Measuring the Variability of Chain Ladder Reserve Estimates

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Abstract

The chain ladder method is a popular heuristic used to estimate ultimate losses. This report, based on “Measuring the Variability of Chain Ladder Reserve Estimates” by Thomas Mack, will present how the method works, its underlying assumptions, and how these can be combined to create reserves and ultimate losses confidence intervals based on the variability of chain ladder estimations. This confidence interval also allows the chain ladder method to be compared to other methods, and provide greater certainty for those methods.

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Section 1: Introduction

This paper starts in Section 2 with a thorough introduction and description of the chain ladder method and its underlying assumptions, as well as its associated age-to-age factors. In Section 3, the age-to-age factors of the chain ladder method are described, as well as their independence in Section 4. Section 5 uses the prior sections to calculate a confidence interval for the ultimate losses. In Section 6, this entire method is then demonstrated with a complete numerical example. Finally, in Section 7, examples of ways to test for is the underlying assumptions of the chain ladder method are demonstrated.

Throughout this paper, a series of estimators shall be used in order to form the ultimate losses confidence interval. These estimators are notated in **bold**, such that, as an example, $\mathbf{f_k}$ is an estimator for the parameter f_k.

Additionally, reported claims total in each development triangle are notated as $C_{j,k}$, where j represents the accident year (with j =1 being the oldest listed year in a development triangle), while k represents the accident period. The term $C_{j,I}$, the reported claims of accident year j at development period I, is considered the ultimate losses, where I is the greatest observed value of k, which is equal to the greatest observed value of j.

Section 2: The Chain Ladder Method

	Reported claims as of (months)				
Accident Year	12	24	36	48	60
1998	37,017,487	43,169,009	45,568,919	46,784,558	47,337,318
1999	38,954,484	46,045,718	48,882,924	50,219,672	
2000	41,155,776	49,371,478	52,358,476		
2001	42,394,069	50,584,112			
2002	44,755,243				

Table 1: Development Triangle Example

A development triangle, as pictured above, consists of a series of paid or reported claims from different accident years, as well as claims totals for each development period. In the above example, the total claims for AY2000 were $37,017,487 on 12/31/1999, 12 months after the start of the period. 24 months after the start of the period, those claims had increased to $43,169,009, and the pattern of cumulative reported claims amounts continue to the right, ending at 12/31/2002. This same process is true for the other accident years as well, continuing to the right until it reaches the diagonal, highlighted in yellow.

At some point, these reported claims amounts reach a point at which they cease to continue developing, as all claims for that accident year have fully resolved. This amount is called the **ultimate losses**. Throughout this paper, the rightmost column of each development triangle is considered the ultimate amount, but this is not necessarily the case in practice.

	Age-to-age factors			
Accident Year	12 to 24 months	24 to 36	36 to 48	48 to 60
1998	1.166	1.056	1.027	1.012
1999	1.182	1.062	1.027	
2000	1.200	1.061		
2001	1.193			

Table 2: Age-to-Age Factors Example

Given a development triangle, the age-to-age factors of development periods can be calculated, as pictured above. These are calculated by taking a reported claims amount, and dividing it by the preceding amount. For example, when the reported claims as of 24 months for AY1998 ($43,169,009) are divided by the reported claims as of 12 months ($37,017,487), the resulting age-to-age factor is 1.166.

On their own, these age-to-age factors do not reveal any information that was not already apparent from the original development triangle. However, they can be useful for estimating the “true” age-to-age factors that the chain ladder method assumes to exist, as described in the next section.

Section 3: Age-to-Age Factors

The core premise of the chain ladder method is the assumption of the existence of age-to-age factors for each development period, which are the same for every accident year. The observed age-to-age factors are usually different from these assumed factors notated as f_k, where k represents which development period it is. For example, the age-to-age factor from 12 to 24 months is notated as f_1, while the age-to-age factor from 48 to 60 months is notated as f_4.

Accident Year	12 months	24 months	Age-to-age factor
2001	100	210	2.10
2002	140	200	1.43
2003	135	180	1.33
2004	150	190	1.27
2005	130		

Table 3: Observed Age-to-Age Factors from 12 to 24 Months

There exist multiple ways of choosing an estimated value of f_k. In the table above, the reported claims as of 12 and 24 months are shown, as well as the observed age-to-age factors. Examples of estimates for the true age-to-age factor include, but are not limited to:

Arithmetic Mean

- $(2.10 + 1.43 + 1.33 + 1.27) \,/\, 4 = 1.53$

Arithmetic Mean (excluding maximum and minimum)

- $(1.43 + 1.33) \,/\, 2 = 1.38$

Geometric Mean

- $\sqrt[4]{(1.54 \cdot 1.43 \cdot 1.33 \cdot 1.27)} = 1.50$

Weighted Average

- $(210 + 200 + 180 + 190) \,/\, (100 + 140 + 135 + 150) = 1.49$

Squared-Weighted Average

- $(100 \cdot 210 + 140 \cdot 200 + 135 \cdot 180 + 150 \cdot 190) \,/\, (100^2 + 140^2 + 135^2 + 150^2) = 1.45$

User-chosen value

Due to changes in how claims are handled over time, or due to observable trends, it is also possible to choose an estimated value of f_k that is based on actuarial judgement. For example, there is a downward trend in age-to-age factors seen in the data, so the mostly recently observed and minimum value, 1.27, could be an appropriate estimator.

These age-to-age factors can be used to predict how claims will continue to develop. In the previous example, if the arithmetic mean of 1.53 was chosen as an estimator, then it would be expected that AY2005 reported claims amount of 130 would increase by a factor of 1.53, to 199. Below is an example of a development triangle that has been entirely filled in with estimated values of future claims amount, based on the weighted average of age-to-age factors.

	Reported claims as of (months)				
Accident Year	12	24	36	48	60
1998	37,017	43,169	45,568	46,784	47,337
1999	38,954	46,045	48,882	50,219	
2000	41,155	49,371	53,358		
2001	42,394	50,584			
2002	44,755				
	12 to 24	24 to 36	36 to 48	48 to 60	
Age-to-age factors	1.186	1.067	1.027	1.012	
	Reported claims as of (months)				
Accident Year	12	24	36	48	60
1998	37,017	43,169	45,568	46,784	47,337
1999	38,954	46,045	48,882	50,219	50,813
2000	41,155	49,371	53,358	54,800	55,448
2001	42,394	50,584	53,950	55,409	56,064
2002	44,755	53,073	56,605	58,135	58,823

Table 4: Development Triangle and Expected Future Claims

These are the expected future claims amount, and any deviation from that can be considered a random disturbance from the “true” age-to-age increase, as estimated by the chosen age-to-age factors. For this reason, at the end of a development period, all unknown future claims amounts (those to the right of the diagonal) can be treated as random variables, and all currently known claims amounts can be treated as scalars, because they are constants.

