cap \{\tilde{X}_{t_i} > \tilde{X}_{t_j}\}) > 0$ and $P(A \cap \{\tilde{X}_{t_i} < \tilde{X}_{t_j}\}) > 0$

If f is increasing, we have

$$\tilde{X}_{t_i} > \tilde{X}_{t_j} \Rightarrow f(\tilde{X}_{t_i}) > f(\tilde{X}_{t_j}) \text{ and}$$

$$\tilde{X}_{t_i} < \tilde{X}_{t_j} \Rightarrow f(\tilde{X}_{t_i}) < f(\tilde{X}_{t_j})$$

Equivalently, the set $\left\{\tilde{X}_{t_i} > \tilde{X}_{t_j}\right\}$ is a subset of $\left\{f(\tilde{X}_{t_i}) > f(\tilde{X}_{t_j})\right\}$ (and similarly for the "$<$" statement).

Thus $P\left(A \cap \left\{f(\tilde{X}_{t_i}) > f(\tilde{X}_{t_j})\right\}\right) > 0$ and $P\left(A \cap \left\{f(\tilde{X}_{t_i}) < f(\tilde{X}_{t_j})\right\}\right) > 0$ both hold as well.

Similarly, if f is decreasing,

$$\tilde{X}_{t_i} > \tilde{X}_{t_j} \Rightarrow f(\tilde{X}_{t_i}) < f(\tilde{X}_{t_j}) \text{ and}$$

$$\tilde{X}_{t_i} < \tilde{X}_{t_j} \Rightarrow f(\tilde{X}_{t_i}) > f(\tilde{X}_{t_j})$$

So $P\left(A \cap \left\{f(\tilde{X}_{t_i}) > f(\tilde{X}_{t_j})\right\}\right) > 0$ and $P\left(A \cap \left\{f(\tilde{X}_{t_i}) < f(\tilde{X}_{t_j})\right\}\right) > 0$ both hold.

The market $\left(1, f(\tilde{X})\right)$ therefore satisfies the condition equivalent to no-arbitrage.

($\Leftarrow$) Let $\tilde{Y} = f(\tilde{X})$, and suppose now that the market $\left(1, f(\tilde{X})\right) = \left(1, \tilde{Y}\right)$ satisfies the no-arbitrage condition, and that f is a strictly monotone function.

As f is strictly monotone, it has an inverse f^{-1}, which is also strictly monotone.

We can therefore apply the part of the theorem which we just proved: If the no-arbitrage property holds in $\left(1, \tilde{Y}\right)$, then the no-arbitrage property holds in $\left(1, f^{-1}(\tilde{Y})\right)$.

Substituting $\tilde{Y} = f(\tilde{X})$, this means if the no-arbitrage property holds in

$\Big(1, f(\tilde{X})\Big)$, then it holds in $\Big(1, f^{-1}(f(\tilde{X}))\Big) = \Big(1, \tilde{X}\Big)$.

□

This theorem provides a method of constructing new arbitrage-free market models from other arbitrage-free markets. It also potentially gives a way to check that the no-arbitrage property holds in a particular model. We can also present an interesting corollary which follows almost directly from the statement of the main result.

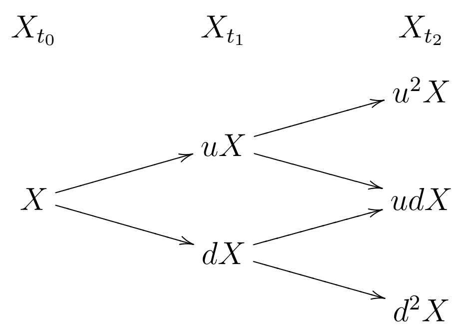
Corollary 4.1. *If M is a martingale and if f is a strictly monotone function, then $(1, f(M))$ is an arbitrage-free market model.*

4.3 Examples

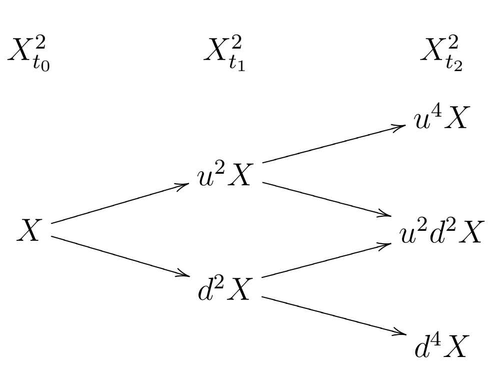
Here we will give a few simple examples to demonstrate how the theorem might be applied.

