Convergence of a Numerical Method for Reconstructing Faults from Boundary Measurements

by

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Abstract

In this thesis we derive the convergence order of a regularized error functional for reconstructing faults from boundary measurements of displacement fields. The convergence was proved to occur as the regularization parameter converges to zero but the convergence order was unknown. This functional is used to solve an inverse problem related to a half-space linear elasticity model. We first discuss this related model and review some basic properties of this functional and then we derive the convergence order for small regularization parameters.

This study is a first, but essential, step toward analyzing the convergence order of related numerical methods. The reconstruction method of faults studied in this thesis was built from a model for real-world faults between tectonic plates that occur in nature. This model was first proposed by geophysicists and was later analyzed by mathematicians who were interested in building efficient numerical methods with proof of convergence.

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Contents

Chapter 1

Introduction and detailed review of preliminary material

1.1 Introduction

The study of subduction zones and interfaces between plates is an important topic in geophysics, as they are intimately related to the onset and evolution of earthquakes. Although it is still impossible to predict when and where the next significant earthquake will occur, a better knowledge of the geophysical structure of regions prone to seismic activity will go a long way toward helping scientists and policy makers assess seismic risk. Aspects of this geophysical structure involved in this risk assessment must include hidden features such as plate boundaries, slippage between boundaries, and accumulated strain. In general, these hidden features cannot be observed or measured directly, they are reconstructed from a model and measurable quantities such as surface displacements. Thanks to techniques such as geodetic and seismological networks, scientists are able to collect measurements on the surface of Earth's crust. These surface measurement of displacement fields can then be used to reconstruct faults and slip fields on those faults. Initially, these model reconstructions were done heuristically, but also occasionally with a deeper understanding of the underlying mathematical challenges in the much simpler case where a fault geometry is assumed and a slip field has to be reconstructed. This leads to a linear inverse problem. Linear inverse problems are related to the well studied mathematical theory of undetermined and ill-posed linear systems. This theory may also be related to mathematical statistics. Tarantola is an investigator who bridged the gap between geophysical models and linear inference [5], section 3.2. [1] is just one of a plethora of geophysics papers using Tarantola's method. [1] shows a study where slip fields are reconstructed from surface measurements of displacement fields and regularization is achieved thanks to a regularization term based on the assumed covariance of the solution. More recently, the combined linear and nonlinear inverse problem consisting of reconstructing slip fields occurring on an *unknown, to be determined*, fault has been studied from a rigorous mathematical perspective [9, 7, 8].

The authors of [7] set forth to study the convergence order of related numerical methods. However, at the time of writing of [7], stability results shown in [8] were not yet available. Using these stability results to derive finer convergence orders is precisely the goal of this thesis. Here is an outline of this thesis. In chapter 1 we review the physical model of interest, and we give a brief introduction to the related inverse problem. We recall results claiming that this inverse problem is uniquely solvable. We introduce a regularized error functional which is used to approximate the solution to our inverse problem. We show that the numerical method using this functional is convergent. Chapter 2 includes the main results of this thesis. We prove that under some conditions we can estimate the convergence order of the minimizer of our regularized error functional as the regularization parameter tends to zero. Chapter 3 contains the conclusion of this thesis. Chapter 4 is an appendix where we cover important lemmas and theorems from functional analysis and specifically, Tikhonov regularization. These results are critical for proving our main theorem.

1.2 Physical significance of the forward and inverse problems

In this thesis we will use the theory of elasticity in the case of linear, homogeneous, isotropic media. First, consider the following forward problem. Let Γ be an open surface included in $\mathbb{R}^{3-}$ and $\boldsymbol{n}$ a continuous normal vector on Γ. Let $\boldsymbol{u}$ be the displacement field in $\mathbb{R}^{3-}$ solving,

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\text{div}\boldsymbol{u} = 0 \text{ in } \mathbb{R}^{3-} \setminus \Gamma, \tag{1.1}$$

$$T_{\boldsymbol{e_3}}\boldsymbol{u} = 0 \text{ on the surface } x_3 = 0, \tag{1.2}$$

$$T_{\boldsymbol{n}}\boldsymbol{u} \text{ is continuous across } \Gamma, \tag{1.3}$$

$$[\boldsymbol{u}] = \boldsymbol{g} \text{ is a given jump across } \Gamma, \tag{1.4}$$

$$\boldsymbol{u}(\boldsymbol{x}) = O(\frac{1}{|\boldsymbol{x}|^2}), \nabla\boldsymbol{u}(\boldsymbol{x}) = O(\frac{1}{|\boldsymbol{x}|^3}), \text{ uniformly as } |\boldsymbol{x}| \to \infty. \tag{1.5}$$

Existence and uniqueness of the field $\boldsymbol{u}$ solving (1.1-1.5) was proved in [9]. Here, (1.1) is the standard equation for linear elasticity with Lamé coefficients μ and λ, where $\mu > 0$, $\lambda > 0$. (1.2) signifies that this displacement field has zero traction on the plane $x_3 = 0$. (1.3) tells us that the stress vector in the normal direction $\boldsymbol{n}$ is continuous across Γ, and (1.4) is the discontinuity condition of the displacement field across the fault Γ given by $\boldsymbol{g}$: it models a slip field on Γ. (1.5) requires this displacement field to have finite energy. It is known from [4, 6] that there exists a unique Green's tensor $\boldsymbol{H}$ such that,

$$\boldsymbol{u}(\boldsymbol{x}) = \int_\Gamma \boldsymbol{H}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{g}(\boldsymbol{y})d\sigma(\boldsymbol{y}), \tag{1.6}$$

for all $\Gamma, \boldsymbol{g}$, and $\boldsymbol{u}$ satisfying (1.1-1.5). We now turn to solving the related inverse problem, that is, reconstructing the fault Γ and the slip field $\boldsymbol{g}$ from measurements of $\boldsymbol{u}$. The following theorem was proved in [9].

Theorem 1.1. *Let Γ_1 and Γ_2 be two bounded open surfaces, with smooth boundary, such that each of them is included in a rectangle contained in $\mathbb{R}^{3-}$. For i in $\{1,2\}$, assume that $\boldsymbol{u}^i$ solves the inverse problem for Γ_i in place of Γ and $\boldsymbol{g}^i$ in place of $\boldsymbol{g}$, where we assume that $\boldsymbol{g}^i$ has $H^1_0(\Gamma_i)$ regularity. Assume that supp $\boldsymbol{g}^i = \overline{\Gamma_i}$. Let V be a non empty open subset of the plane with equation $x_3 = 0$. If $\boldsymbol{u}^1 = \boldsymbol{u}^2$ in V, then $\Gamma_1 = \Gamma_2$, and $\boldsymbol{g}^1 = \boldsymbol{g}^2$.*

In other words, this inverse problem is uniquely solvable. Using integral formula (1.6), we can define a continuous mapping A, mapping $\boldsymbol{g}$ in $H^1_0(\Gamma)$ to $\boldsymbol{u}(x_1, x_2, 0)$ in $L^2(V)$. According to the uniqueness theorem in [9], we find that this mapping is injective, therefore we can define its inverse. However, it is known that such an integral operator A is compact, thus its inverse is unbounded. If we want to apply numerical methods to reconstruct $\boldsymbol{g}$ from the data we collect, in this case, $\boldsymbol{u}(x_1, x_2, 0)$, we need to design a regularization process for $\boldsymbol{g}$, even in the simpler case where Γ is known.