The chain ladder method only uses the most recent claims information, the total claims reported to date, from the diagonal, for each accident year to estimate future claims amount. Additional estimators could be formed by using earlier claims amounts and multiplying by their respective chosen age-to-age factors for each development period. The chain ladder method ignores these other estimators, and thus the relationship for future claims amount can be seen here:

Assumption 1

- $E [C_{i, k+1} | C_{i, 1}, \dots , C_{i, k}] = C_{i, k}f_k$, or equivalently,
- $E [C_{i, k+1} / C_{i, k} | C_{i, 1}, \dots , C_{i, k}] = f_k$

This relationship is an implicit assumption of the chain ladder method, which shows that the estimated value for future claims is not impacted by claims amounts preceding the diagonal. This assumption is not necessarily true for every development triangle, and ways to test if a given development triangle meets this assumption, as well as examples of development triangles that do not meet this or other assumptions are included in Section 7. For the body of this paper, it will be treated that these assumptions are met for the discussed development triangle.

The equation $E [C_{i, k+1} / C_{i, k} | C_{i, 1}, \dots , C_{i, k}] = f_k$ also shows how the expected development from $C_{i, k}$ to $C_{i, k+1}$ is f_k regardless of all previous observed development factors, including the preceding $C_{i, k} / C_{i, k-1}$. Consecutive development factors are uncorrelated; after a small value of $C_{i, k} / C_{i, k-1}$, it is not expected that $C_{i, k+1} / C_{i, k}$ will be larger, and vice versa. This is once again not an assumption that can be met for every development triangle, and will be discussed further in Section 7.

Section 4: Age-to-age Factors & Independence of Accident Years

The true values of age-to-age factors $f_1, \dots f_{I-1}$ can not be inferred through the limited data of a single development triangle, and can only be estimated. The estimator used in this paper is the weighted average, which is an unbiased estimator of f_k. This is the case with the additional assumption that claims amount of different accident years are independent of one another. Mathematically, this is expressed as:

Assumption 2

- The variables $\{C_{i,1}, \dots, C_{i,I}\}$ and $\{C_{j,1}, \dots, C_{j,I}\}$ of different accident years $i \neq j$ are independent.

Because the defining equations of the chain ladder method do not take into account any dependency between accident years, the independence of accident years can thus be taken as another implicit assumption of the chain ladder method. Once again, this is not necessarily true of every development triangle, but shall be assumed to be true until it is discussed in Section 7.

Proof 1: The Weighted Average Estimators are Unbiased

The weighted average estimators $\mathbf{f_1}, \dots, \mathbf{f_{I-1}}$ are unbiased estimators of $f_1, \dots, f_{I-1}$, as shown here, with the definition:

- $\mathbf{f_k} = \sum_{j=1}^{I-k} C_{j,k+1} / \sum_{j=1}^{I-k} C_{j,k}$, $1 \leq k \leq I-1$.
- $E[\,\mathbf{f_k}\,] = E[\,E[\,\mathbf{f_k} \mid B_k\,]\,]$ by the iterative rule of expectations, where B_k represents the set of all reported claims totals. Because these claims can all be considered scalars,
- $E[\,\mathbf{f_k} \mid B_k\,] = \sum_{j=1}^{I-k} E[\,C_{j,k+1} \mid B_k] / \sum_{j=1}^{I-k} C_{j,k}$, but because of the assumption of the independence of accident years, any conditions related to accident years besides $C_{j,k+1}$ can be ignored. Thus,
- $E[\,C_{j,k+1} \mid B_k\,] = E[\,C_{j,k+1} \mid C_{j,1}, \dots, C_{j,k}\,] = C_{j,k} f_k$, which yields
- $E[\,\mathbf{f_k} \mid B_k\,] = \sum_{j=1}^{I-k} C_{j,k} f_k / \sum_{j=1}^{I-k} C_{j,k} = f_k$, and finally
- $E[\,\mathbf{f_k}\,] = E[\,f_k\,] = f_k$, proving it is an unbiased estimator.

The weighted average is not the only unbiased estimator of f_k. Every observed development factor $C_{i, k+1} / C_{i, k}$ is also an unbiased estimator of f_k, as shown below.

- $E [C_{i, k+1} / C_{i, k}] = E [C_{i, k+1} / C_{i, k} | C_{i, 1}, \dots , C_{i, k}]$

$$= E [E [C_{i, k+1} \mid C_{i, 1}, \dots , C_{i, k}] / C_{i, k}]$$

$$= E [C_{i,k} f_k / C_{i,k}]$$

$$= E [f_k]$$

$$= f_k \qquad \blacksquare$$

The reason that the weighted average is chosen over these or any other unbiased estimator is because in point estimation, the preferred estimator is the one that minimizes variance. This is the case if and only if the chosen weights $w_{j, k}$ are inversely proportional to $Var(C_{j, k+1} / C_{j, k} | C_{j,1}, \dots , C_{j,k})$, as proven here:

<u>Proof 2: The Weighted Average of Age-to-Age Factors Minimizes Variance</u>

Given some number of independent unbiased estimators T_i of parameter t with $E [T_i] = t$, the variance of a linear combination of them, T, (with weights w_i adding to 1) is minimal if and only if the weights are inversely proportional to the variance of those estimators. That is to say, $w_i = c / Var(T_i)$.

To minimize $Var(T) = \sum_{i=1}^{I} w_i^2 Var(T_i)$, the extremum must be found. These are where the derivatives of the Lagrangian are equal to 0, such that

- $\frac{\partial}{\partial w_i} (\sum_{i=1}^{I} w_i^2 Var(T_i) + \lambda(1 - \sum_{i=1}^{I} w_i)) = 0$, which yields
- $2w_i Var(T_i) – \lambda = 0$, or
- $w_i = \lambda / (2Var(T_i))$.

With the chain ladder method, $T_i = C_{i, k+1} / C_{i, k}$. To minimize the variance

- $Var(\sum_{i=1}^{I} w_i T_i | C_{i, 1}, \dots , C_{i, k})$,

it can be seen that the minimizing weights are inversely proportional to $Var(T_i | C_{i, 1}, \dots , C_{i, k})$, and since the independence of accident years ensures that $T_i = C_{i, k+1} / C_{i, k}$ are independent, the result is that the minimizing weights are proportional to $Var(C_{j, k+1} / C_{j, k} | C_{j,1}, \dots , C_{j,k})$. ■

Mathematically,

- $Var(C_{j,k+1} / C_{j,k} \mid C_{j,1}, \ldots, C_{j,k}) = \alpha_k^2 / C_{j,k}$,

where α_k^2 is a non-negative proportionality constant independent of j, but possibly dependent on k. Because $C_{j,k}$ is a scalar, the statement can be restated as:

Assumption 3

- $Var(C_{j,k+1} \mid C_{j,1}, \ldots, C_{j,k}) = C_{j,k}\alpha_k^2$

Once again, this equation is an implied assumption of the chain ladder method, and not every development triangle meets this assumption, as will be described in Section 7. Until then, Assumptions 1, 2, and 3 can be used to calculate the uncertainty of the ultimate losses estimator $\mathbf{C_{i,I}}$.

