

# Adaptive distributed observers for a class of linear dynamical systems

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MAHDI HEYDARI

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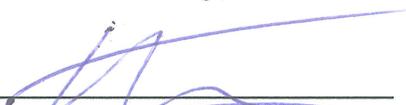
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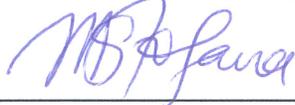
Professor Michael A. Demetriou, Thesis Advisor



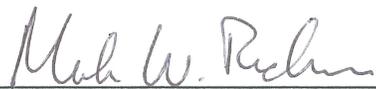
Professor Raghvendra V. Cowlagi, Graduate Committee Member



Professor Nikolaos K. Kazantzis, Graduate Committee Member



Professor Mustapha S. Fofana, Graduate Committee Member



Professor Mark W. Richman, Graduate Committee Representative

# Abstract

The problem of distributed state estimation over a sensor network in which a set of nodes collaboratively estimates the state of continuous-time linear systems is considered. Distributed estimation strategies improve estimation and robustness of the sensors to environmental obstacles and sensor failures in a sensor network. In particular, this dissertation focuses on the benefits of weight adaptation of the interconnection gains in distributed Kalman filters, distributed unknown input observers, and distributed functional observers. To this end, an adaptation strategy is proposed with the adaptive laws derived via a Lyapunov-redesign approach. The justification for the gain adaptation stems from a desire to adapt the pairwise difference of estimates as a function of their agreement, thereby enforcing an interconnection-dependent gain. In the proposed scheme, an adaptive gain for each pairwise difference of the interconnection terms is used in order to address edge-dependent differences in the estimates. Accounting for node-specific differences, a special case of the scheme is presented where it uses a single adaptive gain in each node estimate and which uniformly penalizes all pairwise differences of estimates in the interconnection term. In the case of distributed Kalman filters, the filter gains can be designed either by standard Kalman or Luenberger observers to construct the adaptive distributed Kalman filter or adaptive distributed Luenberger observer. Stability of the schemes has been shown and it is independent of the graph topology and therefore the schemes are applicable to both directed and undirected graphs. The proposed algorithms offer a significant reduction in communication costs associated with information flow by the nodes compared to other distributed Kalman filters. Finally, numerical studies are presented to illustrate the performance and effectiveness of the proposed adaptive distributed Kalman filters, adaptive distributed unknown input observers, and adaptive distributed functional observers.

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# Chapter 1

## Introduction

Wireless sensor network (WSN) refers to a group of spatially dispersed and dedicated sensors, or nodes, that are linked by a wireless medium to monitor and record the physical conditions of the environment in order to perform distributed sensing tasks. Recent advances in micro-electro-mechanical systems (MEMS) technology, wireless communications, and digital electronics have enabled a wide range of applications of WSNs such as environmental monitoring, health and wellness monitoring, military application, biomedical applications, and industrial automation, see [1–3] and references therein. There are many new challenges that have surfaced for the deployment of WSNs, in order to meet the requirements of various applications. One of the challenges is the design of distributed estimation algorithms over WSNs, which has been the subject of many research works in recent years, [4–9] to name a few.

In distributed estimation over networks, a node provides an estimate of a process state using local information and attempts to improve and synchronize its estimate by reaching consensus with the process estimates generated by the other nodes in the network [10]. Distributed estimation strategies improve the estimation and ro-

bustness of the sensors to environmental obstacles and sensor failures in a sensor network compared to the case of a single sensor [4, 11]. They also have advantages over the centralized schemes such as reduced communication bandwidth requirement, increased reliability, and reduced communication cost [4, 11].

A particular problem of interest is the design of distributed estimation algorithms in order to mitigate the uncertainty of each agent's estimation. A prevalent approach concerns the modification of standard Kalman filters by distributed (consensus) protocol, thereby introducing distributed Kalman filters (DKFs).

An early contribution to distributed Kalman filtering [12], required that the global estimate at the previous step be sent from the fusion center to the local sensors. This requirement was relaxed in [13] and [14], where each local processor provided estimates based only on its own measurement and transmitted its estimate and error covariance to the fusion center in order to combine the estimates and associated error covariances and to generate the global estimates. A decentralized Kalman filter has been proposed in [15] for a decentralized control problem and a fully connected network.

Dynamic consensus averaging strategy has been used in [16] and [17], where the nodes of a sensor network use the average of the other sensors' estimates or measurements to construct a distributed Kalman filter. In [17] the associated interconnection weights are related to the covariance matrix of the distributed and non-interacting filters. A distributed Kalman filter algorithm has been proposed in [18] to estimate the state of a sparsely connected large-scale system efficiently. The optimization of the Kalman gain and interconnection weights is considered in [19] for a scalar system. Diffusion strategy has been adopted in [20] and [21] to propose a distributed Kalman filter. In [20] every node of a network shares its data and estimations with its direct neighbors only, and the information is propagated across the network by

the proposed diffusion strategy, while in [21] the covariance data is also incorporated. A gossip-based distributed Kalman filter has been proposed in [22], where each sensor intermittently communicates with a neighbor. The issue of stability and convergence of distributed Kalman filters has been studied in [17, 23, 24] and [25], where the graph is restricted to be undirected to prove the stability of the schemes.

The consensus weight of a network has also designed by other approaches. Weight design of a network has been considered in [26] by using semi-definite convex programming and finding the fastest distributed linear averaging. A distributed minimum variance estimator has been proposed in [27] to track a noisy time-varying signal. The weights of interconnection (consensus) terms were updated adaptively in order to minimize the estimation error variance. For the finite dimensional case, the use of adaptive gains has been applied to the synchronization of complex networks, [28, 29].

Adding another design level addressing robustness of the distributed estimation, the adaptation of the interconnection gains within the consensus protocol proved to be an alternate to optimization of these gains with the obvious savings in computations. A framework for the adaptation of the interconnection gains has been introduced in [7] for infinite dimensional systems with a full connectivity assumption. While the aforementioned works consider optimality of distributed estimation, little attention has been paid toward possible reduction in communication costs due to information exchange amongst nodes. In spite of the fact that distributed Kalman filters reduce estimation error at each agent, they impose huge communication costs and energy requirements on the agents, which make their implementation infeasible in some situations. Therefore, one of the main characteristic of WSNs that must be considered in the design of data collection schemes is the communication cost (thus the energy consumption associated with it).

Descriptor systems (also known as singular systems, or semi-state systems, or differential-algebraic systems, or generalized state-space systems) have also been extensively studied. Descriptor systems can include both dynamic and algebraic equations, as is common in electrical circuits or constrained mechanical systems. Therefore, descriptor systems present a general mathematical framework for the modelling, simulation and control of complex dynamical systems. Such systems have applications in large-scale systems, economic systems, power systems and other areas [30,31]. An important problem in the control design of the descriptor systems is their observer design. The observer design for descriptor systems with known input has been studied in [32–37]. The observer design for systems with unknown inputs has been also widely studied since such systems have applications in the failure detection, fault diagnosis, and synchronization of chaotic systems [38–42]. The observer design for descriptor systems with unknown input has been also studied in [43–45].

Functional observers directly estimate a given linear function of the states without estimating all the systems' states and their existence conditions are weaker than the detectability conditions required by full state observers. Functional observer (FO) design has been considered in [46–51]. The functional observer design for descriptor systems with known input has been proposed by [52]. An approach for the distributed estimation problem using full order functional observers and static feedback has been proposed by [53]. Extending the distributed algorithms to descriptor systems, [54] proposed a distributed algorithm addressing delays.

The adaptation of the interconnection gains in distributed estimation of descriptor systems has not been studied.

## 1.1 Dissertation Contribution

Using the dynamic consensus averaging strategy framework proposed in [17], the first objective of this work is to adaptively adjust the gain of the interconnection term in order to obtain an adaptive interconnection gain and thereby propose an *adaptive-DKF* algorithm for continuous-time linear time-varying (LTV) systems. First the general form of the adaptive-DKF scheme, an *edge-dependent*, is proposed. Then a special case, a *node-dependent*, is proposed. In the edge-dependent case, every node of the network uses different adaptive gains, one for each pairwise disagreement with its neighbors, whereas in the node-dependent case, every node of the network uses a single adaptive gain to penalize all of its pairwise disagreements between state estimates.

A major benefit of the proposed adaptive interconnection weights is the significant reduction in the communication costs associated with information exchange amongst the nodes. Such a communication benefit relies on the assumption that every node is aware of all of its neighbors' sensing models. The proposed adaptive DKFs present a significant savings in the amount of information needed to be transmitted from each node within the network. Additionally the schemes reduce restrictions on the graph topology which makes these schemes applicable for directed graph topology as well.

Continuing, the adaptation of the interconnection gains in distributed estimation of descriptor systems has not been examined. Therefore in this work, the interconnection gain adaptation framework developed by [7] is adapted to the unknown input observer and functional observer design algorithms proposed by [45] and [52], respectively, to arrive at *distributed unknown input observers* and *distributed functional observers* for linear time-invariant descriptor systems, respectively. An

adaptive unknown input observer scheme is proposed, based on an edge-dependent adaptive gain, i.e. every node of the network penalizes the disagreement between its own state estimate and the state estimate of its neighboring nodes differently using separate adaptive gains. The special case where adaptive gains are *node-dependent*, i.e. every given node of the network uses only one adaptive gain to penalize the disagreement between its own state estimate and that of its neighbors', is also presented. Similarly, an adaptive functional observer scheme is proposed, based on an edge-dependent adaptive gain, i.e. every node of the network penalizes the disagreement between its functional estimate and every other functional estimate of its neighboring nodes differently using different adaptive gains. The special case where adaptive gains are node-dependent, i.e. every given node of the network uses only one adaptive gain to penalize the disagreement between its functional estimate and its neighbors' is also presented.

The contribution of this work can be summarized as follows:

1. It proposes novel distributed estimation schemes applicable to Kalman filter sensor networks.
2. It presents adaptive strategies for the interconnected gains of distributed Kalman filters.
3. It reduces significantly the communication costs associated with information exchange amongst the nodes.
4. It proposes distributed schemes applicable to networks whose information exchange is described by directed graphs.
5. It proposes distributed estimation schemes applicable to unknown input observer sensor networks.

6. It presents adaptive strategies for the interconnected gains of distributed unknown input observers.
7. It reduces the estimation error of unknown input observers by applying distributed estimation strategies.
8. It proposes distributed estimation schemes applicable to functional observer sensor networks.
9. It presents adaptive strategies for the interconnected gains of distributed functional observers.
10. It reduces the estimation error of functional observers by applying the distributed estimation strategy.

## 1.2 Dissertation Organization

The remainder of this dissertation is as follows. In the next chapter, the adaptive distributed Kalman filters are proposed and their stability is presented. Communication costs associated with information exchange amongst the nodes has been studied to emphasise the significant reduction in the communication costs.

In Chapter 3, first the preliminaries and formulation of unknown input observers [45, 55] are briefly explained for descriptor linear time-invariant systems. Then, the adaptive distributed unknown input observers (AD-UIO) are proposed and their stability is presented.

In Chapter 4, the formulation of functional observers [50, 52] are briefly explained for descriptor linear time-invariant systems. Then, the adaptive distributed functional observers (ADFO) are proposed and their stability are shown.

Chapter 5 is dedicated to numerical simulations for the proposed adaptive schemes in this dissertation. The numerical simulations for the adaptive distributed Kalman filters, adaptive distributed unknown input observers, and adaptive distributed functional observers are presented in Section 5.1, Section 5.2, and Section 5.3, respectively. Simulation results are provided to demonstrate the performance and effectiveness of the proposed schemes.

Finally, Chapter 6 concludes the dissertation and provides some suggestions for future work. Specially, adaptation of the proposed adaptive schemes to discrete-time distributed Kalman filters is discussed.

## Chapter 2

# Adaptive Distributed Kalman Filters for Linear Time-Varying Systems

In this chapter, two different modifications to distributed Kalman filtering algorithms for sensor networks are proposed. The distributed filters, whether based on Kalman filter or Luenberger observer design, are coupled with terms that penalize the pairwise difference of their estimates. The two adaptive schemes are either node-dependent, in which case all pairwise differences of the state estimates are penalized by the same adaptive weight for every given node uniformly, or edge-dependent in which case the pairwise differences of the state estimates are penalized by different adaptive weights. The significant benefit of the proposed adaptive interconnection weights is described by the communication costs associated with information exchange amongst the nodes.

In a sensor network with distributed estimation, each node can collaborate with its neighbors to improve its own estimation. Therefore, the framework proposed

in [7] for the interconnection gain adaptation is applied to a sensor network of Kalman filters in Theorem 3.1 and Lemma 2.3 to enhance the estimation.

## 2.1 Problem Formulation

The class of systems under consideration is described by the linear time-varying (LTV) system

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (2.1)$$

where  $x(t) \in \mathbb{R}^n$  is the process state and  $w(t) \in \mathbb{R}^n$  represents the process noise which is assumed to be zero-mean white Gaussian noise with covariance matrix  $Q(t)\delta(t - \tau) = E[w(t)w^T(\tau)]$  and distributed by the  $B(t)$  matrix. Process information is obtained from a sensor network containing  $N$  nodes, each of which admits the following sensing model

$$y_i(t) = C_i(t)x(t) + v_i(t), \quad i = 1, \dots, N. \quad (2.2)$$

The matrix  $C_i(t) \in \mathbb{R}^{m_i \times n}$ ,  $m_i < n$ , is the  $i^{\text{th}}$  sensor observation matrix which basically defines its sensing model. The measurement noise of the  $i^{\text{th}}$  sensor is denoted by  $v_i$  which is also assumed to be a zero-mean white Gaussian noise with covariance matrix  $R_i(t)\delta(t - \tau) = E[v_i(t)v_i^T(\tau)]$ .

**Definition 2.1** (Sensor model). *The sensor model is defined as the knowledge of the output matrix  $C_i(t)$  in (2.2).*

The information exchange topology of the sensor network is modeled by a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \{1, \dots, N\}$  and  $\mathcal{E} \subset V \times V$  are the sets of nodes (or vertices) and edges, respectively. The set of incoming neighbors of node  $i$  is defined by  $N_i^I = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$ , where an edge  $(j, i) \in \mathcal{E}$  indicates that a

node  $i$  receives data from node  $j$  and its cardinal number is denoted by  $\aleph_i^I$ . The set of outgoing neighbors of node  $i$  is defined by  $N_i^O = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$ , and its cardinal number is denoted by  $\aleph_i^O$ . Additionally, the graph  $\mathcal{G}$  is also represented by its Laplacian matrix  $\mathcal{L} = \mathcal{D} - \mathcal{A}$ , where  $\mathcal{D}$  and  $\mathcal{A}$  are the degree and adjacency matrices, respectively, [56, 57].

A series of assumptions and a technical lemma will be presented as they are required for the proof of the main theorem in this chapter.

**Assumption 2.1** (Sensor model). *It is assumed that each node  $i$ ,  $\forall i = 1, \dots, N$ , has knowledge of all its neighbors' observation matrices (sensor models)  $C_j(t)$ ,  $\forall j \in N_i^I$ .*

While the above assumption is somewhat conservative, it nonetheless provides a significant reduction in the communication cost associated with the proposed adaptive-DKFs. In the event that all sensors in the network have identical sensing model i.e.  $C_i(t) = C(t)$ ,  $\forall i = 1, \dots, N$ , then Assumption 2.1 becomes redundant. It also should be noted that this knowledge will be satisfied when two nodes  $i$  and  $j$  start communicating with each others and there is no further information transmission required. The proposed theorem and proofs in this chapter are independent of Assumption 2.1 and its contribution is only in reduction of communication costs.

**Assumption 2.2** (Complete observability). *It is assumed that the pairs  $(A(t), C_i(t))$ ,  $i = 1, \dots, N$ , are completely observable.*

**Assumption 2.3** (Complete controllability). *It is assumed that the pair  $(A(t), B(t))$  is completely controllable.*

**Assumption 2.4** (Bounded plant). *The system matrices  $A(t)$ ,  $B(t)$  and  $C_i(t)$ ,  $i = 1, \dots, N$  are appropriately dimensioned real matrices, continuous and bounded over the time interval of interest with  $x(t) \in L^\infty(0, \infty; \mathbb{R}^n)$  for all  $t \geq t_0$ .*

The final assumption concerns the existence of observer gains that would render an associated state observer error stable.

**Assumption 2.5** (Stable distributed decoupled observers). *It is assumed that observer gains  $K_i(t)$  for the pairs  $(A(t), C_i(t))$  exist such that the observers for the noise-free systems*

$$\dot{\hat{x}}_i(t) = \left( A(t) - K_i(t)C_i(t) \right) \hat{x}_i(t) + K_i(t)y_i(t)$$

result in uniformly asymptotically stable error systems  $e_i(t) = x(t) - \hat{x}_i(t)$  described by

$$\dot{e}_i(t) = \left( A(t) - K_i(t)C_i(t) \right) e_i(t), \quad i = 1, \dots, N.$$

The above assumption does not restrict the choice of the observer gain  $K_i(t)$  to be taken from the solution of an associated differential Riccati equation. In fact, it can be based on the observability matrix [58].

The next lemma, taken from [59], provides a time-varying analog to strictly positive real systems.

**Lemma 2.1** ([59]). *Consider an LTV system described by*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned}$$

followed by Assumptions 2.2, 2.3 and 2.4. The system is output strictly passive if there exists a continuous, bounded  $\Pi(t) = \Pi^T(t) \geq \alpha_1 I > 0$  and  $U(t) = U^T(t) \geq$

$\alpha_2 I > 0$ , for all  $t \geq t_0$ , such that

$$\begin{aligned}\dot{\Pi}(t) + A^T(t)\Pi(t) + \Pi(t)A(t) &= -U(t), \\ \Pi(t)B(t) &= C^T(t).\end{aligned}$$

■

Equipped with the above assumptions and lemma, the main result of this chapter can now be introduced which is the design of the *adaptive*-DKF scheme, namely the *edge-dependent* scheme.

In order to minimize repeated long entries, the following compact notation is introduced  $A_i(t) = A(t) - K_i(t)C_i(t)$ , and define the pairwise differences  $\hat{x}_i(t) - \hat{x}_j(t) = \hat{x}_{ij}(t)$ ,  $e_i(t) - e_j(t) = e_{ij}(t)$  with the fact  $\hat{x}_j(t) - \hat{x}_i(t) = e_i(t) - e_j(t) = e_{ij}(t)$ .

## 2.2 Adaptive Distributed Kalman Filters

The proposed adaptive distributed Kalman filters (DKFs) utilize the state estimates obtained by the local (distributed) filters aided by an adjustable weight of the disagreement between them. This adaptive weight is proportional to the distance that a given state estimate  $\hat{x}_i$  has with its neighbors.

The adaptive-DKF based on the edge-dependent strategy is presented first and in which each node  $i$  of the network adaptively adjusts the disagreement between its own state estimate and that of a neighbor node  $j$ ,  $\forall j \in N_i^I$ , using an adaptive gain corresponding the nodes  $i$  and  $j$ .

**Theorem 2.1** (Edge-dependent adaptive strategy). *Consider a sensor network of  $N$  agents with the sensing model (2.2) estimating the states of a LTV dynamical system (2.1). If the following distributed estimation algorithm with a distance-adjusted*

interconnection gain is utilized at each node

$$\dot{\hat{x}}_i(t) = A(t)\hat{x}_i(t) + K_i(t)(y_i(t) - C_i(t)\hat{x}_i(t)) + D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t)C_i(t)\hat{x}_{ji}(t), \quad (2.3a)$$

$$K_i(t) = P_i(t)C_i^T(t)R_i^{-1}(t), \quad (2.3b)$$

$$\dot{P}_i(t) = A(t)P_i(t) + P_i(t)A^T(t) + B(t)Q(t)B^T(t) - K_i(t)R_i(t)K_i^T(t), \quad (2.3c)$$

where the design matrices  $D_i(t)$  are such that the triples  $\{A_i(t), D_i(t), C_i(t)\}$  satisfy the following strictly passive conditions

$$\begin{aligned} \dot{\Pi}_i(t) + A_i^T(t)\Pi_i(t) + \Pi_i(t)A_i(t) &= -U_i(t), \\ \Pi_i(t)D_i(t) &= C_i^T(t) \end{aligned} \quad (2.4)$$

with  $U_i(t) = U_i^T(t) \geq \alpha_3 I > 0$  and  $\Pi_i(t) = \Pi_i^T(t) \geq \alpha_4 I > 0$ , and the adaptive gain matrices  $\Gamma_{ij}(t)$  in (2.3a) are adjusted using the update laws

$$\dot{\Gamma}_{ij}(t) = -(y_i(t) - C_i(t)\hat{x}_i(t)) \left( C_i(t)\hat{x}_{ij}(t) \right)^T, \quad j \in N_i^I, \quad (2.5)$$

then in the system without process and measurement noises, the estimation errors  $e_i(t)$ ,  $i = 1, \dots, N$ ,  $\forall t \geq t_0$ , asymptotically reach zero with all state estimates asymptotically reaching an agreement and all system signals bounded.

**Remark 2.1.** The condition in (2.4) follows from an application of Lemma 2.1 for the triples  $\{A_i(t), D_i(t), C_i(t)\}$ , for all  $i = 1, \dots, N$ . The input matrices  $D_i(t)$  are artificial in the sense that they are generated by  $\Pi_i^{-1}(t)C_i^T(t)$  for all  $i = 1, \dots, N$  with the only requirement being the invertibility of each  $\Pi_i(t)$ .

Instrumental to the proof of the main theorem above, is the following lemma.

**Lemma 2.2.** *Following Theorem 2.1, for a symmetric positive definite  $U_i(t)$ , one has a bounded symmetric positive definite matrix  $\Pi_i(t)$  satisfying the differential Lyapunov equation (2.4) [60].* ■

Now, the proof of Theorem 2.1 can be presented.

