New results in the multiscale analysis on perforated domains and applications

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ABSTRACT

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Multiscale phenomena implicitly appear in every physical model. The understanding of the general behavior of a given model at different scales and how one can correlate the behavior at two different scales is essential and can offer new important information. This thesis describes a series of new techniques and results in the analysis of multi-scale phenomena arising in PDEs on variable geometries. In the Second Chapter of the thesis, we present a series of new error estimate results for the periodic homogenization with nonsmooth coefficients. For the case of smooth coefficients, with the help of boundary layer correctors, error estimates results have been obtained by several authors (Oleinik, Lions, Vogelius, Allaire, Sarkis). Our results answer an open problem in the case of nonsmooth coefficients. Chapter 3 is focused on the homogenization of linear elliptic problems with variable nonsmooth coefficients and variable domains. Based on the periodic unfolding method proposed by Cioranescu, Damlamian and Griso in 2002, we propose a new technique for homogenization in perforated domains. With this new technique classical results are rediscovered in a new light and a series of new results are obtained. Also, among other advantages, the method helps one prove better corrector results. Chapter 4 is dedicated to the study of the limit behavior of a class of Steklov-type spectral problems on the Neumann sieve. This is equivalent with the limit analysis for the DtN-map spectrum on the sieve and has applications in the stability analysis of the earthquake nucleation phase model studied in Chapter 5. In Chapter 5, a Γ-convergence result for a class of contact problems with a slip-weakening friction law, is described. These problems are associated with the modeling of the nucleation phase in earthquakes. Through the Γ-limit we obtain an homogenous friction law as a good approximation for the local friction law and this helps us better understand the global behavior of the model, making use of the micro-scale information. As to our best knowledge, this is the first result proposing a homogenous friction law for this earthquake nucleation model.

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Chapter 1

Introduction

In this Chapter we will briefly introduce the main results presented in the thesis. In each case, we will first describe the problem and its importance for the mathematical community, commenting on the mathematical impact and the applications of our results.

In **Chapter 2**, inspired by the work of Griso [41] and Vogelius and Moskow [59], using suitable boundary layer correctors, we attempt to answer the open question of finding error estimates for the homogenization of elliptic problems with nonsmooth coefficients. We develop a new method which can be generalized to other linear or nonlinear elliptic problems in a divergence form on fixed or variable geometries.

As an example, we consider the classical problem of homogenization, i.e.,

$$\begin{cases} -\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon(x)) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.0.1}$$

where $A \in L^\infty(Y)^{N\times N}$ is symmetric and Y-periodic, $Y =]0,1[^N$, $\Omega \in \mathbf{R}^N$, smooth convex bounded domain, $c|\xi|^2 \leq A_{ij}(y)\xi_i\xi_j \leq C|\xi|^2$ $\forall \xi \in \mathbf{R}^N$. It is well known that (see [11], [8], [73], [51]),

$$u_\epsilon \rightharpoonup u_0 \ \text{ in } H^1(\Omega) \tag{1.0.2}$$

and u_0 verifies

$$\begin{cases} -\nabla \cdot (\mathcal{A}^{hom}\nabla u_0(x)) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{1.0.3}$$

with $\mathcal{A}^{hom}_{ij} = M_Y(A_{ij}(y) + A_{ik}(y)\frac{\partial \chi_j}{\partial y_k})$ where $M_Y(\cdot) = \frac{1}{|Y|}\int_Y \cdot dy$ and $\chi_j \in W_{per}(Y)$ are the solutions of the local problem

$$-\nabla_y \cdot (A(y)(\nabla \chi_j + e_j)) = 0 \tag{1.0.4}$$

and

$$W_{per}(Y) = \{\chi \in H^1_{per}(Y) | M_Y(\chi) = 0\}.$$

In fact, heuristically, if one considers the asymptotic expansion of u_ϵ, i.e.,

$$u_\epsilon(x) = u_0(x) + \epsilon u_1(x, \frac{x}{\epsilon}) + \epsilon^2 u_2(x, \frac{x}{\epsilon}) + ... \tag{1.0.5}$$