Section 5: Variance of Ultimate Losses

The point of the chain ladder method is to find an estimation for the ultimate losses $C_{i,I}$ for accident years i = 2, … I. The chain ladder method creates a point estimate for $C_{i,I}$ by Assumption 1:

- $\mathbf{C_{i,I}} = C_{i,I+1-i} \cdot \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_1}$,

which is an unbiased estimator of $C_{i,I}$, under Assumptions 1 and 2, as shown here:

Proof 3: The Ultimate Losses Estimator is Unbiased

First, it must be shown that the age-to-age factors $\mathbf{f_k}$ are uncorrelated, given the same set B_k as before, with j < k.

- $E[\mathbf{f_j f_k}]$ $= E[E[\mathbf{f_j f_k} \mid B_k]]$ by the iterative rule of expectations.

  $= E[E[\mathbf{f_j f_k} \mid B_k]]$ because $\mathbf{f_j}$ is a scalar for j < k.

  $= E[\mathbf{f_j} f_k]$ due to equation [INSERT NUMBER].

  $= E[\mathbf{f_j}] f_k$ because f_k is a scalar.

  $= f_j f_k$.

This result can be iterated to any number of $\mathbf{f_k}$'s, resulting in

- $E[\mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_{I-1}}] = f_{I+1-i} \cdot \ldots \cdot f_{I-1}$, which yields
- $E[\mathbf{C_{i,I}}]$ $= E[E[\mathbf{C_{i,I}} \mid C_{i,1}, \ldots, C_{i,I+1-I}]]$ by the iterative rule of expectations.

  $= E[E[C_{i,I+1-i} \cdot \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_1} \mid C_{i,1}, \ldots, C_{i,I+1-I}]]$ by the definition of $\mathbf{C_{i,I}}$.

  $= E[C_{i,I+1-i} \cdot E[\mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_1} \mid C_{i,1}, \ldots, C_{i,I+1-I}]]$ because $C_{i,I+1-I}$ is a scalar.

  $= E[C_{i,I+1-i} \cdot E[\mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_1}]]$ because conditions independent of $\mathbf{f_{I+1-i}}, \ldots, \mathbf{f_1}$ can be ignored.

  $= E[C_{i,I+1-i}] \cdot E[\mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_1}]$ because $E[\mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_1}]$ is a scalar.

  $= E[C_{i,I+1-i}] \cdot f_{I+1-i} \cdot \ldots \cdot f_1$ as stated above.

Assumption 1 can be iterated upon as follows:

- $E[C_{i,I}]$ $= E[E[C_{i,I} \mid C_{i,1}, \ldots, C_{i,I+1-I}]]$

  $= E[C_{i,I-1} f_{I-1}]$

$= E [C_{i, I-1}] \cdot f_{I-1}$

= etc.

$= E [C_{i, I+1-i}] \cdot f_{I+1-i} \cdot \ldots \cdot f_1$

$= E [\mathbf{C}_{i, I}]$, proving $\mathbf{C}_{i, I}$ is an unbiased estimator of $C_{i, I}$. ■

However, as stated before, this only results in a point estimate for the ultimate amount. The actual ultimate amount is considered a random variable, and can deviate from the estimated amount. What would be preferred, in addition to $\mathbf{C}_{i, I}$ is, the mean squared error

- $mse(\mathbf{C}_{i, I}) = E [(C_{i,I} - \mathbf{C}_{i, I})^2 | D]$,

where D is the set of all observed claims totals so far.

Calculating the mean squared error based on all observed claims is important, because the goal is to calculate the ultimate losses of the given development triangle based on future randomness, rather than possible deviations from previous observed claims, which are treated as scalars. The mean squared error can also be expressed as such:

- $mse(\mathbf{C}_{i, I}) = Var(C_{i,I} | D) + (E [C_{i,I} | D] - \mathbf{C}_{i, I})^2$

This is because $\mathbf{C}_{i, I}$ is a scalar under the condition that D is all known. The rest of the expression is an inherent definition of variances, namely, that

- $E [X - c]^2 = Var(X) + (E [X] - c)^2$

This calculation for the mean square error is only true if Assumptions 1, 2, and 3 remain true into the future. Development triangles where this is not the case are not discussed in this paper.

The average distance between the estimated ultimate losses and the actual ultimate losses is found through calculating the mean squared error of the ultimate losses. The square root of the estimator $mse(\mathbf{C}_{i, I})$, known as the standard error, is the standard deviation ultimate losses.

The standard error $s.e.(\mathbf{C}_{i, I})$ is equal to the standard error $s.e.(\mathbf{R}_i)$ of the estimator,

- $\mathbf{R}_i = C_{i, I} - C_{i, I+1}$, which estimates the outstanding claims reserve,

- $R_i = C_{i,\,I} - C_{i,\,I+1}$.

The mean squared error of the reserve and ultimate losses are equivalent because,

- $\mathrm{mse}(\mathbf{R_i}) = E\,[\,(\mathbf{R_i} - R_{i,\,I})^2 \mid D\,]$

  $= E\,[\,(C_{i,I} - \mathbf{C_{i,\,I}})^2 \mid D\,]$

  $= \mathrm{mse}(\mathbf{C_{i,\,I}})$.

The derivation for **mse($\mathbf{C_{i,\,I}}$)** shown here:

<u>Proof 4: Derivation of the Mean Squared Error of Ultimate Losses</u>

- $\mathbf{mse(C_{i,\,I})} = \mathrm{Var}(C_{i,\,I} \mid D) + (E\,[\,C_{i,\,I} \mid D\,] - \mathbf{C_{i,\,I}})^2$

The following abbreviations shall be used:

- $E_i\,[\,X\,] = E\,[\,X \mid C_{i,\,1}, \ldots, C_{i,\,I+1-i}\,]$
- $\mathrm{Var}_i(X) = \mathrm{Var}(X \mid C_{i,\,1}, \ldots, C_{i,\,I+1-i})$

Due to the independence of accident years, the condition on D can be ignored, and thus

- $\mathbf{mse(C_{i,\,I})} = \mathrm{Var}_i(C_{i,\,I}) + (E_i\,[\,C_{i,\,I}\,] - \mathbf{C_{i,\,I}})^2$, which can be further broken into
- $\mathrm{Var}_i(C_{i,\,I}) = E_i\,[\,C_{i,\,I}\,] – (E_i\,[\,C_{i,\,I}\,])^2$ by the rule $\mathrm{Var}(x) = E\,[\,X\,] – (E\,[\,X\,])^2$, and
- $E_i\,[\,C_{i,\,k+1}\,] = C_{i,\,I} \cdot f_{I+1-i} \cdot \ldots \cdot f_k$, as described before.