1.3 A regularized error functional for the surface measurement

In this thesis the coordinates of a vector $\boldsymbol{x}$ in $\mathbb{R}^3$ will be denoted by (x_1, x_2, x_3). Let R be a closed rectangle in in the plane with equation $x_3 = 0$, and let B be a closed and bounded subset of $\mathbb{R}^3$, so B is compact in $\mathbb{R}^3$. Denoting $m = (a, b, d)$, we assume that B is such that

$$ax_1 + bx_2 + d < 0,$$

for all $m \in B$, all $(x_1, x_2, 0) \in R$. We now define the set Γ_m for each m in B by the following:

$$\Gamma_m = \{(x_1, x_2, ax_1 + bx_2 + d) : (x_1, x_2, 0) \in R\}.$$

Notice that Γ_m is a parallelogram under the plane with equation $x_3 = 0$, it is the projection of the rectangle R, on the plane with equation $x_3 = ax_1 + bx_2 + d$ in the x_3 direction, see figure 1.1.

Lemma 1.1. *The distance between Γ_m and the plane with equation $x_3 = 0$ is bounded below by the same positive constant for all m in B.*

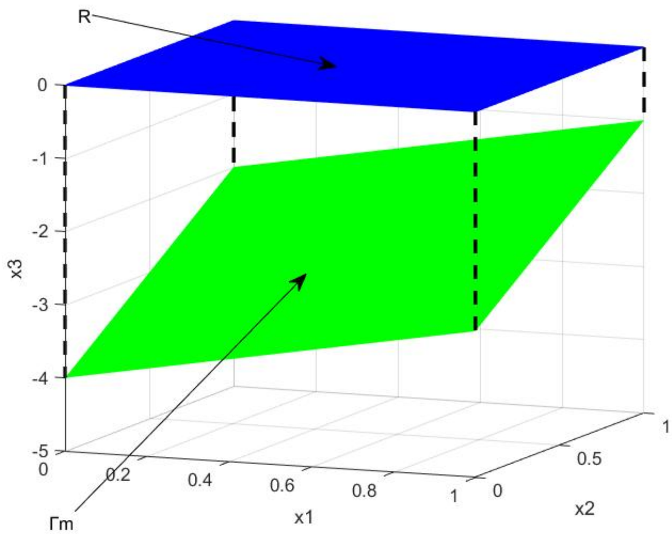

Figure 1.1: A sketch showing Γ_m and the rectangle R in the plane with equation $\{x_3 = 0\}$.

Proof. We define a function ρ as follows:

$$\rho : B \times R \to \mathbb{R},$$

$$\rho(m, (x_1, x_2)) = ax_1 + bx_2 + d.$$

To prove our statement, we show that ρ achieves its maximum value. Then, given our assumptions on B and R this maxium value must be negative. Since both B and R are closed and bounded in R, then both of them are compact so that $B \times R$ is compact. Therefore, the image of ρ is also compact, and thus closed and bounded in $\mathbb{R}$. Thus

$$\sup_{B \times R} \rho = \rho(m_0, (x_1^0, x_2^0))$$

for some m_0 in B and (x_1^0, x_2^0) in R. By the assumption on B, the lemma is proved. $\square$

As mentioned in section 1.2, in this inverse problem, $\boldsymbol{g}$ models a tangential slip field on Γ_m which in turn produces the deformation field $\boldsymbol{u}$. We will use the data $\boldsymbol{u}$ on a subset of the plane with equation $x_3 = 0$ to determine the geometric parameter m and slip field $\boldsymbol{g}$. Recall the definition of $\boldsymbol{H}$, the Green's tensor for linear elasticity in half space as defined in section 1.2. Set $m = (a, b, d)$ and

$$\boldsymbol{H}_m(\boldsymbol{x}, y_1, y_2) = \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}), \quad \text{where } \boldsymbol{y} = (y_1, y_2, ay_1 + by_2 + d).$$

To simplify notations we write $\boldsymbol{g}(y_1, y_2, ay_1 + by_2 + d) = \boldsymbol{g}(y_1, y_2)$. It follows that if $\boldsymbol{u}$ satisfies (1.1-1.5), after emphasizing the dependence of $\boldsymbol{u}$ on $\boldsymbol{x}$, $\boldsymbol{g}$, and m by writing it as $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{g}, m)$, we have

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{g}, m) = \int_R \boldsymbol{H}_m(\boldsymbol{x}, y_1, y_2)\boldsymbol{g}(y_1, y_2)\sigma dy_1 dy_2, \tag{1.7}$$

where m is in B, and σ is the surface element on Γ_m.

Now, let V be a bounded open set of $\{x_3 = 0\}$, fix $\tilde{\boldsymbol{u}}$ in $L^2(V)$, and fix a positive constant C. Let $\boldsymbol{g}$ be in $H_0^1(R)$, m be in B, and $\boldsymbol{u}$ be defined by (1.7). We define the regularized error functional,

$$F_{m,C}(\boldsymbol{g}) = \int_V |(\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{g}, m) - \tilde{\boldsymbol{u}}(\boldsymbol{x})|^2 d\boldsymbol{x} + C \int_R |\nabla \boldsymbol{g}|^2. \tag{1.8}$$

We define the linear operator,

$$A_m : H_0^1(R) \to L^2(V),$$
$$\boldsymbol{g} \to \left(\boldsymbol{x} \to \int_R \boldsymbol{H}_m(\boldsymbol{x}, y_1, y_2)\boldsymbol{g}(y_1, y_2)\sigma dy_1 dy_2 \right). \tag{1.9}$$

One can show that A_m is continuous and compact thanks to lemma 1.1.

Recall that the usual norm on $L^2(V)$ is

$$\|\boldsymbol{u}\|_{L^2(V)} = (\int_V |\boldsymbol{u}(\boldsymbol{x})|^2 d\boldsymbol{x})^{\frac{1}{2}}.$$

We also need to define a norm on $H_0^1(R)$. The usual norm on $H^1(R)$ is

$$\|\boldsymbol{g}\|_{H^1(R)} = \|\boldsymbol{g}\|_{L^2(R)} + \|\nabla \boldsymbol{g}\|_{L^2(R)}.$$

Now thanks to Poincare's inequality, there is an $M > 0$, such that for any $\boldsymbol{g} \in H_0^1(R)$, we have

$$\|\boldsymbol{g}\|_{L^2(R)} \leq M \|\nabla \boldsymbol{g}\|_{L^2(R)}.$$

Thus we can use the equivalent norm on $H_0^1(R)$,

$$\|\boldsymbol{g}\|_{H_0^1(R)} = (\int_R |\nabla \boldsymbol{g}|^2)^{\frac{1}{2}}.$$

Now we can rewrite the functional $F_{m,C}$ as

$$F_{m,C}(\boldsymbol{g}) = \|A_m \boldsymbol{g} - \tilde{\boldsymbol{u}}\|^2_{L^2(V)} + C\|\boldsymbol{g}\|^2_{H_0^1(R)}.$$

Proposition 1.1. *For any fixed m in B and $C > 0$, $F_{m,C}$ achieves a unique minimum $\boldsymbol{h}_{m,C}$ in $H_0^1(R)$.*

Proof. This is a consequence of [3], chapter 16 section 2, see theorems 4.5 and 4.6

in appendix 2 for details. $\square$

Now, we set

$$f_C(m) = F_{m,C}(\boldsymbol{h}_{m,C}). \tag{1.10}$$

Proposition 1.2. *f_C is a Lipschitz continuous function on B. It therefore achieves its minimum value on B.*