*Proof.* The error dynamics of the  $i$ th node is obtained by combining equations (2.1) and (2.3)

$$\dot{e}_i(t) = A_i(t)e_i(t) - D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t)C_i(t)e_{ij}(t). \quad (2.6)$$

In order to examine the stability of the state error equations (2.6) and the adaptation laws (2.5), a Lyapunov-like function is considered

$$V_i(e_i, \Gamma_{ij}) = e_i^T(t)\Pi_i(t)e_i(t) + \sum_{j \in N_i^I} \text{tr} \left( \Gamma_{ij}(t)\Gamma_{ij}^T(t) \right), \quad (2.7)$$

for  $i = 1, \dots, N$ . Then  $\dot{V}_i(e_i, \Gamma_{ij})$  becomes

$$\begin{aligned} \dot{V}_i(e_i, \Gamma_{ij}) &= \dot{e}_i^T(t)\Pi_i(t)e_i(t) + e_i^T(t)\Pi_i(t)\dot{e}_i(t) + e_i^T(t)\dot{\Pi}_i(t)e_i(t) \\ &\quad + \sum_{j \in N_i^I} \text{tr} \left( \dot{\Gamma}_{ij}(t)\Gamma_{ij}^T(t) \right) + \sum_{j \in N_i^I} \text{tr} \left( \Gamma_{ij}(t)\dot{\Gamma}_{ij}^T(t) \right) \\ &= \left( A_i(t)e_i(t) - D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t)C_i(t)e_{ij}(t) \right)^T \Pi_i(t)e_i(t) \\ &\quad + e_i^T(t)\Pi_i(t) \left( A_i(t)e_i(t) - D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t) (C_i(t)e_{ij}(t)) \right) \\ &\quad + e_i^T(t)\dot{\Pi}_i(t)e_i(t) - \sum_{j \in N_i^I} \text{tr} \left( \varepsilon_i(t) \left( C_i(t)\hat{x}_{ij}(t) \right)^T \Gamma_{ij}^T(t) \right) \\ &\quad - \sum_{j \in N_i^I} \text{tr} \left( \Gamma_{ij}(t) \left( \varepsilon_i(t) \left( C_i(t)\hat{x}_{ij}(t) \right)^T \right)^T \right), \end{aligned}$$

where  $\varepsilon_i(t)$ , the  $i$ th *output estimation error* for brevity is defined as

$$\varepsilon_i(t) \triangleq y_i(t) - C_i(t)\hat{x}_i(t) = C_i(t)e_i(t), \quad i = 1, \dots, N.$$

Then  $\dot{V}_i(e_i, \Gamma_{ij})$  can be rewritten as

$$\begin{aligned} \dot{V}_i(e_i, \Gamma_{ij}) &= e_i^T(t)(\dot{\Pi}_i(t) + A_i^T(t)\Pi_i(t) + \Pi_i(t)A_i(t))e_i(t) \\ &\quad \left( D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t)C_i(t)e_{ij}(t) \right)^T \Pi_i(t)e_i(t) \\ &\quad - e_i^T(t)\Pi_i(t) \left( D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t)C_i(t)e_{ij}(t) \right) \\ &\quad - \sum_{j \in N_i^I} \text{tr} \left( \varepsilon_i(t)(C_i(t)\hat{x}_{ij}(t))^T \Gamma_{ij}^T(t) \right) - \sum_{j \in N_i^I} \text{tr} \left( \Gamma_{ij}(t)(\varepsilon_i(t)(C_i(t)\hat{x}_{ij}(t))^T)^T \right). \end{aligned}$$

Since  $\Pi_i(t) \geq \alpha_4 I > 0$ , then one can also define  $D_i(t)$  as

$$D_i(t) = \Pi_i^{-1}(t)C_i^T(t), \quad i = 1, \dots, N. \quad (2.8)$$

Substituting (2.8) and (2.4) into  $\dot{V}_i(e_i, \Gamma_{ij})$  gives

$$\begin{aligned} \dot{V}_i(e_i, \Gamma_{ij}) &= -e_i^T(t)U_i(t)e_i(t) \\ &\quad - \left( \left( \sum_{j \in N_i^I} C_i(t)e_{ij}(t) \right)^T \Gamma_{ij}^T(t)C_i(t)\Pi_i^{-1}(t) \right) \Pi_i e_i(t) \\ &\quad - e_i^T(t)\Pi_i(t) \left( \Pi_i^{-1}(t)C_i^T(t)\Gamma_{ij}(t) \sum_{j \in N_i^I} C_i(t)e_{ij}(t) \right) \\ &\quad + \sum_{j \in N_i^I} \text{tr} \left( \varepsilon_i(t)(C_i(t)e_{ij}(t))^T \Gamma_{ij}^T(t) \right) + \sum_{j \in N_i^I} \text{tr} \left( \Gamma_{ij}(t)C_i(t)e_{ij}(t)\varepsilon_i^T(t) \right) \\ &= -e_i^T(t)U_i(t)e_i(t) \leq 0, \end{aligned}$$

where the identities  $\text{tr}(AB) = \text{tr}(BA)$ ,  $\text{tr}(AB^T) = \text{tr}(BA^T)$  for matrices  $A$  and

$B$ , and  $\text{tr}(x^T y) = \text{tr}(y x^T)$  for column vectors  $x, y$  are used. Since  $U_i(t) = U_i^T(t) \geq \alpha_3 I > 0$ , one has

$$\dot{V}_i(e_i, \Gamma_{ij}) \leq -\alpha_3 \|e_i(t)\|^2 \leq 0, \quad i = 1, \dots, N. \quad (2.9)$$

Using the fact that the plant state is bounded  $x(t) \in L^\infty(0, \infty; \mathbb{R}^n)$  (Assumption 2.4), one has all signals bounded

$$e_i(t), \hat{x}_i(t) \in L^\infty(0, \infty; \mathbb{R}^n), \quad \dot{e}_i(t) \in L^\infty(0, \infty; \mathbb{R}^n), \quad \Gamma_{ij}(t) \in L^\infty(0, \infty; \mathbb{R}^{m \times m}).$$

Additionally, from (2.9) one has that  $e_i(t) \in L^2(0, \infty; \mathbb{R}^n)$  and from (2.5) one has  $\dot{\Gamma}_{ij}(t) \in L^2(0, \infty; \mathbb{R}^{m \times m})$ , and therefore an application of Barbălat's lemma [61, 62] ( $e_i(t) \in L^2 \cap L^\infty, \dot{e}_i(t) \in L^\infty$ ) yields

$$\lim_{t \rightarrow \infty} \|e_i(t)\| = 0, \quad i = 1, \dots, N.$$

□

Please note that for the proof of Theorem 2.1 one does not have to use the collective dynamics in order to establish stability, a condition required for the non-adaptive (standard) case in [17]. The reason is that the coupling terms involving the pairwise differences  $\hat{x}_j(t) - \hat{x}_i(t)$  are canceled out by applying the update laws for  $\Gamma_{ij}(t)$ . Therefore as a notable advantage, the stability of the proposed scheme is not restricted by the graph topology where in [17] the stability was proved only for undirected graphs.

**Remark 2.2.** *The adaptation laws for  $\Gamma_{ij}(t)$  utilize available signals, as both the residuals  $(y_i(t) - C_i(t)\hat{x}_i(t))$  and the pairwise differences  $C_i(t)\hat{x}_{ij}(t)$  can be attained*

since  $y_i(t)$  is measured directly, and  $\hat{x}_i(t)$ ,  $\hat{x}_j(t)$  are generated by each of the distributed filters in (2.3).

**Remark 2.3.** The filter gains  $K_i(t)$  in (2.3a) are not necessarily required to be the standard (non-interconnected) Kalman filter gains. Therefore, one can use a Luenberger observer design [63] instead of Kalman filter design for deterministic systems to construct a new adaptive distributed filter, termed here the adaptive-distributed Luenberger observer and given by

$$\dot{\hat{x}}_i(t) = A(t)\hat{x}_i(t) + L_i(t)(y_i(t) - C_i(t)\hat{x}_i(t)) + D_i(t) \sum_{j \in N_i^I} \Gamma_{ij}(t)C_j(t)\hat{x}_{ji}(t), \quad (2.10)$$

where the  $L_i(t)$  are the Luenberger observer gains. In this case, the state estimation error equations for the non-interconnected case are given  $\dot{e}_i(t) = (A(t) - L_i(t)C_i(t))e_i(t)$ . Following Assumptions 2.2, 2.3 and 2.4, the filter gains  $L_i(t)$  can be chosen to render each of the estimation errors (for the non-interconnected case) uniformly asymptotically stable [63].

**Remark 2.4.** Adding and subtracting  $K_i(t)R_i(t)K_i^T(t)$  to the filter Riccati equation (2.3c) and using (2.3b), then the following is obtained

$$-\dot{P}_i(t) + A_i(t)P_i(t) + P_i(t)A_i^T(t) = -K_i(t)R_i(t)K_i^T(t) - B(t)Q(t)B^T(t).$$

Using the complete observability and controllability conditions (Assumptions 2.2 and 2.3), the error covariance matrices  $P_i(t)$  are symmetric and bounded and there exist positive constants  $\alpha_5$  and  $\alpha_6$  such that [60]

$$0 < \alpha_5 I \leq P_i(t) \leq \alpha_6 I, \quad \forall t > t_0, \quad i = 1, \dots, N.$$

Therefore,  $P_i^{-1}(t)$ ,  $i = 1, \dots, N$  exist and are symmetric positive definite matrices.

Thus, pre- and post- multiplication by  $P_i^{-1}(t)$  gives

$$\begin{aligned} & -P_i^{-1}(t)\dot{P}_i(t)P_i^{-1}(t) + P_i^{-1}(t)A_i(t) + A_i^T(t)P_i^{-1}(t) = \\ & -C_i^T(t)R_i^{-1}(t)C_i(t) - P_i^{-1}(t)B(t)Q(t)B^T(t)P_i^{-1}(t), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \frac{d}{dt}(P_i^{-1}(t)) + P_i^{-1}(t)A_i(t) + A_i^T(t)P_i^{-1}(t) &= -C_i^T(t)R_i^{-1}(t)C_i(t) \\ & -P_i^{-1}(t)B(t)Q(t)B^T(t)P_i^{-1}(t). \end{aligned}$$

This means that  $P_i^{-1}(t)$  satisfies (2.4) and therefore it may be considered as a possible choice of  $\Pi_i(t)$  in the Lyapunov-like function (2.7) required to establish stability.

With this choice, the matrices  $D_i(t)$  can then be expressed in terms of the error covariance matrices  $D_i(t) = P_i(t)C_i^T(t)$ ,  $i = 1, \dots, N$ . While  $P_i^{-1}(t)$  is a good choice for  $\Pi_i(t)$ , one can still use a different  $\Pi_i(t)$  satisfying (2.4) in order to establish stability.

**Remark 2.5.** Following Remark 2.4, one may also choose  $\frac{1}{\gamma}P_i^{-1}(t)$ ,  $\gamma > 0$  as a possible choice for  $\Pi_i(t)$  in (2.7) in order to establish stability. With this choice of  $\Pi_i(t)$ , the matrix  $D_i(t)$  may then be expressed as  $D_i(t) = \gamma P_i(t)C_i^T(t)$ ,  $i = 1, \dots, N$ . Therefore, one can easily adjust the effect of the interconnection term on the estimation by the choice of  $\gamma$ .

**Remark 2.6.** In the event that the Lyapunov matrix  $\Pi_i(t)$  is chosen in terms of the filter Riccati solution as  $\Pi_i(t) = P_i^{-1}(t)$ , then through the appropriate adaptive gains, the matrices  $D_i(t)$  can be conveniently chosen identical to  $K_i(t)$  as  $D_i(t) = P_i(t)C_i^T(t)R_i^{-1}(t)$ ,  $i = 1, \dots, N$ . In this case, the adaptive law for  $\Gamma_{ij}(t)$  is modified

to

$$\dot{\Gamma}_{ij}(t) = -R_i^{-1}(t)(y_i(t) - C_i(t)\hat{x}_i(t))\left(C_i(t)\hat{x}_{ij}(t)\right)^T, \quad j \in N_i^I.$$

**Remark 2.7.** *The adaptive law in (2.5) can be modified as*

$$\dot{\Gamma}_{ij}(t) = -\beta(y_i(t) - C_i(t)\hat{x}_i(t))\left(C_i(t)\hat{x}_{ij}(t)\right)^T, \quad j \in N_i^I, \quad \beta > 0,$$

by choosing the Lyapunov-like function

$$V_i(e_i, \Gamma_{ij}) = e_i^T(t)\Pi_i(t)e_i(t) + \frac{1}{\beta} \sum_{j \in N_i^I} \text{tr} \left( \Gamma_{ij}(t)\Gamma_{ij}^T(t) \right).$$

In Theorem 2.1, the distance between the state estimate at node  $i$  and the estimate at node  $j$ ,  $\forall j \in N_i^I$ , is penalized independently using different  $\Gamma_{ij}(t)$ . Consequently, the gain  $\Gamma_{ij}(t)$  is called *edge-dependent* adaptive gain. In a special case, one can use an identical gain  $\Gamma_{ij}(t)$  for all nodes  $j$ ,  $\forall j \in N_i^I$ . Thus, one can move  $\Gamma_{ij}(t)$  outside the summation and make it an *node-dependent* (also *vertex-dependent*) adaptive gain  $\Gamma_i(t)$ . In order to have a more realistic adaptive scheme, one can move  $\Gamma_i(t)$  inside the summation and make it an *edge-dependent* adaptive gain  $\Gamma_{ij}(t)$ . Therefore, the distance between the state estimate at node  $i$  and the estimate at node  $j$ ,  $\forall j \in N_i^I$ , is penalized with the same adaptive gain  $\Gamma_i(t)$ . The adaptive-DKFs with *node-dependent* adaptive gains of the pairwise differences can now be presented.

**Lemma 2.3** (Node-dependent adaptive strategy). *If the adaptive interconnection*

weights are chosen as node-dependent, then the adaptive-DKF's are given by

$$\begin{aligned}
\dot{\hat{x}}_i(t) &= A(t)\hat{x}_i(t) + K_i(t)(y_i(t) - C_i(t)\hat{x}_i(t)) + D_i(t)\Gamma_i(t) \sum_{j \in N_i^I} C_i(t)\hat{x}_{ji}(t) \\
K_i(t) &= P_i(t)C_i^T(t)R_i^{-1}(t), \\
\dot{P}_i(t) &= A(t)P_i(t) + P_i(t)A^T(t) + B(t)Q(t)B^T(t) - K_i(t)R_i(t)K_i^T(t), \\
D_i(t) &= \Pi_i^{-1}(t)C_i^T(t), \\
\dot{\Gamma}_i(t) &= -\varepsilon_i(t) \sum_{j \in N_i^I} \left( C_i(t)\hat{x}_{ij}(t) \right)^T,
\end{aligned} \tag{2.11}$$

In equation (2.11) above, the matrices  $\Pi_i(t)$  are given in (2.4). However, they can be chosen as  $P_i^{-1}(t)$  or  $\frac{1}{\gamma}P_i^{-1}(t)$  using Remark 2.4 or Remark 2.9, respectively. To proof Lemma 2.3, lyapunov-like function  $V_i(e_i, \Gamma_i) = e_i^T(t)\Pi_i(t)e_i(t) + \text{tr}(\Gamma_i(t)\Gamma_i^T(t))$  can be used. The rest of the proof is very similar to that for Theorem 2.1 and is therefore omitted.

**Remark 2.8** (Node-independent adaptive gains). *When the interconnection weights in Theorem 2.1 are allowed to be node-independent, then the adaptive-DKFs become*

$$\begin{aligned}
\dot{\hat{x}}_i(t) &= A(t)\hat{x}_i(t) + K_i(t)(y_i(t) - C_i(t)\hat{x}_i(t)) + D_i(t)\Gamma(t) \sum_{j \in N_i} C_i(t)\hat{x}_{ji}(t), \\
K_i(t) &= P_i(t)C_i^T(t)R_i^{-1}(t), \\
\dot{P}_i(t) &= A(t)P_i(t) + P_i(t)A^T(t) + B(t)Q(t)B^T(t) - K_i(t)R_i(t)K_i^T(t), \\
D_i(t) &= \Pi_i^{-1}(t)C_i^T(t), \\
\dot{\Gamma}(t) &= -\sum_{i=1}^N \varepsilon_i \sum_{j \in N_i} (C_i\hat{x}_{ij}(t))^T.
\end{aligned} \tag{2.12}$$

### 2.2.1 Special case: time-invariant systems

For the linear time-invariant (LTI) system, the system in (2.1) and (2.2) are given as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \\ y_i(t) &= C_i x(t) + v_i(t), \quad i = 1, \dots, N. \end{aligned} \tag{2.13}$$

In order to present the adaptive distributed Kalman filters for LTI system (2.13), analogous assumptions to assumptions 2.1, 2.2 and 2.4 for LTI matrices  $A$ ,  $B$  and  $C_i$  are required. Assumptions 2.3 and 2.5 for LTI systems become redundant.

**Remark 2.9.** *A consequence of Assumption 2.2 for the LTI system (2.13) is that there exist filter gains  $K_i$  such that  $A_i \triangleq A - K_i C_i$  is Hurwitz [64]. And for  $U_i = U_i^T > 0$  and  $A_i$  Hurwitz, the solution  $\Pi_i$  to the Lyapunov equation*

$$A_i^T \Pi_i + \Pi_i A_i = -U_i, \tag{2.14}$$

*is a symmetric positive definite matrix (spd) [65].*

**Lemma 2.4 (Edge-dependent adaptive-DKF for LTI systems).** *Consider a LTI sensor network in (2.13), then the edge-dependent adaptive-DKF in Theorem 2.1 is simplified as*

$$\begin{aligned} \hat{\hat{x}}_i(t) &= A\hat{x}_i(t) + K_i(y_i(t) - C_i\hat{x}_i(t)) + D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) C_i \hat{x}_{j_i}(t), \\ K_i &= P_i C_i^T R_i^{-1}, \\ 0 &= AP_i + P_i A^T + BQB^T - K_i R_i K_i^T, \\ \dot{\Gamma}_{ij}(t) &= -(y_i(t) - C_i\hat{x}_i(t))(C_i\hat{x}_{j_i}(t))^T, \quad j \in N_i^I, \end{aligned} \tag{2.15}$$

where the matrices  $D_i$  are chosen to satisfy the “artificial” Lur’e condition [61]

$$\begin{aligned} A_i^T \Pi_i + \Pi_i A_i &= -U_i, \\ \Pi_i D_i &= C_i^T, \end{aligned} \tag{2.16}$$

with  $U_i = U_i^T > 0$ . Then the collective dynamics of the errors  $e_i(t)$  (without noise) forms a stable linear system, all the estimators asymptotically reach an agreement (consensus) and all signals in the system are bounded.

## 2.3 Cost of transmit/receive

If knowledge of all sensor models, as required by Assumption 2.1, is considered stringent, then relaxation of such an assumption requires transmission of additional information. To demonstrate this cost better, in this case the node-dependent adaptive-DKFs are rewritten as

$$\hat{\hat{x}}_i(t) = A(t)\hat{x}_i(t) + K_i(t)(y_i(t) - C_i(t)\hat{x}_i(t)) + D_i(t)\Gamma_i(t)C_i(t) \sum_{j \in N_i} \hat{x}_{ji}(t). \tag{2.17}$$

It can be seen that the matrix  $C_i(t)$  in (2.17) above is placed outside the summation, whereas in (2.11) it is kept inside the summation in order to benefit from the advantage of Assumption 2.1. Comparing (2.11) and (2.17), it can be easily observed that the  $i$ th filter in (2.17) needs to receive the  $n$ -dimensional vectors  $\hat{x}_j(t)$ ,  $\forall j \in N_i^I$ , while, the filter in (2.11) needs to receive the  $m_i$ -dimensional vectors  $C_i(t)\hat{x}_j(t)$ ,  $\forall j \in N_i^I$ . The transmitting costs associated with (2.11) and (2.17) can be summarized below:

- In (2.11), i.e. given Assumption 2.1, each node  $i$  multiplies its own state  $\hat{x}_i(t)$  by all  $C_j(t)$ ,  $\forall j \in N_i^O$  and transmits the  $\aleph_i^O$  messages containing the

$m_j$ -dimensional vectors  $C_j(t)\widehat{x}_i(t)$ , to the corresponding  $j$  nodes.

- In (2.17), i.e. without Assumption 2.1, each node  $i$  transmits the  $\aleph_i^O$  messages containing the  $n$ -dimensional vector  $\widehat{x}_i(t)$  to all  $j$  nodes,  $\forall j \in N_i^O$ . Note that here  $n > m_j$ .

To appreciate the reduction in communication costs, consider the simplest case of a scalar measurement  $m_i = m = 1, \forall i = 1, \dots, N$ . If the adaptive-DKF in (2.11) (i.e. with Assumption 2.1) is used, then each node  $i$  must transmit the scalar  $C_j(t)\widehat{x}_i(t)$  to all its neighbors  $j$ ,  $\forall j \in N_i^O$ , and receive the scalar  $C_i(t)\widehat{x}_j(t)$  from all its neighbors  $j$ ,  $\forall j \in N_i^I$  (i.e.  $\aleph_i^I$  scalar data must be received and  $\aleph_i^O$  scalar data must be transmit). However, if the adaptive-DKF in (2.17) (i.e. without Assumption 2.1) is used, then the  $i$ th node must transmit the  $n$ -dimensional vector  $\widehat{x}_i(t)$  to all its neighbors  $j$ ,  $\forall j \in N_i^O$ , and receive the  $n$ -dimensional vectors  $\widehat{x}_j(t)$  from all its neighbors  $j$ ,  $\forall j \in N_i^I$  (i.e.  $\aleph_i^I \times n$ -dimensional vector data must be received and  $\aleph_i^O \times n$ -dimensional vector data must be transmitted).

It must be noted that in the non-adaptive DKF [17], the communication cost is the same as that of the adaptive-DKF in (2.17) (i.e. without Assumption 2.1). In this case, each node  $i$  requires to transmit and receive the same amount of data as in (2.17). In other words, it must transmit the  $n$ -dimensional vector  $\widehat{x}_i(t)$  to all its neighbors  $j$ ,  $j \in N_i^O$  and receive the  $n$ -dimensional vectors  $\widehat{x}_j(t)$  from all its neighbors  $j$ ,  $\forall j \in N_i^I$ .

One way to quantify the advantage of the proposed adaptive-DKFs in the communication costs is via the specific protocol of the wireless sensor network (WSN). It is assumed that the WSN follows the media access control (MAC) protocol based on the Zigbee<sup>®</sup>/IEEE 802.15.4 standard [66]. IEEE 802.15.4 is a standard for low-rate wireless personal area networks (WPANs) to define the physical layer (PHY) and

media access control (MAC). ZigBee<sup>®</sup> is a specification based on the IEEE 802.15.4 protocol which defines higher layer than PHY and MAC. Zigbee<sup>®</sup>/IEEE 802.15.4 is a commonly used standard in WSNs due to its advantages including low-cost, low-power, industrialized standard, security, reliability, and capability with large-scale WSNs. Additionally, it is capable with large number of nodes which make it the most suitable standard for sensors and control devices.

The general format of MAC frame is shown in Figure 2.1 [67]. If it is assumed that the WSN uses the MAC frames with a 5 bytes MAC header and 2 bytes footer [66, 67] and 4 bytes for the transmitting data, then with Assumption 2.1, node  $j$  transmits  $7 + 4m_i$  bytes data to node  $i$  instead of transmitting  $7 + 4n$  bytes without Assumption 2.1 at every communication step. The latter applies to the non-adaptive (standard) DKF [17]. These savings in communication costs are summarized in Table 2.1. It is clear that for the case of  $n \gg m_i$ , this saving is more prominent. Obviously in the case of identical sensor models, i.e.  $C_i = C$ , the proposed schemes propose this reduction in communication cost without having the constraint of Assumption 2.1.

Table 2.1: Communication cost associated with Assumption 2.1

requiring Assumption 2.1?	data transmitted (bytes), $n > m_i$
Yes	$7 + 4m_i$
No	$7 + 4n$

Bytes: 2	1	0-20	variable	2
Frame Control	Sequence number	Address info	Payload	Frame check Payload sequence
MAC header			MAC service data unit	MAC footer
MAC protocol data unit				

Figure 2.1: The general structure of MAC frame [67].