with $u_i(x, y) \in L^2(\Omega, L^2_{per}(Y))$ for $i \in \mathbb{N}$ in problem (1.0.1), and setting equal respective powers of ϵ we obtain convergence (1.0.2) and that the first order corrector $u_1(x, \frac{x}{\epsilon})$ has the following form

$$u_1(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon}) \frac{\partial u_0}{\partial x_j}$$

where the functions $\chi_j \in W_{per}(Y)$ verify problems (1.0.4). As a simple corollary one can obtain the limit problem (1.0.3).

One can easily observe that the series (1.0.5) indicates that one can actually improve the weak convergence result (1.0.2) using a suitable corrector matrix, that is, a matrix C^ϵ such that,

$$\nabla u_\epsilon - C^\epsilon \nabla u_0 \xrightarrow{\epsilon} 0 \text{ strongly in } L^1(\Omega). \tag{1.0.6}$$

Indeed for problem (1.0.1), we have that the matrix of correctors C^ϵ has the following form,

$$C^\epsilon_{ij}(x) = \delta_{ij} + \frac{\partial \chi_j}{\partial y_i}(\frac{x}{\epsilon})$$

and if one assumes for example that $u_0 \in W^{2,\infty}$ or $\chi_j \in W^{1,\infty}(Y)$, one has the convergence (1.0.6) strongly in L^2. As described above, the study of correctors in homogenization helps one to approximate the solution u_ϵ of the initial problem (1.0.1, which is expensive to compute, with a more computationally efficient series, ((1.0.5) in our case). The study of the error estimates is very important in multiscale analysis because it provides the order of accuracy of such an approximation in the norm of suitable functional spaces. The existence of error estimates results in the case of nonsmooth coefficients is therefore very important, as this is the situation in many applied problems (e.g., the case of composite materials).

The following error estimate is classical (see [11], [8], [73], [51]),

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon \chi_j(\frac{\cdot}{\epsilon}) \frac{\partial u_0}{\partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}} \tag{1.0.7}$$

Many results tried to improve (1.0.7) (see [41], [54], [59], [1], [82]). In [41], using the Periodic Unfolding, Griso proved (1.0.7) for general L^∞ coefficients and no assumption on χ_j or u_0. In the rest of the works listed above, the authors tried to improve the order $\epsilon^{\frac{1}{2}}$ in (1.0.7). In order to achieve this, boundary layer terms have been introduced as solutions to

$$-\nabla \cdot (A(\frac{x}{\epsilon}) \nabla \theta_\epsilon) = 0 \text{ in } \Omega \ , \ \theta_\epsilon = \chi_j(\frac{x}{\epsilon}) \frac{\partial u_0}{\partial x_j} \text{ on } \partial\Omega \tag{1.0.8}$$

Assuming $A \in C^\infty(Y)$, Y-periodic matrix and a sufficiently smooth homogenized solution u_0 it has been proved in [11] (see also [54]) that

$$||u^\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j} + \epsilon\theta_\epsilon(\cdot)||_{H_0^1(\Omega)} \leq C\epsilon \tag{1.0.9}$$

$$||u^\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j} + \epsilon\theta_\epsilon(\cdot)||_{L^2(\Omega)} \leq C\epsilon^2. \tag{1.0.10}$$

In [59], the above estimates are proved, assuming $A \in C^\infty(Y)$, Y-periodic matrix and $u_0 \in H^2(\Omega)$ or $u_0 \in H^3(\Omega)$ for (1.0.9) or (1.0.10) respectively.

