Equivalent Resistances of Polytope Networks

A Major Qualifying Project
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Abstract

Polytopes are the generalizations of polygons and polyhedra. In this project, we model polytopes as graphs, with each edge being an identical resistor, and calculate the equivalent resistance between two arbitrary nodes. We explore four methods to solve for equivalent resistances, based on the work of Kirchhoff, van Steenwijk, Lovász, and Nahin. We present results for the Platonic solids, selected Archimedean solids, and all but one of the regular convex four-dimensional polytopes. The project then concludes with an overview of opportunities for future research, including extending these strategies to the Catalan solids.

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1 Introduction

The goal of this project is to determine the equivalent resistance between arbitrary nodes of a polytope whose edges are modeled as identical 1Ω resistors. Resistors networks, involving either a finite or an infinite number of resistors, have been studied for years for their various applications. Polygons and polytopes also provide examples of such networks, when their edges are modeled by identical resistors, and the goal of this project is to expand on an earlier project by Moody [1] by looking at a wider array of methods of solving this problem, and also to carry out a comparison of the effectiveness of the different approaches.

The Platonic solids are made of congruent regular polygons meeting at identical vertices. For example, the cube is made of six squares, with three squares meeting at each of the eight vertices, and the icosahedron is made of twenty equilateral triangles, with five triangles meeting at each of the twelve vertices. There are five Platonic solids, the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, and all are highly symmetric, due to their vertex-, edge-, and face-transivity.

The Archimedean solids are composed of regular polygons meeting at identical vertices, excluding both the Platonic solids and the prisms and anti-prisms. These are much more varied than the Platonic solids, and can have as few as 12 vertices or as many as 120. For example, the cuboctahedron is made from six squares and eight equilateral triangles, with two squares and two triangles meeting at each of the 12 vertices. There are thirteen Archimedean solids, the truncated tetrahedron, cuboctahedron, truncated cube, truncated octahedron, rhombicuboctahedron, truncated cuboctahedron, snub cube, icosidodecahedron, truncated dodecahedron, truncated icosahedron, rhombicosidodecahedron, truncated icosidodecahedron, and snub dodecahedron. The Archimedean solids are vertex-transitive but not face-transitive. The lack of face symmetry does not majorly impact this project, as we are mainly concerned with the vertices and the edges that connect them.

The convex regular polytopes in four dimensions consist of the five analogues of the Platonic solids, along with a sixth polytope (the 24-cell). These are known by the number of three-dimensional cells that make up the polytope, similarly to how the Platonic solids are known by the number of two-dimensional faces that make them up (another name for the cube is the hexahedron, for its six faces). In order of increasing number of cells, they are the 5-cell, 8-cell, 16-cell, 24-cell, 120-cell, and 600-cell, and are made up of that number of tetrahedra, cubes, tetrahedra, octahedra, dodecahedra, and tetrahedra, respectively.

The other objective of this project was to review and discover methods for calculating equivalent resistances of these polytope networks. The first method is based on the use of Kirchhoff's current and loop equations. This process was discussed by Moody [1], so we will only provide a brief overview. The second method is that of van Steenwijk [2], who solved this problem by utilizing the symmetries of the solids to divide the nodes into equipotential planes. After this division, the number of nodes in each layer and the number of edges running between layers is used to finding equivalent resistances.

The third method is the so-called hanging method, developed by Lovász [3], and involves the concept of a hitting time. The hitting time $H(a, b)$ is the average number of steps it takes for a random walk on a connected graph to move from a to b. We then move to a physical model of the graph and prove the equality of hitting times and equivalent resistances between the two nodes. The final method determines hitting times by a computer simulation of random walks on the structure [4], and uses these to calculate the equivalent resistances.

Of the four methods, the first two were discussed in a previous Major Qualifying Project [1], and the second two were adapted this year to be used in solving this problem.

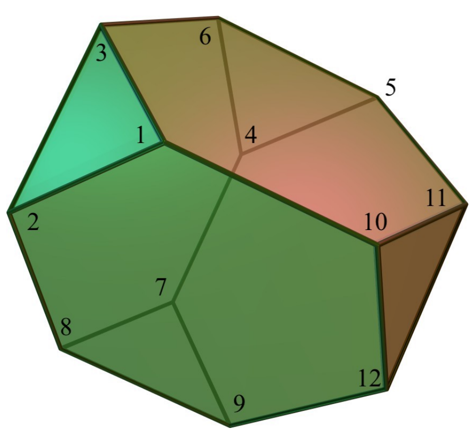
2 Symmetrically Equivalent Layers

For all four methods discussed, we will divide networks into layers of symmetrically equivalent nodes. From a given starting node, all the nodes in a layer will be equipotential, and therefore have the same equivalent resistance. Formally, layers are defined by isomorphisms that send all the nodes in a layer into each other while leaving the starting node invariant.

