Convergence analysis in stochastic nonlinear systems: Asymptotic behaviors in extended time scales and large populations

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Abstract

This thesis focuses on the convergence analysis of stochastic nonlinear systems. Specifically, it delves into the study of their asymptotic behavior, an area of paramount importance for comprehending the long-term dynamics of these systems and the systems with large populations. We concentrate on two key aspects: firstly, the examination of the turnpike property within the context of stochastic control problems, and secondly, the investigation of the convergence of N-player games towards their corresponding mean field games (MFG).

Firstly, we investigate the asymptotic behavior of the systems with long-term dynamics and the convergence is with respect to the time horizon. In the first project, we examine the limiting behavior of a specific class of linear quadratic stochastic optimal control problems and their corresponding value functions as the time horizon approaches infinity. We establish the consistency between the cell problem in weak KAM theory and the static optimization problem from the perspective of the turnpike property. Moreover, we provide the connection between the cell problem and the ergodic cost problem, and then the classical turnpike property and the turnpike property in terms of the cost function are identified.

Different from the first project, next we examine the system complexity. More precisely, we consider the convergence behavior of systems with large populations. MFG has become widely accepted as an approximation for the N-player games, especially when the number of players is large enough. A fundamental question that arises in this context concerns the convergence rate of this approximation.

In the second project, we study the convergence rate of the N-player Linear-Quadratic-Gaussian (LQG) games with a Markov chain as the common noise towards its asymptotic MFG. By postulating a Markovian structure via two auxiliary processes for the first and second moments of the MFG equilibrium and applying the fixed point condition in MFG, we first provide the characterization of the equilibrium measure in MFG with a finite-dimensional Riccati system of ODEs. Additionally, with an explicit coupling of the optimal trajectory of the N-player game driven by N dimensional Brownian motion and MFG counterpart driven by one-dimensional Brownian motion, we obtain the convergence rate $O(N^{-1/2})$ with respect to 2-Wasserstein distance.

The number of states of the common noise considered in the above project is finite, thus it is natural to consider the case when the number of states of the common noise is infinity. In the third project, we focus on exploring the convergence properties of a generic player's trajectory

and empirical measures in an N-player LQG Nash game, where Brownian motion serves as the common noise. The study establishes three distinct convergence rates concerning the representative player and empirical measure. To investigate the convergence, our methodology relies on a specific decomposition of the equilibrium path in the N-player game and utilizes the associated MFG framework.

The basic structure of standard MFG theory assumes symmetry in the connections of the agents but not necessarily in their dynamics. However, asymmetric graph connections in large population games are considered in recent studies. In the network limit, a graphon gives the communication weights. In the last project, we consider the solvability of graphon mean field games. A new type of mean field games PDE system associated with the graphon mean field games is proposed. We establish the existence of solutions via the application of Schauder's fixed point theorem and obtain the uniqueness of solution via the Lasry-Lions monotonicity assumption on the running cost.

Contents

List of Notations

The following notations are used in the dissertation.

Common: for all $i = 1, 2, \cdots, N$,

$W, \tilde{W}, W_i^{(N)}, B$	Brownian motion
$\mathcal{F}, \tilde{\mathcal{F}}, \mathcal{F}^{(N)}$	σ-field
$\mathbb{F}, \tilde{\mathbb{F}}, \mathbb{F}^{(N)}$	Filtration
$\mathbb{P}, \mathbb{P}^{(N)}$	Probability measures
$\Omega, \Omega^{(N)}$	Sample space
$\mathcal{A}, \mathcal{A}_i^{(N)}$	Admissible set for controls
$L^p(\Omega, \mathbb{P})$	Space of random variable with finite p-th moment
$L^p_{\mathbb{F}}([0,T] \times \Omega)$	Space of $\mathbb{F}$-progressively measurable random processes which are integrable on $[0,T]$ in L^p sense
$\mathcal{P}_p(P)$	Collection of measures m on Polish space P having finite p-th moment, i.e., $\int_P \lvert x\rvert^p m(dx) < \infty$
$\mathbb{W}_p(\cdot, \cdot)$	p-Wasserstein metric
δ_x	Dirac measure on the point x
$[m]_k$	k-th moment of the measure m
$\rho(v)$	Empirical measure of the vector $v = (v_1, v_2, \dots, v_N)$ with Dirac measure δ, i.e., $\rho(v) = \frac{1}{N}\sum_{i=1}^N \delta_{v_i}$
$\mathbb{1}_N$	N-dimensional vector with all entries are 1
e_i	N-dimensional natural basis

Chapter 2:

X	Controlled process
u	Control
$\lvert\cdot\rvert$	The Euclidean norm of vectors in $\mathbb{R}^d$
$\lVert\cdot\rVert_2$	The spectral norm of matrices
$\mathbb{S}^{d\times d}$	The space of all $d\times d$ symmetric matrices
$A^\top$	The transpose matrix of A
$A\geq 0$ $(A>0)$	A is a positive semidefinite (definite) matrix
$\sqrt{A}$	The unique $B\geq 0$ satisfying $B^2=A$ when $A\geq 0$ is a positive semidefinite matrix
$\mathbf{O}_d$	The $d\times d$ matrix in which all the entries are 0
$\mathbf{0}_d$	The d-dimensional column vector with all entries are 0
$\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_d)$	The diagonal matrix with entries $\lambda_1,\lambda_2,\dots,\lambda_d$
$\mathcal{U}_{[0,T]}$	Admissible control set defined by $L^2_{\mathbb{F}}([0,T]\times\Omega)$
$\mathcal{U}$	Admissible control set such that the underlying process is square integrable for all $t\geq 0$, i.e., $\mathcal{U}:=\mathcal{U}_{[0,\infty)}$
$\mathcal{U}[\mu_\infty]$	Admissible control set such that for a distribution $\mu_\infty\in\mathcal{P}_2(\mathbb{R})$, the underlying process X satisfying $\lim_{t\to\infty}\mathbb{W}_2(\mathcal{L}(X(t)),\mu_\infty)=0$
$\langle\phi,\mu\rangle$	Inner product of function ϕ on $\mathbb{R}$ and measure $\mu\in\mathcal{P}_2(\mathbb{R})$
$\mathcal{N}(\mu,\nu)$	Normal distribution with mean μ and variance (or covariance matrix) ν

Chapter 3: for all $i=1,2,\cdots,N$,

Y	Continuous-time Markov chain
$\mathcal{Y}$	State space of Markov chain Y
Q	Generator of the continuous-time Markov chain
$\mathcal{L}(X\mid Z)$	Distribution of random variable X conditional on $\sigma(Z)$
$(P,\mathcal{B}(P),d)$	Polish space
$X, X_i^{(N)}$	Controlled processes
$\alpha,\alpha_i^{(N)}$	Control

Chapter 4: for all $i = 1, 2, \cdots, N$,

$X, X_i^{(N)}$	Controlled processes
$\alpha, \alpha_i^{(N)}$	Control
$C_b(\mathbb{R}^d)$	Collection of bounded and continuous functions on $\mathbb{R}^d$
$C_b^1(\mathbb{R}^d)$	Space of function on $\mathbb{R}^d$ whose first order derivative and itself are bounded and continuous
$\mathcal{B}(\mathbb{R}^d)$	Borel set on $\mathbb{R}^d$
f_*m	Pushforward measure of m by function f
$\hat{\alpha}, \hat{\gamma}, \beta, \alpha_i, \gamma_i$	Random variables

Chapter 5:

$\mathbb{T}^d$	d-dimensional torus space
g	Graphon, a symmetric measurable function from $[0,1]^2$ to $\mathbb{R}$
α, β	Nodes in graphon, $\alpha, \beta \in [0,1]$
$\mathbf{a}$	Control
$(S^{1/2}, \rho)$	$S^{1/2}$ is the collection of function $\mu : [0,T] \times [0,1] \mapsto \mathcal{P}_1(\mathbb{T}^d)$ such that $\lvert\mu\rvert_{1/2} = \lvert\mu\rvert_0 + [\mu]_{1/2} < \infty$, where $\lvert\mu\rvert_0 = \sup_{t,\alpha} \int_{\mathbb{T}^d} \lvert x\rvert \mu(t,\alpha,dx)$ and $[\mu]_{1/2} = \sup_{t_1 \neq t_2, \alpha} \frac{\mathbb{W}_1(\mu(t_1,\alpha),\mu(t_2,\alpha))}{\lvert t_1 - t_2\rvert^{1/2}}$, endowed with the metric $\rho(\mu_1,\mu_2) = \sup_{t,\alpha} \mathbb{W}_1(\mu_1(t,\alpha),\mu_2(t,\alpha))$
B_r	A closed convex and compact subset of $(S^{1/2}, \rho)$ defined by $\{\mu \in S^{1/2} : \lvert\mu\rvert_{1/2} \le r\}$

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Chapter 1

Introduction

1.1 Introduction and motivation

In the realm of scientific inquiry, understanding the behavior of complex systems has been a fundamental pursuit, one that transcends disciplinary boundaries and holds profound implications for diverse fields such as biology, physics, economics, and engineering. Within this rich tapestry of complexity, stochastic nonlinear systems occupy a central role, characterized by their intricate dynamics and inherent randomness. These systems, while challenging to decipher, provide valuable insights into the intricate dance of order and chaos that governs our world.

The thesis at hand embarks on a journey into the heart of this complexity, focusing on the convergence analysis of stochastic nonlinear systems. Specifically, it delves into the study of their asymptotic behavior, an area of paramount importance for comprehending the long-term dynamics of these systems and the systems with large populations. The asymptotic behavior not only holds significance from a theoretical standpoint but also bears immense importance in practical applications. Analyzing the asymptotic behavior of systems characterized by long-term dynamics and large populations is often a challenging work. This thesis, more specifically, concentrates on two key aspects: firstly, the examination of the turnpike property within the context of stochastic control problems, and secondly, the investigation of the convergence of N-player games towards their corresponding mean field games (MFG).

1.1.1 Turnpike properties of stochastic control problem

Firstly, we investigate the asymptotic behavior, i.e., turnpike properties, of the systems with long-term dynamics and the convergence is with respect to the time horizon.

The exploration of the turnpike property originated in the field of economics, serving as a tool to analyze the stationary behavior during transient periods in long-horizon control problems. Initially proposed by von Neumann [64], the terminology was later introduced by Dorfman, Samuelson, and Solow [22]. The turnpike property characterizes scenarios where the solution to an optimization problem concentrates on specific static points, evenly spaced along a defined path. Subsequently, the turnpike phenomenon has garnered significant attention in both finite and infinite-dimensional problems within deterministic discrete-time and continuous-time systems, see the book [15] as an excellent survey and a list of numerous references, including [62, 8, 57, 79, 70, 24, 81, 82, 83].

Notably, we highlight the works of [70, 71], delving into continuous-time linear quadratic (LQ) problems for ordinary differential equations (ODE), and the recent publication by Sun, Wang, and Yong [74], which addresses stochastic LQ optimal control problems. Additionally, Sun and Yong, as discussed in [75], have established the exponential, integral, and mean-square turnpike properties for optimal pairs in mean-field linear stochastic differential equations. This is contingent upon the satisfaction of the stabilizability condition for the state equation.

Chapter 2 is mainly based on the paper [49], in which we examine the limiting behavior of a specific class of linear quadratic stochastic optimal control problems and their corresponding value functions as the time horizon approaches infinity. This collaborative work involves Professor Sixian Jin from California State University San Marcos and Professor Qingshuo Song from Worcester Polytechnic Institute. Firstly, we establish the consistency between the cell problem in weak KAM theory and the static optimization problem from the perspective of the turnpike property. Secondly, we provide the connection between the cell problem and the ergodic cost problem, and then the classical turnpike property and the turnpike property in terms of the cost function are identified.

1.1.2 Convergence of mean field games with common noise

Different from the first project, next we examine the system complexity. More precisely, we consider the convergence behavior of systems with large populations.

Mean field games theory was introduced by Lasry and Lions in their seminal paper [55], and by Huang, Caines, and Malhame ([45, 42, 43, 41]). It aims to provide a framework for studying the asymptotic behavior of N-player differential games being invariant under the reshuffling of the players' indices. For a comprehensive overview of recent advancements and relevant applications of MFG theory, it is recommended to refer to the two-volume book by Carmona and Delarue ([16, 17]) published in 2018 and the references provided therein.

Mean field games have been widely accepted as an approximation for the N-player games, particularly when the number of players, N, is large enough. A fundamental question that arises in this context concerns the convergence rate of this approximation. Convergence can be analyzed from different perspectives, such as convergence in value, the trajectory followed by the representative player, or the behavior of the mean field term. Each of these perspectives offers valuable insights into the behavior and characteristics of the MFG approximation. Furthermore, they raise a variety of intriguing questions within this context.

In Chapter 3, we investigate the convergence rate of the N-player game, governed by a Markov chain common noise, towards its asymptotic MFG under the linear-quadratic-Gaussian structure. To achieve this, firstly, we introduce a Markovian structure using two auxiliary processes for the first and second moments of the MFG equilibrium and employ the fixed point condition in MFG. By doing so, we characterize the equilibrium measure in MFG with a finite-dimensional Riccati system of ODEs. Consequently, we obtain the equilibrium path, equilibrium control, and the value function in MFG. Subsequently, we address the N-player game under the LQG structure, and we characterize its equilibrium path, equilibrium control, and the value function through a Riccati system of ODEs with a dimension of $O(N^3)$. Leveraging the N-invariant algebraic structure of this system of ODEs, we establish a dimension reduction result, facilitating a comparison between the equilibrium path in the N-player game and the equilibrium path in the MFG. To demonstrate the convergence between the two equilibrium paths, we embed the equilibrium path in the N-player game to the probability space of the equilibrium path in the MFG using a distribution copy, leading to the achievement of the convergence result and the computation of the convergence rate. Lastly, some numerical examples are presented to demonstrate the convergence result. This chapter is based on the paper [50], which had been accepted by the journal *Nonlinear Analysis: Hybrid Systems* in December 2023. It is a joint work with Dr. Jiaxuan Ye, Ph.D. student Peiyao Lai and Professor Qingshuo Song from Worcester Polytechnic Institute.

Note that the number of states of the common noise considered in the Chapter 3 is finite, it is natural to consider the case when the number of states of the common noise is infinity. Thus, we consider the convergence rate of the mean field game with Brownian motion as its common noise in the next chapter.

Chapter 4 is mainly based on [51], in which we focus on a class of one-dimensional linear-quadratic-Gaussian mean field games with Brownian motion as the common noise. This collaborative work involves Dr. Jiaxuan Ye and Professor Qingshuo Song from Worcester Polytechnic Institute. It is worth noting that the equilibrium path, equilibrium control, and the value function in MFG and the N-player game can be obtained by a similar methodology in Chapter 3. Our main contribution is the establishment of three different convergence rates from the N-player games to the corresponding mean field games. Firstly, we establish that the convergence rate of the p-Wasserstein metric for the distribution of the representative player in the N-player game to the distribution of the generic player in MFG is $O(N^{-1/2})$ for $p \in [1, 2]$. Secondly, it demonstrates that the convergence rate of the p-Wasserstein metric for the empirical measure of the equilibrium path in the N-player game to the equilibrium measure in MFG under the L^p sense is $O(N^{-1/(2p)})$ for $p \in [1, 2]$. Lastly, we show that the convergence rate of the uniform p-Wasserstein metric for the empirical measure of the equilibrium path in the N-player game to the equilibrium measure in MFG under the L^p sense is $O(N^{-1/(2p)})$ for $p \in (1, 2]$, and $O(N^{-1/2} \ln(N))$ for $p = 1$. To investigate these convergence rates, our methodology relies on a specific decomposition of the equilibrium path in the N-player game and utilizes the associated MFG framework.

1.1.3 Graphon mean field games

The basic structure of standard MFG theory assumes a symmetry in the connections of the agents but not necessarily of their dynamics. However, in the recent studies [10, 11, 12], asymmetric graph connections in large population games are considered. Large subpopulations (or clusters) of agents are placed at their particular nodes and communicate with the neighbouring subpopulations via the graph edges. The graphs are heterogeneous with the edges having not necessarily identical weights. In the network limit, a graphon gives the communication weights $g(\alpha, \beta)$, see for instance the introductions to each of [10, 11, 12, 34] for the graphon MFG (GMFG) framework and [58] for graphon theory. Therefore, it is interesting to investigate the large population games with asymmetric graph connections.

Along with [10, 11, 12], in Chapter 5, we consider the solvability of a type of graphon mean field games. A new type of mean field games PDE system associated with the graphon mean field games system is proposed in this project. The graphon mean field games system consists of a collection of parameterized Hamilton-Jacobi-Bellman (HJB) equations and a collection of parameterized Fokker-Planck-Kolmogorov equations. We establish the existence of solutions via the application of Schauder's fixed point theorem and obtain the uniqueness via the Lasry-Lions monotonicity assumption on the running cost. The main difficulty is to obtain the regularity of the solution and the sensitivity of the corresponding HJB equations and Fokker-Planck-Kolmogorov equations. This chapter is mainly based on the paper [9], which is conducted in collaboration with Professor Peter E. Caines from McGill University, Professor Daniel Ho from City University of Hong Kong, Professor Minyi Huang from Carleton University, and Professor Qingshuo Song from Worcester Polytechnic Institute. It had been accepted for publication in the journal *ESAIM: Control, Optimisation and Calculus of Variations* in March 2022.

Chapter 2

Long-time behavior of stochastic LQ control problem

2.1 Introduction

In this chapter, we consider a $\mathbb{R}^d$-valued standard Brownian motion $\{W(t)\}_{t\geq 0}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions. Let $T > 0$ be a given time horizon, $|\cdot|$ be the Euclidean norm of vectors in $\mathbb{R}^d$, and $\|\cdot\|_2$ be the spectral norm of matrices. We use $L^p_{\mathbb{F}}(\Omega \times [0,T])$ to denote the space of all $\mathbb{F}$-progressively measurable random processes $u = \{u(t)\}_{t\in[0,T]}$ satisfying $\mathbb{E}[\int_0^T |u(t)|^p dt] < \infty$. Consider the following diffusion process given by a linear stochastic differential equation (SDE)

$$dX(t) = (AX(t) + u(t) + b)\, dt + \sigma dW(t), \quad X(0) = x, \quad t \geq 0, \tag{2.1.1}$$

where $A \in \mathbb{S}^{d\times d}$ is a $d \times d$ symmetric constant matrix, $b, x \in \mathbb{R}^d$ are constant vectors, and $\sigma \in \mathbb{R}^+$ is a positive constant.

The classical stochastic LQ control problem over the finite time horizon $[0, T]$ is to find an optimal control u^*_T from the space $\mathcal{U}_{[0,T]} := L^2_{\mathbb{F}}(\Omega \times [0,T])$ such that the quadratic cost functional

$$J_T(x; u_T) = \mathbb{E}\left[\int_0^T L(X(t), u_T(t))dt\right] \tag{2.1.2}$$

is minimized for a given initial state $x \in \mathbb{R}^d$, i.e.,

$$V_T(x) := J_T(x; u^*_T) = \inf_{u_T\in\mathcal{U}_{[0,T]}} J_T(x; u_T), \tag{2.1.3}$$

where

$$L(x, u) = \frac{1}{2}\left(x^\top Qx + |u|^2\right) + q^\top x + r^\top u$$

with Q be a positive definite matrix in $\mathbb{S}^{d\times d}$ and $q, r \in \mathbb{R}^d$. The corresponding optimal path is denoted by $X^*_T(t)$ for $t \in [0, T]$. Note that, we could set Q be in $\mathbb{R}^{d\times d}$ in general.

The turnpike property of the above finite time control problem is associated with the following static optimization problem: Determine the point $(\hat{x}, \hat{u})$ to

$$\begin{cases} \text{minimize} \quad F(x,u) := \frac{1}{2}\left(x^\top Q x + |u|^2 + 2q^\top x + 2r^\top u\right) + \frac{1}{2}\sigma^2 \text{trace}(P), \\ \text{subject to} \quad Ax + u + b = \mathbf{0}_d, \end{cases} \tag{2.1.4}$$

where $\mathbf{0}_d$ is the d-dimensional column vector with all entries are 0 and P is a positive definite solution to

$$P^2 - 2AP - Q = \mathbf{O}_d.$$

In the above equation, $\mathbf{O}_d$ is the $d \times d$ matrix in which all the entries are 0. If the underlying control problem is deterministic, i.e., $\sigma = 0$, as it is shown in [70], the turnpike property refers to the following estimation: There exist some $\lambda > 0$ and $K > 0$ independent of t and T such that

$$|X_T^*(t) - \hat{x}| + |u_T^*(t) - \hat{u}| \le K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T].$$

Indeed, this turnpike property reveals that, for a sufficiently large T, one can achieve a good approximation of the optimal trajectory during the majority time period $[\delta T, (1-\delta)T]$ for some $0 < \delta << 1/2$ by simply staying at the stable point $\hat{x}$ of the static optimization problem in the sense

$$|X_T^*(t) - \hat{x}| + |u_T^*(t) - \hat{u}| \le 2Ke^{-\lambda\delta T}, \quad \forall t \in [\delta T, (1-\delta)T], \ \delta \in (0, 1/2).$$

However, extending this turnpike property to the stochastic control problem with $\sigma > 0$ poses challenges. This is because the presence of Brownian noise makes it impossible for any control to freeze the state unchanged at a fixed point. Recently, Sun and Yong (see Theorem 3.2 of [75]) proved the following (stochastic version) turnpike property

$$\mathbb{E}\left[|X_T^*(t) - X^*(t)|^2 + |u_T^*(t) - u^*(t)|^2\right] \le K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T] \tag{2.1.5}$$

by constructing two stochastic processes X^* and u^*, independent to T, satisfying $\mathbb{E}[X^*(t)] = \hat{x}$, $\mathbb{E}[u^*(t)] = \hat{u}$.

In this study, we aim to reexamine the turnpike property, approaching it from a distinct perspective: the cell problem within the framework of weak Kolmogorov–Arnold–Moser (KAM) theory. Specifically, the objective of the cell problem is to seek the solution $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ to the following equation

$$c_0 = H\left(x, -\nabla v(x), -D^2 v(x)\right), \tag{2.1.6}$$

where the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ is given by

$$H(x, \bar{p}, \bar{q}) := \sup_{u \in \mathbb{R}^d}\left\{(Ax + u + b)^\top \bar{p} + \frac{1}{2}\sigma^2 \text{trace}(\bar{q}) - L(x,u)\right\}.$$

Note that uniqueness does not apply to the cell problem, as (v, c_0) is a solution if and only if $(v + k, c_0)$ is also a solution for any $k \in \mathbb{R}$. Therefore, when we refer to the uniqueness of the cell problem, we specifically mean uniqueness in the value of c_0.

Initially, the weak KAM Theory was developed by Fathi [28] and Mather [61], and is linked to the theory of homogenization for Hamilton-Jacobi (HJ) equations developed by Lions, Papanicolaou, and Varadhan in [56]. It provides the connection between a type of control problem and the cell problem, offering a representation of the optimal ergodic cost. In addition to its fundamental role in the theory of homogenization, the weak KAM theory has also been used to study the long-time behavior of dynamic control problems, including the ergodic behavior of the value function and the corresponding HJ equation in the deterministic case (see [63, 7, 26, 27, 77]), as well as the Hamilton-Jacobi-Bellman (HJB) equation in the stochastic case (see [4, 47, 20]).

Compared to the above literature, this Chapter provides a distinct approach to show the turnpike properties in stochastic control theory by using the cell problem in PDE and contributions can be summarized as follows. Our first contribution lies in the formulation of a verification theorem connecting the cell problem to a specific class of infinite time horizon control problems, referred to as the probabilistic cell problem, see Lemma 5. Unlike the typical cell problem explored in the existing literature (e.g., [77]), the underlying cell problem in our context lacks uniqueness due to the non-compactness of the domain. It is the verification theorem, which establishes a tailored sufficient condition to distinguish the right solution to the probabilistic cell problem from multiple solutions of the cell problem. An immediate consequence of the verification theorem is the establishment of a link between the cell problem and the static optimization problem.

Our second contribution provides the connection between the cell problem (2.1.6) and the ergodic cost problem (refer to Remark 3.6.7 in [3]), which involves determining the constant

$$-c_* := \lim_{T\to\infty} \frac{1}{T} V_T(x) = \lim_{T\to\infty} \frac{1}{T} J_T(x; u_T^*). \tag{2.1.7}$$

The importance of this connection is that it unveils a new turnpike property in terms of the cost function in addition to the aforementioned turnpike property of (2.1.5) with respect to the control process and state process:

$$\lim_{T\to\infty} \frac{1}{T} J_T(x; u^*) = -c_*, \tag{2.1.8}$$

where u^* is a T-independent control process obtained from the probabilistic cell problem, see Theorem 9.

The five problems, namely, the Finite Time Stochastic Control Problem (2.1.3), the Static Optimization Problem (2.1.4), the Cell Problem (2.1.6), the Ergodic Cost Problem (2.1.7), and the Probabilistic Cell Problem introduced in this Section 2.1 will be interwoven throughout the remainder of this manuscript in the following manner: In Section 2.2, we prove the verification of the cell problem and further provide the consistency of the cell problem and the static optimization problem. Additionally, we identify the turnpike properties of (2.1.5) and (2.1.8) in Section 2.3. To further elucidate the results obtained in Section 2.3, we provide an illustrative example in Section 2.4. Proofs of certain lemmas are collected in Appendix 2.5.

Throughout this chapter, we use $\mathbb{S}^{d\times d}$ to denote the space of all $d \times d$ symmetric matrices, and $A \geq 0$ ($A > 0$) to denote a positive semidefinite (definite) matrix. If $A \geq 0$, then $\sqrt{A}$ denotes the unique $B \geq 0$ satisfying $B^2 = A$. We also use $\mathbf{O}_d$ denote the $d \times d$ matrix in which all the entries are 0 and $\mathbf{0}_d$ to be the d-dimensional column vector with all entries are 0. We also pose the following assumptions to coefficients:

(Cf) $A \in \mathbb{S}^{d\times d}$ and $Q > 0$ are $d \times d$ symmetric constant matrices, $b, x, q, r \in \mathbb{R}^d$ are constant vectors, and $\sigma \in \mathbb{R}^+$ is a positive constant.

2.2 Cell problem and the verification theorem

We commence our exploration with the solvability of the cell problem (2.1.6) within the framework of weak KAM theory. In prevailing literature, the solution of the cell problem is unique up to the constant. Interestingly, this uniqueness fails in our framework attributed to the absence of compactness in the domain. This necessitates the formulation of a meticulously crafted verification theorem: How can we discern the appropriate solution from the multitude available to accurately characterize the associated optimal control problem?

2.2.1 Existence and nonuniqueness of the cell problem

We recall the cell problem: Find the solution $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ to the following equation

$$c_0 = H\left(x, -\nabla v(x), -D^2 v(x)\right),$$

where the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ is given by

$$H(x, \bar{p}, \bar{q}) := \sup_{u\in\mathbb{R}^d} \left\{ (Ax + u + b)^\top \bar{p} + \frac{1}{2}\sigma^2 \mathrm{trace}(\bar{q}) - L(x, u) \right\} := \sup_{u\in\mathbb{R}^d} H^u(x, \bar{p}, \bar{q}). \tag{2.2.1}$$

Lemma 1. *The cell problem (2.1.6) can be solved by (v, c_0) in the form of*

$$v(x) = \frac{1}{2}x^\top Z_1 x + x^\top Z_2 + Z_3$$

and

$$-c_0 = \frac{1}{2}\sigma^2 \mathit{trace}(Z_1) + b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2, \tag{2.2.2}$$

where $(Z_1, Z_2) \in \mathbb{S}^{d\times d} \times \mathbb{R}^d$ can be any solution pair to the system of equations

$$\begin{cases} Z_1^2 - 2AZ_1 - Q = \mathbf{O}_d, \\ (Z_1 - A)\, Z_2 - Z_1 b + Z_1 r - q = \mathbf{0}_d, \end{cases} \tag{2.2.3}$$

and $Z_3 \in \mathbb{R}$ is an arbitrary constant.

Proof. Assume (v, c_0) solves the cell problem (2.1.6) with v satisfying a quadratic form $v(x) = \frac{1}{2}x^\top Z_1 x + x^\top Z_2 + Z_3$. Note that Z_1 is symmetric, then $\nabla v(x) = Z_1 x + Z_2$ and $D^2 v(x) = Z_1$, and thus the cell problem (2.1.6) can be rewritten by $H(x, -(Z_1 x + Z_2), -Z_1) = c_0$. We first observe that

$$H(x,\bar{p},\bar{q}) = (Ax + \bar{u}(x,\bar{p}) + b)^\top \bar{p} + \frac{1}{2}\sigma^2 \text{trace}(\bar{q}) - L(x, \bar{u}(x,\bar{p})),$$

where

$$\bar{u}(x,\bar{p}) = \bar{p} - r,$$

thus the cell problem can be reduced to

$$\begin{aligned} c_0 = &- (Ax - Z_1 x + b - r - Z_2)^\top (Z_1 x + Z_2) - \frac{1}{2}\sigma^2 \text{trace}(Z_1) - \frac{1}{2} x^\top Q x \\ &- \frac{1}{2} |Z_1 x + Z_2 + r|^2 - q^\top x + r^\top (Z_1 x + Z_2 + r) \end{aligned} \tag{2.2.4}$$

for all $x \in \mathbb{R}^d$. By setting the linear and quadratic terms with respect to x on the right hand side of (2.2.4) to 0, we obtain the system of equations (2.2.3). Combining like terms for the constants in (2.2.4) provides the expression c_0 of (2.2.2). □

The uniqueness of c_0 in the cell problem is established by Theorem 4.2 in [77] for the periodic domain. However, it is noteworthy that the uniqueness does not extend to our setting defined by equation (2.1.6). Recall that, by spectral theorem, if $A > 0$, it admits orthogonal decomposition $A = QDQ^\top$ for some orthogonal matrix Q and diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ with $\lambda_i > 0$. Moreover, any matrix in the form of $B = Q\text{diag}(\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_d)Q^\top$ satisfies $B^2 = A$. In this below, $\sqrt{A}$ is the unique choice of B satisfying $B > 0$ and $B^2 = A$. Also, $\sqrt{A}$ can be represented by $QD^{1/2}Q^\top$.

Lemma 2. *There exists multiple solution pairs (Z_1, Z_2) of (2.2.3) in $\mathbb{S}^{d\times d} \times \mathbb{R}^d$, hence the c_0 of the cell problem (2.1.6) is not unique. Moreover, there exists unique solution pair (Z_1, Z_2) satisfying*

$$D_1 = Z_1 - A > 0.$$

Moreover, Z_1 and D_1 are commutative, hence they share the same eigenvector matrix.

Proof. Taking transpose to the first equation of (2.2.3), we have $Z_1^2 - 2Z_1 A - Q = \mathbf{O}_d$ as Z_1, A and Q are symmetric matrices, which implies that Z_1 and A are commutative, i.e., $Z_1 A = AZ_1$. Thus, $(Z_1 - A)^2 = A^2 + Q$ holds. By spectral theorem, since $A^2 + Q > 0$, there exists multiple solutions to Z_1 and the unique choice of Z_1 to have $D_1 = Z_1 - A > 0$ is

$$Z_1 = A + \sqrt{A^2 + Q}.$$

Accordingly, from the second equation of (2.2.3), Z_2 can be written in terms of Z_1 as

$$Z_2 = (Z_1 - A)^{-1}(Z_1 b - Z_1 r + q).$$

At the end, $D_1 Z_1 = Z_1 D_1 = Z_1^2 - AZ_1$ holds by commutativity of A and Z_1, thus they have the same eigenvector matrix by Page 305 of [73].

□

2.2.2 Verification theorem to probabilistic cell problem

The lack of uniqueness in the determination of c_0 as indicated by Lemma 2 introduces a compelling challenge when striving to identify the optimal value for c_0 in order to substantiate the verification theorem associated with its probabilistic counterparts. In the following discussion, our objective is to delineate the connection between the cell problem (2.1.6) and the probabilistic cell problem (2.2.5)-(2.2.6), which is referred to the verification procedure in the context of the control theory.

2.2.2.1 Probabilistic cell problem

We consider the following probabilistic cell problem. Let $\mathcal{P}_2(\mathbb{R}^d)$ be the Wasserstein space of probability measures μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$ endowed with 2-Wasserstein metric $\mathcal{W}_2(\cdot,\cdot)$ defined by

$$\mathcal{W}_2(\mu_1,\mu_2) = \inf_{\pi\in\Pi(\mu_1,\mu_2)} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^2 d\pi(x,y) \right)^{\frac{1}{2}},$$

where $\Pi(\mu_1,\mu_2)$ is the collection of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with its marginals agreeing with μ_1 and μ_2. Moreover, we denote that $\langle \phi, \mu\rangle$ to be

$$\langle \phi, \mu\rangle = \int_{\mathbb{R}^d} \phi(x)\mu(dx)$$

for all function ϕ valued on $\mathbb{R}^d$ and all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. To proceed, we define $\mathcal{U}$ as the collection of all $\mathbb{F}$ progressively measurable processes such that

- its associated state process $X^u = X$ given by

  $$dX(t) = (AX(t) + u(t) + b)\,dt + \sigma dW(t), \quad X(0) = x, \quad t \geq 0,$$

  is well-defined;

- $\mathbb{E}[|X(t)|^2] < \infty$ for all $t > 0$ and $x \in \mathbb{R}^d$;

- the law of $X(t)$ converges to some distribution $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d)$ in 2-Wasserstein metric $\mathcal{W}_2$, i.e.,

  $$\lim_{t\to\infty} \mathcal{W}_2(Law(X(t)), \mu_\infty) = 0.$$

For convenience, we also use $\mathcal{U}[\mu_\infty]$ for a $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d)$ to denote the collection of $u \in \mathcal{U}$ such that $Law(X^u(t))$ converges to μ_∞.

We define the probabilistic cell problem below: Determine (V, c) such that

$$-c = \inf_{u\in\mathcal{U}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left[L(X(t), u(t))\right] dt, \tag{2.2.5}$$

and

$$V(x)=\inf_{u\in\mathcal{U}[\mu_\infty^*]}\limsup_{T\to\infty}\int_0^T\mathbb{E}\left[L(X(t),u(t))+c\right]dt, \tag{2.2.6}$$

where μ_∞^* is the distribution limit of the optimal path X^* in the ergodic control problem (2.2.5).

Following the convention in stochastic control theory, we denote $V(x)$ as the value function of the probabilistic cell problem, provided it is well-defined. However, unlike the standard control problem, the objective of the probabilistic cell problem is not solely to identify a value function V; rather, it involves determining a pair (V,c) among many solutions of the cell problem (2.1.6).

2.2.2.2 Verification for a general setting

In this part, we will prove a verification for an infinite-time control problem under a general settings and all notations are independent to the rest of this chapter.

The cell problem in the general setting is to find the pair $(v,c_0)\in C^2(\mathbb{R}^d)\times\mathbb{R}$ such that

$$c_0=H\left(x,-\nabla v(x),-D^2v(x)\right):=\sup_a H^a\left(x,-\nabla v(x),-D^2v(x)\right), \tag{2.2.7}$$

where $H^a:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\mapsto\mathbb{R}$ for any $a\in\mathbb{R}^d$ is given by

$$H^a(x,\bar{p},\bar{q})=\hat{b}(x,a)\cdot\bar{p}-\hat{L}(x,a)+\frac{1}{2}\hat{\sigma}^2\mathrm{trace}(\bar{q})$$

for some $\hat{b}:\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}^d$, $\hat{\sigma}\in\mathbb{R}$, and $\hat{L}:\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}$.

Assumption 3. *$\hat{b}$ is Lipschitz continuous on $\mathbb{R}^d\times\mathbb{R}^d$, and $\hat{L}$ is locally Lipschitz continuous and satisfies quadratic growth, i.e., for all $(x,a)\in\mathbb{R}^d\times\mathbb{R}^d$,*

$$\left|\hat{L}(x,a)\right|\leq K\left(1+|x|^2+|a|^2\right)\ \text{for some } K\in\mathbb{R}.$$

At the same time, we define its associated probabilistic cell problem: Consider $\mathbb{R}^d$-valued controlled process X, with a given $\mathbb{R}^d$-Brownian motion $W(t)$, given by

$$dX(t)=\hat{b}\left(X(t),u(t)\right)dt+\hat{\sigma}dW(t),\quad X(0)=x\in\mathbb{R}^d,\quad t\geq 0.$$

The objective is to determine (V,c) such that

$$-c=\inf_{u\in\mathcal{U}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\mathbb{E}\left[\hat{L}(X(t),u(t))\right]dt, \tag{2.2.8}$$

and

$$V(x)=\inf_{u\in\mathcal{U}[\mu_\infty^*]}\limsup_{T\to\infty}\int_0^T\mathbb{E}\left[\hat{L}(X(t),u(t))+c\right]dt, \tag{2.2.9}$$

where μ_∞^* is the distribution limit of the optimal path X^* in the ergodic control problem (2.2.8).

Lemma 4. *For an arbitrary control process $u \in \mathcal{U}[\mu_\infty]$ and any function $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ with a quadratic growth $|\phi(x)| \leq K(1+|x|^2)$ for all $x \in \mathbb{R}^d$, we have*

$$\lim_{t\to\infty} \mathbb{E}\left[\phi(X(t))\right] = \langle \phi, \mu_\infty \rangle.$$

Proof. Since $\mu_t := Law(X(t))$ converges to some $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d)$ in 2-Wasserstein distance, we have

$$\lim_{t\to\infty} \mathbb{E}[|X(t)|^2] = \int_{\mathbb{R}^d} |x|^2 \mu_\infty(dx).$$

By the Skorohod representation theorem, one can find another stochastic process $Y(t)$ in a different probability space, such that $Y(t) \to Y_\infty$ almost surely, as well as, $Law(Y(t)) = \mu_t$ and $Law(Y_\infty) = \mu_\infty$. Hence, by the fact of

$$\phi(Y(t)) \leq K\left(1+|Y(t)|^2\right), \quad \text{and} \quad \mathbb{E}\left[K\left(1+|Y(t)|^2\right)\right] \to \mathbb{E}\left[K\left(1+|Y_\infty|^2\right)\right],$$

one can apply the dominated convergence theorem to $Y(t)$ and obtain

$$\lim_{t\to\infty} \mathbb{E}\left[\phi(X(t))\right] = \lim_{t\to\infty} \mathbb{E}\left[\phi(Y(t))\right] = \mathbb{E}\left[\lim_{t\to\infty} \phi(Y(t))\right] = \langle \phi, \mu_\infty \rangle.$$

□

Lemma 5. *Suppose Assumption 3 holds. Let $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ solves the cell problem (2.2.7). In addition, we assume*

1. *∇v is Lipschitz continuous;*
2. *There exists a unique maximizer of $H^a(x, \bar{p}, \bar{q})$ in the form of*

$$\bar{u}(x, \bar{p}) = \arg\max_a H^a(x, \bar{p}, \bar{q});$$

3. *The distribution of the process $X^*(t)$ controlled by $u^*(t) = \bar{u}(X(t), -\nabla v(X(t)))$ converges to some $\mu^*_\infty \in \mathcal{P}_2(\mathbb{R}^d)$ in 2-Wasserstein distance.*

Then, the pair

$$(V := v - \langle v, \mu^*_\infty \rangle, c_0)$$

solves the probabilistic cell problem (2.2.8)-(2.2.9).

Proof. Since $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ solves the cell problem (2.2.7), for a control $u \in \mathcal{U}$ and its associated state process $X^u = X$, by Itô's formula, we obtain

$$\begin{aligned} v(X(t)) = v(x) + \int_0^t \left(\hat{b}(X(s), u(s)) \cdot \nabla v(X(s)) + \frac{1}{2}\hat{\sigma}^2 \Delta v(X(s)) \right) ds \\ + \int_0^t \hat{\sigma} \nabla v(X(s)) \cdot dW(s). \end{aligned}$$

Fixing $t > 0$ and taking expectation on both sides and note that

$$\mathbb{E}\left[\int_0^t \hat{\sigma}^2 \left|\nabla v(X(s))\right|^2 ds\right] \le \hat{\sigma}^2 K^2 \mathbb{E}\left[\int_0^t \left(1 + |X(s)|^2\right) ds\right] \le \hat{\sigma}^2 K^2 \left(t + \mathbb{E}\left[\int_0^t |X(s)|^2 ds\right]\right)$$

is finite, we have

$$\mathbb{E}\left[v(X(t))\right] = v(x) + \mathbb{E}\left[\int_0^t \left(\hat{b}(X(s), u(s)) \cdot \nabla v(X(s)) + \frac{1}{2}\hat{\sigma}^2 \Delta v(X(s))\right) ds\right].$$

The cell problem (2.2.7) implies that, for all $x, a \in \mathbb{R}^d$,

$$-\hat{b}(x, a) \cdot \nabla v(x) - \hat{L}(x, a) - \frac{1}{2}\hat{\sigma}^2 \Delta v(x) - c_0 \le 0,$$

hence

$$\hat{b}(X(s), u(s)) \cdot \nabla v(X(s)) + \frac{1}{2}\hat{\sigma}^2 \Delta v(X(s)) \ge -\hat{L}(X(s), u(s)) - c_0$$

for all $s \in [0, t]$. Thus

$$v(x) \le \mathbb{E}\left[v(X(t))\right] + \mathbb{E}\left[\int_0^t \left(\hat{L}(X(s), u(s)) + c_0\right) ds\right]. \tag{2.2.10}$$

The inequality (2.2.10) holds for all $u \in \mathcal{U}$ and equality holds if $u = u^*$.

Moreover, due to Lipschitz continuity of ∇v, the value function v satisfies a quadratic growth condition, i.e., $|v(x)| \le K(1 + |x|^2)$ for all $x \in \mathbb{R}^d$. By Lemma 4, we have

$$\lim_{t\to\infty} \mathbb{E}\left[v(X(t))\right] = \langle v, \mu_\infty \rangle,$$

where μ_∞ is the distribution limit of X_t.

Hence, by taking $\limsup_{t\to\infty} \frac{1}{t}$ on both sides of (2.2.10), we have

$$-c_0 \le \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\left[\hat{L}(X(t), u(t))\right] dt.$$

The above inequality holds for all $u \in \mathcal{U}$ and equality holds if $u = u^*$. Thus, we conclude the first identity of the probabilistic cell problem (2.2.8)-(2.2.9).

Moreover, by taking $\limsup_{t\to\infty}$ on both sides of (2.2.10), we have

$$v(x) \le \langle v, \mu_\infty \rangle + \limsup_{t\to\infty} \int_0^t \mathbb{E}\left[\hat{L}(X(s), u(s)) + c_0\right] ds$$

since $\mathbb{E}[|X(t)|^2] < \infty$ for all $t > 0$ and $\hat{L}(x, a)$ is quadratic growth with respect to x and a from Assumption 3. The above inequality holds for arbitrary $u \in \mathcal{U}[\mu_\infty^*]$, and an equality holds for u^*, i.e.,

$$v(x) = \langle v, \mu_\infty^* \rangle + \limsup_{t\to\infty} \int_0^t \mathbb{E}\left[\hat{L}(X^*(s), u^*(s)) + c_0\right] ds.$$

Therefore, we conclude that

$$v(x) - \langle v, \mu_\infty^* \rangle = \inf_{u \in \mathcal{U}[\mu_\infty^*]} \limsup_{t \to \infty} \int_0^t \mathbb{E}\left[\hat{L}(X(s), u(s)) + c_0\right] ds.$$

□

2.2.2.3 Verification for LQG setting

In the following, we provide a complete characterization of the probabilistic cell problem (2.2.5)-(2.2.6) in the LQ setting by the help of cell problem given by Lemma 1.

Theorem 6. *Let $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ be the solution to the cell problem (2.1.6) in the form of*

$$v(x) = \frac{1}{2} x^\top Z_1 x + x^\top Z_2$$

associated to the

$$Z_1 = A + \sqrt{A^2 + Q}, \ Z_2 = (Z_1 - A)^{-1}(Z_1 b - Z_1 r + q). \tag{2.2.11}$$

Then, the pair

$$(V := v - \langle v, \mu_\infty^* \rangle, c_0)$$

with

$$\mu_\infty^* = \mathcal{N}\left(D_1^{-1} D_2, \frac{1}{2}\sigma^2 D_1^{-1}\right), \ D_1 = Z_1 - A, \ D_2 = b - r - Z_2 \tag{2.2.12}$$

solves the probabilistic cell problem (2.2.5)-(2.2.6). Moreover, the optimal path X^* of the probabilistic cell problem (2.2.5)-(2.2.6) is an OU process given by

$$dX^*(t) = (-D_1 X^*(t) + D_2)\, dt + \sigma dW(t), \quad X^*(0) = x, \tag{2.2.13}$$

whose distribution converges to μ_∞^* in 2-Wasserstein metric and the optimal control process u^* admits a feedback form of

$$u^*(t) = \bar{u}\left(X^*(t), -\nabla v\left(X^*(t)\right)\right) = -Z_1 X^*(t) - Z_2 - r, \tag{2.2.14}$$

where

$$\bar{u}(x, \bar{p}) = \bar{p} - r.$$

Proof. From Lemma 1, we know that the cell problem (2.1.6) admits solutions (v, c_0), where v is in the form of

$$v(x) = \frac{1}{2} x^\top Z_1 x + x^\top Z_2 + Z_3.$$

Note that $v - \langle v, \mu_\infty^* \rangle$ is independent to Z_3, hence it's enough to show the results with $Z_3 = 0$. By Lemma 2, there exists unique choice of (Z_1, Z_2) of (2.2.3) such that D_1 is positive definite.

Therefore, it's enough to check all of the assumptions in Lemma 5. First of all, v is a quadratic function and thus its first order derivative ∇v is Lipschitz continuous. Moreover, by the first order condition, the maximizer of H^a of (2.2.1) uniquely exists in the form of (2.2.14).

For the third assumption, we shall check the convergence of the process associated to the optimal control. The explicit solution to (2.2.13) can be written by

$$X^*(t) = \Phi_t x + \int_0^t \Phi_{t-s} ds D_2 + \sigma \int_0^t \Phi_{t-s} dW(s),$$

where Φ_t is a $d \times d$ fundamental matrix satisfying $\Phi_0 = I_d$ and the homogeneous matrix ODE

$$d\Phi_t = -D_1 \Phi_t dt.$$

It is clear that X^* is an Ornstein-Uhlenbeck (OU) process and

$$X^*(t) \sim \mathcal{N}(m_t, \nu_t),$$

where

$$m_t = \Phi_t x + \int_0^t \Phi_{t-s} ds D_2,\ \nu_t = \sigma^2 \int_0^t \Phi_{t-s}^2 ds$$

By Lemma 2, there exists unique choice of the positive definite D_1, and one can write its orthogonal diagonalization $D_1 = \tilde{Q}\Lambda\tilde{Q}^\top$, where $\tilde{Q}$ is an orthogonal matrix and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ is a diagonal matrix with all $\lambda_i > 0$. This implies that D_1 has its inverse in the form of $D_1^{-1} = \tilde{Q}\Lambda^{-1}\tilde{Q}^\top$ and Φ_t can be factored into $\Phi_t = e^{-D_1 t} = \tilde{Q}e^{-\Lambda t}\tilde{Q}^\top$. Moreover, we have

$$\int_0^t \Phi_{t-s} ds = (I_d - \Phi_t) D_1^{-1},\ \int_0^t \Phi_{t-s}^2 ds = \frac{1}{2}(I_d - \Phi_t^2) D_1^{-1}.$$

Therefore, the mean and variance of X_t^* can be rewritten as

$$m_t = \Phi_t x + (I_d - \Phi_t) D_1^{-1} D_2,\ \nu_t = \frac{1}{2}\sigma^2 (I_d - \Phi_t^2) D_1^{-1}.$$

In addition, due to the positive definiteness D_1, $\Phi_t \to 0$ as $t \to \infty$ and there exist

$$m_\infty = D_1^{-1} D_2 \in \mathbb{R}^d \text{ and } \nu_\infty = \frac{1}{2}\sigma^2 D_1^{-1} \in \mathbb{S}^{d\times d}$$

such that m_t converges to m_∞ and ν_t converges to ν_∞. Hence, we obtain the desired result that $X^*(t) \xrightarrow{d} \bar{X}$ as $t \to \infty$, where

$$\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty) := \mu_\infty^* \tag{2.2.15}$$

is a normal random variable. The proof can be concluded from Lemma 5. □

2.2.3 Consistency with the static optimization problem

The static optimization problem (2.1.4) plays an important role in the study of the turnpike properties of the control problem in the deterministic type [70, 79] and the stochastic type [74]. In this subsection, we provide a result to show the equality between the constant $-c_0$ from the cell problem (2.1.6) and the value $F(\hat{x}, \hat{u})$ from the static optimization problem (2.1.4).

Recall that $L(x,u) = \frac{1}{2}(x^\top Qx + |u|^2 + 2q^\top x + 2r^\top u)$ and $F(x,u) = L(x,u) + \frac{1}{2}\sigma^2\text{trace}(P)$, where P is a positive definite solution to $P^2 - 2AP - Q = \mathbf{O}_d$. The solution $(\hat{x}, \hat{u})$ to the static optimization problem (2.1.4) is the one that solves

$$\begin{cases} \text{Minimize} \quad L(x,u), \\ \text{subject to} \quad u = -(Ax+b), \end{cases}$$

since $\frac{1}{2}\sigma^2\text{trace}(P)$ is a constant and it is independent with (x,u).

Lemma 7. *The optimal static value to the static optimization problem (2.1.4) is*

$$F(\hat{x},\hat{u}) = -\frac{1}{2}(Ab+q-Ar)^\top \left(Q+A^2\right)^{-1}(Ab+q-Ar) + \frac{1}{2}b^\top b - r^\top b + \frac{1}{2}\sigma^2 \mathit{trace}(P), \quad (2.2.16)$$

where P is a positive definite solution to

$$P^2 - 2AP - Q = \mathbf{O}_d,$$

and the optimal solution $(\hat{x},\hat{u})$ is given by

$$\begin{aligned} \hat{x} &= -\left(Q+A^2\right)^{-1}(Ab+q-Ar), \\ \hat{u} &= A\left(Q+A^2\right)^{-1}(Ab+q-Ar) - b. \end{aligned}$$

Proof. Plugging $u = -(Ax+b)$, we know that $L(x,u)$ is a quadratic function with respect to x and it is given by

$$L(x,u) = \frac{1}{2}x^\top \left(Q+A^2\right)x + (Ab+q-Ar)^\top x + \frac{1}{2}b^\top b - r^\top b.$$

It is straightforward to obtain the desired result by minimizing the quadratic function under the condition that $Q + A^2 > 0$. $\square$

Now, we are ready to show the consistency between $-c_0$ and $F(\hat{x},\hat{u})$.

Lemma 8. *The constant $-c_0$ of (2.1.6) is identical to the value $F(\hat{x},\hat{u})$ of the static optimization problem (2.1.4) .*

Proof. It's equivalent to verify the equality between the representation of in (2.2.2) and the optimal static value in (2.2.16). Choosing Z_1 be the solution to $Z_1^2 - 2AZ_1 - Q = \mathbf{O}_d$ such that $D_1 = Z_1 - A$ is positive definite, i.e., $Z_1 = A + \sqrt{A^2+Q}$, we obtain that $Z_1 = P$ and $Z_2 = (Z_1-A)^{-1}(Z_1 b + q - Z_1 r)$. To verify that $-c_0 = F(\hat{x},\hat{u})$, we only need to check that

$$b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2 = -\frac{1}{2}(Ab+q-Ar)^\top \left(Q+A^2\right)^{-1}(Ab+q-Ar) + \frac{1}{2}b^\top b - r^\top b.$$

By calculation and applying the representation of Z_2, we have

$$
\begin{aligned}
& b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2 \\
= & -\frac{1}{2}\left((Z_1 - A)^{-1}(Z_1 b + q - Z_1 r) + r\right)^\top \left((Z_1 - A)^{-1}(Z_1 b + q - Z_1 r) + r\right) \\
& \qquad + b^\top (Z_1 - A)^{-1}(Z_1 b + q - Z_1 r) \\
= & -\frac{1}{2}(Z_1 b + q - Z_1 r)^\top \left((Z_1 - A)^{-1}\right)^2 (Z_1 b + q - Z_1 r) - \frac{1}{2} r^\top r \\
& \qquad + (Z_1 b + q - Z_1 r)^\top \left((Z_1 - A)^{-1}\right)^2 (Z_1 b - Ab - Z_1 r + Ar) \\
= & \frac{1}{2}(Z_1 b + q - Z_1 r)^\top \left((Z_1 - A)^{-1}\right)^2 (Z_1 b - Z_1 r - q - 2Ab + 2Ar) - \frac{1}{2} r^\top r.
\end{aligned}
$$

Note that $Z_1 - A = \sqrt{Q + A^2}$, then $((Z_1 - A)^{-1})^2 = (Q + A^2)^{-1}$. Substituting Z_1 by $A + \sqrt{Q + A^2}$, we could obtain

$$
\begin{aligned}
& b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2 \\
= & \frac{1}{2}(Z_1 b + q - Z_1 r)^\top \left(Q + A^2\right)^{-1} (Z_1 b - Z_1 r - q - 2Ab + 2Ar) - \frac{1}{2} r^\top r \\
= & \frac{1}{2}\left(\left(A + \sqrt{Q + A^2}\right)(b - r) + q\right)^\top \left(Q + A^2\right)^{-1} \left(\left(A + \sqrt{Q + A^2}\right)(b - r) - q - 2Ab + 2Ar\right) - \frac{1}{2} r^\top r \\
= & -\frac{1}{2}(Ab + q - Ar)^\top \left(Q + A^2\right)^{-1} (Ab + q - Ar) + \frac{1}{2}(b - r)^\top (b - r) - \frac{1}{2} r^\top r \\
= & -\frac{1}{2}(Ab + q - Ar)^\top \left(Q + A^2\right)^{-1} (Ab + q - Ar) + \frac{1}{2} b^\top b - r^\top b,
\end{aligned}
$$

which yields the desired result. □

2.3 Turnpike property

In this section, we uncover the turnpike property applicable to the optimal trajectory X^* and optimal control u^*, both stemming from the probabilistic cell problem as outlined in (2.2.5)-(2.2.6). Moreover, distinct from the aforementioned turnpike property elucidated by (2.1.5), we also establish the turnpike behavior concerning the cost function:

$$
\lim_{T \to \infty} \frac{1}{T} J_T(x; u^*) = -c_*,
$$

where $-c_* := \lim_{T \to \infty} \frac{1}{T} J_T(x; u_T^*)$ is the ergodic cost defined in (2.1.7). In other words, achieving a near optimality in terms of the average cost over an extended period doesn't require calculating the optimal control u_T^* for every T. Instead, one only needs to compute the optimal control u^* for the probabilistic cell problem.