The estimate (1.0.9) is proved in [1] in the case when $u_0 \in W^{2,\infty}(\Omega)$. Sarkis-Versieux in [82] improved the results obtained in [1] and showed that the estimates (1.0.9) and respectively (1.0.10) still hold in a more general setting, when one has $u_0 \in W^{2,p}(\Omega)$, $\chi_j \in W^{1,q}_{per}$ for (1.0.9), and $u_0 \in W^{3,p}(\Omega)$, $\chi_j \in W^{1,q}_{per}$ for (1.0.10), where $p > N$ and $q > N$ satisfy $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. Obviously, in [82] the right hand side for (1.0.9) and (1.0.10) depend on $||u_0||_{W^{2,p}(\Omega)}$ and respectively $||u_0||_{W^{3,p}(\Omega)}$.

All the error estimates results obtained so far, assumed extra regularity for the u_0 or for the solutions of the cell problems χ_j. In Chapter 2 a new method was developed to help one obtain error estimates for u^ϵ in H^1 norm without assuming any smoothness condition on u_0 or on χ_j. One of the main results, states that, for any dimension N we have,

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j}) + \epsilon\beta_\epsilon(\cdot)||_{H_0^1(\Omega)} \leq C\epsilon||u_0||_{H^2(\Omega)} \tag{1.0.11}$$

where Q_ϵ is a certain regularization operator, and β_ϵ satisfies

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\beta_\epsilon) = 0 \text{ in } \Omega \ , \ \beta_\epsilon = \chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j}) \text{ on } \partial\Omega \tag{1.0.12}$$

Let us define the second order boundary layer corrector as the solution of

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\varphi_\epsilon) = 0 \text{ in } \Omega \ , \ \varphi_\epsilon(x) = \chi_{ij}(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j} \text{ on } \partial\Omega \tag{1.0.13}$$

where $\chi_{ij} \in W_{per}(Y)$ verify,

$$\nabla_y \cdot (A\nabla_y\chi_{ij}) = b_{ij} + \mathcal{A}_0^{ij} \tag{1.0.14}$$

with $\mathcal{A}^{hom}$ defined at (1.0.3), $M_Y(b_{ij}(y)) = -\mathcal{A}_0^{ij}$, and $b_{ij} = -A_{ij} - A_{ik}\frac{\partial\chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j)$.

Considering the second order corrector in the asymptotic expansion of u_ϵ Allaire and Amar proved that, for $u_0 \in W^{3,\infty}(\Omega)$ and $\chi_{ij} \in W^{1,\infty}(Y)$, that

$$||u_\epsilon(\cdot)-u_0(\cdot)-\epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j}+\epsilon\theta_\epsilon(\cdot)-\epsilon^2\chi_{ij}(\frac{\cdot}{\epsilon})\frac{\partial^2 u_0}{\partial x_i\partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}||u_0||_{W^{3,\infty}(\Omega)}. \quad (1.0.15)$$

Following carefully the limit behavior of φ_ϵ we prove that, assuming only that $\chi_j, \chi_{ij} \in W^{1,p}_{per}$ with $p > N$ we have

$$||u_\epsilon(\cdot)-u_0(\cdot)-\epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j}+\epsilon\theta_\epsilon(\cdot)-\epsilon^2\chi_{ij}(\frac{\cdot}{\epsilon})\frac{\partial^2 u_0}{\partial x_i\partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\min\{\frac{3}{2},2-\frac{N}{p}\}}||u_0||_{H^3(\Omega)}. \quad (1.0.16)$$

and this generalizes the result obtained by Allaire and Amar in [1].

In two dimensions, based on a Meyers type regularity result we do have $\chi_j, \chi_{ij} \in W^{1,p}_{per}$ for some $p > 2$ and therefore (1.0.16) remains true without any smoothness assumption whatsoever for this case.

Two immediate applications of the above error estimate results are the rigorous proof of convergence for the Multiscale Finite Element Method proposed by T. Hou and X. Wu in [43] and the proof of the first order corrector for the homogenized eigenvalue associated with the classical problem of homogenization (see [59]) for the general case of nonsmooth coefficients.

The results of this chapter are published in [64].

In **Chapter 3**, we will discuss about the use of the periodic unfolding method developed in [22] for the homogenization of elliptic problems with nonsmooth coefficients in variable domains. The results presented in this chapter are the subject of a joint work with D. Cioranescu, A. Damlamian and G. Griso.