To identify these layers, we investigated three main approaches. The first approach was to take the coordinates of the vertices of a polytope and compute the dot product of the input node with every other node. As long as the polytope is centered on the origin of the coordinate system, the vertices will all be equidistant from the origin. In this case, the dot product will be a stand-in for the cosine of the angle between the vectors going from the origin to the two vertices in question. Since this angle must be between 0 and π, the cosine will be a strictly decreasing function, going from 1 for the dot product of the starting node and itself to -1 for the dot product of the starting node and its antipode (if the antipode exists). In this way, all of the nodes of a polytope may be arranged in order of distance from the starting node. The dot product both accomplishes the same goal and is easier to compute than the Euclidean distance between two nodes.

Just using the dot product, however, is not always enough to uncover all layers. For example, the truncated tetrahedron, shown in Figure 1, will not provide accurate layers just by using the dot product approach. For any node, its three neighbors are equal distances away, but will not be symmetric. In the figure, node 1 is adjacent to nodes 2, 3, and 10, but the equivalent resistance between node 1 and either of nodes 2 and 3 is $17/30\Omega$, but the equivalent resistance between nodes 1 and 10 is $7/10\Omega$.

Figure 1: Truncated Tetrahedron



By the dot product, the nodes would be sorted into the layers given in Table 1, whereas the nodes sorted by equivalent resistance are shown in Table 2. It can clearly be seen that the dot product method does not accurately split nodes 2, 3, and 10. Another method must be used to avoid this undersplitting of layers.

Layer Number	Nodes in Layer
1	1
2	2, 3, 10
3	6, 8, 11, 12
4	5, 9
5	4, 7

Table 1: Layers of Truncated Tetrahedron by Dot Product

Layer Number	Nodes in Layer	Equivalent Resistance (Ω)
1	1	0
2	2, 3	17/30
3	10	7/10
4	6, 8, 11, 12	29/30
5	5, 9	16/15
6	4, 7	11/10

Table 2: Equivalent Resistances of Truncated Tetrahedron

The second approach was to mandate that all nodes in a layer connect to the same number of nodes in other layers. This involves an iterative process where the layer connections of all nodes are found and then checked against the layer connections of all other nodes in a layer. If there are discrepancies, the layer is split and the process is repeated for all nodes. To illustrate this process, we look again at the truncated tetrahedron, starting from the layers found by the dot product method. After finding the layer connections for each node, shown in Table 3, each layer is checked for dissimilarities. As can be seen, layer 2 must be split, as nodes 2 and 3 are connected to layers 1, 2, and 3, but node 10 connects to layers 1, 3, and 3 again. Likewise, layer 3 must be split, as nodes 6 and 8 connect to layers 2, 4, and 5, but nodes 11 and 12 connect to layers 2, 3, and 4. After making these splits, the new layers connections are shown in Table 4. Now it is clear that each of the seven layers have consistent layer connections.

Layer Number	Node Number	Node Connections	Layer Connections
1	1	2, 3, 10	2, 2, 2
2	2	1, 3, 8	1, 2, 3
2	3	1, 2, 6	1, 2, 3
2	10	1, 11, 12	1, 3, 3
3	6	3, 4, 5	2, 4, 5
3	8	2, 7, 9	2, 4, 5
3	11	5, 10, 12	2, 3, 4
3	12	9, 10, 11	2, 3, 4
4	5	4, 6, 11	3, 3, 5
4	9	7, 8, 12	3, 3, 5
5	4	5, 6, 7	3, 4, 5
5	7	4, 8, 9	3, 4, 5

Table 3: Layer Connections of Truncated Tetrahedron after Dot Product

Layer Number	Node Number	Node Connections	Layer Connections
1	1	2, 3, 10	2, 2, 3
2	2	1, 3, 8	1, 2, 4
2	3	1, 2, 6	1, 2, 4
3	10	1, 11, 12	1, 5, 5
4	6	3, 4, 5	2, 6, 7
4	8	2, 7, 9	2, 6, 7
5	11	5, 10, 12	3, 5, 6
5	12	9, 10, 11	3, 5, 6
6	5	4, 6, 11	4, 5, 7
6	9	7, 8, 12	4, 5, 7
7	4	5, 6, 7	4, 6, 7
7	7	4, 8, 9	4, 6, 7

Table 4: Layer Connections of Truncated Tetrahedron after Splitting

However, we now have an oversplitting of layers. Above in Table 2, there were only six layers, with nodes 6, 8, 11, and 12 forming just one layer, but now nodes 6 and 8 are in one layer and nodes 11 and 12 are in another. Physically, we can see the relationship between these four nodes. Referring to Figure 1, we see that the path from node 1 to either of nodes 6 or 8 takes us first along an edge with a triangle on one side and a hexagon on the other, and then along an edge with hexagons on both sides. The path to get from node 1 to either of nodes 11 or 12 is the reverse, first a hexagon-hexagon edge and then a triangle-hexagon edge. This, combined with the fact that all four nodes are just two nodes away on their common hexagonal faces, contributes to the idea that they should be in the same layer, despite their different layer connections. To see if this is really the case, we go back to the original definition of layers.