To accomplish this objective, it becomes imperative to establish a relation between the cost function $J_T(x; u^*)$ with the u^* from the probabilistic cell problem (2.2.5)-(2.2.6), and the corresponding function $V_T(x) = J_T(x, u_T^*)$ originating from the finite time stochastic control problem

(2.1.3).

2.3.1 Main results on turnpike properties

To distinguish the finite time control problem (2.1.3) from the probabilistic cell problem (2.2.5)-(2.2.6), we denote by $\{X_T(t)\}_{0\leq t\leq T}$ as the underlying process controlled by finite time control $\{u_T(t)\}_{0\leq t\leq T}$ in the finite time stochastic control problem, while denote by $\{X(t)\}_{t\geq 0}$ as the underlying process controlled by infinite time control $\{u(t)\}_{t\geq 0}$ in the probabilistic cell problem. In other words, $\{X_T(t)\}_{0\leq t\leq T}$ follows

$$dX_T(t) = (AX_T(t) + u_T(t) + b)\,dt + \sigma dW(t), \quad t \in [0,T]$$

and $\{X(t)\}_{t\geq 0}$ follows

$$dX(t) = (AX(t) + u(t) + b)\,dt + \sigma dW(t), \quad t \geq 0.$$

We also extend the cost functional for the finite time control problem (2.1.3) to an initial condition $X_T(t) = x$, that is

$$J_T(t,x;u_T) := \mathbb{E}\left[\int_t^T L(X_T(s),u_T(s))ds \,\middle|\, X_T(t) = x\right],$$

where $u_T \in \mathcal{U}_{[t,T]}$ is an admissible control between t and T and the value function is

$$V_T(t,x) := J_T\left(t,x;u_T^*\right) = \inf_{u_T\in\mathcal{U}_{[t,T]}} J_T(t,x;u_T)$$

with its optimal control and optimal path denoted by u_T^* and X_T^*.

We recall that the value function $V(x)$ of the probabilistic cell problem is defined in (2.2.5)-(2.2.6) as

$$-c_0 = \inf_{u\in\mathcal{U}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\left[L(X(t),u(t))\right]dt$$

and

$$V(x) := \inf_{u\in\mathcal{U}[\mu_\infty^*]} \limsup_{T\to\infty} \int_0^T \mathbb{E}\left[L\left(X_t,u_t\right) + c_0\right]dt,$$

where μ_∞^* is the distribution limit of the optimal path X^* in the ergodic control problem (2.2.5). The optimal control and optimal path of the probabilistic cell problem is denoted by u^* and X^* respectively, which are characterized by Theorem 6. Note that, the constant c in the above definition is replaced by c_0 of the cell problem corresponding to Z_1 such that D_1 is positive definite according to Theorem 6. Next, we present our main results.

Theorem 9. *Let $J_T(0,x;u^*)$ be the cost functional evaluated along the optimal control of the* *probabilistic cell problem u^* on the finite time horizon $[0,T]$,*

$$J_T(0,x;u^*) := \mathbb{E}\left[\int_0^T L\left(X_T(s),u^*(s)\right)ds\right].$$

For all $x \in \mathbb{R}^d$, the following estimation holds:

$$0 \le J_T(0,x;u^*) - V_T(0,x) = O(1).$$

Moreover, the constant c_0 of the cell problem (2.1.6) is the ergodic cost defined via (2.1.7), i.e.,

$$c_0 = \lim_{T\to\infty}\frac{1}{T}V_T(0,x), \quad \forall x \in \mathbb{R}^d.$$

From the results of Theorem 9, we could establish the turnpike behavior in terms of the average cost function straightforwardly. Note that $V_T(0,x) \le J_T(0,x;u^*) = V_T(0,x) + O(1)$, it follows that

$$\lim_{T\to\infty}\frac{1}{T}J_T(0,x;u^*) = \lim_{T\to\infty}\frac{1}{T}V_T(0,x) = -c_*. \tag{2.3.1}$$

Theorem 10. *For all $x \in \mathbb{R}^d$, there exist some $\lambda > 0$ and $K > 0$ independent of t and T such that*

$$\mathbb{E}\left[|X_T^*(t) - X^*(t)|^2 + |u_T^*(t) - u^*(t)|^2\right] \le K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T].$$

2.3.2 Proofs

The proofs of these two theorems rely on the analytical expressions of the value functions for both the finite time control problem and probabilistic cell problem, and some comparison results between solutions to the Riccati system of ODEs. We first present some preliminary lemmas with their proof posted in Appendix 2.5. The first lemma gives an analytical expression for the value function of the finite time control problem.

Lemma 11. *The value function $V_T(t,x)$ of the finite time control problem has the following form*

$$V_T(t,x) = \frac{1}{2}x^\top \tilde{Z}_1(t)x + x^\top \tilde{Z}_2(t) + \tilde{Z}_3(t),$$

where $\{\tilde{Z}_1(t), \tilde{Z}_2(t), \tilde{Z}_3(t) : t \in [0,T]\}$ solves the Riccati system of ODEs

$$\begin{cases} \dot{\tilde{Z}}_1(t) + 2A\tilde{Z}_1(t) - \tilde{Z}_1^2(t) + Q = \mathbf{O}_d; \\ \dot{\tilde{Z}}_2(t) + A\tilde{Z}_2(t) - \tilde{Z}_1(t)\tilde{Z}_2(t) + \tilde{Z}_1(t)(b-r) + q = \mathbf{0}_d; \\ \dot{\tilde{Z}}_3(t) + \frac{1}{2}\sigma^2 trace(\tilde{Z}_1(t)) + b^\top \tilde{Z}_2(t) - \frac{1}{2}|\tilde{Z}_2(t) + r|^2 = 0, \end{cases} \tag{2.3.2}$$

with terminal conditions $\tilde{Z}_1(T) = \mathbf{O}_d$, $\tilde{Z}_2(T) = \mathbf{0}_d$, $\tilde{Z}_3(T) = 0$. Moreover, the optimal feedback *control of the finite time control problem is given by*

$$u_T^*(t) = -\tilde{Z}_1(t)X_T^*(t) - \tilde{Z}_2(t) - r.$$

Next lemma gives a similar analytical structure for $J_T(t,x;u^*)$, which is the cost functional of the finite time control problem evaluated along the optimal control u^* from the probabilistic cell problem. Recall that (Z_1, Z_2) is the solution to the system of algebraic equations (2.2.3) in which Z_1 is uniquely chosen such that D_1 to be positive definite, see Lemma 2.

Lemma 12. *The cost functional $J_T(t,x;u^*)$ of the finite time control problem evaluated along the optimal control u^* from the probabilistic cell problem has the form*

$$J_T(t,x;u^*) = \frac{1}{2}x^\top f_1(t)x + x^\top f_2(t) + f_3(t),$$

where $\{f_1(t), f_2(t), f_3(t) : t \in [0,T]\}$ solves the Riccati system of ODEs

$$\begin{cases} \dot{f}_1(t) - 2(Z_1 - A)f_1(t) + Q + Z_1^2 = \mathbf{O}_d; \\ \dot{f}_2(t) - (Z_1 - A)f_2(t) + f_1(t)(b - r - Z_2) + Z_1Z_2 + q = \mathbf{0}_d; \\ \dot{f}_3(t) + \frac{1}{2}\sigma^2 trace(f_1(t)) + (b - r - Z_2)^\top f_2(t) + \frac{1}{2}(|Z_2|^2 - |r|^2) = 0, \end{cases} \tag{2.3.3}$$

with terminal conditions $f_1(T) = \mathbf{O}_d$, $f_2(T) = \mathbf{0}_d$, $f_3(T) = 0$.

From the above lemmas, we can observe that

$$J_T(0,x;u^*) - V_T(0,x) = \frac{1}{2}x^\top\Gamma_1(0)x + x^\top\Gamma_2(0) + \Gamma_3(0), \tag{2.3.4}$$

where, for $i = 1,2,3$,

$$\Gamma_i(t) = f_i(t) - \tilde{Z}_i(t), \ i = 1,2,3.$$

If we introduce γ_i by

$$\gamma_i(t) = Z_i - \tilde{Z}_i(t), \ i = 1,2,$$

$\{\Gamma_i : i = 1,2,3\}$ satisfies the following system of ODE

$$\begin{cases} \dot{\Gamma}_1(t) - 2(Z_1 - A)\Gamma_1(t) + \gamma_1^2(t) = \mathbf{O}_d; \\ \dot{\Gamma}_2(t) - (Z_1 - A)\Gamma_2(t) + \Gamma_1(t)(b - r - Z_2) + \gamma_1(t)\gamma_2(t) = \mathbf{0}_d; \\ \dot{\Gamma}_3(t) + \frac{1}{2}\sigma^2\mathrm{trace}(\Gamma_1(t)) + (b - r - Z_2)^\top\Gamma_2(t) + \frac{1}{2}|\gamma_2(t)|^2 = 0, \end{cases} \tag{2.3.5}$$

with the terminal conditions $\Gamma_1(T) = \mathbf{O}_d, \Gamma_2(T) = \mathbf{0}_d, \Gamma_3(T) = 0$. Moreover, $\{\gamma_1, \gamma_2\}$ is the solution to the system of ODEs

$$\begin{cases}\dot{\gamma}_1(t) - 2(Z_1 - A)\gamma_1(t) + \gamma_1^2(t) = \mathbf{O}_d;\\ \dot{\gamma}_2(t) - (Z_1 - A)\gamma_2(t) + \gamma_1(t)\gamma_2(t) + \gamma_1(t)(b - r - Z_2) = \mathbf{0}_d,\end{cases} \tag{2.3.6}$$

with the terminal conditions $\gamma_1(T) = Z_1$ and $\gamma_2(T) = Z_2$.

It is enough to give a proper estimations for the γ_1, γ_2 in (2.3.6) and $\Gamma_1, \Gamma_2, \Gamma_3$ in (2.3.5) to obtain the bound for the difference between $J_T(0, x; u^*)$ and $V_T(0, x)$.

To proceed, we recall that

$$D_1 = Z_1 - A > 0, \quad D_2 = b - r - Z_2.$$

Lemma 13. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ be eigenvalues of D_1 in a descending order. Then, there exists a unique solution $\gamma_1, \gamma_2 \in C^1([0, T])$ of the system of ODEs (2.3.6) satisfying, for some constant $k > 0$,*

$$\|\gamma_1(t)\|_2 \leq ke^{-2\lambda_d(T-t)}, \ |\gamma_2(t)| \leq ke^{-\lambda_d(T-t)}, \quad \forall t \in [0, T]. \tag{2.3.7}$$

Moreover, $\gamma_1(t)$ is positive definite for all $t \in [0, T]$ and γ_1 is an increasing function with respect to Loewner order on $[0, T]$.

Proof. ODE satisfied by γ_1 can be rewritten by

$$\dot{\gamma}_1(t) - 2D_1\gamma_1(t) + \gamma_1^2(t) = \mathbf{O}_d, \quad \gamma_1(T) = Z_1.$$

Let $\tau = T - t$ and denote $\bar{\gamma}_1(\tau) = \gamma_1(t)$, $\bar{\gamma}_1$ is the solution to the ODE

$$\dot{\bar{\gamma}}_1(t) = -2D_1\bar{\gamma}_1(t) + \bar{\gamma}_1^2(t), \quad \bar{\gamma}_1(0) = Z_1.$$

Recall that $D_1 = \tilde{Q}\Lambda\tilde{Q}^\top$, where $\tilde{Q}$ is an orthogonal matrix and Λ is a diagonal matrix with descending diagonal entries λ_i. Since Z_1 and $\tilde{Z}_1(t)$ are symmetric matrices, $\bar{\gamma}_1(t)$ is also symmetric as their difference. Taking transpose to the equation satisfied by $\bar{\gamma}_1$, we have $\dot{\bar{\gamma}}_1(t) = -2\bar{\gamma}_1(t)D_1 + \bar{\gamma}_1^2(t)$, which implies that D_1 and $\bar{\gamma}_1(t)$ are commute, i.e., $D_1\bar{\gamma}_1(t) = \bar{\gamma}_1(t)D_1$. Thus, $\bar{\gamma}_1$ and D_1 share the same eigenvector matrix $\tilde{Q}$ by the results in Page 305 of [73].

Hence, we can write $\bar{\gamma}_1(t) = \tilde{Q}\Sigma(t)\tilde{Q}^\top$ for some $\Sigma(t) = \mathrm{diag}(\sigma_1(t), \sigma_2(t), \ldots, \sigma_d(t))$ for $t \geq 0$. It follows that

$$\dot{\Sigma}(t) = -2\Lambda\Sigma(t) + \Sigma^2(t), \quad \forall t > 0 \tag{2.3.8}$$

with initial condition $\bar{\gamma}_1(0) = \tilde{Q}\Sigma(0)\tilde{Q}^\top = Z_1$. It is equivalent to write

$$\dot{\sigma}_i(t) = -2\lambda_i\sigma_i(t) + \sigma_i^2(t), \quad t > 0, \ i = 1, 2, \ldots, d.$$

Note that $2D_1 - Z_1 = Z_1 - 2A = \sqrt{A^2 + Q} - A > 0$ and $2D_1 - Z_1 = \tilde{Q}(2\Lambda - \Sigma(0))\tilde{Q}^\top$, thus $2\Lambda - \Sigma(0) > 0$, i.e., $\sigma_i(0) < 2\lambda_i$ for $i = 1, 2, \ldots, d$. Therefore, there exists unique solution $\{\sigma_i : i =$

$1, 2, \ldots, d\}$ in the form of

$$0 < \sigma_i(t) = \frac{2\lambda_i}{1 + \left(\frac{2\lambda_i}{\sigma_i(0)} - 1\right) e^{2\lambda_i t}} < \max_{i=1,2,\ldots,d} \left(\frac{2\lambda_i \sigma_i(0)}{2\lambda_i - \sigma_i(0)}\right) e^{-2\lambda_i t} := a_1 e^{-2\lambda_i t}, \quad \forall t \geq 0.$$

Clearly, σ_i is strictly decreasing on $[0, \infty)$, which implies that $\bar{\gamma}_1$ is a strictly decreasing function on $[0, \infty)$ with respect to Loewner order. Moreover, since $\lambda_i > 0$ for all $i = 1, 2, \ldots, d$, $\bar{\gamma}_1(t)$ is positive definite for all $t \geq 0$. Thus, we have

$$\|\bar{\gamma}_1(t)\|_2 = \|\Sigma(t)\|_2 \leq a_1 e^{-2\lambda_d t}, \quad \forall t \geq 0, \tag{2.3.9}$$

which yields the desired result that

$$\|\gamma_1(t)\|_2 \leq a_1 e^{-2\lambda_d (T-t)}, \quad t \in [0, T]. \tag{2.3.10}$$

Moreover, $\gamma_1(t)$ is positive definite for all $t \in [0, T]$ and γ_1 is an increasing function with respect to Loewner order on $[0, T]$.

By (2.3.6), γ_2 satisfies the ODE

$$\dot{\gamma}_2(t) + (\gamma_1(t) - D_1)\gamma_2(t) + \gamma_1(t) D_2 = \mathbf{0}_d, \quad \gamma_2(T) = Z_2.$$

Let $\tau = T - t$ and denote $\bar{\gamma}_2(\tau) = \gamma_2(t)$, $\bar{\gamma}_2$ is the solution to the ODE

$$\dot{\bar{\gamma}}_2(t) = (\bar{\gamma}_1(t) - D_1)\bar{\gamma}_2(t) + \bar{\gamma}_1(t) D_2, \quad \bar{\gamma}_2(0) = Z_2.$$

Denote that $A_1(t) = \bar{\gamma}_1(t) - D_1$ for $t \geq 0$, the explicit form of $\bar{\gamma}_2$ is given by

$$\bar{\gamma}_2(t) = e^{\int_0^t A_1(s)ds} Z_2 + \int_0^t e^{\int_s^t A_1(r)dr} \bar{\gamma}_1(s) D_2 ds \tag{2.3.11}$$

for all $t \geq 0$, which implies that $\gamma_2 \in C^1([0, T])$.

To proceed, we first observe from (2.3.9) that

$$\left\| \int_0^t \Sigma(s) ds \right\|_2 \leq \int_0^t \|\Sigma(s)\|_2 \, ds \leq \frac{a_1}{2\lambda_d}(1 - e^{-2\lambda_d t}) \leq \frac{a_1}{2\lambda_d}.$$

From the fact that $\|e^D\|_2 = e^{\|D\|_2}$ for any $D \geq 0$, it implies that

$$\left\| e^{\int_0^t \Sigma(s)ds} \right\|_2 \leq a_2 := \exp\left\{\frac{a_1}{2\lambda_d}\right\}.$$

Hence, we have

$$\left\| e^{\int_0^t A_1(s)ds} \right\|_2 = \left\| e^{\int_0^t (\Sigma(s) - \Lambda)ds} \right\|_2 \leq \left\| e^{\int_0^t \Sigma(s)ds} \right\|_2 \left\| e^{-\Lambda t} \right\|_2 \leq a_2 e^{-\lambda_d t}. \tag{2.3.12}$$

Therefore, the estimate of (2.3.11) is

$$
\begin{aligned}
|\bar{\gamma}_2(t)| &\leq \left\| e^{\int_0^t A_1(s)ds} \right\|_2 \ |Z_2| + \int_0^t \left\| e^{\int_s^t A_1(r)dr} \right\|_2 \|\bar{\gamma}_1(s)\|_2 \ |D_2| ds \\
&\leq a_2 e^{-\lambda_d t}|Z_2| + \int_0^t a_2 e^{-\lambda_d(t-s)} a_1 e^{-2\lambda_d s} ds |D_2| \\
&\leq \left(a_2|Z_2| + \frac{a_1 a_2 |D_2|}{\lambda_d} \right) e^{-\lambda_d t} := a_3 e^{-\lambda_d t}
\end{aligned}
$$

for all $t \geq 0$. Thus, we obtain that

$$
|\gamma_2(t)| \leq a_3 e^{-\lambda_d(T-t)}, \quad \forall t \in [0, T]. \tag{2.3.13}
$$

□

Lemma 14. *With eigenvalues of D_1 denoted by Lemma 13, The system of ODEs (2.3.5) has a unique solution Γ_1, Γ_2 and $\Gamma_3 \in C^1([0,T])$, and satisfies the following properties for some constant $k > 0$:*

$$
\|\Gamma_1(t)\|_2 \leq k e^{-2\lambda_d(T-t)}, \quad |\Gamma_2(t)| \leq k e^{-\lambda_d(T-t)}, \quad |\Gamma_3(t)| \leq k \tag{2.3.14}
$$

for all $t \in [0, T]$.

Proof. We recall that, Γ_1 satisfies the ODE

$$
\dot{\Gamma}_1(t) - 2D_1\Gamma_1(t) + \gamma_1^2(t) = \mathbf{O}_d, \quad \Gamma_1(T) = \mathbf{O}_d,
$$

which can be written in terms of γ_1 in the form of

$$
\Gamma_1(t) = \int_t^T e^{2D_1(t-s)}\gamma_1^2(s)ds, \quad \forall t \in [0, T].
$$

From (2.3.10), the estimation of Γ_1 is

$$
\|\Gamma_1(t)\|_2 \leq \int_t^T e^{2\lambda_d(t-s)}\|\gamma_1(s)\|_2^2 ds \leq \frac{a_1^2}{2\lambda_d} e^{-2\lambda_d(T-t)} \tag{2.3.15}
$$

for all $t \in [0, T]$. Moreover, from the explicit form of Γ_1, it is clear that $\Gamma_1(t)$ is positive semi-definite for all $t \in [0, T]$. Similarly, the ODE for Γ_2 is

$$
\dot{\Gamma}_2(t) - D_1\Gamma_2(t) + \Gamma_1(t)D_2 + \gamma_1(t)\gamma_2(t) = \mathbf{0}_d, \quad \Gamma_2(T) = \mathbf{0}_d,
$$

which yields an expression in terms of γ_1, γ_2 and Γ_1:

$$
\Gamma_2(t) = \int_t^T e^{D_1(t-s)} \left(\Gamma_1(s)D_2 + \gamma_1(s)\gamma_2(s)\right) ds, \quad \forall t \in [0, T].
$$

By the estimation (2.3.10), (2.3.13), and (2.3.15), for all $t \in [0,T]$,

$$\begin{aligned} |\Gamma_2(t)| &\le \int_t^T \left\| e^{D_1(t-s)} \right\|_2 (\|\Gamma_1(s)\|_2 |D_2| + \|\gamma_1(s)\|_2 |\gamma_2(s)|)\, ds \\ &\le \int_t^T e^{\lambda_d(t-s)} \left(\frac{a_1^2}{2\lambda_d} e^{-2\lambda_d(T-s)} |D_2| + a_1 a_3 e^{-3\lambda_d(T-s)} \right) ds \\ &\le \frac{a_1^2|D_2| + a_1 a_3 \lambda_d}{2\lambda_d^2} e^{-\lambda_d(T-t)} := a_4 e^{-\lambda_d(T-t)}. \end{aligned} \tag{2.3.16}$$

Next, the term Γ_3 satisfies the ODE

$$\dot{\Gamma}_3(t) + \frac{1}{2}\sigma^2 \mathrm{trace}(\Gamma_1(t)) + D_2^\top \Gamma_2(t) + \frac{1}{2}|\gamma_2(t)|^2 = 0, \quad \Gamma_3(T) = 0,$$

which can be rewritten with the above γ_2, Γ_1 and Γ_2:

$$\Gamma_3(t) = \int_t^T \left(\frac{1}{2}\sigma^2 \mathrm{trace}(\Gamma_1(s)) + D_2^\top \Gamma_2(s) + \frac{1}{2}|\gamma_2(s)|^2 \right) ds, \quad \forall t \in [0,T].$$

Thus, we obtain the estimation for Γ_3 with the help of (2.3.13), (2.3.15), and (2.3.16)

$$|\Gamma_3(t)| \le \int_t^T \left(\frac{1}{2} d\sigma^2 \|\Gamma_1(s)\|_2 + |D_2||\Gamma_2(s)| + \frac{1}{2}|\gamma_2(s)|^2 \right) ds \le a_5 \tag{2.3.17}$$

for some constant $a_5 > 0$. $\square$

Now, we are well prepared to prove Theorem 9.

Proof of Theorem 9. From (2.3.14) in Lemma 14, we have

$$\begin{aligned} |J_T(0,x;u^*) - V_T(0,x)| &= \left| \frac{1}{2} x^\top \Gamma_1(0) x + x^\top \Gamma_2(0) + \Gamma_3(0) \right| \\ &\le \frac{1}{2} k e^{-2\lambda_d T} |x|^2 + k e^{-\lambda_d T} |x| + k. \end{aligned}$$

Hence, for all $x \in \mathbb{R}^d$, we obtain that $J_T(0,x;u^*) - V_T(0,x) = O(1)$. Since u_T^* is the optimal control of the finite time control problem, by the definition of $V_T(t,x)$ and $J_T(t,x;u)$, we have $0 \le J_T(0,x;u^*) - V_T(0,x)$ for all $x \in \mathbb{R}^d$. Therefore, we obtain the desired result.

Next, the estimations of γ_1 and γ_2 in (2.3.7) can help us to verify that the identity $V_T(0,x) + c_0 T = o(T)$ holds, where c_0 is given by (2.2.2) and can also be obtained from the solution to the cell problem. Note that $V_T(0,x) = \frac{1}{2} x^\top \tilde{Z}_1(0) x + x^\top \tilde{Z}_2(0) + \tilde{Z}_3(0)$, where $\{\tilde{Z}_1(t), \tilde{Z}_2(t), \tilde{Z}_3(t) : t \in [0,T]\}$ is the solution to the system of Riccati equation (2.3.2). Then, we only need to show that for all $x \in \mathbb{R}^d$

$$\lim_{T\to\infty} \frac{1}{T} \left| \frac{1}{2} x^\top \tilde{Z}_1(0) x + x^\top \tilde{Z}_2(0) + \tilde{Z}_3(0) + c_0 T \right| = 0.$$

From (2.3.7) in Lemma 13, we have the following inequalities

$$\left\|\tilde{Z}_1(0)\right\|_2 = \|Z_1 - \gamma_1(0)\|_2 \le \|Z_1\|_2 + \|\gamma_1(0)\|_2 \le \|Z_1\|_2 + ke^{-2\lambda_d T}$$

and

$$\left|\tilde{Z}_2(0)\right| = |Z_2 - \gamma_2(0)| \le |Z_2| + |\gamma_2(0)| \le |Z_2| + ke^{-\lambda_d T},$$

which implies that

$$\lim_{T\to\infty} \frac{1}{T}\left|\frac{1}{2}x^\top \tilde{Z}_1(0)x + x^\top \tilde{Z}_2(0)\right| = 0, \quad \forall x \in \mathbb{R}^d.$$

Moreover, from the ODE satisfied by $Z_3(t)$ in (2.3.2),

$$\begin{aligned}\tilde{Z}_3(0) &= \int_0^T \left(\frac{1}{2}\sigma^2 \text{trace}\left(\tilde{Z}_1(s)\right) + b^\top \tilde{Z}_2(s) - \frac{1}{2}\left|\tilde{Z}_2(s) + r\right|^2\right) ds \\ &= \int_0^T \left(-\frac{1}{2}\sigma^2 \text{trace}\left(\gamma_1(s)\right) - D_2^\top \gamma_2(s) - \frac{1}{2}\left|\gamma_2(s)\right|^2\right) ds - c_0 T,\end{aligned}$$

where $c_0 = -\frac{1}{2}\sigma^2\text{trace}(Z_1) - b^\top Z_2 + \frac{1}{2}|Z_2 + r|^2$. Then, it follows that

$$\lim_{T\to\infty}\frac{1}{T}\left|\tilde{Z}_3(0) + c_0 T\right| = \lim_{T\to\infty}\frac{1}{T}\left|\int_0^T \left(-\frac{1}{2}\sigma^2 \text{trace}\left(\gamma_1(s)\right) - D_2^\top \gamma_2(s) - \frac{1}{2}\left|\gamma_2(s)\right|^2\right) ds\right|.$$

Using the triangle inequality and the estimations (2.3.7), we have

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\left|\int_0^T \left(-\frac{1}{2}\sigma^2 \text{trace}\left(\gamma_1(s)\right) - D_2^\top \gamma_2(s) - \frac{1}{2}\left|\gamma_2(s)\right|^2\right) ds\right| \\ \le\ &\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\frac{1}{2}d\sigma^2\|\gamma_1(s)\|_2 + |D_2||\gamma_2(s)| + \frac{1}{2}|\gamma_2(s)|^2\right) ds \\ \le\ &\lim_{T\to\infty}\frac{1}{T}\frac{dk\sigma^2 + 4|D_2|k + k^2}{4\lambda_d} = 0,\end{aligned}$$

which implies that $\lim_{T\to\infty}\frac{1}{T}|\tilde{Z}_3(0) + c_0 T| = 0$. Therefore, we obtain the desired result that

$$\lim_{T\to\infty}\frac{1}{T}|V_T(0,x) + c_0 T| = 0, \quad \forall x \in \mathbb{R}^d,$$

i.e., $V_T(0,x) + c_0 T = o(T)$ for all $x \in \mathbb{R}^d$ with c_0 given by (2.2.2). The identical result between c_0 and c_* yields from Definition (2.1.7). $\square$

Recall Theorem 6: From (2.2.14) and (2.2.13), the optimal control u^* of the probabilistic cell problem is

$$u^*(t) = -Z_1 X^*(t) - Z_2 - r,$$

and the optimal path X^* of the probabilistic cell problem is given by

$$dX^*(t) = -D_1 X^*(t)dt + D_2 dt + \sigma dW(t),$$

with $X^*(0) = x$, where D_1 and D_2 are given by (2.2.12). Moreover, the optimal path $X^*(t)$ in the probabilistic cell problem (2.2.5)-(2.2.6) converges in distribution to a normal random variable $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ as $t \to \infty$, where $m_\infty = D_1^{-1} D_2$ and $\nu_\infty = \frac{1}{2}\sigma^2 D_1^{-1}$. Next, we prove the classical turnpike property that is descried in Theorem 10.

Proof of Theorem 10. Here we assume the optimal path X^* in (2.2.13) of the probabilistic cell problem has the initial point $X^*(0) = \bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$, a normal random variable independent to the Brownian motion W, instead of a real-valued vector $x \in \mathbb{R}^d$. To calculate $\mathbb{E}[|X_T^*(t) - X^*(t)|^2]$, we first observe that the optimal control of the finite time control problem is given by

$$u_T^*(t) = -\tilde{Z}_1(t) X_T^*(t) - \tilde{Z}_2(t) - r$$

from the results in Lemma 11. Thus the optimal path for the finite time control problem satisfies

$$dX_T^*(t) = \left(A - \tilde{Z}_1(t)\right) X_T^*(t)dt + \left(b - r - \tilde{Z}_2(t)\right) dt + \sigma dW(t). \tag{2.3.18}$$

Denote that $\delta_T(t) = X_T^*(t) - X^*(t)$, by (2.2.13) and (2.3.18), we have

$$d\delta_T(t) = \left(A - \tilde{Z}_1(t)\right) \delta_T(t)dt + \gamma_1(t) X^*(t)dt + \gamma_2(t)dt$$

with the initial value $\delta_T(0) = X_T^*(0) - X^*(0) = x - \bar{X} \sim \mathcal{N}(x - m_\infty, \nu_\infty)$.

Let $\bar{A}_1(t) = A - \tilde{Z}_1(t) = \gamma_1(t) - D_1$ for all $t \in [0, T]$, then we have $\bar{A}_1(t) = A_1(T - t)$. By the similar method as the estimation in (2.3.12), for $0 \le s \le t \le T$, we have

$$\left\| e^{\int_0^t \bar{A}_1(r)dr} \right\|_2 \le a_2 e^{-\lambda_d t} \text{ and } \left\| e^{\int_s^t \bar{A}_1(r)dr} \right\|_2 \le a_2 e^{-\lambda_d (t-s)}.$$

Applying the integrating factor method, we obtain the explicit form of $\delta_T(t)$ as

$$\delta_T(t) = e^{\int_0^t \bar{A}_1(r)dr} \delta_T(0) + \int_0^t e^{\int_s^t \bar{A}_1(r)dr} \gamma_1(s) X^*(s) ds + \int_0^t e^{\int_s^t \bar{A}_1(r)dr} \gamma_2(s) ds.$$

Therefore, we have the following estimation:

$$\begin{aligned} \mathbb{E}\left[|\delta_T(t)|^2\right] \le 3 \left\| e^{\int_0^t \bar{A}_1(r)dr} \right\|_2^2 \mathbb{E}\left[|\delta_T(0)|^2\right] + 3\mathbb{E}\left[\left| \int_0^t e^{\int_s^t \bar{A}_1(r)dr} \gamma_1(s) X^*(s) ds \right|^2\right] \\ + 3 \left| \int_0^t e^{\int_s^t \bar{A}_1(r)dr} \gamma_2(s) ds \right|^2. \end{aligned}$$

Firstly, by calculation,

$$3 \left\| e^{\int_0^t \bar{A}_1(r)dr} \right\|_2^2 \mathbb{E}\left[|\delta_T(0)|^2\right] \le 3a_2^2 \mathbb{E}\left[\left|x - \bar{X}\right|^2\right] e^{-2\lambda_d t} \le K_1 e^{-2\lambda_d t}$$

for some constant $K_1 \ge 3a_2^2 \mathbb{E}\left[|x - \bar{X}|^2\right]$. Next, using the estimation for γ_2 in (2.3.7) from Lemma

13 and by Hölder's inequality, we get

$$
\begin{aligned}
3\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_2(s)ds\right|^2 &\leq 3\int_0^t \left\|e^{\int_s^t \bar{A}_1(r)dr}\right\|_2^2 ds \int_0^t |\gamma_2(s)|^2 ds \\
\leq 3a_2^2k^2 \int_0^t e^{-2\lambda_d(t-s)}ds \int_0^t e^{-2\lambda_d(T-s)}ds &\leq K_2 e^{-2\lambda_d(T-t)}
\end{aligned}
$$

for some constant $K_2 \geq \frac{3a_2^2k^2}{4\lambda_d^2}$. Lastly, using the Hölder's inequality again, we have

$$
3\mathbb{E}\left[\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_1(s)X^*(s)ds\right|^2\right] \leq 3\int_0^t \left\|e^{\int_s^t \bar{A}_1(r)dr}\right\|_2^2 ds\ \mathbb{E}\left[\int_0^t \|\gamma_1(s)\|_2^2 |X^*(s)|^2 ds\right].
$$

Similar with the calculation of expectation and variance of X^* in Subsection 2.2.2.3, we know the expectation and the variance of $X^*(s)$ in (2.2.13) with the initial $X^*(0) = \bar{X}$ is $\mathbb{E}\left[X^*(s)\right] = D_1^{-1}D_2$ and $\mathbb{V}ar\left(X^*(s)\right) = \frac{1}{2}\sigma^2 D_1^{-1}$ respectively, which implies $\mathbb{E}[|X^*(s)|^2] = a_6$ for all $s > 0$ for some positive constant a_6. Thus, from the estimation of γ_1 in Lemma 13, we obtain the estimation as follows

$$
3\mathbb{E}\left[\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_1(s)X^*(s)ds\right|^2\right] \leq 3k^2a_6a_2^2 \int_0^t e^{-2\lambda_d(t-s)}ds \int_0^t e^{-4\lambda_d(T-s)}ds \leq K_3 e^{-4\lambda_d(T-t)}
$$

for some constant $K_3 \geq \frac{3k^2a_6a_2^2}{8\lambda_d^2}$.

To summarize from the above estimation results, we have

$$
\begin{aligned}
\mathbb{E}\left[|\delta_T(t)|^2\right] &\leq K_1 e^{-2\lambda_d t} + K_2 e^{-2\lambda_d(T-t)} + K_3 e^{-4\lambda_d(T-t)} \\
&\leq K_4\left(e^{-2\lambda_d t} + e^{-2\lambda_d(T-t)}\right)
\end{aligned}
\tag{2.3.19}
$$

for all $t \in [0, T]$ for some positive constant K_4.

Similarly, from the optimal control of the probabilistic cell problem and the optimal control of the finite time control problem, it is clear that

$$
u_T^*(t) - u^*(t) = \gamma_1(t)X^*(t) - \tilde{Z}_1(t)\delta_T(t) + \gamma_2(t),
$$

which yields the inequality

$$
\mathbb{E}\left[|u_T^*(t) - u^*(t)|^2\right] \leq 3\left(\|\gamma_1(t)\|_2^2\,\mathbb{E}\left[|X^*(t)|^2\right] + \left\|\tilde{Z}_1(t)\right\|_2^2 \mathbb{E}\left[|\delta_T(t)|^2\right] + |\gamma_2(t)|^2\right).
$$

Due to the facts that $\gamma_1(t) = Z_1 - \tilde{Z}_1(t)$ and $\|\gamma_1(t)\|_2 \leq ke^{-2\lambda_d(T-t)}$, we have $\|\tilde{Z}_1(t)\|_2 \leq \|Z_1\|_2 + ke^{-2\lambda_d(T-t)} \leq \|Z_1\|_2 + k$ for all $t \in [0, T]$. Applying the estimations (2.3.7) in Lemma 13 and the inequality (2.3.19), we get the estimation

$$
\mathbb{E}\left[|u_T^*(t) - u^*(t)|^2\right] \leq K_5\left(e^{-2\lambda_d t} + e^{-2\lambda_d(T-t)}\right)
$$

for all $t \in [0,T]$ for some $K_5 > 0$ independent of t and T. Therefore, let $\lambda = 2\lambda_d$, the classical turnpike property

$$\mathbb{E}\left[|X_T^*(t) - X^*(t)|^2 + |u_T^*(t) - u^*(t)|^2\right] \leq K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T]$$

is obtained for some $K \geq K_4 + K_5$. $\square$

Remark 15. *In the above proof, we show the turnpike property between $X_T^*(t)$ and the optimal path $X^*(t)$ in (2.2.13) for the probabilistic cell problem taking an initial $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ instead of a real-valued vector $x \in \mathbb{R}^d$. The reason for taking this normal random variable as the initial is that the optimal path $X^*(t)$ starting with any constant initial $x \in \mathbb{R}^d$ converges in distribution to a normal random variable $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ as $t \to \infty$ from (2.2.15). We refer the asymptomatic $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ to be the equilibrium point of $X^*(t)$ in (2.2.13). The proof of Theorem 10 when X^* taking an initial value $x \in \mathbb{R}^d$ follows a similar approach.*

2.4 Example

In this section, we give an example to illustrate the results in Theorem 9 and Theorem 10. In this example, we consider the case when $d = 1$.

Let $A = 0,\ b = 0,\ Q = 1,\ q = 0,\ r = -1$. Then, the underlying process (2.1.1) is reduced to

$$dX(t) = u(t)dt + \sigma dW(t), \quad X_0 = x,$$

and the cost functional (2.1.2) can be simplified as following

$$J_T(0,x;u) = \mathbb{E}\left[\frac{1}{2}\int_0^T \left(X^2(t) + u^2(t) - 2u(t)\right) dt\right].$$

Firstly, we verify the result in Theorem 9. It is clear that $Z_1 = 1$ and $Z_2 = 1$ when we take $Z_1 > 0$ from the results in Lemma 1, and $\{\tilde{Z}_1(t), \tilde{Z}_2(t), \tilde{Z}_3(t) : t \in [0,T]\}$ is the solution to the system of ODEs

$$\begin{cases} \dot{\tilde{Z}}_1(t) - \tilde{Z}_1^2(t) + 1 = 0; \\ \dot{\tilde{Z}}_2(t) - \tilde{Z}_1(t)\tilde{Z}_2(t) + \tilde{Z}_1(t) = 0; \\ \dot{\tilde{Z}}_3(t) + \tilde{Z}_2(t) - \dfrac{1}{2}\tilde{Z}_2^2(t) + \dfrac{1}{2}\sigma^2\tilde{Z}_1(t) - \dfrac{1}{2} = 0; \\ \tilde{Z}_1(T) = \tilde{Z}_2(T) = \tilde{Z}_3(T) = 0. \end{cases}$$

By calculation, we obtain the solution to the above system of ODEs as follows

$$\tilde{Z}_1(t) = \frac{1 - e^{2t-2T}}{1 + e^{2t-2T}}, \quad \tilde{Z}_2(t) = 1 - e^{-\int_t^T \tilde{Z}_1(s)ds} = 1 - \frac{2e^{t-T}}{1 + e^{2t-2T}},$$

and

$$\tilde{Z}_3(t) = \frac{1}{2}\sigma^2(T-t) - \frac{1}{2}\sigma^2 \ln\left(\frac{2}{1 + e^{2t-2T}}\right) + \frac{1}{2} - \frac{1}{1 + e^{2t-2T}}.$$

Similarly, we know that f_1, f_2, f_3 is the solution to

$$\begin{cases} \dot{f}_1(t) - 2f_1(t) + 2 = 0; \\ \dot{f}_2(t) - f_2(t) + 1 = 0; \\ \dot{f}_3(t) + \dfrac{1}{2}\sigma^2 f_1(t) = 0; \\ f_1(T) = f_2(T) = f_3(T) = 0, \end{cases}$$

which gives that

$$f_1(t) = 1 - e^{2t-2T}, \;\; f_2(t) = 1 - e^{t-T}, \;\; f_3(t) = \frac{1}{2}\sigma^2(T-t) - \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 e^{2t-2T}.$$

Thus, we could obtain the difference between $f_i(t)$ and $\tilde{Z}_i(t)$ for $i = 1,2,3$ by

$$\Gamma_1(0) = e^{-2T}\frac{1-e^{-2T}}{1+e^{-2T}}, \quad \Gamma_2(0) = e^{-T}\frac{1-e^{-2T}}{1+e^{-2T}},$$

and

$$\Gamma_3(0) = \frac{1}{4}\sigma^2\left(e^{-2T} - 1 + 2\ln\left(\frac{2}{1+e^{-2T}}\right)\right) + \frac{1}{1+e^{-2T}} - \frac{1}{2}.$$

Note that for $x \in \mathbb{R}$ and T large enough,

$$J_T(0,x;u^*) - V_T(0,x) = \frac{1}{2}\Gamma_1(0)x^2 + \Gamma_2(0)x + \Gamma_3(0) \ge \Gamma_3(0) - \frac{(\Gamma_2(0))^2}{2\Gamma_1(0)} \ge 0.$$

On the other hand side, we have

$$\begin{aligned} &J_T(0,x;u^*) - V_T(0,x) \\ =\;& \frac{e^{-2T}(1-e^{-2T})}{2(1+e^{-2T})}x^2 + \frac{e^{-T}(1-e^{-2T})}{1+e^{-2T}}x + \frac{\sigma^2}{4}\left(e^{-2T} - 1 + 2\ln\left(\frac{2}{1+e^{-2T}}\right)\right) + \frac{1}{1+e^{-2T}} - \frac{1}{2}. \end{aligned}$$

Let $T \to \infty$, we find that

$$\lim_{T\to\infty}\left(J_T(0,x;u^*) - V_T(0,x)\right) = \frac{1}{4}\sigma^2(\ln 4 - 1) + \frac{1}{2},$$

which yields that $J_T(0,x;u^*) - V_T(0,x) = O(1)$ for all $x \in \mathbb{R}$. Thus, we obtain the desired result in Theorem 9 under this example.

Next, we check the result in Theorem 10. It is easy to get that $D_1 = 1$ and $D_2 = 0$ from (2.2.12). Denote $\delta_T(t) = X_T^*(t) - X^*(t)$, we have

$$d\delta_T(t) = -\tilde{Z}_1(t)\delta_T(t)dt + \left(1 - \tilde{Z}_2(t)\right)dt + \left(1 - \tilde{Z}_1(t)\right)X^*(t)dt$$

with the initial value $\delta_T(0) = x - \bar{X} \sim \mathcal{N}(x, \frac{\sigma^2}{2})$. Applying the integrating factor method, we have the explicit form of $\delta_T(t)$ as following

$$\delta_T(t) = \delta_T(0)e^{-\int_0^t \tilde{Z}_1(s)ds} + \int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_2(s)\right)ds$$
$$+ \int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_1(s)\right)X^*(s)ds.$$

Then, we get the estimation

$$\mathbb{E}\left[|\delta_T(t)|^2\right] \leq 3e^{-\int_0^t 2\tilde{Z}_1(s)ds}\mathbb{E}\left[|\delta_T(0)|^2\right] + 3\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_2(s)\right)ds\right)^2$$
$$+ 3\mathbb{E}\left[\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_1(s)\right)X^*(s)ds\right)^2\right].$$

Firstly, since $\delta_T(0) = x - \bar{X} \sim \mathcal{N}(x, \frac{\sigma^2}{2})$, we know that

$$3e^{-\int_0^t 2\tilde{Z}_1(s)ds}\mathbb{E}\left[|\delta_T(0)|^2\right] = 3e^{-2t}\left(\frac{1+e^{2t-2T}}{1+e^{-2T}}\right)^2\left(x^2+\frac{\sigma^2}{2}\right) \leq 12e^{-2t}\left(x^2+\frac{\sigma^2}{2}\right).$$

Next, by the Hölder's inequality and some simplifications, we can estimate the second term as

$$3\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_2(s)\right)ds\right)^2 \leq 12e^{-2(T-t)}.$$

Lastly, using the Hölder's inequality again, we have

$$3\mathbb{E}\left[\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_1(s)\right)X^*(s)ds\right)^2\right] \leq 24e^{-4T}\int_0^t e^{4s}\mathbb{E}\left[(X^*(s))^2\right]ds.$$

Similar with the calculation of expectation and variance of X^* in Subsection 2.2.2.3, we know that

$$\mathbb{E}\left[X^*(s)\right] = \mathbb{E}\left[\bar{X}\right]e^{-s} = 0, \quad \mathbb{V}ar\left(X^*(s)\right) = \frac{\sigma^2}{2}\left(1-e^{-2s}\right) + \mathbb{V}ar\left(\bar{X}\right)e^{-2s} = \frac{\sigma^2}{2},$$

as $\bar{X} \sim \mathcal{N}(0, \frac{\sigma^2}{2})$, which gives $\mathbb{E}[(X^*(s))^2] = \frac{\sigma^2}{2}$. Then

$$3\mathbb{E}\left[\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(r)dr}\left(1-\tilde{Z}_1(s)\right)X^*(s)ds\right)^2\right] \leq 3\sigma^2 e^{-4(T-t)}.$$

Thus, we obtain the desired inequality

$$\mathbb{E}\left[|\delta_T(t)|^2\right] \leq 12\left(x+\frac{\sigma^2}{2}\right)e^{-2t} + 12e^{-2(T-t)} + 3\sigma^2 e^{-4(T-t)}$$
$$\leq K_6\left(e^{-2t}+e^{-2(T-t)}\right)$$

for all $t \in [0,T]$, where $K_6 = \max\{12(x+\frac{\sigma^2}{2}), 12+3\sigma^2\}$. Similarly, since

$$u_T^*(t) - u^*(t) = 1 - \tilde{Z}_2(t) + \left(1 - \tilde{Z}_1(t)\right) X^*(t) - \tilde{Z}_1(t)\delta_T(t),$$

by some analytical calculations and simplifications, we have

$$\mathbb{E}\left[|u_T^*(t) - u^*(t)|^2\right] \leq K_7 \left(e^{-2t} + e^{-2(T-t)}\right)$$

for all $t \in [0,T]$, where $K_7 \geq 12 + 6\sigma^2 + 3K_6$. Therefore, the desired turnpike property is obtained.

2.5 Appendix

Proof of Lemma 11. From the standard dynamic programming principle, we obtain the HJB equation

$$\begin{cases} -\partial_t V_T(t,x) + H\left(x, -\nabla_x V_T(t,x), -D_x^2 V_T(t,x)\right) = 0, \\ V_T(T,x) = 0, \end{cases} \tag{2.5.1}$$

where

$$\begin{aligned} L(x,u) &= \frac{1}{2}\left(x^\top Q x + |u|^2 + 2q^\top x + 2r^\top u\right), \\ H(x,\bar{p},\bar{q}) &= \sup_{u\in\mathbb{R}} \left\{(Ax+u+b)^\top \bar{p} + \frac{1}{2}\sigma^2 \mathrm{trace}(\bar{q}) - L(x,u)\right\}. \end{aligned}$$

Taking derivative to the terms in the supermum with respect to u, and letting it be zero, we have $u + r + \nabla_x V_T(t,x) = 0$. Thus, the optimal feedback control is given by

$$u_T^*(t) = -\left(r + \nabla_x V_T\left(t, X_T^*(t)\right)\right),$$

and the value function $V_T(t,x)$ satisfies

$$\begin{aligned} 0 = \partial_t V_T(t,x) &+ x^\top A \nabla_x V_T(t,x) + (b-r)^\top \nabla_x V_T(t,x) - \frac{1}{2}|\nabla_x V_T(t,x)|^2 \\ &+ \frac{1}{2}\sigma^2 \Delta_x V_T(t,x) + \frac{1}{2}x^\top Q x + q^\top x - \frac{1}{2}|r|^2. \end{aligned} \tag{2.5.2}$$

Next, we give the semi-explicit solution to the HJB equation (2.5.1). Suppose the solution V_T to the HJB equation (2.5.1) is in $C^{1,2}([0,T]\times\mathbb{R}^d)$, we assume the value function has the form

$$V_T(t,x) = \frac{1}{2}x^\top \tilde{Z}_1(t)x + x^\top \tilde{Z}_2(t) + \tilde{Z}_3(t),$$

where $\tilde{Z}_1 : [0,T] \mapsto \mathbb{S}^{d\times d}$, $\tilde{Z}_2 : [0,T] \mapsto \mathbb{R}^d$ and $\tilde{Z}_3 : [0,T] \mapsto \mathbb{R}$ are real-valued functions in

$C^1([0,T])$. Then, it is straightforward to get that

$$\begin{aligned}
&\partial_t V_T(t,x) = \frac{1}{2}x^\top \dot{\tilde{Z}}_1(t)x + x^\top \dot{\tilde{Z}}_2(t) + \dot{\tilde{Z}}_3(t), \\
&\nabla_x V_T(t,x) = \tilde{Z}_1(t)x + \tilde{Z}_2(t), \\
&D_x^2 V_T(t,x) = \tilde{Z}_1(t).
\end{aligned}$$

Plugging the above terms into equation (2.5.2), we have

$$\begin{aligned}
0 = \frac{1}{2}x^\top &\left(\dot{\tilde{Z}}_1(t) + 2A\tilde{Z}_1(t) - \tilde{Z}_1^2(t) + Q\right) x \\
&+ x^\top \left(\dot{\tilde{Z}}_2(t) + A\tilde{Z}_2(t) - \tilde{Z}_1(t)\tilde{Z}_2(t) + \tilde{Z}_1(t)(b-r) + q\right) \\
&+ \dot{\tilde{Z}}_3(t) + \frac{1}{2}\sigma^2 \mathrm{trace}(\tilde{Z}_1(t)) + b^\top \tilde{Z}_2(t) - \frac{1}{2}|\tilde{Z}_2(t) + r|^2
\end{aligned}$$

for all $x \in \mathbb{R}^d$ with $\tilde{Z}_1(T) = \mathbf{O}_d$, $\tilde{Z}_2(T) = \mathbf{0}_d$, and $\tilde{Z}_3(T) = 0$. Setting up the coefficients of linear and quadratic terms with respect to x and constant to 0, we obtain the Riccati system of ODEs (2.3.2). Moreover, from the explicit form of $V_T(t,x)$, the optimal feedback control of the finite time control problem is given by $u_T^*(t) = -(\tilde{Z}_1(t)X_T^*(t) + \tilde{Z}_2(t) + r)$ for all $t \in [0,T]$. □

Proof of Lemma 12. From (2.2.14), the optimal control for the probabilistic cell problem is

$$u^*(t) = -\left(Z_1 X^*(t) + Z_2 + r\right),$$

which is a feedback control. If we take this control in the finite time control problem, then the underlying process becomes

$$\begin{aligned}
dX_T(t) &= \left(AX_T(t) - (Z_1 X_T(t) + Z_2 + r) + b\right) dt + \sigma dW(t) \\
&= -D_1 X_T(t)dt + D_2 dt + \sigma dW(t)
\end{aligned}$$

with $X_T(0) = x$, where D_1 and D_2 are constants given in (2.2.12). By inserting the optimal control of the probabilistic cell problem into the cost functional of the finite time control problem, we obtain $J_T(t,x;u^*)$ equals to

$$\mathbb{E}\left[\int_t^T \left(\frac{1}{2}X_T^\top(s)\left(Q + Z_1^2\right)X_T(s) + X_T^\top(s)(Z_1Z_2 + q) + \frac{1}{2}\left(|Z_2|^2 - |r|^2\right)\right) ds\right].$$

From Feynman-Kac's formula, if $J_T(\cdot,\cdot;u^*) \in C^{1,2}([0,T]\times\mathbb{R}^d)$, it is the solution to the following PDE

$$\begin{cases}
\partial_t J_T(t,x;u^*) + (-D_1 x + D_2)^\top \nabla_x J_T(t,x;u^*) + \frac{1}{2}\sigma^2 \Delta_x J_T(t,x;u^*) \\
\qquad + \frac{1}{2}x^\top \left(Q + Z_1^2\right) x + x^\top \left(Z_1 Z_2 + q\right) + \frac{1}{2}\left(|Z_2|^2 - |r|^2\right) = 0, \\
J_T(T,x;u^*) = 0.
\end{cases} \tag{2.5.3}$$

We consider that the solution to (2.5.3) has the form

$$J_T(t,x;u^*) = \frac{1}{2}x^\top f_1(t)x + x^\top f_2(t) + f_3(t),$$

where $f_1 : [0,T] \mapsto \mathbb{S}^{d\times d}$, $f_2 : [0,T] \mapsto \mathbb{R}^d$ and $f_3 : [0,T] \mapsto \mathbb{R}$ are real-valued functions in $C^1([0,T])$. Then, it is clear that

$$\begin{aligned}
\partial_t J_T(t,x;u^*) &= \frac{1}{2}x^\top \dot{f}_1(t)x + x^\top \dot{f}_2(t) + \dot{f}_3(t) \\
\nabla_x J_T(t,x;u^*) &= f_1(t)x + f_2(t) \\
D_x^2 J_T(t,x;u^*) &= f_1(t).
\end{aligned}$$

Plugging the above terms into the PDE (2.5.3) satisfied by $J_T(t,x;u^*)$, we obtain

$$\begin{aligned}
&\frac{1}{2}x^\top \dot{f}_1(t)x + x^\top \dot{f}_2(t) + \dot{f}_3(t) + (-D_1x + D_2)^\top (f_1(t)x + f_2(t)) + \frac{1}{2}\sigma^2 \mathrm{trace}(f_1(t)) \\
&\qquad + \frac{1}{2}x^\top \left(Q + Z_1^2\right)x + x^\top \left(Z_1Z_2 + q\right) + \frac{1}{2}\left(|Z_2|^2 - |r|^2\right) = 0
\end{aligned}$$

for all $x \in \mathbb{R}^d$ with $f_1(T) = \mathbf{O}_d$, $f_2(T) = \mathbf{0}_d$, and $f_3(T) = 0$. Setting up the coefficients of linear and quadratic terms with respect to x and constant to 0, and plugging D_1, D_2 in (2.2.12) back, we obtain (2.3.3). $\square$

Chapter 3

Convergence rate of LQG mean field games with Markov chain as common noise

3.1 Introduction

In this chapter, we study the convergence rate of equilibrium measures of N-player differential game in the context of Linear-Quadratic (LQ) structure with a common noise to its limiting MFG system. Different from the works mentioned above, the common noise in this chapter is a continuous-time Markov chain (CTMC) instead of Brownian motion, which often models the real-world control problems associated with hybrid systems. Markov chains are widely used to model systems that exhibit randomness and transition between different states. In various real-world scenarios, especially in economics (see [78]), finance (see [85]), biology (see [86]), and engineering (see [84]), the dynamics of systems can be effectively represented as discrete states with probabilistic transitions between them. By using CTMC, the applications aim to model less frequently changing common noises, such as government policies implemented by two different regimes.