$E_i\,[\,C_{i,\,I}^2\,]$ is calculated as follows, for $k \geq I+1-i$:

- $E_i\,[\,C_{i,\,k+1}^2\,] = (E_i\,[\,E\,[\,C_{i,\,k+1}^2 \mid C_{i,\,1}, \ldots, C_{i,\,k}\,]$ by the iterative rule of expectations.

  $= E_i\,[\,\mathrm{Var}(C_{i,\,k+1} \mid C_{i,\,1}, \ldots, C_{i,\,k})\,] + (E\,[\,C_{i,\,k+1} \mid C_{i,\,1}, \ldots, C_{i,\,k}\,])^2$

  $= E_i\,[\,C_{i,\,k}\alpha_k^2 + (C_{i,\,k}f_k)^2\,]$ by Assumptions 1 and 3.

  $= E_i\,[\,C_{i,\,k}\,]\alpha_k^2 + E_i\,[\,C_{i,\,k}^2\,]f_k^2$.

These values for $E_i\,[\,C_{i,\,k+1}\,]$ and $E_i\,[\,C_{i,\,k+1}^2\,]$ can be combined to find

- $E_i\,[\,C_{i,\,I}^2\,] = E_i\,[\,C_{i,\,I-1}\,]\alpha_{I-1}^2 + E_i\,[\,C_{i,\,I-1}^2\,]f_{I-1}^2$

  $= C_{i,\,I-1} \cdot f_{I+1-1} \cdot \ldots \cdot f_{I-2} \cdot \alpha_{I-1}^2 + E_i\,[\,C_{i,\,I-2}\,]\alpha_{I-2}^2 f_{I-1}^2 + E_i\,[\,C_{i,\,I-2}^2\,]f_{I-2}^2 f_{I-1}^2$

  = etc.

$$= C_{i,\,I+1-i} \sum_{k=I+1-i}^{I-1} f_{I+1-i} \cdot \ldots \cdot f_{k-1} \cdot \alpha_k^2 \cdot f_{k+1}^2 \cdot \ldots \cdot f_{I-1}^2 + C_{i,\,I+1-i}^2 \cdot f_{I+1-i}^2 \cdot \ldots \cdot f_{I-1}^2$$

The value of E_i [$C_{i,\,k+1}$] also leads to

- $(E_i [C_{i,\,I}])^2 = C_{i,\,I+1-i}^2 \cdot f_{I+1-i}^2 \cdot \ldots \cdot f_{I-1}^2$

These two most recent equations can be inserted into the equation for $Var_i(C_{i,\,I})$ to get

- $Var_i(C_{i,\,I}) = C_{i,\,I+1-i} \sum_{k=I+1-i}^{I-1} f_{I+1-i} \cdot \ldots \cdot f_{k-1} \cdot \alpha_k^2 \cdot f_{k+1}^2 \cdot \ldots \cdot f_{I-1}^2$.

This sum can then be estimated using the unbiased estimators $\mathbf{f_k}$ and $\boldsymbol{\alpha_k}^2$ to get

- $C_{i,\,I+1-i} \sum_{k=I+1-i}^{I-1} \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_{k-1}} \cdot \boldsymbol{\alpha_k}^2 \cdot \mathbf{f_{k+1}}^2 \cdot \ldots \cdot \mathbf{f_{I-1}}^2 =$

$$= C_{i,\,I+1-i}^2 \cdot \mathbf{f_{I+1-i}}^2 \cdot \ldots \cdot \mathbf{f_{I-1}}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha_k^2 / f_k^2}}{C_{i,I+1-i} \cdot \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_{k-i}}}$$

$$= C_{i,\,I}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha_k^2 / f_k^2}}{\mathbf{C_{i,k}}} ,$$

where $\mathbf{C_{i,\,k}}$ is an unbiased estimator of $C_{i,\,k}$ calculated in the same way as $\mathbf{C_{i,\,I}}$ for k > I+1-i, with $\mathbf{C_{i,\,I+1-i}} = C_{i,\,I+1-I}$.

$(E_i [C_{i,\,I}] - \mathbf{C_{i,\,I}})^2$ can be calculated as follows:

- $(E_i [C_{i,\,I}] - \mathbf{C_{i,\,I}}) = C_{i,\,I+1-i}^2 (f_{I+1-i} \cdot \ldots \cdot f_{I-1} - \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_{I-1}})^2$

Replacing f_k with $\mathbf{f_k}$ will not make for a good estimator, as the result will be 0, when some difference is expected. Instead, this algebraic approach is taken:

- $F \quad = f_{I+1-i} \cdot \ldots \cdot f_{I-1} - \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_{I-1}}$

  $= S_{I+1-I} + \ldots + S_{I-1}$, with
- $S_k \quad = \mathbf{f_{I+1-i}} \cdot \ldots \cdot \mathbf{f_{k-1}} \cdot (f_k - \mathbf{f_k}) \cdot f_{k+1} \cdot \ldots \cdot f_{I-1}$, which yields
- $F^2 \quad = (S_{I+1-I} + \ldots + S_{I-1})^2$

  $= \sum_{k=I+1-i}^{I-1} S_k^2 + 2 \sum_{j,k=I+1-i}^{I-1} S_j S_k$, $j < k$.

To approximate this, S_k^2 is replaced with E [S_k^2 | B_k] and S_jS_k with E [S_kS_j | B_k]. It has been shown that E [$f_k - \mathbf{f_k}$ | B_k] = 0, so E [S_kS_j | B_k] = 0 because all $\mathbf{f_r}$, r < k, are scalars included in B_k. Next,

- $E [(f_k - \mathbf{f_k})^2 | B_k] \quad = Var(\mathbf{f_k} | B_k)$

  $= \sum_{j=1}^{I-k} Var(C_{j,\,k+1} | B_k) / (\sum_{j=1}^{I-k} C_{j,\,k})^2$

$= \sum_{j=1}^{I-k} \text{Var}(C_{j,k+1} \mid C_{j,1}, \ldots, C_{j,k}) / (\sum_{j=1}^{I-k} C_{j,k})^2$

$= \sum_{j=1}^{I-k} C_{j,k}\alpha_k^2 / (\sum_{j=1}^{I-k} C_{j,k})^2$

$= \alpha_k^2 / \sum_{j=1}^{I-k} C_{j,k}$, resulting in

- $E[S_k^2 \mid B_k] = \mathbf{f}_{I+1-i}^2 \cdot \ldots \cdot \mathbf{f}_{k-1}^2 \cdot \boldsymbol{\alpha}_k^2 \cdot f_{k+1}^2 \cdot \ldots \cdot f_{I-1}^2 / \sum_{j=1}^{I-k} C_{j,k}$