# Chapter 3

## Adaptive Distributed Unknown Input Observers for Linear Time-Invariant Descriptor Systems

In this chapter, the local (non-interacting) unknown input observer (UIO) for a linear time invariant descriptor system, as taken from [45], is first summarized and subsequently is modified for distributed (interconnected) systems. The descriptor system is assumed to have multiple outputs provided by a sensor network and for each system corresponding to a different output, a different UIO is designed. The result is a network of distributed and non-interacting UIOs. A special case of the distributed non-interacting UIOs is when the descriptor system is indeed a regular system (the singular matrix in front of the time derivative becomes the identity matrix), and in this case the distributed and non-interacting UIOs reduce to distributed non-interacting observers.

In a sensor network with distributed estimation, each node can collaborate with its neighbors to improve its own estimation. Therefore, the framework proposed in [7] for the interconnection gain adaptation is applied to a sensor network of the unknown input observers in Theorems 3.1 and 3.2 to enhance the estimation.

### 3.1 Preliminaries and Problem statement

Consider the class of systems described by the linear time-invariant (LTI) descriptor system:

$$E^* \dot{x}(t) = A^* x(t) + B^* u(t) + F^* v(t) \quad (3.1)$$

where the state  $x(t) \in \mathbb{R}^n$ , the known input  $u(t) \in \mathbb{R}^k$ , the unknown input  $v(t) \in \mathbb{R}^q$ , the matrices  $E^*, A^* \in \mathbb{R}^{m \times n}$ ,  $B^* \in \mathbb{R}^{m \times k}$ , and  $F^* \in \mathbb{R}^{m \times q}$  are known. Also, it is assumed that  $\text{rank}(E^*) = r \leq n$ . The process information is obtained by a sensor network containing  $N$  nodes, where each node  $i$  has the following sensing model:

$$y_i^*(t) = C_i^* x(t) + G_i^* v, \quad y_i^* \in \mathbb{R}^{p_i}, \quad i = 1, \dots, N. \quad (3.2)$$

where the matrices  $C_i^*$  and  $G_i^*$  are known and of appropriate sizes and it is assumed that  $\text{rank}([C_i^* \quad G_i^*]) = p_i \leq n$ .

**Assumption 3.1.** *It is assumed that  $\text{rank}\left(\begin{bmatrix} F^* \\ G_i^* \end{bmatrix}\right) = q \leq p_i$ .*

**Assumption 3.2.** *It is assumed that*

$$\text{rank}\left(\begin{bmatrix} E^* & A^* & F^* & 0 \\ 0 & E^* & 0 & F^* \\ 0 & C_i^* & G_i^* & 0 \\ 0 & 0 & 0 & G_i^* \end{bmatrix}\right) - \text{rank}\left(\begin{bmatrix} E^* & F^* \\ 0 & G_i^* \end{bmatrix}\right) = n + q.$$

**Assumption 3.3.** *It is assumed that*  $\text{rank}\left(\begin{bmatrix} sE^* - A^* & -F^* \\ C_i^* & G_i^* \end{bmatrix}\right) = n + q, \quad \forall s \in \mathbb{C}, \text{Re}(s) \geq 0.$

There exists a nonsingular matrix  $P$  such that the system described in (3.1) and (3.2) is restricted system equivalent to [45]

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Fv(t) \\ y_i(t) &= C_i x(t) + G_i v, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} PE^* &= \begin{bmatrix} E \\ 0 \end{bmatrix}, & PA^* &= \begin{bmatrix} A \\ A_1 \end{bmatrix}, & PB^* &= \begin{bmatrix} B \\ B_1 \end{bmatrix}, & PF^* &= \begin{bmatrix} F \\ F_1 \end{bmatrix}, \\ y_i(t) &= \begin{bmatrix} -B_1 u \\ y_i^*(t) \end{bmatrix}, & C_i &= \begin{bmatrix} A_1 \\ C_i^* \end{bmatrix}, & G_i &= \begin{bmatrix} F_1 \\ G_i^* \end{bmatrix}, \end{aligned}$$

in which  $E \in \mathbb{R}^{r \times n}$ ,  $\text{rank}(E) = r$ ,  $y_i \in \mathbb{R}^{t_i}$ ,  $C_i \in \mathbb{R}^{t_i \times n}$ ,  $G_i \in \mathbb{R}^{t_i \times q}$ ,  $\text{rank}(G_i) = s_i \leq q$ , and  $t_i = m + p_i - r$ .

Then, (3.3) can be transformed to

$$\begin{aligned}
E\dot{x}(t) &= \Phi_i x(t) + Bu(t) + F_i^{11}y_i^1(t) + F_i^{12}v_i^2(t) \\
y_i^1(t) &= C_i^1 x(t) + v_i^1(t) \\
y_i^2(t) &= C_i^2 x(t),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\begin{bmatrix} y_i^1 \\ y_i^2 \end{bmatrix} &= R_i y_i, & \begin{bmatrix} C_i^1 \\ C_i^2 \end{bmatrix} &= R_i C_i, & \begin{bmatrix} v_i^1 \\ v_i^2 \end{bmatrix} &= S_i^{-1} v, \\
[F_i^{11} \quad F_i^{12}] &= F S_i, & \Phi_i &= A - F_i^{11} C_i^1,
\end{aligned}$$

in which  $v_i^1 \in \mathbb{R}^{s_i}$ ,  $v_i^2 \in \mathbb{R}^{q-s_i}$ ,  $y_i^1 \in \mathbb{R}^{s_i}$ ,  $y_i^2 \in \mathbb{R}^{p_i-s_i}$ ,  $\text{rank}(F_i^{12}) = q - s_i$ , and  $\text{rank}(C_i^2) = t_i - s_i$ .  $R_i$  and  $S_i$  are two nonsingular matrices such that

$$R_i G_i S_i = \begin{bmatrix} I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}.$$

**Theorem 3.1** (UIO for descriptor system [45]). *For each node  $i$  of the system described by (3.1) and (3.2), the reduced-order unknown input observer given by:*

$$\begin{aligned}
\dot{z}_i(t) &= \Lambda_i z_i(t) + L_i^1 y_i^1(t) + L_i^2 y_i^2(t) + H_i u(t) \\
\hat{x}_i(t) &= M_i z_i(t) + N_i y_i^2(t),
\end{aligned} \tag{3.5}$$

*exists and is asymptotically stable.  $z_i(t) \in \mathbb{R}^{n+s_i-t_i}$  and  $\hat{x}_i(t)$  are the observer state and the estimate of  $x(t)$ , respectively. The matrices  $\Lambda_i$ ,  $L_i^1$ ,  $L_i^2$ ,  $H_i$ ,  $M_i$ , and  $N_i$  are*

obtained by

$$\begin{aligned}
M_i &= \begin{bmatrix} Q_i \\ C_i^2 \end{bmatrix}^{-1} \begin{bmatrix} I_{(n+s_i-t_i)} \\ 0 \end{bmatrix} \\
\Delta_i &= (E^T E + C_i^{2T} C_i^2)^{-1} \\
\phi_i &= \begin{bmatrix} I - E \Delta_i E^T \\ -C_i^2 \Delta_i E^T \end{bmatrix} \\
\alpha_i &= I - F_i^{12} (\phi_i F_i^{12})^\dagger \phi_i \\
\Omega_i &= Q_i \Delta_i E^T \alpha_i \Phi_i M_i \\
\Theta_i &= \phi_i \alpha_i \Phi_i M_i \\
T_i &= (Q_i \Delta_i E^T + Z_i \phi_i) \alpha_i \\
N_i &= \begin{bmatrix} T_i E \\ C_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_{(t_i-s_i)} \end{bmatrix} \\
H_i &= T_i B \\
L_i^1 &= T_i F_i^{11} \\
L_i^2 &= T_i \Phi_i N_i
\end{aligned} \tag{3.6}$$

where  $Q_i \in \mathbb{R}^{(n+s_i-t_i) \times n}$  is defined arbitrarily such that  $\begin{bmatrix} Q_i \\ C_i^2 \end{bmatrix}$  to be nonsingular and the matrix  $Z_i$  is chosen arbitrarily such that  $\Lambda_i = \Omega_i + Z_i \Theta_i$  to be Hurwitz. Additionally,  $A^\dagger$  denotes the Moore-Penrose generalized inverse of matrix  $A$  [68].

In the special case of  $E^* = I$  and  $G^* = 0$ , the descriptor system defined in (3.1)

and (3.2) can be rewritten in the form of following full-order LTI system

$$\begin{aligned}\dot{x}(t) &= A^*x(t) + B^*u(t) + F^*v(t) \\ y_i(t) &= C_i^*x(t), \quad y_i \in \mathbb{R}^{p_i}, \quad i = 1, \dots, N.\end{aligned}\tag{3.7}$$

In this case Assumption 3.1 is simplified to  $\text{rank}(F^*) = q \leq p_i$  and Assumption 3.2 becomes equivalent to the condition of  $\text{rank}(C_i^*F^*) = q$ .

**Theorem 3.2** (UIO for full-order systems [55]). *For each node  $i$  of the system described by (3.7), the full-order observer given by*

$$\begin{aligned}\dot{z}_i(t) &= (P_iA^* - K_iC_i^*)z_i(t) + L_iy_i(t) + H_i^*u(t) \\ \hat{x}_i(t) &= z_i(t) + J_iy_i(t),\end{aligned}\tag{3.8}$$

*exists and is asymptotically stable if and only if the pair  $(P_iA^*, C_i^*)$  is observable and  $\text{rank}(C_i^*F^*) = q_i$ . Then, there exists a matrix  $K_i$  such that the matrix  $(P_iA^* - K_iC_i^*)$  is Hurwitz. The matrices  $J_i$ ,  $P_i$ ,  $H_i$  and  $L_i$  are given by*

$$\begin{aligned}J_iC_i^*F^* &= F^* \\ P_i &= I + J_iC_i^* \\ H_i &= P_iB^* \\ L_i &= K_i(I + C_i^*J_i) - P_iA^*J_i.\end{aligned}\tag{3.9}$$

In order to prove the theorems presented in the next section, the following assumption is also required.

**Assumption 3.4** (Bounded plant). *The class of systems (3.1), (3.2) is such that  $x \in L^\infty(0, \infty; \mathbb{R}^n)$  and  $y_i^* \in L^\infty(0, \infty; \mathbb{R}^{p_i})$ .*

Information exchange between nodes of a sensor network is modeled by a directed graph defined in Chapter 2.

Now the main result of this chapter can be introduced which deals with the design of an adaptive distributed UIO (AD-UIOs) scheme, in which the distributed (interacting) UIOs implement an adaptation in their consensus protocol.

## 3.2 Adaptive Distributed Unknown Input Observers (AD-UIO)

The proposed adaptive distributed UIOs using the edge-dependent adaptive gain strategy are presented in this section. The interaction of the distributed UIOs summarized in the previous section take the form of a consensus protocol that adjusts the consensus gains adaptively. Each node  $i$  of the network adaptively adjusts the disagreement between its own state estimate and the state estimates of all the communicating nodes  $j, \forall j \in N_i^I$ . This is achieved via the use of a different adaptive gain corresponding to nodes  $i$  and  $j$ .

**Theorem 3.3** (AD-UIO for descriptor systems). *Consider a sensor network with the sensing model (3.2) estimating the states of the system (3.1). The following distributed estimation algorithm with an edge-dependent interconnected gain is utilized at each node*

$$\begin{aligned} \dot{z}_i(t) &= \Lambda_i z_i(t) + L_i^1 y_i^1(t) + L_i^2 y_i^2(t) + H_i u(t) + \Pi_i^{-1} D_i^T \sum_{j \in N_i^I} \Gamma_{ij}(t) (\hat{x}_j(t) - \hat{x}_i(t)) \\ \hat{x}_i(t) &= M_i z_i(t) + N_i y_i^2(t), \end{aligned} \tag{3.10}$$

where the matrix  $\Pi_i = \Pi_i^T > 0$  is the solution to the Lyapunov equation

$$\Lambda_i^T \Pi_i + \Pi_i \Lambda_i = -U_i, \quad U_i = U_i^T > 0, \quad (3.11)$$

$D_i = C_i^2(T_i E)^+$  and the adaptive gain matrix  $\Gamma_{ij}(t)$  is adjusted using the following adaptive law

$$\dot{\Gamma}_{ij}(t) = -\gamma(D_i z_i(t) - y_i^2(t))(\hat{x}_j(t) - \hat{x}_i(t))^T, \quad i = 1, \dots, N, \quad j \in N_i^I, \quad (3.12)$$

where  $\gamma$  is an arbitrary positive real number. Then, the estimation error, defined as  $e_i(t) \triangleq x(t) - \hat{x}_i(t)$ ,  $i = 1, \dots, N$ , asymptotically reaches to zero and therefore, all the estimators asymptotically reach an agreement and all system signals are bounded.

*Proof.* Let us define  $\epsilon_i(t) \triangleq z_i(t) - T_i E x(t)$ , then the estimation error at the node  $i$  is given by

$$e_i(t) = \hat{x}_i(t) - x(t) = M_i \epsilon_i(t), \quad (3.13)$$

where the fact that  $\begin{bmatrix} T_i E \\ C_i^2 \end{bmatrix} \begin{bmatrix} M_i & N_i \end{bmatrix} = I_n$  is used. Then  $\dot{e}_i(t)$  can be obtained as

$$\begin{aligned} \dot{e}_i(t) &= \Lambda_i \epsilon_i(t) + (\Lambda_i T_i E - T_i \Phi_i) x(t) + (L_i^1 - T_i F_i^{11}) y_i^1(t) + L_i^2 y_i^2(t) \\ &\quad - T_i F_i^{12} v_i^2(t) + (H_i - T_i B) u(t) + \Pi_i^{-1} D_i^T \sum_{j \in N_i^I} \Gamma_{ij}(t) (\hat{x}_j(t) - \hat{x}_i(t)) \\ &= \Lambda_i \epsilon_i(t) + \Pi_i^{-1} D_i^T \sum_{j \in N_i^I} \Gamma_{ij}(t) (\hat{x}_j(t) - \hat{x}_i(t)), \quad i = 1, \dots, N, \end{aligned} \quad (3.14)$$

where (3.6) and the identities  $T_i \Phi_i - \Lambda_i T_i E = L_i^2 C_i^2$  and  $T_i F_i^{12} = 0$  are used [45]. To study the stability of the state error equation (3.14) with the adaptation law defined

in (3.12), a local Lyapunov-like function is considered

$$V_i(\epsilon_i, \Gamma_{ij}) = \epsilon_i^T(t) \Pi_i \epsilon_i(t) + \frac{1}{\gamma} \sum_{j \in N_i^I} \text{tr}(\Gamma_{ij}(t) \Gamma_{ij}^T(t)), \quad i = 1, \dots, N. \quad (3.15)$$

Then  $\dot{V}_i(\epsilon_i, \Gamma_{ij})$  is obtained as

$$\begin{aligned} \dot{V}_i(\epsilon_i, \Gamma_{ij}) &= \dot{\epsilon}_i^T(t) \Pi_i \epsilon_i(t) + \epsilon_i^T(t) \Pi_i \dot{\epsilon}_i(t) + \frac{2}{\gamma} \sum_{j \in N_i^I} \text{tr}(\dot{\Gamma}_{ij}(t) \Gamma_{ij}^T(t)) \\ &= \epsilon_i^T(t) \Lambda_i^T \Pi_i \epsilon_i(t) + \epsilon_i^T(t) \Pi_i \Lambda_i \epsilon_i(t) \\ &\quad + 2 \left( \Pi_i^{-1} D_i^T \sum_{j \in N_i^I} \Gamma_{ij}(t) (\hat{x}_j(t) - \hat{x}_i(t)) \right)^T \Pi_i \epsilon_i(t) \\ &\quad - \frac{2}{\gamma} \sum_{j \in N_i^I} \text{tr} \left( \gamma (D_i z_i(t) - y_i^2(t)) (\hat{x}_j(t) - \hat{x}_i(t))^T \Gamma_{ij}^T(t) \right) \\ &= \epsilon_i^T(t) \left( \Lambda_i^T \Pi_i + \Pi_i \Lambda_i \right) \epsilon_i(t) = -\epsilon_i^T(t) U_i \epsilon_i(t) \end{aligned}$$

where the identities  $\text{tr}(AB) = \text{tr}(BA)$ ,  $\text{tr}(AB^T) = \text{tr}(BA^T)$  for matrices  $A$  and  $B$ , and  $x^T y = \text{tr}(y x^T)$ ,  $x^T y + y^T x = 2x^T y$  for column vectors  $x, y$  are used.

If the smallest eigenvalue of  $U_i$  is denoted by  $\lambda_{\min}(U_i)$ , the derivative of the Lyapunov-like function simplifies to

$$\dot{V}_i(\epsilon_i, \Gamma_{ij}) \leq -\lambda_{\min}(U_i) \|\epsilon_i(t)\|^2 \leq 0, \quad i = 1, \dots, N. \quad (3.16)$$

Using the fact that the plant state is bounded  $x(t) \in L^\infty(0, \infty; \mathbb{R}^n)$  and  $y_i \in L^\infty(0, \infty; \mathbb{R}^{p_i})$  (Assumption 3.1), one has that all signals are bounded

$$\begin{aligned} \epsilon_i(t) &\in L^\infty(0, \infty; \mathbb{R}^{n_i + s_i - t_i}), \quad \hat{x}_i(t) \in L^\infty(0, \infty; \mathbb{R}^n), \\ \dot{\epsilon}_i(t) &\in L^\infty(0, \infty; \mathbb{R}^{n_i + s_i - t_i}), \quad \Gamma_{ij}(t) \in L^\infty(0, \infty; \mathbb{R}^{(t_i - s_i) \times (t_i - s_i)}), \\ &\forall j \in N_i^I. \end{aligned}$$

Additionally, from (3.16) one has that  $\epsilon_i(t) \in L^2(0, \infty; \mathbb{R}^n)$ . One can rewrite (3.12) in the form of

$$\dot{\Gamma}_{ij}(t) = -\gamma D_i \epsilon_i(t) (M_i(\epsilon_j(t) - \epsilon_i(t)))^T,$$

and therefore  $\dot{\Gamma}_{ij}(t) \in L^2(0, \infty; \mathbb{R}^{(t_i-s_i) \times (t_i-s_i)})$ ,  $\forall j \in N_i^I$ . Then from (3.13) one has  $e_i(t) \in L^2(0, \infty; \mathbb{R}^n)$ ,  $e_i(t) \in L^\infty(0, \infty; \mathbb{R}^n)$ ,  $\dot{e}_i(t) \in L^\infty(0, \infty; \mathbb{R}^n)$ . Therefore an application of Barbălat's lemma [61, 62] ( $e_i(t) \in L^2 \cap L^\infty$ ,  $\dot{e}_i(t) \in L^\infty$ ) yields

$$\lim_{t \rightarrow \infty} \|e_i(t)\| = 0, \quad i = 1, \dots, N.$$

□

Please note that the coupling terms involving the pairwise differences  $\hat{x}_j(t) - \hat{x}_i(t)$  are canceled out by applying the proposed adaptation law for  $\Gamma_{ij}(t)$  and therefore, it is not required to use the collective dynamics to prove the stability of Theorem 3.3. Therefore as a notable advantage, the stability of the proposed scheme is not restricted by the graph topology and the proposed scheme can also be applied to networks whose information exchange is described by directed graphs.

The following extensions to the edge-dependent adaptive consensus gains examine the use of node-dependent and uniform gains. Their proofs are a straightforward extension to the one provided in Theorem 3.3 and therefore omitted.

**Remark 3.1.** *The distributed estimation algorithm proposed in Theorem 3.3 can be*

modified to

$$\begin{aligned}\dot{z}_i(t) &= \Lambda_i z_i(t) + L_i^1 y_i^1(t) + L_i^2 y_i^2(t) + H_i u(t) + \Pi_i^{-1} D_i^T \sum_{j \in N_i^I} \Gamma_{ij}(t) C_i^2 (\hat{x}_j(t) - \hat{x}_i(t)) \\ \hat{x}_i(t) &= M_i z_i(t) + N_i y_i^2(t),\end{aligned}\tag{3.17}$$

where the adaptive gain matrix  $\Gamma_{ij}(t)$  is generated by

$$\dot{\Gamma}_{ij}(t) = -\gamma(D_i z_i(t) - y_i^2(t))(C_i^2 \hat{x}_j(t) - C_i^2 \hat{x}_i(t))^T, \quad i = 1, \dots, N, \quad j \in N_i^I.\tag{3.18}$$

Similar convergence results as in Theorem 3.3 can be established for the above modification.

**Remark 3.2.** The following equation is an alternative adaptive law for  $\Gamma_{ij}(t)$  proposed in (3.12)

$$\dot{\Gamma}_{ij}(t) = -\gamma(D_i z_i(t) - y_i^2(t))(C_i^2 \hat{x}_j(t) - C_i^2 \hat{x}_i(t))^T - \gamma \Gamma_{ij}(t), \quad i = 1, \dots, N, \quad j \in N_i^I\tag{3.19}$$

where  $\gamma$  is an arbitrary positive real number.

*Proof.* The proof of Remark 3.2 is very similar to Theorem 3.3. By using the local Lyapunov-like function defined in (3.15),  $\dot{V}_i(\epsilon_i, \Gamma_{ij})$  can be obtained as

$$\dot{V}_i(\epsilon_i, \Gamma_{ij}) = -\epsilon_i^T(t) U_i \epsilon_i(t) - 2 \sum_{j \in N_i^I} \text{tr}(\Gamma_{ij}(t) \Gamma_{ij}^T(t)) \leq 0, \quad i = 1, \dots, N.\tag{3.20}$$

and similar arguments can be used to establish stability. The only difference is that the convergence of the errors  $\epsilon_i$  now becomes exponential.  $\square$

Similar to Theorem 3.3, it can be noted that it is not required to use the collective dynamics to establish stability in Remark 3.2.

**Theorem 3.4** (AD-UIO for full-order systems). *Consider a sensor network described in (3.7). The following distributed estimation algorithm with an edge-dependent interconnection gain is utilized at each node*

$$\begin{aligned}\dot{z}_i(t) &= (P_i A^* - K_i C_i^*) z_i(t) + L_i y_i(t) + H_i^* u(t) + \Pi_i^{-1} C_i^{*T} \sum_{j \in N_i^I} \Gamma_{ij}(t) (\hat{x}_j(t) - \hat{x}_i(t)) \\ \hat{x}_i(t) &= z_i(t) + J_i y_i(t),\end{aligned}\tag{3.21}$$

where the matrix  $\Pi_i = \Pi_i^T > 0$  is the solution to the Lyapunov equation

$$(P_i A^* - K_i C_i^*)^T \Pi_i + \Pi_i (P_i A^* - K_i C_i^*) = -U_i\tag{3.22}$$

for  $U_i = U_i^T > 0$  and the adaptive gain matrix  $\Gamma_{ij}(t)$  is adjusted using the following adaptive law

$$\dot{\Gamma}_{ij}(t) = -\gamma (C_i^* \hat{x}_i(t) - y_i(t)) (\hat{x}_j(t) - \hat{x}_i(t))^T,\tag{3.23}$$

then, the estimation error  $e_i(t)$ ,  $i = 1, \dots, N$ , asymptotically reaches to zero and therefore, all the estimators asymptotically reach an agreement and all system signals are bounded.