In Sections 3.2 and 3.3, using the unfolding method, new proofs are obtained for the homogenization of the Laplace operator with variable coefficients in perforated domains with periodically distributed perforations in volume and along a hyperplane respectively. Although the method will work for any boundary data, Dirichlet data on the boundary of the perforations is assumed for the simplicity of the exposition. Our method extends to more general types of perforated domains were one can simultaneously have hyperplane perforations and bulk perforations in the same model, etc. The homogenization of the Laplace operator with constant coefficients in general perforated domains was first studied by D. Cioranescu and F. Murat, in [26] (see also [58]). Some of the results obtained by us are well known, and have been discussed in many works, including two most recent papers by Calvo Jurado C. and Casado Diaz J. were monotonicity techniques are used (see [15] and references therein), and by G. Dal-Maso and F. Murat were H-convergence techniques are used (see [29]).

The advantage of the unfolding method is that, in the periodic setting, it simplifies the existing proofs and the new formulation of the limit problem allows one to obtain very interesting error estimates, even for the case with nonsmooth coefficients (see [41], [64]) The method has four fundamental steps:

1. Definition of one or more suitable unfolding operators depending on the geometry of the problem

2. Finding the exact L^p-bounds for the unfolding operators

3. Defining the proper test functions in order to capture the contribution of the particular geometry to the limit problem, ex., potential type test function for the perforated domains.

4. Passing at the limit to obtain the unfolding formulation for the limit problem.

One of the main properties of the unfolding operator is that it replaces, integrals on Ω by integrals on the product space $\Omega \times Y$ and weak convergence by strong convergence.

For both, the volume perforations case or the hyperplane perforations case, the problem is formulated as,

$$(\mathcal{P}_1^\epsilon) \begin{cases} u_{\epsilon,\delta} \in H_0^1(D_\epsilon) \\ \displaystyle\int_{D_\epsilon} A^\epsilon(x)\nabla_x u_{\epsilon,\delta}\nabla_x \phi = \int_{D_\epsilon} f\phi \quad , f \in L^2(\Omega) \\ \forall \phi \in H_0^1(D_\epsilon) \end{cases}$$

where Y is the unit cube in $\mathbf{R}^3$ centered in the origin, $\Omega \subset \mathbf{R}^N$, $A \in \mathcal{M}^{N\times N}$ is Y-periodic matrix of coefficients continuous in the origin, $A^\epsilon \doteq A(\frac{x}{\epsilon})$ and D_ϵ is the perforated domain, i.e., the part of Ω outside the perforations. In our analysis we considered the case of small perforations, i.e., the perforations are open sets of diameter $\delta\epsilon$ where $\delta \doteq \delta(\epsilon) < 1$. In this context, the critical scale for the case of volume perforations is $k_1 = \lim\limits_{\epsilon\to 0} \dfrac{\delta^{\frac{N-2}{2}}}{\epsilon}$ and the limit analysis is meaningful only when

$$0 \leq k_1 = \lim_{\epsilon\to 0} \frac{\delta^{\frac{N-2}{2}}}{\epsilon} \quad \text{and} \quad < \infty \tag{1.0.17}$$

Similarly for the case when the perforations are periodically distributed on the hyperplane we have that the critical scale is $k_2 = \lim\limits_{\epsilon\to 0} \dfrac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}$ and the constraint is

$$0 \leq k_2 = \lim_{\epsilon\to 0} \frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}} < \infty. \tag{1.0.18}$$

Two unfolding operators are defined for each of the above problems, one for the periodic oscillations in the coefficient matrix A, and another for the presence of the perforations in the geometry of the problem.