This can replace (ΣS_k^2), with all unknown parameters replaced with their unbiased estimators. This means that F^2 can be estimated by

- $\sum_{k=I+1-i}^{I-1} (\mathbf{f}_{I+1-i}^2 \cdot \ldots \cdot \mathbf{f}_{k-1}^2 \cdot \boldsymbol{\alpha}_k^2 \cdot \mathbf{f}_{k+1}^2 \cdot \ldots \cdot \mathbf{f}_{I-1}^2 / \sum_{j=1}^{I-k} C_{j,k})$

$$= \mathbf{f}_{I+1-i}^2 \cdot \ldots \cdot \mathbf{f}_{k-1}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha}_k^2 / \mathbf{f}_k^2}{\sum_{j=1}^{I-k} C_{j,k}}$$

$$= \mathbf{C}_{i,I}^2 \sum_{k=I+1-i}^{I-1} \frac{\boldsymbol{\alpha}_k^2 / \mathbf{f}_k^2}{\sum_{j=1}^{I-k} C_{j,k}},$$

which leads to the following estimator for mse($\mathbf{C}_{i,I}$):

- $(\text{s.e.}(\mathbf{C}_{i,I}))^2 = \mathbf{C}_{i,I}^2 \sum_{k=I+1-i}^{I-1} \frac{\alpha_k^2}{f_k^2} \left(\frac{1}{C_{i,k}} + \frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\right)$, where

- $\boldsymbol{\alpha}_k^2 = \frac{1}{I-k-1} \sum_{j=1}^{I-k} C_{j,k} \left(\frac{C_{j,k+1}}{C_{j,k}} - \mathbf{f}_k\right)^2, \qquad 1 \le k \le I - 2$

and is an unbiased estimator of α_k^2. ■

This does not reveal an estimator for α_{I-1}^2, and thus one needs to be found separately. If $\mathbf{f}_{I-1} = 1$, i.e. it is predicted that claims have already finished developing, then $\boldsymbol{\alpha}_{I-1}^2 = 0$. Otherwise, it is seen that the series $\boldsymbol{\alpha}_1^2, \boldsymbol{\alpha}_2^2, \ldots, \boldsymbol{\alpha}_{I-1}^2, \boldsymbol{\alpha}_{I-2}^2$, is usually decreasing. Or, $\boldsymbol{\alpha}_{I-1}^2$ can be set with the equation

- $\boldsymbol{\alpha}_{I-3}^2 / \boldsymbol{\alpha}_{I-2}^2 = \boldsymbol{\alpha}_{I-2}^2 / \boldsymbol{\alpha}_{I-1}^2$, with $\boldsymbol{\alpha}_{I-3}^2 > \boldsymbol{\alpha}_{I-2}^2$

These two means of estimating $\boldsymbol{\alpha}_{I-1}^2$ can be combined into the following:

- $\boldsymbol{\alpha}_{I-3}^2 = \min(\boldsymbol{\alpha}_{I-2}^4 / \boldsymbol{\alpha}_{I-3}^2, \min(\boldsymbol{\alpha}_{I-2}^2, \boldsymbol{\alpha}_{I-3}^2))$

With all of this, a confidence interval for the reserve R_i can now be calculated. It can be assumed by the central limit theorem that the reserve follows a Normal distribution if the number of claims is sufficiently high enough. An X% confidence interval for the reserve would thus be

- ($\mathbf{R_i}$ - Z_X s.e.($\mathbf{R_i}$) , $\mathbf{R_i}$ + Z_X s.e.($\mathbf{R_i}$)) ,

where Z is the corresponding standard score to create an X% confidence interval.

Because the chosen value for Z_X is usually greater than or equal to 2 ($Z_X = 2$ creates a 95% confidence interval for R_i), a symmetric Normal distribution for R_i is not appropriate if s.e.($\mathbf{R_i}$) > $\mathbf{R_i}$ / 2, and can lead to a negative lower limit for the reserve, which though not always impossible, is typically not the case.

When this is the case, it is better to use a Lognormal distribution to find a confidence interval for R_i, with parameters μ_i and σ_i^2 such that

- $\exp(\mu_i + \sigma_i^2 / 2) = \mathbf{R_i}$ and
- $\exp(2\mu_i + \sigma_i^2)(\exp(\sigma_i^2) - 1) = (\text{s.e.}(\mathbf{R_i}))^2$,

  so that the mean and variances of the distributions are the same. It follows that,

- $\sigma_i^2 = \ln(1 + (\text{s.e.}(\mathbf{R_i}))^2 / \mathbf{R_i}^2)$ and
- $\mu_i = \ln(\mathbf{R_i}) - \sigma_i^2 / 2$.

  This results in the new confidence interval,

- ($\exp(\mu_i - Z_X\sigma_i)$, $\exp(\mu_i + Z_X\sigma_i)$), which will no longer have a negative lower bound.

In the same way as shown before, this method can be used when calculating the confidence interval for $R = R_2 + \dots R_I$ with estimator $\mathbf{R} = \mathbf{R_2} + \dots + \mathbf{R_I}$. While $R_2 , \dots , R_I$ are independent of each other, their respective estimators $\mathbf{R_2} , \dots , \mathbf{R_I}$ are not, as they are all dependent on the same age-to-age factors $\mathbf{f_k}$. Thus, s.e.($\mathbf{R}$) ≠ s.e.($\mathbf{R_2}$) + … + s.e.($\mathbf{R_I}$). Instead, s.e.($\mathbf{R}$) is derived as follows:

Proof 5: Derivation of the Standard Error of the Reserve

First, mse(R) must be determined.

- $$\text{mse}(\textstyle\sum_{i=2}^{I} \mathbf{R_i}) = E [(\sum_{i=2}^{I} \mathbf{R_i} - \sum_{i=2}^{I} R_i)^2 \mid D]$$

  $$= E [(\textstyle\sum_{i=2}^{I} \mathbf{C_{i,I}} - \sum_{i=2}^{I} C_i)^2 \mid D]$$

  $$= \text{Var}\textstyle\sum_{i=2}^{I} C_{i,I} \mid D) + (E [\sum_{i=2}^{I} C_{i,I} \mid D] - \sum_{i=2}^{I} \mathbf{C_{i,I}})^2 .$$

The independence of accident years results in

- $\mathrm{Var}(\sum_{i=2}^{I} C_{i,I} \mid D) = \mathrm{Var}(\sum_{i=2}^{I} C_{i,I} \mid C_{i,I}, \ldots, C_{i,I+1-i})$, which has previously been calculated.