*Proof.* The proof of Theorem 3.4 is very similar to Theorem 3.3 and is omitted due to the similarity. Note, since  $M_i = I$  then  $e_i = \epsilon_i$ . Therefore, the Lyapunov-like function (3.15) with  $\epsilon_i$  replaced by  $e_i$  can be used for establishing stability. □

In Theorems 3.3 and 3.4, the differences  $\hat{x}_i - \hat{x}_j$  between the estimation at node  $i$  and the estimation at node  $j$ ,  $\forall j \in N_i$ , are separately penalized using different  $\Gamma_{ij}(t)$ . Thus, the gains  $\Gamma_{ij}(t)$  are called *edge-dependent* adaptive gains. As a special case, one can use an identical gain  $\Gamma_{ij}(t)$  for all nodes neighboring  $j$ ,  $\forall j \in N_i$ . Thus,

one can move  $\Gamma_{ij}(t)$  outside the summation in (3.10) or (3.21) and make it *node-dependent* (also *vertex-dependent*) adaptive gain  $\Gamma_i(t)$ . For this case, the differences between the estimates of node  $i$  and all of nodes  $j$ ,  $\forall j \in N_i$ , are uniformly penalized. The resulting AD-UIOs with *node-dependent* adaptive gains are summarized in the lemma below.

**Lemma 3.1** (Node-dependent AD-UIO). *If the adaptive interconnection weights are defined to be node-dependent, then the AD-UIO for descriptor systems is given by*

$$\begin{aligned}\dot{z}_i(t) &= \Lambda_i z_i(t) + L_i^1 y_i^1(t) + L_i^2 y_i^2(t) + H_i u(t) + \Pi_i^{-1} D_i^T \Gamma_i(t) \sum_{j \in N_i^I} (\hat{x}_j(t) - \hat{x}_i(t)) \\ \hat{x}_i(t) &= M_i z_i(t) + N_i y_i^2(t), \\ \dot{\Gamma}_i(t) &= -\gamma (D_i z_i(t) - y_i^2(t)) \sum_{j \in N_i^I} (\hat{x}_j(t) - \hat{x}_i(t))^T, \quad i = 1, \dots, N, \quad j \in N_i^I,\end{aligned}\tag{3.24}$$

and the AD-UIO for full-order systems is given by

$$\begin{aligned}\dot{z}_i(t) &= (P_i A^* - K_i C_i^*) z_i(t) + L_i y_i(t) + H_i^* u(t) + \Pi_i^{-1} C_i^{*T} \Gamma_i(t) \sum_{j \in N_i^I} (\hat{x}_j(t) - \hat{x}_i(t)), \\ \hat{x}_i(t) &= z_i(t) + J_i y_i(t), \\ \dot{\Gamma}_i(t) &= -\gamma (C_i^* \hat{x}_i(t) - y_i(t)) \sum_{j \in N_i^I} (\hat{x}_j(t) - \hat{x}_i(t))^T, \quad i = 1, \dots, N, \quad j \in N_i^I,\end{aligned}\tag{3.25}$$

and the estimation error  $e_i(t)$ ,  $i = 1, \dots, N$ , asymptotically reaches to zero.

*Proof.* To proof Lemma 3.1, the Lyapunov-like functions  $V_i(\epsilon_i, \Gamma_i) = \epsilon_i^T(t) \epsilon_i(t) + \frac{1}{\gamma} \text{tr}(\Gamma_i(t) \Gamma_i^T(t))$  and  $V_i(e_i, \Gamma_i) = e_i^T(t) e_i(t) + \frac{1}{\gamma} \text{tr}(\Gamma_i(t) \Gamma_i^T(t))$  can be used for the node-dependent AD-UIOs in (3.24) and (3.25), respectively. The rest of the proof is similar to the proof of Theorem 3.3 and therefore it is omitted.  $\square$

## Chapter 4

# Adaptive Distributed Functional Observers for Linear Time-Invariant Descriptor Systems

In this chapter, we first summarize the results of an functional observer (FO) for a linear time invariant system in descriptor form, as taken from [52]. The descriptor system is assumed to have multiple outputs provided by a sensor network and for each system corresponding to a different output, a different FO is designed. The result is a network of distributed and non-interacting FOs. A special case of the distributed non-interacting FOs is when the descriptor system is indeed a regular system (the singular matrix in front of the time derivative becomes the identity matrix), and in this case the distributed and non-interacting FOs reduce to distributed non-interacting observers.

In a sensor network with distributed estimation, each node can collaborate with

its neighbors to improve its own estimation. Therefore, the framework proposed in [7] for the interconnection gain adaptation is applied to a sensor network of the functional observers in Theorems 4.1 and 4.2 to enhance the estimation.

## 4.1 Preliminaries and Problem formulation

Consider the following linear time-invariant (LTI) descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (4.1)$$

where the state  $x(t) \in \mathbb{R}^n$ , the known input  $u(t) \in \mathbb{R}^k$ , the matrices  $A$  and  $B$  are known and of appropriate dimensions. State measurements are obtained by a sensor network containing  $N$  nodes, where each node  $i$  has the following sensing model

$$y_i(t) = C_i x(t), \quad y_i \in \mathbb{R}^{p_i}, \quad i = 1, \dots, N. \quad (4.2)$$

where  $C_i$  is the observation matrix of node  $i$  and assumed to have row rank of  $p_i$ . Let the linear function required to be estimated  $z(t)$  be given by

$$z(t) = Lx(t), \quad (4.3)$$

where  $z(t) \in \mathbb{R}^r$ ,  $r \leq n$  and  $L$  is a known  $r \times n$  constant matrix with  $\text{rank}(L) = r$ . Additionally, it is assumed that  $\text{rank}(E) = m \leq n$ . Furthermore,  $E^\perp$  is a maximal row rank matrix of matrix  $E$  such that  $E^\perp E = 0$ , [68].

The non-interacting FO scheme for descriptor system [52] is first summarized in the following theorem and subsequently will be modified in Section 4.2 for interconnected systems.

**Theorem 4.1** (FO for descriptor systems [52]). *The functional observer for each node  $i$  of the descriptor system described in (4.1), (4.2) and (4.3) is given by*

$$\begin{aligned}\dot{\zeta}_i(t) &= N_i \zeta_i(t) + J_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix} + H_i u(t) \\ \widehat{z}_i(t) &= M_i \zeta_i(t) + Q_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix}, \quad i = 1, \dots, N,\end{aligned}\tag{4.4}$$

where  $\zeta_i(t) \in \mathbb{R}^{q_i}$  is the observer state, and  $\widehat{z}_i(t) \in \mathbb{R}^r$  is the estimate of  $z(t)$ . The matrices  $N_i$ ,  $J_i$ ,  $H_i$ ,  $M_i$  and  $Q_i$  are obtained by

$$\begin{aligned}\Delta_i &= \begin{bmatrix} E \\ E^\perp A \\ \bar{C}_i \end{bmatrix}, \quad \Omega_i = \begin{bmatrix} R \\ E^\perp A \\ \bar{C}_i \end{bmatrix}, \quad \alpha_i = R_i \Delta_i^\dagger \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \alpha_i^1 = R_i \Delta_i^\dagger \begin{bmatrix} 0 \\ I \end{bmatrix}, \\ \beta_i &= (I - \Delta_i \Delta_i^\dagger) \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \beta_i^1 = (I - \Delta_i \Delta_i^\dagger) \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \Sigma_i = \begin{bmatrix} \Omega_i \\ \beta_i A \end{bmatrix}, \quad \Theta_i = \alpha_i A, \\ A_i^1 &= \Theta_i \Sigma_i^\dagger \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \quad A_i^2 = \Theta_i \Sigma_i^\dagger \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \quad A_i^3 = \Theta_i \Sigma_i^\dagger \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \\ B_i^1 &= (I - \Sigma_i \Sigma_i^\dagger) \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \quad B_i^2 = (I - \Sigma_i \Sigma_i^\dagger) \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \quad B_i^3 = (I - \Sigma_i \Sigma_i^\dagger) \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix},\end{aligned}$$

$$\begin{aligned}
N_i &= A_i^1 - Z_i B_i^1, \quad K_i^1 = A_i^2 - Z_i B_i^2, \quad Y_i = A_i^3 - Z_i B_i^3, \\
T_i &= \alpha_i - Y_i \beta_i, \quad K_i = \alpha_i^1 - Y_i \beta_i^1, \quad J_i = K_i^1 + N_i K_i, \\
\begin{bmatrix} M_i & K_i^2 \end{bmatrix} &= L \Omega_i^\dagger + Z_i^1 (I - \Omega_i \Omega_i^\dagger), \quad Q_i = K_i^2 + M_i K_i, \quad H_i = T_i B
\end{aligned} \tag{4.5}$$

where  $R_i \in \mathbb{R}^{q_i \times n}$  is a full row rank matrix such that  $\text{rank} \left( \begin{bmatrix} R_i \\ \Delta_i \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} L \\ \Delta_i \\ L_i \\ \Omega_i \end{bmatrix} \right) =$

$\text{rank}(\Omega_i)$ ,  $Z_i^1$  is an arbitrary matrix, and  $Z_i$  is defined such that the matrix  $N_i$  to be Hurwitz. Also,  $A^\dagger$  denotes the Moore-Penrose generalized inverse matrix of  $A$ , [68]. Moreover, the functional observer given in (4.5) exists and is asymptotically stable. Finally, the matrix  $Z_i$  exists such that the matrix  $N_i$  is Hurwitz if and only if the following two conditions are satisfied

1. 
$$\text{rank} \left( \begin{bmatrix} \Sigma_i \\ \Theta_i \end{bmatrix} \right) = \text{rank}(\Sigma_i), \tag{4.6}$$

2. 
$$\text{rank} \left( \begin{bmatrix} sR_i - \alpha_i A \\ E^\perp A \\ C_i \\ \beta_i A \end{bmatrix} \right) = \text{rank}(\Sigma_i), \quad \forall s \in \mathbb{C}, \quad \text{Re}(s) \geq 0. \tag{4.7}$$

Furthermore, for such functional observer, the following two conditions are satisfied

1. 
$$N_i T_i E - T_i A + J_i \begin{bmatrix} E^\perp A \\ C_i \end{bmatrix} = 0, \tag{4.8}$$

2.

$$\left[ \begin{array}{c|c} M_i & Q_i \end{array} \right] \begin{bmatrix} T_i E \\ E^\perp A \\ C_i \end{bmatrix} = L. \quad (4.9)$$

In the special case of  $E = I$ , the descriptor system defined in (4.1), (4.2), and (4.3) reduces to the following full-order (standard) LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y_i(t) &= C_i x(t), \quad i = 1, \dots, N, \\ z(t) &= Lx(t). \end{aligned} \quad (4.10)$$

**Theorem 4.2** (FO for full-order systems [50]). *For each node  $i$ , the functional observer defined in Theorem 4.1 for the full-order system (4.10) simplifies to*

$$\begin{aligned} \dot{\zeta}_i(t) &= N_i w_i(t) + J_i y_i(t) + H_i u(t) \\ \hat{z}_i(t) &= \zeta_i(t) + Q_i y_i(t), \quad i = 1, \dots, N, \end{aligned} \quad (4.11)$$

where matrices  $N_i$ ,  $J_i$ ,  $H_i$ , and  $Q_i$  are obtained by

$$\begin{aligned} \bar{A} &= A(I - L^\dagger L), \quad \bar{C}_i = C_i(I - L^\dagger L), \quad \Sigma_i = \begin{bmatrix} C_i \bar{A} \\ \bar{C}_i \end{bmatrix}, \\ F_i &= LAL^\dagger - L\bar{A}\Sigma_i^\dagger \begin{bmatrix} C_i AL^\dagger \\ C_i L^\dagger \end{bmatrix}, \quad G_i = (I - \Sigma_i \Sigma_i^\dagger) \begin{bmatrix} C_i AL^\dagger \\ C_i L^\dagger \end{bmatrix}, \\ N_i &= F_i - Z_i G_i, \quad \begin{bmatrix} Q_i & K_i \end{bmatrix} = L\bar{A}\Sigma_i^\dagger + Z_i(I - \Sigma_i \Sigma_i^\dagger), \\ T_i &= L - Q_i C_i, \quad J_i = K_i + N_i Q_i, \quad H_i = T_i B, \end{aligned} \quad (4.12)$$

and where the matrix  $Z_i$  is defined such that the matrix  $N_i$  is Hurwitz. Moreover, the functional observer given in (4.11) exists and is asymptotically stable, and the matrix  $Z_i$  exists such that the matrix  $N_i$  is Hurwitz if and only if the following two conditions are satisfied

1.

$$\text{rank} \begin{pmatrix} \begin{bmatrix} LA \\ C_i A \\ C_i \\ L \end{bmatrix} \end{pmatrix} = \text{rank} \begin{pmatrix} \begin{bmatrix} C_i A \\ C_i \\ L \end{bmatrix} \end{pmatrix}, \quad (4.13)$$

2.

$$\text{rank} \begin{pmatrix} \begin{bmatrix} sL - LA \\ C_i A \\ C_i \end{bmatrix} \end{pmatrix} = \text{rank} \begin{pmatrix} \begin{bmatrix} C_i A \\ C_i \\ L \end{bmatrix} \end{pmatrix}, \quad \forall s \in \mathbb{C}, \quad \text{Re}(s) \geq 0. \quad (4.14)$$

Certain definitions pertaining to the observability properties of the descriptor system (4.1) are essential for establishing convergence of the proposed interacting FOs.

**Definition 4.1** (Partial impulse observable [52]). *The triplet  $(C, E, A)$  is partial impulse observable with respect to  $L$  if  $y(t)$  is impulse free for  $t \geq 0$ , only if  $Lx(t)$  is impulse free for  $t \geq 0$ .*

The following assumption is essential for the convergence of the estimation error of FO for descriptor systems to zero asymptotically [52].

**Assumption 4.1** (Partial impulse observable). *It is assumed that the descriptor system in (4.1), (4.2) is partial impulse observable with respect to  $L$  [52].*

The following assumption is also required for the proof of the theorems presented in the next section.

**Assumption 4.2** (Bounded plant). *The class of systems (4.1), (4.10) is such that  $x \in L^\infty(0, \infty; \mathbb{R}^n)$ ,  $y_i \in L^\infty(0, \infty; \mathbb{R}^{p_i})$  and  $z \in L^\infty(0, \infty; \mathbb{R}^r)$ .*

Information exchange between nodes of a sensor network is modeled by a directed graph defined in Chapter 2.

Now, the main result of this chapter can be introduced which deals with the design of an adaptive distributed FO (ADFO) scheme.

## 4.2 Adaptive Distributed Functional Observers (ADFO)

First, the ADFO scheme based on the edge-dependent adaptive gain strategy is presented in the following theorem, in which each node  $i$  of the network adaptively adjust the disagreement between its functional estimates and the functional estimate  $\widehat{z}_j$  of a node  $j$ ,  $\forall j \in N_i^I$ , using an adaptive gain corresponding the nodes  $i$  and  $j$ . For brevity, the pairwise difference of functional estimate of node  $i$ ,  $\widehat{z}_i$ , and the functional estimate of node  $j$ ,  $\widehat{z}_j$ , is denoted as  $\widehat{z}_{ij} = \widehat{z}_i - \widehat{z}_j$ .

**Theorem 4.3** (ADFO for descriptor systems). *Consider a sensor network described in (4.1), (4.2), and (4.3). The following distributed estimation algorithm with a*

distance-adjusted *interconnection gain is utilized at each node*

$$\begin{aligned}\dot{\zeta}_i(t) &= N_i \zeta_i(t) + J_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix} + H_i u(t) + \Pi_i^{-1} M_i^T D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t), \\ \dot{\hat{z}}_i(t) &= M_i \zeta_i(t) + Q_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix}, i = 1, \dots, N,\end{aligned}\tag{4.15}$$

where  $D_i = (CL_i^\dagger)^T$  and the adaptive gain matrix  $\Gamma_{ij}(t)$  is adjusted using the adaptive law

$$\dot{\Gamma}_{ij}(t) = -\gamma (D_i^T \hat{z}_i(t) - y_i(t)) \hat{z}_{ji}^T(t) D_i, \quad i = 1, \dots, N, \quad j \in N_i^I \tag{4.16}$$

where  $\gamma$  is an arbitrary positive real number. The matrix  $\Pi_i$  is the solution to the Lyapunov equation

$$N_i^T \Pi_i + \Pi_i N_i = -U_i, \quad i = 1, \dots, N, \tag{4.17}$$

and is a symmetric positive definite matrix for  $U_i = U_i^T > 0$  and  $N_i$  Hurwitz [65]. Then, the estimation error, defined as  $e_i(t) \triangleq \hat{z}_i(t) - z(t)$ ,  $i = 1, \dots, N$ , asymptotically reaches zero and therefore, all the estimators asymptotically reach an agreement and all system signals are bounded.

The ADFO scheme proposed in (4.4) is similar to the non-interacting FO scheme in (4.15) and only a coupling term  $\Pi_i^{-1} M_i^T D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t)$  is added to the first equation. The matrices  $N_i$ ,  $J_i$ ,  $H_i$ ,  $M_i$  and  $Q_i$  in Theorem 4.3 are obtained from (4.5) and therefore the conditions in (4.6), (4.7), (4.8) and (4.9) are also satisfied in Theorem 4.3.

*Proof.* The estimation error for node  $i$  is written as

$$e_i(t) = \widehat{z}_i(t) - z(t) = \widehat{z}_i(t) - Lx(t) = M_i\epsilon_i(t), \quad (4.18)$$

where  $\epsilon_i(t) \triangleq \zeta_i(t) - T_iEx(t)$ . Then, the error dynamics  $\epsilon_i(t)$  of the  $i$ th node is obtained by combining (4.3) and (4.4)

$$\begin{aligned} \dot{\epsilon}_i(t) &= N_i\epsilon_i(t) + (N_iP_i + J_iC_i - P_iA)x(t) + (H_i - P_iB)u(t) \\ &\quad + \Pi_i^{-1}M_i^T D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) D_i^T \widehat{z}_{ji}(t) \\ &= N_i\epsilon_i(t) + \Pi_i^{-1}M_i^T D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) D_i^T \widehat{z}_{ji}(t), \quad i = 1, \dots, N, \end{aligned} \quad (4.19)$$

where (4.5), (4.8), and (4.9) have been used which are satisfied for the observer defined in Theorem 4.1 [52]. In order to study the stability of the state error equation (4.19) and the adaptation law (4.16), a local Lyapunov-like function is considered

$$V_i(\epsilon_i, \Gamma_{ij}) = \epsilon_i^T(t) \Pi_i \epsilon_i(t) + \frac{1}{\gamma} \sum_{j \in N_i^I} \text{tr}(\Gamma_{ij}(t) \Gamma_{ij}^T(t)), \quad i = 1, \dots, N. \quad (4.20)$$

Then  $\dot{V}_i(\epsilon_i, \Gamma_{ij})$  is obtained as

$$\begin{aligned}
\dot{V}_i(\epsilon_i, \Gamma_{ij}) &= \dot{\epsilon}_i^T(t) \Pi_i \epsilon_i(t) + \epsilon_i^T(t) \Pi_i \dot{\epsilon}_i(t) + \frac{2}{\gamma} \sum_{j \in N_i^I} \text{tr} \left( \dot{\Gamma}_{ij}(t) \Gamma_{ij}^T(t) \right) \\
&= \epsilon_i^T(t) N_i^T \Pi_i \epsilon_i(t) + \epsilon_i^T(t) \Pi_i N_i \epsilon_i(t) \\
&\quad + 2 \left( \Pi_i^{-1} M_i^T D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t) \right)^T \Pi_i \epsilon_i(t) \\
&\quad - \frac{2}{\gamma} \sum_{j \in N_i^I} \text{tr} \left( \gamma \left( D_i^T \hat{z}_i(t) - y_i(t) \right) \hat{z}_{ji}^T(t) D_i \Gamma_{ij}^T(t) \right) \\
&= \epsilon_i^T(t) N_i^T \Pi_i \epsilon_i(t) + \epsilon_i^T(t) \Pi_i N_i \epsilon_i(t) + 2 \sum_{j \in N_i^I} \left( \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t) \right)^T D_i^T M_i \epsilon_i(t) \\
&\quad - 2 \sum_{j \in N_i^I} \left( \left( \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t) \right)^T \left( D_i^T \hat{z}_i(t) - y_i(t) \right) \right)
\end{aligned}$$

where the identities  $\text{tr}(AB) = \text{tr}(BA)$ ,  $\text{tr}(AB^T) = \text{tr}(BA^T)$  for matrices  $A$  and  $B$ , and  $x^T y = \text{tr}(y x^T)$ ,  $x^T y + y^T x = 2x^T y$  for column vectors  $x, y$  are used. Using (4.18), on can obtain

$$\begin{aligned}
\dot{V}_i(\epsilon_i, \Gamma_{ij}) &= \epsilon_i^T(t) N_i^T \Pi_i \epsilon_i(t) + \epsilon_i^T(t) \Pi_i N_i \epsilon_i(t) \\
&\quad + 2 \sum_{j \in N_i^I} \left( \left( \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t) \right)^T D_i^T \left( \hat{z}_i(t) - z(t) \right) \right) \\
&\quad - 2 \sum_{j \in N_i^I} \left( \left( \Gamma_{ij}(t) D_i^T \hat{z}_{ji}(t) \right)^T \left( D_i^T \hat{z}_i(t) - y_i(t) \right) \right) \\
&= \epsilon_i^T(t) \left( N_i^T \Pi_i + \Pi_i N_i \right) \epsilon_i(t) = -\epsilon_i^T(t) U_i \epsilon_i(t)
\end{aligned}$$

If the smallest eigenvalue of  $U_i$  is denoted by  $\lambda_{\min}(U_i)$ , the derivative of the Lyapunov-like function (4.20) simplifies to

$$\dot{V}_i(\epsilon_i, \Gamma_{ij}) \leq -\lambda_{\min}(U_i) \|\epsilon_i(t)\|^2 \leq 0, \quad i = 1, \dots, N. \quad (4.21)$$

Using the fact that the plant state is bounded  $z(t) \in L^\infty(0, \infty; \mathbb{R}^r)$  and  $y_i \in L^\infty(0, \infty; \mathbb{R}^{p_i})$  (Assumption 4.2), one has that all signals are bounded

$$\begin{aligned} \epsilon_i(t) &\in L^\infty(0, \infty; \mathbb{R}^{q_i}), & \widehat{z}_i(t) &\in L^\infty(0, \infty; \mathbb{R}^r), \\ \dot{\epsilon}_i(t) &\in L^\infty(0, \infty; \mathbb{R}^{q_i}), & \Gamma_{ij}(t) &\in L^\infty(0, \infty; \mathbb{R}^{p_i \times p_i}), & \forall j \in N_i^I. \end{aligned}$$

Additionally, from (4.21) one has that  $\epsilon_i(t) \in L^2(0, \infty; \mathbb{R}^n)$ . One can rewrite (4.16) in the form of

$$\dot{\Gamma}_{ij}(t) = -\gamma (D_i^T \widehat{z}_i(t) - y_i(t)) (\epsilon_j(t) - \epsilon_i(t))^T M_i^T D_i,$$

and therefore  $\dot{\Gamma}_{ij}(t) \in L^2(0, \infty; \mathbb{R}^{p_i \times p_i})$ ,  $\forall j \in N_i^I$ . Then from (4.18) one has  $e_i(t) \in L^2(0, \infty; \mathbb{R}^n)$ ,  $e_i(t) \in L^\infty(0, \infty; \mathbb{R}^n)$ ,  $\dot{e}_i(t) \in L^\infty(0, \infty; \mathbb{R}^n)$ . Therefore an application of Barbălat's lemma [61] ( $e_i(t) \in L^2 \cap L^\infty$ ,  $\dot{e}_i(t) \in L^\infty$ ) yields

$$\lim_{t \rightarrow \infty} \|e_i(t)\| = 0, \quad i = 1, \dots, N.$$

□

It is interesting to note that for the proof of Theorem 4.3 one does not have to use the collective dynamics to establish stability. The reason is that the coupling terms involving the pairwise differences  $\widehat{z}_j(t) - \widehat{z}_i(t)$  are canceled out by applying the proposed adaptation law for  $\Gamma_{ij}(t)$ . Therefore as a notable advantage, the stability of the proposed scheme is not restricted by the graph topology and the proposed scheme is also working for directed graphs.