Proof. It is known that Green's tensor $\boldsymbol{H}_m$ for $(\boldsymbol{x}, \boldsymbol{y})$ in $\overline{\mathbb{R}^{3-}} \times \mathbb{R}^{3-}$ such that $\boldsymbol{x} \neq \boldsymbol{y}$ is C^∞ in a neighbourhood of $(\boldsymbol{x}, \boldsymbol{y})$. Thus by lemma 1.1 $\boldsymbol{H}_m$ and its first derivatives are uniformly bounded, for x in V and (y_1, y_2) in R, since the distance from V to R is strictly positive, and V and R are bounded. Thus $\boldsymbol{H}_m$ is Lipschitz continuous for m in B and (y_1, y_2) in R, so there exists a positive constant L_0 such that,

$$|\boldsymbol{H}_m(\boldsymbol{x}, y_1, y_2) - \boldsymbol{H}_{m'}(\boldsymbol{x}, y_1, y_2)| \leq L_0|m - m^{'}|,$$

for all $\boldsymbol{x}$ in V, any m, $m^{'}$ in B and all (y_1, y_2) in R, and since the surface element σ is a smooth function of m there exists a positive constant L such that,

$$|\boldsymbol{H}_m(\boldsymbol{x}, y_1, y_2)\sigma_m - \boldsymbol{H}_{m'}(\boldsymbol{x}, y_1, y_2)\sigma_{m'}| \leq L|m - m^{'}|, \tag{1.11}$$

for all $\boldsymbol{x}$ in V, any m, $m^{'}$ in B and all (y_1, y_2) in R. Now, we also observe that

$$\begin{aligned}
&|F_{m,C}(\boldsymbol{h}_{m,C}) - F_{m',C}(\boldsymbol{h}_{m,C})| = |\|A_m\boldsymbol{h}_{m,C} - \tilde{\boldsymbol{u}}\|^2 - \|A_{m'}\boldsymbol{h}_{m,C} - \tilde{\boldsymbol{u}}\|^2| \\
&= |\|A_m\boldsymbol{h}_{m,C}\|^2 - \|A_{m'}\boldsymbol{h}_{m,C}\|^2 + 2\langle A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C}, \tilde{\boldsymbol{u}}\rangle| \\
&\leq |\|A_m\boldsymbol{h}_{m,C}\|^2 - \|A_{m'}\boldsymbol{h}_{m,C}\|^2| + 2|\langle A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C}, \tilde{\boldsymbol{u}}\rangle|.
\end{aligned}$$

Notice that, by the triangle inequality,

$$\begin{aligned}
&|\|A_m\boldsymbol{h}_{m,C}\|^2 - \|A_{m'}\boldsymbol{h}_{m,C}\|^2| \\
&= (\|A_m\boldsymbol{h}_{m,C}\| + \|A_{m'}\boldsymbol{h}_{m,C}\|)|(\|A_m\boldsymbol{h}_{m,C}\| - \|A_{m'}\boldsymbol{h}_{m,C}\|)| \\
&\leq (\|A_m\boldsymbol{h}_{m,C}\| + \|A_{m'}\boldsymbol{h}_{m,C}\|)\|A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C}\|.
\end{aligned}$$

And thanks to Cauchy-Schwartz inequality,

$$2|\langle A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C}, \tilde{\boldsymbol{u}}\rangle| \leq 2\|A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C}\|\|\tilde{\boldsymbol{u}}\|.$$

We conclude that

$$\begin{aligned}
&|F_{m,C}(\boldsymbol{h}_{m,C}) - F_{m',C}(\boldsymbol{h}_{m,C})| \\
&\leq (\|A_m\boldsymbol{h}_{m,C}\| + \|A_{m'}\boldsymbol{h}_{m,C}\| + 2\|\tilde{\boldsymbol{u}}\|)\|A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C}\|.
\end{aligned}$$

Notice that,

$$A_m\boldsymbol{h}_{m,C} - A_{m'}\boldsymbol{h}_{m,C} = \int_R (\boldsymbol{H}_m\sigma_m - \boldsymbol{H}_{m'}\sigma_{m'})\boldsymbol{h}_{m,C} dy_1 dy_2.$$

Thus, using (1.11),

$$\|A_m \boldsymbol{h}_{m,C} - A_{m'} \boldsymbol{h}_{m,C}\| \leq LK_1 |m - m'| \|\boldsymbol{h}_{m,C}\|, \tag{1.12}$$

where K_1 is some constant independent of the choice of m and m' in B. Altogether we now have

$$|F_{m,C}(\boldsymbol{h}_{m,C}) - F_{m',C}(\boldsymbol{h}_{m,C})| \leq K_2 \|\boldsymbol{h}_{m,C}\| (\|\boldsymbol{h}_{m,C}\| + \|\tilde{\boldsymbol{u}}\|) |m - m'|,$$

where K_2 is some constant independent of the choice of m and m' in B.

By definition, $\boldsymbol{h}_{m',C}$ is the minimizer of $F_{m',C}$, so $F_{m',C}(\boldsymbol{h}_{m',C}) \leq F_{m',C}(\boldsymbol{h}_{m,C})$, therefore

$$K_2 \|\boldsymbol{h}_{m,C}\| (\|\boldsymbol{h}_{m,C}\| + \|\tilde{\boldsymbol{u}}\|) |m - m'| \geq F_{m',C}(\boldsymbol{h}_{m,C}) - F_{m,C}(\boldsymbol{h}_{m,C}) \geq F_{m',C}(\boldsymbol{h}_{m',C}) - F_{m,C}(\boldsymbol{h}_{m,C}).$$

Here the choices of m and m' are arbitrary, so we can also show that

$$F_{m,C}(\boldsymbol{h}_{m,C}) - F_{m',C}(\boldsymbol{h}_{m',C}) \leq K_2 \|\boldsymbol{h}_{m,C}\| (\|\boldsymbol{h}_{m,C}\| + \|\tilde{\boldsymbol{u}}\|) |m - m'|.$$

Thus

$$|F_{m',C}(\boldsymbol{h}_{m',C}) - F_{m,C}(\boldsymbol{h}_{m,C})| \leq K_2 \|\boldsymbol{h}_{m,C}\| (\|\boldsymbol{h}_{m,C}\| + \|\tilde{\boldsymbol{u}}\|) |m - m'|.$$

Now, we are going to find the upper bound of $\|\boldsymbol{h}_{m,C}\|$. Notice that $\boldsymbol{h}_{m,C}$ is the minimizer of $F_{m,C}$, thus $F_{m,C}(\boldsymbol{h}_{m,C}) \leq F_{m,C}(\mathbf{0})$, therefore

$$F_{m,C}(\boldsymbol{h}_{m,C}) = \|A_m \boldsymbol{h}_{m,C} - \tilde{\boldsymbol{u}}\|^2 + C\|\boldsymbol{h}_{m,C}\|^2 \leq \|\tilde{\boldsymbol{u}}\|^2 = F_{m,C}(\mathbf{0}).$$

Hence,

$$C\|\boldsymbol{h}_{m,C}\|^2 \leq \|\tilde{\boldsymbol{u}}\|^2.$$

We have

$$\|\boldsymbol{h}_{m,C}\| \leq C^{-\frac{1}{2}} \|\tilde{\boldsymbol{u}}\|.$$

Finally,

$$|F_{m',C}(\boldsymbol{h}_{m',C}) - F_{m,C}(\boldsymbol{h}_{m,C})| \leq 2K_2 C^{-1} \|\tilde{\boldsymbol{u}}\|^2 |m - m'|,$$

for $0 < C < 1$. So we proved that f_C is a Lipschitz continuous function on B, but B is compact, thus f_C achieves its minimum value on B. □