The third method that we investigated was attempting to find the physical isomorphisms that send the nodes of a layer into each other while leaving the starting node invariant. For the three-dimensional polytopes (the polyhedra), the only transformations that leave a point invariant are rotations about an axis containing the point and reflections across a plane containing the point. The axis of rotation and plane of reflection must pass through both the center of the polyhedron and the starting node, which fixes the axis but still leaves one degree of freedom for the plane. For the truncated tetrahedron, the only isomorphism is a reflection about the plane containing the center and nodes 1 and 10. This isomorphism leads to the layers shown in Table 5, which are the same as those achieved through the layer connection method. Since there is no isomorphism that sends nodes 6 and 8 into 11 and 12 while leaving node 1 invariant, we see that the identical equivalent resistances are a coincidence.

Layer Number	Nodes in Layer	Equivalent Resistance (Ω)
1	1	0
2	2, 3	17/30
3	10	7/10
4	6, 8	29/30
5	11, 12	29/30
6	5, 9	16/15
7	4, 7	11/10

Table 5: Layers of Truncated Tetrahedron by Isomorphisms

In this project, we will find layers by first splitting the nodes using the dot product method and then mandating that the layer connections within a layer be identical. For the truncated tetrahedron, this leads to the layers given in Table 5.

The last ingredient we will need is the $l \times l$ layer matrix L, whose entries L_{ij} are equal to the number of nodes in layer j that are connected to each node in layer i. For the truncated tetrahedron, the layer matrix is given in Equation 1. The layer matrix will usually have the property that the i^{th} row will be the same as the $(l-i+1)^{th}$ row, but reversed. This does not hold for the truncated tetrahedron because it lacks antipodes for all its vertices.

$$L = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{1}$$

3 Kirchhoff's Circuit Laws

The first of Kirchhoff's circuit laws dictates that the sum of currents in a network meeting at a node is zero. In other words, the sum of currents flowing into a node is equal to the sum of currents flowing out of the node. Kirchhoff's second law states that the sum of potential differences around a closed loop is also zero. As a loop is traversed, positive potential differences arise from moving through a voltage source from negative to positive or from passing a resistor while moving against the current. Alternatively, negative potential differences occur when moving through a voltage source from positive to negative or from passing a resistor in the direction of the current. The results of both laws for the network shown in Figure 2 are as follows:

Kirchhoff's Current Law: The sum of currents flowing into the right node, i_1, must equal the sum of currents flowing of the node, $i_2 + i_3$. This leads to the equation $i_1 = i_2 + i_3$.

Kirchhoff's Voltage Law: In the first loop, moving clockwise from the right node, there is first a negative potential difference from moving through the resistor R_2 with current i_2, then a positive difference from moving through the source ϵ_1 from negative to positive, and finally a negative difference from moving through the resistor R_1 with current i_1. This leads to the equation $-i_2R_2 + \epsilon_1 - i_1R_1 = 0$.

In the second loop, again moving clockwise from the right node, there is first a negative difference from moving through the resistor R_3 with current i_3, then another negative difference from moving through the source ϵ_2 from positive to negative, another negative difference from doing the same with the source ϵ_1, and finally a positive difference from moving through the resistor R_2 against the current i_2. This leads to the equation $-i_3R_3 - \epsilon_2 - \epsilon_1 + i_2R_2 = 0$.

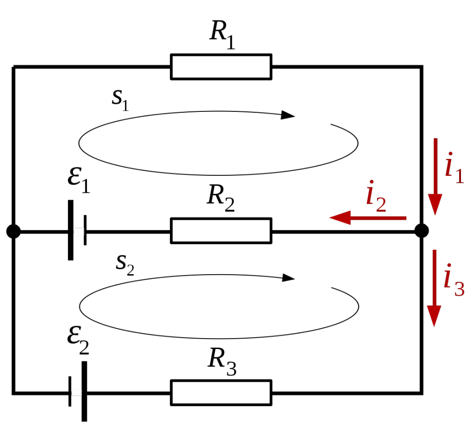


Figure 2: Example Network for Kirchhoff's Laws

Using these equations, if given the values of any five of the eight unknowns, would enable us to solve for the remaining three unknowns. We can use this method to solve for the equivalent resistances of a polytope by first solving for the currents running between layers, assuming that current always flows from layers with lower number to those with higher numbers. We insert a current of 1A into the starting node and take a current of $\frac{1}{H-1}$A out of every other node, where H is the total number of nodes in the network. We can create an equation from Kirchhoff's current law for a node each of the layers, though one of them will not be linearly independent. For the truncated tetrahedron, the currents running between layers are $I_{12}, I_{13}, I_{24}, I_{35}, I_{46}, I_{47}, I_{56}, I_{67}$. In layers 2, 5, and 7, resistors are running between two nodes in the same layer, but since all nodes in a layer are equipotential, there is no current flowing across those resistors and we can neglect them. Since there are eight currents, we need eight linearly independent equations, six of which we get from the current equations, shown below.

$$\begin{aligned}
&1 = 2I_{12} + I_{13} \\
&I_{12} = I_{24} + 1/11 \\
&I_{13} = 2I_{35} + 1/11 \\
&I_{24} = I_{46} + I_{47} + 1/11 \\
&I_{35} = I_{56} + 1/11 \\
&I_{46} + I_{56} = I_{67} + 1/11 \\
&I_{47} + I_{67} = 1/11
\end{aligned} \tag{2}$$