Example I

Consider a two-period binomial model for the stock price, where the risk-free interest rate is $r = 0$. At time t_0, the price of the stock is X_{t_0}; for simplicity let us denote $X_{t_0} = X$. At time t_1, the price can change to either uX or dX, where $d < u$. Similarly, in each period, the price $X_{t_{i+1}}$ is either uX_{t_i} or dX_{t_i}. The no-arbitrage condition for this model is $0 < d < 1 + r < u$, i.e. $0 < d < 1 < u$ for our assumption of $r = 0$. So the model can be diagrammed in this way:

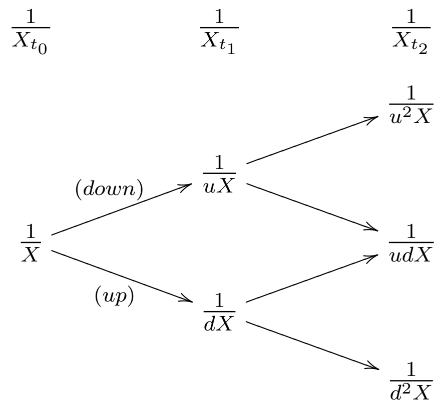


Now let $f(x) = x^2$ and consider the tree corresponding to the price process $f(X_t) = X_t^2$ (since we generally take the price process to be strictly positive, we don't have any concerns about monotonicity; the function is strictly increasing for $x > 0$). Note that we still have a multiperiod binomial model, and the new up and down factors are u^2 and d^2 (these factors are still constant between periods, but in general this may not always be the case). The no-arbitrage condition for a general binomial model is that there is no arbitrage in any of the individual single-period submodels; naturally if $d < 1 < u$ then $d^2 < 1 < u^2$. So the no-arbitrage condition is preserved in this model when the risky price process is squared.



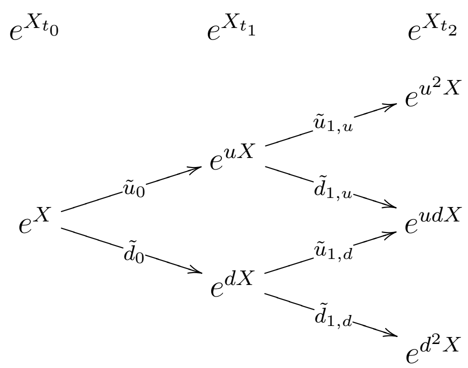
Example II

Now we look at a case where the function f is strictly decreasing. For instance, consider the function $f(x) = \frac{1}{x}$. The new up and down factors are $1/d$ and $1/u$, respectively. Since $0 < d < 1 < u$, this implies $0 < 1/u < 1 < 1/d$. So again, the no-arbitrage condition is satisfied.



Example III

As a final example, consider the function $f(x) = e^x$. Composing X_t with this function again gives a two-period binomial model, but the up and down factors are no longer constant between nodes. However, we will still find that the no-arbitrage condition will hold for the model; i.e. the individual up and down factors at each node will still satisfy $\tilde{d} < 1 < \tilde{u}$. Let us denote by $\tilde{u}_0$ and $\tilde{d}_0$ the new up and down factors in the first period. Additionally let $\tilde{u}_{1,u}$ denote the up factor in the second period provided that the "up" path was taken in the first period, and so on (as shown in the diagram).



We can compute the new up and down factors at each node directly:

$$\tilde{u}_0 = \frac{e^{uX}}{e^X} = e^{(u-1)X} > 1 \text{ since } u - 1 > 0$$
$$\tilde{d}_0 = \frac{e^{dX}}{e^X} = e^{(d-1)X} < 1 \text{ since } d - 1 < 0$$

$$\tilde{u}_{1,u} = \frac{e^{u^2X}}{e^{uX}} = e^{u(u-1)X} > 1 \text{ since } u - 1 > 0$$
$$\tilde{d}_{1,u} = \frac{e^{udX}}{e^{uX}} = e^{u(d-1)X} < 1 \text{ since } d - 1 < 0$$
$$\tilde{u}_{1,d} = \frac{e^{udX}}{e^{dX}} = e^{d(u-1)X} > 1 \text{ since } u - 1 > 0$$
$$\tilde{d}_{1,d} = \frac{e^{d^2X}}{e^{dX}} = e^{d(d-1)X} < 1 \text{ since } d - 1 < 0$$

The no-arbitrage property holds at each node on the binomial tree, so the model is arbitrage-free. This argument can easily be extended to a binomial model with any number of periods.