LQ control problems have been widely recognized in the stochastic control theory due to their broad applications. More importantly, LQ structure leads to solvability in a closed form, namely the Ricatti system, and this usually sheds light on many fundamental properties of the control theory. For this reason, LQ structure has also been studied in MFG with or without common noises for its importance. The related literature include major and minor Linear-Quadratic-Gaussian (LQG) Mean Field Games system ([39, 65, 30]); social optimal in LQG Mean Field Games ([44, 29]); the LQG Mean Field Games with different model settings ([5, 37, 6, 38]); and LQG Graphon Mean Field Games ([35]). Recently, LQ Mean Field Games with a Brownian motion as the common noise have also been studied in [1, 76] with restrictions of the dependence of measure on its mean alone. Moreover, some literature considers various topics of Mean Field control and game problems with Markov chain common noise, see [59, 66, 67].

A fundamental question in this regard is the convergence rate of the N-player game to the desired MFG system. A well-known result is about the convergence rate of value functions of the generic player, which can be shown $O(N^{-1})$, see for instance [13, 14, 16, 17, 46]. In particular, [46] establishes the convergence rate of value functions in the sense of

$$J_1^N(\hat{\alpha}_1, \hat{\alpha}_{-1}) \leq J_1^N(\alpha_1, \hat{\alpha}_{-1}) + O(N^{-1}),$$

where J_1^N is the value of the first player in the N-player game and $\hat{\alpha}$ is the Nash equilibrium decentralized control process for the Mean Field Game problem.

In contrast, the convergence rate of equilibrium measures is another challenging question due to the complication of the correlation structures among N players. To be more concrete, we examine the behavior of the $\hat{X}_{it}^{(N)}$, who represents the equilibrium state of the i-th player at time t in the N-player game defined within the probability space $\left(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{F}^{(N)}, \mathbb{P}^{(N)}\right)$. Additionally, we denote $\hat{X}_t$ as the equilibrium path at time t derived from the associated MFG defined in the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The question pertains to the convergence of $\hat{X}_{1t}^{(N)}$ as follows:

(Q) The $\mathbb{W}_p$-convergence rate of the representative equilibrium path,

$$\mathbb{W}_p\left(\mathcal{L}\left(\hat{X}_{1t}^{(N)}\right), \mathcal{L}\left(\hat{X}_t\right)\right) = O\left(N^{-?}\right).$$

Here, $\mathbb{W}_p$ denotes the p-Wasserstein metric.

The existing literature extensively explores the convergence rate in this context. For (Q), Theorem 2.4.9 of the monograph [14] establishes a convergence rate of $O(N^{-1/2})$ using the $\mathbb{W}_1$ metric. More recently, [48] addresses (Q) by introducing displacement monotonicity and controlled common noise, and Theorem 2.23 applies the maximum principle of forward-backward propagation of chaos to achieve the same convergence rate. It is important to note that these results are not applicable to the LQG framework, primarily due to the assumption concerning the linear growth of the cost functional.

The main result of this chapter establishes that the equilibrium measures exhibit a convergence rate of 1/2 concerning the 2-Wasserstein distance. The precise statement of this result can be found in Theorem 21. In comparison to the aforementioned literature, two primary distinctions emerge. Firstly, within the framework of Mean Field Games, the common noise is modeled as a Continuous-Time Markov Chain. Secondly, a significant difference lies in the cost function's behavior, as it does not possess linear growth within the context of the LQG framework.

To obtain the desired convergence rate in this chapter, the first building block is the characterization of the equilibrium measure of the limiting MFG by a finite-dimensional ODE system. The key step leading us to a desired finite-dimensional system is that, instead of searching for the infinite-dimensional function directly, we postulate a Markovian structure via two auxiliary processes (3.3.1) governed by its finite-dimensional coefficient functions, which exhibits the distinct feature of Markov chain common noise relatives to the Brownian motion counterpart.

The next stage towards the convergence rate is to compare the limiting MFG system to an N-player game. In contrast to the characterization of the MFG system, it is relatively routine to solve the N-player game due to its LQ structure. Therefore, the convergence rate problem can be recasted to the following question about a coupling of the two following processes: For two equilibrium processes $\hat{X}$ of MFG in Ω and $\hat{X}_1^{(N)}$ of the N-player game in $\Omega^{(N)}$, finding a random process Z^N in Ω whose distribution is identical to $\hat{X}_1^{(N)}$ satisfying the estimate in the form of $\mathbb{E}[|\hat{X}_t - Z_t^N|^2] = O\left(N^{-?}\right)$. For this purpose, we first show an N-invariant algebraic structure of the seemingly intractable κN^3 dimensional ODE system (3.4.1), which originated from [46, Huang and Yang] as a dimensional reduction in the system with Brownian common noise. Thanks to this N-invariant structure, the complex ODE system (3.4.1) can be reduced to the ODE system (3.4.5) whose dimension agrees with the ODE (3.2.12) of MFG system. Moreover, $\hat{X}_1^{(N)}$ can be represented as a stochastic flow driven by two Brownian motions $W_1^{(N)}$ and $W_{-1}^{(N)} := \frac{1}{\sqrt{N-1}} \sum_{i=2}^{N} W_i^{(N)}$, which enables us to embed the equilibrium process $\hat{X}_1^{(N)}$ to any probability space having only two Brownian motions.

The rest of this chapter is outlined as follows: Section 3.2 presents a precise formulation of the problem and two main results. Section 3.3 is devoted to the derivation of our first result: the equilibrium of MFG. In Section 3.4, we show in detail the convergence of the N-player game to MFG, which yields our second main result. Section 3.5 demonstrates the convergence by some numerical examples. The conclusion and some possible future works are summarised in Section 3.6. Section 3.7 is an appendix that collects some related facts to support our main theme.

3.2 Problem setup and main results

First, we collect common notations used in this chapter in Subsection 3.2.1. Then, we set up problems on MFG and the N-player game separately in Subsections 3.2.2 and 3.2.3. The main results are presented in Subsection 3.2.4 and some interpretations of our main results are added in Subsection 3.2.5.

3.2.1 Notations

Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t : 0 \le t \le T\}, \mathbb{P})$ be a completed filtered probability space satisfying the usual conditions, on which W and B are two independent standard Brownian motions, and Y is a continuous time Markov chain (CTMC) independent of (W, B) taking values in a finite state space $\mathcal{Y} = \{1, 2, \ldots, \kappa\}$ with a generator

$$Q = (q_{i,j})_{i,j \in \mathcal{Y}} \tag{3.2.1}$$

satisfying $q_{i,j} \ge 0$ for all $i \ne j \in \mathcal{Y}$ and $\sum_{i \ne j} q_{i,j} + q_{i,i} = 0$ for each $i \in \mathcal{Y}$. In the above, the Brownian motion B does not play any role in MFG problem formulation until the convergence proof of the N-player game to MFG.

By $L^p := L^p(\Omega, \mathbb{P})$, we denote the space of random variables X on $(\Omega, \mathcal{F}_T, \mathbb{P})$ with finite p-th moment with norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$. We also denote by $L^p_{\mathbb{F}} := L^p_{\mathbb{F}}([0,T] \times \Omega)$ the space of all $\mathbb{F}$-progressively measurable random processes $\alpha = (\alpha_t)_{0 \le t \le T}$ satisfying

$$\mathbb{E}\left[\int_0^T |\alpha_t|^p dt\right] < \infty.$$

For any polish (complete separable metric) space $(P, \mathcal{B}(P), d)$, we use δ_x to denote the Dirac measure on the point $x \in P$. Then, the collection of all probabilities m on $(P, \mathcal{B}(P), d)$ having finite k-th moment is denoted by $\mathcal{P}_k(P)$, i.e.

$$[m]_k := \int x^k m(dx) < \infty, \quad \forall m \in \mathcal{P}_k(P).$$

The equilibrium of MFG with the common noise yields the conditional distribution. For real-valued random variables X and Z in $(\Omega, \mathcal{F}_T, \mathbb{P})$, we denote the distribution of X conditional on $\sigma(Z)$ by $\mathcal{L}(X|Z)$, or equivalently

$$\mathcal{L}(X|Z)(A) = \mathbb{E}[I_A(X)|Z], \quad \forall A \in \mathcal{F}_T.$$

Note that $\mathcal{L}(X|Z)(A) : \Omega \mapsto \mathbb{R}$ is a $\sigma(Z)$-measurable random variable, therefore, $\mathcal{L}(X|Z)$ is $\sigma(Z)$-measurable random probability distribution with k-th moment $[\mathcal{L}(X|Z)]_k = \mathbb{E}[X^k|Z]$, if it exists. We refer to more details on the conditional distribution in Volume II of [17]. The next proposition provides an embedding approach to prove a convergence in distribution, which will be used later in the convergence of the N-player game to MFG.

Proposition 16. *Suppose $(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)})$ is a complete probability space. Let $X^{(N)}$ and X be random variables of $\Omega^{(N)} \mapsto P$ and $\Omega \mapsto P$, respectively. Then, $X^{(N)}$ is convergent in distribution to X, denoted by $X^{(N)} \Rightarrow X$, if there exists $Z^N : \Omega \mapsto P$ satisfying $\mathcal{L}(Z^N) = \mathcal{L}(X^{(N)})$, such that $Z^N \to X$ holds almost surely, i.e.*

$$\lim_{N \to \infty} d(Z^N, X) = 0, \text{ almost surely in } \mathbb{P},$$

where d represents the metric assigned to the space P.

In this chapter, we formulate the N-player game in the completed filtered probability space

$$(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{F}^{(N)} := \{\mathcal{F}_t^{(N)} : 0 \le t \le T\}, \mathbb{P}^{(N)}),$$

and $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the same generator given by (3.2.1) and $W^{(N)} = (W_i^{(N)} : i = 1, 2, \ldots, N)$ is an N-dimensional standard Brownian motion. We assume $Y^{(N)}$ and $W^{(N)}$ are independent of each other.

For better clarity, we use the superscript (N) for a random variable to emphasize the probability space $\Omega^{(N)}$ it belongs to. For example, Proposition 16 denotes a random variable in $\Omega^{(N)}$ by $X^{(N)}$, while its distribution copy in Ω by Z^N, but not by $Z^{(N)}$.

3.2.2 The equilibrium of MFG

In this section, we define the equilibrium of MFG associated with a generic player's stochastic control problem in the probability setting Ω, see Section 3.2.1.

Given a random measure flow $m : (0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, consider a generic player who wants to minimize her expected accumulated cost on $[0, T]$:

$$J(y, x, \alpha) = \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_s^2 + F(Y_s, X_s, m_s)\right) ds + G(Y_T, X_T, m_T)\Big| Y_0 = y, X_0 = x\right] \tag{3.2.2}$$

with some given cost functions $F, G : \mathcal{Y} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ and underlying random processes $(Y, X) : [0, T] \times \Omega \mapsto \mathcal{Y} \times \mathbb{R}$. Among three processes (Y, X, m), the generic player can control the process X via α in the form of

$$X_t = X_0 + \int_0^t \left(\tilde{b}_1(Y_s, s)X_s + \tilde{b}_2(Y_s, s)\alpha_s\right) ds + W_t, \quad \forall t \in [0, T], \tag{3.2.3}$$

where $\tilde{b}_1(\cdot, \cdot)$ and $\tilde{b}_2(\cdot, \cdot)$ are two deterministic functions. We assume that the initial state X_0 is independent of Y. The Brownian motion W is the individual noise of the generic player, the process Y of (3.2.1) represents the common noise, and $m = (m_t)_{0 \le t \le T}$ is a given random density flow normalized up to total mass one.

The objective of the control problem for the generic player is to find its optimal control $\hat{\alpha} \in \mathcal{A} := L^4_{\mathbb{F}}$ to minimize the total cost, i.e.

$$V[m](y, x) = J[m](y, x, \hat{\alpha}) \le J[m](y, x, \alpha), \quad \forall \alpha \in \mathcal{A}. \tag{3.2.4}$$

Associated with the optimal control $\hat{\alpha}$, we denote the optimal path by $\hat{X} = (\hat{X}_t)_{0 \le t \le T}$. To introduce MFG Nash equilibrium, it is often convenient to highlight the dependence of the optimal path and optimal control of the generic player and its associated value on the underlying density flow m, which are denoted by

$$\hat{X}_t[m], \hat{\alpha}_t[m], \text{ and } V[m],$$

respectively. Now, we present the definition of the equilibrium below, see also Volume II-P127 of [17] for a general setup with a common noise.

Definition 17. *Given an initial distribution $\mathcal{L}(X_0) = m_0 \in \mathcal{P}_2(\mathbb{R})$, a random measure flow $\hat{m} = \hat{m}(m_0)$ is said to be an MFG equilibrium measure if it satisfies the fixed point condition*

$$\hat{m}_t = \mathcal{L}(\hat{X}_t[\hat{m}]|Y), \ \forall 0 < t \le T, \quad \textit{almost surely in } \mathbb{P}. \tag{3.2.5}$$

The path $\hat{X}$ and the control $\hat{\alpha}$ associated to $\hat{m}$ is called the MFG equilibrium path and equilibrium control, respectively. The value function of the control problem associated with the equilibrium measure $\hat{m}$ is called as MFG value function, denoted by

$$U(m_0, y, x) = V[\hat{m}](y, x). \tag{3.2.6}$$

The flowchart of MFG diagram is given in Figure 3.1. It is noted from the optimality condition (3.2.4) and the fixed point condition (3.2.5) that

$$J[\hat{m}](y, x, \hat{\alpha}) \le J[\hat{m}](y, x, \alpha), \quad \forall \alpha$$

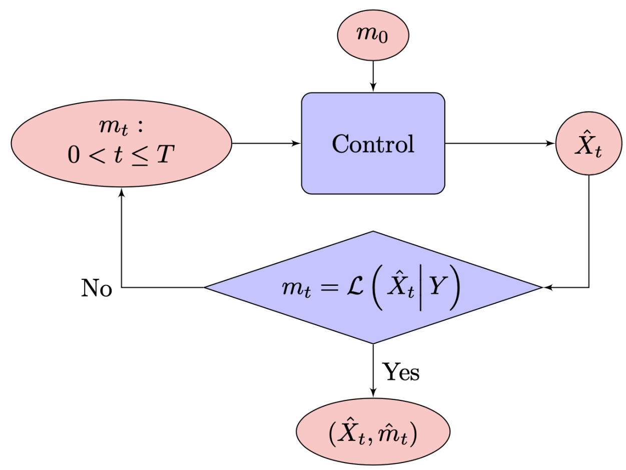


Figure 3.1: MFG diagram.

holds for the equilibrium measure $\hat{m}$ and its associated equilibrium control $\hat{\alpha}$, while it is not

$$J[\hat{m}](y, x, \hat{\alpha}) \leq J[m](y, x, \alpha), \quad \forall \alpha, m.$$

Otherwise, this problem turns into a McKean-Vlasov control problem discussed in [66]. Furthermore, it's important to note that the Continuous-Time Markov Chain Y serves a role as common noise. This is due to the fact that the mean field term is conditioned on the distribution of Y.

3.2.3 Equilibrium of the N-player game

The discrete counterpart of MFG is an N-player game, which is formulated below in the probability space $\Omega^{(N)}$, see Section 3.2.1 for more details on the probability setup.

Recall that, $W_{it}^{(N)}$ and $W_{jt}^{(N)}$ are independent Brownian motions for $j \neq i$ and they are called individual noises in the N-player game. The common noise $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the generator given by (3.2.1). Let the player i follow the dynamic, for $i = 1, 2, \ldots, N$,

$$dX_{it}^{(N)} = \left(\tilde{b}_1(Y_t^{(N)}, t)X_{it}^{(N)} + \tilde{b}_2(Y_t^{(N)}, t)\alpha_{it}^{(N)}\right) dt + dW_{it}^{(N)}, \quad X_{i0}^{(N)} = x_i^{(N)}. \tag{3.2.7}$$

The cost function for player i associated to the control $\alpha^{(N)} = (\alpha_i^{(N)} : i = 1, 2, \ldots, N)$ is

$$\begin{aligned} J_i^N(y, x^{(N)}, \alpha^{(N)}) = \mathbb{E}\Bigg[\int_0^T \left(\frac{1}{2}|\alpha_{it}^{(N)}|^2 + F(Y_t^{(N)}, X_{it}^{(N)}, \rho(X_t^{(N)}))\right) dt + \\ G(Y_T^{(N)}, X_{iT}^{(N)}, \rho(X_T^{(N)}))\Big| X_0^{(N)} = x^{(N)}, Y_0^{(N)} = y\Bigg], \end{aligned} \tag{3.2.8}$$

where $x^{(N)} = (x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)})$ is an $\mathbb{R}^N$-valued random vector in $\Omega^{(N)}$ to denote the initial state for N player, $\alpha_i^{(N)} \in \mathcal{A}^{(N)} := L^4_{\mathbb{F}^{(N)}}$, and

$$\rho(x^{(N)}) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^{(N)}}$$

is the empirical measure of a vector $x^{(N)}$ with Dirac measure δ. We use the notation for the control $\alpha^{(N)} = (\alpha_i^{(N)}, \alpha_{-i}^{(N)}) = (\alpha_1^{(N)}, \alpha_2^{(N)}, \ldots, \alpha_N^{(N)})$.

Definition 18. *1. The value function of player i for $i = 1, 2, \ldots, N$ of the Nash game is defined by $V^N = (V_i^N : i = 1, 2, \ldots, N)$ satisfying the equilibrium condition*

$$V_i^N(y, x^{(N)}) \quad = J_i^N(y, x^{(N)}, \hat{\alpha}_i^{(N)}, \hat{\alpha}_{-i}^{(N)}) \leq J_i^N(y, x^{(N)}, \alpha_i^{(N)}, \ \hat{\alpha}_{-i}^{(N)}), \tag{3.2.9}$$

for all $\alpha_i^{(N)} \in \mathcal{A}^{(N)}$.

2. The equilibrium path of the N-player game is the random path $\hat{X}_t^{(N)} = (\hat{X}_{1t}^{(N)}, X_{2t}^{(N)}, \ldots, \hat{X}_{Nt}^{(N)})$ driven by (3.2.7) associated to the control $\hat{\alpha}_t^{(N)}$ satisfying the equilibrium condition of (3.2.9).

3.2.4 The main result with quadratic cost structures

We consider the following two functions $F, G : \mathcal{Y} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ in the cost functional (3.2.2):

$$F(y, x, m) = h(y) \int_{\mathbb{R}} (x - z)^2 m(dz), \tag{3.2.10}$$

and

$$G(y, x, m) = g(y) \int_{\mathbb{R}} (x - z)^2 m(dz), \tag{3.2.11}$$

for some $h, g : \mathcal{Y} \mapsto \mathbb{R}^+$. In this case, the F and G terms in (3.2.8) of the N-player game can be written by

$$F(Y_t^{(N)}, X_{it}^{(N)}, \rho(X_t^{(N)})) = \frac{h(Y_t^{(N)})}{N} \sum_{j=1}^{N} (X_{it}^{(N)} - X_{jt}^{(N)})^2,$$

and

$$G(Y_T^{(N)}, X_{iT}^{(N)}, \rho(X_T^{(N)})) = \frac{g(Y_T^{(N)})}{N} \sum_{j=1}^{N} (X_{iT}^{(N)} - X_{jT}^{(N)})^2,$$

respectively.

Remark 19. *First, we note that F and G possess the quadratic structures in x. Secondly, the coefficients $h(y)$ and $g(y)$ provide the sensitivity to the mean field effects, which depend on the current CTMC state. For another remark, let us consider the scenario where the number of states is 2 and sensitivities are invariant, say*

$$h(0) = h(1) = h, \ g(0) = g(1) = 0.$$

Then the cost function and hence the entire problem is free from the common noise. Interestingly, as shown in the Appendix 3.7.1, there is no global solution for MFG when $h < 0$, while there is a global solution when $h > 0$.

Moreover, the uniqueness of Mean Field Game can be achieved under the displacement monotonicity condition. It is easy to check that (3.2.10)-(3.2.11) satisfy the displacement monotonicity condition. Note that

$$F_x(y,x,m) = 2h(y)(x - [m]_1), \quad G_x(y,x,m) = 2g(y)(x - [m]_1),$$

which gives that

$$\mathbb{E}\left[\left(F_x(y,X_1,m_{X_1}) - F_x(y,X_2,m_{X_2})\right)(X_1 - X_2)\right] = 2h(y)\left(\mathbb{E}\left[(X_1 - X_2)^2\right] - (\mathbb{E}[X_1] - \mathbb{E}[X_2])^2\right) \geq 0$$

for all $y \in \mathcal{Y}$ if $h > 0$ on $\mathcal{Y}$, where m_{X_1} and m_{X_2} is the law of X_1 and X_2 respectively. Similarly, we can obtain that

$$\mathbb{E}\left[\left(G_x(y,X_1,m_{X_1}) - G_x(y,X_2,m_{X_2})\right)(X_1 - X_2)\right] \geq 0$$

for all $y \in \mathcal{Y}$ if $g > 0$ on $\mathcal{Y}$. Therefore, we require positive values for all sensitivities for simplicity. It is of course an interesting problem to investigate the explosion when some sensitivities are negative.

Wrapping up the above discussions, we impose the following assumptions:

(A0) $\tilde{b}_1(y,\cdot), \tilde{b}_2(y,\cdot) : [0,T] \mapsto \mathbb{R}$ are continuous functions for all $y \in \mathcal{Y}$.

(A1) The cost functions are given by (3.2.10)-(3.2.11) with $h, g > 0$; The initial X_0 of MFG satisfies $\mathbb{E}[X_0^2] < \infty$.

(A2) In addition to (A1), the initial $x^{(N)} = (x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)})$ of the N-player game is a vector of i.i.d. random variables in $\Omega^{(N)}$ with the same distribution as the initial $\mathcal{L}(X_0)$ of MFG.

Our objective for this chapter is to understand the Nash equilibrium of MFG and its connection to the N-player game equilibrium:

(P1) With Assumptions (A0), (A1), and (A2), obtain the convergence rate of $(\hat{X}_{1t}^{(N)}, Y^{(N)})$ from the N-player game of Definition 18 to $(\hat{X}_t, Y)$ from MFG of Definition 17 in distribution.

To answer (P1), it is critical to have a solid understanding of the joint distribution $(\hat{X}_t, Y)$ for the underlying MFG, which yields another question:

(P2) With Assumptions (A0) and (A1), characterize the MFG equilibrium path $\hat{X}$, as well as associated equilibrium measure $\hat{m}$ along the Definition 17;

For our first main result, we first answer (P2) via the following Riccati system for unknowns

$(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$:

$$\begin{cases} a_y' + 2\tilde{b}_{1y} a_y - 2\tilde{b}_{2y}^2 a_y^2 + \sum_{i=1}^{\kappa} q_{y,i} a_i + h_y = 0, \\ b_y' + \left(2\tilde{b}_{1y} - 4\tilde{b}_{2y}^2 a_y\right) b_y + \sum_{i=1}^{\kappa} q_{y,i} b_i + h_y = 0, \\ c_y' + a_y + b_y + \sum_{i=1}^{\kappa} q_{y,i} c_i = 0, \\ k_y' - 2\tilde{b}_{2y}^2 a_y^2 + 4\tilde{b}_{2y}^2 a_y b_y + 2\tilde{b}_{1y} k_y + \sum_{i=1}^{\kappa} q_{y,i} k_i = 0, \\ a_y(T) = b_y(T) = g_y \ , \ c_y(T) = k_y(T) = 0, \end{cases} \tag{3.2.12}$$

where $h_y = h(y)$, $g_y = g(y)$ for $y \in \mathcal{Y}$. Next, we present our first main result about the equilibrium path, the equilibrium control, and the value function in MFG.

Theorem 20 (MFG)**.** *Under (A0)-(A1), there exists a unique solution $(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$ for the Riccati system (3.2.12). With these solutions, the MFG equilibrium path $\hat{X} = \hat{X}[\hat{m}]$ is given by*

$$d\hat{X}_t = \left(\tilde{b}_1(Y_t, t)\hat{X}_t - 2\tilde{b}_2^2(Y_t, t) a_{Y_t}(t) \left(\hat{X}_t - \hat{\mu}_t\right)\right) dt + dW_t, \quad \hat{X}_0 = X_0, \tag{3.2.13}$$

with equilibrium control

$$\hat{\alpha}_t = -2\tilde{b}_2(Y_t, t) a_{Y_t}(t) \left(\hat{X}_t - \hat{\mu}_t\right), \tag{3.2.14}$$

where

$$d\hat{\mu}_t = \tilde{b}_1(Y_t, t)\hat{\mu}_t dt, \quad \hat{\mu}_0 = \mathbb{E}[X_0].$$

Moreover, the value function U is

$$U(m_0, y, x) = a_y(0)x^2 - 2a_y(0)x[m_0]_1 + k_y(0)[m_0]_1^2 + b_y(0)[m_0]_2 + c_y(0), \quad y \in \mathcal{Y}.$$

The proof of theorem 20 is based on the Markovian structure of the equilibrium and the fixed point condition of the MFG problem, and it is provided in Subsection 3.3.3. The next theorem establishes the convergence result and answers the problem (P1) with the convergence rate $\frac{1}{2}$.

Theorem 21 (Convergence rate)**.** *Under Assumptions (A0)-(A1)-(A2), the joint law $(\hat{X}_{1t}^{(N)}, Y_t^{(N)})$ of the N-player game converges in distribution to that of the MFG equilibrium $(\hat{X}_t, Y_t)$ for any $t \in (0, T]$ at the convergence rate*

$$\mathbb{W}_2 \left(\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)}), \mathcal{L}(\hat{X}_t, Y_t)\right) = O\left(N^{-\frac{1}{2}}\right), \quad \text{as } N \to \infty.$$

The proof of Theorem 21 is given in Subsection 3.4.3 since it needs the comparison between the equilibrium path $\hat{X}_{1t}^{(N)}$ in the N-player game and the equilibrium path $\hat{X}_t$ in MFG.

3.2.5 Remarks on the main results

One can interpret the main results in plain words: For the N-player game with dynamic (3.2.7) and cost structure (3.2.8) for large N, the equilibrium control of the generic player can be effectively approximated by steering itself toward the population center $\hat{\mu}_t$ depending only on the function $\tilde{b}_1(\cdot)$ and the entire past of the common noise, whose velocity is dependent on only the function $\tilde{b}_2(\cdot)$ and the entire past of the common noise. The effectiveness can be quantified by the convergence rate of 1/2 for the one-dimensional MFG under LQ structure and CMTC common noise. A natural question is whether the convergence rate can be generalized to more general settings.

We may utilize the above outcome to depict a simplified scenario of traffic flow control, where individual vehicles strive to minimize their own costs while taking into account their impact on overall traffic dynamics. We could denote the dynamics of vehicle i within a city by $X_i^{(N)}$, and employ the common noise $Y^{(N)}$ to represent traffic conditions across various districts or intersections. Each vehicle's behavior is influenced not only by its driving habits and preferences but also by the traffic situation in different districts or intersections of the city. Additionally, vehicles must factor in their proximity to traffic centers and endeavor to find more efficient routes to their destinations.

This chapter focuses on the one-dimensional problem to avoid unnecessary symbol complexity. The main convergence rate 1/2 still holds for multidimensional problems using the same coupling procedure. For convenience to check, we summarize the computation involved in multidimensional problems in Appendix 3.7.5.

The current coupling procedure can also be adapted with suitable modifications to the LQ Mean Field Game problems with Brownian common noise, see [51]. In particular, the reduction of the $O(N^3)$-dimensional ODE can be conducted similarly and the convergence rate is still maintained as 1/2. However, the dependence of the mean and variance process on the common noise and subsequent calculations are significantly different from the current chapter, see Definition 4 of [51].

Indeed, choosing the CTMC common noise instead of Brownian motion does not simplify the underlying problem, since it preserves the path-dependence feature of the equilibrium measure. On the contrary, the advantage of CTMC common noise is that the applications aim to model less frequently changing environment settings, such as government policies implemented by multiple different regimes. Due to its realistic applications, stochastic control theory perturbed by CTMC is extensively studied in the context of hybrid control problems, see books [60, 80] and the references therein.

We close this section with a remark on the uniqueness. The uniqueness of Mean Field Game can be achieved under Lasry-Lions monotonicity [55] or displacement monotonicity [33] and our setting in Section 3.2.2 satisfies the displacement monotonicity. Thus, the convergence of Theorem 21 implies that the unique equilibrium path of the N-player game converges to the unique equilibrium paths of the limiting MFG, which is characterized by Theorem 20.

3.3 Main results of MFG

This section is devoted to the proof of the first main result Theorem 20 on the MFG solution. First, we outline the scheme based on the Markovian structure of the equilibrium by reformulating the

MFG problem in Subsection 3.3.1. Next, we solve the underlying control problem in Subsection 3.3.2 and provide the corresponding Riccati system. Finally, Subsection 3.3.3 proves Theorem 20 by checking the fixed point condition of MFG problem.

3.3.1 Overview

By Definition 18, to solve for the equilibrium measure, one shall search the infinite dimensional space of the random measure flows $m : (0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, until a measure flow satisfies the fixed point condition $m_t = \mathcal{L}(\hat{X}_t|Y), \forall t \in (0,T]$, see Figure 3.1, which requires to check the following infinitely many conditions:

$$[m_t]_k = \mathbb{E}[\hat{X}_t^k|Y], \quad \forall k = 1, 2, \ldots,$$

if they exist.

The first observation is that the cost functions F and G in (3.2.10)-(3.2.11) are dependent on the measure m only via the first two moments:

$$F(y,x,m) = h(y)(x^2 - 2x[m]_1 + [m]_2),$$
$$G(y,x,m) = g(y)(x^2 - 2x[m]_1 + [m]_2).$$

Therefore, the underlying stochastic control problem for MFG can be entirely determined by the input given by $\mathbb{R}^2$ valued random process $\mu_t = [m_t]_1$ and $\nu_t = [m_t]_2$, which implies that the fixed point condition can be effectively reduced to check two conditions only:

$$\mu_t = \mathbb{E}[\hat{X}_t|Y], \ \nu_t = \mathbb{E}[\hat{X}_t^2|Y].$$

This observation effectively reduces our search from the space of random measure-valued processes $m : (0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$ to the space of $\mathbb{R}^2$-valued random processes $(\mu, \nu) : (0,T] \times \Omega \mapsto \mathbb{R}^2$.

Note that, if underlying MFG have no common noise Y, then (μ, ν) is a deterministic mapping $[0,T] \mapsto \mathbb{R}^2$ and the above observation is enough to reduce the original infinite-dimensional MFG into a finite-dimensional system. However, the following example shows that this is not the case for MFG with a common noise and it becomes the main drawback to characterizing MFG via a finite-dimensional system.

Example 1. *To illustrate, we consider the following uncontrolled mean field dynamics: Let the mean field term $\mu_t := \mathbb{E}[\hat{X}_t|Y]$, where the underlying dynamic is given by*

$$d\hat{X}_t = -\mu_t Y_t dt + dW_t.$$

- μ_t *is path dependent on Y, i.e.,*

$$\mu_t = \mu_0 \exp\Big\{ - \int_0^t Y_s ds \Big\}.$$

This implies that no finite dimensional system is possible to characterize the process μ_t, since the $(t, Y) \mapsto \mu_t$ is a function on an infinite dimensional domain.

- μ_t *is Markovian, i.e.,*
$$d\mu_t = -Y_t \mu_t dt.$$
It might be possible to characterize μ_t *via a function* $(t, Y_t, \mu_t) \mapsto \frac{d\mu_t}{dt}$ *on a finite dimensional domain.*

To solidify the above idea, we need to postulate the Markovian structure for the first and second moments of the MFG equilibrium. More precisely, our search for the equilibrium will be confined to the space $\mathcal{M}$ of measure flows whose first and second moment exhibits Markovian structure.

Definition 22. *The space* $\mathcal{M}$ *is the collection of all* $\mathcal{F}_t^Y$*-adapted measure flows* $m : [0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$*, whose first moment* $[m_t]_1 := \mu_t$ *and second moment* $[m_t]_2 := \nu_t$ *satisfy*

$$\begin{aligned} \mu_t &= \mu_0 + \int_0^t (w_0(Y_s, s)\mu_s + w_1(Y_s, s))\, ds, \\ \nu_t &= \nu_0 + \int_0^t \big(w_2(Y_s, s)\mu_s + w_3(Y_s, s)\nu_s + w_4(Y_s, s)\mu_s^2 + w_5(Y_s, s)\big)\, ds, \end{aligned} \tag{3.3.1}$$

for all $t \in [0,T]$ *and for some smooth deterministic functions* $(w_i : i = 0, 1, \ldots, 5)$.

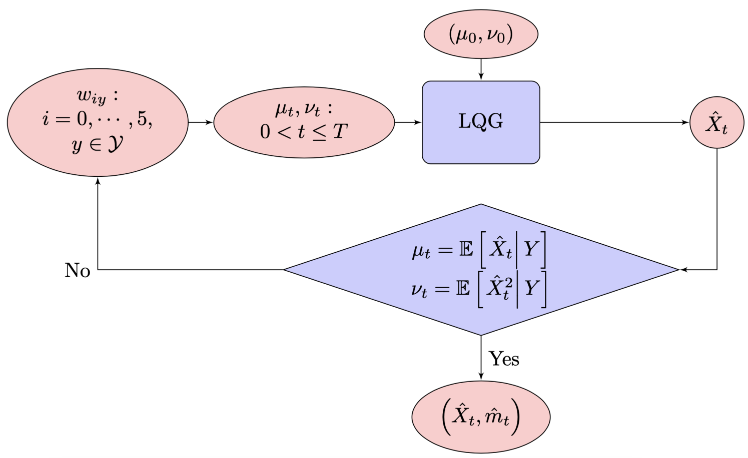


Figure 3.2: Equivalent MFG diagram.

The flowchart for our equilibrium is depicted in Figure 3.2. Subsection 3.3.2 covers the derivation of the Riccati system for the LQG system with a given population measure flow $m \in \mathcal{M}$, which provides the key building block to MFG. In Subsection 3.3.3, we check the fixed point condition and provide a finite-dimensional characterization of MFG, which gives the first main result Theorem 20.

3.3.2 The generic player's control with a given population measure

The advantage of the generic player's control problem associated with $m \in \mathcal{M}$ is that its optimal path can be characterized via the following classical stochastic control problem:

- (P3) Given smooth functions $w = (w_i : i = 0, 1, \ldots, 5)$, find the optimal value $\bar{V} = \bar{V}[w]$

$$\bar{V}(y, x, t, \bar{\mu}, \bar{v}) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\alpha_s^2 + \bar{F}(Y_s, X_s, \mu_s, \nu_s)\right) ds \right.$$
$$\left. + \bar{G}(Y_T, X_T, \mu_T, \nu_T) \middle| Y_t = y, X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu}\right]$$

underlying $\mathbb{R}^4$-valued processes (Y, X, μ, ν) defined through (3.2.1)-(3.2.3)-(3.3.1) with the finite dimensional cost functions: $\bar{F}, \bar{G} : \mathbb{R}^4 \mapsto \mathbb{R}$ given by

$$\bar{F}(y, x, \bar{\mu}, \bar{\nu}) = h(y)(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

$$\bar{G}(y, x, \bar{\mu}, \bar{\nu}) = g(y)(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

where $\bar{\mu}, \bar{\nu}$ are scalars, while μ, ν are used as processes.

Lemma 23. *Given $m \in \mathcal{M}$ associated with $w = (w_i : i = 0, 1, \ldots, 5)$, the player's value (3.2.4) under assumption (A1) is*

$$U[m_0](y, x) = \bar{V}(y, x, 0, [m_0]_1, [m_0]_2),$$

and the optimal control has a feedback form

$$\hat{\alpha}_t = \bar{\alpha}(Y_t, X_t, t, \mu_t, \nu_t)$$

underlying the processes (Y, X, μ, ν) defined through (3.2.1)-(3.2.3)-(3.3.1), whenever there exists a feedback optimal control $\bar{\alpha}$ for the problem (P3).

Proof. Due to the quadratic cost structure in (3.2.10)-(3.2.11), we have enough regularity to all concerned value functions and the details are omitted. □

Next, we turn to the solution to the control problem (P3).

3.3.2.1 HJB equation

For the simplicity of notations, for each $i \in \{0, 1, 2, 3, 4, 5\}$ and $y \in \mathcal{Y}$, denote the function $(x, t, \bar{\mu}, \bar{\nu}) \mapsto v(y, x, t, \bar{\mu}, \bar{\nu})$ as v_y, and denote $t \mapsto w_i(y, t)$ as w_{iy}. We apply similar notations for other functions whenever they have a variable $y \in \mathcal{Y}$. Formally, under enough regularity conditions, the value function $\bar{V}$ defined in (P3) is the solution v of the following coupled HJBs

$$\begin{cases} \partial_t v_y + \tilde{b}_{1y} x \partial_x v_y - \dfrac{1}{2}\left(\tilde{b}_{2y}\partial_x v_y\right)^2 + \dfrac{1}{2}\partial_{xx} v_y + \partial_\mu v_y \left(w_{0y}\bar{\mu} + w_{1y}\right) + \\ \qquad \partial_\nu v_y \left(w_{2y}\bar{\mu} + w_{3y}\bar{\nu} + w_{4y}\bar{\mu}^2 + w_{5y}\right) + \displaystyle\sum_{i=1}^{\kappa} q_{y,i} v_i + \bar{F}_y = 0, \\ v_y(x, T, \mu_T, \nu_T) = \bar{G}_y(x, \mu_T, \nu_T), \ y \in \mathcal{Y}. \end{cases} \tag{3.3.2}$$

Furthermore, the optimal control has to admit the feedback form of

$$\hat{\alpha}(t) = -\tilde{b}_2(Y_t, t)\partial_x v(Y_t, \hat{X}_t, t, \mu_t, \nu_t). \tag{3.3.3}$$

Next, we identify what conditions are needed for equating the control problem (P3) and HJB equation. Denote

$$\mathbb{S} = \left\{ v \in C^\infty : \begin{array}{l} (1+|x|^2)^{-1}(|v| + |\partial_t v|) + \\ \qquad (1+|x|)^{-1}(|\partial_x v| + |\partial_\mu v| + |\partial_\nu v|) + |\partial_{xx} v| < K, \\ \qquad \forall (y, x, t, \mu, \nu), \text{ for some } K \end{array} \right\}.$$

Lemma 24. *(Verification theorem) Consider the control problem (P3) with some given smooth w. Suppose there exists a solution $v \in \mathbb{S}$ of (3.3.2). Then, $v_y(x, t, \bar\mu, \bar\nu) = \bar V(y, x, t, \bar\mu, \bar\nu)$ holds, and an optimal control is provided by (3.3.3).*

Proof. We first prove the verification theorem. Since $v \in \mathbb{S}$, for any admissible $\alpha \in L^4_{\mathbb{F}}$, the process X^α is well defined and one can use Dynkin's formula given by Lemma 34 to write

$$\mathbb{E}\left[v(Y_T, X_T, T, \mu_T, \nu_T)\right] = v(y, x, t, \bar\mu, \bar\nu) + \mathbb{E}\left[\int_t^T \mathcal{G}^{\alpha(s)} v(Y_s, X_s, s, \mu_s, \nu_s) ds\right],$$

where

$$\mathcal{G}^a f(y, x, s, \bar\mu, \bar\nu) = \Big(\partial_t + \left(\tilde b_{1y} x + \tilde b_{2y} a\right)\partial_x + \frac{1}{2}\partial_{xx} + \mathcal{Q} + (w_{0y}\bar\mu + w_{1y})\,\partial_{\bar\mu} + \\ \left(w_{2y}\bar\mu + w_{3y}\bar\nu + w_{4y}\bar\mu^2 + w_{5y}\right)\partial_{\bar\nu}\Big) f(y, x, s, \bar\mu, \bar\nu).$$

Note that HJB actually implies that

$$\inf_a \left\{\mathcal{G}^a v + \frac{1}{2}a^2\right\} = -\bar F,$$

which again implies

$$-\mathcal{G}^a v \le \frac{1}{2}a^2 + \bar F.$$

Hence, we obtain that for all $\alpha \in L^4_{\mathbb{F}}$,

$$\begin{aligned} & v(y, x, t, \bar\mu, \bar\nu) \\ &= \mathbb{E}\left[\int_t^T -\mathcal{G}^{\alpha(s)} v(Y_s, X_s, s, \mu_s, \nu_s) ds\right] + \mathbb{E}\left[v(Y_T, X_T, T, \mu_T, \nu_T)\right] \\ &\le \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\alpha^2(s) + \bar F(Y_s, X_s, \mu_s, \nu_s)\right) ds\right] + \mathbb{E}\left[\bar G(Y_T, X_T, \mu_T, \nu_T)\right] \\ &= J(y, x, t, \alpha, \bar\mu, \bar\nu). \end{aligned}$$

In the above, if α is replaced by $\hat\alpha$ given by the feedback form (3.3.3), then since $\partial_x v$ is Lipschitz continuous in x, there exists corresponding optimal path $\hat X \in L^4_{\mathbb{F}}$. Thus, $\hat\alpha$ is also in $L^4_{\mathbb{F}}$. One can repeat all above steps by replacing X and α by $\hat X$ and $\hat\alpha$, and $\le$ sign by $=$ sign to conclude that v is indeed the optimal value.

∎

3.3.2.2 LQG solution

Note that, the costs $\bar{F}$ and $\bar{G}$ of (P3) are quadratic functions in $(x, \bar{\mu}, \bar{\nu})$, while the drift function of the process ν of (3.3.1) is not linear in $(x, \bar{\mu}, \bar{\nu})$. Therefore, the control problem (P3) does not fall into the standard LQG control framework. Nevertheless, similar to the LQG solution, we guess the value function as a quadratic function in the form of

$$v_y(x,t,\bar{\mu},\bar{\nu}) = a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t), \quad y \in \mathcal{Y}. \tag{3.3.4}$$

With the above setup, for $t \in [0,T]$, the optimal control is

$$\hat{\alpha}_t = -\tilde{b}_2(Y_t,t)\partial_x v(Y_t,\hat{X}_t,t,\mu_t,\nu_t) = -\tilde{b}_2(Y_t,t)\left(2a_{Y_t}(t)\hat{X}_t + d_{Y_t}(t) + f_{Y_t}(t)\mu_t\right), \tag{3.3.5}$$

and the optimal path $\hat{X}$ is

$$d\hat{X}_t = \left(\tilde{b}_1(Y_t,t)\hat{X}_t - \tilde{b}_2^2(Y_t,t)\left(2a_{Y_t}(t)\hat{X}_t + d_{Y_t}(t) + f_{Y_t}(t)\mu_t\right)\right)dt + dW_t. \tag{3.3.6}$$

Denote the following ODE systems for $y \in \mathcal{Y}$,

$$\begin{cases} a_y' + 2\tilde{b}_{1y}a_y - 2\tilde{b}_{2y}^2 a_y^2 + \sum_{i=1}^{\kappa} q_{y,i}a_i + h_y = 0, \\ d_y' + \tilde{b}_{1y}d_y - 2\tilde{b}_{2y}^2 a_y d_y + f_y w_{1y} + \sum_{i=1}^{\kappa} q_{y,i}d_i = 0, \\ e_y' - \tilde{b}_{2y}^2 d_y f_y + 2k_y w_{1y} + e_y w_{0y} + b_y w_{2y} + \sum_{i=1}^{\kappa} q_{y,i}e_i = 0, \\ f_y' + \tilde{b}_{1y}f_y - 2\tilde{b}_{2y}^2 a_y f_y + f_y w_{0y} + \sum_{i=1}^{\kappa} q_{y,i}f_i - 2h_y = 0, \\ k_y' - \frac{1}{2}\tilde{b}_{2y}^2 f_y^2 + 2k_y w_{0y} + b_y w_{4y} + \sum_{i=1}^{\kappa} q_{y,i}k_i = 0, \\ b_y' + b_y w_{3y} + \sum_{i=1}^{\kappa} q_{y,i}b_i + h_y = 0, \\ c_y' + a_y - \frac{1}{2}\tilde{b}_{2y}^2 d_y^2 + e_y w_{1y} + b_y w_{5y} + \sum_{i=1}^{\kappa} q_{y,i}c_i = 0, \end{cases} \tag{3.3.7}$$

with terminal conditions

$$\begin{aligned} &a_y(T) = g_y,\ b_y(T) = g_y,\ c_y(T) = 0,\ d_y(T) = 0, \\ &e_y(T) = 0,\ f_y(T) = -2g_y,\ k_y(T) = 0. \end{aligned} \tag{3.3.8}$$

Lemma 25. *Suppose there exists a unique solution $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y})$ to the ODE* *system* (3.3.7)-(3.3.8) *on* $[0, T]$. *Then the value function of (P3) is*

$$\begin{aligned}
&\bar{V}(y, x, t, \bar{\mu}, \bar{\nu}) = v_y(x, t, \bar{\mu}, \bar{\nu}) \\
&= a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t)
\end{aligned} \tag{3.3.9}$$

for $y \in \mathcal{Y}$ *and the optimal control and optimal path are given by* (3.3.5) *and* (3.3.6), *respectively.*

Proof. With the form of value function v_y given in (3.3.4) and the first and second moment of the conditional population density given in (3.3.1), we have

$$\begin{aligned}
&\partial_t v_y = a'_y(t)x^2 + d'_y(t)x + e'_y(t)\bar{\mu} + f'_y(t)x\bar{\mu} + k'_y(t)\bar{\mu}^2 + b'_y(t)\bar{\nu} + c'_y(t), \\
&\partial_x v_y = 2xa_y(t) + d_y(t) + f_y(t)\bar{\mu}, \\
&\partial_{xx} v_y = 2a_y(t), \\
&\partial_{\bar{\mu}} v_y = e_y(t) + f_y(t)x + 2k_y(t)\bar{\mu}, \\
&\partial_{\bar{\nu}} v_y = b_y(t),
\end{aligned}$$

for $y \in \mathcal{Y}$. Plugging them back to the coupled HJBs in (3.3.2), we get a system of ODEs in (3.3.7) by equating x, $\bar{\mu}$, $\bar{\nu}$-like terms in each equation.

Therefore, any solution $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y})$ of ODE system (3.3.7) leads to the solution of HJB (3.3.2) in the form of the quadratic function given by (3.3.9). Since the $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y})$ are differentiable functions on the closed set $[0, T]$, they are also bounded, and the function v meets regularity conditions required by Lemma 24 to conclude the desired result. $\square$

3.3.3 Fixed point condition and the proof of Theorem 20

Going back to the ODE system (3.3.7), there are 7κ equations, while we have total 13κ deterministic functions of $[0, T] \times \mathbb{R}$ to be determined to characterize MFG. Those are

$$(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y \in \mathcal{Y}) \text{ and } (w_{iy} : i = 0, 1, \dots 5,\ y \in \mathcal{Y}).$$

In the following, we identify the missing 6κ equations by checking the fixed point condition:

$$\mu_s = \mathbb{E}\left[\hat{X}_s \middle| Y\right],\ \nu_s = \mathbb{E}\left[\hat{X}_s^2 \middle| Y\right], \quad \forall s \in [0, T], \tag{3.3.10}$$

where μ and ν are two auxiliary processes $(\mu, \nu)[w]$ defined in (3.3.1), see Figure 3.2. This leads to a complete characterization of the equilibrium for the MFG posed by (P2).

Note that based on the dynamic of the optimal $\hat{X}$ defined in (3.3.6), the fixed point condition (3.3.10) implies that the first moment $\hat{\mu}_s := \mathbb{E}[\hat{X}_s \big| Y]$ and the second moment $\hat{\nu}_s := \mathbb{E}[\hat{X}_s^2 \big| Y]$ of the optimal path conditioned on Y satisfy

$$\begin{cases}
\hat{\mu}_s = \bar{\mu} + \displaystyle\int_t^s \Big(\Big(\tilde{b}_1(Y_r, r) - \tilde{b}_2^2(Y_r, r) \left(2a_{Y_r}(r) + f_{Y_r}(r)\right) \Big) \hat{\mu}_r - \tilde{b}_2^2(Y_r, r) d_{Y_r}(r) \Big) dr, \\
\hat{\nu}_s = \bar{\nu} + \displaystyle\int_t^s \Big(1 + 2\tilde{b}_1(Y_r, r)\hat{\nu}_r - \tilde{b}_2^2(Y_r, r) \left(4a_{Y_r}(r)\hat{\nu}_r + 2d_{Y_r}(r)\hat{\mu}_r + 2f_{Y_r}(r)\hat{\mu}_r^2\right) \Big) dr,
\end{cases} \tag{3.3.11}$$

for $s \geq t$. Note that under the optimal control in (3.3.5), comparing the terms in (3.3.1) and (3.3.11), we obtain another 6κ equations:

$$\begin{aligned} &w_{0y} = \tilde{b}_{1y} - 2\tilde{b}_{2y}^2 a_y - \tilde{b}_{2y}^2 f_y,\ w_{1y} = -\tilde{b}_{2y}^2 d_y,\ w_{2y} = -2\tilde{b}_{2y}^2 d_y, \\ &w_{3y} = -4\tilde{b}_{2y}^2 a_y + 2\tilde{b}_{1y},\ w_{4y} = -2\tilde{b}_{2y}^2 f_y,\ w_{5y} = 1, \end{aligned} \tag{3.3.12}$$

for $y \in \mathcal{Y}$. Using further algebraic structures, one can reduce the ODE system of 13κ equations composed by (3.3.7) and (3.3.12) into a system of 4κ equations of the form (3.2.12) for the MFG characterization in Theorem 20.

Proof of Theorem 20. Since a_y $(y \in \mathcal{Y})$ has the same expressions as (3.2.12), its existence, uniqueness and boundedness are shown in Lemma 38. Given a_y $(y \in \mathcal{Y})$ and smooth bounded w's,

$$(b_y, d_y, e_y, f_y : y \in \mathcal{Y})$$

is a coupled linear system, and their existence, uniqueness and boundedness is shown by Theorem 12.1 in [2]. Similarly, given $(b_y, d_y, f_y : y \in \mathcal{Y})$, $(k_y, c_y : y \in \mathcal{Y})$ is a linear system, and their existence and uniqueness is also guaranteed by Theorem 12.1 in [2].

The ODE system (3.3.7) can be rewritten by

$$\begin{cases} a_y' + 2\tilde{b}_{1y} a_y - 2\tilde{b}_{2y}^2 a_y^2 + \sum_{i=1}^{\kappa} q_{y,i} a_i + h_y = 0, \\ d_y' + \tilde{b}_{1y} d_y - 2\tilde{b}_{2y}^2 a_y d_y - \tilde{b}_{2y}^2 f_y d_y + \sum_{i=1}^{\kappa} q_{y,i} d_i = 0, \\ e_y' - \tilde{b}_{2y}^2 d_y f_y - 2\tilde{b}_{2y}^2 k_y d_y + e_y \left(\tilde{b}_{1y} - 2\tilde{b}_{2y}^2 a_y - \tilde{b}_{2y}^2 f_y\right) - 2\tilde{b}_{2y}^2 b_y d_y + \sum_{i=1}^{\kappa} q_{y,i} e_i = 0, \\ f_y' + \tilde{b}_{1y} f_y - 2\tilde{b}_{2y}^2 a_y f_y + f_y \left(\tilde{b}_{1y} - 2\tilde{b}_{2y}^2 a_y - \tilde{b}_{2y}^2 f_y\right) + \sum_{i=1}^{\kappa} q_{y,i} f_i - 2h_y = 0, \\ k_y' - \frac{1}{2}\tilde{b}_{2y}^2 f_y^2 + 2k_y \left(\tilde{b}_{1y} - 2\tilde{b}_{2y}^2 a_y - \tilde{b}_{2y}^2 f_y\right) - 2\tilde{b}_{2y}^2 b_y f_y + \sum_{i=1}^{\kappa} q_{y,i} k_i = 0, \\ b_y' + b_y \left(-4\tilde{b}_{2y}^2 a_y + 2\tilde{b}_{1y}\right) + \sum_{i=1}^{\kappa} q_{y,i} b_i + h_y = 0, \\ c_y' + a_y - \frac{1}{2}\tilde{b}_{2y}^2 d_y^2 - 2\tilde{b}_{2y}^2 d_y e_y + b_y + \sum_{i=1}^{\kappa} q_{y,i} c_i = 0, \end{cases}$$

with the terminal conditions

$$a_y(T) = g_y,\ b_y(T) = g_y,\ c_y(T) = 0,\ d_y(T) = 0,\ e_y(T) = 0,\ f_y(T) = -2g_y,\ k_y(T) = 0.$$

Since a_y, b_y $(y \in \mathcal{Y})$ has the same expressions as (3.2.12), its existence, uniqueness and boundedness are shown in Lemma 38. Meanwhile, with the given $(a_y, b_y : y \in \mathcal{Y})$, we denote $l_y = 2a_y + f_y$, and then

$$l_y' + 2\tilde{b}_{1y} l_y - \tilde{b}_{2y}^2 l_y^2 + \sum_{i=1}^{\kappa} q_{y,i} l_i = 0 \ , \ l_y(T) = 0.$$

By Lemma 36 and Lemma 37 in Appendix, there exists a unique solution for l_y $(y \in \mathcal{Y})$, which is $l_y = 0, y \in \mathcal{Y}$. This gives $f_y = -2a_y$ and $d_y' + \tilde{b}_{1y} d_y + \sum_{i=1}^{\kappa} q_{y,i} d_i = 0$, which implies $d_y = 0, y \in \mathcal{Y}$. Then, the equation for e_y can be simplified as $e_y' + \tilde{b}_{1y} e_y + \sum_{i=1}^{\kappa} q_{y,i} e_i = 0$, which indicates that $e_y = 0, y \in \mathcal{Y}$. For k_y, c_y, with the given of $(a_y, b_y : y \in \mathcal{Y})$, we have

$$k_y' + 2\tilde{b}_{1y} k_y - 2\tilde{b}_{2y}^2 a_y^2 + 4\tilde{b}_{2y}^2 a_y b_y + \sum_{i=1}^{\kappa} q_{y,i} k_i = 0 \ , \ k_y(T) = 0,$$

$$c_y' + a_y + b_y + \sum_{i=1}^{\kappa} q_{y,i} c_i = 0 \ , \ c_y(T) = 0.$$

The existence and uniqueness of the solution for k_y, c_y $(y \in \mathcal{Y})$ are yielded by Theorem 12.1 in [2].