**Remark 4.1.** *The following equation is an alternative adaptive law for  $\Gamma_{ij}(t)$  pro-*

posed in (4.16)

$$\dot{\Gamma}_{ij}(t) = -\gamma (D_i^T \widehat{z}_i(t) - y_i(t)) \widehat{z}_{ji}^T(t) D_i - \gamma \Gamma_{ij}(t), \quad i = 1, \dots, N, \quad j \in N_i^I \quad (4.22)$$

where  $\gamma$  is an arbitrary positive real number.

*Proof.* The proof of Remark 4.1 is very similar to Theorem 4.3. By using the local Lyapunov-like function defined in (4.20),  $\dot{V}_i(\epsilon_i, \Gamma_{ij})$  can be obtained as

$$\dot{V}_i(\epsilon_i, \Gamma_{ij}) = -\epsilon_i^T(t) U_i \epsilon_i(t) - 2 \sum_{j \in N_i^I} \text{tr} (\Gamma_{ij}(t) \Gamma_{ij}^T(t)) \leq 0, \quad i = 1, \dots, N. \quad (4.23)$$

which completes the proof. □

Similar to Theorem 4.3, it can be noted that it is not required to use the collective dynamics to establish stability in Remark 4.1.

**Theorem 4.4** (ADFO for full-order systems). *Consider a sensor network described in (4.10). The following distributed estimation algorithm with a distance-adjusted interconnection gain is utilized at each node*

$$\dot{\zeta}_i(t) = N_i \zeta_i(t) + J_i y_i(t) + H_i u(t) + \Pi_i^{-1} D_i \sum_{j \in N_i^I} \Gamma_{ij}(t) D_i^T \widehat{z}_{ji}(t), \quad (4.24)$$

$$\widehat{z}_i(t) = \zeta_i(t) + Q_i y_i(t),$$

where the adaptive gain matrix  $\Gamma_{ij}(t)$  is adjusted using the following adaptive law

$$\dot{\Gamma}_{ij}(t) = -\gamma (D_i^T \widehat{z}_i(t) - y_i(t)) \widehat{z}_{ji}^T(t) D_i, \quad (4.25)$$

then, the estimation error  $e_i(t)$ ,  $i = 1, \dots, N$ , asymptotically reaches to zero and therefore, all the estimators asymptotically reach an agreement and all system signals

are bounded.

*Proof.* The proof of Theorem 4.4 is similar to Theorem 4.3 and is omitted due to their similarities. Note, since  $M_i = I$  then  $e_i = \epsilon_i$  and therefore the Lyapunov-like function (4.20) with  $\epsilon_i$  replaced by  $e_i$  can be used for the stability proof.  $\square$

In Theorems 4.3 and 4.4, the distance between the estimation at node  $i$  and the estimation at node  $j$ ,  $\forall j \in N_i^I$ , is penalized independently using different  $\Gamma_{ij}(t)$ . Thus, the gain  $\Gamma_{ij}(t)$  is called *edge-dependent* adaptive gain. In a special case, one can use an identical gain  $\Gamma_{ij}(t)$  for all nodes  $j$ ,  $\forall j \in N_i^I$ . Thus, one can move  $\Gamma_{ij}(t)$  outside the summation in (4.15) or (4.24) and make it an *node-dependent* (also *vertex-dependent*) adaptive gain  $\Gamma_i(t)$ . Therefore, distances between node  $i$  and all nodes  $j$ ,  $\forall j \in N_i^I$ , are penalized identically. The ADFOs with *node-dependent* adaptive gains is summarized in the following lemma.

**Lemma 4.1** (Node-dependent ADFO). *If the adaptive interconnection weights are defined to be node-dependent, then the ADFO for descriptor systems is given by*

$$\begin{aligned}
\dot{\zeta}_i(t) &= N_i \zeta_i(t) + J_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix} + H_i u(t) + \Pi_i^{-1} M_i^T D_i \Gamma_i(t) D_i^T \sum_{j \in N_i^I} \widehat{z}_{ji}(t), \\
\widehat{z}_i(t) &= M_i \zeta_i(t) + Q_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix}, \\
\dot{\Gamma}_i(t) &= -\gamma (D_i^T \widehat{z}_i(t) - y_i(t)) \sum_{j \in N_i^I} \widehat{z}_{ji}^T(t) D_i,
\end{aligned} \tag{4.26}$$

and the ADFO for full-order systems is given by

$$\begin{aligned}
\dot{\zeta}_i(t) &= N_i \zeta_i(t) + J_i y_i(t) + H_i u(t) + \Pi_i^{-1} D_i \Gamma_i(t) D_i^T \sum_{j \in N_i^I} \widehat{z}_{ji}(t), \\
\widehat{z}_i(t) &= \zeta_i(t) + Q_i y_i(t), \\
\dot{\Gamma}_i(t) &= -\gamma (D_i^T \widehat{z}_i(t) - y_i(t)) \sum_{j \in N_i^I} \widehat{z}_{ji}^T(t) D_i,
\end{aligned} \tag{4.27}$$

and the estimation error  $e_i(t)$ ,  $i = 1, \dots, N$ , asymptotically reaches to zero.

*Proof.* To prove Lemma 4.1, the Lyapunov-like functions  $V_i(\epsilon_i, \Gamma_i) = \epsilon_i^T(t) \epsilon_i(t) + \frac{1}{\gamma} \text{tr}(\Gamma_i(t) \Gamma_i^T(t))$  and  $V_i(e_i, \Gamma_i) = e_i^T(t) e_i(t) + \frac{1}{\gamma} \text{tr}(\Gamma_i(t) \Gamma_i^T(t))$  can be used for the node-dependent ADFOs in (4.26) and (4.27), respectively. The rest of the proof is similar to that for Theorem 4.3 and is therefore omitted.  $\square$

# Chapter 5

## Numerical Studies

This chapter is dedicated to numerical simulations for the proposed adaptive schemes in this dissertation. The chapter is therefore divided into the following sections:

- Section 5.1 presents four numerical simulations to demonstrate the performance and effectiveness of the proposed adaptive distributed Kalman filter schemes.
- Section 5.2 presents a numerical simulation to demonstrate the performance and effectiveness of the proposed adaptive distributed unknown input observer scheme.
- Section 5.3 presents a numerical simulation to demonstrate the performance and effectiveness of the proposed adaptive distributed functional observer scheme.

## 5.1 Adaptive-DKF for LTV systems

In this section, four numerical simulations are presented to demonstrate the performance of the proposed adaptive-DKF schemes. Consider the linear time-varying (LTV) system in (2.1) with

$$A = \begin{bmatrix} -0.15 + 0.4 \sin(t) & -2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

having initial condition  $x_0 = (15 \ -10)^T$ .

In the first simulation, the observation matrices are chosen to be either  $C_i = C^1 \triangleq [1 \ 0]$  or  $C_i = C^2 \triangleq [0 \ 1]$ ,  $i = 1, \dots, N$ , at random, but each set of neighbors  $N_i$  of node  $i$  contains nodes with both types of matrices. The process noise covariance is chosen as  $Q = \mathbf{I}_{2 \times 2}$ . The measurement noise covariances are also chosen as  $R_i = 0.1i$ ,  $i = 1, \dots, N$ . An undirected WSN with  $N = 50$  nodes is chosen as shown schematically in Figure 5.1 where the nodes are randomly located. The reason for choosing an undirected WSN in the first simulation is to have the ability of comparing with the standard DKF [17] which was restricted to undirected graphs. Finally, the initial estimate of each node  $i$ ,  $\hat{x}_i(0)$ , is chosen to be different from the initial estimate of the other nodes, i.e.  $\hat{x}_i(0) \neq \hat{x}_j(0)$ ,  $i, j = 1, \dots, N$ ,  $i \neq j$ .

The node-average estimation error

$$\|e\|_{node} = \frac{1}{N} \|e\|_2, \quad \|e\|_2^2 = \sum_{i=1}^N e_i^T e_i,$$

for the adaptive-DKFs proposed in Theorem 2.1 (the edge-dependent gain case) and in Lemma 2.3 (the node-dependent case) along with the non-adaptive (standard) DKF from [17] are compared in Figure 5.2. It can be observed that the

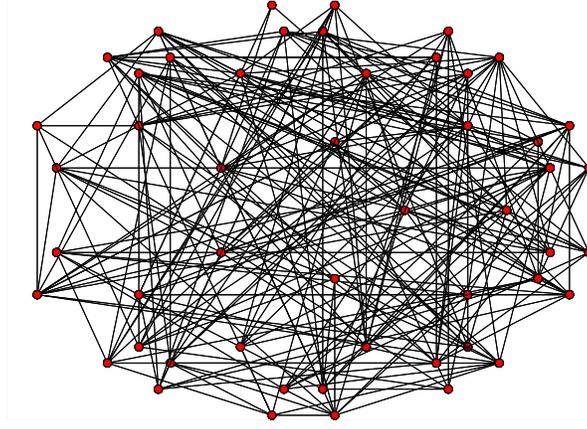


Figure 5.1: A sensor network with randomly located nodes consisting of  $N=50$  nodes and 229 links

adaptive-DKFs from Theorem 2.1 and Lemma 2.3 have a better performance than the standard DKF [17] and the adaptive-DKFs from Theorem 2.1 exhibit the best performance.

The following measure for the performance of the state estimates that is independent of the network topology [17] is considered to assess the performance of the proposed adaptive strategy

$$\delta_i(t) \triangleq \hat{x}_i(t) - \frac{1}{N} \sum_{j=1}^N \hat{x}_j(t), \quad (5.1)$$

$$\|\delta(t)\| = \sqrt{\sum_{i=1}^N |\delta_i(t)|_{\mathbb{R}^n}^2} = \sqrt{\sum_{i=1}^N \delta_i^T(t) \delta_i(t)}. \quad (5.2)$$

Figure 5.3 compares the disagreement  $\|\delta\|$  for the adaptive-DKFs proposed in Theorem 2.1 and in Lemma 2.3 and the non-adaptive DKF from [17]. Once again, it can be observed that the adaptive-DKFs from Theorem 2.1 and Lemma 2.3 exhibit a better performance than the standard DKFs [17] and the adaptive-DKFs from

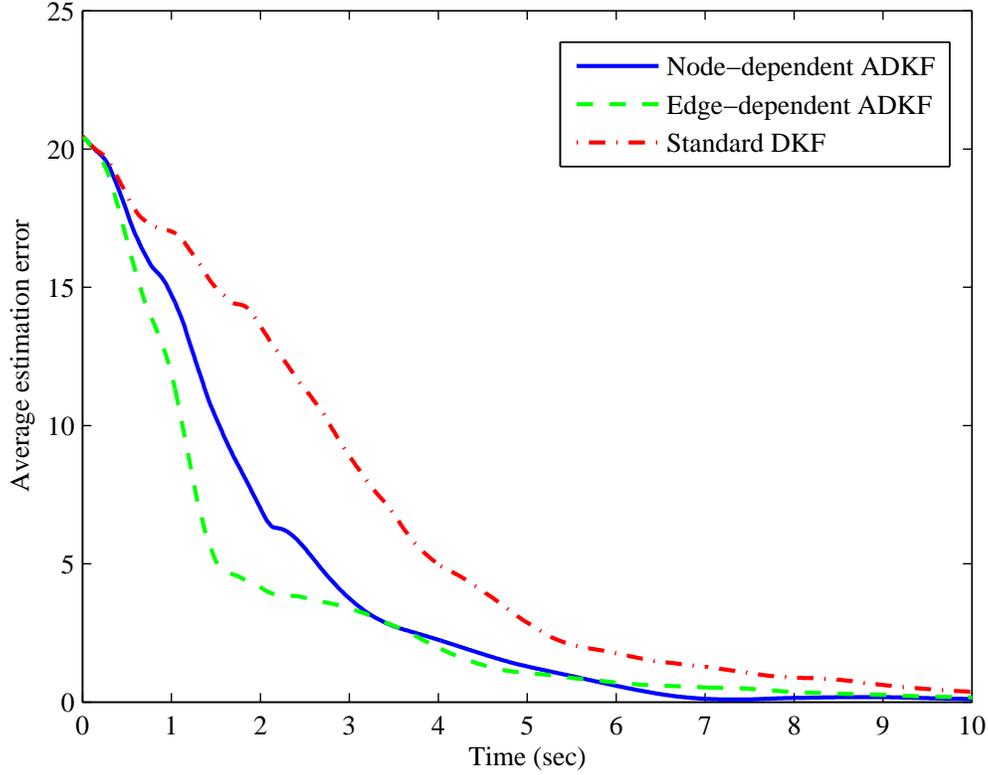


Figure 5.2: Comparison of estimation error of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF), and the standard DKF [17] for the first simulation.

Theorem 2.1 have the fastest convergence rate.

To provide a level of the associated communication cost, the proposed adaptive-DKFs in Theorem 2.1 and in Lemma 2.3 required to transmit 5038 bytes per time unit, while, the non-adaptive DKFs [17] required to transmit 6870 bytes per time unit. This represents a 36% increase over the proposed strategies. It should be emphasized that with the higher system dimension  $n$ , this reduction in the communication cost become more prominent.

In the second and third simulations, the effect of the sensor network graph has been studied on the response of the adaptive-DKFs. Therefore, two undirected

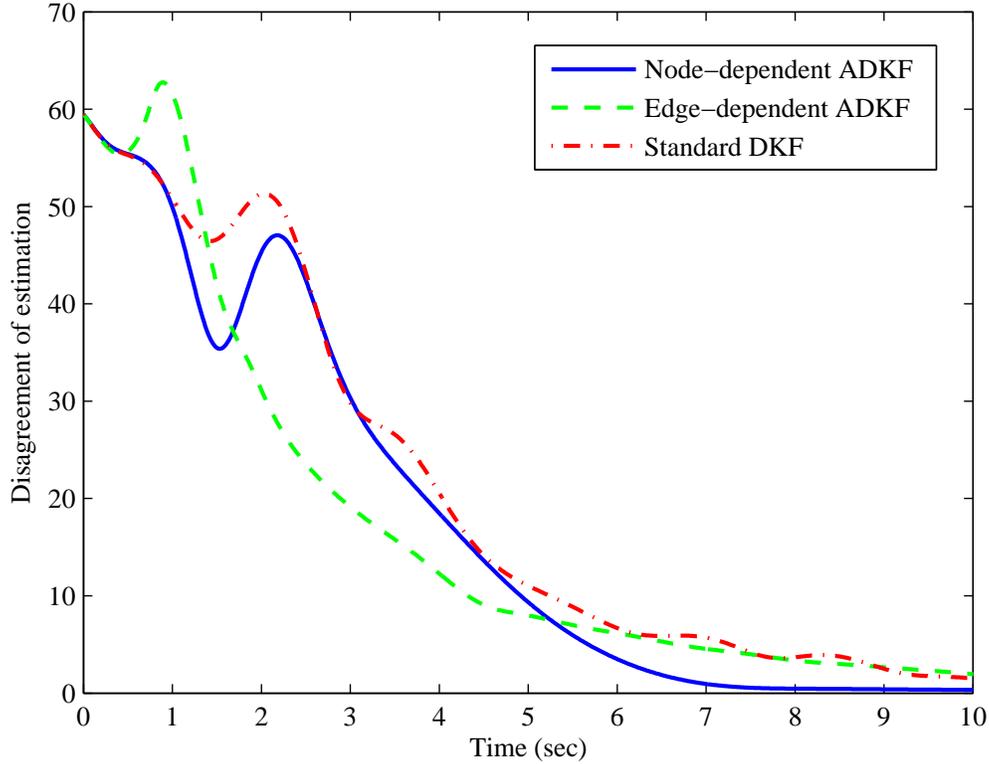


Figure 5.3: Comparison of disagreement  $\|\delta\|$  of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF), and the standard DKF [17] for the first simulation.

WSNs with  $N = 6$  nodes are chosen for the second and third simulations as shown schematically in Figures 5.4 and 5.5, respectively. The observation matrices are chosen as  $C_1 = C_3 = C_5 = C^1 \triangleq [1 \ 0]$  and  $C_2 = C_4 = C_6 = C^2 \triangleq [0 \ 1]$ . The measurement noise covariances are also chosen as  $R_1 = R_2 = R_4 = R_5 = 0.1$ ,  $R_3 = 5$ , and  $R_6 = 10$ . The rest of system parameters are the same as the first simulation. Similarly, the initial estimate of each node  $i$ ,  $\hat{x}_i(0)$ , is chosen to be different from the initial estimate of the other nodes in the second and third simulations also.

The node-average estimation error  $\|e\|_{node}$  for the adaptive-DKFs proposed in Theorem 2.1 (the edge-dependent gain case) and in Lemma 2.3 (the node-dependent case) along with the non-adaptive (standard) DKF from [17] are compared in the

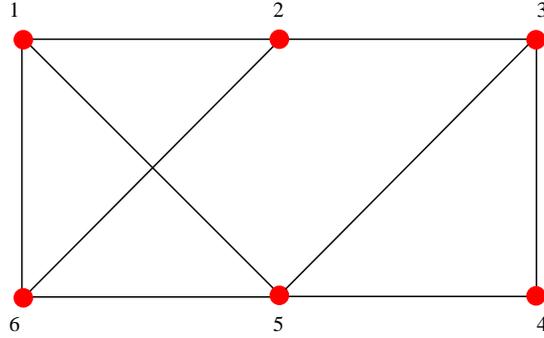


Figure 5.4: A sensor network consisting of  $N=6$  nodes used in the second simulation.

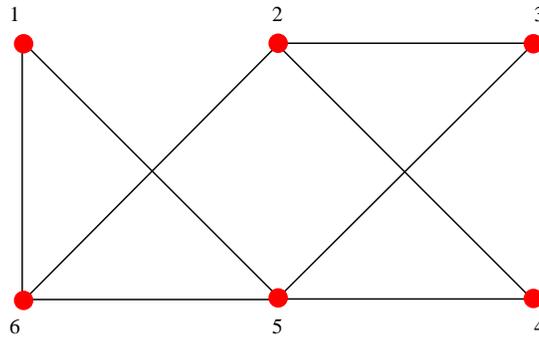


Figure 5.5: A sensor network consisting of  $N=6$  nodes used in the third simulation.

second and third simulations in Figures 5.6 and 5.7 respectively. Figures 5.8 and 5.9 compare the disagreement  $\|\delta\|$  for the adaptive-DKFs proposed in Theorem 2.1 and in Lemma 2.3 and the non-adaptive DKF from [17] for the second and third simulations, respectively. It can be observed that the adaptive-DKFs from Theorem 2.1 and Lemma 2.3 have a better performance than the standard DKF [17] and the adaptive-DKFs from Theorem 2.1 exhibit the best performance in the second and third simulations as well.

In the fourth simulation, a directed WSN with  $N = 5$  nodes is chosen as shown schematically in Figure 5.10. The observation matrices are chosen as  $C_1 = C_3 = C_5 = C^1 \triangleq [1 \ 0]$  and  $C_2 = C_4 = C^2 \triangleq [0 \ 1]$ . The measurement noise covariances are also chosen as  $R_i = 0.25i$ ,  $i = 1, \dots, N$ . The rest of system parameters are the

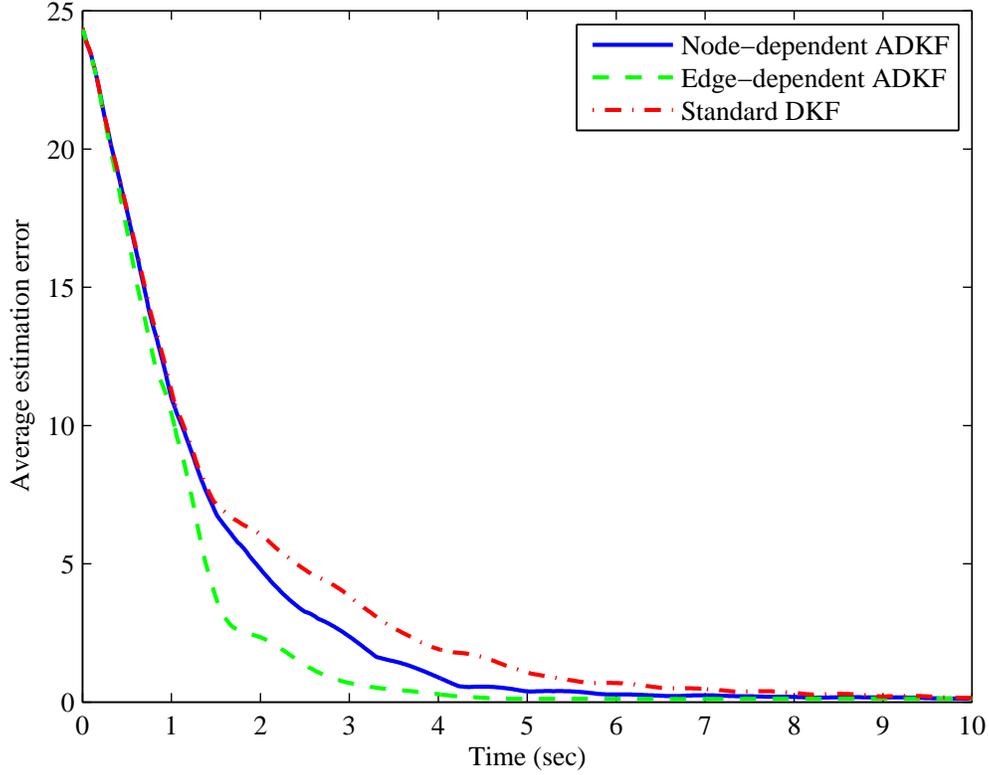


Figure 5.6: Comparison of estimation error of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF), and the standard DKF [17] for the second simulation.

same as the first simulation. Similarly, the initial estimate of each node  $i$ ,  $\hat{x}_i(0)$ , is chosen to be different from the initial estimate of the other nodes in the fourth simulation also.