In the limit we obtain the unfolded formulation for the limit problem, i.e., a weak formulation on $\Omega \times Y$ for the triplet $(u_0, \hat{u}, U) \in H_0^1(\Omega) \times L^2(\Omega, W_{per}(Y)) \times L^2(\Omega, K^2(\mathbf{R}^N))$ or $(u_0, \hat{u}, U) \in H_0^1(\Omega) \times L^2(\Omega, W_{per}(Y)) \times L^2(\Sigma, K^2(\mathbf{R}^N))$ depending whether we are in volume perforations case or the hyperplane perforation case respectively, where u_0 is the homogenized limit of u^ϵ and $K^2(\mathbf{R}^N)$ is the usual capacity space defined in [38]. Our formulation of the limit problem is new in the literature and together with the fact that it provides the limit equation for u_0, with the help of

$\hat{u}$ and U offers us the possibility to obtain new corrector results for these models as well as error estimates for the solutions.

Sections 3.4 and 3.5 describe the multiscale analysis of the Neumann sieve model. For this, we constructed a new unfolding operator (see [63]), to capture the contribution of the sieve into the limit problem. The geometry of the model is described by a domain Ω cut in two parts by a hyperplane Σ which, for the simplicity of the exposition is assumed to be a subset of the plane $\Pi = \{x_N = 0\}$. A periodical 2-dimensional network of size ϵ is considered on Σ, and an open set (hole in the sieve) is brought by homothety of ratio $\delta\epsilon$, with $\delta \doteq \delta(\epsilon) < 1$, from a fixed open set $S \subset\subset]0,1[^2$ in each cell of the network. The reunion of all the holes is denoted by $S_{\epsilon,\delta}$. For the PDE problem the set $S_{\epsilon,\delta}$ is considered part of the domain and Neumann homogenous boundary condition are imposed on the sieve outside $S_{\epsilon,\delta}$. When the Sieve has a certain thickness $h(\epsilon) > 0$ we have the thick Neumann sieve model. We only considered the case when $h(\epsilon) \leq \epsilon$ the other situations being trivial. Depending on the limit behavior of the ratio $\dfrac{\delta^{N-2}}{\epsilon}$ we obtain different limit equations. In order to obtain the limit problems for these models in [63] we define the bl-Unfolding Operator, which characterizes the geometry of the models, and acts only on a thin layer of size ϵ around the hyperplane Σ. A similar approach will work for much more complex boundary layer problems.

Mathematically, if we define Ω_+ to be the part of the domain Ω above Σ and similarly for Ω_-, then our functional space $V_{\epsilon,\delta}$ will be the space of functions in $H^1(\Omega_+ \cup \Omega_-)$ which are continues over the holes $S_{\epsilon,\delta}$ (see (1.0.21), Section 2). For $f \in L^2(\Omega)$ the ϵ-problem is

$$\int_{D_\epsilon^{ns}} A(\frac{x}{\epsilon})\nabla u^\epsilon \nabla\psi dx = \int_{D_\epsilon^{ns}} f\psi \text{ for all } \psi \in V_{\epsilon,\delta} \tag{1.0.19}$$

with A is Y-periodic matrix continuous in the origin, $D_\epsilon^{ns} \doteq \Omega_{\epsilon,\delta}^{bl} = \Omega_+ \cup \Omega_- \cup S_{\epsilon,\delta}$ for the case of no thickness and

$$D_\epsilon^{ns} \doteq \Omega_{\epsilon,\delta}^{ns} = \Omega \setminus F_{\epsilon\delta}$$

for the thick sieve, where $F_{\epsilon\delta}$ is the thick sieve assumed to be symetric and sufficiently smooth. The sieve is not consider a part of the domain D_ϵ^{ns} and on it we impose a zero normal derivative. Due to the fact that we are considering small holes $S_{\epsilon,\delta}$ we will have a critical scale which will dictate the limit behavior of the model, i.e., we have

$$0 \leq k_2 = \lim_{\epsilon \longrightarrow 0} \frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}} < \infty. \tag{1.0.20}$$

With the help of boundary layer unfolding operator defined at [63], we are able to study the limit behaviour of problems (1.0.19 with respect to ϵ. Although the thin Neumann sieve has been studied before by many authors (see [30], [60], [61], [73]) we give a formulation of the limit problem in the product space, which is perfectly taylored to successfully address the question of correctors partially answered in [72].