Theorem 1.2. *Assume that $\tilde{\boldsymbol{u}} = A_{\tilde{m}} \tilde{\boldsymbol{h}}$ for some $\tilde{m}$ in B and some $\tilde{\boldsymbol{h}}$ in $H_0^1(R)$. Let C_n be a sequence of positive numbers converging to zero. Let m_n be any sequence in B such that $f_{C_n}(m_n)$ minimizes $f_{C_n}(m)$ for m in B and set $f_{C_n}(m_n) = F_{m_n,C_n}(\boldsymbol{h}_{m_n,C_n})$. Then m_n converges to $\tilde{m}$, $\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{h}}$ in $H_0^1(R)$, and $A_{m_n}\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{u}}$ in $L^2(V)$.*

Proof. This theorem was proved in [9]. In this thesis we give a more detailed proof

of this theorem because the minutia of this proof are important in the derivation of our main result.
We first note that

$$\begin{aligned}\int_V |A_{m_n}\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}|^2 + C_n \int_R |\nabla \boldsymbol{h}_{m_n,C_n}|^2 &= f_{C_n}(m_n) \\ &\leq f_{C_n}(\tilde{m}) \\ &= F_{\tilde{m},C_n}(\boldsymbol{h}_{\tilde{m},C_n}) \\ &\leq F_{\tilde{m},C_n}(\tilde{\boldsymbol{h}}) \\ &= C_n \int_R |\nabla \tilde{\boldsymbol{h}}|^2.\end{aligned} \tag{1.13}$$

From there, as C_n converges to zero, we observe that $A_{m_n}\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{u}}$ in $L^2(V)$, and the sequence $||\nabla \boldsymbol{h}_{m_n,C_n}||_{L^2(V)}$ is bounded, so $\boldsymbol{h}_{m_n,C_n}$ is bounded in $H_0^1(R)$. Now we want to show that m_n converges to $\tilde{m}$, we argue by contradiction. Assume that m_n does not converges to $\tilde{m}$. As m_n is a sequence in a compact set B, we can extract a subsequence m_{n_k}, such that m_{n_k} converges to some m^* in B with $m^* \neq \tilde{m}$. As $\boldsymbol{h}_{m_{n_k},C_{n_k}}$ is bounded in $H_0^1(R)$, a subsequence is weakly convergent to some h^* in $H_0^1(R)$, so at this stage we can redefine the sequence m_{n_k} so that it converges to m^* and $\boldsymbol{h}_{m_{n_k},C_{n_k}}$ is weakly convergent to h^* in $H_0^1(R)$.
Next ,we want to show that $A_{m_{n_k}}$ converges to A_{m^*} in operator norm, that is,

$$\sup_{\varphi \in H_0^1(R), \|\varphi\|=1} |(A_{m_{n_k}} - A_{m^*})\varphi| \to 0,$$

as n_k tends to infinity. To do that, we have to indicate explicitly the continuous dependence of the surface element σ on the geometry parameter m. Notice that

$$\begin{aligned}|(A_{m_{n_k}} - A_{m^*})\varphi| &= |\int_R (\boldsymbol{H}_{m_{n_k}}\sigma_{m_{n_k}} - \boldsymbol{H}_{m^*}\sigma_{m^*})\varphi \mathrm{d}y_1 \mathrm{d}y_2| \\ &\leq \int_R \sup_{(\boldsymbol{x},y_1,y_2)\in \overline{V}\times R} |(\boldsymbol{H}_{m_{n_k}}\sigma_{m_{n_k}} - \boldsymbol{H}_{m^*}\sigma_{m^*})||\varphi| \mathrm{d}y_1 \mathrm{d}y_2 \\ &= J \sup_{(\boldsymbol{x},y_1,y_2)\in \overline{V}\times R} |(\boldsymbol{H}_{m_{n_k}}\sigma_{m_{n_k}} - \boldsymbol{H}_{m^*}\sigma_{m^*})| \|\varphi\|_{H_0^1(R)},\end{aligned}$$

where J is some constant independent of m^* and m_{n_k}. Therefore

$$\int_V |(A_{m_{n_k}} - A_{m^*})\varphi|^2 \leq (J \sup_{(\boldsymbol{x},y_1,y_2)\in \overline{V}\times R} |(\boldsymbol{H}_{m_{n_k}}\sigma_{m_{n_k}} - \boldsymbol{H}_{m^*}\sigma_{m^*})| \|\varphi\|_{H_0^1(R)})^2 |V|.$$

Here $|V|$ is the area of V. We know that the function $(\boldsymbol{x}, y_1, y_2, m) \to \boldsymbol{H}_m(\boldsymbol{x}, y_1, y_2)\sigma_m$

is continuous on the compact set $\overline{V} \times R \times B$, thus

$$\sup_{(\boldsymbol{x},y_1,y_2)\in\overline{V}\times R} |(\boldsymbol{H}_{m_{n_k}}\sigma_{m_{n_k}} - \boldsymbol{H}_{m^*}\sigma_{m^*})|$$

converges to zero as k tends to infinity and $A_{m_{n_k}}$ converges to A_{m^*} in operator norm. Next, we notice that

$$\begin{aligned}\|A_{m_{n_k}}\boldsymbol{h}_{m_{n_k},C_{n_k}} - A_{m^*}\boldsymbol{h}^*\| \leq \|(A_{m_{n_k}} - A_{m^*})\boldsymbol{h}_{m_{n_k},C_{n_k}}\| \\ + \|A_{m^*}(\boldsymbol{h}_{m_{n_k},C_{n_k}} - \boldsymbol{h}^*)\|.\end{aligned}$$

Both of these two terms tend to zero as n_k tends to infinity. Thus $A_{m_{n_k}}\boldsymbol{h}_{m_{n_k},C_{n_k}}$ converges to $A_{m^*}\boldsymbol{h}^*$ strongly (for more details see appendix 1, theorem 4.4). Notice that from (1.13) we have

$$\int_V |A_{m_{n_k}}\boldsymbol{h}_{m_{n_k},C_{n_k}} - \tilde{\boldsymbol{u}}|^2 + C_{n_k}\int_R |\nabla \boldsymbol{h}_{m_{n_k},C_{n_k}}|^2 \leq C_{n_k}\int_R |\nabla\tilde{\boldsymbol{h}}|^2.$$

Hence if we take limit on both sides of the above inequality , we have

$$\int_V |A_{m^*}\boldsymbol{h}^* - \tilde{\boldsymbol{u}}|^2 = 0.$$

Thus $A_{m^*}\boldsymbol{h}^* = \tilde{\boldsymbol{u}}$. But we assumed that $m^* \neq \tilde{m}$, so this result contradicts theorem 1.1. So we can conclude that m_n converges to $\tilde{m}$.