The node-average estimation error  $\|e\|_{node}$  and the disagreement  $\|\delta\|$  for the adaptive-DKFs proposed in Theorem 2.1 (the edge-dependent gain case) and in Lemma 2.3 (the node-dependent case) are compared in Figures 5.11 and 5.12 respectively. It can be observed that the adaptive-DKFs from Theorem 2.1 converge faster than the adaptive-DKFs of Lemma 2.3 in the fourth simulation as well. The results are not compared with the standard DKF [17] in the fourth simulation, since

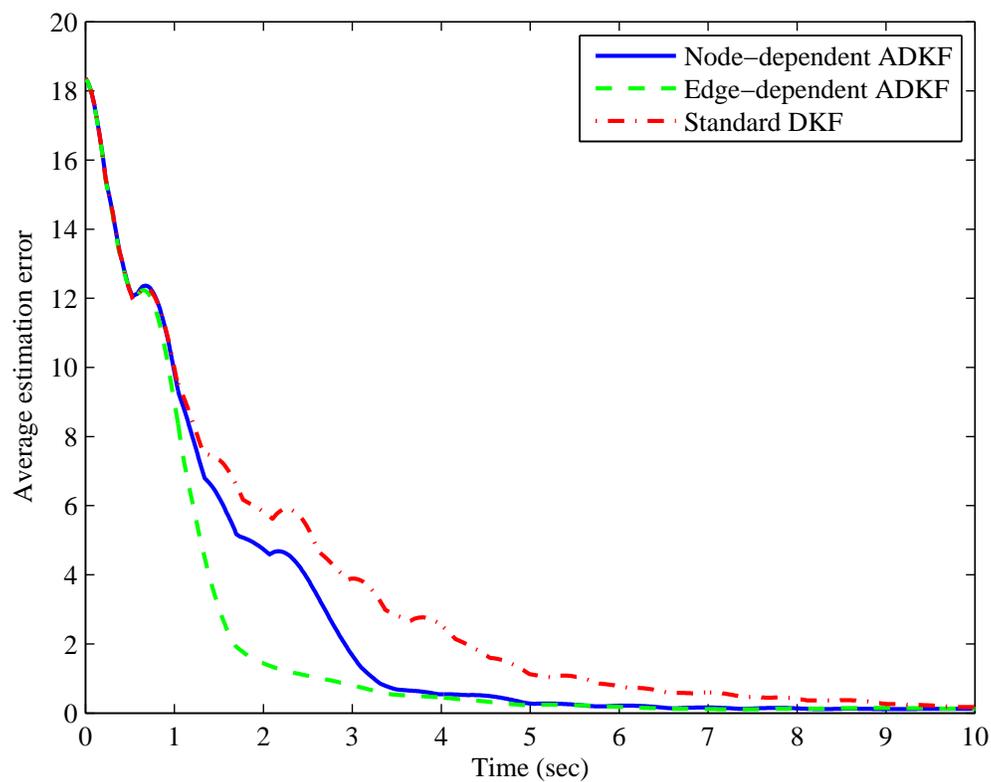


Figure 5.7: Comparison of estimation error of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF), and the standard DKF [17] for the third simulation.

it is not applicable to the case of directed graphs.

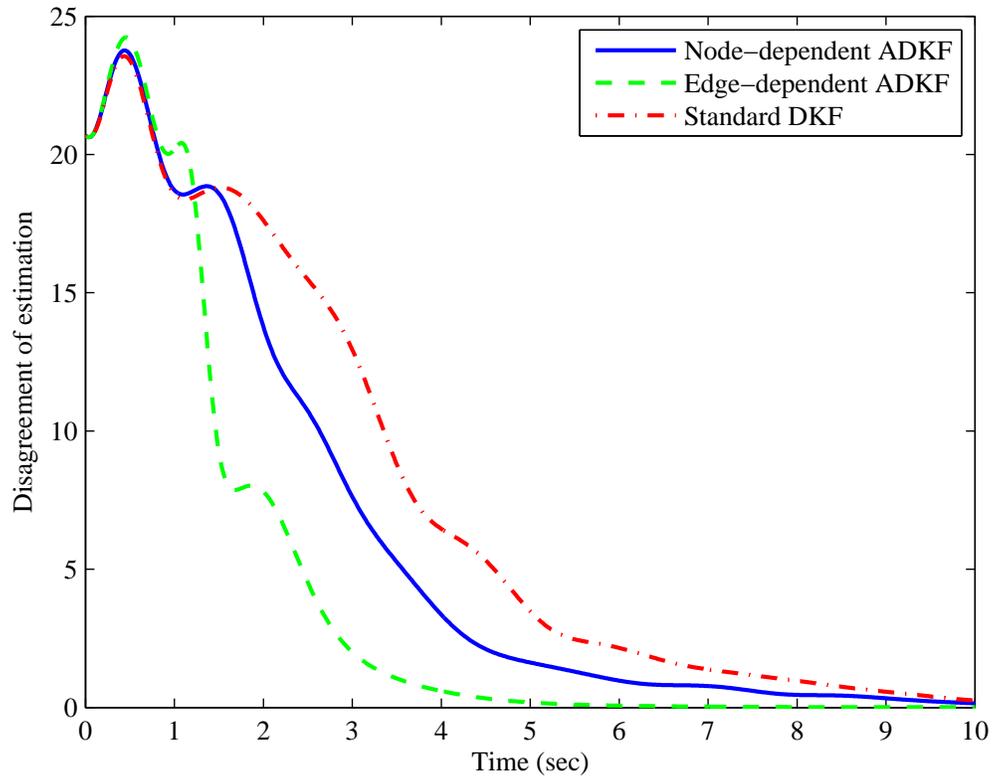


Figure 5.8: Comparison of disagreement  $\|\delta\|$  of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF), and the standard DKF [17] for the second simulation.

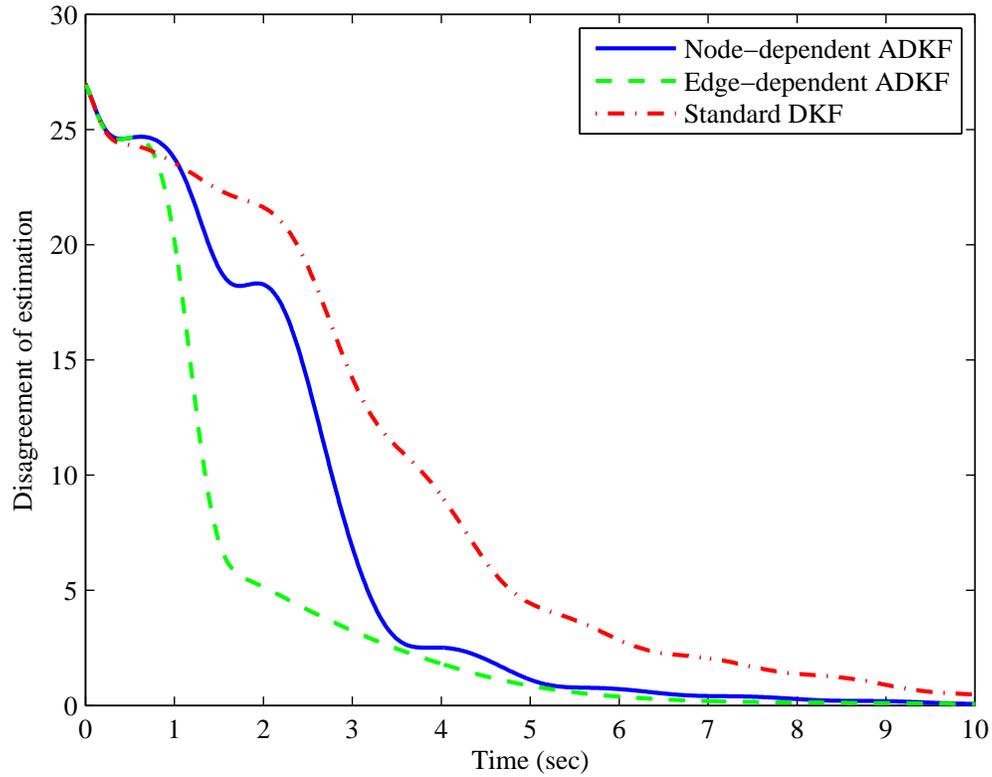


Figure 5.9: Comparison of disagreement  $\|\delta\|$  of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF), and the standard DKF [17] for the third simulation.

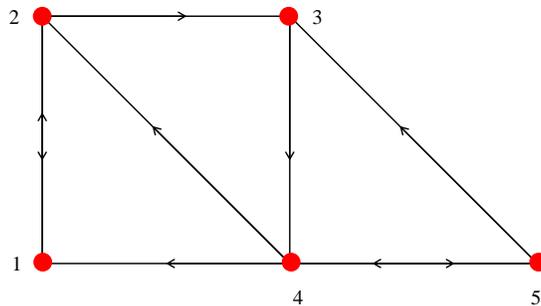


Figure 5.10: A directed sensor network consisting of  $N=5$  nodes

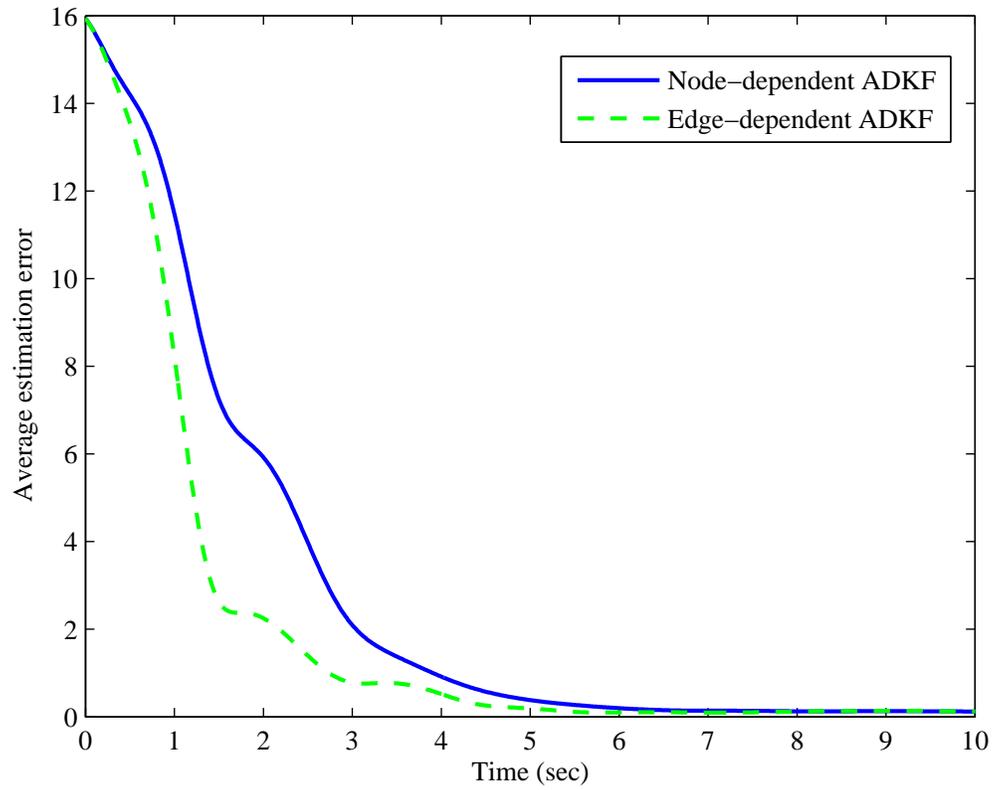


Figure 5.11: Comparison of estimation error of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF) for the fourth simulation.

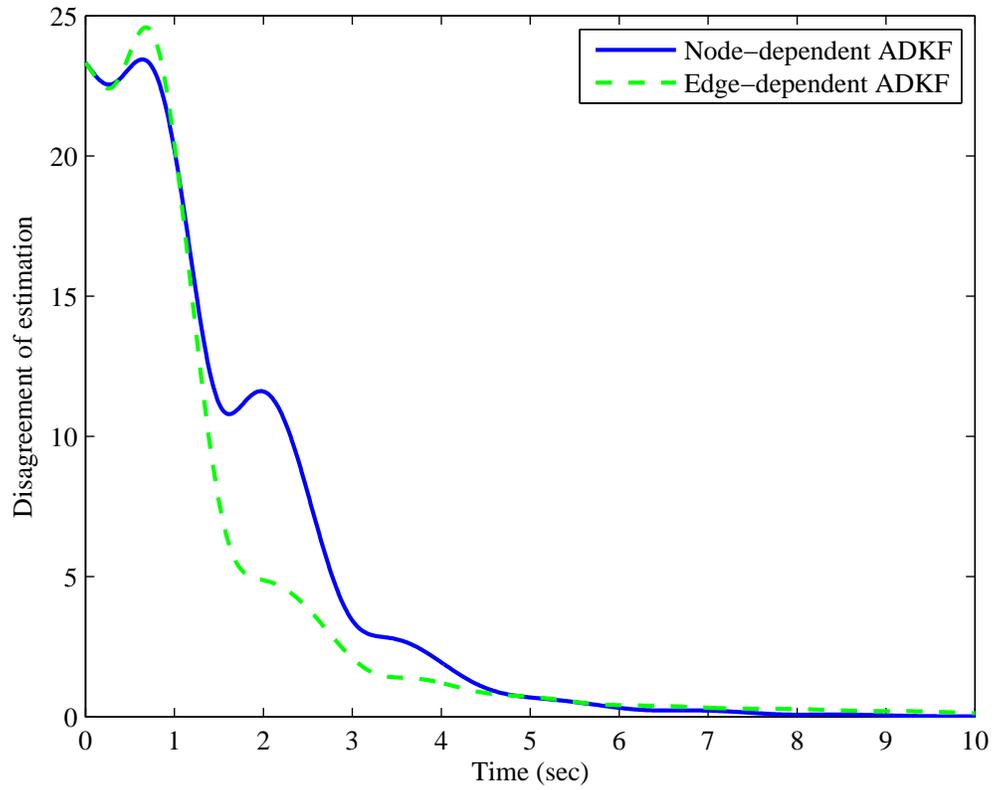


Figure 5.12: Comparison of disagreement  $\|\delta\|$  of adaptive-DKFs proposed by Theorem 2.1 (edge-dependent ADKF) and Lemma 2.3 (node-dependent ADKF) for the fourth simulation.

## 5.2 AD-UIO for LTI descriptor systems

In this section, a numerical simulation is presented to demonstrate the performance of the proposed adaptive distributed unknown input observer scheme. The linear time-invariant (LTI) system in (3.1) is considered with [43,45]

$$E^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^* = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$F^* = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T$$

and initial condition  $x_0 = (15, 30, 40, -30)^T$ . The known input  $u$  and the unknown input  $v$  are chosen to be

$$u = \begin{bmatrix} \cos(2t) \\ 5\mathbf{1}(t-2) - 3\mathbf{1}(t-5) \end{bmatrix}, \quad v = 5 \sin(t).$$

where  $\mathbf{1}(t)$  is the unit step function. The dynamics of the system is shown in Figure 5.13. A sensor network of 6 nodes is chosen as shown in Figure 5.14. The sensor model in (3.2) is chosen as

$$C_1^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad C_2^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad C_3^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C_4^* = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_5^* = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_6^* = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and  $G_i^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $i = 1, \dots, 6$ .

To construct the observers  $P = I_4$ ,  $R_1 = R_3 = R_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ ,  $R_2 = R_4 = R_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ , and  $Z_i = \begin{bmatrix} 0 & 0 & 3 & 3 & 0 & 3 \end{bmatrix}$ ,  $i = 1, \dots, 6$  are chosen.

Then, the matrices  $\Lambda_i$ ,  $L_i^1$ ,  $L_i^2$ ,  $H_i$ ,  $M_i$ ,  $N_i$ ,  $C_i^2$ ,  $i = 1, \dots, 6$  in (3.10) are obtained as

$$\Lambda_1 = -1, \quad L_1^1 = 0, \quad L_1^2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Lambda_2 = -1, \quad L_2^1 = 0, \quad L_2^2 = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & -1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Lambda_3 = -1.5, \quad L_3^1 = 0, \quad L_3^2 = \begin{bmatrix} 1 & 0.75 & -1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1.5 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Lambda_4 = -1, \quad L_4^1 = 0, \quad L_4^2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Lambda_5 = -2, \quad L_5^1 = 0, \quad L_5^2 = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}, \quad H_5 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -2 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Lambda_6 = -2.5, \quad L_6^1 = 0, \quad L_6^2 = \begin{bmatrix} -1.5 & 1.5 & -4.75 \end{bmatrix}, \quad H_6 = \begin{bmatrix} 0 & -1.5 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad N_6 = \begin{bmatrix} 0 & 1 & -1.5 \\ 0 & 0 & 1 \\ 0 & 0 & 1.5 \\ 1 & 0 & 0 \end{bmatrix},$$

$$C_1^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad C_2^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad C_3^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

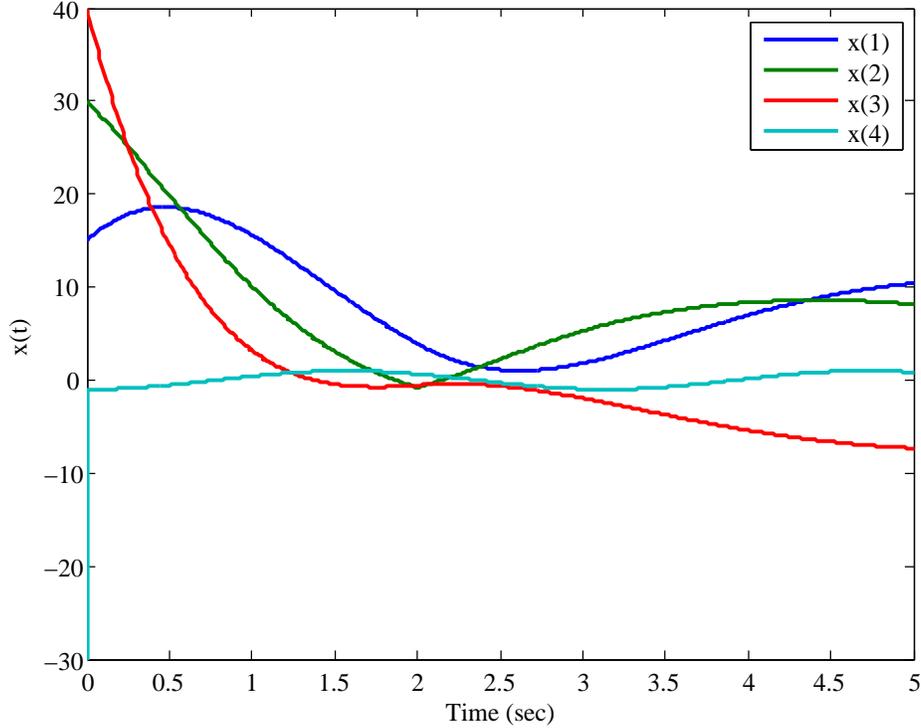


Figure 5.13: Dynamics of the linear system assumed for simulation.

$$C_4^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_5^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_6^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The parameter  $\gamma$  is chosen as 0.01. Additionally, the initial  $z_i(0)$  is chosen in each node independent of the other nodes as

$$z_1(0) = 60, z_2(0) = -30, z_3(0) = -50, z_4(0) = 40, z_5(0) = 30, \text{ and } z_6(0) = -20.$$

The estimation errors of the edge-dependent AD-UIOs proposed by Theorem 3.3 and the non-interacting UIOs in Theorem 3.1 for all nodes are compared in Figures 5.15, 5.16, 5.17, and 5.18. It can be seen that the AD-UIOs in Theorem 3.3 significantly improve the estimation of the non-interacting UIOs in Theorem 3.1.

As a measure of the agreement between the state estimates  $\hat{x}_i$ , the deviation

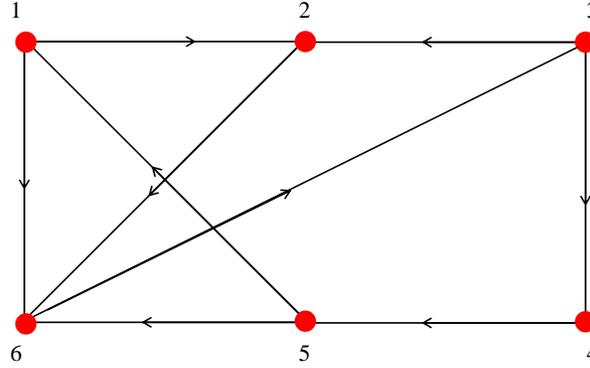


Figure 5.14: A directed sensor network with 6 nodes

from the mean (disagreement) is considered similar to (5.1) as

$$\delta_i(t) = \hat{x}_i(t) - \frac{1}{6} \sum_{j=1}^6 \hat{x}_j(t) = e_i(t) - \frac{1}{6} \sum_{j=1}^6 e_j(t), \quad i = 1, \dots, 6, \quad \delta_i(t) \in \mathbb{R}^n. \quad (5.3)$$

The norm of the deviation from the mean,  $\|\delta_i(t)\|$  of all 6 nodes for the edge-dependent AD-UIOs presented in Theorem 3.3 and for the non-interacting UIOs of Theorem 3.1 are presented in Figure 5.19. It can be seen that the AD-UIOs of Theorem 3.3 exhibit significant improvement over the non-interacting UIOs of Theorem 3.1.

To further emphasize the difference in performance between the proposed AD-UIOs and the non-interacting UIOs, the aggregate estimation error norms  $|e| = |(e_1, e_2, e_3, e_4, e_5, e_6)^T|$  are depicted in Figure 5.20. Additionally, the  $L^2(0, 5; \mathbb{R}^{nN})$  norm  $\|e\|_2$  of the aggregate state error  $e$ , is presented in Table 5.1 to highlight the performance improvement due to the proposed distributed strategy.