Since only six of these are linearly independent, we must also consider two loop equations. These can be generated using a script written by Jeremy Moody in 2013 [1]. His program, written in Java, has been translated to MATLAB for this project. It works by finding two paths between a pair of layers that pass through different layers on the way and noting that the potential differences caused by them must be equal. For example, to get from layer 1 to layer 6 in the truncated tetrahedron, one could take the layer path $1 \to 3 \to 5 \to 6$ or $1 \to 2 \to 4 \to 6$. Since all the resistors are identical, we know that $I_{13} + I_{35} + I_{56} = I_{12} + I_{24} + I_{46}$. The other loop we can use are the two paths to get from layer 4 to layer 7: $4 \to 7$ and $4 \to 6 \to 7$, yielding the equation $I_{47} = I_{46} + I_{67}$. With all eight equations in hand, we can solve for the currents running between each of the layers, and obtain the results

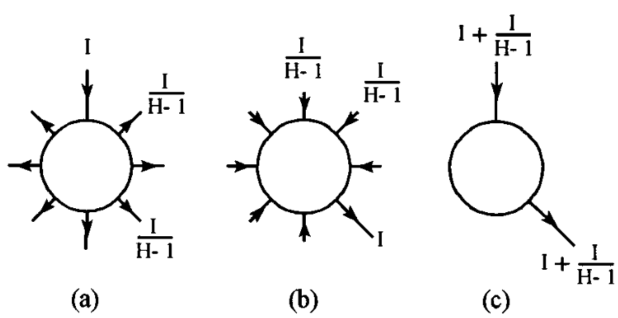
$$\begin{aligned}
&I_{12} = 17/55A \\
&I_{13} = 21/55A \\
&I_{24} = 12/55A \\
&I_{35} = 8/55A \\
&I_{46} = 3/55A \\
&I_{47} = 4/55A \\
&I_{56} = 3/55A \\
&I_{67} = 1/55A
\end{aligned} \tag{3}$$

At this point we consider a different situation where a current of 1A is taken out of a node in the layer we are trying to find the equivalent resistance to and a current of $\frac{1}{H-1}$A is inserted into every other node. When the new system is solved, we have the same values for currents between layers. Superposing the two situations, as seen in Figure 3, we find the the potential difference between the input node and the layer of interest is equal to $2\sum I_{ij}$, where the sum of currents takes a path from the starting node to the ending layer. Since all resistors have the same value of 1Ω, we omit it from the sum. The superposed system has a current of $1+\frac{1}{H-1} = \frac{H}{H-1}$A entering the starting node and coming out of a node in the ending layer, so the equivalent resistance is found by

$$R_i = 2\sum I_{ij}\frac{H-1}{H}. \tag{4}$$

It is now trivial to calculate the equivalent resistances given in Table 5.

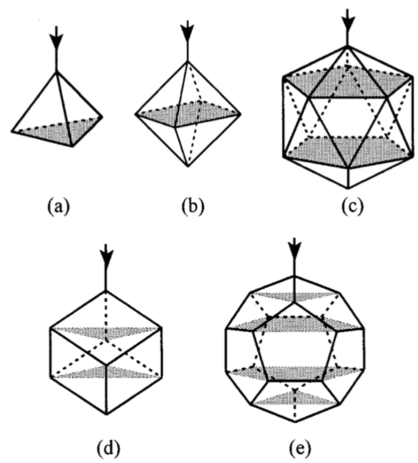
Figure 3: Superposition of currents [2]



4 van Steenwijk's Symmetry Method

By as early as the 1990s, the equivalent resistances had been derived for all five Platonic solids. In 1998, van Steenwijk [2] published a paper outlining his method, which involves noting that, from the perspective of any arbitrary node, the other vertices of the regular solids form equipotential layers, as shown in Figure 4.

Figure 4: Equipotential layers of Platonic solids [2]



The method is very similar to that of Kirchhoff's circuit laws, in that a current of 1A is inserted into one of the nodes (shown as an arrow in Figure 4), and a current of $\frac{1}{H-1}$ is taken out of the remaining $H-1$ vertices. The potential of each of the i equipotential planes is then

$$V_i = \sum_{j=2}^{i} \frac{I_j}{n_j} = \sum_{j=2}^{i} \frac{1}{n_j} \left(1 - \frac{1}{H-1} \sum_{k=2}^{j-1} q_k \right) \tag{5}$$

where I_j denotes the total current flowing between layers $j-1$ and j, n_j is the number of resistors crossing between layers $j-1$ and j, and q_j is the number of vertices in layer j, where the input node is in layer 1 and the bottom layer is N. We have also omitted the 1Ω resistor values. Since the network is symmetric, the fraction $\frac{I_j}{n_j}$ divides the total current moving between layers into the current through just one resistor. This current then gets multiplied by the resistor value to obtain the potential drop over that layer. Since an equal amount of current is taken out at all nodes other than the starting node, and all remaining current is passed on to the next layer, we can calculate the current moving from one layer to the next to be