Continuing,

- $(E[\sum_{i=2}^{I} C_{i,I} \mid D] - \sum_{i=2}^{I} \mathbf{C_{i,I}})^2 = (\sum_{i=2}^{I} (E[C_{i,I} \mid D] - \mathbf{C_{i,I}}))^2$

$$= \sum_{2 \le i,\ j \le I} (E[C_{i,I} \mid D] - \mathbf{C_{i,I}}) \cdot (E[C_{j,I} \mid D] - \mathbf{C_{j,I}})$$

$$= \sum_{2 \le i,\ j \le I} C_{i,I+1-i} C_{j,I+1-j} F_i F_j$$

$$= \sum_{i=2}^{I} (C_{i,I+1-i} F_i)^2 + 2\mathrm{sum}\, C_{i,I+1-i} C_{j,I+1-j} F_i F_j$$

Where F_i follows the same definition as F from before. From there, when compared with the equation from before:

- $\mathrm{mse}(\mathbf{R_i}) = \mathrm{Var}(C_{i,I} \mid C_{i,I}, \ldots, C_{i,I+1-i}) + (C_{i,I+1-i} F_i)^2$, it can be seen that
- $\mathrm{mse}(\sum_{i=2}^{I} \mathbf{R_i}) = \sum_{i=2}^{I} \mathrm{mse}(\mathbf{R_i}) + \sum_{2 \le i,\ j \le I} 2 C_{i,I+1-i} C_{j,I+1-j} F_i F_j$.

From there, all that remains is an estimator for F_iF_j, which is analogous to the estimator for F^2, resulting in the estimator,

- $\sum_{k=I+1-i}^{I-1} \mathbf{f_{I+1-j}} \cdot \ldots \cdot \mathbf{f_{I-i}} \cdot \mathbf{f_{I+1-i}}^2 \cdot \ldots \cdot \mathbf{f_{k-1}}^2 \cdot \boldsymbol{\alpha_k}^2 \cdot \mathbf{f_{k+1}}^2 \cdot \mathbf{f_{I-1}}^2 / \sum_{n=1}^{I-k} C_{n,k}$, which leads to
- $(\mathrm{s.e.}(\mathbf{R}))^2 = \sum_{i=2}^{I} \{ (\mathrm{s.e.}(\mathbf{R_i}))^2 + \mathbf{C_{i,I}}(\sum_{j=i+1}^{I} \mathbf{C_{j,I}}) \sum_{k=I+1-i}^{I-1} \frac{2\alpha_k^2 / f_k^2}{\sum_{n=1}^{I-k} C_{n,k}} \}$. ■

With the equation for the standard error of the total reserve R, it is now possible to create a confidence for the overall reserve of a development triangle, as shall be presented in the following section.

Section 6: Numerical Example

This section is an example of completing this entire process for a 5x5 development triangle. The result of such a process would not seem to indicate that the ultimate losses has been reached after only five years, but a development of such a small size has been used for the sake of simplicity. Additionally, it shall be treated from the start that this development triangle meets Assumptions 1, 2, and 3. The methods described in Section 7 could be used to test for this.

This is the development triangle of incurred losses in dollars:

	Development Period (k)				
AY (i)	1	2	3	4	5
1	5012	8269	10907	11805	13539
2	1506	4285	5396	10666	
3	3410	8992	13873		
4	5655	11555			
5	1092				

Table 5: Numerical Example Development Triangle

This development triangle results in the following age-to-age factors:

	Development Period (years)			
AY	1-2	2-3	3-4	4-5
1	1.6498	1.3190	1.0823	1.1469
2	2.8453	1.2593	1.9766	
3	2.6370	1.5428		
4	2.0433			

Table 6: Numerical Example Observed Age-to-Age Factors

Below are the chosen values for $\mathbf{f_k}$, all equal to the weighted average of age-to-age factors, as well as the squared-weighted average and simple average for comparison.

k	1	2	3	4
Squared weighted	2.0269	1.4204	1.2582	1.1469
Weighted	2.1242	1.4005	1.3783	1.1469

Simple Average	2.2939	1.3737	1.5295	1.1469
$\mathbf{f_k}$	2.1242	1.4005	1.3783	1.1469

Table 7: Numerical Example Estimators of Age-to-Age Factors

The values for $\boldsymbol{\alpha_k}^2$ are as follows, following the equations

- $\boldsymbol{\alpha_k}^2 = \frac{1}{I-k-1}\sum_{j=1}^{I-k} C_{j,k} \left(\frac{C_{j,k+1}}{C_{j,k}} - \mathbf{f_k}\right)^2$, and
- $\boldsymbol{\alpha_4}^2 = \min(\boldsymbol{\alpha_3}^4 / \boldsymbol{\alpha_2}^2, \min(\boldsymbol{\alpha_3}^2, \boldsymbol{\alpha}^2))$

k	1	2	3	4
$\boldsymbol{\alpha_k}^2$	711.0931	107.4921	1443.6524	107.4921

Table 8: Numerical Example Proportionality Constants

This leads to the following results for the ultimate losses, the outstanding reserve $\mathbf{R_i}$, the standard error s.e.($\mathbf{R_i}$), and the ratio of the standard error to the reserve:

AY (i)	Ultimate Losses	Reserve = $\mathbf{R_i}$	s.e.($\mathbf{R_i}$)	s.e.($\mathbf{R_i}$) / $\mathbf{R_i}$
2	12232.70	1566.70	1432	91%
3	21930.36	8057.36	6897	86%
4	25582.35	14027.35	7969	57%
5	5135.51	4043.51	3508	87%
Overall	64880.92	27694.92	16348	59%

Table 9: Numerical Example Reserve and Standard Error

The ratio s.e.($\mathbf{R_i}$) / $\mathbf{R_i}$ is greater than 50% in all instances, including for the overall reserve **R**, so it is more appropriate to use a Lognormal distribution to form a confidence interval for the reserve. Consequently, it is found that

- $\sigma^2 = \ln(1 + (\text{s.e.}(\mathbf{R}))^2 / \mathbf{R}^2) = 0.2989$, and
- $\mu = \ln(\mathbf{R}) - \sigma^2 / 2 = 10.0795$.

σ_i^2 and μ_i can also be found in this way for each accident, substituting **R** with $\mathbf{R_i}$. With this, the following 90% confidence intervals can be formed:

AY (i)	90% Confidence Interval

2	(426, 3139)
3	(2478, 15118)
4	(7699, 19322)
5	(1214, 7686)
Overall	(14586, 38997)

Table 10: Numerical Example 90% Confidence Interval

As an additional example, here are the resulting tables for a development triangle where the observed age-to-age factors are nearly identical, which results in confidence intervals that are extremely close to the expected ultimate losses. (Note: A development triangle with completely identical age-to-age factors in each accident year can lead to issues in formulas involving dividing by zero.)