We have shown in each example that our main result holds by first principles. The advantage of the theorem is that it can be applied directly to much more complicated cases; for instance, the function may be too cumbersome to apply to every possible value of X_{t_i} in the model, or the model itself may be more difficult to work with than a simple binomial model. In such examples, our theorem provides a sure way to construct new arbitrage-free models.

5 No-Arbitrage with Short-Sale Restriction

In this section we discuss no arbitrage conditions when shortsale is not allowed in the market. Let $(1, \tilde{X})$ be the discounted market and again we assume all the trading in this market can happen only at finitely many randomly fixed dates $t_1, t_2, \ldots, t_n$. On each trading date t_i, the investor chooses his long position $\tilde{H}_{t_i}$ in the risky asset $\tilde{X}$ and keeps it until time t_{i+1}. So a typical trading strategy with shortsale restriction is given by $\tilde{H} = (\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_{t_{n-1}})$, where each $\tilde{H}_i, i = 1, \ldots, n-1$ is a nonnegative random variable. We denote by $\mathcal{H}^+$ the class of trading strategies $\tilde{H}$ with shortsale restriction. The gains process corresponding to the trading strategy $\tilde{H} = (\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_{t_{n-1}})$ is given by

$$\tilde{G}_{t_i}^{\tilde{H}} = \tilde{H}_{t_0}(\tilde{X}_{t_1} - \tilde{X}_{t_0}) + \ldots + \tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}), \quad i = 1, 2, \ldots, n.$$

Definition 5.1. *We say that the discounted market $(1, \tilde{X})$ does not admit arbitrage with shortsale restriction if there exists no $\tilde{H}$ in $\mathcal{H}^+$, such that the corresponding gains process satisfies*

$$\tilde{G}_{t_n}^{\tilde{H}} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}_{t_n}^{\tilde{H}} > 0 \text{ for some } \omega.$$

In the following lemma we provide a condition which is necessary and sufficient for the no-arbitrage property of the market $(1, \tilde{X})$.

Lemma 5.1. *The market $(1, \tilde{X})$ does not admit arbitrage with shortsale re-*

striction if and only if the following condition $(\star\star)$ *is satisfied*

Condition $(\star\star)$***:*** *For any two trading dates* $t_i < t_j$ *and any atom* A *in the partition of* Ω *that belongs to* $\mathcal{F}_{t_i}$*, either*

$$X_{t_i} = X_{t_j} \ \textit{for all}\ \omega\ \textit{in}\ A,\ \textit{or}$$

$$X_{t_i} > X_{t_j} \ \textit{for some}\ \omega\ \textit{in}\ A$$

Before proving this emma, we discuss the condition $(\star\star)$ for the case of supermartingales. If $\tilde{X}$ is a supermartingale, then $E[X_{t_j}|\mathcal{F}_i] \leq X_{t_i}$. Therefore for any $A \in \mathcal{F}_{t_i}$, we have $E[1_A X_{t_i}] = E[1_A X_{t_j}]$ and this is equivalent to the condition $(\star\star)$. So we can state the following result.

If $\tilde{X}$ *is a supermartingale, the market* $(1, \tilde{X})$ *does not admit arbitrage with shortsale restriction.*

Proof. (**Proof of Lemma 5.1**) ($\Rightarrow$) By way of contradiction, assume the market $(1, \tilde{X})$ does not admit arbitrage, and the condition $(\star\star)$ does not hold, then there are two trading dates $t_i < t_j$ and an atom $A \in \mathcal{F}_{t_i}$, such that $X_{t_j} \geq X_{t_i}$ for all $\omega \in A$ and $X_{t_j} > X_{t_i}$ for some ω in A. Let $\tilde{H} = (\tilde{H}_{t_1}, \tilde{H}_{t_2}, \ldots, \tilde{H_{t_{n-1}}})$ be such that $H_{t_k} = 0$ for $k = 1, 2, \ldots, i-1, j, j+1, \ldots, n-1$ and $H_{t_l} = 1_A$ for $l = i, i+1, \ldots, j-1$. Then $\tilde{G}^{\tilde{H}}_{t_n} = 1_A(X_{t_j} - X_{t_i})$ and

$$\tilde{G}^{\tilde{H}}_{t_n} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}^{\tilde{H}}_{t_n} > 0 \text{ for some } \omega.$$

So $\tilde{H}$ is an arbitrage opportunity. Contradiction.