$$1 - \frac{1}{H-1}\sum_{k=2}^{j-1} q_k.$$

We then apply the same superposition as in Kirchhoff's method, taking current of 1A out of a node in the i^{th} layer and feeding currents of $\frac{1}{H-1}$A into each of the other nodes. This superposition leads to the equation

$$2V_i = \left(1 + \frac{1}{H-1}\right) R_i.$$

Rearranged, the equation becomes

$$R_i = \frac{2(H-1)V_i}{H} = \frac{2}{H}\sum_{j=2}^{i} \frac{1}{n_j}\left(H - 1 - \sum_{k=2}^{j-1} q_k\right).$$

Noting that $q_1 = 1$, the equation can be rewritten to yield

$$R_i = \frac{2}{H}\sum_{j=2}^{i} \frac{1}{n_j}\left(H - \sum_{k=1}^{j-1} q_k\right). \tag{6}$$

A full chart of resistances and other relevant information is shown in Table 6.

	H	E	N	$\{n_2, \ldots, n_N\}$ $\{q_1, \ldots, q_N\}$	R_2	R_3	R_4	R_5	R_6
Tetrahedron	4	6	2	$\{3\}$ $\{1, 3\}$	$\frac{1}{2}$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
Octahedron	6	12	3	$\{4, 4\}$ $\{1, 4, 1\}$	$\frac{5}{12}$	$\frac{1}{2}$	$\ldots$	$\ldots$	$\ldots$
Icosahedron	12	30	4	$\{5, 10, 5\}$ $\{1, 5, 5, 1\}$	$\frac{11}{30}$	$\frac{7}{15}$	$\frac{1}{2}$	$\ldots$	$\ldots$
Cube	8	12	4	$\{3, 6, 3\}$ $\{1, 3, 3, 1\}$	$\frac{7}{12}$	$\frac{3}{4}$	$\frac{5}{6}$	$\ldots$	$\ldots$
Dedecahedron	20	30	6	$\{3, 6, 6, 6, 3\}$ $\{1, 3, 6, 6, 3, 1\}$	$\frac{19}{30}$	$\frac{9}{10}$	$\frac{16}{15}$	$\frac{17}{15}$	$\frac{7}{6}$

Table 6: Equivalent Resistances of the Platonic Solids

There are a few special cases that van Steenwijk covers in his paper. First, the equivalent resistance between two adjacent vertices is

$$R_2 = \frac{2}{H}\frac{1}{n_2}(H - q_1) = \frac{H-1}{E},$$

since the product of the number of vertices, H, with the number of edges coming out of a node n_2, is equal to twice the number of edges, E. Second, the equivalent resistance between opposite vertices (if they exist) is

$$R_N = \sum_{j=2}^{N} \frac{1}{n_j}.$$

Finally, the difference between equivalent resistances of opposite and next-to-opposite vertices is

$$\Delta R_e = R_N - R_{N-1} = \frac{1}{E}.$$

Though van Steenwijk's symmetry method is very useful for the Platonic solids, it has two large drawbacks. First, there are several pieces of information that need to be collected about a network before using the method. The number of nodes and edges is not difficult to obtain, but what is difficult is determining how many nodes are on each layer, and how many edges connect each of the layers. The task of assigning nodes to layers will be addressed shortly. The second drawback is that the method only works if there is no layer skipping. That is, all the current must flow from the first layer to the second, and then to the third, and so forth. Some networks, such as the truncated tetrahedron, involve edges that go straight from the second layer to the fourth, for example. In this case, the calculation of the total current flowing between layers becomes much more difficult.

5 Lovász's Hanging Method

Both this method and the next will involve random walks. A random walk on a graph is a sequence of nodes that begins with a specified node. A step is then taken from the first node to one of its neighboring nodes, and then a step to one of the neighbors of that node, and so forth. From any given node, we assume that there is an equal probability that it will step to any of its neighbors. The random walk can either be infinite or have some terminating condition, such as stepping to a certain node or taking some total number of steps. For example, a random walk of length 7 on the truncated tetrahedron shown in Figure 1 starting at node 1 could be $(1, 3, 6, 5, 4, 7, 4)$.

At this point, we introduce the concept of a hitting time. The hitting time between two nodes a and b is the average number of steps it takes to get from the former to the latter, and is denoted by $H(a, b)$. In general $H(a, b) \neq H(b, a)$. Lovász [3] gives a concise formula for the hitting time between two nodes, given in Equation 7. The degree of a node, $\deg(a)$, is equal to the number of neighbors of that node, and $v \in N(a)$ refers to the neighbors of node a. We define the hitting time of a node to itself to be zero, or $H(a, a) = H(b, b) = 0$, since it takes zero steps to go from a node to itself.

$$H(a, b) = 1 + \frac{1}{\deg(a)} \sum_{v \in N(a)} H(v, b) \tag{7}$$

We will now discuss the rubber band model. In this model, we treat the edges of a network not as resistors, but as rubber bands, or ideal springs with a natural length of zero. In this case the force that the rubber band exerts is directly proportional to its length. The next action we take is to attach to each node a weight equal to the degree of the node. We then pick an input node, suspend it, and let the rest of the network hang freely under the influence of gravity. An example of this model for the cube is shown in Figure 5, which has been horizontally distorted to show the different nodes. Since every node the cube has a degree of three, a weight of three is attached to each node.