	Development Period (k)				
AY (i)	1	2	3	4	5
1	5012	8269	10907	11805	13539
2	5013	8270	10908	11806	
3	5014	8271	10909		
4	5015	8272			
5	5016				

Table 11: Development Triangle with Close Values

	Development Period (years)			
AY	1-2	2-3	3-4	4-5
1	1.6498	1.3190	1.0823	1.1469
2	1.6497	1.3190	1.0823	
3	1.6496	1.3189		
4	1.6495			

Table 12: Observed Age-to-Age Factors with Close Values

k	1	2	3	4
Squared weighted	1.6496	1.3190	1.0823	1.1469
Weighted	1.6496	1.3190	1.0823	1.1469
Simple Average	1.6496	1.3190	1.0823	1.1469

$\mathbf{f_k}$	1.6496	1.3190	1.0823	1.1469

Table 13: Estimators of Age-to-Age Factors with Close Values

k	1	2	3	4
$\boldsymbol{\alpha_k^2}$	0.000105226	0.000008202	0.000000155	0.000000003

Table 14: Proportionality Constants with Close Values

AY (i)	Ultimate Losses	Reserve = $\mathbf{R_i}$	s.e.($\mathbf{R_i}$)	s.e.($\mathbf{R_i}$) / $\mathbf{R_i}$
2	13540.15	1734.15	0.008064	0.0005%
3	13541.44	2632.44	0.057655	0.0022%
4	13543.47	5271.47	0.366469	0.0070%
5	13547.77	8531.77	1.328065	0.0156%
Overall	54172.82	18169.82	1.408877	0.0078%

Table 15: Reserve and Standard Error with Close Values

The ratio s.e.($\mathbf{R_i}$) / $\mathbf{R_i}$ is less than 50% in all instances, including for the overall reserve **R**, so it is Normal distribution shall be used to form a 90% confidence interval for the reserve, as follows:

AY (i)	90% confidence interval
2	(1734.1336, 1734.1369)
3	(2632.3406, 2632.4354)
4	(5270.8658, 5271.4685)
5	(8529.5805, 8531.7648)
Overall	(18167.4985, 18169.8158)

Table 16: 90% Confidence Interval with Close Values

A Lognormal distribution produces similarly narrow confidence intervals. These narrow confidence intervals indicate that the chain ladder method estimations are accurate.

Section 7: Assumption Testing

This final section is about testing whether a development triangle meets the assumptions necessary to be appropriate for the chain ladder method. To repeats the assumptions, they are as follows:

- $E [C_{i, k+1} | C_{i, 1}, \dots , C_{i, k}] = C_{i, k}f_k$
- The variables $\{C_{i, 1}, \dots , C_{i,I}\}$ and $\{C_{j, 1}, \dots , C_{j,I}\}$ of different accident years $i \neq j$ are independent
- $Var(C_{j, k+1} | C_{j, 1}, \dots , C_{j, k}) = C_{j, k}\alpha_k^2$

The first test is to determine if there is a calendar year effect that would indicate that different accident years are not independent of each other. A calendar year effect would influence one of the diagonals of the development triangle, as well as the sets of development factors associated with that diagonal. What this means is that if the reported claims of a diagonal are larger than usual, then the preceding development factors are also larger than usual, and the subsequent development factors are smaller than usual, and vice versa.

The observed age-to-age factors of the columns of a development triangle can thus be categorized as either larger (L), or smaller (S). The median of these development factors will divide these two groups, and shall by any further part of this test. It is expected that each diagonal of the- age-to-age factors triangle should have an approximately equal number of larger and smaller development, since every non-median development factor has a 50% chance of being either larger or smaller. If this is not the case, then that would serve as evidence for there being a calendar year influence. Mathematically, if $Z_j = \min(L_j, S_j)$ is significantly less than $(L_j + S_j) / 2$, where L_j and S_j are the number of larger and smaller development factors in column j respectively, then this would indicate a calendar year effect.

The result of a generalized derivation is as follows, with $n = L_j + S_j$ and $m = ((n-1) / 2)$, truncated:

- $E [Z_j] = \frac{n}{2} - \binom{n-1}{m} \frac{n}{2^n}$
- $Var(Z_j) = \frac{n(n-1)}{4} - \binom{n-1}{m} \frac{n(n-1)}{2^n} + E [Z_j] - (E [Z_j])^2$.

To test the entire triangle as a whole, the analysis can be applied to the sum $Z = Z_2 + \dots + Z_{I-1}$. Z_1 is excluded, because with only one element in the diagonal, Z_1 will always be equal to 0, and is thus not a random variable. Similarly, any diagonal for which $L_j + S_j \leq 1$ will also be excluded from the sum. The null hypothesis of this test will be that the different Z_j's are uncorrelated, and thus

- $E[Z] = E[Z_2] + \ldots + E[Z_{I-1}]$
- $Var(Z) = Var(Z_2) + \ldots + Var(Z_{I-1})$,

with Z having an approximately Normal distribution. The null hypothesis will be rejected if Z falls outside of the interval $(E[Z] - 2\sqrt{Var(Z)}, E[Z] + 2\sqrt{Var(Z)})$, the equivalence of an error probability of 5%.

Here is an example of applying this to a triangle of development factors.

	Development Period								
AY	1	2	3	4	5	6	7	8	9
1	1.61	1.32	1.08	1.15	1.20	1.11	1.03	1.00	1.01
2	4.42	1.26	1.98	1.29	1.13	1.01	1.04	1.03	
3	2.63	1.54	1.16	1.16	1.19	1.03	1.26		
4	2.04	1.36	1.35	1.10	1.11	1.04			
5	8.85	1.66	1.40	1.17	1.01				
6	4.36	1.82	1.11	1.23					
7	7.27	2.72	1.12						
8	5.18	1.89							
9	1.79								

Table 17: Calendar Year Effect Test – Observed Age-to-Age Factors

The different development factors are ranked as follows:

	Development period								
AY	1	2	3	4	5	6	7	8	9
1	S	S	S	S	L	L	*	S	*
2	L	S	L	L	*	S	L	L	
3	S	S	*	S	L	S	S		

4	S	S	L	S	S	L			
5	L	L	L	L	S				
6	*	L	S	L					
7	L	L	S						
8	L	L							
9	S								

Table 18: Calendar Year Effect Test – Ranked Age-to-Age Factors

J	S_j	L_j	Z_j	n	M	E [Z_j]	Var(Z_j)
2	1	1	1	2	0	0.5	0.25
3	3	0	0	3	1	0.75	0.1875
4	3	1	1	4	1	1.25	0.4375
5	1	3	1	4	1	1.25	0.4375
6	1	3	1	4	1	1.25	0.4375
7	2	4	2	6	2	2.065	0.6211
8	4	4	4	8	3	2.90625	0.8037
9	4	4	4	8	3	2.90625	0.8037
Total			14			12.875	3.9785 = 1.9946^2

Table 19: Calendar Year Effect Test – Count of Ranked Age-to-Age Factors

The 95% confidence interval for $Z = \Sigma Z_j$ for this triangle is (12.875 - 2(1.9946), 12.875 + 2(1.9946)) = (8.886, 16.864), which includes the test statistic Z = 14, so the null hypothesis is not rejected. This indicates that there is not enough of a calendar year effect to reject using the chain ladder method.