We now want to show that $\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{h}}$ in $H_0^1(R)$. As we now know that $A_{m_n}\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{u}}$, since A_{m_n} is norm convergent to $A_{\tilde{m}}$ and $\boldsymbol{h}_{m_n,C_n}$ is bounded in $H_0^1(R)$, we can claim that $A_{\tilde{m}}\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{u}}$. Let $\boldsymbol{v}$ be in $L^2(V)$. We have

$$\langle \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}, A_{\tilde{m}}^*\boldsymbol{v}\rangle = \langle A_{\tilde{m}}\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}, \boldsymbol{v}\rangle \to 0.$$

We also know that $A_{\tilde{m}}$ is injective, so the range of $A_{\tilde{m}}^*$ is dense (see theorem 4.1 in appendix 1 for a proof), thus $\boldsymbol{h}_{m_n,C_n}$ converges weakly to $\tilde{\boldsymbol{h}}$ in $H_0^1(R)$. Now from (1.13) we observe that $||\boldsymbol{h}_{m_n,C_n}|| \leq ||\tilde{\boldsymbol{h}}||$, so

$$\begin{aligned}\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\|^2 &= \|\boldsymbol{h}_{m_n,C_n}\|^2 - 2\langle \boldsymbol{h}_{m_n,C_n}, \tilde{\boldsymbol{h}}\rangle + \|\tilde{\boldsymbol{h}}\|^2 \\ &\leq \|\tilde{\boldsymbol{h}}\|^2 - 2\langle \boldsymbol{h}_{m_n,C_n}, \tilde{\boldsymbol{h}}\rangle + \|\tilde{\boldsymbol{h}}\|^2 \\ &= 2\langle \tilde{\boldsymbol{h}} - \boldsymbol{h}_{m_n,C_n}, \tilde{\boldsymbol{h}}\rangle,\end{aligned} \tag{1.14}$$

which converges to zero due to the weak convergence of $\boldsymbol{h}_{m_n,C_n}$. □

Chapter 2

Estimates of the convergence order

In this chapter we derive estimates for the convergence of m_n, a minimizing sequence of geometry parameters for f_{C_n} as defined in theorem 1.2 and estimates for the corresponding minimizing field $\boldsymbol{h}_{m_n,C_n}$ as the regularization parameter C_n tends to zero.

2.1 A preliminary result

In this section we review and prove in details a result from [7].

Proposition 2.1. *Let $\tilde{\boldsymbol{u}}, m_n, \boldsymbol{h}_{m_n,C_n}$ be as in theorem 1.2. The following convergence rate estimates hold*

$$\|A_{m_n}\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}\| \leq C_n^{\frac{1}{2}}\|\tilde{\boldsymbol{h}}\|, \tag{2.1}$$

$$\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\| \leq \sqrt{2\|\boldsymbol{v}\|\|\tilde{\boldsymbol{h}}\|}(C_n^{\frac{1}{4}} + (LK_1)^{\frac{1}{2}}|m_n - \tilde{m}|^{\frac{1}{2}}). \tag{2.2}$$

Here for (2.2) *we assume that $\tilde{\boldsymbol{h}}$ is in the image of $A^*_{\tilde{m}}$ with $\tilde{\boldsymbol{h}} = A^*_{\tilde{m}}\boldsymbol{v}$, L was previously introduced previously in* (1.11)*, and K_1 was introduced in* (1.12)*.*

Proof. (2.1) is clear due to (1.13). To show (2.2) we observe that

$$\begin{aligned}
|\langle \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{h}}\rangle| &= |\langle \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}, A^*_{\tilde{m}}\boldsymbol{v}\rangle| \\
&= |\langle A_{\tilde{m}}\boldsymbol{h}_{m_n,C_n} - A_{\tilde{m}}\tilde{\boldsymbol{h}}, \boldsymbol{v}\rangle| \\
&= |\langle A_{\tilde{m}}\boldsymbol{h}_{m_n,C_n} - A_{m_n}\boldsymbol{h}_{m_n,C_n} + A_{m_n}\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}, \boldsymbol{v}\rangle| \\
&\leq |\langle A_{\tilde{m}}\boldsymbol{h}_{m_n,C_n} - A_{m_n}\boldsymbol{h}_{m_n,C_n}, \boldsymbol{v}\rangle| + |\langle A_{m_n}\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}, \boldsymbol{v}\rangle|.
\end{aligned}$$

From estimate (1.12) we observe that

$$|\langle A_{\tilde{m}}\boldsymbol{h}_{m_n,C_n} - A_{m_n}\boldsymbol{h}_{m_n,C_n}, \boldsymbol{v}\rangle| \leq LK_1\|\tilde{\boldsymbol{h}}\|\|\boldsymbol{v}\||\tilde{m} - m_n|.$$

And thanks to Cauchy-Schwartz inequality and (2.1),

$$|\langle A_{m_n}\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}, \boldsymbol{v}\rangle| \leq C_n^{\frac{1}{2}}\|\tilde{\boldsymbol{h}}\|\|\boldsymbol{v}\|.$$

Combining with (1.14) we have

$$\begin{aligned}
\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\| &\leq \sqrt{2\langle \tilde{\boldsymbol{h}} - \boldsymbol{h}_{m_n,C_n}, \tilde{\boldsymbol{h}}\rangle} \\
&\leq \sqrt{2(LK_1\|\tilde{\boldsymbol{h}}\||\tilde{m} - m_n|\|\boldsymbol{v}\| + C_n^{\frac{1}{2}}\|\tilde{\boldsymbol{h}}\|\|\boldsymbol{v}\|)} \\
&= \sqrt{2\|\boldsymbol{v}\|\|\tilde{\boldsymbol{h}}\|}\sqrt{C_n^{\frac{1}{2}} + LK_1|m_n - \tilde{m}|} \\
&\leq \sqrt{2\|\boldsymbol{v}\|\|\tilde{\boldsymbol{h}}\|}(C_n^{\frac{1}{4}} + (LK_1)^{\frac{1}{2}}|m_n - \tilde{m}|^{\frac{1}{2}}).
\end{aligned} \tag{2.3}$$

□

As we can see here, if we can determine the convergence order of m_n to $\tilde{m}$, we will be able to estimate the convergence order of $\boldsymbol{h}_{m_n,C_n}$. To do so we need to recall a theorem from [8].

Theorem 2.1. *Let B be defined as in section 1.3, m in B, and A_m be defined as in (1.9). Fix a non-zero $\tilde{\boldsymbol{h}}$ in $H_0^1(R)$ and $\tilde{m}$ in B. Assume that $\tilde{\boldsymbol{h}}$ satisfies one of the two following additional assumptions:*

(i). $\tilde{\boldsymbol{h}}$ is one-directional, that is, $\tilde{\boldsymbol{h}}$ is parallel to a fixed tangential vector.
(ii). $\tilde{\boldsymbol{h}}$ is the gradient of a function φ in $H^2(R)$.

Then there exists a positive constant C_0 depending on $\tilde{m}$ and $\tilde{\boldsymbol{h}}$ but not on m such that

$$\inf_{\boldsymbol{h}\in H_0^1(R)} \|A_m\boldsymbol{h} - A_{\tilde{m}}\tilde{\boldsymbol{h}}\|_{L^2(V)} \geq C_0|m - \tilde{m}| \tag{2.4}$$

for all m in B.

Proof. This is proved in [8], theorem 4.1. □

2.2 Convergence order

The following theorem is the main new result of this thesis. By combining proposition 2.1 and theorem 2.1, we are able to estimate the convergence order of m_n. This is an important result in practice since only m_n can be computed and then by playing with the regularization constant C_n, one can gauge how close m_n is to $\tilde{m}$, in other words one can claim with some degree of confidence how close one is to having determined the geometry of the planar fault Γ_m. We also include sharper convergence estimates further in this section, but these sharper estimates can be skipped in a first reading.