Note that in this case, since $2a_y + f_y = 0$ and $d_y = 0$ for $y \in \mathcal{Y}$, from (3.3.11) we have

$$\hat{\mu}_s = \bar{\mu} + \int_t^s \tilde{b}_1(Y_r, r) \hat{\mu}_r \ dr$$

for all $s \in [t, T]$. Then

$$\hat{\nu}_s = \bar{\nu} + \int_t^s \left(1 + 2\tilde{b}_1(Y_r, r)\hat{\nu}_r - 4\tilde{b}_2^2(Y_r, r) a_{Y_r}(r) \hat{\nu}_r + 4\tilde{b}_2^2(Y_r, r) a_{Y_r}(r) \hat{\mu}_r^2 \right) dr.$$

Plugging $d_y = 0$ for $y \in \mathcal{Y}$ back to (3.3.5), we obtain the optimal control by

$$\hat{\alpha}_s = -2\tilde{b}_2^2(Y_s, s) a_{Y_s}(s) \left(\hat{X}_s - \hat{\mu}_s \right).$$

Since we have $d_y = 0$ for $y \in \mathcal{Y}$, the value function can be simplified from (3.3.4) to

$$v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t) x^2 - 2a_y(t) x \bar{\mu} + k_y(t) \bar{\mu}^2 + b_y(t) \bar{\nu} + c_y(t).$$

By the equivalence Lemma 23, it yields the value function U of Theorem 20 . Moreover, since $f_y = -2a_y$ and $k_y \neq 0$, the ODE system (3.3.7) together with (3.3.12) can be reduced into (3.2.12). From the Lemma 38, the existence and uniqueness of $(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$ in (3.2.12) is guaranteed.

□

3.4 The N-player game and its convergence to MFG

In this section, we show the convergence of the N-player game to MFG. To simplify the presentation, we may omit the superscript (N) for the processes in the probability space $\Omega^{(N)}$, whenever there is no confusion. First, we solve the N-player game in Subsection 3.4.1, which provides a Riccati system consisting of $O(N^3)$ equations. Subsection 3.4.2 reduces the corresponding Riccati system into an ODE system whose dimension is independent of N. This becomes the key building block of the convergence rate obtained in Subsection 3.4.3. To obtain the convergence rate, Subsection 3.4.3 provides an explicit embedding of some processes in $\Omega^{(N)}$ into the probability space Ω. Note that, $\Omega^{(N)}$ is much richer than Ω since $\Omega^{(N)}$ contains N Brownian motions while Ω has only two Brownian motions. Therefore, careful treatment has to be carried out to some processes of our interest, otherwise, such an embedding is in general implausible.

3.4.1 Characterization of the N-player game by Riccati system

The N-player game is indeed an N-coupled stochastic LQG problem by its very own definition, see Subsection 3.2.3. Therefore, the solution can be derived via Riccati system with the existing LQG theory given below: For $i = 1, 2, \ldots, N$, $y \in \mathcal{Y}$,

$$
\begin{cases}
A'_{iy} + 2\tilde{b}_{1y} e_i e_i^\top A_{iy} - 2\tilde{b}_{2y}^2 A_{iy}^\top e_i e_i^\top A_{iy} + \sum_{j\neq i}^{N} \left(2\tilde{b}_{1y} e_j e_j^\top A_{iy} - 4\tilde{b}_{2y}^2 A_{jy}^\top e_j e_j^\top A_{iy} \right) \\
\qquad\qquad + \sum_{j=1}^{\kappa} q_{y,j} A_{ij} + \frac{h_y}{N} \sum_{j\neq i}^{N} (e_i - e_j)(e_i - e_j)^\top = 0, \\
B'_{iy} + \sum_{j\neq i}^{N} \left(\tilde{b}_{1y} e_j e_j^\top B_{iy} - 2\tilde{b}_{2y}^2 A_{iy}^\top e_j e_j^\top B_{jy} - 2\tilde{b}_{2y}^2 A_{jy}^\top e_j e_j^\top B_{iy} \right) \\
\qquad\qquad + \tilde{b}_{1y} e_i e_i^\top B_{iy} - 2\tilde{b}_{2y}^2 A_{iy}^\top e_i e_i^\top B_{iy} + \sum_{j=1}^{\kappa} q_{y,j} B_{ij} = 0, \\
C'_{iy} - \frac{1}{2}\tilde{b}_{2y}^2 B_{iy}^\top e_i e_i^\top B_{iy} - \sum_{j\neq i}^{N} \tilde{b}_{2y}^2 B_{jy}^\top e_j e_j^\top B_{iy} + \sum_{j=1}^{N} tr(A_{jy}) + \sum_{j=1}^{\kappa} q_{y,j} C_{ij} = 0, \\
A_{iy}(T) = \frac{g_y}{N}\Lambda_i, \ B_{iy}(T) = 0 \cdot \mathbb{1}_N, \ C_{iy}(T) = 0,
\end{cases}
\tag{3.4.1}
$$

where the solutions consist of $N \times N$ symmetric matrices A_{iy}'s, N-dimensional vectors B_{iy}'s, and $C_{iy} \in \mathbb{R}$. In the above, $\mathbb{1}_N$ is the N-dimensional vector with all entries are 1, Λ_i's are $N \times N$ matrices with diagonal 1 except $(\Lambda_i)_{ii} = N - 1$, $(\Lambda_i)_{ij} = (\Lambda_i)_{ji} = -1$ for any $j \neq i$ and the rest entries as 0, and e_i's are the N-dimensional natural basis.

Lemma 26. *Suppose $(A_{iy}, B_{iy}, C_{iy} : i = 1, 2, \ldots, N, \ y \in \mathcal{Y})$ is the solution of (3.4.1). Then, the value functions of the N-player game defined by (3.2.9) are*

$$
V_i(y, x^{(N)}) = (x^{(N)})^\top A_{iy}(0) x^{(N)} + (x^{(N)})^\top B_{iy}(0) + C_{iy}(0), \quad i = 1, 2, \ldots, N.
$$

Moreover, the path and the control under the equilibrium are

$$
d\hat{X}_{it} = \left(\tilde{b}_1(Y_t, t)\hat{X}_{it} - \tilde{b}_2^2(Y_t, t) \left(2(A_{iY_t})_i^\top \hat{X}_t + (B_{iY_t})_i \right) \right) dt + dW_{it}, \quad i = 1, 2, \ldots, N, \tag{3.4.2}
$$

and

$$
\hat{\alpha}_{it} = -\tilde{b}_2(Y_t, t) \left(2(A_{iY_t})_i^\top \hat{X}_t + (B_{iY_t})_i \right),
$$

where $(A)_i$ denotes the i-th column of matrix A, $(B)_i$ denotes the i-th entry of vector B and $\hat{X}_t = [\hat{X}_{1t}, \hat{X}_{2t}, \ldots, \hat{X}_{Nt}]^\top$.

Proof. It is standard that, under enough regularities, the value function $V(y, x^{(N)}) = (V_1, V_2, \ldots, V_N)(y, x^{(N)})$ of the N-player game can be lifted to the solution $v_{iy}(x^{(N)}, t)$ of the following system of HJB equations, for $i = 1, 2, \ldots, N$ and $y \in \mathcal{Y}$,

$$\begin{cases} \partial_t v_{iy} + \tilde{b}_{1y} x_i \partial_i v_{iy} - \dfrac{1}{2}\left(\tilde{b}_{2y}\partial_i v_{iy}\right)^2 + \displaystyle\sum_{j\neq i}^{N}\left(\tilde{b}_{1y}x_j - \tilde{b}_{2y}^2\partial_j v_{jy}\right)\partial_j v_{iy} \\ \qquad\qquad + \dfrac{1}{2}\Delta v_{iy} + \displaystyle\sum_{j=1}^{\kappa} q_{y,j} v_{ij} + \dfrac{h_y}{N}\sum_{j\neq i}^{N}\left(\left(e_i - e_j\right)^\top x^{(N)}\right)^2 = 0, \\ v_{iy}(x^{(N)}, T) = \dfrac{g_y}{N}\displaystyle\sum_{j\neq i}^{N}\left((e_i - e_j)^\top x^{(N)}\right)^2. \end{cases} \tag{3.4.3}$$

Then, the value functions V of the N-player game defined by (3.2.9) is $V_i(y, x^{(N)}) = v_{iy}(x^{(N)}, 0)$ for all $i = 1, 2, \ldots, N$. Moreover, the path and the control under the equilibrium are

$$d\hat{X}_{it} = \left(\tilde{b}_1(Y_t, t)\hat{X}_{it} - \tilde{b}_2^2(Y_t, t)\partial_i v_{iY_t}(\hat{X}_t, t)\right)dt + dW_{it}, \quad i = 1, 2, \ldots, N,$$

and

$$\hat{\alpha}_{it} = -\tilde{b}_2(Y_t, t)\partial_i v_{iY_t}(\hat{X}_t, t).$$

The proof is the application of Dynkin's formula and the details are omitted here. Due to its LQG structure, the value function leads to a quadratic function of the form

$$v_{iy}(x^{(N)}, t) = (x^{(N)})^\top A_{iy}(t) x^{(N)} + (x^{(N)})^\top B_{iy}(t) + C_{iy}(t).$$

For each $i = 1, 2, \ldots, N$, after plugging V_{iy} into (3.4.3), and matching the coefficient of variables, we get the desired results. $\square$

3.4.2 Reduced Riccati form for the equilibrium

So far, the N-player game and MFG have been characterized by Lemma 26 and Theorem 20, respectively. One of our main objectives is to investigate the convergence of the generic optimal path $\hat{X}_{1t}^{(N)}$ of the N-player game generated (3.4.1)-(3.4.2) to the optimal path $\hat{X}_t$ of MFG generated by (3.2.12)-(3.2.13).

Note that $\hat{X}_t$ relies only on κ functions $(a_y : y \in \mathcal{Y})$ from the simple ODE system (3.2.12) while $\rho(\hat{X}_t^{(N)})$ depends on $O(N^3)$ functions from $(A_{iy} : i = 1, 2, \ldots, N,\ y \in \mathcal{Y})$ solved from a huge Riccati system (3.4.1). Therefore, it is almost a hopeless task for a meaningful comparison between these two processes without gaining further insight into the complex structure of the Riccati system (3.4.1).

To proceed, let us first observe some hidden patterns from a numerical result for the solution of Riccati (3.4.1). The following matrix shows A_{20} at $t = 1$ for $N = 5$ with the same parameters as in Figure 3.3 and Figure 3.4 in Section 3.5.1:

$$A_{20}(1) = \begin{bmatrix} 0.1319 & -0.1924 & 0.0202 & 0.0202 & 0.0202 \\ -0.1924 & 0.7696 & -0.1924 & -0.1924 & -0.1924 \\ 0.0202 & -0.1924 & 0.1319 & 0.0202 & 0.0202 \\ 0.0202 & -0.1924 & 0.0202 & 0.1319 & 0.0202 \\ 0.0202 & -0.1924 & 0.0202 & 0.0202 & 0.1319 \end{bmatrix}.$$

Interestingly enough, we observe that the entire 25 entries of $A_{20}(1)$ indeed consists of 4 distinct values. Moreover, similar computation with different values of N only yields a larger table depending on N, but always consists of 4 values. Inspired by this accidental discovery from the above numerical example, we may want to believe and prove a pattern of the matrix A_{iy} in the following form:

$$(A_{iy})_{pq} = \begin{cases} a_{1y}(t), & \text{if } p = q = i, \\ a_{2y}(t), & \text{if } p = q \neq i, \\ a_{3y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ a_{4y}(t), & \text{otherwise,} \end{cases} \tag{3.4.4}$$

for $y \in \mathcal{Y}$. The next result justifies the above pattern: the N^2 entries of the matrix A_{iy} can be embedded to a 2κ-dimensional vector space no matter how big N is.

Lemma 27. *There exists a unique solution (a^N_{1y}, a^N_{2y}) from the ODE system(3.4.5)*

$$\begin{cases} a'_{1y} + 2\tilde{b}_{1y}a_{1y} - \dfrac{2(N+1)}{N-1}\tilde{b}^2_{2y}a^2_{1y} + \displaystyle\sum_{j=1}^{\kappa} q_{y,j}a_{1j} + \dfrac{N-1}{N}h_y = 0, \\ a'_{2y} + 2\tilde{b}_{1y}a_{2y} + \dfrac{2}{(N-1)^2}\tilde{b}^2_{2y}a^2_{1y} - \dfrac{4N}{N-1}\tilde{b}^2_{2y}a_{1y}a_{2y} + \displaystyle\sum_{j=1}^{\kappa} q_{y,j}a_{2j} + \dfrac{h_y}{N} = 0, \\ a_{1y}(T) = \dfrac{N-1}{N}g_y, \ a_{2y}(T) = \dfrac{g_y}{N}, \end{cases} \tag{3.4.5}$$

for $y \in \mathcal{Y}$. Moreover, the path and the control of player i under the equilibrium are

$$d\hat{X}^{(N)}_{it} = \left(\tilde{b}_1(Y^{(N)}_t, t)\hat{X}^{(N)}_{it} - 2\tilde{b}^2_2(Y^{(N)}_t, t)a^N_{1Y^{(N)}_t}(t)\left(\hat{X}^{(N)}_{it} - \frac{1}{N-1}\sum_{j\neq i}^{N}\hat{X}^{(N)}_{jt}\right)\right)dt + dW^{(N)}_{it}, \tag{3.4.6}$$

and

$$\hat{\alpha}^{(N)}_{it} = -2\tilde{b}_2(Y^{(N)}_t, t)a^N_{1Y^{(N)}_t}(t)\left(\hat{X}^{(N)}_{it} - \frac{1}{N-1}\sum_{j\neq i}^{N}\hat{X}^{(N)}_{jt}\right)$$

for $i = 1, 2, \ldots, N$.

Proof. It is obvious to see that in the Riccati system (3.4.1), $B_{iy} = 0$ for all $i = 1, 2, \ldots, N$ and

$y \in \mathcal{Y}$. Note that in this case, for $i = 1, 2, \ldots, N$, the optimal control is given by

$$\hat{\alpha}_i^{(N)} = -2\tilde{b}_2(Y_t^{(N)}, t) \sum_{j=1}^{N} (A_{iY_t^{(N)}})_{ij} \hat{X}_{jt}^{(N)} = -2\tilde{b}_2(Y_t^{(N)}, t) \left(A_{iY_t^{(N)}}\right)_i^\top \hat{X}_t^{(N)}.$$

Plugging the pattern (3.4.4) into the differential equation of A_{iy}, we have

$$a'_{1y} + 2\tilde{b}_{1y}a_{1y} - 2\tilde{b}_{2y}^2 a_{1y}^2 - 4(N-1)\tilde{b}_{2y}^2 a_{3y}^2 + \sum_{j=1}^{\kappa} q_{y,j}a_{1j} + \frac{N-1}{N}h_y = 0,$$

$$a'_{2y} + 2\tilde{b}_{1y}a_{2y} - 2\tilde{b}_{2y}^2 a_{3y}^2 - 4\tilde{b}_{2y}^2 \left(a_{1y}a_{2y} + (N-2)a_{3y}a_{4y}\right) + \sum_{j=1}^{\kappa} q_{y,j}a_{2j} + \frac{h_y}{N} = 0,$$

$$a'_{3y} + 2\tilde{b}_{1y}a_{3y} - 2\tilde{b}_{2y}^2 a_{1y}a_{3y} - 4\tilde{b}_{2y}^2 \left(a_{1y}a_{3y} + (N-2)a_{3y}^2\right) + \sum_{j=1}^{\kappa} q_{y,j}a_{3j} - \frac{h_y}{N} = 0,$$

$$a'_{3y} + 2\tilde{b}_{1y}a_{3y} - 2\tilde{b}_{2y}^2 a_{1y}a_{3y} - 4\tilde{b}_{2y}^2 \left(a_{2y}a_{3y} + (N-2)a_{3y}a_{4y}\right) + \sum_{j=1}^{\kappa} q_{y,j}a_{3j} - \frac{h_y}{N} = 0,$$

$$a'_{4y} + 2\tilde{b}_{1y}a_{4y} - 2\tilde{b}_{2y}^2 a_{3y}^2 - 4\tilde{b}_{2y}^2 \left(a_{2y}a_{3y} + a_{1y}a_{4y} + (N-3)a_{3y}a_{4y}\right) + \sum_{j=1}^{\kappa} q_{y,j}a_{4j} = 0,$$

which gives $a_{1y} + (N-2)a_{3y} = a_{2y} + (N-2)a_{4y}$ since two expressions for a_{3y} should be identical. This implies that $(a_{1y} + (N-2)a_{3y})' = (a_{2y} + (N-2)a_{4y})'$ or

$$\begin{aligned}
& - 2\tilde{b}_{1y}a_{1y} + 2\tilde{b}_{2y}^2 a_{1y}^2 + 4(N-1)\tilde{b}_{2y}^2 a_{3y}^2 - \frac{N-1}{N}h_y - \sum_{j=1}^{\kappa} q_{y,j}a_{1j} \\
& + (N-2)\Big(- 2\tilde{b}_{1y}a_{3y} + 2\tilde{b}_{2y}^2 a_{1y}a_{3y} + 4\tilde{b}_{2y}^2 \left(a_{2y}a_{3y} + (N-2)a_{3y}a_{4y}\right) - \sum_{j=1}^{\kappa} q_{y,j}a_{3j} + \frac{h_y}{N}\Big) \\
= & - 2\tilde{b}_{1y}a_{2y} + 2\tilde{b}_{2y}^2 a_{3y}^2 + 4\tilde{b}_{2y}^2 \left(a_{1y}a_{2y} + (N-2)a_{3y}a_{4y}\right) - \sum_{j=1}^{\kappa} q_{y,j}a_{2j} - \frac{h_y}{N} \\
& + (N-2)\Big(- 2\tilde{b}_{1y}a_{4y} + 2\tilde{b}_{2y}^2 a_{3y}^2 + 4\tilde{b}_{2y}^2 \left(a_{1y}a_{4y} + a_{2y}a_{3y} + (N-3)a_{3y}a_{4y}\right) - \sum_{j=1}^{\kappa} q_{y,j}a_{4j}\Big).
\end{aligned}$$

After combining terms and substituting $a_{2y} + (N-2)a_{4y}$ with $a_{1y} + (N-2)a_{3y}$, we get $a_{1y}^2 + (N-2)a_{1y}a_{3y} - (N-1)a_{3y}^2 = 0$, which yields $a_{3y} = a_{1y}$ or $a_{3y} = -\frac{1}{N-1}a_{1y}$. Note that $a_{3y} \neq a_{1y}$ due to their different differential equations. Hence, we can conclude that $a_{3y} = -\frac{1}{N-1}a_{1y}$. In conclusion, for $i = 1, 2, \ldots, N$, A_{iy} $(y \in \mathcal{Y})$ has the following expressions:

$$(A_{iy})_{pq} = \begin{cases} a_{1y}(t), & \text{if } p = q = i, \\ a_{2y}(t), & \text{if } p = q \neq i, \\ -\dfrac{1}{N-1} a_{1y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ \dfrac{1}{(N-1)(N-2)} a_{1y}(t) - \dfrac{1}{N-2} a_{2y}(t), & \text{otherwise.} \end{cases}$$

The existence and uniqueness of (3.4.1) is equivalent to the existence and uniqueness of (3.4.5). For a_{1y}, the existence and uniqueness can be deduced from Lemma 36 and 37. Given a_{1y}'s, a_{2y}'s are linear equations, thus their existence and uniqueness are guaranteed by Theorem 12.1 in [2]. Together with previous discussions, we conclude the results. □

3.4.3 Convergence

Based on the current progress, let us reiterate our goal (P1) for the convergence. Our objective is the convergence of the joint distribution $\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)})$ of the N-player game generated (3.4.5)-(3.4.6) in the probability space $\Omega^{(N)}$ to the distribution $\mathcal{L}(\hat{X}_t, Y_t)$ of MFG generated by (3.2.12)-(3.2.13) in the probability space Ω. More precisely, we want to find a number $\eta > 0$ satisfying

$$\mathbb{W}_2 \left(\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)}), \mathcal{L}(\hat{X}_t, Y_t) \right) = O\left(N^{-\eta}\right), \tag{3.4.7}$$

where $\mathbb{W}_2$ is the 2-Wasserstein metric. This procedure is given in the following two steps:

1. We will construct a process Z^N in the probability space Ω, who provides exact copy of the joint distribution in the sense of

$$\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)}) = \mathcal{L}(Z_t^N, Y_t), \quad \forall t \in [0, T].$$

   Note that, the (3.4.6) shows that $\hat{X}_{1t}^{(N)}$ correlates to N many Brownian motions $\{W_i^{(N)} : i = 1, 2, \ldots, N\}$ from a much richer space $\Omega^{(N)}$ while Ω is a much smaller space having only two Brownian motions W and B. Therefore, such an embedding essentially requires to represent $\hat{X}_{1t}^{(N)}$ by two independent Brownian motions and is in general not possible. However, due to the symmetric structure of MFG (or the nature of the mean field effect), the embedding is possible and the details are provided in Lemma 28.

2. By Proposition 16, we can use distribution copy (Z^N, Y) in Ω to write

$$\mathbb{W}_2^2 \left(\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)}), \mathcal{L}(\hat{X}_t, Y_t) \right) \leq \mathbb{E}\left[\left| Z_t^N - \hat{X}_t \right|^2 \right]. \tag{3.4.8}$$

   To obtain the estimate of the above right hand side, we shall compare the (3.4.9) of Z^N and (3.2.13) of $\hat{X}$, and it becomes essential to obtain the convergence rate of the ODE system (3.4.5) towards the ODE system (3.2.12). The details are provided in Lemma 29.

Lemma 28. *Let $\{X_0^i : i \in \mathbb{N}\}$ be i.i.d. random variables in Ω independent to (W, B, Y) with $X_0^1 = X_0$. Let Z^N be the solution of*

$$Z_t^N = X_0 + \int_0^t \tilde{b}_1(Y_s, s) Z_s^N ds - \int_0^t 2\tilde{b}_2^2(Y_s, s)\hat{a}_{1Y_s}^N(s)\left(Z_s^N - \bar{X}_s^N\right) ds + W_t, \tag{3.4.9}$$

where

$$d\bar{X}_t^N = \tilde{b}_1(Y_t, t)\bar{X}_t^N dt + \frac{\sqrt{N-1}}{N} dB_t + \frac{1}{N} dW_t, \quad \bar{X}_0^N = \frac{1}{N}\sum_{i=1}^N X_0^i,$$

and

$$\hat{a}_{1y}^N = \frac{N}{N-1} a_{1y}^N.$$

In the above, a_{1y}^N is from the ODE system(3.4.5). Then, (Z_t^N, Y_t) in $(\Omega, \mathcal{F}_T, \mathbb{P})$ has the same distribution as $(\hat{X}_{1t}^{(N)}, Y_t^{(N)})$ in $(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)})$.

Proof. Continued from the Lemma 27, player i's path in the N-player game follows

$$\begin{aligned}\hat{X}_{it}^{(N)} = x_i^{(N)} &+ \int_0^t \tilde{b}_1(Y_s^{(N)}, s)\hat{X}_{is}^{(N)} ds \\ &- \int_0^t 2\tilde{b}_2^2(Y_s^{(N)}, s) a_{1Y_s^{(N)}}^N(s)\left(\hat{X}_{is}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^N \hat{X}_{js}^{(N)}\right) ds + W_{it}^{(N)}.\end{aligned}$$

With the notation

$$\bar{X}_s^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{X}_{is}^{(N)},$$

one can rewrite the path by

$$\begin{aligned}\hat{X}_{it}^{(N)} = x_i^{(N)} &+ \int_0^t \tilde{b}_1(Y_s^{(N)}, s)\hat{X}_{is}^{(N)} ds \\ &- \int_0^t 2\tilde{b}_2^2(Y_s^{(N)}, s)\hat{a}_{1Y_s^{(N)}}^N(s)\left(\hat{X}_{is}^{(N)} - \bar{X}_s^{(N)}\right) ds + W_{it}^{(N)}.\end{aligned} \tag{3.4.10}$$

By adding up the above equations (3.4.10) indexed by $i = 1$ to N, one can have

$$\begin{aligned}\bar{X}_t^{(N)} &= \bar{x}^{(N)} + \int_0^t \tilde{b}_1(Y_s^{(N)}, s)\bar{X}_s^{(N)} ds + \frac{1}{N}\sum_{i=1}^N W_{it}^{(N)} \\ &= \bar{x}^{(N)} + \int_0^t \tilde{b}_1(Y_s^{(N)}, s)\bar{X}_s^{(N)} ds + \frac{\sqrt{N-1}}{N}\left(\sqrt{N-1}\bar{W}_{-it}^{(N)}\right) + \frac{1}{N} W_{it}^{(N)},\end{aligned} \tag{3.4.11}$$

where $\bar{W}_{-it}^{(N)} := \frac{1}{N-1}\sum_{j\neq i} W_{jt}^{(N)}$.

Next, we define solution maps of (3.4.10) and (3.4.11):

$$\bar{G}_t(x, \phi, W_1, W_2) = \mathcal{E}_t(\phi)\left(x + \int_0^t \mathcal{E}_s(-\phi)d(W_{1s} + W_{2s})\right) \tag{3.4.12}$$

and

$$G_t(x, \phi_1, \phi_2, \phi_3, W) = x\mathcal{E}_t(\phi_1 - \phi_2) + \mathcal{E}_t(\phi_1 - \phi_2)\int_0^t \mathcal{E}_s(-\phi_1 + \phi_2)\left(\phi_2(s)\phi_3(s)ds + dW_s\right), \tag{3.4.13}$$

where

$$\mathcal{E}_t(\phi) = \exp\left\{\int_0^t \phi_s ds\right\}.$$

Now, we can rewrite $\bar{X}_t^{(N)}$ of (3.4.11) and $\hat{X}_{1t}^{(N)}$ of (3.4.10) as

$$\bar{X}_t^{(N)} = \bar{G}_t\left(\frac{1}{N}\sum_{i=1}^N x_i^{(N)}, \tilde{b}_1(Y_\cdot^{(N)}, \cdot), \frac{\sqrt{N-1}}{N}\left(\sqrt{N-1}\bar{W}_{-1}^{(N)}\right), \frac{1}{N}W_1^{(N)}\right),$$

and

$$\hat{X}_{1t}^{(N)} = G_t\left(x_1^{(N)}, \tilde{b}_1(Y_\cdot^{(N)}, \cdot), 2\tilde{b}_2(Y_\cdot^{(N)}, \cdot)\hat{a}_1^N(Y_\cdot^{(N)}, \cdot), \bar{X}^{(N)}(\cdot), W_1^{(N)}\right)$$

Meanwhile, $(Z^N, \bar{X}^N)$ of (3.4.9) can also be written in the form of

$$\bar{X}_t^N = \bar{G}_t\left(\frac{1}{N}\sum_{i=1}^N X_0^i, \tilde{b}_1(Y_\cdot, \cdot), \frac{\sqrt{N-1}}{N}B, \frac{1}{N}W\right),$$

and

$$Z_t^N = G_t\left(X_0, \tilde{b}_1(Y_\cdot, \cdot), 2\tilde{b}_2(Y_\cdot, \cdot)\hat{a}_1^N(Y_\cdot, \cdot), \bar{X}^N(\cdot), W\right) \tag{3.4.14}$$

Finally, the fact that the distribution of (Z^N, Y) in the space Ω is identical distribution to $(\hat{X}_1^{(N)}, Y^{(N)})$ in $\Omega^{(N)}$ comes from the followings:

- $\tilde{b}_1, \tilde{b}_2, \hat{a}_1^N$ are deterministic functions.
- The random processes $(\sqrt{N-1}\bar{W}_{-1}^{(N)}, W_1^{(N)}, Y^{(N)})$ are independent mutually in $\Omega^{(N)}$, while the random elements (B, W, Y) are also independent triples. Moreover, two random triples have identical joint distributions.
- Initial states are generated from identical joint distributions $\{x_i^{(N)} : i = 1, 2, \dots, N\}$ and $\{X_0^i : i = 1, 2, \dots, N\}$.

Therefore, (Z^N, Y) and $(\hat{X}_1^{(N)}, Y^{(N)})$ have the same distributions. This completes the proof. $\square$

In view of (3.4.8), we shall estimate the second moment $\mathbb{E}[|Z_t^N - \hat{X}_t|^2]$. First, we can rewrite $\hat{X}$ of (3.2.13) using above representations via G_t:

$$\hat{X}_t = G_t\left(X_0, \tilde{b}_1(Y_\cdot, \cdot), 2\tilde{b}_2(Y_\cdot, \cdot)a(Y_\cdot, \cdot), \hat{\mu}(\cdot), W\right),$$

which leads to a better comparison with Z^N in the form of (3.4.14). To proceed, a Lipschitz properties of G_t are useful for the estimate of the second moment, whose proof is relegated to the Appendix 3.7.3. Throughout the proof of the next lemma, we will use K in various places as a generic constant which varies from line to line.

Lemma 29. *The convergence rate under the Wasserstein metric $\mathbb{W}_2(\cdot,\cdot)$ is*

$$\mathbb{W}_2\left(\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)}), \mathcal{L}(\hat{X}_t, Y_t)\right) = O\left(N^{-\frac{1}{2}}\right).$$

Proof. In view of (3.4.8), we start with

$$\begin{aligned}
&\mathbb{W}_2^2\left(\mathcal{L}(\hat{X}_{1t}^{(N)}, Y_t^{(N)}), \mathcal{L}(\hat{X}_t, Y_t)\right) \leq \mathbb{E}\left[\left|Z_t^N - \hat{X}_t\right|^2\right] \\
&= \mathbb{E}\left[\left|G_t\left(X_0, \tilde{b}_1(Y_\cdot,\cdot), 2\tilde{b}_2(Y_\cdot,\cdot)\hat{a}_1^N(Y_\cdot,\cdot), \bar{X}^N(\cdot), W\right) - G_t\left(X_0, \tilde{b}_1(Y_\cdot,\cdot), 2\tilde{b}_2(Y_\cdot,\cdot)a(Y_\cdot,\cdot), \hat{\mu}(\cdot), W\right)\right|^2\right] \\
&:= \mathbb{E}\left[|I_1(t) - I_2(t)|^2\right].
\end{aligned}$$

Applying the Lipschitz continuity of $(\phi_2,\phi_3) \mapsto G_t(x,\phi_1,\phi_2,\phi_3,W)$ by Appendix 3.7.3 on the conditional expectation $\mathbb{E}\left[|I_1(t) - I_2(t)|^2\middle| Y\right]$, we have

$$\begin{aligned}
\mathbb{E}\left[|Z_t^N - \hat{X}_t|^2\right] &\leq K\mathbb{E}\left[\sup_{0\leq t\leq T}\left(2\tilde{b}_2(Y_t,t)\hat{a}_{1Y_t}^N(t) - 2\tilde{b}_2(Y_t,t)a_{Y_t}(t)\right)^2 + \sup_{0\leq t\leq T}\left(\bar{X}^N(t) - \hat{\mu}(t)\right)^2\right] \\
&\leq K\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{b}_2(Y_t,t)\right|^2 \sup_{0\leq t\leq T}\left|\hat{a}_{1Y_t}^N(t) - a_{Y_t}(t)\right|^2 + \sup_{0\leq t\leq T}\left|\bar{X}^N(t) - \hat{\mu}(t)\right|^2\right] \\
&\leq K\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\hat{a}_{1Y_t}^N(t) - a_{Y_t}(t)\right|^2 + \sup_{0\leq t\leq T}\left|\bar{X}^N(t) - \hat{\mu}(t)\right|^2\right].
\end{aligned}$$

From the dynamic of $\bar{X}^N$ and $\hat{\mu}$,

$$\begin{cases}
d\left(\bar{X}_t^N - \hat{\mu}_t\right) = \tilde{b}_1(Y_t,t)\left(\bar{X}_t^N - \hat{\mu}_t\right)dt + \dfrac{\sqrt{N-1}}{N}dB_t + \dfrac{1}{N}dW_t, \\
\bar{X}_0^N - \hat{\mu}_0 = \dfrac{1}{N}\displaystyle\sum_{i=1}^N X_0^i - \hat{\mu}_0,
\end{cases}$$

which can be written in terms of $\bar{G}_t$ of (3.4.12):

$$\bar{X}^N(t) - \hat{\mu}(t) = \bar{G}_t\left(\frac{1}{N}\sum_{i=1}^N X_0^i - \hat{\mu}_0, \tilde{b}_1(Y_\cdot,\cdot), \frac{\sqrt{N-1}}{N}B, \frac{1}{N}W\right).$$

Using the fact of $\left|\tilde{b}_{1y}\right|_\infty < \infty$ and Ito's isometry, this yields the following estimation:

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\bar{X}^N(t)-\hat{\mu}(t)\right|^2\right] \le K\left(\frac{1}{N}+\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N X_0^i-\hat{\mu}_0\right|^2\right]\right).$$

Note that, by central limit theorem, we have

$$N\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N X_0^i-\hat{\mu}_0\right|^2\right]=\mathbb{E}\left[\left|\frac{\sum_{i=1}^N(X_0^i-\hat{\mu}_0)}{\sqrt{N}}\right|^2\right]\to Var(X_0^1)<\infty,\quad N\to\infty,$$

and we conclude that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\bar{X}^N(t)-\hat{\mu}(t)\right|^2\right]=O(N^{-1}). \tag{3.4.15}$$

Next we investigate the boundness of

$$\sup_{0\le t\le T}\left|\hat{a}^N_{1Y_t}(t)-a_{Y_t}(t)\right|^2.$$

From (3.4.5) and $\hat{a}^N_{1y}=\frac{N}{N-1}a^N_{1y}$, we have

$$\begin{cases}\left(\hat{a}^N_{1y}\right)'+2\tilde{b}_{1y}\hat{a}^N_{1y}-\dfrac{2(N+1)}{N}\tilde{b}^2_{2y}\left(\hat{a}^N_{1y}\right)^2+\displaystyle\sum_{i=1}^{\kappa}q_{y,i}\hat{a}^N_{1i}+h_y=0\\ \hat{a}^N_{1y}(T)=g_y.\end{cases}$$

Define $u_y=a_y-\hat{a}^N_{1y}$, let $\tau=T-t$ and denote $u_y(\tau):=u_y(T-t)$, we have

$$\begin{cases}u_y'(\tau)=2\tilde{b}_{1y}(\tau)u_y(\tau)-2\tilde{b}^2_{2y}(\tau)\left(a_y(\tau)+\hat{a}^N_{1y}(\tau)\right)u_y(\tau)\\ \qquad\qquad+\dfrac{2}{N}\tilde{b}^2_{2y}(\tau)\left(\hat{a}^N_{1y}(\tau)\right)^2+\displaystyle\sum_{i=1}^{\kappa}q_{y,i}u_i(\tau)\\ u_y(0)=0,\end{cases} \tag{3.4.16}$$

which gives that

$$u_y(\tau)=\int_0^\tau\left(2\tilde{b}_{1y}(s)u_y(s)-2\tilde{b}^2_{2y}(s)\left(a_y(s)+\hat{a}^N_{1y}(s)\right)u_y(s)+\frac{2}{N}\tilde{b}^2_{2y}(s)\left(\hat{a}^N_{1y}(s)\right)^2+\sum_{i=1}^{\kappa}q_{y,i}u_i(s)\right)ds.$$

Thus for $\tau\in[0,T]$,

$$\begin{aligned}|u_y(\tau)|\le\int_0^\tau\Bigg(&2\left|\tilde{b}_{1y}\right|_\infty|u_y(s)|+2\left|\tilde{b}_{2y}\right|^2_\infty\left(|a_y|_\infty+\left|\hat{a}^N_{1y}\right|_\infty\right)|u_y(s)|\\ &+\frac{2}{N}\left|\tilde{b}_{2y}\right|^2_\infty\left|\hat{a}^N_{1y}\right|^2_\infty+\sum_{i=1}^{\kappa}|q_{y,i}||u_i(s)|\Bigg)ds.\end{aligned}$$

Let $\left(\left|\tilde{b}_{1y}\right|_{\infty}, \left|\tilde{b}_{2y}\right|_{\infty}, |a_y|_{\infty}, \left|\hat{a}_{1y}^N\right|_{\infty}, \sup_{i\in\mathcal{Y}} |q_{y,i}|\right) \leq K_1$, then

$$|u_y(\tau)| \leq \frac{2}{N} K_1^4 T + \int_0^\tau \left(\left(2K_1 + 4K_1^3\right) |u_y(s)| + K_1 \sum_{i=1}^{\kappa} |u_i(s)| \right) ds.$$

By adding up the above equation indexed by $y = 1$ to κ, one can have

$$\sum_{y=1}^{\kappa} |u_y(\tau)| \leq \frac{2\kappa K_1^4 T}{N} + \left(2K_1 + 4K_1^3 + \kappa K_1\right) \int_0^\tau \sum_{y=1}^{\kappa} |u_y(s)| ds.$$

Let $K_2 = 2\kappa K_1^4 T$ and $K_3 = 2K_1 + 4K_1^3 + \kappa K_1$, by the Grönwall's inequality,

$$\sum_{y=1}^{\kappa} |u_y(\tau)| \leq \frac{K_2}{N} e^{K_3 \tau} \leq \frac{K_2}{N} e^{K_3 T}, \quad \forall \tau \in [0, T],$$

which implies that

$$\sum_{y=1}^{\kappa} |u_y(\tau)| \leq \frac{K}{N}, \quad \forall \tau \in [0, T].$$

Thus, we have

$$\sup_{0\leq t\leq T} \left|\hat{a}_{1Y_t}^N(t) - a_{Y_t}(t)\right|^2 \leq \frac{K}{N^2}, \quad \text{almsot surely.} \tag{3.4.17}$$

Therefore, the convergence is obtained from (3.4.15) and (3.4.17):

$$\mathbb{W}_2^2 \left(\mathcal{L}(Z_t^N), \mathcal{L}(\hat{X}_t)\right) \leq K\mathbb{E} \left[\sup_{0\leq t\leq T} \left|\hat{a}_{1Y_t}^N(t) - a_{Y_t}(t)\right|^2 + \sup_{0\leq t\leq T} \left|\bar{X}^N(t) - \hat{\mu}(t)\right|^2 \right] = O(N^{-1}).$$

□

3.5 Numerical results

3.5.1 Simulations of Riccati system, the value function and optimal control of the generic palyer

We have derived a 4κ dimensional Riccati ODE system (3.2.12) to determine the parameter functions

$$(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$$

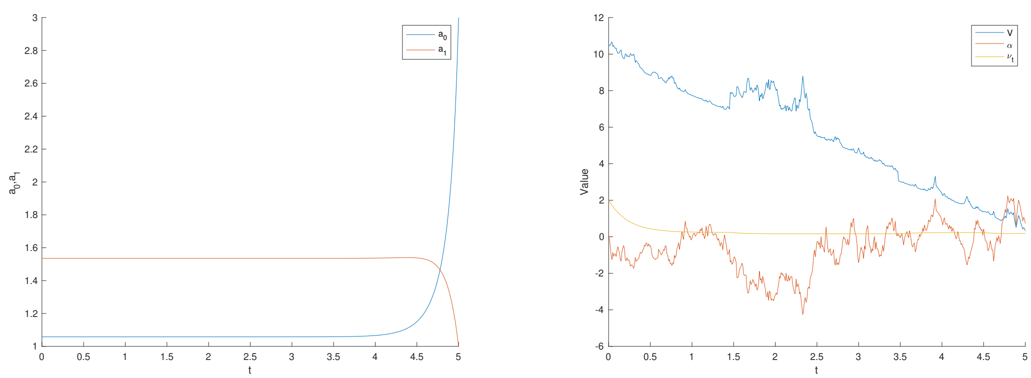
needed for the characterization of the equilibrium and the value function. Meanwhile, we also show the solvability of the Riccati ODE system in Section 3.3.

As mentioned earlier, different from the MFG characterization with the common noise, the derived Riccati system is essentially finite-dimensional. In this subsection, we present a numerical experiment and show some numerical results for solving the Riccati system to demonstrate its computational advantages.

For the illustration purpose, assume the finite time horizon is given with $T = 5$ and the coefficients of the dynamic equation are listed below:

$$
\begin{aligned}
&\mathcal{Y} = \{0, 1\},\\
&Q = \begin{bmatrix} -0.5 & 0.5 \\ 0.6 & -0.6 \end{bmatrix},\\
&\tilde{b}_1(\cdot,\cdot) = 0,\ \tilde{b}_2(\cdot,\cdot) = 1,\\
&h_0 = 2,\ h_1 = 5,\ g_0 = 3,\ g_1 = 1,\\
&\mu_0 = 0, \nu_0 = 2.
\end{aligned}
$$

Firstly, using the forward Euler's method with the step size $\delta = 10^{-2}$, we can obtain trajectories of $(a_y, b_y, c_y : y \in \mathcal{Y})$, which is the solution of ODE system (3.2.12). Next, using the trajectories of the parameter functions and Markov chain Y_t, we can achieve the simulations for $\hat{\alpha}_t$ and $\hat{X}_t$. The Matlab code can be found at `https://github.com/JiaminJIAN/Regime_switching_MFG`.



(a) $a_y(t)$

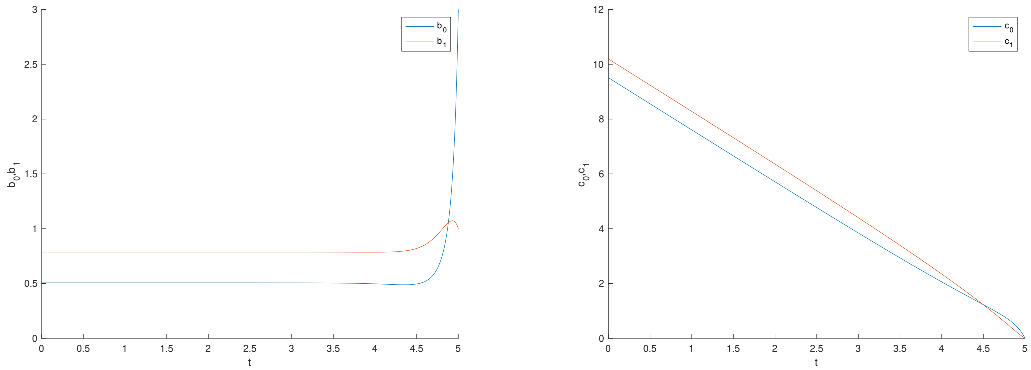
(b) Value function V, optimal control α, and conditional second moment ν

Figure 3.3: Simulations for a_y, V, α and ν.

As shown in Figure 3.3, people tend to centralize since the conditional second moment of the population density ν_t is always decreasing.

3.5.2 Convergence of the N-player game

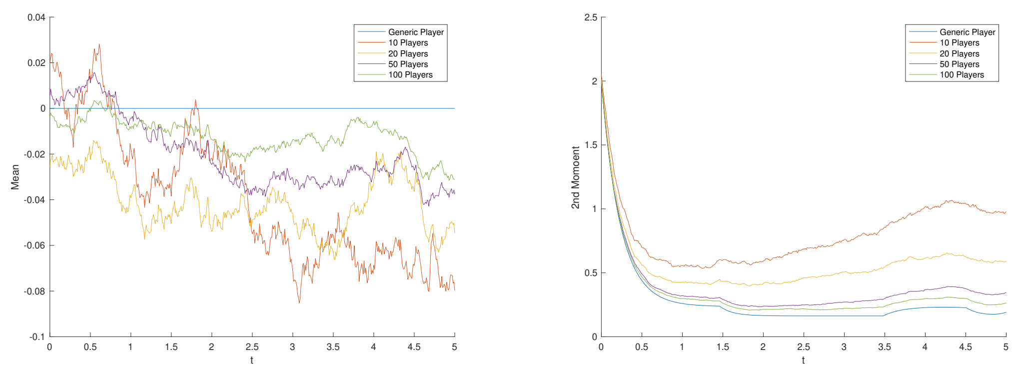
In Section 3.4, we showed that the generic player's path for the N-player game is convergent to the generic player's path for MFG. In this subsection, we demonstrate the convergence of the conditional first moment, conditional second moment, and the value functions of the N-player game to the corresponding terms of the generic player in the Mean Field Game setup by using some numerical examples.



(a) $b_y(t)$ (b) $c_y(t)$

Figure 3.4: Simulations for b_y and c_y.

The following figures show the value functions, $\mu^{(N)}$ and $\nu^{(N)}$ under $N \in \{10, 20, 50, 100\}$ with the same parameters' settings as in Figure 3.3 and Figure 3.4 in Section 3.5.1. We can clearly see the convergence to the solution of the generic player in Figure 3.5 and Figure 3.6.



(a) μ_t: conditional mean of the population density (b) ν_t: conditional 2nd moment of the population density

Figure 3.5: Simulations for μ_t and ν_t.

3.6 Conclusion

This chapter investigates the convergence rate of the N-player game, governed by a Markov chain common noise, towards its asymptotic MFG under the LQG structure. To achieve this, firstly, we introduce a Markovian structure using two auxiliary processes for the first and second moments of the MFG equilibrium and employ the fixed point condition in MFG. By doing so, we characterize the equilibrium measure in MFG with a finite-dimensional Riccati system of ODEs. Consequently, we obtain the equilibrium path, equilibrium control, and the value function in MFG.

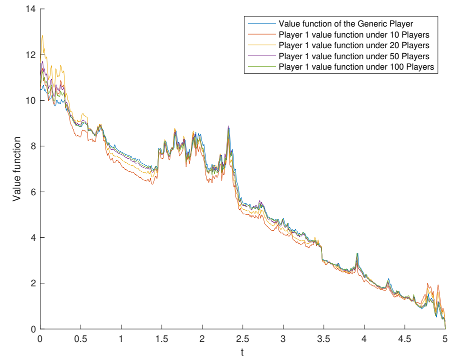


Figure 3.6: Simulation of player 1's optimal value function V.

Subsequently, we address the N-player game under the LQG structure, and we characterize its equilibrium path, equilibrium control, and the value function through a Riccati system of ODEs with a dimension of $O(N^3)$. Leveraging the N-invariant algebraic structure of this system of ODEs, we establish a dimension reduction result, facilitating a comparison between the equilibrium path $\hat{X}_1^{(N)}$ in the N-player game and the equilibrium path $\hat{X}$ in the MFG.

To demonstrate the convergence between the two equilibrium paths, we embed $\hat{X}_1^{(N)}$ from $\Omega^{(N)}$ to Ω using a distribution copy $Z^N \in \Omega$, leading to the achievement of the convergence result and the computation of the convergence rate. Lastly, some numerical examples are presented to demonstrate the convergence result.

In the future, firstly, we can consider the MFG in more general settings, such as with time delays and Poisson jumps. Next, except for considering the LQG structure, we could consider the convergence of MFG with common noise under more general structures. Furthermore, in this chapter, we require positive values for all sensitivities in the cost functional. We find that there is no global solution for MFG when the coefficient of the cost functional is negative, while there is a global solution when the coefficient is positive. So, it is also an interesting problem to investigate the explosion when some sensitivities are negative.

3.7 Appendix

3.7.1 Some explicit solutions on LQG-MFG

In this part, we only provide explicit solutions to some LQG-MFG without the common noise. The methodology could be the utilization of the standard Stochastic Maximum Principle or Dynamic Programming approach, and all proofs will be omitted.

Suppose the position of a generic player X_t follows

$$dX_t = \alpha_t dt + \sigma dW_t,\ X_0 \sim \mathcal{N}(0,1).$$

The goal of the generic player is to minimize the running cost

$$\inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_t^2 + h\int_{\mathbb{R}} (X_t - y)^2 m(t,dy)\right) dt\right],$$

subject to

$$m_t = \mathcal{L}aw(X_t), \quad \forall t \in [0,T],$$

where $h \in \mathbb{R}$ is a constant.

Denote

$$V(x,t) = \inf_{\alpha} \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\alpha_s^2 + h\int_{\mathbb{R}} (X_s - y)^2 m(s,dy)\right) ds \middle| X_t = x\right].$$

Note that the model can be characterized by Hamilton-Jacobian-Bellman equation coupled by Fokker-Planck-Kolmogorov equation:

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2 \partial_{xx} V - \frac{1}{2}(\partial_x V)^2 + F(x,m) = 0, & (t,x) \in [0,T]\times\mathbb{R}, \\ \partial_t m - \frac{1}{2}\sigma^2\partial_{xx} m - \partial_x(m\partial_x V) = 0, & (t,x)\in[0,T]\times\mathbb{R}, \\ m_0 \sim \mathcal{N}(0,1), V(x,T) = 0, & x \in \mathbb{R}, \end{cases}$$

where $F(x,m) = h\int_{\mathbb{R}}(x-y)^2 m(dy)$.

The monotonicity condition on the source term F in the variable m plays a crucial role in the uniqueness of the MFG system. A monotone function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be increasing if it satisfies $(f(x_1) - f(x_2))(x_1 - x_2) \geq 0$, and decreasing if $-f$ is increasing. This definition can be generalized to an infinite dimensional function $F(x,m)$.

Definition 30. *The real function F on $\mathbb{R}\times\mathcal{P}_2(\mathbb{R})$ is said to be monotone, if, for all $m \in \mathcal{P}_2(\mathbb{R})$, the mapping $\mathbb{R} \ni x \mapsto F(x,m)$ is at most of quadratic growth, and for all m_1, m_2 it satisfies*

$$\int_{\mathbb{R}} (F(x,m_1) - F(x,m_2))\, d(m_1 - m_2)(x) \geq 0.$$

F is said to be anti-monotone, if $(-F)$ is monotone.

According to [13], if F is monotone, then MFG have at most one solution. Interestingly, the monotonicity of F is dependent on the sign of h.

Lemma 31. *$F(x,m) = h\int_{\mathbb{R}}(x-y)^2 m(dy)$ is monotone if $h < 0$, and anti-monotone if $h > 0$.*

A natural question is how the MFG system behaves differently to the monotonicity of F?

3.7.1.1 Case I: $h > 0$

Lemma 32. *For $h > 0$, there exists a solution (may not be unique) to the MFG system in the form of $V(x,t) = f_1(t)x^2 + f_3(t)$ and $m(t) \sim \mathcal{N}(0, \gamma(t))$, where*

$$
\begin{aligned}
f_1(t) &= \sqrt{\frac{h}{2}} \frac{1 - e^{-2\sqrt{2h}(T-t)}}{1 + e^{-2\sqrt{2h}(T-t)}}, \\
\gamma(t) &= e^{-\int_0^t 4f_1(s)ds} \left(1 + \int_0^t \sigma^2 e^{\int_0^s 4f_1(u)du} ds\right), \\
f_3(t) &= \int_t^T (\sigma^2 f_1(s) + h\gamma(s))ds.
\end{aligned}
$$

3.7.1.2 Case II: $h < 0$

Lemma 33. *For $h < 0$, there exists a unique solution in $(t_0, T]$ to the MFG system in the form of $V(x,t) = g_1(t)x^2 + g_3(t)$ and $m(t) \sim \mathcal{N}(0, \lambda(t))$, where*

$$
\begin{aligned}
g_1(t) &= -\sqrt{-\frac{h}{2}} \tan\left(\sqrt{-2h}(T-t)\right), \\
\lambda(t) &= e^{-\int_0^t 4g_1(s)ds} \left(1 + \int_0^t \sigma^2 e^{\int_0^s 4g_1(u)du} ds\right), \\
g_3(t) &= \int_t^T (\sigma^2 g_1(s) + h\lambda(s))ds, \\
t_0 &= max\left(0, T - \frac{1}{\sqrt{-2h}} \frac{\pi}{2}\right).
\end{aligned}
$$

3.7.1.3 Remark

When $h > 0$, the cost is anti-monotone, and there exists at least one global solution. When $h < 0$, the cost is monotone, and there exists at most one solution. Unfortunately, this solution lives in a short period. Lemma 33 coincides with the notes in Section 3.8 of [16] saying that due to the opposite time evolution of the system of HJB-FPK, the existence of the solution may exist for only a short period.