The thick Neumann sieve model has been studied in [35] only for the particular case of an uniform sieve. The multiscale analysis of the nonuniform sieve is new and our method offers the perspective for the study of error estimates (see[41],[63]).

The results of this chapter are published in [63] and [23].

In **Chapter 4** we discuss about the homogenization of a class of Steklov type spectral problems associated to linear operators on the Neumann Sieve model. Using G-convergence results we were able to present, in [66], a general technique for the asymptotic analysis of such problems associated to the Laplace operator. It is shown in [66] that the limit analysis of the Steklov problems for the Laplace operator on the Neumann sieve is equivalent with the description of the asymptotic behavior for the spectra of the DtN map associated to the Neumann Sieve.

Our technique can be generalized to the nonlinear case but it highly depends on the Hilbertian functional setting; therefore the study of similar problems in general spaces will require a different approach.

The method developed by us in [66] was later applied in [47], to obtain the limit problem of a Steklov problem associated to the linear elasticity operator on the Neumann sieve. This problem appeared in the context of an earthquake initiation model, and the limit analysis of its first eigenvalue provided interesting information about the stability of the minimum for the associated energy functional.

The geometry of the problem is described by a plane Σ that separates a three dimensional domain Ω in two subdomains Ω_+ and Ω_-. On Σ, two dimensional small sets (holes) of diameter $\delta\epsilon$ where $\delta \doteq \delta(\epsilon) < 1$, are ϵ-periodically distributed. The two dimensional holes are brought by homothety of ratio $\delta\epsilon$ and translation with integers multiple of ϵ from a fixed open set $S \subset\subset]0,1[^2$. If we denote by $S_{\epsilon,\delta}$ their union and define

$$V = \{u \in H^1(\Omega_+) \cup H^1(\Omega_-) \mid u = 0 \text{ on } \partial\Omega\} \text{ and } V_{\epsilon,\delta} = \{u \in V \mid [u] = 0 \text{ on } S_{\epsilon,\delta}\} \tag{1.0.21}$$

where $[u] = u^+ - u^-$ with $u^+ = u$ on Ω_+ and $u^- = u$ on Ω_-, then our problem is:

$$\begin{cases} -\Delta u^\epsilon = 0 & \text{in} \quad \Omega_+ \cup \Omega_- \cup S_{\epsilon,\delta} \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = \lambda^\epsilon [u^\epsilon] & \text{on} \quad \Sigma - S_{\epsilon,\delta} \\ u^\epsilon = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.0.22}$$

This problem is in fact the Steklov eigenvalue problem associated to the Neumann Sieve model first considered in [30], [60] (see also [6]). Problem (1.0.22) can also be considered as the spectral problem associated to a heat conduction problem where imperfectly conducting interfaces are present (see Sanchez-Palencia [73], Lipton and Vernescu [55] and Belyaev et al. [10]). Homogenization of a Stekloff type problem for perforated domains with three dimensional ϵ sized holes distributed in the entire domain has been studied in Vanninathan [59], using multiscale analysis and Tartar's method.

Using G-convergence techniques together with the homogenization result obtained by Damlamian in [30] we prove in [66] that the limit problem for (1.0.22),

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega_+ \cup \Omega_- \\ \dfrac{\partial u^+}{\partial n} = -\dfrac{\partial u^-}{\partial n} = \left(\lambda - \dfrac{C}{4}\right)[u] & \text{on} \quad \Sigma \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $C = 0$ if $\delta(\epsilon) \ll \epsilon$, C is the capacity in $\mathbf{R}^3$ of the holes if $\delta(\epsilon) \approx \epsilon$ or $C = \infty$ if $\delta(\epsilon) \gg \epsilon$ and λ is a limit point of a sequence of eigenvalues $\{\lambda^\epsilon\}_{\epsilon>0}$ of (1.0.22).