($\Leftarrow$) Assume the condition ($\star\star$) does hold. By way of contradiction assume $(1, \tilde{X})$ admits arbitrage. Then there is a trading strategy $\tilde{H} = (\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_{t_{n-1}}) \in \mathcal{H}^+$, such that the corresponding gains process satisfies

$$\tilde{G}^{\tilde{H}}_{t_n} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}^{\tilde{H}}_{t_n} > 0 \text{ for some } \omega.$$

Let i be the minimum of all k that satisfy

$$\tilde{G}^{\tilde{H}}_{t_k} \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{G}^{\tilde{H}}_{t_k} > 0 \text{ for some } \omega.$$

Case I: $i = 1$

Then

$$\tilde{H}_{t_0}(\tilde{X}_{t_1} - \tilde{X}_{t_0}) \geq 0, \text{ for all } \omega, \text{ and } \quad \tilde{H}_{t_0}(\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0 \text{ for some } \omega.$$

Let $F = \{\tilde{H}_{t_0} > 0\}$ then F is not empty. Otherwise, we would have $\tilde{H}_{t_0} \equiv 0$ and $\tilde{G}^{\tilde{H}}_{t_1} \equiv 0$ which contradicts the minimality of $i = 1$. Also we have $F \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0\} \neq \emptyset$. Otherwise we would also have $\tilde{G}^{\tilde{H}}_{t_1} \equiv 0$.

Now, since $\tilde{H}_{t_0}$ is $\mathcal{F}_{t_0}$-measurable, we have $F \in \mathcal{F}_{t_0}$. F is the union of disjoint atoms and one of these atoms, say A, has the property $A \cap \{\omega : (\tilde{X}_{t_1} - \tilde{X}_{t_0}) > 0\} \neq \emptyset$ (because F has this property). So $\tilde{X}_{t_1} \geq \tilde{X}_{t_0}$ for all ω

in A and $\tilde{X}_{t_1} > \tilde{X}_{t_0}$ for some ω in A. This contradicts the condition $(\star\star)$.

Case II: $i > 1$

Then, either $\tilde{G}_{t_{i-1}} \equiv 0$, or $\tilde{G}_{t_{i-1}} < 0$ for some ω.

(i) If $\tilde{G}_{t_{i-1}} \equiv 0$, then $\tilde{H}_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) \geq 0$ for all ω and $\tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}) > 0$ for some ω. This is the same as the **Case I** above. So we have a contradiction again.

(ii) If $\tilde{G}_{t_{i-1}} < 0$ for some ω,

Let $F = \{\tilde{G}_{t_{i-1}} < 0\}$, then since $\tilde{G}_{t_i} \geq 0$ on F we have $\tilde{H}_{t_{i-1}}(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}) > 0$ on F. Therefore $\tilde{X}_{t_i} > \tilde{X}_{t_{i-1}}$ on F. Note that $F \in \mathcal{F}_{t_{i-1}}$ because $\tilde{G}_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$ measurable. F is the union of the atoms of Ω that are in $\mathcal{F}_{t_{i-1}}$. Let A be one of the atoms whose union is F. Then on A we have $\tilde{X}_{t_i} > \tilde{X}_{t_{i-1}}$. This contradicts the condition $(\star\star)$. This completes the proof.

□

The following theorem states that the no arbitrage property is invariant under the composition with a strictly increasing function.

Theorem 5.1. *The market* $(1, \tilde{X})$ *does not admit arbitrage with shortsale restriction if and only if for any strictly monotone determinstic function* f *the market* $(1, f(\tilde{X}))$ *does not admit arbitrage with shortsale restriction.*

Proof. Its easy to check that the condition $(\star)$ for the price process $\tilde{X}$ is equivalent to the condition $(\star)$ for the price process $f(\tilde{X})$. Then Lemma 5.1 applies.

□

Since supermartingales do not admit arbitrage with shortsale restrictions, we can state the following result

Corollary 5.1. *Let $\tilde{X}$ be a supermartingale, then for any strictly monotone function f, the market $(1, f(\tilde{X}))$ does not admit arbitrage with short sale restriction.*

6 Conclusion

We have shown that if the discounted market model admits no arbitrage, then the no-arbitrage property is preserved when the discounted price process of the risky asset is composed with a strictly monotone function. The result gives a method of constructing arbitrage-free market models given only a little information; it can also be used to determine if a specific model is arbitrage-free (provided enough information is given). Indeed, it is important to study market models having the no-arbitrage property, since we want to be able to fairly price derivative securities, and we want to disallow the possibility of investors making riskless profits. The result as presented in our model is fairly simplistic, but it is possible to extend it to more general models.

There are, however, a few issues we have not yet discussed. In particular, we expect that there should be a relation between the martingale measure Q of the discounted market and the measure Q_f of the market under composition with the function f. Furthermore, we did not explicitly prove the results for the change of numéraire and the Dalang-Morton-Willinger Theorem in our setting, but instead relied on the more general results given in various past articles.

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