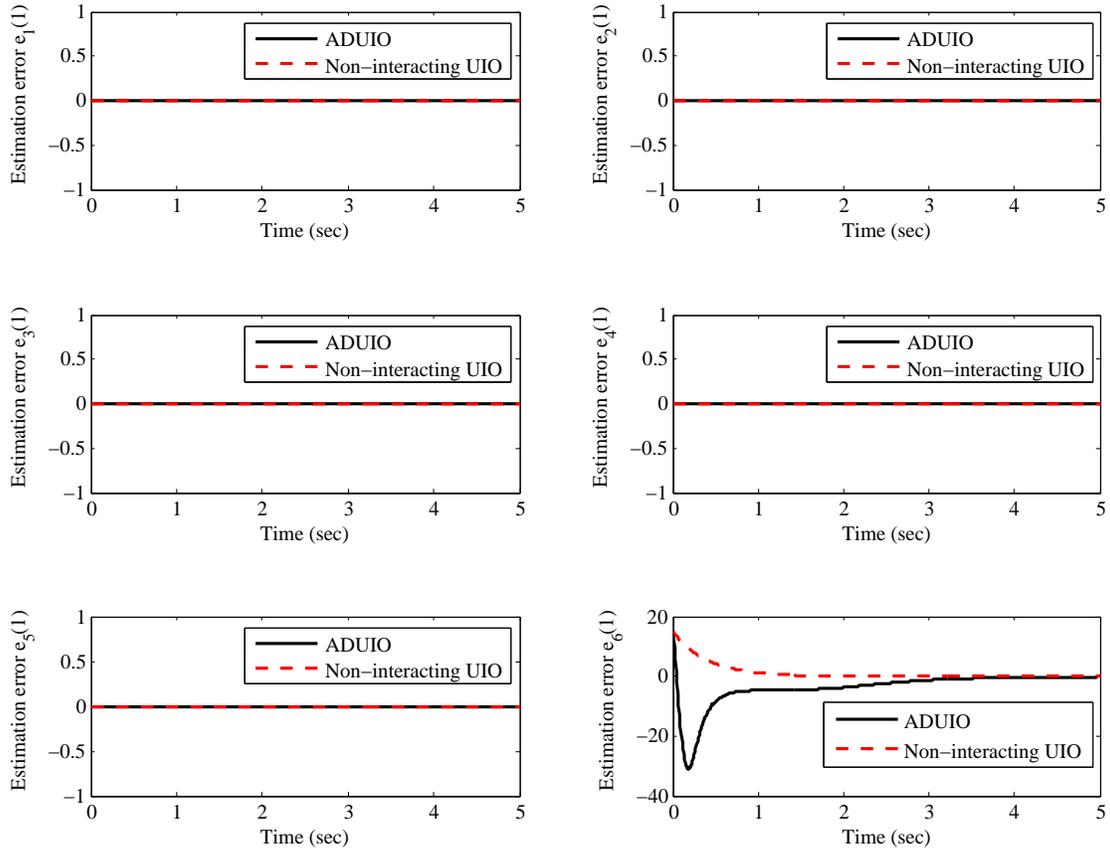


Figure 5.15: Comparison of the first state of estimation error  $e_i = \hat{x}_i - x$  of the AD-UIO proposed by Theorem 3.3 (—) and the non-interacting UIO in Theorem 3.1 (- -) in all 6 nodes.

Table 5.1: Comparison of  $\|e\|_2$  between the edge-dependent AD-UIO in Theorem 3.3 and the non-interacting UIO in Theorem 3.1.

norm	Non-interacting UIO	Adaptive Distributed UIO
$\ e\ _2$	117.22	69.92

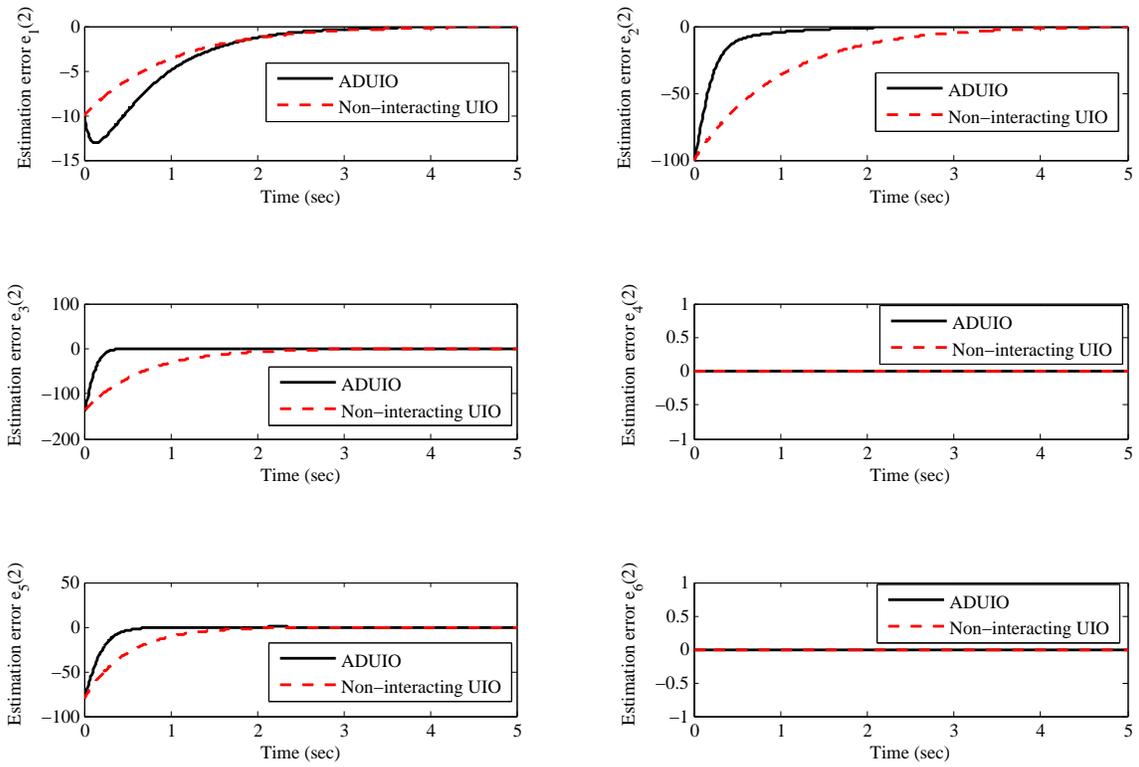


Figure 5.16: Comparison of the second state of estimation error  $e_i = \hat{x}_i - x$  of the AD-UIO proposed by Theorem 3.3 (-) and the non-interacting UIO in Theorem 3.1 (- -) in all 6 nodes.

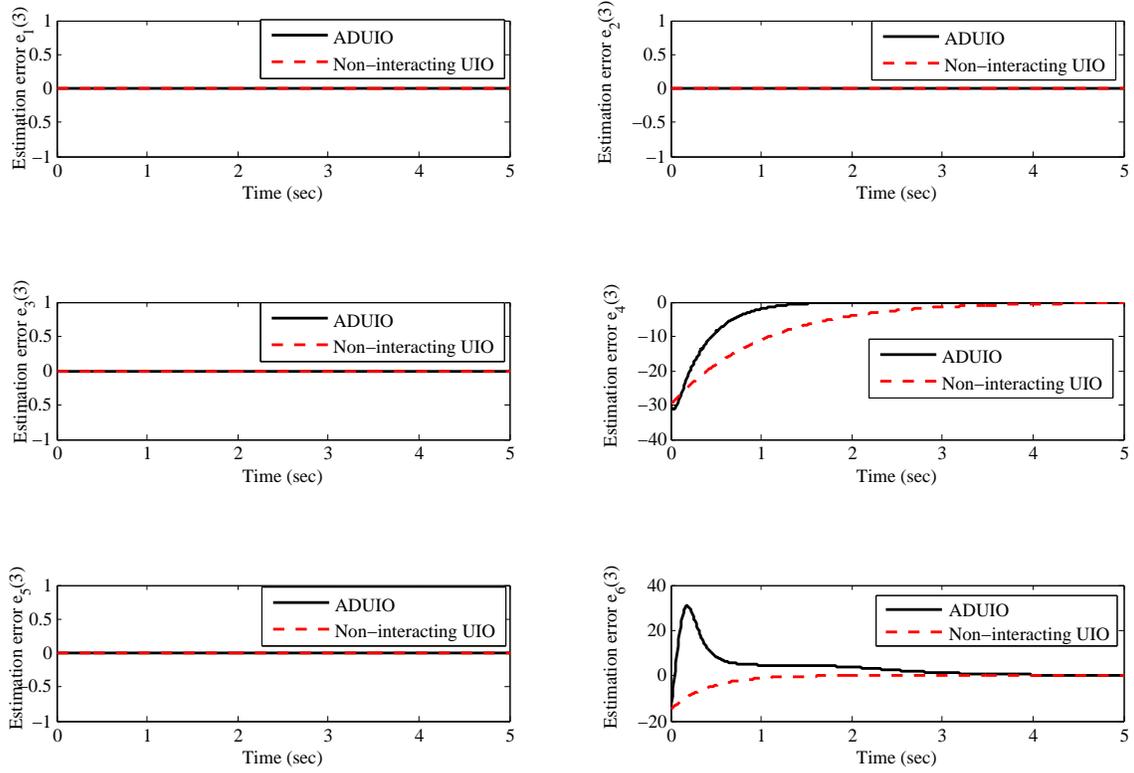


Figure 5.17: Comparison of the third state of estimation error  $e_i = \hat{x}_i - x$  of the AD-UIO proposed by Theorem 3.3 (—) and the non-interacting UIO in Theorem 3.1 (- -) in all 6 nodes.

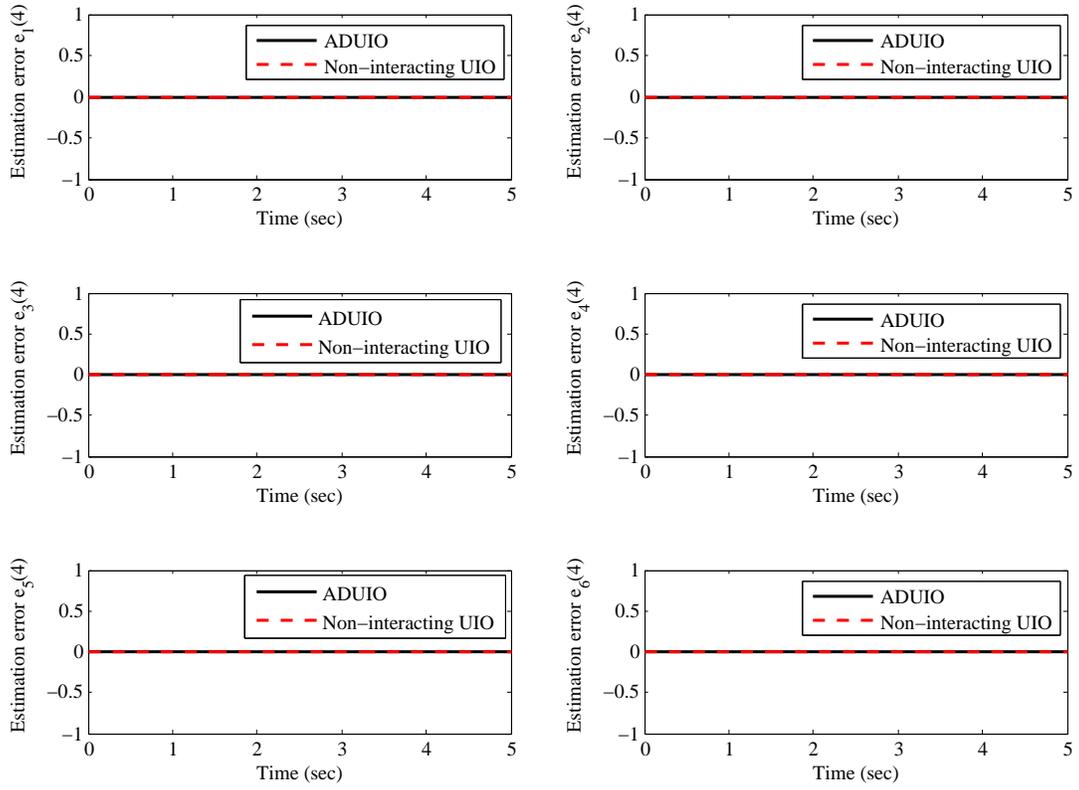


Figure 5.18: Comparison of the fourth state of estimation error  $e_i = \hat{x}_i - x$  of the AD-UIO proposed by Theorem 3.3 (-) and the non-interacting UIO in Theorem 3.1 (- -) in all 6 nodes.

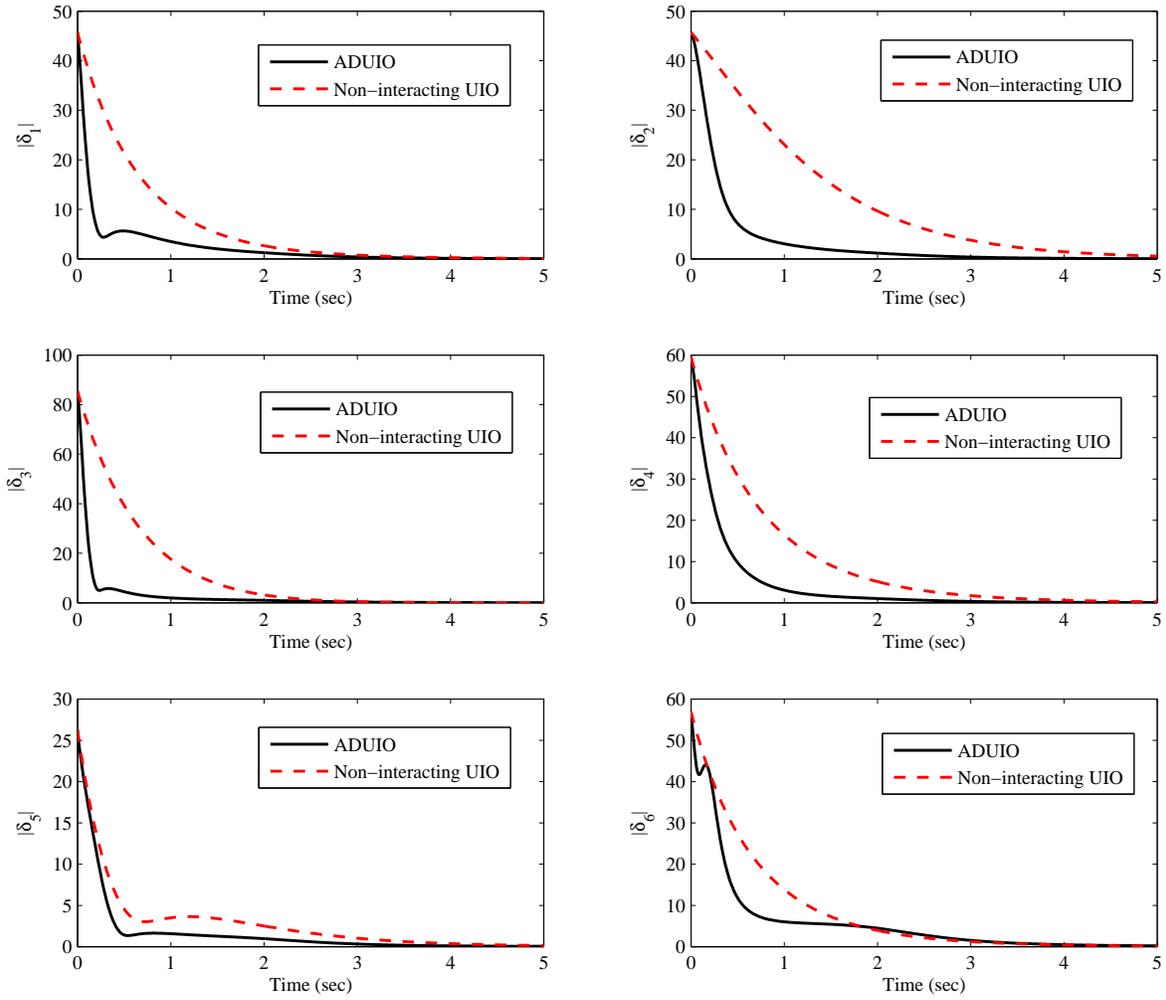


Figure 5.19: Evolution of the norms of the deviations from the mean  $\|\delta_i(t)\|$  in (5.3), of the AD-UIO proposed in Theorem 3.3 (-) and the non-interacting UIO in Theorem 3.1 (- -).

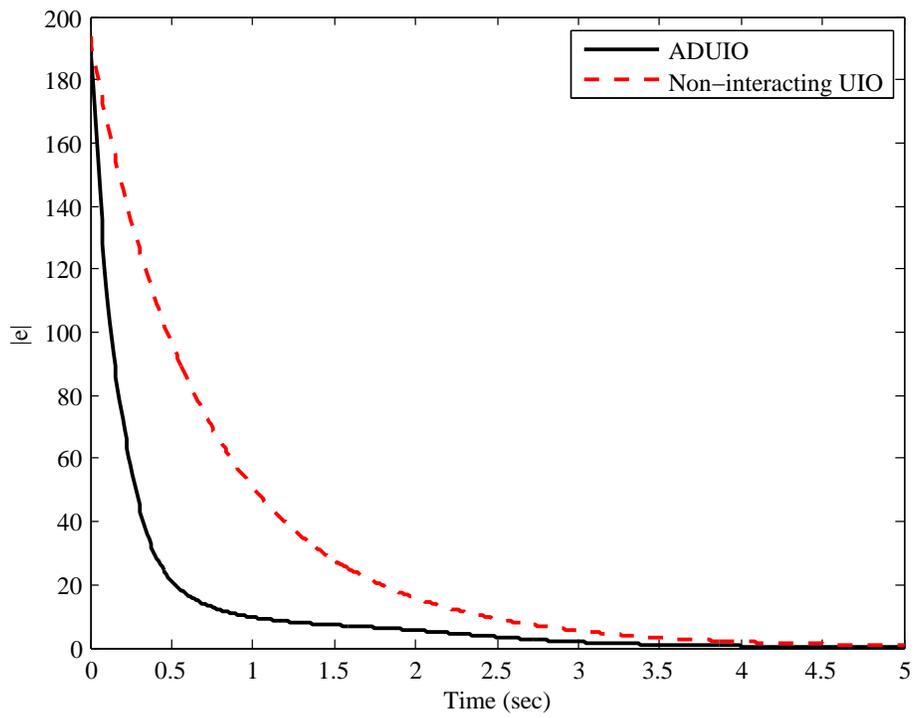


Figure 5.20: Comparison of estimation error norm  $|e|$  of the AD-UIO proposed by Theorem 3.3 (—) and the non-interacting UIO in Theorem 3.1 (---).

### 5.3 ADFO for LTI descriptor systems

In this section, a numerical simulation is presented to demonstrate the performance of the proposed adaptive distributed functional observer scheme. The linear time-invariant (LTI) system in (4.1) is considered with

$$A = \begin{bmatrix} 1 & 0 & -0.25 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and initial condition  $x_0 = (15, 30, 30, -16)^T$ . The matrix  $L$  in (4.3) is chosen as  $L = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ . A sensor network of 4 nodes is chosen as shown in Figure 5.21. Different sensor models are considered for each node in (4.2) as

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

For each of the sensor models, conditions (4.6) and (4.7) must be verified. One has, for  $i = 1, \dots, 4$ ,

$$\text{rank} \left( \begin{bmatrix} \Sigma_i \\ \Theta_i \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} sR_i - \alpha_i A \\ E^\perp A \\ C_i \\ \beta_i A \end{bmatrix} \right) = \text{rank}(\Sigma_i) = 4.$$

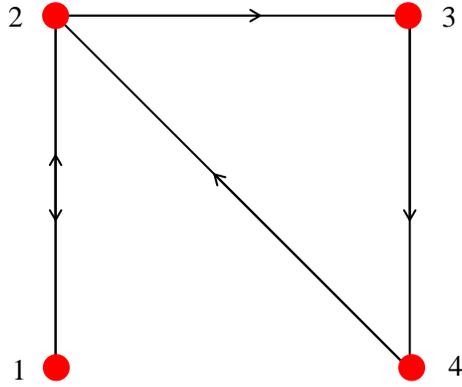


Figure 5.21: A directed sensor network with 4 nodes

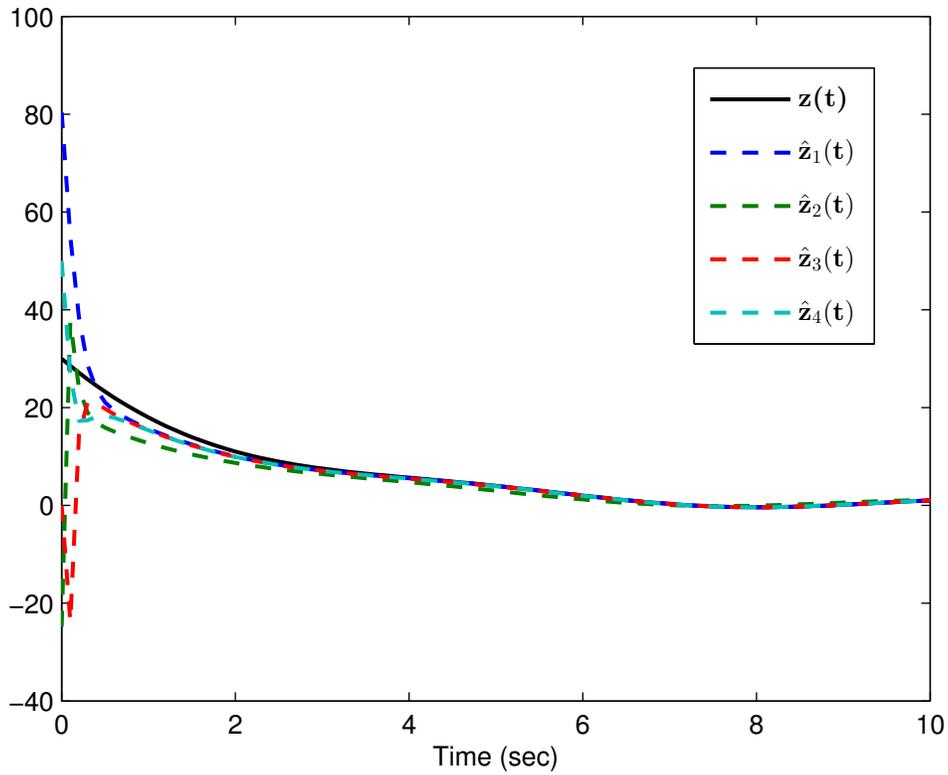


Figure 5.22: Comparison of estimated functional  $\hat{z}_i$ ,  $i = 1, \dots, 4$  and the actual functional  $z$  using the edge-dependent ADFOs in Theorem 4.3.

Therefore the essential conditions in (4.6) and (4.7) are satisfied, which allow us to design FOs with any of the 4 sensor models. The input signal  $u$  is chosen to be  $u = (-\sin(t), \cos(t))^T$ . The estimates of the edge-dependent ADFOs proposed

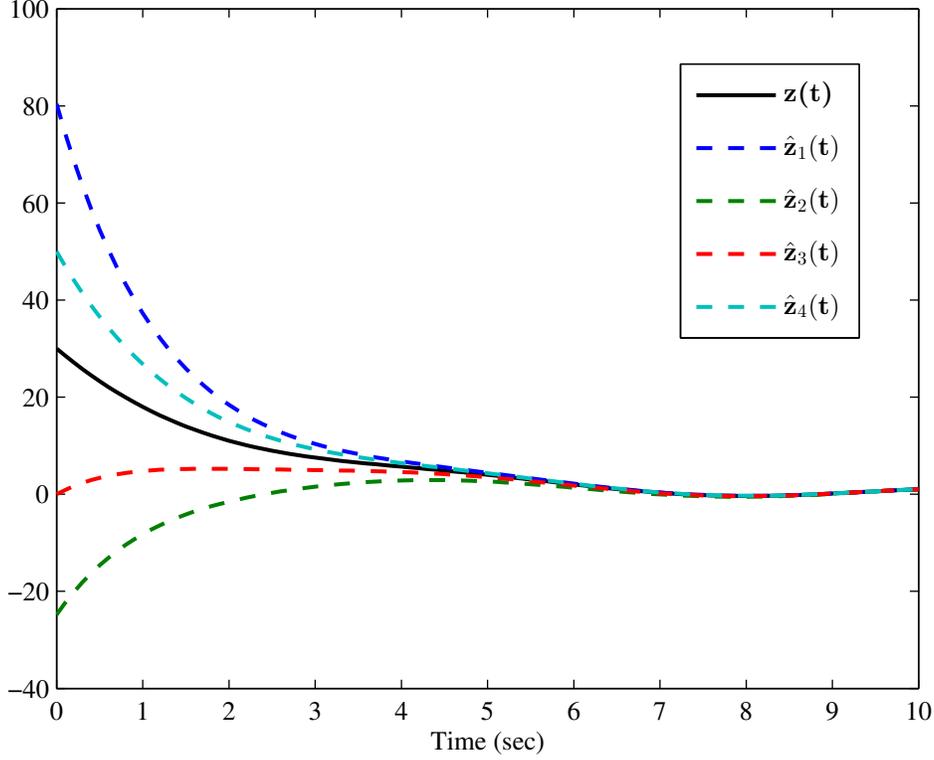


Figure 5.23: Comparison of estimated functional  $\hat{z}_i$ ,  $i = 1, \dots, 4$  and the actual functional  $z$  using the non-interacting FOs in Theorem 4.1.

by Theorem 4.3 and the estimates of the non-interacting FOs in Theorem 4.1 for all nodes are shown in Figures 5.22 and 5.23, respectively. It can be seen that the ADFOs in Theorem 4.3 highly enhance the estimation of the non-interacting FOs in Theorem 4.1.