3.7.2 Dynkin's formula for a regime-switching diffusion with a quadratic function

Since the running cost (3.2.10) has a quadratic growth in the state variable, the value function $V[\hat{m}](y,x,t)$ is expected to possess similar growth. Next, we present a version of Dynkin's formula for the functions of quadratic growth, which is sufficient for our purpose. Throughout this subsection, we will use K in various places as a generic constant that varies from line to line. The notions of this subsection are independent of other parts of the chapter.

Lemma 34. *Let X be the $\mathbb{R}^d$-valued process satisfying*

$$X_t = X_0 + \int_0^t \left(\tilde{b}_1(Y_s, s)X_s + \tilde{b}_2(Y_s, s)\alpha_s\right) ds + \int_0^t \sigma(s)dW_s,$$

where Y is CTMC with a generator

$$Y \sim Q = (q_{ij})_{i,j=1,2,\ldots,\kappa},$$

Suppose $\sigma(\cdot)$, $\tilde{b}_1(y,\cdot)$ and $\tilde{b}_2(y,\cdot)$ are continuous functions on $[0,T]$ for every $y \in \mathcal{Y} := \{1, 2, \ldots, \kappa\}$. If $X_0 \in L^4$, $\alpha \in L^4_{\mathbb{F}}$ and $f : \mathcal{Y} \times R^d \times \mathbb{R} \mapsto \mathbb{R}$ satisfies, for some large K

$$\sup_{y\in\mathcal{Y},t\in[0,T]} \{|f(y,x,t)| + (1+|x|)|\nabla f(y,x,t)| + (1+|x|)^2|\Delta f(y,x,t)| + |\partial_t f(y,x,t)|\} \leq K(|x|^2+1),$$

then the following identity holds for all $t \in [0,T]$:

$$\mathbb{E}\left[f(Y_t, X_t, t)\right] = \mathbb{E}\left[f(Y_0, X_0, 0)\right] + \mathbb{E}\left[\int_0^t (\partial_t + \mathcal{L}^{\alpha_s} + \mathcal{Q})f(Y_s, X_s, s)ds\right],$$

where

$$\mathcal{L}^a f(y,x,s) = \left(\frac{1}{2} Tr\left(\sigma_s\sigma_s^\top \Delta\right) + \left(\tilde{b}_{1y}x + \tilde{b}_{2y}a\right) \cdot \nabla_x\right) f(y,x,s)$$

and

$$\mathcal{Q}f(y,x,s) = \sum_{i=1}^n q_{y,i} f(i,x,s).$$

Proof. It's enough to show that the local martingale defined by Itô's formula

$$M_t^f = f(Y_t, X_t, t) - f(Y_0, X_0, 0) - \int_0^t (\partial_t + \mathcal{L}^{\alpha_s} + \mathcal{Q})f(Y_s, X_s, s)ds \tag{3.7.1}$$

is uniformly integrable, hence is a true martingale.

First, note that from the assumptions on X_0 and α, we have

$$\begin{aligned}
\mathbb{E}\left[\|X_t\|^4\right] &\leq K\mathbb{E}\left[\|X_0\|^4 + \int_0^t \|\tilde{b}_1(Y_s,s)X_s + \tilde{b}_2(Y_s,s)\alpha_s\|^4 ds + \int_0^t \|\sigma_s W_s\|^4 ds\right] \\
&\leq K\mathbb{E}\left[\|X_0\|^4 + \int_0^t \|X_s\|^4 ds + \int_0^t \|\alpha_s\|^4 ds + \int_0^t \|\sigma_s W_s\|^4 ds\right] \\
&\leq K + K\int_0^t \mathbb{E}\left[\|X_s\|^4\right] ds,
\end{aligned}$$

where K is a generic constant that varies from line to line. Then, by the Grönwall's inequality,

$$\mathbb{E}\left[\|X_t\|^4\right] \leq Ke^{Kt} \leq K,$$

which implies that $\{X_t : 0 \leq t \leq T\}$ is L^4 bounded uniformly in t.

On the other hand, since $x \mapsto f(y,x,t)$ is at most quadratic growth uniformly in (y,t), we conclude that $f(Y_t, X_t, t)$ is uniformly L^2 bounded from the fact

$$\sup_{t\in[0,T]} \mathbb{E}\left[f^2(Y_t,X_t,t)\right] \le K \sup_{t\in[0,T]} \mathbb{E}\left[\|X_t\|^4\right] + K \le K.$$

The uniform L^2-boundedness of $\int_0^t \partial_t f(Y_s,X_s,s)ds$ follows from our assumption on $\partial_t f$. Similarly, since $\mathcal{Q}f$ has a quadratic growth uniformly in y and t, and

$$\left\{\int_0^t \mathcal{Q}f(Y_s,X_s,s)ds : 0 \le t \le T\right\}$$

is L^2 bounded. At last, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \mathcal{L}^{\alpha_s} f(Y_s,X_s,s)ds\right)^2\right] \\
&\le K\mathbb{E}\left[\int_0^t \left(\left(\tilde{b}_1(Y_s,s)X_s + \tilde{b}_2(Y_s,s)\alpha_s\right)\cdot\nabla f + \frac{1}{2}\mathrm{Tr}\left(\sigma_s\sigma_s^\top \Delta f\right)\right)^2 (Y_s,X_s,s)ds\right] \\
&\le K\mathbb{E}\left[\int_0^t \|\tilde{b}_1(Y_s,s)X_s + \tilde{b}_2(Y_s,s)\alpha_s\|^2 \|\nabla f\|^2 (Y_s,X_s,s)ds\right] \\
&\qquad + K\mathbb{E}\left[\int_0^t \frac{1}{4}\|\mathrm{Tr}\left(\sigma_s\sigma_s^\top \Delta f\right)\|^2 (Y_s,X_s,s)ds\right] \\
&\le K\mathbb{E}\left[\int_0^t \|\alpha_s\|^4 ds\right] + K\mathbb{E}\left[\int_0^t \|X_s\|^4 ds\right] + K\mathbb{E}\left[\int_0^t |\nabla f|^4 (Y_s,X_s,s)ds\right] \\
&\qquad + K\mathbb{E}\left[\int_0^t \frac{1}{4}\|\mathrm{Tr}\Delta f\|^2 \left(Y_s,X_s,s\right)ds\right].
\end{aligned}$$

Since ∇f is linear growth in x, the second term $\sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t \|\nabla f\|^4(Y_s,X_s,s)ds\right]$ is finite. Together with assumptions on Δf and α, we have uniform L^2-boundedness of $\int_0^t \mathcal{L}^{\alpha_s} f(Y_s,X_s,s)ds$.

As a result, each term of the right-hand side of (3.7.1) is uniform L^2-bounded in t, and thus M_t^f belongs to $L^2_{\mathbb{F}}$ and this implies the uniform integrability. □

3.7.3 Proof of the property of G

Lemma 35. *Define*

$$\mathcal{E}_t(\phi) = \exp\left\{\int_0^t \phi_s ds\right\},$$

and

$$G_t(x,\phi_1,\phi_2,\phi_3,W) = x\mathcal{E}_t(\phi_1-\phi_2) + \mathcal{E}_t(\phi_1-\phi_2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2)\left(\phi_2(s)\phi_3(s)ds + dW_s\right),$$

where x is a given constant, ϕ_1, ϕ_2, ϕ_3 are RCLL functions on $[0, T]$. Then

$$\mathbb{E}\left[\left|G_t(x^1, \phi_1, \phi_2^1, \phi_3^1, W) - G_t(x^2, \phi_1, \phi_2^2, \phi_3^2, W)\right|^2\right]$$
$$\leq K\left(|x^1 - x^2|^2 + \sup_{0\leq t\leq T}|\phi_2^1(t) - \phi_2^2(t)|^2 + \sup_{0\leq t\leq T}|\phi_3^1(t) - \phi_3^2(t)|^2\right).$$

Proof. Firstly, it can be shown that $G(\cdot, \phi_1, \phi_2, \phi_3, W)$ is Lipschitz continuous with respect to x

$$\begin{aligned}\mathbb{E}\left[\left|G_t(x^1, \phi_1, \phi_2, \phi_3, W) - G(x^2, \phi_1, \phi_2, \phi_3, W)\right|\right] &\leq \left|x^1\mathcal{E}_t(\phi_1 - \phi_2) - x^2\mathcal{E}_t(\phi_1 - \phi_2)\right| \\ &\leq \mathcal{E}_t(\phi_1 - \phi_2)|x^1 - x^2| \\ &\leq K(|\phi_1|_\infty, |\phi_2|_\infty, T)|x^1 - x^2|.\end{aligned}$$

Next, we have

$$\begin{aligned}&\mathbb{E}\left[\left|G_t(x, \phi_1, \phi_2, \phi_3^1, W) - G(x, \phi_1, \phi_2, \phi_3^2, W)\right|^2\right] \\ &= \left|\mathcal{E}_t(\phi_1 - \phi_2)\int_0^t \mathcal{E}_s(\phi_1 - \phi_2)\phi_2(s)(\phi_3^1(s) - \phi_3^2(s))ds\right|^2 \\ &\leq \mathcal{E}_t(2\phi_1 - 2\phi_2)\left(\int_0^t \mathcal{E}_s(\phi_1 - \phi_2)|\phi_2(s)||(\phi_3^1(s) - \phi_3^2(s))|ds\right)^2 \\ &\leq K(|\phi_1|_\infty, |\phi_2|_\infty, T)\left(\int_0^T |\phi_3^1(s) - \phi_3^2(s)|ds\right)^2 \\ &\leq K(|\phi_1|_\infty, |\phi_2|_\infty, T)\sup_{0\leq t\leq T}\left|\phi_3^1(t) - \phi_3^2(t)\right|^2.\end{aligned}$$

Similarly, for $\phi_2^1(\cdot), \phi_2^2(\cdot) \in C([0, T])$,

$$\begin{aligned}&\mathbb{E}\left[\left|G_t(x, \phi_1, \phi_2^1, \phi_3, W) - G(x, \phi_1, \phi_2^2, \phi_3, W)\right|^2\right] \\ &\leq K\left|x\mathcal{E}_t(\phi_1 - \phi_2^1) - x\mathcal{E}_t(\phi_1 - \phi_2^2)\right|^2 \\ &\quad + K\left|\mathcal{E}_t(\phi_1 - \phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^1)\phi_2^1(s)\phi_3(s)ds - \mathcal{E}_t(\phi_1 - \phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)\phi_2^2(s)\phi_3(s)ds\right|^2 \\ &\quad + K\mathbb{E}\left[\left|\mathcal{E}_t(\phi_1 - \phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^1)dW_s - \mathcal{E}_t(\phi_1 - \phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1 + \phi_2^2)dW_s\right|^2\right] \\ &:= K(J_1 + J_2 + J_3).\end{aligned}$$

Note that by the mean-value theorem and the continuity of ϕ_1, ϕ_2^1 and ϕ_2^2 on $[0, T]$, we can get

$$\begin{aligned}J_1 &= \left|x\mathcal{E}_t(\phi_1 - \phi_2^1) - x\mathcal{E}_t(\phi_1 - \phi_2^2)\right|^2 \\ &= x^2\left(e^{\int_0^t(\phi_1(s) - \phi_2^1(s))ds} - e^{\int_0^t(\phi_1(s) - \phi_2^2(s))ds}\right)^2 \\ &\leq K\left(x, \left|\phi_2^1\right|_\infty, \left|\phi_2^2\right|_\infty, T\right)e^{\int_0^t 2\phi_1(s)ds}\left|\phi_2^1 - \phi_2^2\right|_\infty^2 \\ &\leq K\left(x, |\phi_1|_\infty, \left|\phi_2^1\right|_\infty, \left|\phi_2^2\right|_\infty, T\right)\left|\phi_2^1 - \phi_2^2\right|_\infty^2,\end{aligned}$$

and

$$
\begin{aligned}
J_3 &= \mathbb{E}\left[\left|\mathcal{E}_t(\phi_1-\phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)dW_s - \mathcal{E}_t(\phi_1-\phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)dW_s\right|^2\right] \\
&= \mathbb{E}\left[\left|\mathcal{E}_t(\phi_1-\phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)dW_s - \mathcal{E}_t(\phi_1-\phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)dW_s\right.\right. \\
&\qquad \left.\left. +\ \mathcal{E}_t(\phi_1-\phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)dW_s - \mathcal{E}_t(\phi_1-\phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)dW_s\right|^2\right] \\
&\le 2\mathcal{E}_t(2\phi_1-2\phi_2^1)\int_0^t \left(\mathcal{E}_s(-\phi_1+\phi_2^1)-\mathcal{E}_s(-\phi_1+\phi_2^2)\right)^2 ds \\
&\qquad + 2\left(\mathcal{E}_t(\phi_1-\phi_2^1)-\mathcal{E}_t(\phi_1-\phi_2^2)\right)^2\int_0^t \mathcal{E}_s(-2\phi_1+2\phi_2^2)ds \\
&\le K\left(|\phi_1|_\infty, \left|\phi_2^1\right|_\infty, \left|\phi_2^2\right|_\infty, T\right)\left|\phi_2^1-\phi_2^2\right|_\infty^2 .
\end{aligned}
$$

Lastly, using the similar argument, we have

$$
\begin{aligned}
J_2 &= \left|\mathcal{E}_t(\phi_1-\phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds - \mathcal{E}_t(\phi_1-\phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)\phi_2^2(s)\phi_3(s)ds\right|^2 \\
&= \left|\mathcal{E}_t(\phi_1-\phi_2^1)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds - \mathcal{E}_t(\phi_1-\phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds\right. \\
&\qquad \left. +\ \mathcal{E}_t(\phi_1-\phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds - \mathcal{E}_t(\phi_1-\phi_2^2)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)\phi_2^2(s)\phi_3(s)ds\right|^2 \\
&\le 2\left|\left(\mathcal{E}_t(\phi_1-\phi_2^1)-\mathcal{E}_t(\phi_1-\phi_2^2)\right)\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds\right|^2 \\
&\qquad + 2\left|\mathcal{E}_t(\phi_1-\phi_2^2)\left(\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds - \int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)\phi_2^2(s)\phi_3(s)ds\right)\right|^2 \\
&\le K\left(|\phi_1|_\infty, \left|\phi_2^1\right|_\infty, \left|\phi_2^2\right|_\infty, |\phi_3|_\infty, T\right)\left|\phi_2^1-\phi_2^2\right|_\infty^2 \\
&\qquad + 2\left|\mathcal{E}_t(\phi_1-\phi_2^2)\left(\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^1)\phi_2^1(s)\phi_3(s)ds - \int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)\phi_2^1(s)\phi_3(s)ds\right)\right. \\
&\qquad \left. +\ \mathcal{E}_t(\phi_1-\phi_2^2)\left(\int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)\phi_2^1(s)\phi_3(s)ds - \int_0^t \mathcal{E}_s(-\phi_1+\phi_2^2)\phi_2^2(s)\phi_3(s)ds\right)\right|^2 \\
&\le K\left(|\phi_1|_\infty, \left|\phi_2^1\right|_\infty, \left|\phi_2^2\right|_\infty, |\phi_3|_\infty, T\right)\left|\phi_2^1-\phi_2^2\right|_\infty^2 .
\end{aligned}
$$

Sum up the above inequalities for J_1, J_2 and J_3, then

$$
\begin{aligned}
&\mathbb{E}\left[\left|G_t(x,\phi_1,\phi_2^1,\phi_3,W) - G(x,\phi_1,\phi_2^2,\phi_3,W)\right|^2\right] \\
&\le K\left(x, |\phi_1|_\infty, \left|\phi_2^1\right|_\infty, \left|\phi_2^2\right|_\infty, |\phi_3|_\infty, T\right)\left|\phi_2^1-\phi_2^2\right|_\infty^2 .
\end{aligned}
$$

Thus, we can obtain the desired result.

□

3.7.4 Proof of the existence and uniqueness of the ODE system

Consider the following ODE system

$$\begin{cases} a_y' + C_1\tilde{b}_{1y}a_y - C_2\tilde{b}_{2y}^2 a_y^2 + \sum_{i=1}^{\kappa} q_{y,i}a_i + h_y = 0, \\ a_y(T) = g_y, \end{cases} \tag{3.7.2}$$

for $y \in \mathcal{Y} = \{1, 2, \dots, \kappa\}$, where C_1, C_2, h_y, g_y are in $\mathbb{R}^+$. We need to show the existence and uniqueness of the solution to (3.7.2). Define $T_y^{(N)}$ as

$$T_y^{(N)}[a](t) = \left[\left(g_y + \int_t^T \left(h_y + C_1\tilde{b}_{1y}(s)a_y(s) - C_2\tilde{b}_{2y}^2(s)a_y^2(s) + \sum_{i=1}^{\kappa} q_{y,i}a_i(s)\right) ds\right) \wedge N\right] \vee 0,$$

where $a = [a_1, a_2, \dots, a_\kappa]^\top$. Let $D = \{f \in C([0,T]) : 0 \leq \sup_{t\in[0,T]} f(t) \leq N\}$. Note that $T_y^{(N)}(y \in \mathcal{Y})$ maps D^κ to D^κ.

Lemma 36. *For fixed N, there exists a unique solution in $C([0,T])$ to*

$$a = T_y^{(N)}[a]. \tag{3.7.3}$$

Proof. Denote the norm $\|f\|_k = \left\|e^{kt}\max_{y\in\mathcal{Y}}|f_y|\right\|_\infty$, where k needs to be determined later and f is a κ dimensional vector with entry of $f_y, y \in \mathcal{Y}$, which is equivalent to the infinite norm. Define the iteration rule $a_y^{(n+1)} = T_y^{(N)}\left[a_y^{(n)}\right]$ for $y \in \mathcal{Y}$. Note that

$$\begin{aligned}
&\left\|e^{kt}\left(a_y^{(n+1)}(t) - a_y^{(n)}(t)\right)\right\|_\infty \\
&\leq \sup_{t\in[0,T]} e^{kt}\int_t^T \left(C_1\left|\tilde{b}_{1y}\right|_\infty \left|a_y^{(n)}(s) - a_y^{(n-1)}(s)\right| + C_2\left|\tilde{b}_{2y}\right|_\infty^2 \left|\left(a_y^{(n)}(s)\right)^2 - \left(a_y^{(n-1)}(s)\right)^2\right|\right. \\
&\qquad\qquad \left. + \sum_{i=1}^{\kappa} q_{y,i}\left|a_i^{(n)}(s) - a_i^{(n-1)}(s)\right|\right) ds \\
&\leq \sup_{t\in[0,T]} e^{kt}\int_t^T \left(C_1\left|\tilde{b}_{1y}\right|_\infty \left|a_y^{(n)}(s) - a_y^{(n-1)}(s)\right| + 2NC_2\left|\tilde{b}_{2y}\right|_\infty^2 \left|a_y^{(n)}(s) - a_y^{(n-1)}(s)\right|\right. \\
&\qquad\qquad \left. + \sum_{i=1}^{\kappa} q_{y,i}\left|a_i^{(n)}(s) - a_i^{(n-1)}(s)\right|\right) ds \\
&\leq \sup_{t\in[0,T]} e^{kt}\int_t^T e^{-ks}\left(C_1\left|\tilde{b}_{1y}\right|_\infty + 2NC_2\left|\tilde{b}_{2y}\right|_\infty^2 + \kappa\max_{i\in\mathcal{Y}}|q_{y,i}|\right)\left\|a^{(n)} - a^{(n-1)}\right\|_k ds \\
&\leq \frac{C_1\left|\tilde{b}_{1y}\right|_\infty + 2NC_2\left|\tilde{b}_{2y}\right|_\infty^2 + \kappa\max_{i\in\mathcal{Y}}|q_{y,i}|}{k}\left\|a^{(n)} - a^{(n-1)}\right\|_k.
\end{aligned}$$

Choose $k > C_1 \left|\tilde{b}_{1y}\right|_\infty + 2NC_2 \left|\tilde{b}_{2y}\right|_\infty^2 + \kappa \max_{i\in\mathcal{Y}} |q_{y,i}|$, then

$$\left\|a^{(n+1)} - a^{(n)}\right\|_k \le \frac{C_1 \left|\tilde{b}_{1y}\right|_\infty + 2NC_2 \left|\tilde{b}_{2y}\right|_\infty^2 + \kappa \max_{i\in\mathcal{Y}} |q_{y,i}|}{k} \left\|a^{(n)} - a^{(n-1)}\right\|_k,$$

which gives us a contraction mapping from D^κ to D^κ. Hence, by the Banach fixed point theorem, there exists a unique solution to (3.7.3). □

Next, we want to show that for large enough N, the solution to (3.7.3) is also the solution to (3.7.2).

Lemma 37. *For*

$$N \ge e^{KT} \left(\sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right),$$

where $K := C_1 \max_{y\in\mathcal{Y}} \left|\tilde{b}_{1y}\right|_\infty + \max_{i\in\mathcal{Y}} \sum_{y=1}^{\kappa} |q_{y,i}|$, *the solution* $a^{(N)}$ *to* (3.7.3) *satisfies the inequalities*

$$0 \le g_y + \int_t^T \left(h_y + C_1 \tilde{b}_{1y}(s) a_y^{(N)}(s) - C_2 \tilde{b}_{2y}^2(s) \left(a_y^{(N)}(s)\right)^2 + \sum_{i=1}^{\kappa} q_{y,i} a_i^{(N)}(s) \right) ds \le N \qquad (3.7.4)$$

for all $t \in [0, T]$*, where* $y \in \mathcal{Y}$*.*

Proof. For simplicity of notations, a_y is used instead of $a_y^{(N)}$ for $y \in \mathcal{Y}$ if there is no confusion.

First, for $y \in \mathcal{Y}$, we prove the positiveness of a_y by contradiction. Suppose a_y $(y \in \mathcal{Y})$ are not positive functions on $[0, T]$. Since a_1 is continuous and $a_1(T) = g_1 > 0$, there exists some $\tau_1 \in [0, T]$ as the closest time to T such that $a_1(\tau_1) = 0$. Note that finding such a τ_1 is possible. Let $t_n \in [0, T]$ be a non-decreasing sequence such that $a_1(t_n) = 0$, there exists some τ_1 such that $t_n \to \tau_1 < T$ as $n \to \infty$ since a_1 is continuous and $a_1(T) = g_1 > 0$. By the continuity of a_1, we have $a_1(\tau_1) = 0$, which gives the desirable point τ_1. Then for all $t \in (\tau_1, T]$, $a_1(t) > 0$ and it implies that $a_1'(\tau_1) > 0$. In this case, plugging $t = \tau_1$ to (3.7.2), we have

$$a_1'(\tau_1) = -h_1 - \sum_{i\neq 1}^{\kappa} q_{1,i} a_i(\tau_1) > 0,$$

which implies there is some $y \in \mathcal{Y}$ and $y \neq 1$ such that $a_y(\tau_1) < 0$. Without loss of generality, we let $a_2(\tau_1) < 0$. Since a_2 is continuous on $[0, T]$ and $a_2(T) = g_2 > 0$, from the intermediate value theorem, there exists some $\tau_2 \in (\tau_1, T)$ such that $a_2(\tau_2) = 0$ and $a_2'(\tau_2) > 0$. This indicates that $a_2'(\tau_2) = -h_2 - \sum_{i\neq 2}^{\kappa} q_{2,i} a_i(\tau_2) > 0$ by plugging $t = \tau_2$ back to (3.7.2), and it implies that there is some $y \in \mathcal{Y}$ and $y \neq 1, 2$ such that $a_y(\tau_2) < 0$ since we already know $a_1(\tau_2) > 0$. Without loss of generality, we can let $a_3(\tau_2) < 0$. By induction with the same argument, there is a $\tau_\kappa \in (\tau_{\kappa-1}, T)$

such that $a_\kappa(\tau_\kappa) = 0$ and $a'_\kappa(\tau_\kappa) > 0$, which gives

$$a'_\kappa(\tau_\kappa) + h_\kappa + \sum_{i \neq \kappa}^{\kappa} q_{\kappa,i} a_i(\tau_\kappa) = 0.$$

But it contradicts with the fact that

$$a'_\kappa(\tau_\kappa) > 0, \;\; h_\kappa > 0, \;\; q_{\kappa,i} > 0, \;\; a_i(\tau_\kappa) > 0$$

for $i \in \{1, 2, \ldots, \kappa - 1\}$. Thus the positiveness of a_y on $[0, T]$ for all $y \in \mathcal{Y}$ is obtained.

Next, we prove the upper boundness for the integral in (3.7.4). Note that for all $t \in [0, T]$ and $y \in \mathcal{Y}$, let $\tau = T - t$, we have

$$a'_y(\tau) = h_y + C_1 \tilde{b}_{1y}(\tau) a_y(\tau) - C_2 \tilde{b}^2_{2y}(\tau) a^2_y(\tau) + \sum_{i=1}^{\kappa} q_{y,i} a_i(\tau),$$

and thus

$$\begin{aligned}
\sum_{y=1}^{\kappa} a'_y(\tau) &= \sum_{y=1}^{\kappa} h_y + C_1 \sum_{y=1}^{\kappa} \tilde{b}_{1y}(\tau) a_y(\tau) - C_2 \sum_{y=1}^{\kappa} \tilde{b}^2_{2y}(\tau) a^2_y(\tau) + \sum_{y=1}^{\kappa} \sum_{i=1}^{\kappa} q_{y,i} a_i(\tau) \\
&\leq \sum_{y=1}^{\kappa} h_y + C_1 \max_{y \in \mathcal{Y}} \left| \tilde{b}_{1y} \right|_\infty \sum_{y=1}^{\kappa} a_y(\tau) + \sum_{y=1}^{\kappa} \sum_{i=1}^{\kappa} |q_{y,i}| a_i(\tau) \\
&\leq \sum_{y=1}^{\kappa} h_y + \sum_{i=1}^{\kappa} \left(C_1 \max_{y \in \mathcal{Y}} \left| \tilde{b}_{1y} \right|_\infty + \sum_{y=1}^{\kappa} |q_{y,i}| \right) a_i(\tau) \\
&\leq \sum_{y=1}^{\kappa} h_y + K \sum_{i=1}^{\kappa} a_i(\tau),
\end{aligned}$$

where

$$K := C_1 \max_{y \in \mathcal{Y}} \left| \tilde{b}_{1y} \right|_\infty + \max_{i \in \mathcal{Y}} \sum_{y=1}^{\kappa} |q_{y,i}|$$

with $\sum_{y=1}^{\kappa} a_y(T) = \sum_{y=1}^{\kappa} g_y$. By Grönwall's inequality, for all $\tau \in [0, T]$,

$$\sum_{y=1}^{\kappa} a_y(\tau) \leq e^{KT} \left(\sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right).$$

Hence $a_y(t) \leq e^{KT} \left(\sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right)$ for all $t \in [0, T]$ and $y \in \mathcal{Y}$. Hence, when

$$e^{KT} \left(\sum_{y=1}^{\kappa} g_y + T \sum_{y=1}^{\kappa} h_y \right) \leq N,$$

(3.7.4) holds. $\square$

Lemma 38. *With the given of $h_y, g_y \in \mathbb{R}^+$, $y \in \mathcal{Y}$, there exists a unique solution to the Riccati system* (3.2.12).

Proof. The existence, uniqueness and boundedness of the solution to a_y ($y \in \mathcal{Y}$) are shown in Lemma 36 and Lemma 37. Given ($a_y : y \in \mathcal{Y}$), the coefficient functions b_y ($y \in \mathcal{Y}$) form a linear ordinary differential equation system. Their existence and uniqueness are guaranteed by Theorem 12.1 in [2]. Similarly, with the given of ($a_y, b_y : y \in \mathcal{Y}$), the coefficient functions c_y, k_y ($y \in \mathcal{Y}$) also form a linear ordinary differential equation system. Applying the Theorem 12.1 in [2], we can obtain the existence and uniqueness of c_y, k_y ($y \in \mathcal{Y}$). $\square$

3.7.5 Multidimensional problem

In this subsection, we consider the multidimensional problem, which is a straightforward extension of the previous one-dimensional setup. The same type of Ricatti system to characterize the equilibrium and the value function is obtained, and we have a similar result as the Theorem 20.

Suppose that X_t, W_t and α_t take values in $\mathbb{R}^d$, and all components of W_t are independent. Suppose that the dynamic of the generic player is given by

$$X_t = X_0 + \int_0^t \left(\tilde{b}_1(Y_s, s)X_s + \tilde{b}_2(Y_s, s)\alpha_s\right) ds + W_t.$$

Consider the cost function

$$\begin{aligned}
&J[m](y, x, t, \bar{\mu}, \bar{\nu}) \\
&= \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\|\alpha_s\|_2^2 + h(Y_s)\int_{\mathbb{R}^d} \|X_s - z\|_2^2 m(dz)\right) ds \right.\\
&\qquad \left. + g(Y_T)\int_{\mathbb{R}^d} \|X_T - z\|_2^2 m(dz) \middle| X_t = x, Y_t = y, \mu_t = \bar{\mu}, \nu_t = \bar{\nu}\right] \\
&= \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\alpha_s^\top \alpha_s + h(Y_s)\left(X_s^\top X_s - 2\mu_s^\top X_s + \nu_s \cdot \mathbb{1}_d\right)\right) ds \right.\\
&\qquad \left. + g(Y_T)\left(X_T^\top X_T - 2\mu_T^\top X_T + \nu_T \cdot \mathbb{1}_d\right) \middle| X_t = x, Y_t = y, \mu_t = \bar{\mu}, \nu_t = \bar{\nu}\right],
\end{aligned}$$

where m is the joint density function in $\mathbb{R}^d$, and μ, ν take value in $\mathbb{R}^d$. For $y \in \mathcal{Y}$, define the Riccati system

$$
\begin{cases}
a_y' + 2\tilde{b}_{1y}a_y - 2\tilde{b}_{2y}^2 a_y^2 + \sum_{i=1}^{\kappa} q_{y,i}a_i + h_y = 0, \\
b_y' + \left(2\tilde{b}_{1y} - 4\tilde{b}_{2y}^2 a_y\right) b_y + \sum_{i=1}^{\kappa} q_{y,i}b_i + h_y = 0, \\
c_y' + da_y + db_y + \sum_{i=1}^{\kappa} q_{y,i}c_i = 0, \\
k_y' - 2\tilde{b}_{2y}^2 a_y^2 + 4\tilde{b}_{2y}^2 a_y b_y + 2\tilde{b}_{1y}k_y + \sum_{i=1}^{\kappa} q_{y,i}k_i = 0, \\
a_y(T) = b_y(T) = g_y\ ,\ c_y(T) = k_y(T) = 0.
\end{cases}
\tag{3.7.5}
$$

Theorem 39 (Verification theorem for MFG)**.** *There exists a unique solution* $(a_y, b_y, c_y, k_y : y \in \mathcal{Y})$ *for the Riccati system* (3.7.5)*. With these solutions, for* $t \in [0, T]$*, the MFG equilibrium path follows* $\hat{X} = \hat{X}[\hat{m}]$ *is given by*

$$
d\hat{X}_t = \left(\tilde{b}_1(Y_t, t)\hat{X}_t - 2\tilde{b}_2^2(Y_t, t)a_{Y_t}(t)\left(\hat{X}_t - \hat{\mu}_t\right)\right)dt + dW_t, \quad \hat{X}_0 = X_0,
$$

with equilibrium control $\hat{\alpha}_t = -2\tilde{b}_2(Y_t, t)a_{Y_t}(t)\left(\hat{X}_t - \hat{\mu}_t\right)$*, where*

$$
d\hat{\mu}_t = \tilde{b}_1(Y_t, t)\hat{\mu}_t dt, \quad \hat{\mu}_0 = \mathbb{E}[X_0].
$$

Moreover, the value function U *is*

$$
U(m_0, y, x) = a_y(0)x^\top x - 2a_y(0)x^\top [m_0]_1 + k_y(0)[m_0]_1^\top [m_0]_1 + b_y(0)[m_0]_2^\top \mathbb{1}_d + c_y(0)
$$

for $y \in \mathcal{Y}$*.*

The proof is similar to the one-dimensional problem, and we don't show the details here.

Chapter 4

Convergence rate of LQG mean field games with Brownian motion as common noise

4.1 Introduction

In this chapter, we examine the behavior of the triangular array $\hat{X}_t^{(N)} = (\hat{X}_{it}^{(N)} : 1 \leq i \leq N)$ as $N \to \infty$, where $\hat{X}_{it}^{(N)}$ represents the equilibrium state of the i-th player at time t in the N-player game, defined within the probability space $\left(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{F}^{(N)}, \mathbb{P}^{(N)}\right)$. Additionally, we denote $\hat{X}_t$ as the equilibrium path at time t derived from the associated MFG, defined in the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Considering the identical but not independent distribution $\mathcal{L}(\hat{X}_{it}^{(N)})$, the first question pertains to the convergence of $\hat{X}_{1t}^{(N)}$, which represents the generic path. It can be framed as follows:

(Q1) The $\mathbb{W}_p$-convergence rate of the representative equilibrium path,

$$\mathbb{W}_p\left(\mathcal{L}\left(\hat{X}_{1t}^{(N)}\right), \mathcal{L}\left(\hat{X}_t\right)\right) = O\left(N^{-?}\right).$$

Here, $\mathbb{W}_p$ denotes the p-Wasserstein metric.

The existing literature extensively explores the convergence rate in this context. For (Q1), Theorem 2.4.9 of the monograph [14] establishes a convergence rate of $O(N^{-1/2})$ using the $\mathbb{W}_1$ metric. More recently, [48] addresses (Q1) by introducing displacement monotonicity and controlled common noise, and Theorem 2.23 applies the maximum principle of forward-backward propagation of chaos to achieve the same convergence rate. Within the LQG framework, [50] also provides a convergence rate of 1/2 for the representative player.

The second question pertains to the convergence of the mean-field term, which is equivalent to the convergence of the empirical measure $\rho(\hat{X}_t^{(N)}) = \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_{it}^{(N)}}$ of N players. Given the Brownian motion, denoted as $\tilde{W}_t$, to be the common noise, the problem lies in determining the rate of convergence of the empirical measures to the MFG equilibrium measure

$$\hat{m}_t = \mathcal{L}\left(\hat{X}_t \middle| \mathcal{F}_t^{\tilde{W}}\right), \quad \forall t \in (0, T].$$

Thus, the second question can be stated as follows:

(Q2) The $\mathbb{W}_p$-convergence rate of empirical measures in L^p sense,

$$\left(\mathbb{E}\left[\mathbb{W}_p^p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \mathcal{F}_t^{\tilde{W}}\right)\right)\right]\right)^{\frac{1}{p}} = O\left(N^{-?}\right).$$

As for (Q2), Theorem 3.1 of [21] provides an answer, stating that the empirical measures exhibit a convergence rate of $O(N^{-1/(2p)})$ in the $\mathbb{W}_p$ distance for $p \in [1, 2]$. In [21], they also explore a related question that is both similar and more intriguing, which concerns the uniform $\mathbb{W}_p$-convergence rate:

(Q3) The t-uniform $\mathbb{W}_p$-convergence rate of empirical measures in L^p sense,

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} \mathbb{W}_p^p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \mathcal{F}_t^{\tilde{W}}\right)\right)\right]\right)^{\frac{1}{p}} = O\left(N^{-?}\right).$$

The answer provided by Theorem 3.1 in [21] reveals that the uniform convergence rate, as formulated in (Q3), is considerably slower compared to the convergence rate mentioned in (Q2). Specifically, the convergence rate for (Q3) is $O\left(N^{-1/(d+8)}\right)$ when $p = 2$, where d represents the dimension of the state space.

In this chapter, we specifically focus on a class of one-dimensional Linear-Quadratic-Gaussian (LQG) Mean Field Nash Games with Brownian motion as the common noise. It is important to note that the assumptions made in the aforementioned papers except [50] only account for linear growth in the state and control elements for the running cost, thus excluding the consideration of LQG. It is also noted that differences between [50] and the current chapter lie in various aspects: (1) The problem setting in this chapter considers Brownian motion as the common noise, whereas [50] employs a Markov chain. This discrepancy leads to significant differences in the subsequent analysis; (2) The work in [50] does not address the questions posed in (Q2) and (Q3).

Our main contribution is the establishment of the convergence rate of all three questions in the above in LQG framework. Firstly, this chapter establishes that the convergence rate of the p-Wasserstein metric for the distribution of the representative player is $O(N^{-1/2})$ for $p \in [1, 2]$. Secondly, it demonstrates that the convergence rate of the p-Wasserstein metric for the empirical measure in the L^p sense is $O(N^{-1/(2p)})$ for $p \in [1, 2]$. Lastly, this chapter shows that the convergence rate of the uniform p-Wasserstein metric for the empirical measure in the L^p sense is $O(N^{-1/(2p)})$ for $p \in (1, 2]$, and $O(N^{-1/2}\ln(N))$ for $p = 1$.

It is worth noting that the convergence rates obtained for (Q1) and (Q2) in the LQG framework align with the results found in existing literature, albeit under different conditions. Additionally, it is revealed that the uniform convergence rate of (Q3) may be slower than that of (Q2), which is consistent with the observations made by [21] from a similar perspective. Interestingly, when considering the specific case where $p = 2$ and $d = 1$, the uniform convergence rate of (Q3) is established as $O(N^{-1/9})$ according to [21], while it is determined to be $O(N^{-1/4})$ within our framework that incorporates the LQG structure.

Regarding (Q2), if the states $(\hat{X}_{it}^{(N)} : 1 \leq i \leq N)$ were independent, the convergence rate could be determined as $1/(2p)$ based on Theorem 1 of [32] and Theorem 5.8 of [16], which provide convergence rates for empirical measures of independent and identically distributed sequences. However, in the mean-field game, the states $\hat{X}_{it}^{(N)}$ are not independent of each other, despite having identical distributions. The correlation is introduced mainly by two factors: One is the system coupling arising from the mean-field term and the other is the common noise. Consequently, determining the convergence rate requires understanding the contributions of these two factors to the correlation among players.

In our proof, we rely on a specific decomposition (refer to Lemma 53 and the proof of the main theorem) of the underlying states. This decomposition reveals that the states can be expressed as a sum of a weakly correlated triangular array and a common noise. By analyzing the behavior of these components, we can address the correlation and establish the convergence rate.

Additionally, it is worth mentioning that a similar technique of dimension reduction in N-player LQG games have been previously utilized in [46] and related papers to establish decentralized Nash equilibria and the convergence rate in terms of value functions.

The remainder of the chapter is organized as follows: Section 4.2 outlines the problem setup and presents the main result. The proof of the main result, which relies on two propositions, is provided in Section 4.3. We establish the proof for these two propositions in Section 4.4 and Section 4.5. Some lemmas are given in the Appendix.

4.2 Problem setup and main results

4.2.1 The formulation of equilibrium in mean field games

In this section, we present the formulation of the MFG in the sample space Ω.

Let $T > 0$ be a given time horizon. We assume that $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion constructed on the probability space $(\bar{\Omega}, \bar{\mathcal{F}} = \bar{\mathcal{F}}_T, \bar{\mathbb{P}}, \bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t\}_{t\geq 0})$. Similarly, the process $\tilde{W} = \{\tilde{W}_t\}_{t\geq 0}$ is a standard Brownian motion constructed on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}} = \tilde{\mathcal{F}}_T, \tilde{\mathbb{P}}, \tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t\geq 0})$. We define the product structure as follows:

$$\Omega = \bar{\Omega} \times \tilde{\Omega}, \quad \mathcal{F}, \quad \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \quad \mathbb{P},$$

where $(\mathcal{F}, \mathbb{P})$ is the completion of $(\bar{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \bar{\mathbb{P}} \otimes \tilde{\mathbb{P}})$ and $\mathbb{F}$ is the complete and right continuous augmentation of $\{\bar{\mathcal{F}}_t \otimes \tilde{\mathcal{F}}_t\}_{t\geq 0}$.

Note that, W and $\tilde{W}$ are two Brownian motions from separate sample spaces $\bar{\Omega}$ and $\tilde{\Omega}$, they are independent of each other in their product space Ω. In our manuscript, W is called individual or idiosyncratic noise, and $\tilde{W}$ is called common noise, see their different roles in the problem formulation later defined via fixed point condition (4.2.4). To proceed, we denote by $L^p := L^p(\Omega, \mathbb{P})$

the set of random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite p-th moment with norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$ and by $L^p_{\mathbb{F}} := L^p_{\mathbb{F}}(\Omega \times [0,T])$ the space of all $\mathbb{R}$ valued $\mathbb{F}$-progressively measurable random processes α such that

$$\mathbb{E}\left[\int_0^T |\alpha_t|^p dt\right] < \infty.$$

Let $\mathcal{P}_p(\mathbb{R})$ denote the Wasserstein space of probability measures μ on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} |x|^p d\mu(x) < \infty$ endowed with p-Wasserstein metric $\mathbb{W}_p(\cdot,\cdot)$ defined by

$$\mathbb{W}_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{\mathbb{R}\times\mathbb{R}} |x-y|^p d\pi(x,y)\right)^{\frac{1}{p}},$$

where $\Pi(\mu,\nu)$ is the collection of all probability measures on $\mathbb{R}\times\mathbb{R}$ with its marginals agreeing with μ and ν.

Let $X_0 \in L^2$ be a random variable that is independent with W and $\tilde{W}$. For any control $\alpha \in L^2_{\mathbb{F}}$, consider the state $X = \{X_t\}_{t\geq 0}$ of the generic player governed by a stochastic differential equation (SDE)

$$dX_t = \alpha_t dt + dW_t + d\tilde{W}_t \tag{4.2.1}$$

with the initial value X_0, where the underlying process $X : [0,T]\times\Omega \mapsto \mathbb{R}$. Given a random measure flow $m : (0,T]\times\Omega \mapsto \mathcal{P}_2(\mathbb{R})$, the generic player wants to minimize the expected accumulated cost on $[0,T]$:

$$J(x,\alpha) = \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_s^2 + F(X_s, m_s)\right) ds \,\middle|\, X_0 = x\right] \tag{4.2.2}$$

with some given cost function $F : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$.

The objective of the control problem for the generic player is to find its optimal control $\hat{\alpha} \in \mathcal{A} := L^4_{\mathbb{F}}$ to minimize the total cost, i.e.,

$$V[m](x) = J[m](x,\hat{\alpha}) \leq J[m](x,\alpha), \quad \forall \alpha \in \mathcal{A}. \tag{4.2.3}$$

Associated to the optimal control $\hat{\alpha}$, we denote the optimal path by $\hat{X} = \{\hat{X}_t\}_{t\geq 0}$.

Next, to introduce the MFG Nash equilibrium, it is useful to emphasize the dependence of the optimal path and optimal control of the generic player, as well as its associated value, on the underlying measure flow m. These quantities are denoted as $\hat{X}_t[m]$, $\hat{\alpha}_t[m]$, $J[m]$, and $V[m]$, respectively.

We now present the definitions of the equilibrium measure, equilibrium path, and equilibrium control. Please also refer to page 127 of [17] for a general setup with a common noise.

Definition 40. *Given an initial distribution $\mathcal{L}(X_0) = m_0 \in \mathcal{P}_2(\mathbb{R})$, a random measure flow $\hat{m} = \hat{m}(m_0)$ is said to be an MFG equilibrium measure if it satisfies the fixed point condition*

$$\hat{m}_t = \mathcal{L}\left(\hat{X}_t[\hat{m}] \,\middle|\, \tilde{\mathcal{F}}_t\right), \ \forall 0 < t \leq T, \quad \textit{almost surely in } \mathbb{P}. \tag{4.2.4}$$

The path $\hat{X}$ and the control $\hat{\alpha}$ associated with $\hat{m}$ are called the MFG equilibrium path and equilibrium *control, respectively.*

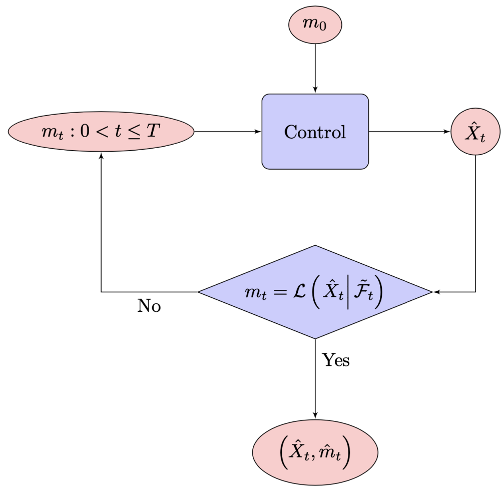


Figure 4.1: MFG diagram 2.

The flowchart of the MFG diagram is given in Figure 4.1. It is noted from the optimality condition (4.2.3) and the fixed point condition (4.2.4) that

$$J[\hat{m}](x, \hat{\alpha}) \leq J[\hat{m}](x, \alpha), \quad \forall \alpha$$

holds for the equilibrium measure $\hat{m}$ and its associated equilibrium control $\hat{\alpha}$, while it is not

$$J[\hat{m}](x, \hat{\alpha}) \leq J[m](x, \alpha), \quad \forall \alpha, m.$$

Otherwise, this problem turns into a McKean-Vlasov control problem, which is essentially different from the current Mean Field Games setup. Readers refer to [19, 18] to see the analysis of this different model as well as some discussion of the differences between these two problems.

4.2.2 The formulation of Nash equilibrium in the N-player game

In this subsection, we set up the N-player game and define the Nash equilibrium of the N-player game in the sample space $\Omega^{(N)}$. Firstly, let $W^{(N)} = (W_i^{(N)} : i = 1, 2, \ldots, N)$ be an N-dimensional standard Brownian motion constructed on the space $(\bar{\Omega}^{(N)}, \bar{\mathcal{F}}^{(N)}, \bar{\mathbb{P}}^{(N)}, \bar{\mathbb{F}}^{(N)} = \{\bar{\mathcal{F}}_t^{(N)}\}_{t\geq 0})$ and $\tilde{W} = \{\tilde{W}_t\}_{t\geq 0}$ be the common noise in MFG defined in Section 4.2.1 on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The probability space for the N-player game is $\left(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{F}^{(N)}, \mathbb{P}^{(N)}\right)$, which is constructed via the product structure with

$$\Omega^{(N)} = \bar{\Omega}^{(N)} \times \tilde{\Omega}, \quad \mathcal{F}^{(N)}, \quad \mathbb{F}^{(N)} = \left\{\mathcal{F}_t^{(N)}\right\}_{t\geq 0}, \quad \mathbb{P}^{(N)}.$$

where $(\mathcal{F}^{(N)}, \mathbb{P}^{(N)})$ is the completion of $(\bar{\mathcal{F}}^{(N)} \otimes \tilde{\mathcal{F}}, \bar{\mathbb{P}}^{(N)} \otimes \tilde{\mathbb{P}})$ and $\mathbb{F}^{(N)}$ is the complete and right continuous augmentation of $\{\bar{\mathcal{F}}_t^{(N)} \otimes \tilde{\mathcal{F}}_t\}_{t\geq 0}$.

Consider a stochastic dynamic game with N players, where each player $i \in \{1, 2, \dots, N\}$ controls a state process $X_i^{(N)} = \{X_{it}^{(N)}\}_{t\geq 0}$ in $\mathbb{R}$ given by

$$dX_{it}^{(N)} = \alpha_{it}^{(N)} dt + dW_{it}^{(N)} + d\tilde{W}_t, \quad X_{i0}^{(N)} = x_i^{(N)} \tag{4.2.5}$$

with a control $\alpha_i^{(N)}$ in an admissible set $\mathcal{A}^{(N)} := L^4_{\mathbb{F}^{(N)}}$ and random initial state $x_i^{(N)}$.

Given the strategies $\alpha_{-i}^{(N)} = (\alpha_1^{(N)}, \dots, \alpha_{i-1}^{(N)}, \alpha_{i+1}^{(N)}, \dots, \alpha_N^{(N)})$ from other players, the objective of player i is to select a control $\alpha_i^{(N)} \in \mathcal{A}^{(N)}$ to minimize her expected total cost given by

$$J_i^N \left(x^{(N)}, \alpha_i^{(N)}; \alpha_{-i}^{(N)}\right) = \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\left(\alpha_{it}^{(N)}\right)^2 + F\left(X_{it}^{(N)}, \rho\left(X_t^{(N)}\right)\right)\right) dt \,\middle|\, X_0^{(N)} = x^{(N)}\right], \tag{4.2.6}$$

where $x^{(N)} = (x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})$ is a $\mathbb{R}^N$-valued random vector in $\Omega^{(N)}$ to denote the initial state for N players, and

$$\rho\left(x^{(N)}\right) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^{(N)}}$$

is the empirical measure of the vector $x^{(N)}$ with Dirac measure δ. We use the notation $\alpha^{(N)} := (\alpha_i^{(N)}, \alpha_{-i}^{(N)}) = (\alpha_1^{(N)}, \alpha_2^{(N)}, \dots, \alpha_N^{(N)})$ to denote the control from N players as a whole. Next, we give the equilibrium value function and equilibrium path in the sense of the Nash game.

Definition 41. *1. The value function of player i for $i = 1, 2, \dots, N$ of the Nash game is defined by $V^N = (V_i^N : i = 1, 2, \dots, N)$ satisfying the equilibrium condition*

$$V_i^N\left(x^{(N)}\right) := J_i^N\left(x^{(N)}, \hat{\alpha}_i^{(N)}; \hat{\alpha}_{-i}^{(N)}\right) \leq J_i^N\left(x^{(N)}, \alpha_i^{(N)}; \hat{\alpha}_{-i}^{(N)}\right), \tag{4.2.7}$$

for all $\alpha_i^{(N)} \in \mathcal{A}^{(N)}$.

2. The equilibrium path of the N-player game is the N-dimensional random path $\hat{X}_t^{(N)} = (\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \dots, \hat{X}_{Nt}^{(N)})$ driven by (4.2.5) associated to the control $\hat{\alpha}_t^{(N)}$ satisfying the equilibrium condition of (4.2.7).

4.2.3 Main result

We consider three convergence questions on the N-player game defined in $\Omega^{(N)}$: The first one is the convergence of the representative path $\hat{X}_{it}^{(N)}$, the second one is the convergence of the empirical measure $\rho(\hat{X}_t^{(N)})$, while the last one is the t-uniform convergence of the empirical measure $\rho(\hat{X}_t^{(N)})$. To be precise, we shall assume the following throughout this chapter:

Assumption 42.

- $\mathbb{E}[|X_0|^q] < \infty$ *for some* $q > 4$.
- *The initials $X_{i0}^{(N)}$ of the N-player game are i.i.d. random variables in $\Omega^{(N)}$ with the same distribution as $\mathcal{L}(X_0)$ in the MFG.*

Note that the equilibrium path $\hat{X}_t^{(N)} = (\hat{X}_{it}^{(N)} : i = 1, 2, \ldots, N)$ is a vector-valued stochastic process. Due to the Assumption 42, the game is invariant to index reshuffling of N players and the elements in $(\hat{X}_{it}^{(N)} : i = 1, 2, \ldots, N)$ have identical distributions, but they are not independent of each other.

So, the first question on the representative path is indeed about $\hat{X}_{1t}^{(N)}$ in $\Omega^{(N)}$ and we are interested in how fast it converges to $\hat{X}_t$ in Ω in distribution:

(Q1) The $\mathbb{W}_p$-convergence rate of the representative equilibrium path,

$$\mathbb{W}_p\left(\mathcal{L}\left(\hat{X}_{1t}^{(N)}\right), \mathcal{L}\left(\hat{X}_t\right)\right) = O\left(N^{-?}\right).$$

The second question is about the convergence of the empirical measure $\rho(\hat{X}_t^{(N)})$ of the N-player game defined by

$$\rho\left(\hat{X}_t^{(N)}\right) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\hat{X}_{it}^{(N)}}.$$

We are interested in how fast this converges to the MFG equilibrium measure given by

$$\hat{m}_t = \mathcal{L}\left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right), \quad \forall t \in (0, T].$$

(Q2') The $\mathbb{W}_p$-convergence rate of empirical measures,

$$\mathbb{W}_p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right)\right) = O\left(N^{-?}\right).$$

Note that the left-hand side of the above equality is a random quantity and one shall be more precise about what the Big O notation means in this context. Indeed, by the definition of the empirical measure, $\rho(\hat{X}_t^{(N)})$ is a random distribution measurable by σ-algebra generated by the random vector $\hat{X}_t^{(N)}$. On the other hand, $\mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t)$ is a random distribution measurable by the σ-algebra $\tilde{\mathcal{F}}_t$. Therefore, from the construction of the product probability space $\Omega^{(N)}$ in Section 4.2.2, both random distributions $\rho(\hat{X}_t^{(N)})$ and $\mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t)$ are measurable with respect to $\mathcal{F}_t^{(N)} = \bar{\mathcal{F}}_t^{(N)} \otimes \tilde{\mathcal{F}}_t$. Consequently, $\mathbb{W}_p(\rho(\hat{X}_t^{(N)}), \mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t))$ is a random variable in the probability space $(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{P}^{(N)})$ and we will focus on a version of (Q2') in the L^p sense:

(Q2) The $\mathbb{W}_p$-convergence rate of empirical measures in L^p sense for each $t \in [0, T]$,

$$\left(\mathbb{E}\left[\mathbb{W}_p^p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right)\right)\right]\right)^{\frac{1}{p}} = O\left(N^{-?}\right).$$

In addition, we also study the following related question:

(Q3) The t-uniform $\mathbb{W}_p$-convergence rate of empirical measures in L^p sense,

$$\left(\mathbb{E}\left[\sup_{0\le t\le T} \mathbb{W}_p^p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right)\right)\right]\right)^{\frac{1}{p}} = O\left(N^{-?}\right).$$

In this chapter, we will study the above three questions (Q1), (Q2), and (Q3) in the framework of LQG structure with Brownian motion as a common noise with the following function F in the cost functional (4.2.2).