This type of behavior was first observed in the work of Cioranescu and Murat [26] where the same problem but with three dimensional holes periodically distributed in the entire domain or on a hyperplane was studied.

We show that for a (eigenvalue, eigenvector) pair of the ϵ-problem $(\lambda_n^\epsilon, u_n^\epsilon)$ we have $\lambda_n^\epsilon \xrightarrow{\epsilon} \lambda_n$ and $u_n^\epsilon \xrightarrow{\epsilon} u_n$ where (λ_n, u_n) is an (eigenvalue, eigenvector) pair of (1.0.23) and the later converge is up to a subsequence in general. More precisely, for λ_i eigenvalue of the limit problem with multiplicity m_i, we show that the sequence of subspaces generated by $\{u_i^\epsilon, ..., u_{i+m_i-1}^\epsilon\}$ Mosco-converge (see [6] for definition and properties) in $L^2(\Omega)$ to the eigenspace $\{u_i, ..., u_{i+m_i-1}\}$, associated to λ_i.

The case $\lim\limits_{\epsilon\to 0} \dfrac{\delta(\epsilon)}{\epsilon} = 0$ is particularly interesting because it is the only case when the problem (1.0.22) fits into the general class of spectral problems analyzed be Oleinik, Jikov and Kozlov in [51], Chapter 11.

The results presented in this chapter have been published in [**?**] and [65].

Chapter 5 is dedicated to the study of the limit behaviour of an earthquake initiation model. In a joint work with Bogdan Vernescu, and Ioan Ionescu we considered the three dimensional shearing of an elastic domain which contains an internal boundary (the fault) located on a plane (the fault plane). The contact on the fault is described through a slip weakening friction (i.e. the friction force decreases with the slip). This friction law is used in the geophysical context of earthquakes modeling; experimental studies [67] pointed out the good agreement of this model with experimental data. The Geometry of the physical model is represented by an open and bounded domain $\Omega \subset \mathbf{R}^3$ cut in two by the hyperplane $\Pi = \{x_3 = 0\}$. Ω_+ will denote the part of Ω above the plane Π. and Γ_d we denote the exterior boundary of Ω_+. On each square of an ϵ-lattice constructed on the plane Π we consider a 2-dimensional set (barrier) of diameter $\delta\epsilon$ where $\delta \doteq \delta(\epsilon) < 1$. The term barrier denotes here a patch on the fault plane where no slip occurs. Let $\Sigma = \Pi \cap \Omega$ and denote by Γ_t^ϵ the union of all the barriers inside Ω. On the fault outside the barriers, i.e., on $\Gamma_f^\epsilon = \Sigma \setminus \Gamma_t^\epsilon$, we consider a friction law. The mathematical description of the above physical model is:

find the displacement field $u^\epsilon : \Omega_+ \to \mathbb{R}^3$ solution of

$$\sigma(u^\epsilon) = \mathcal{A}\epsilon(u^\epsilon), \quad \operatorname{div}(\mathcal{A}\epsilon(u^\epsilon)) = 0 \quad \text{in} \quad \Omega_+, \tag{1.0.23}$$

$$u^\epsilon = 0 \quad \text{on} \quad \Gamma_d, \qquad \sigma_{33}(u^\epsilon) = 0,\ u_\tau^\epsilon = 0 \quad \text{on} \quad \Gamma_t^\epsilon, \tag{1.0.24}$$

$$\sigma_{33}(u^\epsilon) = 0, \quad \begin{cases} \sigma_\tau(u^\epsilon) = -S_\perp\mu(|u_\tau^\epsilon|)\dfrac{u_\tau^\epsilon}{|u_\tau^\epsilon|} - \tau^\infty & \text{if} \quad u_\tau^\epsilon \neq 0 \\ |\sigma_\tau(u^\epsilon) + \tau^\infty| \leq S\mu(0) & \text{if} \quad u_\tau^\epsilon = 0. \end{cases} \quad \text{on } \Gamma_f^\epsilon, \tag{1.0.25}$$