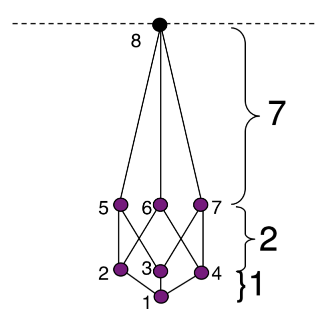


Figure 5: Rubber Band Model of a Cube

We will now show that the vertical distance of any node to the suspended node is equal to its hitting time. Starting from the hitting time equation, we multiply both sides by the degree of node a and obtain

$$\deg(a)H(a,b) = \deg(a) + \sum_{v \in N(a)} H(v,b).$$

Since the sum is over all the neighbors of node a, it is clear that there are $\deg(a)$ terms in the sum, equal to the number of copies of $H(a,b)$ that we are adding together on the left side. We can thus subtract over the sum and obtain

$$\sum_{v \in N(a)} [H(a,b) - H(v,b)] = \deg(a) = W_a, \tag{8}$$

where W_a is the weight attached to node a, and is thus equal to $\deg(a)$. We then split the neighbors of a into those that are above the node and those that are below. The neighbors that are above exert an upward spring force on a, and the neighbors that are below exert a downward spring force. We can ignore any neighbors that are at the same height, since they exert no force at all. Since the rubber bands have a natural length of zero, the difference in height between two nodes is exactly proportional to the force exerted. Equation 8 can then be rearranged such that the upward forces are on the left side and the downward forces are on the right side. The upward forces are

$$\sum F_+ = \sum_{v \in N(a)} [H(a,b) - H(v,b)]$$

such that $H(a,b) > H(v,b)$. The downward forces are

$$\sum F_- = \sum_{v \in N(a)} [H(v,b) - H(a,b)] + W_a$$

such that $H(a,b) < H(v,b)$. These two equations are equal to each other, which can only be true if the hitting time is equal to the distance from the highest, suspended node.

While this is an interesting result if we want to physically model a network, it is not actually necessary to solve for the equivalent resistances. If we imagine that the random walk is being taken by a theoretical positive charge carrier, we can see how the hitting time and the equivalent resistance are both measures for how "hard" it is for charge to go from one point to another. Tetali [5] shows how the hitting times from node x to node y and vice versa are found by

$$H(x,y) = mR_{xy} + \frac{1}{2}\sum_z \deg(z)\,[R_{yz} - R_{xz}]$$

and

$$H(y,x) = mR_{xy} - \frac{1}{2}\sum_{z} \deg(z)\left[R_{yz} - R_{xz}\right],$$

where m is the total number of edges in the network and $R_{xy} = R_{yx}$ is the equivalent resistance between nodes x and y. The networks that we are dealing with are vertex-transitive, so the term $[R_{yz} - R_{xz}]$ will go to zero when summed over all nodes z. Thus we are just left with

$$R_{xy} = \frac{H(x,y)}{m} = \frac{H(y,x)}{m}. \tag{9}$$

We can create a system of equations just using the hitting time equation and solve for the hitting time of every node. The equivalent resistance between any node and the suspended one will be equal to the hitting time of that node, up to a scaling factor. Given that we are using 1Ω resistors, we will need to divide the hitting times by the total number of edges of the network. For the cube shown in Figure 5, the equations are as follows, where the second argument in $H(a,b)$ has been omitted since $b = 8$ for all terms:

$$\begin{aligned}
H(1) &= 1 + \tfrac{1}{3}\left[H(2) + H(3) + H(4)\right]\\
H(2) &= 1 + \tfrac{1}{3}\left[H(1) + H(5) + H(6)\right]\\
H(3) &= 1 + \tfrac{1}{3}\left[H(1) + H(5) + H(7)\right]\\
H(4) &= 1 + \tfrac{1}{3}\left[H(1) + H(6) + H(7)\right]\\
H(5) &= 1 + \tfrac{1}{3}\left[H(2) + H(3) + H(8)\right]\\
H(6) &= 1 + \tfrac{1}{3}\left[H(2) + H(4) + H(8)\right]\\
H(7) &= 1 + \tfrac{1}{3}\left[H(3) + H(4) + H(8)\right]
\end{aligned}$$