Assumption 43. *Let the function $F : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ be given in the form of*

$$F(x, m) = k \int_{\mathbb{R}} (x - z)^2 m(dz) = k(x^2 - 2x[m]_1 + [m]_2) \tag{4.2.8}$$

for some $k > 0$, where $[m]_1, [m]_2$ are the first and second moment of the measure m.

The main result of this chapter is presented below. Let us recall that q denotes the parameter defined in Assumption 42.

Theorem 44. *Under Assumptions 42-43, for any $p \in [1, 2]$, we have*

1. *The $\mathbb{W}_p$-convergence rate of the representative equilibrium path is $1/2$, i.e.,*

$$\mathbb{W}_p \left(\mathcal{L}\left(\hat{X}_{1t}^{(N)}\right), \mathcal{L}\left(\hat{X}_t\right) \right) = O\left(N^{-\frac{1}{2}}\right).$$

2. *The $\mathbb{W}_p$-convergence rate of empirical measures in L^p sense is*

$$\mathbb{E}\left[\mathbb{W}_p^p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right)\right)\right] = O\left(N^{-\frac{1}{2}}\right).$$

3. *The uniform $\mathbb{W}_p$-convergence rate of empirical measures in L^p sense is*

$$\mathbb{E}\left[\sup_{0 \le t \le T} \mathbb{W}_p^p\left(\rho\left(\hat{X}_t^{(N)}\right), \mathcal{L}\left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right)\right)\right] = \begin{cases} O\left(N^{-\frac{1}{2}} \ln(N)\right), & \text{if } p = 1, \\ O\left(N^{-\frac{1}{2}}\right), & \text{if } 1 < p \le 2. \end{cases}$$

We would like to provide some additional remarks on our main result. Firstly, the cost function F defined in (4.2.6) applies to the running cost for the i-th player in the N-player game, and it takes the form:

$$F\left(X_{it}^{(N)}, \rho\left(X_t^{(N)}\right)\right) = \frac{k}{N} \sum_{j=1}^{N} \left(X_{it}^{(N)} - X_{jt}^{(N)}\right)^2. \tag{4.2.9}$$

Interestingly, if $k < 0$, although F does satisfy the Lasry-Lions monotonicity ([13]) as demonstrated in Appendix 6.1 of [50], there is no global solution for MFG due to the concavity in x. On the contrary, when $k > 0$, F satisfies the displacement monotonicity proposed in [33] as shown by the following derivation:

$$\mathbb{E}\left[(F_x(X_1, \mathcal{L}(X_1)) - F_x(X_2, \mathcal{L}(X_2)))(X_1 - X_2)\right] = 2k\left(\mathbb{E}\left[(X_1 - X_2)^2\right] - (\mathbb{E}[X_1 - X_2])^2\right) \ge 0.$$

4.3 Proof of the main result with two propositions

Our objective is to investigate the relations between $(\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \ldots, \hat{X}_{Nt}^{(N)})$ and $\hat{X}_t$ described in (Q1), (Q2), and (Q3). In this part, we will give the proof of Theorem 44 based on two propositions whose proofs will be given later.

Proposition 45. *Under Assumptions 42-43, the MFG equilibrium path $\hat{X} = \hat{X}[\hat{m}]$ is given by*

$$d\hat{X}_t = -2a(t)\left(\hat{X}_t - \hat{\mu}_t\right)dt + dW_t + d\tilde{W}_t, \quad \hat{X}_0 = X_0, \tag{4.3.1}$$

where a is the solution of

$$a'(t) - 2a^2(t) + k = 0, \quad a(T) = 0, \tag{4.3.2}$$

and $\hat{\mu}$ is

$$\hat{\mu}_t := \mathbb{E}\left[\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right] = \mathbb{E}[X_0] + \tilde{W}_t.$$

Moreover, the equilibrium control follows

$$\hat{\alpha}_t = -2a(t)\left(\hat{X}_t - \hat{\mu}_t\right). \tag{4.3.3}$$

Proposition 46. *Suppose Assumptions 42-43 hold. For the N-player game, the path and the control of player i under the equilibrium are given by*

$$d\hat{X}_{it}^{(N)} = -2a^N(t)\left(\hat{X}_{it}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^{N}\hat{X}_{jt}^{(N)}\right)dt + dW_{it}^{(N)} + d\tilde{W}_t, \tag{4.3.4}$$

and

$$\hat{\alpha}_{it}^{(N)} = -2a^N(t)\left(\hat{X}_{it}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^{N}\hat{X}_{jt}^{(N)}\right)$$

respectively for $i = 1, 2, \ldots, N$, where a^N is the solution of

$$a' - \frac{2(N+1)}{N-1}a^2 + \frac{N-1}{N}k = 0, \quad a(T) = 0. \tag{4.3.5}$$

4.3.1 Preliminaries

We first recall the convergence rate of empirical measures of i.i.d. sequence provided in Theorem 1 of [32] and Theorem 5.8 of [16].

Lemma 47. *Let $d = 1$ or 2. Suppose $\{X_i : i \in \mathbb{N}\}$ is a sequence of d dimensional i.i.d. random variables with $\mathbb{E}[|X_1|^q] < \infty$ for some $q > 4$. Then, the empirical measure*

$$\rho^N(X) = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i}$$

satisfies

$$\mathbb{E}\left[\mathbb{W}_p^p\left(\rho^N(X),\mathcal{L}(X_1)\right)\right]=\begin{cases}O\left(N^{-1/2}\right), & \text{if } p\in(1,2],\\ O\left(N^{-1/2}\right), & \text{if } p=1, d=1,\\ O\left(N^{-1/2}\ln N\right), & \text{if } p=1, d=2.\end{cases}$$

Next, we give the definition of some notations that will be used in the following part. Denote $C_b(\mathbb{R}^d)$ to be the collection of bounded and continuous functions on $\mathbb{R}^d$, and let $C_b^1(\mathbb{R}^d)\subset C_b(\mathbb{R}^d)$ be the space of functions on $\mathbb{R}^d$ whose first order derivative is also bounded and continuous.

Lemma 48. *Suppose m_1, m_2 are two probability measures on $\mathcal{B}(\mathbb{R}^d)$ and $f\in C_b^1(\mathbb{R}^d,\mathbb{R})$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel set on $\mathbb{R}^d$. Then,*

$$\mathbb{W}_p(f_*m_1, f_*m_2)\le |Df|_0\mathbb{W}_p(m_1,m_2),$$

where f_*m_j is the pushforward measure for $j=1,2$, and $|Df|_0=\sup_{x\in\mathbb{R}^d}\max\{|\partial_{x_i}f(x)|: i=1,2,\dots,d\}$.

Proof. We define a function $F(x,y)=(f(x),f(y)):\mathbb{R}^{2d}\mapsto\mathbb{R}^2$. Note that, for any $\pi\in\Pi(m_1,m_2)$, $F_*\pi\in\Pi(f_*m_1,f_*m_2)$, i.e.,

$$F_*\Pi(m_1,m_2)\subset\Pi(f_*m_1,f_*m_2).$$

Therefore, we have the following inequalities:

$$\begin{aligned}\mathbb{W}_p^p(f_*m_1,f_*m_2)&=\inf_{\pi'\in\Pi(f_*m_1,f_*m_2)}\int_{\mathbb{R}^2}|x-y|^p\pi'(dx,dy)\\ &\le\inf_{\pi'\in F_*\Pi(m_1,m_2)}\int_{\mathbb{R}^2}|x-y|^p\pi'(dx,dy)\\ &=\inf_{\pi\in\Pi(m_1,m_2)}\int_{\mathbb{R}^{2d}}|f(x)-f(y)|^p\pi(dx,dy)\\ &\le|Df|_0^p\inf_{\pi\in\Pi(m_1,m_2)}\int_{\mathbb{R}^{2d}}|x-y|^p\pi(dx,dy)\\ &=|Df|_0^p\mathbb{W}_p^p(m_1,m_2).\end{aligned}$$

□

Lemma 49. *Let $\{X_i: i\in\mathbb{N}\}$ be a sequence of d dimensional random variables in $(\Omega,\mathcal{F},\mathbb{P})$. Let $f\in C_b^1(\mathbb{R}^d)$. We also denote by $f(X)$ the sequence $\{f(X_i): i\in\mathbb{N}\}$. Then*

$$\mathbb{W}_p\left(\rho^N(f(X)),\mathcal{L}(f(X_1))\right)\le|Df|_0\mathbb{W}_p\left(\rho^N(X),\mathcal{L}(X_1)\right),\quad \textit{almost surely}$$

where $|Df|_0=\sup_{x\in\mathbb{R}^d}\max\{|\partial_{x_i}f(x)|: i=1,2,\dots,d\}$.

Proof. For any sequence $\{c_i: i\in\mathbb{N}\}$ in $\mathbb{R}^d$, the empirical measure $\rho^N(c):=\frac{1}{N}\sum_{i=1}^N\delta_{c_i}$ satisfies

$$\rho^N(f(c))=f_*\rho^N(c),$$

since

$$\langle \phi, \rho^N(f(c)) \rangle = \frac{1}{N}\sum_{i=1}^{N} \phi(f(c_i)) = \langle \phi \circ f, \rho^N(c) \rangle, \quad \forall \phi \in C_b(\mathbb{R}^d).$$

This implies that

$$\rho^N(f(X)) = f_* \rho^N(X), \quad \text{almost surely.}$$

On the other hand, we also have

$$\mathcal{L}(f(X_1))(A) = \mathbb{P}(f(X_1) \in A) = \mathbb{P}(X_1 \in f^{-1}(A)) = f_*\mathcal{L}(X_1)(A), \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

Therefore, the conclusion follows by applying Lemma 48. □

4.3.2 Empirical measures of a sequence with a common noise

We are going to apply lemmas from the previous subsection to study the convergence of empirical measures of a sequence with a common noise in the following sense.

Definition 50. *We say a sequence of random variables $X = \{X_i : i \in \mathbb{N}\}$ is a sequence with a common noise, if there exists a random variable β such that*

- *$X - \beta = \{X_i - \beta : i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables,*
- *β is independent to $X - \beta$.*

By this definition, a sequence with a common noise is i.i.d. if and only if β is a deterministic constant.

Example 2. *Let $q > 4$ be a given constant and $X = \{X_i : i \in \mathbb{N}\}$ be a 1-dimensional sequence of L^q random variables with a common noise term β, where*

$$X_i - \beta = \gamma_i + \lambda\alpha_i.$$

In above, $\{(\alpha_i, \gamma_i) : i \in \mathbb{N}\}$ is a sequence of 2-dimensional i.i.d. random variables independent to β, and λ is a given non-negative constant. Let $\rho^N(X)$ be the empirical measure defined by

$$\rho^N(X) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i}.$$

The first question is

(Qa) In Example 2, where does $\rho^N(X)$ converge to?

For any test function $\phi \in C_b(\mathbb{R})$,

$$\langle \phi, \rho^N(X) \rangle = \frac{1}{N}\sum_{i=1}^{N} \phi(X_i) = \frac{1}{N}\sum_{i=1}^{N} \phi(\gamma_i + \lambda\alpha_i + \beta).$$

Since β is independent to (α_i, γ_i), by Example 4.1.5 of [23] together with the Law of Large Numbers, we have

$$\frac{1}{N}\sum_{i=1}^{N}\phi(\gamma_i+\lambda\alpha_i+c)\to\mathbb{E}[\phi(\gamma_1+\lambda\alpha_1+c)]=\mathbb{E}[\phi(\gamma_1+\lambda\alpha_1+\beta)|\beta=c],\quad\forall c\in\mathbb{R}.$$

Therefore, we conclude that

$$\begin{aligned}\langle\phi,\rho^N(X)\rangle&\to\mathbb{E}[\phi(\gamma_1+\lambda\alpha_1+\beta)|\beta],\quad\beta-a.s.\\&=\langle\phi,\mathcal{L}(\gamma_1+\lambda\alpha_1+\beta|\beta)\rangle,\quad\beta-a.s.\end{aligned}$$

Hence, the answer for the (Qa) is

- $\rho^N(X)\Rightarrow\mathcal{L}(X_1|\beta)$, β-a.s. More precisely, since all random variables are square-integrable, the weak convergence implies, for all $p\in[1,2]$,

$$\mathbb{W}_p\left(\rho^N(X),\mathcal{L}\left(X_1|\beta\right)\right)\to 0,\quad\beta-a.s.$$

The next question is

(Qb) In Example 2, what's the convergence rate in the sense $\mathbb{E}\left[\mathbb{W}_p^p\left(\rho^N(X),\mathcal{L}\left(X_1|\beta\right)\right)\right]$?

Since β is independent to $\gamma_1+\lambda\alpha_1$, by Example 4.1.5 of [23], we have

$$\mathbb{E}[\phi(\gamma_1+\lambda\alpha_1+\beta)|\beta=c]=\mathbb{E}[\phi(\gamma_1+\lambda\alpha_1+c)],\quad\forall\phi\in C_b(\mathbb{R}),c\in\mathbb{R},$$

or equivalently, if one takes $c=\beta(\omega)$,

$$\mathcal{L}(X_1|\beta)(\omega)=\mathcal{L}(\gamma_1+\lambda\alpha_1+\beta|\beta)(\omega)=\mathcal{L}(\gamma_1+\lambda\alpha_1+c).$$

On the other hand, with $c=\beta(\omega)$,

$$\rho^N(X)(\omega)=\rho^N(X(\omega))=\frac{1}{N}\sum_{i=1}^{N}\delta_{\gamma_i(\omega)+\lambda\alpha_i(\omega)+c}.$$

From the above two identities, with $c=\beta(\omega)$, we can write

$$\mathbb{W}_p\left(\rho^N(X)(\omega),\mathcal{L}(X_1|\beta=c)(\omega)\right)=\mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\gamma_i(\omega)+\lambda\alpha_i(\omega)+c},\mathcal{L}(\gamma_1+\lambda\alpha_1+c)\right).\tag{4.3.6}$$

Now we can conclude (Qb) in the next lemma.

Lemma 51. *Let $p\in[1,2]$ be a given constant. For a sequence $X=\{X_i:i\in\mathbb{N}\}$ with a common noise β as of Example 2, we have*

$$\mathbb{E}\left[\mathbb{W}_p^p\left(\rho^N(X),\mathcal{L}(X_1|\beta)\right)\right]=O\left(N^{-\frac{1}{2}}\right).$$

Proof. Originally, $X_i = \gamma_i + \lambda\alpha_i + \beta$ of Example 2 are dependent due to the common term β. We apply (4.7.2) in Lemma 60 in Appendix to (4.3.6) and obtain

$$\begin{aligned}\mathbb{W}_p\left(\rho^N(X)(\omega), \mathcal{L}(X_1|\beta)(\omega)\right) &= \mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^N \delta_{\gamma_i(\omega)+\lambda\alpha_i(\omega)+\beta(\omega)}, \mathcal{L}(\gamma_1 + \lambda\alpha_1 + \beta(\omega))\right) \\ &= \mathbb{W}_p\left(\rho^N(\gamma(\omega) + \lambda\alpha(\omega)), \mathcal{L}(\gamma_1 + \lambda\alpha_1)\right).\end{aligned}$$

Now, the convergence of empirical measures is equivalent to the ones of i.i.d. sequence $\{\gamma_i + \lambda\alpha_i : i \in \mathbb{N}\}$. The conclusion follows from Lemma 47. $\square$

Next, we present the uniform convergence rate by combining Lemma 49.

Lemma 52. *In Example 2, we use $X(\lambda)$ to denote X to emphasize its dependence on λ. Then,*

$$\mathbb{E}\left[\sup_{\lambda\in[0,1]} \mathbb{W}_p^p\left(\rho^N(X(\lambda)), \mathcal{L}\left(X_1(\lambda)|\beta\right)\right)\right] = \begin{cases} O\left(N^{-\frac{1}{2}}\ln(N)\right), & \text{if } p = 1, \\ O\left(N^{-\frac{1}{2}}\right), & \text{if } 1 < p \le 2. \end{cases}$$

Proof. Note that, by (4.7.2) in Lemma 60 in Appendix,

$$\mathbb{W}_p^p\left(\rho^N(X(\lambda)), \mathcal{L}\left(X_1(\lambda)|\beta\right)\right) = \mathbb{W}_p^p\left(\rho^N(\gamma + \lambda\alpha), \mathcal{L}\left(\gamma_1 + \lambda\alpha_1\right)\right).$$

Next, applying Lemma 49 with $f(x, y) = x + \lambda y$, we obtain

$$\begin{aligned}\sup_{\lambda\in[0,1]} \mathbb{W}_p^p\left(\rho^N(\gamma + \lambda\alpha), \mathcal{L}\left(\gamma_1 + \lambda\alpha_1\right)\right) &\le \sup_{\lambda\in[0,1]} \max\{1, \lambda^p\}\mathbb{W}_p^p\left(\rho^N((\gamma, \alpha)), \mathcal{L}\left((\gamma_1, \alpha_1)\right)\right) \\ &= \mathbb{W}_p^p\left(\rho^N((\gamma, \alpha)), \mathcal{L}\left((\gamma_1, \alpha_1)\right)\right).\end{aligned}$$

At last, using Lemma 47 for the 2-dimensional i.i.d. sequence $\{(\gamma_i, \alpha_i) : i \in \mathbb{N}\}$, we obtain the desired conclusion. $\square$

4.3.3 Generalization of the convergence to triangular arrays

Unfortunately, $(\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \ldots, \hat{X}_{Nt}^{(N)})$ of the N-player's game does not have a clean structure with a common noise term β given in Example 2. Therefore, we need a generalization of the convergence result in Example 2 to a triangular array. To proceed, we provide the following lemma.

Lemma 53. *Let $\lambda > 0$, $q > 4$, and*

$$X_i^N(\lambda) = \gamma_i^N + \lambda\alpha_i^N + \Delta_i^N(\lambda) + \beta, \text{ and } \hat{X}(\lambda) = \hat{\gamma} + \lambda\hat{\alpha} + \beta,$$

where

- *$(\gamma^N, \alpha^N) = \{(\gamma_i^N, \alpha_i^N) : i \in \mathbb{N}\}$ is a sequence of 2-dimensional i.i.d. random variables with distribution identical to $\mathcal{L}((\hat{\gamma}, \hat{\alpha}))$ with $(\hat{\gamma}, \hat{\alpha}) \in L^q$ for some $q > 4$,*
- *$\beta \in L^q$ is independent to the random variables $(\gamma_i^N, \alpha_i^N, \hat{\gamma}, \hat{\alpha})$,*

- $\max_{i=1,2,\dots,N} \mathbb{E}\left[\sup_{\lambda\in[0,1]} |\Delta_i^N(\lambda)|^2\right] = O(N^{-1})$.

Let $\rho^N(X^N)$ be the empirical measure given by

$$\rho^N(X^N) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}.$$

Then, we have the following three results: For $p \in [1,2]$,

$$\mathbb{W}_p\left(\mathcal{L}\left(X_1^N(\lambda)\right), \mathcal{L}\left(\hat{X}(\lambda)\right)\right) = O\left(N^{-\frac{1}{2}}\right), \tag{4.3.7}$$

$$\sup_{\lambda\in[0,1]} \mathbb{E}\left[\mathbb{W}_p^p\left(\rho^N\left(X^N(\lambda)\right), \mathcal{L}\left(\hat{X}(\lambda)\middle|\,\beta\right)\right)\right] = O\left(N^{-\frac{1}{2}}\right), \tag{4.3.8}$$

and

$$\mathbb{E}\left[\sup_{\lambda\in[0,1]} \mathbb{W}_p^p\left(\rho^N\left(X^N(\lambda)\right), \mathcal{L}\left(\hat{X}(\lambda)\middle|\,\beta\right)\right)\right] = \begin{cases} O\left(N^{-\frac{1}{2}}\ln(N)\right), & \text{if } p = 1, \\ O\left(N^{-\frac{1}{2}}\right), & \text{if } p > 1. \end{cases} \tag{4.3.9}$$

Proof. We will omit the dependence of λ if there is no confusion, for instance, we use X in lieu of $X(\lambda)$. Since $\mathcal{L}(\hat{X}) = \mathcal{L}(X_1^N - \Delta_1^N)$, the first result (4.3.7) directly follows from

$$\mathbb{W}_p^p\left(\mathcal{L}\left(X_1^N\right), \mathcal{L}\left(\hat{X}\right)\right) \le \mathbb{E}\left[\left|\Delta_1^N\right|^p\right] \le \left(\mathbb{E}\left[\left|\Delta_1^N\right|^2\right]\right)^{\frac{p}{2}} = O\left(N^{-\frac{p}{2}}\right).$$

Next, we set $Y_i^N(\lambda) = \gamma_i^N + \lambda\alpha_i^N + \beta$. By the definition of empirical measures, we have

$$\mathbb{W}_p^p\left(\rho^N\left(X^N\right), \rho^N\left(Y^N\right)\right) \le \frac{1}{N}\sum_{i=1}^N \left|X_i^N - Y_i^N\right|^p = \frac{1}{N}\sum_{i=1}^N \left|\Delta_i^N\right|^p. \tag{4.3.10}$$

From the third condition on Δ_i^N, we obtain

$$\mathbb{E}\left[\mathbb{W}_p^p\left(\rho^N\left(X^N\right), \rho^N\left(Y^N\right)\right)\right] = O\left(N^{-\frac{p}{2}}\right).$$

By Lemma 51, we also have

$$\mathbb{E}\left[\mathbb{W}_p^p\left(\rho^N\left(Y^N\right), \mathcal{L}\left(\hat{X}\middle|\,\beta\right)\right)\right] = O\left(N^{-\frac{1}{2}}\right).$$

In the end, (4.3.8) follows from the triangle inequality together with the fact that $p \ge 1$. Finally, for the proof of (4.3.9), we first use (4.3.10) to write

$$\mathbb{W}_p^p\left(\rho^N\left(X^N(\lambda)\right), \mathcal{L}\left(\hat{X}(\lambda)\middle|\,\beta\right)\right) \le 2^{p-1}\left(\mathbb{W}_p^p\left(\rho^N\left(Y^N(\lambda)\right), \mathcal{L}\left(\hat{X}(\lambda)\middle|\,\beta\right)\right) + \frac{1}{N}\sum_{i=1}^N \left|\Delta_i^N(\lambda)\right|^p\right).$$

Applying Lemma 52 and the third condition on $\Delta_i^N(\lambda)$, we can conclude (4.3.9). $\square$

4.3.4 Proof of Theorem 44

For simplicity, let us introduce the following notations:

$$\mathcal{E}_t(a) = \exp\left\{\int_0^t a(s)ds\right\}, \quad \mathcal{E}_t(a, M) = \int_0^t \mathcal{E}_s(a)dM_s$$

for a deterministic function $a(\cdot)$ and a martingale $M = \{M_t\}_{t\geq 0}$. With these notations, one can write the solution to the Ornstein–Uhlenbeck process

$$dX_t = -a_t X_t dt + dM_t$$

for a determinant function a in the form of

$$\mathcal{E}_t(a)X_t = X_0 + \mathcal{E}_t(a, M). \tag{4.3.11}$$

For MFG equilibrium, we define

$$\tilde{X}_t = \hat{X}_t - \hat{\mu}_t.$$

According to (4.3.1) in Proposition 45, $\tilde{X}$ satisfies the following equation:

$$\tilde{X}_t = \tilde{X}_0 - \int_0^t 2a_s\tilde{X}_s ds + W_t.$$

Next, we express the solution of the above SDE in the form of

$$\tilde{Y}_t := \mathcal{E}_t(2a)\tilde{X}_t = \tilde{X}_0 + \mathcal{E}_t(2a, W).$$

Note that $\tilde{Y}$ and $\hat{\mu}$ are independent. Therefore, $\hat{X}$ admits a decomposition of two independent processes as

$$\hat{X}_t = \tilde{X}_t + \hat{\mu}_t.$$

Furthermore, we have

$$\hat{Y}_t := \mathcal{E}_t(2a)\hat{X}_t = \tilde{X}_0 + \mathcal{E}_t(2a, W) + \mathcal{E}_t(2a)\left(\hat{\mu}_0 + \tilde{W}_t\right).$$

In the N-player game, we define the following quantities:

$$\bar{X}_t^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{X}_{it}^{(N)}, \quad \bar{W}_t^{(N)} = \frac{1}{N}\sum_{i=1}^N W_{it}^{(N)},$$

and

$$\tilde{X}_{it}^{(N)} = \hat{X}_{it}^{(N)} - \bar{X}_t^{(N)}.$$

It is worth noting that, by Proposition 46, we have

$$\hat{X}_{it}^{(N)} = \hat{X}_{i0}^{(N)} - \int_0^t 2\frac{N}{N-1}a^N(s)\left(\hat{X}_{is}^{(N)} - \frac{1}{N}\sum_{j=1}^{N}\hat{X}_{js}^{(N)}\right)ds + W_{it}^{(N)} + \tilde{W}_t$$

for all $i = 1, 2, \ldots, N$, then the mean-field term satisfies

$$\bar{X}_t^{(N)} = \bar{X}_0^{(N)} + \bar{W}_t^{(N)} + \tilde{W}_t$$

and the i-th player's path deviated from the mean-field path can be rewritten by

$$\tilde{X}_{it}^{(N)} = \tilde{X}_{i0}^{(N)} - \int_0^t 2\hat{a}^N(s)\tilde{X}_{is}^{(N)}ds + W_{it}^{(N)} - \bar{W}_t^{(N)},$$

where

$$\hat{a}^N = \frac{N}{N-1}a^N.$$

Next, we introduce

$$\hat{Y}_{it}^{(N)} = \mathcal{E}_t\left(2\hat{a}^N\right)\hat{X}_{it}^{(N)}, \quad \tilde{Y}_{it}^{(N)} = \mathcal{E}_t\left(2\hat{a}^N\right)\tilde{X}_{it}^{(N)}, \quad \bar{Y}_t^{(N)} = \mathcal{E}_t\left(2\hat{a}^N\right)\bar{X}_t^{(N)}.$$

Consequently, we obtain the following relationships:

$$\tilde{Y}_{it}^{(N)} = \tilde{X}_{i0}^{(N)} + \mathcal{E}_t\left(2\hat{a}^N, W_i^{(N)} - \bar{W}^{(N)}\right),$$

$$\bar{Y}_t^{(N)} = \mathcal{E}_t\left(2\hat{a}^N\right)\left(\bar{W}_t^{(N)} + \tilde{W}_t + \bar{X}_0^{(N)}\right),$$

and

$$\hat{Y}_{it}^{(N)} = \bar{Y}_{it}^{(N)} + \tilde{Y}_{it}^{(N)}.$$

To compare the process $\hat{Y}_{it}^{(N)}$ with the target process

$$\begin{aligned}\hat{Y}_t &= \tilde{X}_0 + \mathcal{E}_t\left(2a, W\right) + \mathcal{E}_t(2a)\left(\hat{\mu}_0 + \tilde{W}_t\right) \\ &= \tilde{X}_0 + \lambda_t Z_t + \mathcal{E}_t(2a)\left(\hat{\mu}_0 + \tilde{W}_t\right),\end{aligned} \tag{4.3.12}$$

where

$$\lambda_t = \left(\int_0^t \mathcal{E}_s(4a)ds\right)^{1/2},$$

and

$$Z_t = \lambda_t^{-1}\mathcal{E}_t\left(2a, W\right) \sim \mathcal{N}(0,1),$$

we write $\hat{Y}_{it}^{(N)}$ by

$$\begin{aligned}\hat{Y}_{it}^{(N)} &= \tilde{X}_{i0}^{(N)} + \mathcal{E}_t\left(2a, W_i^{(N)}\right) + \Delta_{it}^{(N)} + \mathcal{E}_t(2a)\left(\hat{\mu}_0 + \tilde{W}_t\right) \\ &= \tilde{X}_{i0}^{(N)} + \lambda_t Z_{it}^{(N)} + \Delta_{it}^{(N)} + \mathcal{E}_t(2a)\left(\hat{\mu}_0 + \tilde{W}_t\right),\end{aligned} \tag{4.3.13}$$

where

$$Z_{it}^{(N)} = \lambda_t^{-1}\mathcal{E}_t\left(2a, W_i^{(N)}\right) \sim \mathcal{N}(0,1),$$

and

$$\begin{aligned}\Delta_{it}^{(N)} = &\left(\mathcal{E}_t\left(2\hat{a}^N, W_i^{(N)}\right) - \mathcal{E}_t\left(2a, W_i^{(N)}\right)\right) \\ &- \mathcal{E}_t\left(2\hat{a}^N, \bar{W}^{(N)}\right) \\ &+ \left(\mathcal{E}_t\left(2\hat{a}^N\right) - \mathcal{E}_t(2a)\right)\left(\hat{\mu}_0 + \tilde{W}_t\right) \\ &+ \mathcal{E}_t\left(2\hat{a}^N\right)\left(\bar{X}_0^{(N)} - \hat{\mu}_0 + \bar{W}_t^{(N)}\right) \\ := &\, I_{it}^{(N)} + II_t^{(N)} + III_t^{(N)} + IV_t^{(N)}.\end{aligned} \tag{4.3.14}$$

To apply Lemma 53 to the processes of (4.3.13) and (4.3.12), we only need to show the second moment on $\sup_{t\in[0,T]} \Delta_{it}^{(N)}$ is $O(N^{-1})$ for each $i = 1, 2, \ldots, N$. In the following analysis, we will utilize the explicit solution of the ODE:

- Let $c, d > 0$ be two constants. The solution of

  $$v'(t) - c^2 v^2(t) + d^2 = 0, \quad v(T) = 0$$

  is

  $$v(t) = \frac{d}{c} \cdot \frac{1 - e^{2dc(t-T)}}{1 + e^{2dc(t-T)}}. \tag{4.3.15}$$

We will employ this solution to derive the second-moment estimations of $\sup_{t\in[0,T]} \Delta_{it}^{(N)}$.

1. From (4.3.15), we have an estimation of

   $$\left|a^N(t) - a(t)\right| = \frac{k|T-t|}{N} + o\left(N^{-1}\right). \tag{4.3.16}$$

   Therefore, we have

   $$\left|\mathcal{E}_t(2\hat{a}^N) - \mathcal{E}_t(2a)\right| = \frac{2t(T-t)}{N} + o\left(N^{-1}\right) \tag{4.3.17}$$

and thus by Burkholder-Davis-Gundy (BDG) inequality

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left(I_{it}^{(N)}\right)^2\right] &= \mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t \left(\mathcal{E}_s\left(2\hat{a}^N\right) - \mathcal{E}_s\left(2a\right)\right) dW_{is}^{(N)}\right)^2\right] \\
&\le C\mathbb{E}\left[\left(\int_0^T \left(\mathcal{E}_s\left(2\hat{a}^N\right) - \mathcal{E}_s\left(2a\right)\right) dW_{is}^{(N)}\right)^2\right] \quad \text{for some constant } C > 0 \\
&= C\int_0^T \left(\mathcal{E}_s\left(2\hat{a}^N\right) - \mathcal{E}_s\left(2a\right)\right)^2 ds \\
&= O\left(N^{-2}\right).
\end{aligned}$$

2. Since $\hat{a}^N$ is uniformly bounded by $\sqrt{k/2}$, $II_t^{(N)}$ is a martingale with its quadratic variance

$$[II^{(N)}]_T = \frac{1}{N}\int_0^T \mathcal{E}_s(4\hat{a}^N)ds = O\left(N^{-1}\right).$$

   So, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(II_t^{(N)}\right)^2\right] = O\left(N^{-1}\right).$$

3. From the estimation (4.3.17), we also have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(III_t^{(N)}\right)^2\right] = O\left(N^{-2}\right).$$

4. By the assumption of i.i.d. initial states, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(IV_t^{(N)}\right)^2\right] = \mathcal{E}_T\left(4\hat{a}^N\right)\left(Var\left(\bar{X}_0^{(N)}\right) + \mathbb{E}\left[\sup_{t\in[0,T]}\left(\bar{W}_t^{(N)}\right)^2\right]\right) = O\left(N^{-1}\right).$$

As a result, we have the following expression:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(\Delta_{it}^{(N)}\right)^2\right] = O\left(N^{-1}\right), \quad \forall i = 1, 2, \ldots, N. \tag{4.3.18}$$

By combining equations (4.3.12), (4.3.13), and (4.3.18), we can conclude Theorem 44 by applying Lemma 53.

4.4 Proposition 45: Derivation of the MFG path

This section is dedicated to proving Proposition 45, which provides insights into the MFG solution. To proceed, in Subsection 4.4.1, we begin by reformulating the MFG problem, assuming a Markovian structure for the equilibrium. Then, in Subsection 4.4.2, we solve the underlying control problem and derive the corresponding Riccati system. Finally, in Subsection 4.4.3, we examine the fixed-point condition of the MFG problem, leading to the conclusion.

4.4.1 Reformulation

To determine the equilibrium measure, as defined in Definition 41, one needs to explore the infinite-dimensional space of random measure flows $m : (0,T] \times \Omega \to \mathcal{P}_2(\mathbb{R})$ until a measure flow satisfies the fixed-point condition $m_t = \mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t)$ for all $t \in (0,T]$, as illustrated in Figure 4.1.

The first observation is that the cost function F in (4.2.8) is only dependent on the measure m through the first two moments with the quadratic cost structure, which is given by

$$F(x,m) = k(x^2 - 2x[m]_1 + [m]_2).$$

Consequently, the underlying stochastic control problem for MFG can be entirely determined by the input given by the $\mathbb{R}^2$ valued random processes $\mu_t = [m_t]_1$ and $\nu_t = [m_t]_2$, which implies that the fixed point condition can be effectively reduced to merely checking two conditions:

$$\mu_t = \mathbb{E}\left[\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right],\ \nu_t = \mathbb{E}\left[\hat{X}_t^2 \middle| \tilde{\mathcal{F}}_t\right].$$

This observation effectively reduces our search from the space of random measure-valued processes $m : (0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$ to the space of $\mathbb{R}^2$-valued random processes $(\mu,\nu) : (0,T] \times \Omega \mapsto \mathbb{R}^2$.

It is important to note that if the underlying MFG does not involve common noise, the aforementioned observation is adequate to transform the original infinite-dimensional MFG into a finite-dimensional system. In this case, the moment processes (μ,ν) become deterministic mappings $[0,T] \to \mathbb{R}^2$. However, the following example demonstrates that this is not applicable to MFG with common noise, which presents a significant drawback in characterizing LQG-MFG using a finite-dimensional system.

Example 3. *To illustrate this point, let's consider the following uncontrolled mean field dynamics: Let the mean field term $\mu_t := \mathbb{E}[\hat{X}_t|\tilde{\mathcal{F}}_t]$, where the underlying dynamic is given by*

$$d\hat{X}_t = -\mu_t \tilde{W}_t dt + dW_t + d\tilde{W}_t, \quad \hat{X}_0 = X_0.$$

Here are two key observations:

- *μ_t is path dependent on entire path of $\tilde{W}$, i.e.,*

$$\mu_t = \mu_0 e^{-\int_0^t \tilde{W}_s ds} + e^{-\int_0^t \tilde{W}_s ds} \int_0^t e^{\int_0^s \tilde{W}_r dr} d\tilde{W}_s.$$

  This implies that the $(t,\tilde{W}) \mapsto \mu_t$ is a function on an infinite dimensional domain.

- *μ_t is Markovian, i.e.,*

$$d\mu_t = -\mu_t \tilde{W}_t dt + d\tilde{W}_t.$$

  It is possible to express the μ_t via a SDE with finite-dimensional coefficient functions of (t,μ_t).

To make the previous idea more concrete, we propose the assumption of a Markovian structure for the first and second moments of the MFG equilibrium. In other words, we restrict our search for equilibrium to a smaller space $\mathcal{M}$ of measure flows that capture the Markovian structure of the first and second moments.

Definition 54. *The space $\mathcal{M}$ is the collection of all $\tilde{\mathcal{F}}_t$-adapted measure flows $m : [0,T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, whose first moment $[m_t]_1 := \mu_t$ and second moment $[m_t]_2 := \nu_t$ satisfy a system of SDE*

$$\begin{aligned} \mu_t &= \mu_0 + \int_0^t (w_1(s)\mu_s + w_2(s))\, ds + \tilde{W}_t, \\ \nu_t &= \nu_0 + \int_0^t \left(w_3(s)\mu_s + w_4(s)\nu_s + w_5(s)\mu_s^2 + w_6(s)\right) ds + 2\int_0^t \mu_s d\tilde{W}_s, \end{aligned} \tag{4.4.1}$$

for some smooth deterministic functions $(w_i : i = 1, 2, \dots, 6)$ for all $t \in [0,T]$.

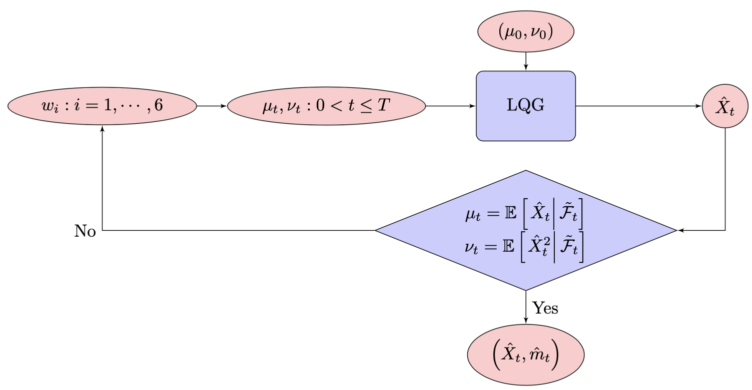


Figure 4.2: Equivalent MFG diagram 2.

The MFG problem originally given by Definition 40 can be recast as the following combination of stochastic control problem and fixed point condition:

- RLQG(Revised LQG):

  Given smooth functions $w = (w_i : i = 1, 2, \dots, 6)$, we want to find the value function $\bar{V} = \bar{V}[w] : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and optimal path $(\hat{X}, \hat{\mu}, \hat{\nu})[w]$ from the following control problem:

  $$\bar{V}(t, x, \bar{\mu}, \bar{\nu}) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^T \left(\frac{1}{2}\alpha_s^2 + \bar{F}(X_s, \mu_s, \nu_s)\right) ds \middle| X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu}\right]$$

  with the underlying process X of (4.2.1) and (μ, ν) of (4.4.1) and with the cost functions: $\bar{F} : \mathbb{R}^3 \mapsto \mathbb{R}$ given by

  $$\bar{F}(x, \bar{\mu}, \bar{\nu}) = k(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

  where $\bar{\mu}, \bar{\nu}$ are scalars, while μ, ν are used as processes.

- RFP(Revised fixed point condition):

Determine w satisfying the following fixed point condition:

$$\hat{\mu}_s = \mathbb{E}\left[\hat{X}_s\middle|\tilde{\mathcal{F}}_s\right] \text{ and } \hat{\nu}_s = \mathbb{E}\left[\hat{X}_s^2\middle|\tilde{\mathcal{F}}_s\right], \quad \forall s \in [0,T]. \tag{4.4.2}$$

The equilibrium measure is then $\mathcal{N}(\hat{\mu}_t, \hat{\nu}_t - \hat{\mu}_t^2)$.

Remark 55. *It is important to highlight that the Markovian structure for the first and second moments of the MFG equilibrium in this manuscript differs significantly from that presented in [50]. In [50], the processes μ_t and ν_t are pairs of processes with finite variation, while in our case, they are quadratic variation processes.*

Specifically, in [50], the coefficient functions depend on the common noise Y, whereas in (4.4.1), the coefficient functions $(w_i : i = 1, 2, \ldots, 6)$ are independent of the common noise $\tilde{W}$. Instead, the first and second moments of the MFG equilibrium are only influenced by the common noise through an additive term.

4.4.2 The generic player's control with a given population measure

This section is devoted to the control problem RLQG parameterized by w.

4.4.2.1 HJB equation

To simplify the notation, let's denote each function $w_i(t)$ as w_i for $i \in \{1, 2, \ldots, 6\}$. Assuming sufficient regularity conditions, and according to the dynamic programming principle (refer to [69] for more details), the value function $\bar{V}$ defined in the RLQG problem can be obtained as a solution v of the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \partial_t v + \inf_{a\in\mathbb{R}}\left(a\partial_x v + \frac{1}{2}a^2\right) + (w_1\bar{\mu} + w_2)\,\partial_{\bar{\mu}} v + \left(w_3\bar{\mu} + w_4\bar{\nu} + w_5\bar{\mu}^2 + w_6\right)\partial_{\bar{\nu}} v + \partial_{xx} v + \frac{1}{2}\partial_{\bar{\mu}\bar{\mu}} v \\ \qquad + \partial_{x\bar{\mu}} v + 2\bar{\mu}^2\partial_{\bar{\nu}\bar{\nu}} v + 2\bar{\mu}\partial_{\bar{\mu}\bar{\nu}} v + 2\bar{\mu}\partial_{x\bar{\nu}} v + k(x^2 - 2\bar{\mu}x + \bar{\nu}) = 0, \\ v(T, x, \mu_T, \nu_T) = 0. \end{cases}$$

Therefore, the optimal control has to admit the feedback form of

$$\hat{\alpha}(t) = -\partial_x v\left(t, \hat{X}_t, \mu_t, \nu_t\right), \tag{4.4.3}$$

and then the HJB equation can be reduced to

$$\begin{cases} \partial_t v - \frac{1}{2}(\partial_x v)^2 + (w_1\bar{\mu} + w_2)\,\partial_{\bar{\mu}} v + \left(w_3\bar{\mu} + w_4\bar{\nu} + w_5\bar{\mu}^2 + w_6\right)\partial_{\bar{\nu}} v + \partial_{xx} v + \frac{1}{2}\partial_{\bar{\mu}\bar{\mu}} v \\ \qquad + \partial_{x\bar{\mu}} v + 2\bar{\mu}^2\partial_{\bar{\nu}\bar{\nu}} v + 2\bar{\mu}\partial_{\bar{\mu}\bar{\nu}} v + 2\bar{\mu}\partial_{x\bar{\nu}} v + k(x^2 - 2\bar{\mu}x + \bar{\nu}) = 0, \\ v(T, x, \mu_T, \nu_T) = 0. \end{cases} \tag{4.4.4}$$

Next, we identify what conditions are needed for equating the control problem RLQG and the above HJB equation. Denote $\mathcal{S}$ to be the set of v such that $v \in C^\infty$ satisfies

$$\left(1+|x|^2\right)^{-1}(|v|+|\partial_t v|)+(1+|x|+|\mu|)^{-1}(|\partial_x v|+|\partial_\mu v|) \\ +(|\partial_{xx} v|+|\partial_{x\mu} v|+|\partial_{\mu\mu} v|+|\partial_\nu v|)<K$$

for all (t,x,μ,ν) for some positive constant K.

Lemma 56. *Consider the control problem RLQG with some given smooth functions $w=(w_i : i=1,2,\ldots,6)$.*

1. *(Verification theorem) Suppose there exists a solution $v \in \mathcal{S}$ of (4.4.4). Then $v(t,x,\bar{\mu},\bar{\nu})=\bar{V}(t,x,\bar{\mu},\bar{\nu})$, and an optimal control is provided by (4.4.3).*

2. *Suppose that the value function $\bar{V}$ belongs to $\mathcal{S}$, and then $\bar{V}(t,x,\bar{\mu},\bar{\nu})$ solves HJB equation (4.4.4). Moreover, $\hat{\alpha}$ of (4.4.3) is an optimal control.*

Proof. 1. First, we prove the verification theorem. Since $v \in \mathcal{S}$, for any admissible $\alpha \in \mathcal{H}^4_{\mathbb{F}}$, the process X^α is well defined and one can apply Itô's formula to obtain

$$\mathbb{E}\left[v(T,X_T,\mu_T,\nu_T)\right]=v(t,x,\bar{\mu},\bar{\nu})+\mathbb{E}\left[\int_t^T \mathcal{G}^{\alpha(s)}v(s,X_s,\mu_s,\nu_s)ds\right],$$

where

$$\mathcal{G}^a f(s,x,\bar{\mu},\bar{\nu})=\Big(\partial_t+a\partial_x+\partial_{xx}+(w_1\bar{\mu}+w_2)\,\partial_{\bar{\mu}}+\left(w_3\bar{\mu}+w_4\bar{\nu}+w_5\bar{\mu}^2+w_6\right)\partial_{\bar{\nu}} \\ +\frac{1}{2}\partial_{\bar{\mu}\bar{\mu}}+2\bar{\mu}^2\partial_{\bar{\nu}\bar{\nu}}+\partial_{x\bar{\mu}}+2\bar{\mu}\partial_{\bar{\mu}\bar{\nu}}+2\bar{\mu}\partial_{x\bar{\nu}}\Big)f(s,x,\bar{\mu},\bar{\nu}).$$

Note that the HJB equation actually implies that

$$\inf_a\left\{\mathcal{G}^a v+\frac{1}{2}a^2\right\}=-\bar{F},$$

which again yields

$$-\mathcal{G}^a v \le \frac{1}{2}a^2+\bar{F}.$$

Hence, we obtain that for all $\alpha \in \mathcal{H}^4_{\mathbb{F}}$,

$$\begin{aligned}
&v(t,x,\bar{\mu},\bar{\nu})\\
=\;&\mathbb{E}\left[\int_t^T -\mathcal{G}^{\alpha(s)}v(s,X_s,\mu_s,\nu_s)ds\right]+\mathbb{E}\left[v(T,X_T,\mu_T,\nu_T)\right]\\
\le\;&\mathbb{E}\left[\int_t^T\left(\frac{1}{2}\alpha^2(s)+\bar{F}(X_s,\mu_s,\nu_s)\right)ds\right]\\
=\;&J(t,x,\alpha,\bar{\mu},\bar{\nu}).
\end{aligned}$$

In the above, if α is replaced by $\hat{\alpha}$ given by the feedback form (4.4.3), then since $\partial_x v$ is Lipschitz continuous in x, there exists corresponding optimal path $\hat{X} \in \mathcal{H}^4_{\mathbb{F}}$. Thus, $\hat{\alpha}$ is also in $\mathcal{H}^4_{\mathbb{F}}$. One can repeat all the above steps by replacing X and α by $\hat{X}$ and $\hat{\alpha}$, and $\leq$ sign by $=$ sign to conclude that v is indeed the optimal value.

2. The opposite direction of the verification theorem follows by taking $\theta \to t$ for the dynamic programming principle, for all stopping time $\theta \in [t, T]$,

$$\begin{aligned}&\bar{V}(t, x, \bar{\mu}, \bar{\nu})\\ &= \mathbb{E}\left[\int_t^\theta \left(\frac{1}{2}\alpha_s^2 + \bar{F}(X_s, \mu_s, \nu_s)\right) ds + \bar{V}(\theta, X_\theta, \mu_\theta, \nu_\theta) \middle| X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu}\right],\end{aligned}$$

which is valid under our regularity assumptions on all the partial derivatives.

□

4.4.2.2 LQG solution

It is worth noting that the costs $\bar{F}$ of RLQG are quadratic functions in $(x, \bar{\mu}, \bar{\nu})$, while the drift function of the process ν of (4.4.1) is not linear in $(x, \bar{\mu}, \bar{\nu})$. Therefore, the stochastic control problem RLQG does not fit into the typical LQG control structure. Nevertheless, similarly to the LQG solution, we guess the value function to be a quadratic function in the form of

$$v(t, x, \bar{\mu}, \bar{\nu}) = a(t)x^2 + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t) + e(t)x + f(t)\bar{\mu} + g(t)x\bar{\mu}. \tag{4.4.5}$$

Under the above setup for the value function v, for $t \in [0, T]$, the optimal control is given by

$$\hat{\alpha}_t = -\partial_x v(t, \hat{X}_t, \mu_t, \nu_t) = -2a(t)\hat{X}_t - e(t) - g(t)\mu_t, \tag{4.4.6}$$

and the optimal path $\hat{X}$ is

$$\begin{cases} d\hat{X}_t = \left(-2a(t)\hat{X}_t - e(t) - g(t)\mu_t\right) dt + dW_t + d\tilde{W}_t, \\ \hat{X}_0 = X_0. \end{cases} \tag{4.4.7}$$

To proceed, we introduce the following Riccati system of ODEs for $t \in [0, T]$,

$$\begin{cases} a' - 2a^2 + k = 0, \\ b' - \dfrac{1}{2}g^2 + 2bw_1 + cw_5 = 0, \\ c' + cw_4 + k = 0, \\ d' - \dfrac{1}{2}e^2 + fw_2 + cw_6 + 2a + b + g = 0, \\ e' - 2ae + w_2 g = 0, \\ f' - eg + w_1 f + 2bw_2 + cw_3 = 0, \\ g' - 2ag + w_1 g - 2k = 0, \end{cases} \tag{4.4.8}$$

with terminal conditions

$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = g(T) = 0. \tag{4.4.9}$$

Lemma 57. *Suppose there exists a unique solution* (a, b, c, d, e, f, g) *to the Riccati system of ODEs* (4.4.8)-(4.4.9) *on* $[0, T]$. *Then the value function of (RMFG) is given by*

$$\begin{aligned} &\bar{V}(t, x, \bar{\mu}, \bar{\nu}) = v(t, x, \bar{\mu}, \bar{\nu}) \\ &= a(t)x^2 + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t) + e(t)x + f(t)\bar{\mu} + g(t)x\bar{\mu} \end{aligned} \tag{4.4.10}$$

for $t \in [0, T]$ *and the optimal control and optimal path are given by* (4.4.6) *and* (4.4.7), *respectively.*

Proof. With the form of value function v given in (4.4.5) and the conditional first and second moment of $\hat{X}_t$ under the σ-algebra $\tilde{\mathcal{F}}_t$ given in (4.4.1), we have

$$\begin{aligned} &\partial_t v = a'(t)x^2 + e'(t)x + b'(t)\bar{\mu}^2 + f'(t)\bar{\mu} + g'(t)x\bar{\mu} + c'(t)\bar{\nu} + d'(t), \\ &\partial_x v = 2xa(t) + e(t) + g(t)\bar{\mu}, \\ &\partial_{xx} v = 2a(t), \\ &\partial_{\bar{\mu}} v = 2b(t)\bar{\mu} + f(t) + g(t)x, \\ &\partial_{\bar{\nu}} v = c(t), \\ &\partial_{\bar{\mu}\bar{\mu}} v = 2b(t), \\ &\partial_{x\bar{\mu}} v = g(t), \\ &\partial_{\bar{\mu}\bar{\nu}} v = \partial_{\bar{\nu}\bar{\nu}} v = \partial_{x\bar{\nu}} v = 0. \end{aligned}$$

Plugging them back to the HJB equation in (4.4.4), we get a system of ODEs in (4.4.8) by equating x, $\bar{\mu}$, $\bar{\nu}$-like terms in each equation with the terminal conditions given in (4.4.9).

Therefore, any solution (a, b, c, d, e, f, g) of a system of ODEs (4.4.8) leads to the solution of HJB (4.4.4) in the form of the quadratic function given by (4.4.10). Since the (a, b, c, d, e, f, g) are differentiable functions on the closed set $[0, T]$, they are also bounded, and thus the regularity conditions needed for $v \in \mathcal{S}$ is valid. Finally, we invoke the verification theorem given by Lemma 56 to conclude the desired result. ☐

4.4.3 Fixed point condition and the proof of Proposition 45

Returning to the ODE system (4.4.8), there are 7 equations, whereas we need to determine a total of 13 deterministic functions of $[0, T] \times \mathbb{R}$ to characterize MFG. These are

$$(a, b, c, d, e, f, g) \quad \text{and} \quad (w_i : i = 1, 2, \dots, 6).$$

In this below, we identify the missing 6 equations by checking the fixed point condition of RFP. This leads to a complete characterization of the equilibrium for MFG in Definition 40.