Theorem 2.2. *Let $F_{m,C}$ be defined as in* (1.8) *and f_C as in* (1.10). *Let C_n be a sequence of positive numbers converging to zero. Let m_n be a sequence in B such that $f_{C_n}(m_n) = F_{m_n,C_n}(\boldsymbol{h}_{m_n,C_n})$ is the minimum of $f_{C_n}(m) = F_{m,C}(\boldsymbol{h}_{m,C})$ for m in B, then there exists a constant C^* such that,*

$$|m_n - \tilde{m}| \leq C^* C_n^{\frac{1}{2}} \|\tilde{\boldsymbol{h}}\|. \tag{2.5}$$

In addition, if $\tilde{\boldsymbol{h}}$ is in the range of $A^*_{\tilde{m}}$, then

$$\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\| = O(C_n^{\frac{1}{4}}). \tag{2.6}$$

Remark: Due to the definition of $A_{\tilde{m}}$ we know from theorem 1.1 that $A_{\tilde{m}}$ is injective. Thus the range of $A^*_{\tilde{m}}$ is dense in $L^2(V)$ (see Appendix, theorem 4.1). The assumption that $\tilde{\boldsymbol{h}}$ is in the range of $A^*_{\tilde{m}}$ can be interpreted as a regularity condition.

Proof. Starting from (2.4) we set $C^* = C_0^{-1}$ to obtain,

$$|m_n - \tilde{m}| \leq C^* \inf_{\boldsymbol{h} \in H_0^1(R)} \|A_{m_n}\boldsymbol{h} - A_{\tilde{m}}\tilde{\boldsymbol{h}}\|_{L^2(V)},$$

thus using (2.1), we arrive at (2.5).
Combining (2.2) and (2.5), we have

$$\begin{aligned}
\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\| &\leq \sqrt{2\|\boldsymbol{v}\|\|\tilde{\boldsymbol{h}}\|}(C_n^{\frac{1}{4}} + (LK_1)^{\frac{1}{2}}|m_n - \tilde{m}|^{\frac{1}{2}}) \\
&\leq \sqrt{2\|\boldsymbol{v}\|\|\tilde{\boldsymbol{h}}\|}(C_n^{\frac{1}{4}} + (LK_1)^{\frac{1}{2}}\|\tilde{\boldsymbol{h}}\|^{\frac{1}{2}}(C^*)^{\frac{1}{2}}C_n^{\frac{1}{4}}) \\
&= \sqrt{2\|\boldsymbol{v}\|\|\tilde{\boldsymbol{h}}\|}(1 + \sqrt{C^* LK_1\|\tilde{\boldsymbol{h}}\|})C_n^{\frac{1}{4}} \\
&= O(C_n^{\frac{1}{4}}),
\end{aligned}$$

indicating a convergence order is at least one fourth. $\square$

The estimates from theorem 2.2 can be bootstrapped to obtain the following result.

Theorem 2.3. *With the same notations and assumptions as in theorem 2.2*

$$|m_n - \tilde{m}| = o(C_n^{\frac{1}{2}}), \tag{2.7}$$

and if $\tilde{\boldsymbol{h}}$ is in the range of $A^*_{\tilde{m}}$, then

$$\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\| = o(C_n^{\frac{1}{4}}), \tag{2.8}$$

and

$$|m_n - \tilde{m}| = o(C_n^{\frac{5}{8}}). \tag{2.9}$$

Remark: Further sharper estimates can be investigated if we assume more regularity on $\tilde{\boldsymbol{h}}$. For example, a reasonable assumption would be to take $\tilde{\boldsymbol{h}}$ is in the range of $A^*_{\tilde{m}} A_{\tilde{m}} A^*_{\tilde{m}}$. Alternatively, another reasonable assumption on $\tilde{\boldsymbol{h}}$ is to set it equal to a finite linear combination of a few of the first eigenvectors of $A^*_{\tilde{m}} A_{\tilde{m}}$.

Proof. To prove estimate (2.7) we first rearrange the terms in inequality (1.13) in such away to obtain

$$\int_V |A_{m_n} \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}|^2 \leq C_n(\int_R |\nabla \tilde{\boldsymbol{h}}|^2 - \int_R |\nabla \boldsymbol{h}_{m_n,C_n}|^2). \tag{2.10}$$

But theorem 1.2 asserts that $\boldsymbol{h}_{m_n,C_n}$ converges to $\tilde{\boldsymbol{h}}$ in $H^1_0(R)$, so the right hand side of (2.10) is $o(C_n)$ so in light of (2.4), estimate (2.7) is proved. Next, assuming that $\tilde{\boldsymbol{h}}$ is in the range of $A^*_{\tilde{m}}$, there is a $\boldsymbol{v}$ in $L^2(V)$ such that $\tilde{\boldsymbol{h}} = A^*_{\tilde{m}} \boldsymbol{v}$, so as

$$|\langle \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{h}} \rangle| \leq |\langle A_{\tilde{m}} \boldsymbol{h}_{m_n,C_n} - A_{m_n} \boldsymbol{h}_{m_n,C_n}, \boldsymbol{v} \rangle| + |\langle A_{m_n} \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}, \boldsymbol{v} \rangle|,$$

and given that from (2.7),

$$\|A_{\tilde{m}} - A_{m_n}\| = O(|m_n - \tilde{m}|) = o(C_n^{\frac{1}{2}}),$$

and (2.10) shows that $\|A_{m_n} \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}\| = o(C_n^{\frac{1}{2}})$, we infer that $|\langle \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{h}} \rangle|$ is of order $o(C_n^{\frac{1}{2}})$. As explained in the proof of proposition 2.1, as $\|\boldsymbol{h}_{m_n,C_n}\| \leq \|\tilde{\boldsymbol{h}}\|$,

$$\|\boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{h}}\| \leq \sqrt{2\langle \tilde{\boldsymbol{h}} - \boldsymbol{h}_{m_n,C_n}, \tilde{\boldsymbol{h}} \rangle},$$

and thus (2.8) is proved. Finally, to show (2.9), combining (2.8) and (2.10) we find that

$$\|A_{m_n} \boldsymbol{h}_{m_n,C_n} - \tilde{\boldsymbol{u}}\| = o(C_n^{\frac{5}{8}}),$$

so by (2.4), (2.9) is proved. $\square$

Chapter 3

Conclusion

In this thesis, we studied a regularization error functional that can be used to solve a nonlinear inverse problem for half space elasticity minus a fault, that has to be determined from boundary measurements. We have proved that we can estimate the convergence order of the minimizer of our regularized error functional as the regularization parameter tends to zero. Unlike in classical Tikhonov regularization we also had to contend with a nonlinear parameter which models the geometry of the fault, and we also found an estimate of the convergence order of that parameter as the regularization constant tends to zero.

In future work, it would be interesting to see if the estimate of the order of convergence can be further improved if more regularity is assumed on the data. However, more importantly, our priority should be to derive the convergence order for the fully discrete numerical method related to the error functional. Such a study was undertaken in previous work [7] but stability results were unknown at the time of the writing of that paper, so only convergence of the numerical method could be proved, without order estimates.

Chapter 4

Appendix: basic functional analysis and Tikhonov regularization

In the proof of theorem 1.2, we refer to the following theorem:

Theorem 4.1. *Let X and Y be two Hilbert spaces. Let $A : X \to Y$ be an injective linear operator, then $A^*(Y)$ is dense in X.*

In order to prove this theorem, we need to review some results in functional analysis.

Lemma 4.1. *Let V be a subset of a Hilbert space X, then the orthogonal space of V is closed.*

Proof. Let y be in $\overline{V^\perp}$, then there exists a sequence y_n in $V^\perp$ such that y_n converges to y. Choose an arbitrary x in V, since the inner product function is continuous, we have

$$\langle y, x\rangle = \langle \lim_{n\to\infty} y_n, x\rangle = \lim_{n\to\infty} \langle y_n, x\rangle = 0.$$

This shows that y is in $V^\perp$, so $V^\perp$ is closed. □

Lemma 4.2. *If A and B are two subsets of Hilbert space X such that $A \subset B$, then $B^\perp \subset A^\perp$.*

Proof. Let x be in $B^\perp$, then for any y in B we have $\langle x, y\rangle = 0$. But since $A \subset B$, then for a any z in A, we also have $\langle x, z\rangle = 0$. Thus x is in $A^\perp$, which implies $B^\perp \subset A^\perp$. □

Lemma 4.3. $V^\perp = (\overline{V})^\perp$.