Here is an example of a development triangle which does not pass the test for a calendar year effect, following the same steps as before:

	Development Period								
AY	1	2	3	4	5	6	7	8	9
1	1.61	1.32	1.08	1.15	1.20	1.11	1.04	1.00	1.01
2	4.42	1.26	1.98	1.29	1.13	1.01	1.26	1.03	
3	2.63	1.72	1.16	1.17	1.01	1.03	1.03		
4	2.04	1.36	1.35	1.10	1.11	1.04			
5	8.85	1.66	1.12	1.16	1.19				
6	4.36	1.82	1.11	1.23					
7	7.27	1.54	1.40						
8	1.79	1.89							
9	5.18								

Table 20: Calendar Year Effect Failed Test – Observed Age-to-Age Factors

The different development factors are ranked as follows:

	Development period								
AY	1	2	3	4	5	6	7	8	9
1	S	S	S	S	L	L	*	S	*
2	L	S	L	L	*	S	L	L	
3	S	L	*	L	S	S	S		
4	S	S	L	S	S	L			
5	L	L	S	S	L				
6	*	L	S	L					
7	L	S	L						
8	S	L							
9	L								

Table 21: Calendar Year Effect Failed Test – Ranked Age-to-Age Factors

J	S_j	L_j	Z_j	n	m	$E[Z_j]$	$Var(Z_j)$
2	1	1	1	2	0	0.5	0.25
3	3	0	0	3	1	0.75	0.1875
4	2	2	2	4	1	1.25	0.4375

5	1	3	1	4	1	1.25	0.4375
6	0	4	0	4	1	1.25	0.4375
7	4	2	2	6	2	2.065	0.6211
8	7	1	1	8	3	2.90625	0.8037
9	1	7	1	8	3	2.90625	0.8037
Total			8			12.875	3.9785 = 1.9946^2

Table 22: Calendar Year Effect Failed Test – Count of Ranked Age-to-Age Factors

The 95% confidence interval for $Z = \Sigma Z_j$ for this triangle is (12.875 - 2(1.9946), 12.875 + 2(1.9946)) = (8.886, 16.864), the same as the previous development triangle. This time, however, Z = 8, which is outside of this confidence interval. Thus, the null hypothesis is rejected, and it is assumed that there is a calendar year effect for this development triangle, indicating that the chain ladder method would not be appropriate to use.

Assumptions 1 and 3 can be checked by using a regression model. For development period k, the known values $C_{i,k}$, $1 \leq i \leq I$, can create a regression model of the form,

- $Y_i = c + x_i b + \varepsilon_i$,

where $E[\varepsilon_i] = 0$. In the case c = 0 and $b = f_k$, then $Y_i = C_{i,k+1}$ and $x_i = C_{i,k}$. $b = f_k$ can then be estimated using the least squares method by minimizing $\sum_{i=1}^{I-k}(C_{i,k+1} - C_{i,k}f_k)$. Setting the derivative of the sum with respect to f_k equal to 0 results in

- $f_{k0} = \sum_{i=1}^{I-k} C_{i,k}C_{i,k+1} / \sum_{i=1}^{I-k} C_{i,k}^2$

f_{k0} is not equal to the previous estimator f_k, and is in fact the squared-weighted average of the development factors, as opposed to the squared-weighted average. Therefore, f_{k0} assumes that

- $Var(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k})$ is proportional to 1

such that $Var(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k})$ is the same for all observed values, which is not in agreement with the third assumption of the chain ladder method. This is because the least-squares method assumes that $Var(Y_i) = Var(\varepsilon_i)$, independent of i. If this is dependent on i, then a weighted least-squares method should be used, minimizing the sum $\sum_{i=1}^{I} w_i(Y_i - c - x_i b)^2$ with weights w_i inversely proportional to $Var(Y_i)$. If the weights w_i are instead proportional to $1 / C_{i,k}$, this results in the estimator $f_{k1} = \mathbf{f_k}$. Finally, using weights proportional to $1 / C_{i,k}^2$ results in:

- $f_{k2} = \frac{1}{I-k}\sum_{i=1}^{I-k}\frac{C_{i,k+1}}{C_{i,k}}$,

  which is the arithmetic average of development factors.

  With these three estimators, the following regression plots can then be formed,

- $C_{i,k+1} - C_{i,k}f_{k0}$ versus $C_{i,k}$
- $\frac{(C_{i,k+1} - C_{i,k}f_{k1})}{\sqrt{C_{i,k}}}$ versus $C_{i,k}$
- $\frac{(C_{i,k+1} - C_{i,k}f_{k1})}{C_{i,k}}$ versus $C_{i,k}$,

and then checked to see which residual plot exhibits the most random behavior, for all values of k for which a sufficient number of data points exist. If the regression plot for f_{k1} exhibits nonrandom behavior for multiple values of k, while the other two plots do not, then it would make sense to instead to replace $\mathbf{f_k} = f_{k1}$ with f_{k0} or f_{k2} instead.

Section 8: Conclusion

The chain ladder method has applications beyond just providing a confidence interval for ultimate losses. The confidence interval can also be compared to alternate estimates for ultimate losses, such as the Expected Claims, Banktender, and Cape Cod methods, all of which consider additional information about reported premiums that the chain ladder method does not, as well as other computerized estimators. If for example, the resulting estimates for these methods fall within the confidence interval provided by the chain ladder method, then that serves as further evidence that those could be appropriate estimates for the ultimate losses.

Additionally, the weaknesses of the chain ladder method must be addressed. The estimators for the final few age-to-age factors f_I, f_{I-1}, etc. are based on very few observed claims amounts. Similarly, the estimated ultimate losses for the most recent accident year has a large range, due to the immaturity of the claims, and the point estimate for such claims is highly variable. The results of the chain ladder method must still be given actuarial judgement, even if the development triangle met all of the given assumptions. If changes in future claims processes occur, then the estimates of the chain ladder method may become obsolete anyways.

In spite of these weaknesses, the chain ladder method is still a simple method, and can be intuitively explained to those not in the field.

Bibliography

Mack, T.H. (1999). Measuring the Variability of Chain Ladder Reserve Estimates.

Appendix

The following is a link to complete, development triangles, currently with the values used Section 6. The formulas used in these excel sheets provide complete development triangles, reserve estimates, and confidence intervals.

Measuring the Variability of Chain Ladder Reserve Estimates Development Triangles.xlsx