Lemma 58. *With the dynamic of the optimal path* $\hat{X}$ *defined in* (4.4.7), *the fixed point condition* (4.4.2) *implies that the first moment* $\hat{\mu}_s := \mathbb{E}[\hat{X}_s|\tilde{\mathcal{F}}_s]$ *and the second moment* $\hat{\nu}_s := \mathbb{E}[\hat{X}_s^2|\tilde{\mathcal{F}}_s]$ *of the* *optimal path conditioned on $\tilde{\mathcal{F}}_t$ satisfy*

$$\begin{cases} \hat{\mu}_s = \bar{\mu} + \int_t^s \left((-2a(r) - g(r))\,\hat{\mu}_r - e(r)\right) dr + \tilde{W}_s, \\ \hat{\nu}_s = \bar{\nu} + \int_t^s \left(2 - 4a(r)\hat{\nu}_r - 2e(r)\hat{\mu}_r - 2g(r)\hat{\mu}_r^2\right) dr + \int_t^s 2\hat{\mu}_r d\tilde{W}_r, \end{cases} \tag{4.4.11}$$

for $s \geq t$, and thus the coefficient functions $w = (w_i : i = 1, 2, \ldots, 6)$ in (4.4.1) satisfy the following equations:

$$w_1 = -2a - g,\ w_2 = -e,\ w_3 = -2e,\ w_4 = -4a,\ w_5 = -2g,\ w_6 = 2, \quad \forall t \in [0, T]. \tag{4.4.12}$$

Proof. With the dynamic of the optimal path $\hat{X}$ given by (4.4.7), we have

$$\hat{X}_t = X_0 + \int_0^t \left(-2a(s)\hat{X}_s - e(s) - g(s)\hat{\mu}_s\right) ds + W_t + \tilde{W}_t,$$

and since the functions a, e, g are continuous on $[0, T]$, then we can change of order of integration and expectation and it yields

$$\begin{aligned} \hat{\mu}_t &= \mathbb{E}\left[\hat{X}_t \middle| \tilde{\mathcal{F}}_t\right] \\ &= \mathbb{E}\left[X_0 \middle| \tilde{\mathcal{F}}_t\right] + \int_0^t \left(-2a(s)\hat{\mu}_s - e(s) - g(s)\hat{\mu}_s\right) ds + \mathbb{E}\left[W_t + \tilde{W}_t \middle| \tilde{\mathcal{F}}_t\right] \\ &= \mathbb{E}\left[X_0 \middle| \tilde{\mathcal{F}}_t\right] + \int_0^t \left(-2a(s)\hat{\mu}_s - e(s) - g(s)\hat{\mu}_s\right) ds + \tilde{W}_t. \end{aligned}$$

Similarly, applying Itô's formula, we obtain

$$\hat{X}_t^2 = X_0^2 + \int_0^t \left(2 - 4a(s)\hat{X}_s^2 - 2e(s)\hat{X}_s - 2g(s)\hat{\mu}_s\hat{X}_s\right) ds + \int_0^t 2\hat{X}_s dW_s + \int_0^t 2\hat{X}_s d\tilde{W}_s,$$

and it follows that

$$\begin{aligned} \hat{\nu}_t &= \mathbb{E}\left[\hat{X}_t^2 \middle| \tilde{\mathcal{F}}_t\right] \\ &= \mathbb{E}\left[X_0^2 \middle| \tilde{\mathcal{F}}_t\right] + \int_0^t \left(2 - 4a(s)\hat{\nu}_s - 2e(s)\hat{\mu}_s - 2g(s)\hat{\mu}_s^2\right) ds + \mathbb{E}\left[\int_0^t 2\hat{X}_s dW_s + \int_0^t 2\hat{X}_s d\tilde{W}_s \middle| \tilde{\mathcal{F}}_t\right] \\ &= \mathbb{E}\left[X_0^2 \middle| \tilde{\mathcal{F}}_t\right] + \int_0^t \left(2 - 4a(s)\hat{\nu}_s - 2e(s)\hat{\mu}_s - 2g(s)\hat{\mu}_s^2\right) ds + \int_0^t 2\hat{\mu}_s d\tilde{W}_s. \end{aligned}$$

Thus the desired result in (4.4.11) is obtained. Next, comparing the terms in (4.4.1) and (4.4.11), to satisfy the fixed point condition in MFG, we require another 6 equations in (4.4.12) for the coefficient functions $w = (w_i : i = 1, 2, \ldots, 6)$. □

Using further algebraic structures, one can reduce the ODE system of 13 equations composed by (4.4.8) and (4.4.12) into a system of 4 equations.

Proof of Proposition 45. Let the smooth and bounded functions $\{w_i : i = 1, 2, \ldots, 6\}$ be given, the functions (a, b, c, d, e, f, g) in (4.4.8) is a coupled linear system, and thus their existence, uniqueness and boundedness is shown by Theorem 12.1 in [2].

Plugging the 6 equations in (4.4.12) to the ODE system (4.4.8), we obtain

$$\begin{cases} a' - 2a^2 + k = 0, \\ b' - \dfrac{1}{2}g^2 - 4ab - 2bg - 2cg = 0, \\ c' - 4ac + k = 0, \\ d' - \dfrac{1}{2}e^2 - ef + 2c + 2a + b + g = 0, \\ e' - 2ae - eg = 0, \\ f' - eg - 2af - gf - 2be - 2ce = 0, \\ g' - 4ag - g^2 - 2k = 0, \end{cases}$$

with the terminal conditions

$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = g(T) = 0.$$

Let $l = 2a + g$, and it is easy to obtain

$$l'(t) - l^2(t) = 0, \quad l(T) = 0,$$

which implies that $l(t) = 2a(t) + g(t) = 0$ for all $t \in [0, T]$. This gives the result that $g = -2a$ and it yields $e' = 0$. Then with $e(T) = 0$, we have $e(t) = 0$ for all $t \in [0, T]$ and thus one can obtain $f' = 0$, which indicates that $f(t) = 0$ for all $t \in [0, T]$ as $f(T) = 0$. Therefore the ODE system (4.4.8) can be simplified to the following form about $(a(t), b(t), c(t), d(t) : t \in [0, T])$:

$$\begin{cases} a'(t) - 2a^2(t) + k = 0, \\ b'(t) - 2a^2(t) + 4a(t)c(t) = 0, \\ c'(t) - 4a(t)c(t) + k = 0, \\ d'(t) + b(t) + 2c(t) = 0, \end{cases} \tag{4.4.13}$$

with the terminal conditions

$$a(T) = b(T) = c(T) = d(T) = 0. \tag{4.4.14}$$

The unique solvability of the Riccati system (4.4.13)-(4.4.14) is proven in Lemma 61 in the Appendix. Note that the solution a of (4.3.2) is consistent with the solution of the Riccati system given by equations (4.4.13)-(4.4.14).

In this case, since $2a + g = 0$ and $e = 0$ for all $t \in [0, T]$, it follows that $\hat{\mu}_s = \bar{\mu} + \tilde{W}_s$ for all $s \in [t, T]$ from the fixed point result (4.4.11). Similarly,

$$\hat{\nu}_s = \bar{\nu} + \int_t^s \left(2 + 4a(r)\hat{\mu}_r^2 - 4a(r)\hat{\nu}_r\right) dr + \int_t^s 2\hat{\mu}_r \, d\tilde{W}_r, \quad \forall s \in [t, T].$$

Plugging $e = 0$ and $\hat{\mu}_s = \bar{\mu} + \tilde{W}_r$ back to (4.4.6), we obtain the optimal control by

$$\hat{\alpha}_s = 2a(s)(\bar{\mu} + \tilde{W}_s - \hat{X}_s).$$

Moreover, since $e = f = 0$ and $g = -2a$ for $s \in [t, T]$, the value function can be simplified from (4.4.5) to

$$v(t, x, \bar{\mu}, \bar{\nu}) = a(t)x^2 - 2a(t)x\bar{\mu} + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t).$$

This concludes Proposition 45.

□

4.5 The N-Player Game

This section focuses on proving Proposition 46 regarding the corresponding N-player game. For simplicity, we can omit the superscript (N) when referring to the processes in the sample space $\Omega^{(N)}$.

To begin, we address the N-player game in Subsection 4.5.1, where we solve it and obtain a Riccati system containing $O(N^3)$ equations. Subsequently, we reduce the relevant Riccati system to an ODE system in Subsection 4.5.2, which has a dimension independent of N. This simplified system forms the fundamental component of the convergence result.

4.5.1 Characterization of the N-player game by Riccati system

It is important to emphasize that based on the problem setting in Subsection 4.2.2 and the running cost for each player specified in (4.2.9), the N-player game can be classified as an N-coupled stochastic LQG problem. As a result, the value function and optimal control for each player can be determined by means of the following Riccati system:

For $i = 1, 2, \ldots, N$, consider

$$\begin{cases} A_i' - 2A_i^\top e_i e_i^\top A_i - 4\displaystyle\sum_{j \neq i}^N A_j^\top e_j e_j^\top A_i + \frac{k}{N}\sum_{j \neq i}^N (e_i - e_j)(e_i - e_j)^\top = 0, \\ B_i' - 2A_i^\top e_i e_i^\top B_i - 2\displaystyle\sum_{j \neq i}^N \left(A_i^\top e_j e_j^\top B_j + A_j^\top e_j e_j^\top B_i\right) = 0, \\ C_i' - \dfrac{1}{2}B_i^\top e_i e_i^\top B_i - \displaystyle\sum_{j \neq i}^N B_j^\top e_j e_j^\top B_i + 2tr(A_i) = 0, \\ A_i(T) = B_i(T) = C_i(T) = 0, \end{cases} \tag{4.5.1}$$

where A_i is $N \times N$ symmetric matrix, B_i is N-dimensional vector, $C_i \in \mathbb{R}$ is a real constant, and e_i is the i-th natural basis in $\mathbb{R}^N$ for each $i = 1, 2, \ldots, N$.

Lemma 59. *Suppose $(A_i, B_i, C_i : i = 1, 2, \ldots, N)$ is the solution of the Riccati system (4.5.1). Then, the value functions of the N-player game defined by (4.2.7) is*

$$V_i\left(x^{(N)}\right) = \left(x^{(N)}\right)^\top A_i(0) x^{(N)} + \left(x^{(N)}\right)^\top B_i(0) + C_i(0), \quad i = 1, 2, \ldots, N.$$

Moreover, the path and the control under the equilibrium are given by

$$d\hat{X}_{it}^{(N)} = \left(-2(A_i(t))_i^\top \hat{X}_t^{(N)} - (B_i(t))_i\right) dt + dW_{it}^{(N)} + d\tilde{W}_t, \tag{4.5.2}$$

and

$$\hat{\alpha}_{it}^{(N)} = -2(A_i(t))_i^\top \hat{X}_t^{(N)} - (B_i(t))_i$$

for each $i = 1, 2, \ldots, N$, where $(A)_i$ denotes the i-th column of matrix A, $(B)_i$ denotes the i-th entry of vector B and $\hat{X}_t^{(N)} = [\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \ldots, \hat{X}_{Nt}^{(N)}]^\top$.

Proof. From the dynamic programming principle, it is standard that, under enough regularities, the players' value function $V(x^{(N)}) = (V_1, V_2, \ldots, V_N)(x^{(N)})$ can be lifted to the solution $v_i(t, x^{(N)})$ of the following system of HJB equations, for $i = 1, 2, \ldots, N$,

$$\begin{cases} \partial_t v_i + \inf\limits_{a_{it} \in \mathbb{R}} \left(a_{it} \partial_{x_i} v_i + \frac{1}{2} a_{it}^2\right) + \sum\limits_{j \neq i}^N a_{jt} \partial_{x_j} v_i + \Delta v_i + \frac{k}{N} \sum\limits_{j \neq i}^N \left((e_i - e_j)^\top x^{(N)}\right)^2 = 0, \\ v_i\left(T, x^{(N)}\right) = 0. \end{cases}$$

Note that with $a_{it} = -\partial_{x_i} v_i\left(t, x^{(N)}\right)$ for each $i = 1, 2, \ldots, N$, the term in the infimum attains the optimal value and thus the HJB equation can be reduced to

$$\begin{cases} \partial_t v_i - \frac{1}{2} \left(\partial_{x_i} v_i\right)^2 - \sum\limits_{j \neq i}^N \partial_{x_j} v_j \partial_{x_j} v_i + \Delta v_i + \frac{k}{N} \sum\limits_{j \neq i}^N \left((e_i - e_j)^\top x^{(N)}\right)^2 = 0, \\ v_i\left(T, x^{(N)}\right) = 0. \end{cases} \tag{4.5.3}$$

Then, the value functions V of the N-player game defined by (4.2.7) is $V_i(x^{(N)}) = v_i(0, x^{(N)})$ for all $i = 1, 2, \ldots, N$. Moreover, the path and the control under the equilibrium are given by

$$d\hat{X}_{it}^{(N)} = -\partial_{x_i} v_i\left(t, \hat{X}_t^{(N)}\right) dt + dW_{it}^{(N)} + d\tilde{W}_t,$$

and

$$\hat{\alpha}_{it}^{(N)} = -\partial_{x_i} v_i\left(t, \hat{X}_t^{(N)}\right)$$

for $i = 1, 2, \ldots, N$. The proof is the application of Itô's formula and the details are omitted here.

Due to its LQG structure, the value function leads to a quadratic function of the form

$$v_i\left(t, x^{(N)}\right) = \left(x^{(N)}\right)^\top A_i(t) x^{(N)} + \left(x^{(N)}\right)^\top B_i(t) + C_i(t).$$

Plugging V_i into (4.5.3), and matching the coefficient of variables, we get the Riccati system of ODEs in (4.5.1) and the desired results are obtained. □

4.5.2 Proof of Proposition 46: Reduced Riccati form for the equilibrium

At present, MFG and the corresponding N-player game can be characterized by Proposition 45 and Lemma 59, respectively. One of our primary objectives is to examine the convergence of the representative optimal path $\hat{X}_{1t}^{(N)}$ generated by the N-player game defined in (4.5.1)-(4.5.2) to the optimal path $\hat{X}_t$ of the MFG described in Proposition 45.

It should be noted that $\hat{X}_t$ is solely dependent on the function $a(t)$, as indicated in the ODE (4.3.2). In contrast, $\hat{X}_{1t}^{(N)}$ depends on $O(N^3)$ many functions derived from the solutions of a substantial Riccati system (4.5.1) involving matrices $(A_{it}, B_{it} : i = 1, 2, \ldots, N)$. Consequently, comparing these two processes meaningfully becomes an exceedingly challenging task without gaining further insight into the intricate structure of the Riccati system (4.5.1).

Proof of Proposition 46. Inspired from the setup in [50] and [46], we may seek a pattern for the matrix A_i in the following form:

$$(A_i)_{pq} = \begin{cases} a_1(t), & \text{if } p = q = i, \\ a_2(t), & \text{if } p = q \neq i, \\ a_3(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ a_4(t), & \text{otherwise.} \end{cases} \tag{4.5.4}$$

The next result justifies the above pattern: the N^2 entries of the matrix A_i can be embedded to a 2-dimensional vector space no matter how big N is.

For the Riccati system (4.5.1), with the given of A_i and suppose each function in A_i is continuous on $[0, T]$, it is obvious to see that $B_i = 0$ for all $t \in [0, T]$ and for all $i = 1, 2, \ldots, N$. Note that in this case, for $i = 1, 2, \ldots, N$, the optimal control is given by

$$\hat{\alpha}_i = -2 \sum_{j=1}^{N} (A_i)_{ij} \hat{X}_{jt}^{(N)} = -2 \left(A_i\right)_i^\top \hat{X}_t^{(N)},$$

where $(A)_i$ is the i-th column of matrix A.

Plugging the pattern (4.5.4) into the differential equation of A_i, we obtain the following system of ODEs:

$$\begin{cases} a_1' - 2a_1^2 - 4(N-1)a_3^2 + \dfrac{N-1}{N}k = 0, \\ a_2' - 2a_3^2 - 4a_1a_2 - 4(N-2)a_3a_4 + \dfrac{k}{N} = 0, \\ a_3' - 2a_1a_3 - 4a_1a_3 - 4(N-2)a_3^2 - \dfrac{k}{N} = 0, \\ a_3' - 2a_1a_3 - 4a_2a_3 - 4(N-2)a_3a_4 - \dfrac{k}{N} = 0, \\ a_4' - 2a_3^2 - 4a_2a_3 - 4a_1a_4 - 4(N-3)a_3a_4 = 0 \end{cases}$$

with the terminal conditions

$$a_1(T) = a_2(T) = a_3(T) = a_4(T) = 0.$$

It is worth noting that there are two ODEs for a_3, and the two expressions should be equal, thus

$$a_1a_3 + (N-2)a_3^2 = a_2a_3 + (N-2)a_3a_4,$$

which implies that $(a_1 + (N-2)a_3)' = (a_2 + (N-2)a_4)'$ or

$$\begin{aligned} &2a_1^2 + 2(N-2)a_1a_3 + 4(N-1)a_3^2 + 4(N-2)a_2a_3 + 4(N-2)^2a_3a_4 - \frac{k}{N} \\ &= 2(N-1)a_3^2 + 4a_1a_2 + 4(N-2)(a_2a_3 + a_3a_4 + a_1a_4) + 4(N-2)(N-3)a_3a_4 - \frac{k}{N}. \end{aligned}$$

After combining terms and substituting $a_2 + (N-2)a_4$ with $a_1 + (N-2)a_3$, we get

$$a_1^2 + (N-2)a_1a_3 - (N-1)a_3^2 = 0,$$

which yields $a_3 = a_1$ or $a_3 = -\frac{1}{N-1}a_1$. Note that, since a_1 and a_3 satisfies different differential equations, it follows that $a_3 \neq a_1$. Hence, we can conclude that $a_3 = -\frac{1}{N-1}a_1$. Next, from the equation $a_1 + (N-2)a_3 = a_2 + (N-2)a_4$, we have

$$a_4 = \frac{1}{N-2}a_1 + a_3 - \frac{1}{N-2}a_2.$$

In conclusion, for $i = 1, 2, \ldots, N$, A_i has the following expressions:

$$(A_i)_{pq} = \begin{cases} a_1(t), & \text{if } p = q = i, \\ a_2(t), & \text{if } p = q \neq i, \\ -\dfrac{1}{N-1}a_1(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ \dfrac{1}{(N-1)(N-2)}a_1(t) - \dfrac{1}{N-2}a_2(t), & \text{otherwise,} \end{cases}$$

where a_1 and a_2 satisfies the system of ODEs (4.5.5)

$$\begin{cases} a_1' - \dfrac{2(N+1)}{N-1}a_1^2 + \dfrac{N-1}{N}k = 0, \\ a_2' + \dfrac{2}{(N-1)^2}a_1^2 - \dfrac{4N}{N-1}a_1 a_2 + \dfrac{k}{N} = 0, \\ a_1(T) = a_2(T) = 0. \end{cases} \tag{4.5.5}$$

The existence and uniqueness of A_i in (4.5.1) are equivalent to the existence and uniqueness of (4.5.5). Firstly, the existence, uniqueness, and boundness of a_1 in (4.5.5) is from the same argument for a in (4.4.13), which is shown as the proof of Lemma 61 in Appendix. The explicit solution of a_1 is given by

$$a_1(t) = \sqrt{\frac{k}{2}\frac{(N-1)^2}{N(N+1)}} \frac{1 - e^{-2\sqrt{2}\sqrt{\frac{N+1}{N}k}(T-t)}}{1 + e^{-2\sqrt{2}\sqrt{\frac{N+1}{N}k}(T-t)}}$$

for all $t \in [0,T]$. Next, with the given of a_1, the existence, uniqueness, and boundness of a_2 in (4.5.5) is guaranteed by Theorem 12.1 in [2]. Therefore, we can express the equilibrium paths and associated controls as the following:

$$d\hat{X}_{it}^{(N)} = -2a_1^N(t)\left(\hat{X}_{it}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^{N}\hat{X}_{jt}^{(N)}\right)dt + dW_{it}^{(N)} + d\tilde{W}_t, \tag{4.5.6}$$

and

$$\hat{\alpha}_{it}^{(N)} = -2a_1^N(t)\left(\hat{X}_{it}^{(N)} - \frac{1}{N-1}\sum_{j\neq i}^{N}\hat{X}_{jt}^{(N)}\right)$$

respectively for $i = 1, 2, \ldots, N$, where a_1^N is the solution to the ODE for a_1 in (4.5.5). This concludes Proposition 46. □

4.6 Further remark

We have now established Proposition 45 concerning the MFG in Section 4.4 and Proposition 46 regarding the N-player game in Section 4.5. With these propositions proven, we are now able to conclude the proof of Theorem 44, which was presented in Section 4.3.4.

4.7 Appendix

Lemma 60. *Let $\mathbb{W}_p$ be the p-Wasserstein metric. If X and Y are two real-valued random variables and c is a constant, then*

$$\mathbb{W}_p(\mathcal{L}(X), \mathcal{L}(Y)) = \mathbb{W}_p(\mathcal{L}(X+c), \mathcal{L}(Y+c)). \tag{4.7.1}$$

Moreover, if $\alpha = \{\alpha_i : i \in \mathbb{N}\}$ is a sequence of random variables, then

$$\mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha_i+c}, \mathcal{L}(Y+c)\right) = \mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha_i}, \mathcal{L}(Y)\right). \tag{4.7.2}$$

Proof. By definition of the p-Wasserstein metric, we have:

$$\mathbb{W}_p(\mathcal{L}(X), \mathcal{L}(Y)) = \left(\inf_{\pi\in\Pi(\mathcal{L}(X),\mathcal{L}(Y))}\int_{\mathbb{R}^2}|x-y|^p d\pi(x,y)\right)^{\frac{1}{p}},$$

where $\Pi(\mathcal{L}(X), \mathcal{L}(Y))$ is the set of all joint probability measures with marginals $\mathcal{L}(X)$ and $\mathcal{L}(Y)$. Similarly,

$$\mathbb{W}_p(\mathcal{L}(X+c), \mathcal{L}(Y+c)) = \left(\inf_{\pi\in\Pi(\mathcal{L}(X+c),\mathcal{L}(Y+c))}\int_{\mathbb{R}^2}|x-y|^p d\pi(x,y)\right)^{\frac{1}{p}},$$

where $\Pi(\mathcal{L}(X+c), \mathcal{L}(Y+c))$ is the set of all joint probability measures with marginals $\mathcal{L}(X+c)$ and $\mathcal{L}(Y+c)$.

Now, consider the mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Phi(x,y) = (x+c, y+c)$. For any $\pi \in \Pi(\mathcal{L}(X), \mathcal{L}(Y))$, the pushforward measure of π under Φ belongs to $\Pi(\mathcal{L}(X+c), \mathcal{L}(Y+c))$, i.e., $\pi' = \Phi_*\pi \in \Pi(\mathcal{L}(X+c), \mathcal{L}(Y+c))$. Thus, we have

$$\Phi_*\Pi(\mathcal{L}(X), \mathcal{L}(Y)) \subset \Pi(\mathcal{L}(X+c), \mathcal{L}(Y+c)).$$

Moreover, Φ is bijective and measure preserving, then

$$\int_{\mathbb{R}^2}|x-y|^p d\pi'(x,y) = \int_{\mathbb{R}^2}|(x+c)-(y+c)|^p d\pi(x,y) = \int_{\mathbb{R}^2}|x-y|^p d\pi(x,y).$$

Therefore, we know that

$$\begin{aligned}
\mathbb{W}_p^p(\mathcal{L}(X), \mathcal{L}(Y)) &= \inf_{\pi\in\Pi(\mathcal{L}(X),\mathcal{L}(Y))}\int_{\mathbb{R}^2}|x-y|^p d\pi(x,y) \\
&= \inf_{\pi\in\Pi(\mathcal{L}(X),\mathcal{L}(Y))}\int_{\mathbb{R}^2}|x-y|^p d\Phi_*\pi(x,y) \\
&= \inf_{\pi'\in\Phi_*\Pi(\mathcal{L}(X),\mathcal{L}(Y))}\int_{\mathbb{R}^2}|x-y|^p d\pi'(x,y) \\
&\geq \mathbb{W}_p^p(\mathcal{L}(X+c), \mathcal{L}(Y+c)).
\end{aligned}$$

by the definition of the p-Wasserstein metric. If we apply the above inequality to $X' = X+c$, $Y' = Y+c$, and $c' = -c$, the opposite inequality is provided. Thus, it completes the proof of (4.7.1).

Next, we note that
$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha_i+c}=\mathcal{L}(\alpha_u+c|\alpha),$$
where u be a uniform random variable on $\{1,2,\dots,N\}$ independent to α. Using (4.7.1), we conclude (4.7.2) from
$$\begin{aligned}\mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha_i+c},\mathcal{L}(Y+c)\right)&=\mathbb{W}_p\left(\mathcal{L}(\alpha_u+c|\alpha),\mathcal{L}(Y+c)\right)\\&=\mathbb{W}_p\left(\mathcal{L}(\alpha_u|\alpha),\mathcal{L}(Y)\right)\\&=\mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha_i},\mathcal{L}(Y)\right).\end{aligned}$$
□

Lemma 61. *Under the Assumption 43, there exists a unique solution* $(a(t),b(t),c(t),d(t):t\in[0,T])$ *for the Riccati system of ODEs* (4.4.13)-(4.4.14) *and the solution can given explicitly by*
$$\begin{cases}a(t)=\sqrt{\dfrac{k}{2}}\dfrac{1-e^{-2\sqrt{2k}(T-t)}}{1+e^{-2\sqrt{2k}(T-t)}},\\ b(t)=\displaystyle\int_t^T\left(4a(s)c(s)-2a^2(s)\right)ds,\\ c(t)=k\displaystyle\int_t^T e^{\int_t^s-4a(r)dr}ds,\\ d(t)=\displaystyle\int_t^T\left(b(s)+2c(s)\right)ds.\end{cases}$$

Proof. Firstly, with the given of $k>0$, we can solve the ODE
$$a'(t)-2a^2(t)+k=0,\quad a(T)=0$$
explicitly by the method of separating variables. Note that with the differential form, we have
$$\frac{da}{\left(\sqrt{2}a-\sqrt{k}\right)\left(\sqrt{2}a+\sqrt{k}\right)}=\frac{1}{2\sqrt{k}}\left(\frac{1}{\sqrt{2}a-\sqrt{k}}-\frac{1}{\sqrt{2}a+\sqrt{k}}\right)da=dt.$$
It follows that
$$\ln\left(\left|\frac{\sqrt{2}a-\sqrt{k}}{\sqrt{2}a+\sqrt{k}}\right|\right)=2\sqrt{2k}t+C_1$$
for some constant C_1 by taking integration on both sides. Thus by calculation, we obtain
$$a(t)=\sqrt{\frac{k}{2}}\frac{1-C_2e^{2\sqrt{2k}t}}{1+C_2e^{2\sqrt{2k}t}}$$

for some constant C_2 to be determined. Since $a(T) = 0$, it yields that $C_2 = e^{-2\sqrt{2k}T}$ and thus

$$a(t) = \sqrt{\frac{k}{2}} \frac{1 - e^{-2\sqrt{2k}(T-t)}}{1 + e^{-2\sqrt{2k}(T-t)}}.$$

It is easy to verify that $a(\cdot)$ is in $C^\infty([0,T])$ and is bounded. With the given of a, the functions (b,c,d) in the Riccati system (4.4.13)-(4.4.14) is a coupled linear system, and thus their existence, uniqueness, and boundedness are given by Theorem 12.1 in [2]. $\square$

Chapter 5

On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions

5.1 Introduction

Mean Field Game (MFG) theory establishes Nash equilibirum conditions for large populations of asymptotically negligible non-cooperating agents via an analysis of the infinite limit population (Huang, Caines, and Malhame [40, 42, 45]; Lasry and Lions [55]). The resulting PDEs (Partial Differential Equations) consist of a backward Hamilton-Jacobi-Bellman (HJB) equation and a forward Fokker-Planck-Kolmogorov (FPK) equation for each generic agent. These equations are linked by the state distribution of a generic agent which is called the mean field of the system.

In this Chapter, our objective is to establish the unique solvability of the GMFG equation in an appropriate function space. The GMFG equations consist of a collection of parameterized Hamilton-Jacobi- Bellman equations, $HJB(\alpha), \alpha \in [0,1]$, and a collection of parameterized Fokker-Planck-Kolmogorov equations, $FPK(\alpha)$ with $\alpha \in [0,1]$. The solution of a set of GMFG equations is a parameterized pair (v,μ), where $v[\alpha] = v(t,\alpha,x)$ solves the $HJB(\alpha)$ equation and $\mu[\alpha] = \mu(t,\alpha,x)$ solves the $FPK(\alpha)$ equation. The coupling of the system PDEs in this chapter has the following features (see [12] for a more general framework subject to different hypotheses):

- $FPK(\alpha)$ depends upon $HJB(\alpha)$ through its first order coefficient ∇v.
- $HJB(\alpha)$ depends upon $FPK(\alpha')$ for all $\alpha' \in [0,1]$ through the graphon g acting on $\mu[\alpha']$; this is the major difference from MFG.

The GMFG equations with a constant graphon reduce to the classical MFG system as a special case, and the original methods to establish solvability of the classical MFG equations are helpful in the present case. In [45] and [68], a Banach fixed point analysis is used depending on a contraction argument; this is based on assumptions on the Lipschitz continuity of the functions appearing in the MFG equations and their derivatives, and yields uniquenss as well as existence. This approach is used in the parallel study [12] of the solvability of the GMFG equations. On the other hand, [13] and [72] carry out the existence analysis using the Schauder fixed point theorem based upon regularity assumptions and then obtain uniqueness via a monotonicity assumption on the running cost.

In this work, similar to the aforementioned analyses, we will establish the existence of solutions via the application of a fixed point theorem. Our existence proof adopts Schauder's argument on the fixed point theorem and is more closely relevant to [36], [13], and [72] in this sense. Unlike [36] on the solvability in Sobolev space, our solvability is to answer the existence in Hölder space along the lines of [13] and [72]. Nevertheless, different from all aforementioned papers, our proof on the continuity of the gradient of the value function with respect to the coefficient functions relies on probabilistic estimates rather than the theory of viscosity solutions. The main advantage of our approach is that we can conclude the local Lipschitz continuity of the solution map, which is stronger than continuity and beneficial to the subsequent analysis of the GMFG.

Having said that, the major difficulty generalizing existence from the MFG case to the GMFG case is to obtain the regularity of the solution with respect to the variable α, which is essential for the existence result by the Schauder's fixed point theorem. To be more illustrative, for instance, to obtain a uniform first order estimate of $|\nabla v(t, \alpha, x) - \nabla v(t, \alpha', x)|$ for the solution v of the HJB equation, one has to compare the solutions from two different HJBs parameterized by α and α'. This leads to a study of the sensitivity with respect to coefficient functions of corresponding PDEs. Therefore, the local Lipschitz continuity of the HJB solution map becomes essential for this procedure.

The chapter is organized as follows. Section 5.2 gives the problem set up. Section 5.3 presents the regularity of parabolic PDE and applies this to the FPK. Section 5.4 presents the existence result and Section 5.5 treats uniqueness. Section 5.6 presents a summary and extensions of the main result. For better clarity, all notations used in this chapter have been collected and explained in the Appendix Section 5.7.

5.2 Problem setup

Let $\mathbb{T}^d$ be a d-torus. $\mathcal{P}_1(\mathbb{T}^d)$ is the Wasserstein space of probability measures on $\mathbb{T}^d$ satisfying

$$\int_{\mathbb{T}^d} |x| d\mu(x) < \infty$$

endowed with 1-Wasserstein metric $\mathbb{W}_1(\cdot, \cdot)$ defined by

$$\mathbb{W}_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y| d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the collection of all probability measures on $\mathbb{T}^d \times \mathbb{T}^d$ with its marginals agreeing with μ and ν.

We consider the following large system of multi-agent problems. A generic agent can be identified by its state pair $(\alpha, x) \in [0,1] \times \mathbb{T}^d$, where α is the cluster index and x is a $\mathbb{T}^d$ valued state. The weights of connections between clusters are given by a symmetric measurable function $g : [0,1]^2 \mapsto \mathbb{R}$, which is commonly referred to a graphon [58]. The population density at the cluster α at time t will be given by $\mu(t, \alpha) \in \mathcal{P}_1(\mathbb{T}^d)$.

Example. Two examples of graphons are given in the following discussion, while the reader is referred to [58] for the fundamental theory of this subject. A uniform graphon which corresponds to the limit of a sequence of Erdös-Rényi graphs with parameter p, $0 \le p \le 1$, is given by

$$g(\alpha, \alpha') = p, \quad \forall \alpha, \alpha' \in [0,1] \tag{5.2.1}$$

and the uniform attachment graph limit has the graphon

$$g(\alpha, \alpha') = 1 - \max\{\alpha, \alpha'\}, \quad \forall \alpha, \alpha' \in [0,1]. \tag{5.2.2}$$

$\square$

A running cost incurred to the generic agent of (α, x) with a feedback control exertion $\mathbf{a} : [0,T] \times [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$ at time t is given by

$$\ell(\mu, g, \mathbf{a}, t, \alpha, x) = \frac{1}{2}|\mathbf{a}(t, \alpha, x)|^2 + \ell_1(\mu, g, t, \alpha, x) \tag{5.2.3}$$

for some given function $\ell_1(\cdot, \cdot, \cdot, \cdot, \cdot)$. The following cost can be considered as an example for ℓ_1

$$\ell_1(\mu, g, t, \alpha, x) = \int_0^1 \int_{\mathbb{T}^d} \ell_2(x, y)\mu(t, \alpha', dy) g(\alpha, \alpha') \, d\alpha' \tag{5.2.4}$$

for some $\ell_2 : \mathbb{T}^d \times \mathbb{T}^d \mapsto \mathbb{R}$.

Let $b : [0,T] \times [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}$ and $m_0 : [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}^+$ be two given smooth enough functions. By ∇b, we denote the gradient of b on the domain $\mathbb{T}^d$, which is mapping $[0,T] \times [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$. Finding a solution of the GMFG equations consists of solving for the unknown triples $(v, \mathbf{a}^*, \mu)$:

- the value function $v : [0,T] \times [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}$,
- optimal control $\mathbf{a}^* : [0,T] \times [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$,
- and the density $\mu : [0,T] \times [0,1] \times \mathbb{T}^d \mapsto \mathbb{R}^+$,

satisfying the α parameterized family

$$\begin{cases} \partial_t v + (\nabla b + \mathbf{a}^*) \cdot \nabla v + \frac{1}{2}\Delta v + \ell(\mu, g, \mathbf{a}^*) = 0 \\ \mathbf{a}^*(t, \alpha, x) = \arg\min_{a \in \mathbb{R}^d} \left\{ a \cdot \nabla v(t, \alpha, x) + \frac{1}{2}|a|^2 \right\} \\ \partial_t \mu = -\mathrm{div}_x((\nabla b + \mathbf{a}^*)\mu) + \frac{1}{2}\Delta\mu \\ v(T, \alpha, x) = 0, \quad \mu(0, \alpha, x) = m_0(\alpha, x). \end{cases} \tag{5.2.5}$$

In the first and third equation of (5.2.5), each term is a function of (t, α, x) without further specification. In particular, the $\ell(\mu, g, \mathbf{a}^*)$ shall be understood as a mapping

$$(t, \alpha, x) \mapsto \ell(\mu, g, \mathbf{a}^*)(t, \alpha, x) := \ell(\mu, g, \mathbf{a}^*, t, \alpha, x).$$

Our goal in this chapter is to establish existence, uniqueness for the solution of (5.2.5) in an appropriate solution space. We close this section with a brief illustration of the probabilistic formulation on the GMFG for the motivational purpose. A generic player in GMFG is identified by a pair $(\alpha, x) \in [0,1] \times \mathbb{T}^d$, where α is geographical information and x is a state. The population density at index α at time t is denoted by $\mu(t, \alpha) \in \mathcal{P}_1(\mathbb{T}^d)$ and the relation between two generic players in α and α' is given by a graphon $g(\alpha, \alpha')$. Given a population density $(t, \alpha) \mapsto \mu(t, \alpha)$ and a graphon $(\alpha, \alpha') \mapsto g(\alpha, \alpha')$, a generic player exerts its optimal strategy of the following stochastic control problem described below. State evolution of the generic player at α follows a controlled stochastic differential equation:

$$X_t^\alpha = X_0^\alpha + \int_0^t \left(\nabla b\left(s, \alpha, X_s^\alpha\right) + \mathbf{a}\left(s, \alpha, X_s^\alpha\right)\right) ds + W_t^\alpha, \tag{5.2.6}$$

where the drift is formed by a control process $\mathbf{a}$ and a conservative vector field ∇b, W^α is a Brownian motion in a filtered probability space independent to W^β for any $\beta \neq \alpha$, and X_0^α is an initial random variable with a given distribution $m_0(\alpha)$. In the above, the left hand side is understood as the coset of $\mathbb{Z}^d$ that contains the right hand side by a mapping $\pi(x) = x + \mathbb{Z}^d$. We use $X^\alpha[\mathbf{a}]$ to denote the process with the dependence on $\mathbf{a}$. The objective of the generic player at α with a given population density flow μ is to minimize the total cost incurred during $[0, T]$ of the form

$$J^\alpha(\mathbf{a}, \mu) = \mathbb{E}\left[\int_0^T \ell\left(\mu, g, \mathbf{a}, t, \alpha, X_t^\alpha[\mathbf{a}]\right) dt\right]$$

over a reasonably rich enough control space of $\mathbf{a}$. Note that the optimal strategy $\mathbf{a}^*$ depends on μ. Given an initial distribution m_0, the goal of the GMFG is to find the Nash equilibrium μ^* and the corresponding $\mathbf{a}^*$, i.e. the pair $(\mu^*, \mathbf{a}^*)$ satisfies

$$J^\alpha(\mathbf{a}^*, \mu^*) \leq J^\alpha(\mathbf{a}, \mu^*), \ \forall \mathbf{a} \ \text{ and } \ \mu^*(t, \alpha) \sim X_t^\alpha[\mathbf{a}^*], \ \forall (t, \alpha).$$

Indeed, the above formulation poses a class of mean field game problems indexed by $\alpha \in [0,1]$ and couplings between mean field games are imposed by the running cost ℓ via graphon g. For more detailed discussion and various applications are referred to [10, 11, 12].

5.3 Some regularity results

We are going to present sensitivity results of the parabolic PDE and FPK equations with respect to their coefficients separately, which eventually serve for the proof of fixed point theorem as key elements. Throughout the chapter, we will use $\Psi(\cdot)$ in various places as a generic positive function increasing with respect to its variables. Morevoer, all function spaces and relevant norms are sorted out in Section 5.7.

5.3.1 Parabolic equations

Consider the equation

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u - cu + f, & \text{on } (0,T)\times\mathbb{T}^d \\ u(0,x) = 0, & \text{on } x\in\mathbb{T}^d. \end{cases} \tag{5.3.1}$$

We will denote the solution map by $u = u[c,f]$ whenever it is necessary to emphasize its dependence on the coefficient functions.

5.3.1.1 Preliminaries on solvability

If the coefficients c and f are Hölder in both variables (t,x), then there exists a unique classical solution. Recall that $\Psi(\cdot)$ is a generic function mentioned in the first paragraph of Section 3.

Lemma 62. *If $c, f \in C^{\delta/2,\delta}([0,T]\times\mathbb{T}^d)$ holds for some $\delta\in(0,1)$, then there exists unique solution $u\in C^{1+\delta/2,2+\delta}([0,T]\times\mathbb{T}^d)$ of (5.3.1) satisfying*

$$|u|_{1+\delta/2,2+\delta} \le \Psi(|c|_{\delta/2,\delta}, |f|_{\delta/2,\delta}).$$

Moreover, $v(t,x) := u(T-t,x)$ has a probabilistic representation $v[c,f]$ of the form

$$v(t,x) = v[c,f](t,x) := \mathbb{E}\left[\int_t^T \exp\left\{-\int_t^s c\left(r, X^{t,x}(r)\right)dr\right\} f\left(s, X^{t,x}(s)\right)ds\right], \tag{5.3.2}$$

where

$$X^{t,x}(s) = x + W(s) - W(t) \tag{5.3.3}$$

for some Brownian motion W.

Proof. The solvability and its Hölder estimate is from Theorem 8.7.2 and Theorem 8.7.3 of [53], Theorem IV.5.1 of [54]. The probabilistic representation $v[c,f]$ is from Feynman-Kac formula, see [31]. $\square$

In the above, we remark that, (5.3.3) reads by $X^{t,x}(s) = \pi(x + W(s) - W(t))$, where π is the generic mapping $\mathbb{R}^d \mapsto \mathbb{R}^d/\mathbb{Z}^d$. Later we also need to use the following definition of weak solution, see [25].

Definition 63. *A function $u \in L^2([0,T], H^1(\mathbb{T}^d))$ is a weak solution of (5.3.1) if u satisfies*

$$\begin{cases} \int_{\mathbb{T}^d} \phi(-\partial_t u - cu + f)dx = \frac{1}{2}\int_{\mathbb{T}^d} \nabla\phi \cdot \nabla u dx, \quad \forall \phi \in H^1(\mathbb{T}^d) \\ u(0,x) = 0, \quad \text{on } x \in \mathbb{T}^d. \end{cases} \tag{5.3.4}$$

We have the following uniqueness with the same assumptions as in Lemma 62.

Lemma 64. *If $c, f \in C^{\delta/2,\delta}([0,T] \times \mathbb{T}^d)$ holds for some $\delta \in (0,1)$, then there exists unique weak solution of (5.3.1) in $L^2([0,T], H^1(\mathbb{T}^d))$.*

Proof. By Lemma 62, there exists a classical solution u. Together with the compactness of the domain, it yields $u \in L^2([0,T], H^1(\mathbb{T}^d))$. By Theorem 7.4 of [25], uniqueness in $L^2([0,T], H^1(\mathbb{T}^d))$ holds if $c \in L^\infty$ and $f \in L^2$, and this is valid, since all coefficients are continuous on the compact domain. $\square$

5.3.1.2 First order regularity and sensitivity of the solution map

Although Lemma 62 has an estimation on $|u|_{1,2}$, it is controlled by an upper bound relevant to the Hölder norm of coefficients in the t variable, which is not desirable, see Section 5.4.5 for further remarks. Next, we will develop an upper bound independent of t-Hölder norm of the coefficients. To proceed, we define a linear operator

$$Lu = \partial_t u - \frac{1}{2}\Delta u. \tag{5.3.5}$$

The first result is on an estimate of $|u|_0 = \sup_{[0,T]\times\mathbb{T}^d} |u(t,x)|$.

Lemma 65. *If $c, f \in C^{\delta/2,\delta}([0,T] \times \mathbb{T}^d)$, then u of (5.3.1) satisfies $|u|_0 \le e^{|c|_0 T}|f|_0 T$.*

Proof. If $c = 0$, then with $u_1 = |f|_0 t$,

$$Lu_1 - f = |f|_0 - f \ge 0.$$

If $c \neq 0$, then with $u_2 = \frac{|f|_0(e^{|c|_0 t} - 1)}{|c|_0}$,

$$\begin{aligned} (L + c)u_2 &= |f|_0 e^{|c|_0 t}\left(1 + \frac{c}{|c|_0}\right) - \frac{c}{|c|_0}|f|_0 \\ &= |f|_0(e^{|c|_0 t} - 1)\left(1 + \frac{c}{|c|_0}\right) + |f|_0 \\ &\ge f. \end{aligned}$$

Note that both u_1 and u_2 are no greater than $e^{|c|_0 T}|f|_0 T$, and finally the comparison principle yields the result. $\square$

Next we will have the first order estimate independent to the Hölder norm in t of the coefficients. It also gives sensitivity of the solution map with respect to the coefficients.

Lemma 66. *Let c, f be in $C^{\delta,1}([0,T] \times \mathbb{T}^d)$ for some $\delta \in (0,1)$. Then the solution u of (5.3.1) belongs to $C^{1,2}([0,T] \times \mathbb{T}^d)$ with*

$$|u|_{0,1} \le \Psi(|c|_{0,1} + |f|_{0,1}).$$

Furthermore, the solution map $u = u[c,f]$ satisfies

$$|u[c_1, f_1] - u[c_2, f_2]|_0 \le \Psi(K)(|c_1 - c_2|_0 + |f_1 - f_2|_0)$$

for $K := |c_1|_0 + |c_2|_0 + |f_1|_0 + |f_2|_0$.

Proof. u of (5.3.1) can be written by $u(t,x) = v[c,f](T-t,x)$ with its probabilistic representation of (5.3.2). By setting $X^i := X^{t,x_i}$ of (5.3.3), we have

$$X^1_s - X^2_s = x_1 - x_2, \ \forall s \ge t.$$

If we define

$$\Lambda^i_s = e^{-\int_t^s c(r, X^i(r))dr},$$

then

$$v[c,f](t,x_i) = \mathbb{E}\left[\int_t^T \Lambda^i_s f(s, X^i(s))ds\right].$$

We first note that, by mean value theorem,

$$\left|\int_t^s c\left(r, X^1(r)\right)dr - \int_t^s c\left(r, X^2(r)\right)dr\right| \le T|c|_{0,1}|x_1 - x_2|.$$

Once again by mean value theorem and the fact of $|-\int_t^s c(r, X^i(r))dr| \le T|c|_0$, we obtain

$$\begin{aligned}
\left|\Lambda^1_s - \Lambda^2_s\right| &\le e^{T|c|_0}\left|\int_t^s c(r, X^1(r))dr - \int_t^s c(r, X^2(r))dr\right| \\
&\le \Psi(|c|_{0,1})|x_1 - x_2|
\end{aligned}$$

with probability one for $\Psi = T|c|_{0,1}e^{T|c|_0}$. Therefore, we have

$$\begin{aligned}
|v[c,f](t,x_1) - v[c,f](t,x_2)| &\le \mathbb{E}\left[\int_t^T \left|\Lambda^1_s f\left(s, X^1(s)\right) - \Lambda^2_s f\left(s, X^2(s)\right)\right| ds\right] \\
&\le \mathbb{E}\left[\int_t^T \left(|\Lambda^1|_0 \left|f\left(s, X^1(s)\right) - f\left(s, X^2(s)\right)\right| + |f|_0 \left|\Lambda^1_s - \Lambda^2_s\right|\right) ds\right],
\end{aligned}$$

which gives

$$\begin{aligned}
&|v[c,f](t,x_1)-v[c,f](t,x_2)| \\
&\le \mathbb{E}\left[\int_t^T \Psi(|c|_0)|\nabla f|_0 \left|X^1(s)-X^2(s)\right| ds\right] + \mathbb{E}\left[\int_t^T |f|_0 \left|\Lambda_s^1-\Lambda_s^2\right| ds\right] \\
&\le T\Psi(|c|_0)|\nabla f|_0|x_1-x_2| + T|f|_0|\Psi(|c|_{0,1})|x_1-x_2| \\
&\le \Psi(|c|_{0,1}+|f|_{0,1})|x_1-x_2|.
\end{aligned}$$

This implies that $|\nabla v|_0 \le \Psi(|c|_{0,1}+|f|_{0,1})$. Together with Lemma 65, we conclude that

$$|u|_{0,1} \le \Psi(|c|_{0,1}+|f|_{0,1}). \tag{5.3.6}$$

Next, we estimate $|u[c,f_1]-u[c,f_2]|_0$. For any (t,x), we set $\Lambda_s = e^{-\int_t^s c(r,X(r))dr}$, and note that

$$\begin{aligned}
|v[c,f_1](t,x)-v[c,f_2](t,x)| &\le \mathbb{E}\left[\int_t^T |\Lambda_s f_1(s,X_s)-\Lambda_s f_2(s,X_s)|ds\right] \\
&\le |f_1-f_2|_0 \mathbb{E}\left[\int_t^T |\Lambda_s|ds\right] \\
&\le Te^{T|c|_0}|f_1-f_2|_0.
\end{aligned}$$

This concludes that

$$|u[c,f_1]-u[c,f_2]|_0 \le \Psi(|c|_0)|f_1-f_2|_0. \tag{5.3.7}$$

In the following, we estimate $|u[c_1,f]-u[c_2,f]|_0$. By setting $\Lambda_s^i = e^{-\int_t^s c_i(r,X(r))dr}$, we have

$$|\Lambda_s^1-\Lambda_s^2| \le e^{T(|c_1|_0+|c_2|_0)}\int_t^s |c_1(r,X_r)-c_2(r,X_r)|dr \le e^{T(|c_1|_0+|c_2|_0)}|c_1-c_2|_0 T$$

with probability one. Therefore,

$$\begin{aligned}
|v[c_1,f](t,x)-v[c_2,f](t,x)| &\le \mathbb{E}\left[\int_t^T \left|\Lambda_s^1 f(s,X_s)-\Lambda_s^2 f(s,X_s)\right| ds\right] \\
&\le |f|_0 \mathbb{E}\left[\int_t^T \left|\Lambda_s^1-\Lambda_s^2\right| ds\right] \\
&\le T^2 e^{T(|c_1|_0+|c_2|_0)}|f|_0|c_1-c_2|_0.
\end{aligned}$$

This implies that

$$|u[c_1,f]-u[c_2,f]|_0 \le \Psi(|c_1|_0+|c_2|_0+|f|_0)|c_1-c_2|_0. \tag{5.3.8}$$

The conclusion yields from (5.3.6), (5.3.7), (5.3.8). □

5.3.1.3 Second order regularity and first order sensitivity

Next, we will see that under better regularity of c and f in x, we can improve regularity and sensitivity. Formally, if u of (5.3.1) is smooth enough, one can take derivatives of the equation to conclude that $\bar{u}_j = \partial_j u$ is the solution of the following equation depending on c, f and u of (5.3.1)

$$\begin{cases} \partial_t \bar{u}_j = \frac{1}{2}\Delta \bar{u}_j - c\bar{u}_j - u\partial_j c + \partial_j f, & \text{on } (0,T)\times\mathbb{T}^d \\ \bar{u}_j(0,x) = 0, & \text{on } x \in \mathbb{T}^d. \end{cases} \tag{5.3.9}$$

However, (5.3.9) is valid only if $u \in C^{1,3}$ is given a priori.

Lemma 67. *If $c, f \in C^{\delta,2}([0,T]\times\mathbb{T}^d)$ for some $\delta \in (0,1)$, then the solution u of (5.3.1) is in $C^{1,3}([0,T]\times\mathbb{T}^d)$ and $\bar{u}_j = \partial_j u$ is the unique solution of (5.3.9).*

Proof. By Lemma 64, u satisfies, for any $\phi \in H^2(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} \phi(-\partial_t u - cu + f)dx = \frac{1}{2}\int_{\mathbb{T}^d} \nabla\phi \cdot \nabla u dx.$$

Now, if we replace the test function ϕ by $\partial_i\phi$ in the above variational form, then we have

$$\int_{\mathbb{T}^d} \partial_i\phi(-\partial_t u - cu + f)dx = \frac{1}{2}\int_{\mathbb{T}^d} \nabla\partial_i\phi \cdot \nabla u dx.$$

Using integration by parts, we can show that $\bar{u}_j$ solves the variational form of (5.3.9) for any $\phi \in H^2(\mathbb{T}^d)$. Since $H^2(\mathbb{T}^d)$ is a dense subset in $H^1(\mathbb{T}^d)$, $\bar{u}_j$ is indeed a unique weak solution of (5.3.9).

Lastly, since the $\nabla c, \nabla f \in C^{\delta,1}([0,T]\times\mathbb{T}^d)$, we conclude that $\bar{u}_j$ is indeed a classical solution from Lemma 62. This also implies that $u \in C^{1,3}([0,T]\times\mathbb{T}^d)$. □

Lemma 68. *Let $c, f \in C^{\delta,2}([0,T]\times\mathbb{T}^d)$. Then the solution u of (5.3.1) belongs to $C^{1,3}([0,T]\times\mathbb{T}^d)$ with*

$$|u[c,f]|_{0,2} \leq \Psi(|c|_{0,2} + |f|_{0,2}).$$

Furthermore, the solution map $u = u[c,f]$ of (5.3.1) satisfies

$$|u[c_1,f_1] - u[c_2,f_2]|_{0,1} \leq \Psi(K)(|c_1 - c_2|_{0,1} + |f_1 - f_2|_{0,1})$$

for

$$K := |c_1|_{0,1} + |c_2|_{0,1} + |f_1|_{0,1} + |f_2|_{0,1}.$$

Proof. By Lemma 67, $\bar{u}_j = \partial_j u$ is the classical solution of (5.3.9), which satisfies

$$\bar{u}_j = u[c, \bar{f}],$$

where

$$\bar{f} = -u\partial_j c + \partial_j f.$$

Applying Lemma 66, we have $|\bar{u}_j|_{0,1} < \Psi(|c|_{0,1} + |\bar{f}|_{0,1})$. Note that, $|\bar{f}|_{0,1}$ is controlled by $|u|_{0,1} + |\partial_j c|_{0,1} + |\partial_j f|_{0,1}$, which implies that $|\bar{f}|_{0,1} \leq \Psi(|c|_{0,2} + |f|_{0,2})$ due to Lemma 66. Hence, we conclude that $|u[c,f]|_{0,2} \leq \Psi(|c|_{0,2} + |f|_{0,2})$.

At last, applying Lemma 66 on $u[c, \bar{f}]$ again, we have

$$|u[c_1, \bar{f}_1] - u[c_2, \bar{f}_2]|_0 \leq \Psi(K)(|c_1 - c_2|_0 + |\bar{f}_1 - \bar{f}_2|_0)$$

for $K = |c_1|_0 + |\bar{f}_1|_0 + |c_2|_0 + |\bar{f}_2|_0$, which similarly concludes the desired result. $\square$

5.3.1.4 Summary on regularity and sensitivity

Now we may summarize and generalize the results above to a PDE with non-zero initial conditions. Consider equation

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u - cu + f, & \text{on } (0, T) \times \mathbb{T}^d \\ u(0, x) = \psi(x), & \text{on } x \in \mathbb{T}^d. \end{cases} \tag{5.3.10}$$

To proceed, we recall the following notations:

- $C^{\delta,n}_{0,n'}$ be the space of all functions $f \in C^{\delta,n}([0, T] \times \mathbb{T}^d)$ with the topology induced by the norm $|\cdot|_{0,n'}$.
- $C^{1,3}_{0,1}([0, T] \times \mathbb{T}^d)$ is the space of all $u \in C^{1,3}([0, T] \times \mathbb{T}^d)$ topologized by $|\cdot|_{0,1}$.

For more details, we refer it to Section 5.7.

Theorem 69. *The solution map $u : [c, f, \psi] \mapsto u[c, f, \psi]$ given by (5.3.10) is a locally Lipschitz continuous map*

$$C^{\delta,2}_{0,1} \times C^{\delta,2}_{0,1} \times C^4_3 \mapsto C^{1,3}_{0,1}.$$

Proof. It is enough to show that

$$|u[c_1, f_1, \psi_1] - u[c_2, f_2, \psi_2]|_{0,1} \leq \Psi(K)(|c_1 - c_2|_{0,1} + |f_1 - f_2|_{0,1} + |\psi_1 - \psi_2|_3)$$

for $K = |c_1|_{0,1} + |c_2|_{0,1} + |f_1|_{0,1} + |f_2|_{0,1} + |\psi_1|_3 + |\psi_2|_3$. Indeed, setting $\tilde{u}(t, x) = u(t, x) - \psi(x)$, we have

$$\tilde{u} = u\left[c, f + \frac{1}{2}\Delta\psi - c\psi, 0\right]$$

for the solution map $u[\cdot, \cdot, \cdot]$ defined via (5.3.10), and observe that the desired result is a consequence of Lemma 68. $\square$

Note that the local Lipschitz continuity of Theorem 69 automatically yields its local boundedness, i.e.,

$$|u[c, f, \psi]|_{0,1} \leq \Psi(|c|_{0,1} + |f|_{0,1} + |\psi|_3) \tag{5.3.11}$$

for some positive increasing function Ψ. The following Harnack type inequality will be useful.