Proof. First, since $V \subset \overline{V}$, then $(\overline{V})^\perp \subset V^\perp$. Conversely, let x be in $V^\perp$, and y be in $\overline{V}$, by continuity of inner product, we have $\langle x, y\rangle = 0$. So x is in $(\overline{V})^\perp$, thus $V^\perp \subset (\overline{V})^\perp$. Therefore we have $V^\perp = (\overline{V})^\perp$. □

We now define orthogonal projections.

Definition 4.1. The linear operator P from X to X is an orthogonal projection if $P = P^2$ and $P = P^*$.

Remark: By definition, $P = P^*$ is equivalent to the statement: for any x, y in X, $\langle Px, y\rangle = \langle x, Py\rangle$.

Proposition 4.1. *Let P be an orthogonal projection from X to X, then for any x and y in X, we have the following:*

$$\begin{aligned}
&(i)\langle Px, (I-P)y\rangle = 0,\\
&(ii)\|Px + (I-P)y\|^2 = \|Px\|^2 + \|(I-P)y\|^2,\\
&(iii)Im(P) = N(I-P),\\
&(iv)I-P \text{ is also an orthogonal projection},\\
&(v)\|P\| \text{ is equal to 1 or 0.}
\end{aligned}$$

Proof. (i) By definition, $P = P^2$, $P = P^*$, thus

$$\langle Px, (I-P)y\rangle = \langle x, P^*(I-P)y\rangle = \langle x, P(I-P)y\rangle = \langle x, Py - Py\rangle = 0.$$

(ii) is clear by (i). To prove (iii), first fix an arbitrary y in $Im(P)$, then there exists x in X, such that $y = Px$. Thus

$$(I-P)y = (I-P)Px = (P-P^2)x = 0.$$

So $Im(P) \subset N(I-P)$. On the other hand, fix an arbitrary x in $N(I-P)$, we have

$$(I-P)x = x - Px = 0.$$

So $x = Px$, thus $x \in Im(P)$, therefore $N(I-P) \subset Im(P)$, hence $Im(P) = N(I-P)$.

To prove (iv), notice that

$$(I-P)^2 = I^2 - 2P + P^2 = I - P,$$

$$(I-P)^* = I^* - P^* = I - P.$$

By definition, $I - P$ is also an orthogonal projection.

Finally, let x be in X, from (ii) we observe that

$$\|Px\|^2 + \|(I-P)x\|^2 = \|Px + (I-P)x\|^2 = \|x\|^2.$$

Thus $\|Px\| \leq \|x\|$, which implies

$$\|P\| \leq 1.$$

Now, assume that $P \neq 0$, Then there exists y in X, such that $Py \neq 0$, thus if we set $z = Py$, we have $Pz = P^2y = Py = z$, so we have $\|P\| \geq 1$, hence we conclude that $\|P\| = 1$. Otherwise, $P = 0$, $\|P\| = 0$. $\square$

We now introduce what is arguably the most important theorem about Hilbert spaces, namely the orthogonal projection on closed subspaces theorem.

Theorem 4.2. *(Projection Theorem) Let X be a Hilbert space and V be a closed subspace of X. Then the space X can be decomposed to the direct sum*

$$X = V \oplus V^{\perp},$$

meanning that any element $x \in X$ can be written as,

$$x = y + z, \; y \in V \textit{ and } z \in V^{\perp}.$$

This decomposition is unique. The mapping $P_V : X \to X$, such that $P_V x = y$ is the orthogonal projection from X to V and the mapping $I - P_V$ is the orthogonal projection from X to $V^{\perp}$.

Note that $P_{V^{\perp}} = I - P_V$ since $I - P_V$ is the orthogonal projection from X to $V^{\perp}$. For more a detailed proof of this fundamental theorem, see Yosida's textbook [10], chapter III, section 1.

Lemma 4.4. *If V is a subspace of X, then $(V^{\perp})^{\perp} = \overline{V}$.*

Proof. We first want to show that $\overline{V} \subset (V^{\perp})^{\perp}$. Let x be arbitrary in V, then for any y in $V^{\perp}$, $\langle x, y\rangle = 0$, so $x \in (V^{\perp})^{\perp}$, thus $V \subset (V^{\perp})^{\perp}$. By lemma 4.1, $(V^{\perp})^{\perp}$ is closed, so $\overline{V} \subset (V^{\perp})^{\perp}$. There remains to prove that $(V^{\perp})^{\perp} \subset \overline{V}$. Choose x in $(V^{\perp})^{\perp}$, by projection theorem 4.2, there exits y in $\overline{V}$ and z in $\overline{V}^{\perp}$, such that $x = y + z$. By lemma 4.3, $\overline{V}^{\perp} = V^{\perp}$, so $z \in V^{\perp}$, hence

$$0 = \langle x, z\rangle = \langle y + z, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \langle z, z\rangle.$$

So we have $z = 0$. Thus, for any x in $(V^{\perp})^{\perp}$, there exists y in $\overline{V}$ such that $x = y$, which implies that $(V^{\perp})^{\perp} \subset \overline{V}$. $\square$

Corollary. If V is a closed subspace, then$(V^{\perp})^{\perp} = V$.

Lemma 4.5. *Let V be a subspace of a Hilbert space X, then V is dense in X if and only if $V^{\perp} = \{0\}$.*

Proof. First we assume that V is dense in X. Let x be in $V^\perp$, then there exists a sequence x_n in V, such that x_n converges to x, and $\langle x_n, x\rangle$=0 for all n. Using the fact that the function of inner product is continuous, we have

$$\langle x, x\rangle = \langle \lim_{n\to\infty} x_n, x\rangle = \lim_{n\to\infty}\langle x_n, x\rangle = 0.$$

Therefore x will be identically zero, which implies $V^\perp = \{0\}$.

Conversely, suppose $V^\perp = \{0\}$, then $(V^\perp)^\perp = X$. But by lemma 4.4, $(V^\perp)^\perp = \overline{V}$, so $\overline{V} = X$, therefore V is dense in X. □

We now give the proof of theorem 4.1.

Proof. Let x be in the orthogonal space of $A^*(Y)$, then for any y in Y, we have $\langle x, A^*y\rangle = 0$, so $\langle Ax, y\rangle = 0$. Set $y = Ax$, we have $\langle Ax, Ax\rangle = 0$, which implies $Ax = 0$. So $x \in N(A)$. But A is injective, so $N(A) = \{0\}$. Hence $(A^*(Y))^\perp = \{0\}$. By lemma 4.5 we show that $A^*(Y)$ is dense in X. □

We also use some important results about compact operators in the proof of theorem 1.2, we give more details here.