We omit $H(8)$ since we know that it is equal to zero, and we can also drop it from the equations of every node that connects to it. We can rearrange the system to obtain

$$\begin{bmatrix}
1 & -1/3 & -1/3 & -1/3 & 0 & 0 & 0\\
-1/3 & 1 & 0 & 0 & -1/3 & -1/3 & 0\\
-1/3 & 0 & 1 & 0 & -1/3 & 0 & -1/3\\
-1/3 & 0 & 0 & 1 & 0 & -1/3 & -1/3\\
0 & -1/3 & -1/3 & 0 & 1 & 0 & 0\\
0 & -1/3 & 0 & -1/3 & 0 & 1 & 0\\
0 & 0 & -1/3 & -1/3 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} H(1)\\ H(2)\\ H(3)\\ H(4)\\ H(5)\\ H(6)\\ H(7) \end{bmatrix}
=
\begin{bmatrix} 1\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1 \end{bmatrix}$$

Solving the system, we find that

$$\begin{aligned}
H(1) &= 10\\
H(2) &= 9\\
H(3) &= 9\\
H(4) &= 9\\
H(5) &= 7\\
H(6) &= 7\\
H(7) &= 7
\end{aligned}$$

Or, dividing by the total number of edges in the cube, 12, we have

$$\begin{aligned}
&R_{1,8} = 5/6\Omega \\
&R_{2,8} = R_{3,8} = R_{4,8} = 3/4\Omega \\
&R_{5,8} = R_{6,8} = R_{7,7} = 7/12\Omega
\end{aligned}$$

This method works, but is fairly computationally expensive, as there is one variable for every node. A much easier computation involves taking advantage of the layers present in the network. Since nodes 2, 3, and 4 are in the same layer, they have identical layer connections and equivalent resistances, and likewise for nodes 5, 6, and 7. We can then simplify the equations, letting $H(2,3,4) = H(2) = H(3) = H(4)$ and $H(5,6,7) = H(5) = H(6) = H(7)$:

$$\begin{aligned}
&H(1) = 1 + \tfrac{1}{3}\left[3H(2,3,4)\right] \\
&H(2,3,4) = 1 + \tfrac{1}{3}\left[H(1) + 2H(5,6,7)\right] \\
&H(5,6,7) = 1 + \tfrac{1}{3}\left[2H(2,3,4) + H(8)\right]
\end{aligned}$$

We can rearrange these equations into the matrix equation

$$\begin{bmatrix} 1 & -1 & 0 \\ -1/3 & 1 & -2/3 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} H(1) \\ H(2,3,4) \\ H(5,6,7) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving the system, we find that

$$\begin{aligned}
&H(1) = 10 \\
&H(2,3,4) = 9 \\
&H(5,6,7) = 7
\end{aligned}$$

Or, dividing by the total number of edges in the cube, 12, we have

$$\begin{aligned}
&R_{1,8} = 5/6\Omega \\
&R_{2,8} = R_{3,8} = R_{4,8} = 3/4\Omega \\
&R_{5,8} = R_{6,8} = R_{7,7} = 7/12\Omega
\end{aligned}$$

This produces the exact same results as before with only one variable for each layer, omitting the starting node. This method can be very useful and not computationally taxing, but relies on the network being vertex-transitive, which is not generally true.

6 Nahin's Random Walk Simulation

The final method that we investigated was adapted from the random walk simulations that Nahin [4] wrote about 2009, with two major changes. The first change is that, where Nahin was trying to find the potentials of various nodes in a node, we are trying to find equivalent resistances. The other change is that Nahin did not take advantage of the layers of the network, which greatly reduces the number of computations needed.

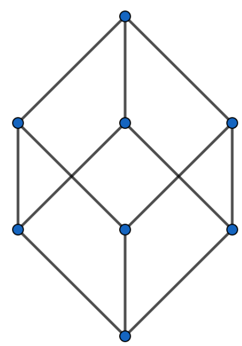


Figure 6: Graph of Cube

The first step is to compute the probability of stepping from each layer in a network to every other layer. Because we are taking a simple random walk where all the resistors have identical values, that probability is equal to the number of edges running between the two layers over the total number of edges connected to the first layer. For example, if we number the layers of the cube shown in Figure 6 from top to bottom as 1 through 4, we can compute the transition matrix P with entries P_{ij} being the probability of stepping from layer i to layer j as

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This essentially sets up a Markov chain, which has been related to the equivalent resistance problems extensively in other papers [6]. Once we have our transition matrix, Nahin details a way to run the random walk simulation, which can be easily adapted to keep count of the number of steps taken. The results for the cube, as well as their deviations from the actual resistances, are shown in Table 7 for a run of five million trials per layer.

Layer Number	Result (Ω)	Actual (Ω)	Actual (Ω)	Error
2	.5832	.5833	7/12	.0224%
3	.7502	.75	3/4	.0220%
4	.8326	.8333	5/6	.0922%

Table 7: Random Walk Results for Cube

For just five millions trials per layer, all of the networks tested both provided accurate results (within 0.1%) and ran fairly quickly. The Platonic solids all ran in under 30 seconds and the Archimedean solids in under 10 minutes. The regular four-dimensional solids had very wide range, with the 5-cell taking under a second and the 120-cell, with 45 layers, taking three and a half hours.