Corollary 70. *If $f \equiv 0$, $\psi = e^b$ for some $c, b \in C^{\delta,2}([0, T] \times \mathbb{T}^d)$, then the solution u of (5.3.10) satisfies the inequality*

$$e^{-(|b|_0 + |c|_0 T)} < u(t, x) < e^{|b|_0 + |c|_0 T}, \quad \forall (t, x) \in [0, T] \times \mathbb{T}^d.$$

Proof. The inequalities follow from the representation for $v(t,x) = u(T-t,x)$ in the form of

$$v(t,x) = \mathbb{E}\left[\exp\left\{-\int_t^T c\left(r, X^{t,x}(r)\right) dr\right\} \psi\left(X^{t,x}(T)\right)\right],$$

where X is given by (5.3.3). $\square$

5.3.2 The FPK equation

We study the weak solution of FPK equation on $[0,T) \times \mathbb{T}^d$:

$$\begin{cases} \partial_t \nu(t,x) = -\mathrm{div}_x(b(t,x)\nu(t,x)) + \frac{1}{2}\Delta\nu(t,x) \\ \nu(0,x) = m_0(x). \end{cases} \tag{5.3.12}$$

We adopt the conventional notation of

$$\langle m, \psi \rangle := \int_{\mathbb{T}^d} \psi(x) m(dx)$$

for any $m \in \mathcal{P}_1(\mathbb{T}^d)$ and $\psi : \mathbb{T}^d \mapsto \mathbb{R}$ whenever it is well defined.

Definition 71. *ν is said to be a weak solution of FPK (5.3.12), if it satisfies, for any $\phi \in C_c^\infty([0,T] \times \mathbb{T}^d)$*

$$\langle m_0, \phi(0,x)\rangle + \int_0^T \langle \nu_t, (\partial_t + \mathcal{L})\phi \rangle dt = 0,$$

where

$$\mathcal{L} = b \cdot \nabla + \frac{1}{2}\Delta.$$

We denote the solution map of (5.3.12) by $\nu = \nu[b, m_0]$. We recall that $C([0,T], \mathcal{P}_1(\mathbb{T}^d))$ is the space of all continuous mappings $\nu : [0,T] \mapsto \mathcal{P}_1(\mathbb{T}^d)$ with a metric given by

$$dist(\nu_1, \nu_2) = \sup_t \mathbb{W}_1(\nu_1(t), \nu_2(t)),$$

where $\mathbb{W}_1$ is 1-Wasserstein metric for $\mathcal{P}_1$.

Theorem 72. *Let $m_0 \in \mathcal{P}_1(\mathbb{T}^d)$. Then the solution map $b \mapsto \nu[b, m_0]$ of (5.3.12) is a locally Lipschitz continuous mapping from $C([0,T] \times \mathbb{T}^d)$ to $C([0,T], \mathcal{P}_1(\mathbb{T}^d))$. In particular, if $|b_1|_0 + |b_2|_0 < K$ then*

$$\sup_t \mathbb{W}_1(\nu_1(t), \nu_2(t)) \le \Psi(K)|b_1 - b_2|_0.$$

Moreover, $\nu = \nu[b, m_0]$ satisfies,

$$\mathbb{W}_1(\nu(t), \nu(s)) \le (1 + \sqrt{T}|b|_0)|t-s|^{1/2}, \tag{5.3.13}$$

$$\sup_t \int_{\mathbb{T}^d} |x| \nu(t, dx) \le \int_{\mathbb{T}^d} |x| m_0(dx) + |b|_0 T + \sqrt{T}. \tag{5.3.14}$$

Proof. If $|b|_0 < \infty$ and $m_0 \in \mathcal{P}_1$, then

$$X(t) = X(0) + \int_0^t b(s, X_s)ds + W(t), \ X(0) \sim m_0$$

has a unique solution. An application of Itô's formula and the definition of the weak solution verifies that $\nu(t) = Law(X(t))$ is the weak solution of (5.3.12), see [13]. (5.3.13) also follows from [13].

Next, (5.3.14) follows from

$$\sup_t \mathbb{E}\left[|X(t)|\right] \leq \mathbb{E}[|X(0)|] + |b|_0 T + \sqrt{T}.$$

Let's assume $|b_1|_0 + |b_2|_0 < K$ and ν_1 and ν_2 are corresponding solutions of (5.3.12). We denote by X_1 and X_2 the solutions of the SDE above. Note that

$$\begin{aligned}\mathbb{E}[|X_1(t) - X_2(t)|] &\leq \mathbb{E}\left[\int_0^t |b_1(s, X_1(s)) - b_2(s, X_2(s))| ds\right] \\ &\leq |b_1 - b_2|_0 T + K \int_0^t \mathbb{E}[|X_1(s) - X_2(s)|] ds.\end{aligned}$$

So, we can use the Gronwall's inequality to have

$$\mathbb{E}[|X_1(t) - X_2(t)|] \leq |b_1 - b_2|_0 T e^{KT}.$$

Therefore, we can have local Lipschitz of $b \mapsto \nu[b, m_0]$ from

$$\mathbb{W}_1(\nu_1(t), \nu_2(t)) \leq \mathbb{E}[|X_1(t) - X_2(t)|] \leq |b_1 - b_2|_0 T e^{KT}.$$

□

5.4 Existence

We now return to the GMFG scheme. First observe that, by using the cost of the form (5.2.3), the triple $(v, \mathbf{a}^*, \mu)$ is the solution of (5.2.5) if and only if the pair $(\tilde{v} := v - b, \mu)$ is the solution of HJB equation

$$\begin{cases} \partial_t \tilde{v} - \frac{1}{2}|\nabla \tilde{v}|^2 + \frac{1}{2}\Delta \tilde{v} + \tilde{\ell}_1(\mu, g) = 0 \\ \tilde{v}(T, \alpha, x) = -b(T, \alpha, x) \end{cases} \tag{5.4.1}$$

coupled with FPK equation

$$\begin{cases} \partial_t \mu = \mathrm{div}_x(\mu \nabla \tilde{v}) + \frac{1}{2}\Delta \mu \\ \mu(0, \alpha, x) = m_0(\alpha, x), \end{cases} \tag{5.4.2}$$

where $\tilde{\ell}_1$ is

$$\tilde{\ell}_1(t,\alpha,x) = \ell_1(t,\alpha,x) + \left(\partial_t b + \frac{1}{2}|\nabla b|^2 + \frac{1}{2}\Delta b\right)(t,\alpha,x). \tag{5.4.3}$$

Next, we outline our approach to the existence as follows. We define an operator

$$\nu = \Phi(\mu) = \Phi_2 \circ \Phi_1(\mu),$$

where

1. $\nabla\tilde{v} = \Phi_1(\mu)$, where $\tilde{v}$ is the solution of (5.4.4) with a given μ:

$$\begin{cases} \partial_t \tilde{v} - \frac{1}{2}|\nabla\tilde{v}|^2 + \frac{1}{2}\Delta\tilde{v} + \tilde{\ell}_1(\mu, g) = 0 \\ \tilde{v}(T,\alpha,x) = -b(T,\alpha,x) \end{cases} \tag{5.4.4}$$

2. $\nu = \Phi_2(\bar{v})$ be the function solving (5.4.5) with a given $\bar{v}$:

$$\begin{cases} \partial_t \nu = \mathrm{div}_x(\bar{v}\nu) + \frac{1}{2}\Delta\nu \\ \nu(0,\alpha,x) = m_0(\alpha,x). \end{cases} \tag{5.4.5}$$

The existence of the solution for the GMFG can be accomplished by the Schauder's fixed point theorem in an appropriate space to be mentioned below.

To proceed, we recall that d_1 is the Wasserstein metric on $\mathcal{P}_1(\mathbb{T}^d)$. We define the space $S^{1/2}$ as the collection of $\mu : [0,T] \times [0,1] \mapsto \mathcal{P}_1(\mathbb{T}^d)$ such that

$$|\mu|_{1/2} = |\mu|_0 + [\mu]_{1/2} < \infty,$$

where

$$|\mu|_0 = \sup_{t,\alpha} \int_{\mathbb{T}^d} |x|\mu(t,\alpha,dx)$$

and

$$[\mu]_{1/2} = \sup_{t_1 \neq t_2, \alpha} \frac{d_1(\mu(t_1,\alpha), \mu(t_2,\alpha))}{|t_1 - t_2|^{1/2}}.$$

Note that, $S^{1/2}$ is metrizable by

$$\rho(\mu_1,\mu_2) = \sup_{t,\alpha} \mathbb{W}_1(\mu_1(t,\alpha), \mu_2(t,\alpha)), \tag{5.4.6}$$

and we denote the space $S^{1/2}$ by $(S^{1/2},\rho)$ whenever we need to emphasize its underlying metric. Note that $B_r := \{\mu \in S^{1/2} : |\mu|_{1/2} \leq r\}$ is a closed convex compact subset of $(S^{1/2},\rho)$ by generalized version of Arzelà–Ascoli theorem, see P232 of [52].

It is often useful by the duality representation of Wasserstein metric to write

$$\rho(\mu_1, \mu_2) = \sup_{t,\alpha, Lip(f)\le 1} \int_{\mathbb{T}^d} f(x) d(\mu_1(t,\alpha) - \mu_2(t,\alpha))(x) \tag{5.4.7}$$

where $Lip(f)$ is the Lipschitz constant of the function f. Similarly, if $\mu \in B_r$ and $f \in C^1$, then

$$\begin{aligned} \int_{\mathbb{T}^d} f(y) d(\mu(t_1,\alpha) - \mu(t_2,\alpha))(y) &\le |\nabla f|_0 \mathbb{W}_1(\mu(t_1,\alpha), \mu(t_2,\alpha)) \\ &\le r|\nabla f|_0 |t_1 - t_2|^{1/2}. \end{aligned} \tag{5.4.8}$$

5.4.1 Assumptions

To proceed, we define a space $C^{\delta,0,m}_{0,0,m'}$ as the collection of all functions in $C^{\delta,0,m}([0,T]\times[0,1]\times\mathbb{T}^d,\mathbb{R})$ equipped with a $C^{0,0,m'}([0,T]\times[0,1]\times\mathbb{T}^d,\mathbb{R})$ norm. For instance, if $f \in C^{0.5,0,2}_{0,0,2}$, then we write its norm as

$$|f|^{0.5,0,2}_{0,0,2} = |f|_{0,0,2} = |f|_0 + \sum_i |\partial_{x_i} f|_0 + \sum_{ij} |\partial_{x_i x_j} f|_0.$$

For more details, we refer to Section 5.7.

Assumption 73. *$b : [0,T]\times[0,1]\times\mathbb{T}^d \mapsto \mathbb{R}^d$, $g : [0,1]^2 \mapsto \mathbb{R}$, and $m_0 : [0,1]\times\mathbb{T}^d \mapsto \mathbb{R}^d$ are infinitely smooth functions in all variables.*

We pose the following assumptions for the cost function ℓ_1. Throughout the chapter, since g will be a priori given function, we will suppress g by writing

$$\ell_1(\mu, g, t, \alpha, x) = \ell_1(\mu, t, \alpha, x)$$

if this does not cause any confusion. For convenience, we will write

$$\ell_1[\mu](t,\alpha,x) = \ell_1(\mu,t,\alpha,x) = \ell_1(\mu,g,t,\alpha,x).$$

Assumption 74. *The mapping $\mu \mapsto \ell_1[\mu]$ is a bounded and Lipschitz continuous mapping from $S^{1/2}$ to $C^{0.5,0,2}_{0,0,1}$, that is, for any $\mu \in S^{1/2}$, $\ell_1[\mu]$ belongs to $C^{0.5,0,2}$ and*

$$|\ell_1[\mu]|_{0,0,1} < M, \quad |\ell_1[\mu_1] - \ell_1[\mu_2]|_{0,0,1} \le M\rho(\mu_1,\mu_2),$$

for some $M > 0$ independent to the choice of μ.

We check that the assumptions are valid for a class of examples given in Lemma 75.

Lemma 75. *Suppose $\ell_2 \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d, \mathbb{R})$ and g are given smooth enough. Then, the cost ℓ_1 of (5.2.4) satisfies Assumption 74.*

Proof. Let $d = 1$ for the simplicity. For $\mu \in S^{1/2}$, we have

$$|\ell_1[\mu]|_0 \le |\ell_2|_0 |g|_0,$$

$$|\partial_x \ell_1[\mu]|_0 \leq |\partial_x \ell_2|_0 |g|_0,$$
$$|\partial_{xx} \ell_1[\mu]|_0 \leq |\partial_{xx} \ell_2|_0 |g|_0,$$

and

$$\begin{aligned}
|\ell_1(\mu, t_1, \alpha, x) - \ell_1(\mu, t_2, \alpha, x)| &\leq \int_0^1 \int_{\mathbb{T}^d} |\ell_2(x, y)(\mu(t_1, \alpha', dy) - \mu(t_2, \alpha', dy))g(\alpha, \alpha')|d\alpha' \\
&\leq \int_0^1 |\partial_y \ell_2|_0 d_1(\mu(t_1, \alpha'), \mu(t_2, \alpha'))g(\alpha, \alpha')d\alpha' \\
&\leq |\partial_y \ell_2|_0 |g|_0 |\mu|_{1/2} |t_1 - t_2|^{1/2}.
\end{aligned}$$

This implies that $\ell_1[\mu] \in C^{1/2,0,2}$ with estimation

$$|\ell_1[\mu]|_{1/2,0,2} \leq |\ell_2|_{2,0} |g|_0 (1 + |\mu|_{1/2}). \tag{5.4.9}$$

Note that (5.4.9) does not give a uniform upper bound due to the μ-dependence on the right hand side of the inequality. Nevertheless, we have a uniform upper bound for the weaker norm $|\cdot|_{0,0,1}$:

$$|\ell_1[\mu]|_{0,0,1} \leq |\ell_2|_{1,0} |g|_0, \ \forall \mu \in S^{1/2}.$$

For $\mu_1, \mu_2 \in S^{1/2}$, we have

$$\begin{aligned}
\ell_1(\mu_1, t, \alpha, x) - \ell_1(\mu_2, t, \alpha, x) &= \int_0^1 \int_{\mathbb{T}^d} \ell_2(x, y)(\mu_1(t, \alpha', dy) - \mu_2(t, \alpha', dy))g(\alpha, \alpha')d\alpha' \\
&\leq |\partial_y \ell_2|_0 d_1(\mu_1(t, \alpha), \mu_2(t, \alpha))|g|_0.
\end{aligned}$$

This implies that

$$|\ell_1[\mu_1] - \ell_1[\mu_2]|_0 \leq |\partial_y \ell_2|_0 |g|_0 \rho(\mu_1, \mu_2).$$

Similarly, we obtain

$$|\partial_x \ell_1[\mu_1] - \partial_x \ell_1[\mu_2]|_0 \leq |\partial_y \partial_x \ell_2|_0 |g|_0 \rho(\mu_1, \mu_2).$$

Therefore, we have Lipschitz continuity

$$|\ell_1[\mu_1] - \ell_1[\mu_2]|_{0,0,1} \leq |\ell_2|_{1,1} |g|_0 \rho(\mu_1, \mu_2),$$

and this implies Assumption 74 with $M = |\ell_2|_{1,1} |g|_0$. □

5.4.2 Operator Φ_1

Recall that $\nabla \tilde{v} = \Phi_1(\mu)$, where $\tilde{v}$ is the solution of (5.4.4) with given μ. By Hopf-Cole transform $\tilde{v}$ is the solution of (5.4.4) if and only if

$$w = \exp\{-\tilde{v}\} \tag{5.4.10}$$

is the solution of

$$\begin{cases} \partial_t w + \frac{1}{2}\Delta w - w\tilde{\ell}_1[\mu] = 0 & \text{on } (0,T)\times[0,1]\times\mathbb{T}^d \\ w(T,\alpha,x) = e^{b(T,\alpha,x)} & \text{on } [0,1]\times\mathbb{T}^d. \end{cases} \tag{5.4.11}$$

In addition, we have the following relation by chain rule:

$$\nabla\tilde{v} = -\frac{\nabla w}{w},\ \Delta\tilde{v} = \frac{-w\Delta w + |\nabla w|^2}{w^2}.$$

Since w-term appears in the denominator, Harnack type inequality in Corollary 70 ensures that $\nabla\tilde{v}$ and $\Delta\tilde{v}$ are well defined.

5.4.2.1 Estimates of parameterized PDEs

We define

$$w = G(f) \tag{5.4.12}$$

by the solution of

$$\begin{cases} \partial_t w + \frac{1}{2}\Delta w - wf = 0 & \text{on } (0,T)\times[0,1]\times\mathbb{T}^d \\ w(T,\alpha,x) = e^{b(T,\alpha,x)} & \text{on } [0,1]\times\mathbb{T}^d. \end{cases} \tag{5.4.13}$$

Note that $w = G(\tilde{\ell}_1[\mu])$ is the solution of (5.4.11).

Lemma 76. *The mapping G is a locally Lipschitz continuous mapping from $C^{0.5,0,2}_{0,0,1}$ to $C^{1,0,2}_{0,0,1}$.*

Proof. Let $f \in C^{0.5,0,2}$ and $w = G(f)$. By Theorem 69, we have $w(\alpha) \in C^{1,3}$. If $\alpha \to \alpha_0$, then $f(\alpha) \to f(\alpha_0)$ holds pointwisely. Together with Dominated Convergence Theorem on the probabilistic representation of w, one can conclude $w(t,\alpha,x) \to w(t,\alpha_0,x)$ whenever $\alpha \to \alpha_0$. Therefore, w belongs to $C^{1,0,3}$.

Given $f_1, f_2 \in C^{0.5,0,2}$ and $w_i = G(f_i)$ with

$$K(\alpha) = |f_1(\alpha)|_{0,1} + |f_2(\alpha)|_{0,1} + |e^{b(T,\alpha)}|_3,$$

we can use local Lipschitz continuity of Theorem 69 to obtain local Lipschitz of G,

$$\begin{aligned} |w_1 - w_2|_{0,0,1} &= \sup_\alpha |w_1(\alpha) - w_2(\alpha)|_{0,1} \\ &\le \sup_\alpha \Psi(K(\alpha))|f_1(\alpha) - f_2(\alpha)|_{0,1} \\ &\le \Psi(\sup_\alpha K(\alpha))|f_1 - f_2|_{0,0,1}. \end{aligned}$$

In the above, we used the monotonicity of $\Psi(\cdot)$ to switch Ψ and sup. Since $\sup_\alpha K(\alpha) \le \Psi(|f_1|_{0,0,1} + |f_2|_{0,0,1} + |b|_3)$, we can rewrite the above estimations as

$$|w_1 - w_2|_{0,0,1} \le \Psi(|f_1|_{0,0,1} + |f_2|_{0,0,1} + |b|_3)|f_1 - f_2|_{0,0,1}.$$

$\square$

5.4.2.2 Φ_1 estimate

Lemma 77. *Φ_1 is a uniformly bounded and Lipschitz continuous mapping from $(S^{1/2}, \rho)$ to $C^0([0,T]\times[0,1]\times\mathbb{T}^d, \mathbb{R}^d)$.*

Proof. If $\mu \in S^{1/2}$, then $\ell_1[\mu] \in C^{0.5,0,2}$ with $|\ell_1[\mu]|_{0,0,1} < M$ by Assumption 74. We recall that

$$\tilde{\ell}_1 = \ell_1 + \left(\partial_t b + \frac{1}{2}|\nabla b|^2 + \frac{1}{2}\Delta b\right).$$

Due to the smoothness of b and compactness of its domain, we still have $\tilde{\ell}_1[\mu] \in C^{0.5,0,2}$ with $|\tilde{\ell}_1[\mu]|_{0,0,1} < \Psi(M)$. Together with local Lipschitz continuity of $G(\cdot)$ in Lemma 76, it implies uniform boundedness of $w = G(\tilde{\ell}_1[\mu])$, i.e.

$$|w|_{0,0,1} < \Psi(M).$$

Moreover, Corollary 70 says that the reciprocal of $w = G(\tilde{\ell}_1[\mu])$ is bounded in the sense $|w^{-1}|_0 < \Psi(|\tilde{\ell}_1[\mu]|_0)$. Therefore, we have

$$|w|_{0,0,1} + |w^{-1}|_0 < \Psi(M).$$

Next, we can prove that Φ_1 is uniformly bounded in C^0:

$$|\Phi_1(\mu)|_0 = |\nabla\tilde{v}|_0 = |w^{-1}\nabla w|_0 \le |w^{-1}|_0|\nabla w|_0 \le |w^{-1}|_0|w|_{0,0,1} \le \Psi(M).$$

Finally, we can show the global Lipschitz for Φ_1 by the following estimates:

$$\begin{aligned}
|\Phi_1(\mu_1) - \Phi_1(\mu_2)|_0 &= \left|w_1^{-1}\nabla w_1 - w_2^{-1}\nabla w_2\right|_0 \\
&= \left|\frac{w_2\nabla w_1 - w_1\nabla w_2}{w_1 w_2}\right|_0 \\
&\le \Psi(M)(|w_2|_0|\nabla w_1 - \nabla w_2|_0 + |\nabla w_2|_0|w_1 - w_2|_0) \\
&\le \Psi(M)|w_1 - w_2|_{0,0,1} \\
&\le \Psi(M)\left|\tilde{\ell}_1[\mu_1] - \tilde{\ell}_1[\mu_2]\right|_{0,0,1} \\
&\le \Psi(M)\rho(\mu_1, \mu_2).
\end{aligned}$$

In the last two steps, we used Lipschitz continuity obtained by Lemma 76 and Assumption 74. $\square$

5.4.3 Operator Φ_2

Next, we will show the properties associated to Φ_2 mapping from $C^0([0,T]\times[0,1]\times\mathbb{T}^d, \mathbb{R}^d)$ to $S^{1/2}$.

Lemma 78. *Φ_2 is a locally Lipschitz continuous mapping from $C^0([0,T]\times[0,1]\times\mathbb{T}^d, \mathbb{R}^d)$ to $(S^{1/2}, \rho)$. Moreover, $|\Phi_2(\bar{v})|_{1/2} \le \Psi(|\bar{v}|_0)$ for all $\bar{v} \in C^0([0,T]\times[0,1]\times\mathbb{T}^d, \mathbb{R}^d)$ for some monotonically increasing positive function Ψ.*

Proof. Given $\bar{v} \in C^0([0,T] \times [0,1] \times \mathbb{T}^d, \mathbb{R}^d)$ and $\nu = \Phi_2(\bar{v})$, applying (5.3.14) of Theorem 72, it yields that

$$\begin{aligned}|\nu|_0 = \sup_{t,\alpha} \int_{\mathbb{T}^d} |x| \nu(t,\alpha,dx) &= \sup_{\alpha} \sup_{t} \int_{\mathbb{T}^d} |x| \nu(t,\alpha,dx) \\ &\leq \sup_{\alpha} \left(\int_{\mathbb{T}^d} |x| m_0(\alpha,dx) + |\bar{v}(\alpha)|_0 T + \sqrt{T} \right) \\ &\leq \Psi(|\bar{v}|_0).\end{aligned}$$

Next, we show the following equicontinuity property again by (5.3.13) of Theorem 72:

$$\begin{aligned}\sup_{t_1 \neq t_2, \alpha} \mathbb{W}_1(\nu(t_1,\alpha), \nu(t_2,\alpha)) &\leq \sup_{\alpha} \left(1 + \sqrt{T} |\bar{v}(\alpha)|_0\right) |t_1 - t_2|^{1/2} \\ &\leq \Psi(|\bar{v}|_0) |t_1 - t_2|^{1/2}.\end{aligned}$$

This proves $\nu \in S^{1/2}$ with

$$|\nu|_{1/2} \leq \Psi(|\bar{v}|_0).$$

For the continuity of Φ_2, given $\bar{v}_1, \bar{v}_2 \in C^0([0,T] \times [0,1] \times \mathbb{T}^d, \mathbb{R}^d)$, we set $\nu_i = \Phi_2(\bar{v}_i)$ for $i = 1, 2$. Then, we use the local Lipschitz continuity in Theorem 72 to obtain local Lipschitz continuity of Φ_2 as follows:

$$\begin{aligned}\rho(\nu_1, \nu_2) &= \sup_{t,\alpha} \mathbb{W}_1(\nu_1(t,\alpha), \nu_2(t,\alpha)) \\ &= \sup_{\alpha} \sup_{t} \mathbb{W}_1(\nu_1(t,\alpha), \nu_2(t,\alpha)) \\ &= \sup_{\alpha} \Psi(|\bar{v}_1(\alpha)|_0 + |\bar{v}_2(\alpha)|_0) |\bar{v}_1(\alpha) - \bar{v}_2(\alpha)|_0 \\ &\leq \Psi(|\bar{v}_1|_0 + |\bar{v}_2|_0) |\bar{v}_1 - \bar{v}_2|_0.\end{aligned}$$

□

5.4.4 Existence by the Schauder's fixed point theorem

Theorem 79. *Suppose Assumptions 73 - 74 are valid. Then there exists a solution of (5.2.5) in the space $C^{1,0,2}([0,T] \times [0,1] \times \mathbb{T}^d, \mathbb{R}) \times C([0,T] \times [0,1], \mathcal{P}_1(\mathbb{T}^d))$.*

Proof. It is enough to show that $\Phi_2 \circ \Phi_1$ has a fixed point in $S^{1/2}$. Recall that B_r is a convex closed and compact subset of $S^{1/2}$. For simplicity, we denote by $\hat{B}_r$ the closed ball of radius r in $C^0([0,T] \times [0,1] \times \mathbb{T}^d, \mathbb{R}^d)$.

1. By Lemma 77, there exists some positive increasing function Ψ_1 independent to r, such that the mapping

$$\Phi_1 : B_r \mapsto \hat{B}_{\Psi_1(M)}$$

is continuous.

2. By Lemma 78, there exists some positive increasing function Ψ_2 such that the mapping

$$\Phi_2 : \hat{B}_{\Psi_1(M)} \mapsto B_{\Psi_2\circ\Psi_1(M)}$$

is continuous.

Now we take

$$r = \Psi_2(\Psi_1(M))$$

and we have

$$\Phi_2 \circ \Phi_1 : B_r \mapsto B_r$$

is a continuous mapping and this yields the existence of a fixed point for Φ by the Schauder's fixed point theorem. □

In the above, we have indeed proved the existence in the space $C^{1,0,2}([0,T]\times[0,1]\times\mathbb{T}^d,\mathbb{R})\times S^{1/2}$.

5.4.5 Further remarks on the fixed point theorem

In connection with GMFG, we explain why Theorem 69 establishes locally Lipschitz continuity of the solution map $u : [c, f, \psi] \mapsto u[c, f, \psi]$ of (5.3.10) in the sense of

$$C^{\delta,2}_{0,1} \times C^{\delta,2}_{0,1} \times C^4_3 \mapsto C^{1,3}_{0,1} \tag{5.4.14}$$

instead of

$$C^{\delta,2} \times C^{\delta,2} \times C^4 \mapsto C^{1,3}. \tag{5.4.15}$$

For the illustration purpose, if we freeze c, ψ of the solution map u, then local Lipschitz continuity in the sense of (5.4.14) implies local boundedness

$$|u|_{0,1} \le \Psi(|f|_{0,1}),$$

while local Lipschitz continuity in the sense of (5.4.15) implies local boundedness

$$|u|_{0,1} \le |u|_{1,3} \le \Psi(|f|_{\delta,2}).$$

The main difference of these two local boundedness properties is that, the first one controls u by f with 0-norm in t-variable while the second one does by f with δ-norm in t-variable, which is not desirable. The main reason is that the running cost $|\ell_1[\mu]|_{1/2,0,1} \le \Psi(|\mu|_{1/2})$ of (5.4.9) does not have uniform bound in μ, while $|\ell_1[\mu]|_{0,0,1}$ does. For this reason, we included the regularity results for parabolic PDE solutions by dropping t-regularity while increasing x-regularity as a tradeoff.

Recall that, we have established the existence of a fixed point of a mapping $\Phi = \Phi_2 \circ \Phi_1$ for $\Phi_1 : \mu \mapsto \nabla\tilde{v}$ and $\Phi_2 : \nabla\tilde{v} \mapsto \nu$. Our approach is along the the Schuader's fixed point theorem with estimates

$$\Phi_1 : B_r \mapsto \hat{B}_{\Psi_1(M)}, \quad \Phi_2 : \hat{B}_{\Psi_1(M)} \mapsto B_{\Psi_2\circ\Psi_1(M)}.$$

In the above, it is crucial that the Φ_1 is upper bounded by $\Psi_1(M)$ independent to r, and this can be inferred from local boundedness of (5.4.14) together with uniform boundedness of $|\ell_1[\mu]|_{0,0,1}$.

In contrast, if we use local boundedness in the sense of (5.4.15), then we have estimations in the form of

$$\Phi_1 : B_r \mapsto \hat{B}_{\Psi_1(r)}, \quad \Phi_2 : \hat{B}_{\Psi_1(r)} \mapsto B_{\Psi_2 \circ \Psi_1(r)}.$$

Since the norm of the running cost $|\ell_1[\mu]|_{1,0,3}$ depends on μ, Φ_1 can not be uniformly bounded. As a result, the choice of $r = \Psi_1(r)$ is infeasible.

5.5 Uniqueness of GMFG

Assumption 80. *There exists some $\alpha \in [0,1]$ satisfying*

$$\int_{\mathbb{T}^d} (\ell_1(\mu_1, g, t, \alpha, x) - \ell_1(\mu_2, g, t, \alpha, x))(\mu_1 - \mu_2)(t, \alpha, dx) > 0,$$

for all $\mu_1 \neq \mu_2 \in C([0,T] \times [0,1], \mathcal{P}_1(\mathbb{T}^d))$ and $t \in [0,T]$.

Theorem 81. *([13], [72]) Suppose Assumptions 73 - 74 and 80 are valid. Then, there exists a unique solution of (5.2.5) in the space $C^{1,0,2}([0,T] \times [0,1] \times \mathbb{T}^d, \mathbb{R}) \times C([0,T] \times [0,1], \mathcal{P}_1(\mathbb{T}^d))$.*

Proof. For $i = 1, 2$, let (v_i, μ_i) be two different solution pairs, and set

$$\bar{v} = v_1 - v_2, \ \bar{\mu} = \mu_1 - \mu_2.$$

Note that $\bar{v}(T, \alpha, x) = \bar{\mu}(0, \alpha, x) = 0$ for all (α, x) by their given initial-terminal data. We also write $\ell_1[\mu_i] = \ell_1[\mu_i, g]$ for short. Then $\bar{v}$ satisfies

$$\partial_t \bar{v} + \nabla b \cdot \nabla \bar{v} + \frac{1}{2}\Delta \bar{v} - \frac{1}{2}|\nabla v_1|^2 + \frac{1}{2}|\nabla v_2|^2 + \ell_1[\mu_1] - \ell_1[\mu_2] = 0$$

and $\bar{\mu}$ satisfies

$$-\partial_t \bar{\mu} - \mathrm{div}(\nabla b \bar{\mu}) + \frac{1}{2}\Delta \bar{\mu} + \mathrm{div}(\nabla v_1 \mu_1) - \mathrm{div}(\nabla v_2 \mu_2) = 0.$$

The above two equations can be rewritten as

$$\langle \partial_t \bar{v} + \nabla b \cdot \nabla \bar{v} + \frac{1}{2}\Delta \bar{v}, \bar{\mu} \rangle + \langle -\frac{1}{2}|\nabla v_1|^2 + \frac{1}{2}|\nabla v_2|^2 + \ell_1[\mu_1] - \ell_1[\mu_2], \bar{\mu} \rangle = 0$$

and

$$\langle \partial_t \bar{v} + \nabla b \cdot \nabla \bar{v} + \frac{1}{2}\Delta \bar{v}, \bar{\mu} \rangle + \langle \bar{v}, \mathrm{div}(\nabla v_1 \mu_1) - \mathrm{div}(\nabla v_2 \mu_2) \rangle = 0.$$

By subtracting above two equations, and utilizing the identities

$$\langle \mathrm{div}(\nabla v_1 \mu_1), \bar{v} \rangle = -\langle |\nabla v_1|^2, \mu_1 \rangle + \langle \nabla v_1 \cdot \nabla v_2, \mu_1 \rangle$$

and

$$\langle \mathrm{div}(\nabla v_2 \mu_2), \bar{v} \rangle = \langle |\nabla v_2|^2, \mu_2 \rangle - \langle \nabla v_1 \cdot \nabla v_2, \mu_2 \rangle,$$

we obtain

$$\langle \frac{1}{2}(\mu_1 + \mu_2), |\nabla \bar{v}|^2 \rangle + \langle \ell_1[\mu_1] - \ell_1[\mu_2], \bar{\mu} \rangle = 0.$$

The first term is non-negative and the second term is strictly positive for some $\alpha \in [0, 1]$ by (A3), which implies a contradiction. □

5.6 Concluding remarks

Our main result of Theorem 81 provides existence and uniqueness of the GMFG equation under some assumptions. One limitation of the current setting is that the running cost in the current setup allows to use Hopf-Cole transformation, which is essential to the subsequent analysis on regularities. To deal with the full generalization of the running cost, one must adopt different approaches and it will be in our future research direction. It is also desirable to generalize the result to the whole domain $\mathbb{R}^d$ instead of compact domain $\mathbb{T}^d$. Another limitation is that, the current setting requires the continuity of the graphon. Note that some graphons are not necessarily continuous. Nevertheless, the continuity condition of the graphon can be relaxed in the following sense by similar arguments with additional complexity of notations, which is sketched below briefly.

To proceed, we define $\hat{C}^0$ as the collection of bounded measurable functions $f : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}$, and we denote its norm as

$$|f|_0 = \sup_{[0,T]\times[0,1]\times\mathbb{T}^d} |f(t, \alpha, x)|.$$

With $\hat{C}^{\delta,0,2}$, we denote the set of functions $f \in \hat{C}^0$ with finite norm

$$|f|_{\delta,0,2} = |f|_0 + \sup_{t_1<t_2,\alpha,x} \frac{|f(t_1, \alpha, x) - f(t_2, \alpha, x)|}{|t_1 - t_2|^\delta} + \sum_i |\partial_i f|_0 + \sum_{ij} |\partial_{ij} f|_0.$$

By the above definition $\hat{C}^{\delta,0,2}$ allows the discontinuity in α.

Assumption 82. *1. b and m_0 are infinitely smooth in their domains.*

2. The graphon g is bounded measurable on $[0, 1]^2$ with

$$|g|_0 = \sup_{[0,1]^2} |g(\alpha, \alpha')| < \infty.$$

We recall that B_r is defined in $S^{1/2}$. We use $\hat{C}^{\delta,0,2}_{0,0,2}$ to denote the same set $\hat{C}^{\delta,0,2}$ with the norm $|\cdot|_{0,0,2}$, i.e.

$$|f|_{0,0,2} = |f|_0 + \sum_i |\partial_i f|_0 + \sum_{ij} |\partial_{ij} f|_0.$$

Assumption 83. *The mapping $\mu \mapsto \ell_1[\mu]$ is a bounded and Lipschitz continuous mapping from $S^{1/2}$ to $\hat{C}^{0.5,0,2}_{0,0,1}$, that is, for any $\mu \in S^{1/2}$, $\ell_1[\mu]$ belongs to $\hat{C}^{0.5,0,2}$ and*

$$|\ell_1[\mu]|_{0,0,1} < M, \quad |\ell_1[\mu_1] - \ell_1[\mu_2]|_{0,0,1} \le M\rho(\mu_1, \mu_2),$$

for some $M > 0$ independent to the choice of μ.

We also define $\hat{C}^{m,0,n}$ as the collection of $f \in \hat{C}^0$ with continuous bounded m-th derivatives in t and n-th derivatives x. For instance, for $f \in \hat{C}^{1,0,2}$, we have finite norm

$$|f|_{1,0,2} = |f|_0 + |\partial_t f|_0 + \sum_i |\partial_i f|_0 + \sum_{ij} |\partial_{ij} f|_0.$$

Now we present a result in parallel to Theorem 81. The proof is similar and so omitted.

Corollary 84. *Suppose Assumptions 82 - 83 and 80 are valid. Then there exists a unique solution of (5.2.5) in the space $\hat{C}^{1,0,2}([0,T] \times [0,1] \times \mathbb{T}^d, \mathbb{R}) \times \hat{C}([0,T] \times [0,1], \mathcal{P}_1(\mathbb{T}^d))$.*

5.7 Appendix

In this appendix, we will summarize the notations of Hölder space used in this chapter. For this purpose, we will define the following functionals for a function u from a product normed space $S = X \times Y$ to $\mathbb{R}^d$ whenever it is well defined.

- $|u|_0 = \sup_S |u(x,y)|$.
- For nonnegative integers l, m, define

  $$|u|_{l,m} = \sum_{i=0}^{l} \sum_{|\alpha|=i} |D_x^\alpha u|_0 + \sum_{i=0}^{m} \sum_{|\alpha|=i} |D_y^\alpha u|_0.$$

  In the above, α is a multi-index for differential operators. For instance, $|\alpha| = \sum_{i=1}^{d_1} |\alpha_i|$ for a multi-index $\alpha = (\alpha_i : i = 1, 2, \ldots, d_1)$.
- For positive numbers $l', m' \in (0,1)$, define

  $$[u]_{l',m'} = [u]_{l',0} + [u]_{0,m'},$$

  where

  $$[u]_{l',0} = \sup_{x_1 \neq x_2, y} \frac{|u(x_1,y) - u(x_2,y)|}{|x_1 - x_2|^{l'}},$$

  and

  $$[u]_{0,m'} = \sup_{x, y_1 \neq y_2} \frac{|u(x,y_1) - u(x,y_2)|}{|y_1 - y_2|^{m'}}.$$
- For nonnegative integers l, m and positive number $l' \in (0,1)$, define

  $$|u|_{l+l',m} = |u|_{l,m} + \sum_{|\alpha|=l} [D_x^\alpha u]_{l',0}.$$

- For nonnegative integers l, m and positive numbers $l', m' \in (0,1)$, define

$$|u|_{l+l',m+m'} = |u|_{l,m} + \sum_{|\alpha|=l} [D_x^\alpha u]_{l',m'} + \sum_{|\alpha|=m} [D_y^\alpha u]_{l',m'}.$$

One can check that the following spaces are Banach spaces:

- $C^{l,m}(X \times Y; \mathbb{R}^d) := \{u : |u|_{l,m} < \infty\}$,
- $C^{l+l',m}(X \times Y; \mathbb{R}^d) := \{u : |u|_{l+l',m} < \infty\}$,
- $C^{l+l',m+m'}(X \times Y; \mathbb{R}^d) := \{u : |u|_{l+l',m+m'} < \infty\}$.

In this chapter, we also involve the space $C^{l',0,m}$ of functions with a domain $S = X \times Y \times Z$, whose norm is defined as

$$|u|_{l',0,m} = |u|_{0,0,m} + [D_z^m u]_{l',0,0},$$

where

$$|u|_{0,0,m} = \sum_{i=0}^{m} \sum_{|\alpha|=i} |D_z^\alpha u|_0, \text{ and } [u]_{l',0,0} = \sup_{x_1 \neq x_2, y, z} \frac{|u(x_1, y, z) - u(x_2, y, z)|}{|x_1 - x_2|^{l'}}.$$

In this chapter, our functions involve state domain taking values in d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For $x \in \mathbb{R}^d$, let $\pi(x)$ be the coset of $\mathbb{Z}^d$ that contains x, i.e.

$$\pi(x) = x + \mathbb{Z}^d.$$

A canonical metric on $\mathbb{T}^d$ can be induced from the Euclidean metric by

$$|\pi(x) - \pi(y)|_{\mathbb{T}^d} = \inf\{|x - y - z| : z \in \mathbb{Z}^d\}.$$

For the illustration purpose, we provide a list of representative Hölder spaces used throughout the chapter:

- $C^{\delta/2,\delta}([0,T] \times \mathbb{T}^d)$ is a space of functions $u(t,x)$ with a norm defined by

  $$|u|_{\delta/2,\delta} = |u|_0 + [u]_{\delta/2,\delta},$$

  where $[u]_{\delta/2,\delta}$ is a seminorm defined by

  $$[u]_{\delta/2,\delta} = \sup_{t_1 \neq t_2, x} \frac{|u(t_1, x) - u(t_2, x)|}{|t_1 - t_2|^{\delta/2}} + \sup_{t, x_1 \neq x_2} \frac{|u(t, x_1) - u(t, x_2)|}{|x_1 - x_2|^\delta}.$$

  This definition may be slightly different from different resources. For instance, the definition given by [53] for the seminorm is

  $$[u]'_{\delta/2,\delta} = \sup_{(t_1,x_1) \neq (t_2,x_2) \in [0,T] \times \mathbb{T}^d} \frac{|u(t_1, x_1) - u(t_2, x_2)|}{(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^\delta}.$$

Indeed, two norms induced by $[u]_{\delta/2,\delta}$ and $[u]'_{\delta/2,\delta}$ are equivalent, which can be seen from below:

$$[u]_{\delta/2,\delta} = [u]_{\delta/2,0} + [u]_{0,\delta} \leq 2[u]'_{\delta/2,\delta},$$

and

$$\begin{aligned}
[u]'_{\delta/2,\delta} &\leq \sup_{(t_1,x_1)\neq(t_2,x_2)} \frac{|u(t_1,x_1)-u(t_2,x_1)|+|u(t_2,x_1)-u(t_2,x_2)|}{(|t_1-t_2|^{1/2}+|x_1-x_2|)^\delta} \\
&\leq \sup_{t_1\neq t_2} \frac{|u(t_1,x_1)-u(t_2,x_1)|}{|t_1-t_2|^{\delta/2}} + \sup_{x_1\neq x_2} \frac{|u(t_2,x_1)-u(t_2,x_2)|}{|x_1-x_2|^\delta} \\
&\leq [u]_{\delta/2,0} + [u]_{0,\delta} = [u]_{\delta/2,\delta}.
\end{aligned}$$

- $C^{0,1}([0,T]\times\mathbb{T}^d)$ is a space of functions $u(t,x)$ with a norm

$$|u|_{0,1} = |u|_0 + \sum_{i=1\ldots d} |\partial_{x_i} u|_0,$$

and $C^{\delta,1}([0,T]\times\mathbb{T}^d)$ is a space of functions $u(t,x)$ with a norm

$$|u|_{\delta,1} = |u|_{0,1} + \sum_{i=1\ldots d} [\partial_{x_i} u]_{\delta,0}.$$

- $C^{1,2}([0,T]\times\mathbb{T}^d)$ is a space of functions $u(t,x)$ with a norm

$$|u|_{1,2} = |u|_0 + |\partial_t u|_0 + \sum_{i=1\ldots d} |\partial_{x_i} u|_0 + \sum_{i,j=1\ldots d} |\partial_{x_i x_j} u|_0.$$

- $C^{1+\delta/2,2+\delta}([0,T]\times\mathbb{T}^d)$ is a space with a norm

$$|u|_{1+\delta/2,2+\delta} = |u|_{1,2} + [\partial_t u]_{\delta/2,\delta} + \sum_{i,j=1\ldots d} [\partial_{x_i x_j} u]_{\delta/2,\delta}.$$

- $C^{0,2}([0,T]\times\mathbb{T}^d)$ is a space with a norm

$$|u|_{0,2} = |u|_0 + \sum_{i=1\ldots d} |\partial_{x_i} u|_0 + \sum_{i,j=1\ldots d} |\partial_{x_i x_j} u|_0.$$

$C^{\delta,2}([0,T]\times\mathbb{T}^d)$ is a space with a norm

$$|u|_{\delta,2} = |u|_{0,2} + \sum_{i,j=1\ldots d} [\partial_{x_i x_j} u]_{\delta,0}.$$

We use $C^{\delta,2}_{0,2}([0,T]\times\mathbb{T}^d)$ to denote the space of all functions in $C^{\delta,2}([0,T]\times\mathbb{T}^d)$ topologized by the norm $|\cdot|_{0,2}$. Such a space $C^{\delta,2}_{0,2}([0,T]\times\mathbb{T}^d)$ is not complete. However, every $|\cdot|_{\delta,2}$-norm bounded ball in $C^{\delta,2}_{0,2}([0,T]\times\mathbb{T}^d)$ is precompact since $C^{\delta,2}([0,T]\times\mathbb{T}^d)$ is compactly embedded into $C^{0,2}([0,T]\times\mathbb{T}^d)$.

- $C^{1,3}_{0,1}([0,T]\times\mathbb{T}^d)$ is the space of all $u\in C^{1,3}([0,T]\times\mathbb{T}^d)$ topologized by $|\cdot|_{0,1}$, i.e.

$$|u|_{0,1} = |u|_0 + \sum_{i=1\ldots d} |\partial_{x_i} u|_0.$$

- $C^{0,0,2}([0,T]\times[0,1]\times\mathbb{T}^d)$ is the space of all $u(t,\alpha,x)$ having finite norm of

$$|u|_{0,0,2} = |u|_0 + \sum_{i=1\ldots d} |\partial_{x_i} u|_0 + \sum_{i,j=1\ldots d} |\partial_{x_i x_j} u|_0.$$

- $C^{\delta,0,2}([0,T]\times[0,1]\times\mathbb{T}^d)$ is the space of all $u(t,\alpha,x)$ having finite norm of

$$|u|_{\delta,0,2} = |u|_{0,0,2} + \sum_{i,j=1\ldots d} [\partial_{x_i x_j} u]_{\delta,0,0}.$$

- $C^{\delta,0,2}_{0,0,2}([0,T]\times[0,1]\times\mathbb{T}^d)$ is the space of all $u(t,\alpha,x)$ having finite norm of $|u|_{\delta,0,2}$, but topologized by $|\cdot|_{0,0,2}$.

Chapter 6

Conclusion and future work

6.1 Conclusion

In this thesis, our focus centers on the convergence analysis of stochastic nonlinear systems and we explore the examination of their asymptotic behavior, a critical area for understanding the prolonged dynamics of these systems and those with substantial populations. This thesis emphasizes two key aspects: firstly, the examination of the turnpike property within the context of stochastic control problems, and secondly, the investigation of the convergence of N-player games towards their corresponding mean field games.

Firstly, our investigation delves into the asymptotic behavior of systems exhibiting long-term dynamics, focusing on convergence concerning the time horizon. In Chapter 2, we scrutinize the limiting behavior of a specific category of linear quadratic stochastic optimal control problems and their associated value functions as the time horizon extends to infinity. We provide a distinct approach to show the turnpike properties in stochastic control theory by using the cell problem within the realm of weak KAM theory in PDE and contributions can be summarized as follows.

(1) Our first contribution lies in the formulation of a verification theorem connecting the cell problem to a specific class of infinite time horizon control problems, referred to as the probabilistic cell problem, see Lemma 5. Unlike the typical cell problem explored in the literature (e.g., [77]), the underlying cell problem in our context lacks uniqueness due to the non-compactness of the domain. The above verification theorem establishes a connection between the cell problem and the static optimization problem.

(2) Our second contribution provides the connection between the cell problem (2.1.6) and the ergodic cost problem. This involves determining the constant

$$-c_* := \lim_{T\to\infty} \frac{1}{T} V_T(x) = \lim_{T\to\infty} \frac{1}{T} J_T(x; u_T^*).$$

This connection identifies not only the aforementioned turnpike property of (2.1.5) with respect to the control process and state process, but also unveils a new turnpike property in terms of the cost function:

$$\lim_{T\to\infty} \frac{1}{T} J_T(x; u^*) = -c_*,$$

where u^* is a control process obtained from the probabilistic cell problem independent to the length of the terminal time T, see Theorem 9.

Next, we examine the system complexity and consider the convergence behavior of stochastic nonlinear systems with large populations. More precisely, we concentrate on the fundamental question arising in MFG and delves into the examination of convergence rate from the N-player games to the associated MFG within different settings.

In Chapter 3, we investigate the convergence rate of the N-player game, governed by a continuous-time Markov chain as the common noise, towards its asymptotic MFG under LQG structure. The main results and contributions are summarized as follows.

(1) Firstly, we introduce a Markovian structure using two auxiliary processes for the first and second moments of the MFG equilibrium and employ the fixed point condition in MFG. By doing so, we characterize the equilibrium measure in MFG with a finite-dimensional Riccati system of ODEs. Consequently, we obtain the equilibrium path, equilibrium control, and the value function in MFG.

(2) Subsequently, we address the N-player game under the LQG structure, and we characterize its equilibrium path, equilibrium control, and the value function through a Riccati system of ODEs with a dimension of $O(N^3)$. Leveraging the N-invariant algebraic structure of this system of ODEs, we establish a dimension reduction result, facilitating a comparison between the equilibrium path in the N-player game and the equilibrium path in the MFG.

(3) To demonstrate the convergence between the two equilibrium paths, we embed the equilibrium path in the N-player game to the probability space of the equilibrium path in the MFG using a distribution copy, leading to the achievement of the convergence result and the computation of the convergence rate. We obtain the convergence rate $O(N^{-1/2})$ with respect to 2-Wasserstein distance.

(4) Lastly, some numerical examples are presented to demonstrate the convergence result.

In Chapter 3, the number of states of the common noise is finite, thus next we consider the case when the number of states of the common noise is infinity. We investigate the convergence rate of the N-player game with Brownian motion as its common noise in the following chapter.

We focus on a class of one-dimensional LQG mean field games with Brownian motion as the common noise in Chapter 4. It is worth noting that the equilibrium path, equilibrium control, and the value function in MFG and the N-player game can be obtained by a similar methodology as Chapter 3. Our main contribution is the establishment of three different convergence rates from the N-player games to the corresponding mean field games:

- Firstly, we establish that the convergence rate of the p-Wasserstein metric for the distribution of the representative player in the N-player game to the distribution of the generic player in MFG is $O(N^{-1/2})$ for $p \in [1, 2]$;

- Secondly, it demonstrates that the convergence rate of the p-Wasserstein metric for the empirical measure of the equilibrium path in the N-player game to the equilibrium measure in MFG under the L^p sense is $O(N^{-1/(2p)})$ for $p \in [1, 2]$;

- Lastly, we show that the convergence rate of the uniform p-Wasserstein metric for the empirical measure of the equilibrium path in the N-player game to the equilibrium measure in MFG under the L^p sense is $O(N^{-1/(2p)})$ for $p \in (1, 2]$, and $O(N^{-1/2} \ln(N))$ for $p = 1$.

To achieve the above convergence rates, the methodology relies on a specific decomposition of the equilibrium path in the N-player game and the associated MFG framework. We establish the convergence results for the empirical measure of a non-i.i.d. sequence of random variables and generalize the result to triangular arrays, which provides a desired structure to complete the establishment of the convergence rates.

In Chapter 5, we investigate the large population games with asymmetric graph connections. We consider the solvability of a type of graphon mean field games. A new type of mean field games PDE system associated with the graphon mean field games system, see (5.2.5), is proposed in this project. The graphon mean field games system consists of a collection of parameterized HJB equations and a collection of parameterized Fokker-Planck-Kolmogorov equations. We establish the existence of solutions via the application of Schauder's fixed point theorem and obtain the uniqueness via the Lasry-Lions monotonicity assumption on the running cost. The main difficulty is to obtain the regularity of the solution and the sensitivity of the corresponding HJB equations and Fokker-Planck-Kolmogorov equations.

6.2 Future work

There are multiple short and long-term research directions that we would like to explore in the future based on the results in this thesis.

In Chapter 2, the diffusion term is a constant in our model setting and it is independent of the control term. We could try to involve the control in the diffusion term and reexamine the corresponding turnpike properties. Next, we only investigate the model under the LQG structure. The general model will be considered in the future. For example, we could formulate the problem within a general setting and give some regularity conditions to drift term, diffusion term and the running cost in the cost functional. Moreover, with the resurgent of interests of mean field games and mean field models, we could also consider the ergodic property of the mean field control problems.

For the convergence of the N-player game to the associated MFG investigated in Chapter 3 and Chapter 4, we propose the following future ideas. Firstly, we could consider the mean field game under more general settings about the dynamic processes, such as with time delays, Poisson jumps, etc. Next, except for considering the LQG structure, we could examine the convergence of mean field games with common noise under more general structures. It is important to note that the assumptions made in the aforementioned papers usually account for linear growth in the state and control elements for the running cost, or they only focus on the linear quadratic structure.

The convergence of mean field games with common noise under a more general setting should obtain more attention. Furthermore, in our two previous works, we require positive values for all sensitivity parameters in the cost functional. We find that there is no global solution for MFG when the coefficient of the cost functional is negative, while there is a global solution when the coefficient is positive. So, it is also an interesting problem to investigate the explosion when some sensitivities take negative value.

One limitation of the current setting in Chapter 5 is that the running cost in the current setup allows to use Hopf-Cole transformation, which is essential to the subsequent analysis of regularities. To deal with the full generalization of the running cost, one must adopt different approaches and it will be in our future research direction. It is also desirable to generalize the result to the whole domain $\mathbb{R}^d$ instead of the compact domain $\mathbb{T}^d$. In addition, the convergence of graphon mean field game is not addressed in our work. The establishment of the convergence rate from the N-subpopulation game to the corresponding graphon mean field game will be considered in our future works.

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