The estimation errors are compared for the ADFOs proposed by Theorem 4.3 and the non-interacting FOs in Theorem 4.1 in Figure 5.24 for all 4 nodes. It can be seen that the ADFOs in Theorem 4.3 are performing better than the non-interacting FOs in Theorem 4.1. The estimation error norms  $|e| = |(e_1, e_2, e_3, e_4)^T|$  and  $L^2(0, 10)$  norm of  $e$ ,  $\|e\|_2$  are also compared in Figure 5.25 and Table 5.2, respectively, to demonstrate the improvement caused by the proposed distributed strategy.

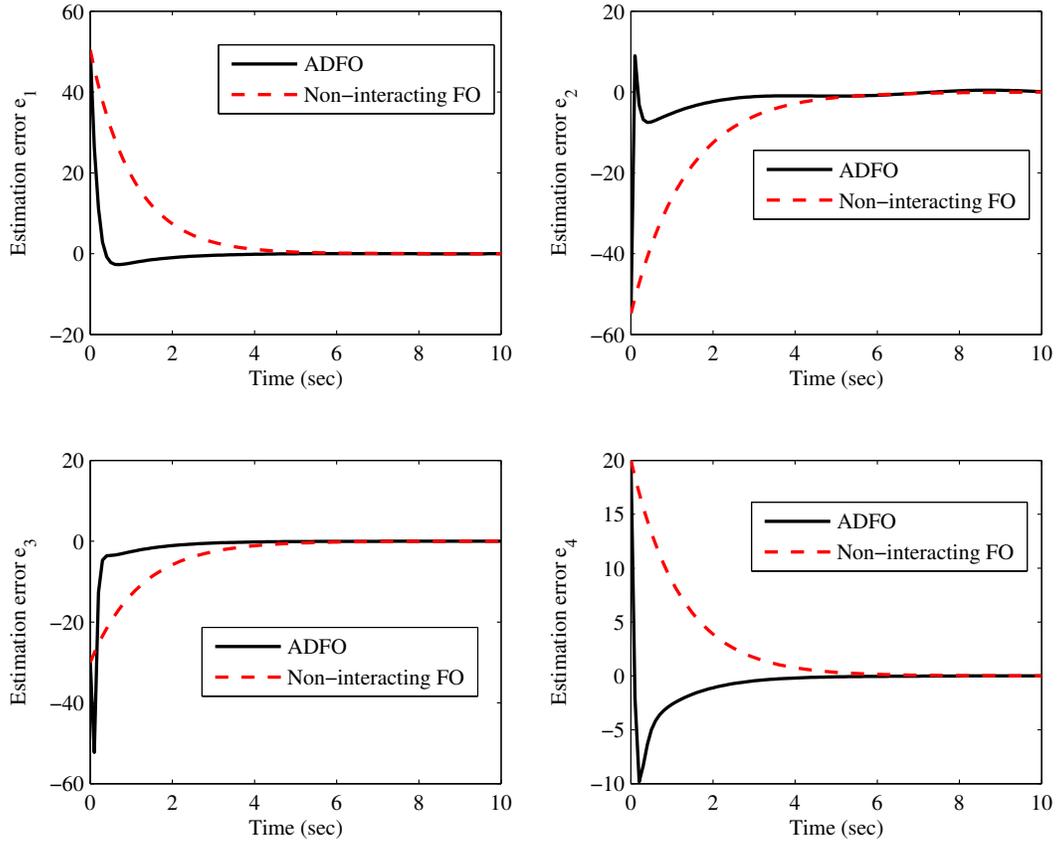


Figure 5.24: Comparison of estimation error  $e_i = \hat{z}_i - z$  of the ADFO proposed by Theorem 4.3 (—) and the non-interacting FO in Theorem 4.1 (---) in all 4 nodes.

Table 5.2: Comparison of  $\|e\|_2$  between the edge-dependent ADFO in Theorem 4.3 and the non-interacting FO in Theorem 4.1.

norm	Non-interacting FO	Adaptive Distributed FO
$\ e\ _2$	64.5	28.8

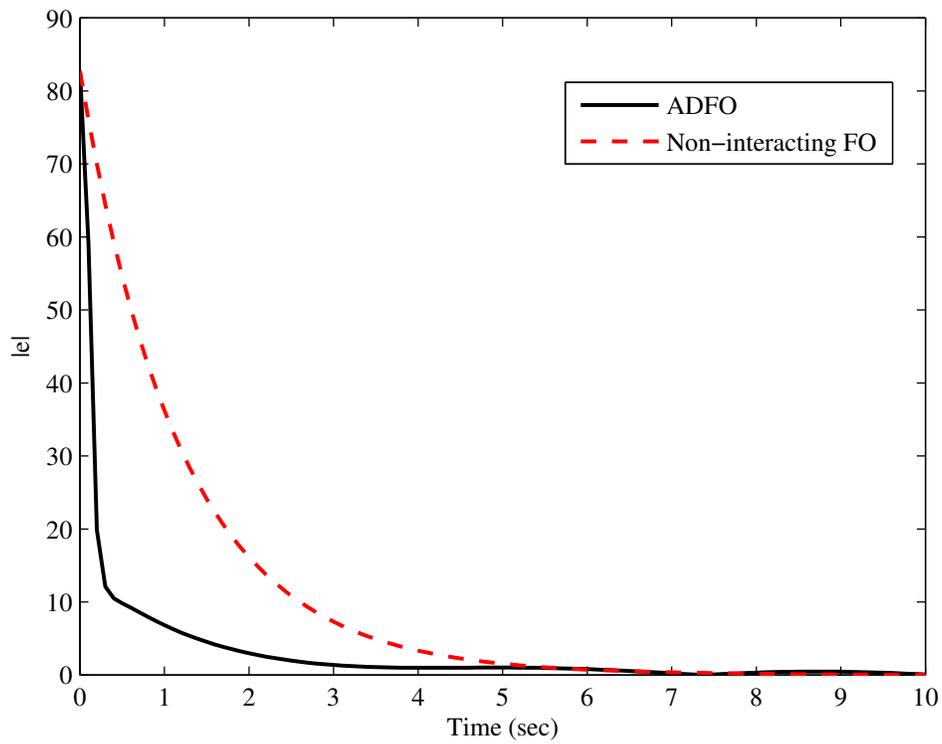


Figure 5.25: Comparison of estimation error norm  $|e|$  of the ADFO proposed by Theorem 4.3 (—) and the non-interacting FO in Theorem 4.1 (---).

# Chapter 6

## Conclusion and Future Work

### 6.1 Conclusions

This research was motivated by the need for adaptive distributed estimation algorithms for sensor networks. The research proposed adaptive distributed strategies for distributed Kalman filters in linear time varying systems, as well as for distributed unknown input observers and distributed functional observers in linear time-invariant descriptor systems. The problem took the form of interconnection gain adaptation. Such a time variation of the interconnection gain aimed at furnishing a time varying penalty gain that is proportional to the level of agreement between different state estimates in the cases of distributed Kalman filters and distributed unknown input observers and to the level of agreement between different functional estimates in the case of distributed functional observers.

The adaptive weights were derived using a Lyapunov-redesign method and were dependent on the level of disagreement between a given node estimate and its communicating neighbors'. The schemes used adaptive gains for each pairwise difference

in the coupling term, which were adjusted in proportion to the pairwise differences of the estimates. A special case where a single adaptive gain is used in each node to uniformly penalize all pairwise differences of the estimates in the coupling term was also presented.

It has been shown that the adaptive distributed scheme can be applied to Luenberger observers in order to construct adaptive distributed Luenberger observers. The distributed unknown input observers and distributed functional observers were also shown for the special case of full-order systems.

The proposed adaptive distributed Kalman filters introduced a significant reduction in communication costs associated with information flow by the nodes compared to the standard (non-adaptive) distributed Kalman filters while demonstrating a similar performance. Also, as an important consequence of this reduction in communication costs, there would be a significant saving in battery power and bandwidth as well.

The proposed distributed unknown input observers and distributed functional observers demonstrated significant improvement compared with the non-interacting unknown input observer and functional observer cases, respectively.

Furthermore, the stability of the proposed schemes is independent of the graph topology and therefore the schemes are applicable to both directed and undirected graphs. This advantage relaxes the limitation of many existing distributed estimation schemes for application to directed graphs.

## 6.2 Future Work

The goal of this work was the development of adaptive distributed estimation algorithms for linear continuous-time systems. Therefore, a highly desirable devel-

opment on the proposed algorithms is to apply the scheme on linear discrete-time systems, especially for the distributed Kalman filter case.

Another modification is to apply the schemes to nonlinear Kalman filters and observers to propose nonlinear adaptive distributed estimation algorithms. Also, considering systems with time-delay in inputs, measurement or communication is another possible extension for the presented schemes. In order to have a better modelling of communication in sensor networks, one can consider asynchronous or intermittent communication.

A very interesting problem to be done in future is to obtain the optimal interconnection gains in (2.3), (3.10), and (4.15) in order to achieve optimal distributed estimation algorithms. One possible method to obtain them is by considering the dual problem which is the optimal distributed synchronization of multi-agent systems.

The preliminary study for application of the proposed adaptive scheme on linear discrete-time systems in order to obtain discrete-time adaptive distributed Kalman filters is discussed in the following section. The stability of the proposed discrete-time scheme should be proven in future.

### 6.2.1 Discrete-time adaptive distributed Kalman filters

Consider a class of discrete-time systems described by the following linear time-varying (LTV) form which are equivalent to the system in (2.1) and (2.2):

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \\ y_i(k) &= C_i(k)x(k) + v_i(k), \quad i = 1, \dots, N. \end{aligned} \tag{6.1}$$

where the process noise  $w(k)$  and the measurement noise  $v_i(k)$  are assumed to be zero-mean Gaussian noise with covariance matrices  $Q(k)\delta(k-l) = E[w(k)w^T(l)]$  and  $R_i(k)\delta(k-l) = E[v_i(k)v_i^T(l)]$ , respectively.

In order to present the adaptive distributed discrete-time Kalman filters for LTV system (6.1), the following analogous assumptions to assumptions 2.2, 2.3 and 2.4 for LTV matrices  $A(k)$ ,  $B(k)$  and  $C_i(k)$  are required. The analogous assumption to assumption 2.1 is also required in order to benefit from the significant reduction in the communication cost.

**Assumption 6.1** (Uniform observability). *The pairs  $(A(k), C_i(k))$  are uniform observable for all  $t \geq t_0$ ,  $\forall i = 1, \dots, N$ .*

**Assumption 6.2** (Uniform controllability). *The pairs  $(A(k), B(k))$  are uniform controllable for all  $t \geq t_0$ .*

**Assumption 6.3** (uniformly bounded plant).  *$A(k)$ ,  $B(k)$  and  $C_i(k)$  are appropriately dimensioned real matrices, and uniformly bounded over the time interval of interest such that  $x(k) \in l^\infty$  for all  $t \geq t_0$ .*

**Assumption 6.4** (Existence of observer matrix). *The pairs  $(A(k), C_i(k))$  are uniform observable and the pairs  $(A(k), B(k))$  are uniform controllable ( $A(k)$ ,  $B(k)$  and  $C_i(k)$  are bounded) if and only if there exist a bounded matrix  $K(k)$  such that  $x(k+1) = (A(k) - K(k)C(k))x(k)$  is exponentially stable [69, 70].*

Note that in the case that system is linear time-invariant, the Assumptions 6.1 - 6.4 are simplified to the pairs  $(A, C_i)$  being observable, all of the eigenvalues of matrix  $A$  being inside the unit disk and  $x(k) \in l^\infty$ .

**Lemma 6.1.** *The system in (6.1) is uniformly exponentially stable if and only if there exists an if there exists a continuous, bounded  $\Pi(k) = \Pi^T(k)$  and  $U(k) =$*

$U^T(k) \geq \alpha_1 I > 0$ , for all  $t \geq t_0$ , such that [63]

$$0 < \alpha_2 I \leq \Pi(k) \leq \alpha_3 I,$$

$$A^T(k)\Pi(k+1)A(k) - \Pi(k) = -U(k).$$

■

In this section, the proposed algorithms in the Chapter 2 are adopted for discrete-time LTV systems. The following lemma proposes the adaptive distributed discrete-time KF based on the edge-dependent adaptive gain strategy.

**Lemma 6.2** (edge-dependent adaptive gain). *When the interconnection weights are allowed to be edge-dependent, then the adaptive-DKFs for discrete-time LTV systems in (6.1) become*

$$\begin{aligned} \hat{x}_i(k) &= \bar{x}_i(k) + K_i(k) (y_i(k) - C_i(k)\bar{x}_i(k)) \\ &\quad + \gamma D_i(k) \sum_{j \in N_i^I} (\Gamma_{ij}(k) C_i(k) (\bar{x}_j(k) - \bar{x}_i(k))), \end{aligned} \quad (6.2a)$$

$$K_i(k) = P_i(k) C_i^T(k) \left( R_i + C_i(k) P_i(k) C_i^T(k) \right)^{-1}, \quad (6.2b)$$

$$M_i(k) = \left( I - K_i(k) C_i(k) \right) P_i(k), \quad (6.2c)$$

$$P_i(k+1) = A M_i(k) A^T + B(k) Q B^T(k), \quad (6.2d)$$

$$\bar{x}_i(k+1) = A(k) \hat{x}_i(k), \quad (6.2e)$$

$$\begin{aligned} \Gamma_{ij}(k+1) &= \Gamma_{ij}(k) - \gamma \left( y_i(k) - C_i(k) \hat{x}_i(k) \right) \times \\ &\quad \left( C_i(k) A(k) \hat{x}_i(k) - C_i(k) A(k) \hat{x}_j(k) \right)^T (k), \quad j \in N_i^I. \end{aligned} \quad (6.2f)$$

where the matrices  $D_i(k)$  are designed such that the set  $\{A_i(k), C_i(k), D_i(k)\}$  satisfy

the following strictly passive condition

$$\begin{aligned} A_i^T(k)\Pi_i(k+1)A_i(k) - \Pi_i(k) &= -U_i(k) \\ A_i^T(k)\Pi_i(k+1)D_i(k) &= C_i^T(k) \end{aligned} \tag{6.3}$$

where  $U_i(k) = U_i^T(k) \geq \alpha_{4i}I > 0$ ,  $\Pi_i(k) = \Pi_i^T(k) > 0$ ,  $A_i \triangleq \left( A(k) - K_i(k)C_i(k)A(k) \right)$ , and  $\gamma$  is a relatively small positive constant which is chosen in the order of discretization time-step. Then the error dynamics  $e_i(k) \triangleq x(k) - \hat{x}_i(k)$  in the system without noise forms a stable linear system, all the estimators asymptotically reach an agreement and all system signals are bounded.

Similarly, the adaptive distributed discrete-time KF based on the node-dependent adaptive gain strategy is proposed in the following lemma.

**Lemma 6.3** (node-dependent adaptive gain). *If the adaptive interconnection weights are chosen as node-dependent, then the adaptive distributed discrete-time KF's are given by*

$$\begin{aligned} \hat{x}_i(k) &= \bar{x}_i(k) + K_i(k) \left( y_i(k) - C_i\bar{x}_i(k) \right) \\ &\quad + \gamma D_i(k) \Gamma_i(k) \sum_{j \in N_i^I} (C_i\bar{x}_j(k) - C_i\bar{x}_i(k)), \\ K_i(k) &= P_i(k) C_i^T(k) \left( R_i + C_i(k) P_i(k) C_i^T(k) \right)^{-1}, \\ M_i(k) &= \left( I - K_i(k) C_i(k) \right) P_i(k), \\ P_i(k+1) &= A M_i(k) A^T + B(k) Q B^T(k), \\ \bar{x}_i(k+1) &= A(k) \hat{x}_i(k), \\ \Gamma_i(k+1) &= \Gamma_i(k) - \gamma \left( y_i(k) - C_i(k) \hat{x}_i(k) \right) \times \\ &\quad \sum_{j \in N_i^I} \left( C_i(k) A(k) \hat{x}_i(k) - C_i(k) A(k) \hat{x}_j(k) \right)^T (k). \end{aligned} \tag{6.4}$$

The following Lemma 6.4 and Lemma 6.5 are instrumental to the proof of Lem-

ma 6.2.

**Lemma 6.4.** *The state estimation errors are defined as  $e_i(k) \triangleq x(k) - \hat{x}_i(k)$ . Following Assumption 6.4, the system  $e_i(k+1) = (A(k) - K_i(k)C_i(k))e_i(k)$  is asymptotically stable [23].* ■

**Lemma 6.5.** *Following Lemma 6.1 and Lemma 6.4, for  $U_i(k)$  a symmetric positive definite matrix, the solution to the Lyapunov equation  $\Pi_i(k)$  in (6.3) exists. Therefore, one can define  $D_i(k)$  as*

$$D_i(k) = \Pi_i^{-1}(k+1)A_i^{-T}(k)C_i^T(k), i = 1, \dots, N. \quad (6.5)$$

■

**Proposition 6.1.** *Following Assumptions 6.1, 6.2, 6.3, the error covariance matrix  $P_i(k+1)$  is positive definite and  $P_i^{-1}(k+1)$  is bounded [70]. Thus, one can multiply (6.2d) by the inverse of  $P_i(k+1)$  and add  $A_iP_i^{-1}(k+1)A_i^T - P_i^{-1}(k) - I$  to obtain*

$$\begin{aligned} A_iP_i^{-1}(k+1)A_i^T - P_i^{-1}(k) &= P_i^{-1}(k+1)\left(AM_i(k)A^T + B(k)QB^T(k)^T\right) \\ &\quad + A_iP_i^{-1}(k+1)A_i^T - P_i^{-1}(k) - I \end{aligned}$$

*This allows one to use  $P_i^{-1}$  as  $\Pi_i$ . Therefore, the  $D_i(k)$  may be chosen as*

$$D_i(k) = P_i(k+1)A_i^{-T}(k)C_i^T(k), \quad i = 1, \dots, N. \quad (6.6)$$

*Note that  $\frac{1}{\alpha_5}P_i^{-1}$ ,  $\alpha_5 > 0$ , is also a possible candidate for  $\Pi_i$ .* ■

The filter gains  $K_i(t)$  in (6.2a) are not necessarily required to be the standard (non-interconnected) Kalman filter gains. Therefore, one can use a Luenberger observer design [63] instead of Kalman filter design for deterministic systems to

construct a new adaptive distributed filter, termed here the *adaptive-distributed Luenberger observer*. The adaptive distributed discrete-time Luenberger observer based on the edge-dependent adaptive gain strategy is proposed in the following lemma.

**Lemma 6.6** (adaptive distributed discrete-time Luenberger observer). *Consider a sensor network with the sensing model presented in (6.1). If the following distributed estimation algorithm with an adaptively edge-dependent interconnection gain is utilized at each node*

$$\begin{aligned} \hat{x}_i(k+1) &= A(k)\hat{x}_i(k) + L_i(k)(y_i(k) - C_i(k)\hat{x}_i(k)) \\ &+ \gamma D_i(k) \sum_{j \in N_i^f} \Gamma_{ij}(k) (C_i(k)(\hat{x}_j(k) - \hat{x}_i(k))), \end{aligned} \quad (6.7)$$

where the  $L_i(k)$  are the Luenberger gains and the matrices  $D_i(k)$  are designed such that the set  $\{A_{2i}(k), C_i(k), D_i(k)\}$  satisfy the following strictly passive condition

$$\begin{aligned} A_{2i}^T(k) \Pi_i(k+1) A_{2i}(k) - \Pi_i(k) &= -U_i(k) \\ A_{2i}^T(k) \Pi_i(k+1) D_i(k) &= C_i^T(k) \end{aligned} \quad (6.8)$$

with  $U_i(k) = U_i^T(k) \geq \alpha_{6i} I > 0$ , and  $\Pi_i(k) = \Pi_i^T(k) > 0$  and where  $A_{2i}(k) \triangleq A(k) - L_i(k)C_i(k)$  and the adaptive gain matrix  $\Gamma_{ij}(k)$  is obtained using the following adaptive law

$$\Gamma_{ij}(k+1) = \Gamma_{ij}(k) - \gamma (y_i(k) - C_i(k)\hat{x}_i(k)) \left( C_i(k)\hat{x}_i(k) - C_i(k)\hat{x}_j(k) \right)^T, \quad (6.9)$$

where  $\gamma$  is a relatively small positive constant which is chosen in the order of discretization time-step.

The following Lemma 6.7 and Lemma 6.8 are essential to be stated prior to proof

of the Lemma 6.6.

**Lemma 6.7.** *Following Assumption 6.4, there exist filter gains  $L_i(k)$  such that the system  $e_i(k+1) = (A(k) - L_i(k)C_i(k))e_i(k)$  is asymptotically stable. ■*

**Lemma 6.8.** *Following Lemma 6.1 and Lemma 6.7, for  $U_i(k)$  a symmetric positive definite matrix, the solution to the Lyapunov equation  $\Pi_i(k)$  in (6.8) exists. Since  $\Pi_i(k) > 0$ , then one can define  $D_i(k)$  as*

$$D_i(k) = \Pi_i^{-1}(k+1)A_{2i}^{-T}(k)C_i^T(k), i = 1, \dots, N. \quad (6.10)$$

■

Now the proof of Lemma 6.2, Lemma 6.3, and Lemma 6.6, can be stated. A possible method to establish the stability of the error dynamics in them is to use similar Lyapunov-like functions to Chapter 2. Therefore, the following Lyapunov-like function is a candidate for Lemma 6.2 and Lemma 6.6

$$V_i(k) = e_i^T(k)\Pi_i(k)e_i(k) + \text{tr} \left( \sum_{j \in N_i^I} (\Gamma_{ij}(k)\Gamma_{ij}^T(k)) \right),$$

$$i = 1, \dots, N,$$

and the following Lyapunov-like function is a candidate for Lemma 6.3

$$V_i(k) = e_i^T(k)\Pi_i(k)e_i(k) + \text{tr} \left( \Gamma_i(k)\Gamma_i^T(k) \right),$$

$$i = 1, \dots, N.$$

The proof of the Lemma 6.2, Lemma 6.3, and Lemma 6.6 can be done as a possible future contribution to this work.

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