7 Future Research

There are three main topics of future research that follow directly from this project. The first is to find a better method of splitting nodes into layers. As discussed in the section on Symmetrical Equivalent Layers, just computing the dot products between nodes can sometimes lead to undersplitting layers, but considering layer connections can lead to oversplitting. It would be very useful to have another method that can predict when nodes with different layer connections will still have the same resistance, as this would greatly simplify calculations for some networks. For example, the truncated icosidodecahedron, one of the Archimedean solids, has 120 vertices, each with their own layer when considering layer connections. When just looking at the dot product, there are 62 layers, but the final calculation reveals 75 unique resistances. It seems logical that there should be some method to predict the resistance layers before the computation, or that there be some proof that equal resistances are just a coincidence.

The second area of future research is to extend some or all of these methods to the Catalan solids. The Catalan solids are the dual polyhedra of the Archimedean solids, and are notable in that their vertices do not all have equal degree, and thus the solids are not vertex-transitive. This would seem to rule out Lovász's method, which is otherwise the most efficient method, with one variable for every layer. The direct proportionality between hitting time and equivalent resistance does not necessarily hold if the network is not vertex-transitive. Since van Steenwijk's method required that no edges skip a layer, we are left with just using Kirchhoff's circuit laws. Moody [1] details an approach to adapt that method to the Catalan solids but only calculates results for the rhombic dodecahedron and rhombic triacontahedron. A useful addition to a later paper would be to include results for all of the Archimedean and Catalan solids, along with either labeled pictures of the polyhedra, similar to Figure 1, or labeled version of layer diagrams like Figure 6.

The final topic of future research is to investigate why certain networks have a higher or lower range of equivalent resistances than other networks. For example, the resistances for the truncated cube, shown in Appendix B, range from 0.75 to 1.5667, whereas the resistances for the 600-Cell, shown in Appendix C, range from 0.16528 to 0.21164. Is there some measure of the connectivity of a graph that would allow insight into what range of resistances could be expected for a given network? It is obvious that the more connected a graph is, the lower the resistance between any two points is, since there are more possible paths for current to flow, but research into a possible qualitative measure of this property is warranted.

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A Equivalent Resistances of the Platonic Solids

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	3	1/2	0.5

Tetrahedron

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	3	7/12	0.58333
3	3	3/4	0.75
4	1	5/6	0.83333

Cube

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	4	5/12	0.41667
3	1	1/2	0.5

Octahedron

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	3	19/30	0.63333
3	6	9/10	0.9
4	6	16/15	1.0667
5	3	17/15	1.1333
6	1	7/6	1.1667

Dodecahedron

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	5	11/30	0.36667
3	5	7/15	0.46667
4	1	1/2	0.5

Icosahedron

B Equivalent Resistances of Selected Archimedean Solids

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	2	17/30	0.56667
3	1	7/10	0.7
4	2	29/30	0.96667
5	2	29/30	0.96667
6	2	16/15	1.0667
7	2	11/10	1.1

Truncated Tetrahedron

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	4	11/24	0.45833
3	2	7/12	0.58333
4	4	5/8	0.625
5	1	2/3	0.66667

Cuboctahedron

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	1	3/4	0.75
3	2	7/12	0.58333
4	2	13/12	1.0833
5	2	13/12	1.0833
6	2	77/60	1.2833
7	2	4/3	1.3333
8	2	83/60	1.3833
9	2	13/10	1.3
10	2	29/20	1.45
11	2	29/20	1.45
12	1	91/60	1.5167
13	2	31/20	1.55
14	1	47/30	1.5667

Truncated Cube

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	2	281/560	0.50179
3	2	767/1680	0.45655
4	1	51/80	0.6375
5	2	257/420	0.6119
6	2	1133/1680	0.6744
7	2	1133/1680	0.6744
8	2	1229/1680	0.73155
9	2	1229/1680	0.73155
10	1	421/560	0.75179
11	2	323/420	0.76905
12	2	63/80	0.7875
13	2	1343/1680	0.7994
14	1	57/70	0.81429

Rhombicuboctahedron

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	4	29/60	0.48333
3	4	61/90	0.67778
4	4	127/180	0.70556
5	2	7/9	0.77778
6	2	7/9	0.77778
7	4	49/60	0.81667
8	4	38/45	0.84444
9	4	157/180	0.87222
10	1	8/9	0.88889

Icosidodecahedron

C Equivalent Resistances of Selected Regular Four-Dimensional Convex Solids

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	4	2/5	0.4

5-Cell

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	4	15/32	0.46875
3	6	7/12	0.58333
4	4	61/96	0.63542
5	1	2/3	0.66667

8-Cell

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	6	7/24	0.29167
3	1	1/3	0.33333

16-Cell

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	8	23/96	0.23958
3	6	11/40	0.275
4	8	139/480	0.28958
5	1	3/10	0.3

24-Cell

Layer	Nodes in Layer	Resistance (Exact)	Resistance (Decimal)
2	12	119/720	0.16528
3	20	14293/75600	0.18906
4	12	737/3780	0.19497
5	30	1903/9450	0.20138
6	12	37/180	0.20556
7	20	5231/25200	0.20758
8	12	3179/15120	0.21025
9	1	40/189	0.21164

600-Cell