Theorem 4.3. *Let X and Y be two normed spaces and $T : X \to Y$ be a compact linear operator. Assume that x_n is a sequence in X, converges weakly to some x in X, then Tx_n converges strongly to Tx in Y.*

Proof. See Friedman's book [2], chapter 5, section 1 (page 186). □

Theorem 4.4. *Let X and Y be two normed spaces, and $T_n : X \to Y$ be a sequence of bounded linear operator, converging to a compact operator T in operator norm. Let x_n be bounded sequence in X converging to x in X weakly, then T_nx_n converges to Tx strongly in Y.*

Proof. Notice that

$$\begin{aligned}\|T_nx_n - Tx\| &\le \|T_nx_n - Tx_n\| + \|Tx_n - Tx\| \\ &\le \|T_n - T\|\|x_n\| + \|Tx_n - Tx\|.\end{aligned}$$

The first term converges to zero because x_n is bounded and T_n converges to T in operator norm. For the second term, as T is compact, we apply theorem 4.3 to show the convergence. Therefore, Tx_n converges to Tx strongly in Y. □

We now turn to Tikhonov regularization. Although this regularization process is well known in linear algebra, thanks to functional analysis techniques, this process can also apply to Hilbert spaces. Here, we follow the approach advocated in [3], chapter 16.

Definition 4.2. Let X and Y be two norm spaces, and let A mapping X to Y be an injective bounded linear operator. Let $C > 0$, then a family of bounded linear operators R_C mapping X to Y is called a regularization scheme for the operator A, if

$$\lim_{C \to 0} R_C A\boldsymbol{g} = \boldsymbol{g}, \ \boldsymbol{g} \in X.$$

The parameter C is called the regularization parameter.

In our problem, we use Tikhonov regularization to approximate our solution. We provide more details of this process here.

Theorem 4.5. *Let $A : X \to Y$ be a bounded linear operator and let $C > 0$. Then for each $\boldsymbol{u}$ in Y there exists a unique $\boldsymbol{g}_C$ in X such that*

$$\|A\boldsymbol{g}_C - \boldsymbol{u}\|^2 + C\|\boldsymbol{g}_C\|^2 = \inf_{\boldsymbol{g} \in X} \{\|A\boldsymbol{g} - \boldsymbol{u}\|^2 + C\|\boldsymbol{g}\|^2\}.$$

The minimizer $\boldsymbol{g}_C$ is given by the unique solution of the following equation:

$$C\boldsymbol{g}_C + A^* A\boldsymbol{g}_C = A^*\boldsymbol{u}.$$

Proof. Here is a proof from [3]. We first show that why the solution $\boldsymbol{g}_C$ is unique. Consider the operator $T_C := CI + A^*A$, since

$$C\|\boldsymbol{g}\|^2 \leq C\|\boldsymbol{g}\|^2 + \|A\boldsymbol{g}\|^2 = \langle T_C\boldsymbol{g}, \boldsymbol{g}\rangle,$$

then T_C is strictly coercive, so by Lax-Milgram's Theorem (this theorem is stated and proved for example in [3] chapter 13), the solution exists and is unique. Now we will show that $\boldsymbol{g}_C$ is exactly the minimizer. In fact, for any $\boldsymbol{g}$ in X, it is true that:

$$\begin{aligned} \|A\boldsymbol{g} - \boldsymbol{u}\|^2 + C\|\boldsymbol{g}\|^2 = {} & \|A\boldsymbol{g}_C - \boldsymbol{u}\|^2 + C\|\boldsymbol{g}_C\|^2 \\ & + \langle \boldsymbol{g} - \boldsymbol{g}_C, C\boldsymbol{g}_C + A^*(A\boldsymbol{g}_C - \boldsymbol{u})\rangle \\ & + \|A(\boldsymbol{g} - \boldsymbol{g}_C)\|^2 + C\|\boldsymbol{g} - \boldsymbol{g}_C\|^2. \end{aligned}$$

If $\boldsymbol{g}_C$ satisfies the previous equation, then the equation above will be minimized. □

We now state a lemma, which will be used to prove

$$R_C := (CI + AA^*)^{-1}A^*$$

is a regularization scheme.

Lemma 4.6. *Let X be a Banach space, and $\mathcal{L}(X)$ be the space of linear operator from X to X. The space*

$$V = \{S \in \mathcal{L}(X) : S \textit{ is invertible with bounded inverse}\}$$

is open in $\mathcal{L}(X)$. The mapping

$$T \to T^{-1}$$

is continuous.

Proof. Fix S in V, we will prove that any linear operator T in $\mathcal{L}(X)$ satisfying

$$\|S - T\| < \frac{1}{2\|S^{-1}\|} \tag{4.1}$$

is also in V, so V is open in $\mathcal{L}(X)$.

First, we observe that

$$\|I - S^{-1}T\| = \|S^{-1}(S - T)\| \leq \|S^{-1}\|\|S - T\| < \frac{1}{2}.$$

So the series

$$\sum_{n=0}^{\infty}(I - S^{-1}T)^n \tag{4.2}$$

is convergent. Set $A = I - S^{-1}T$, we notice that

$$(I - A)\sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = I.$$

Similarly,

$$\sum_{n=0}^{\infty} A^n(I - A) = I.$$

So $I - A = S^{-1}T$ is invertible, $T^{-1}S$ is given by (4.2), and in particular T is invertible. From (4.2) we infer that,

$$\|T^{-1}S\| \leq \frac{1}{1 - \|I - S^{-1}T\|} < 2.$$

From here we have

$$\|T^{-1}\| = \|T^{-1}SS^{-1}\| \leq \|T^{-1}S\|\|S^{-1}\| \leq 2\|S^{-1}\|.$$

Therefore, T^{-1} is bounded, So V is open.

Now we observe that

$$\|S^{-1} - T^{-1}\| \leq \|S^{-1}\|\|T - S\|\|T^{-1}\| \leq 2\|S^{-1}\|^2\|S - T\|,$$

for all S in V and all T satisfying (4.1) and we conclude that the mapping $T \to T^{-1}$

is continuous. □

Theorem 4.6. *Let* $A : X \to Y$ *be an injective bounded linear operator. Then*

$$R_C := (CI + AA^*)^{-1} A^*$$

describes a regularization scheme with

$$||R_C|| \leq \frac{||A||}{C}.$$

Proof. This is proved by [3], chapter 16, section 2. □

Bibliography

[1] Cavalie, O., Pathier, E., Radiguet, M., Vergnolle, M., Cotte, N., Walpersdorf, A., ... & Cotton, F. (2013). *Slow slip event in the Mexican subduction zone: Evidence of shallower slip in the Guerrero seismic gap for the 2006 event revealed by the joint inversion of InSAR and GPS data*, Earth and Planetary Science Letters, 367, 52-60.

[2] A. Friedman, *Foundations of modern analysis*, Dover Publications, Inc., 1982.

[3] R. Kress, *Linear Integral Equations* (3rd Ed.), vol. 82, Springer, 2014.

[4] Y. Okada, *Internal deformation due to shear and tensile faults in a half-space*, Bulletin of the Seismological Society of America, **82**, 1018-1040, 1992.

[5] A. Tarantola, (2005), *Inverse problem theory and methods for model parameter estimation*, (Vol. 89). SIAM.

[6] D. Volkov, *A double layer surface traction free green's tensor*, SIAM J. Appl. Math., **69**, 1433-1456, 2009. Springer, 1980.

[7] D. Volkov, J. C. Sandiumenge, *A stochastic approach to reconstruction of faults in elastic half space*, Inverse Problem and Imaging Volume 13, No. 3, 2019, 479-511.

[8] D. Volkov, F. Triki, *Stability estimates for the fault inverse problem*, Inverse Problem **35** (2019) 075007.

[9] D. Volkov, C. Voisin, I. R. Ionescu, *Reconstruction of faults in elastic half space from surface measurements*, Inverse Problem **33** (2017) 055018 (27pp).

[10] K. Yosida, *Functional Analysis*, Springer, 1980.