where $\mathcal{A}$ is the fourth order elastic tensor, $\sigma(u^\epsilon)$ is the over stress tensor , $\epsilon(u^\epsilon) = \frac{1}{2}(\nabla u^\epsilon + \nabla^T u^\epsilon)$ is the small strain tensor, $\sigma_\tau(u^\epsilon) = -(\sigma_{13}(u^\epsilon), \sigma_{23}(u^\epsilon), 0)$ is the tangential over-stress, $\sigma_{33}(u^\epsilon)$ is the normal over-stress, $u^\epsilon_\tau = (u^\epsilon_1, u^\epsilon_2, 0)$ is the tangential displacement, and $\tau^\infty =: -(\sigma^\infty_{13}, \sigma^\infty_{23}, 0)$ and $-S_\perp =: \sigma^\infty_{33}$ are the tangential and the normal pre-stress acting on Γ^ϵ_f. The friction coefficient $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ in (1.0.25) is a Lipschitz function with respect to the slip as pointed out in [67]. The symmetry of the displacement field with respect to the fault plane (see for instance [39] for the geophysical meaning) gives an important simplification of the problem: the normal over stress on the fault vanishes. The fact that the normal stress has a weak variation during the dynamic rupture was already observed in direct computations [5, 57] as well as in the inversion of seismological data [27].

The main problem of the existing local model is that due to the small parameter ϵ, problem (1.0.23), (1.0.24), (1.0.25) becomes computationally inefficient. In this section, using Γ-convergence techniques, (see [6] for definition and properties of Γ-convergence) for the sequence of the associated energy functionals, $\mathcal{W}_\epsilon$, we obtain a homogeneous friction law as a good approximation of the existing local law.

An important consequence of the symmetry assumption is the fact that we can associate to the physical problem a nonconvex minimization problem for the energy function. Solutions of (1.0.23), (1.0.24), (1.0.25) are local minimum points for

$$\mathcal{W}_\epsilon(v) = \frac{1}{2}\|v\|^2_V + \int_{\Sigma_0} S_\perp H(|v_\tau|) - f(v), \quad \forall v \in V_{\epsilon,\delta}, \tag{1.0.26}$$

where $f(v) = -\int_\Sigma \tau^\infty \cdot v_\tau$ and

$$V_\epsilon := \{v \in [H^1(\Omega_+)]^3 / v = 0 \quad \text{on} \quad \Gamma_d, \quad v_\tau = 0 \quad \text{on} \quad \Gamma^\epsilon_t\}. \tag{1.0.27}$$

and H is the antiderivative of the friction coefficient.

The macroscopic behavior of a fault with small-scale heterogeneity of rupture resistance (small scale barriers) is difficult to relate to the local properties of the fault. A formal measure of the friction on the fault itself would just be a local particular law, that is varying with the position along the fault. The problem is then to find a homogeneous friction law as a good replacement of the local friction law.

Mathematically the problem is related to the homogenization of the Neumann Sieve problem for the Laplacian studied by several authors [30, 26, 6, 21]. In the geophysical context the problem was studied (see [18, 17, 71]) in two dimensions (anti-plane geometry) to obtain the rescaling of the weakening rate through a spectral analysis.

The Neumann Sieve problem associated to the linear elasticity operator was studied by Lobo and Perez [56, ?]. An extension to the non-linear case of the Neumann Sieve has been studied by Ansini in [3]. We use Γ-convergence to obtain the limit functional of the sequence $\mathcal{W}_\epsilon$. Our approach is based on the adaptation of a very interesting separation lemma due to Braides and Ansini (see [4]) which is designed to isolate the contribution of the perforations in the limit process. This lemma helps one to prove Γ-liminf inequality and offers an ingenuous way to construct the necessary recovery sequence for the Γ-limsup inequality. Through the Γ-limit of the sequence $\mathcal{W}_\epsilon$ defined at (1.0.26) we propose an equivalent friction law used on a homogeneous fault as a good replacement for the local friction law on the heterogeneous fault. The main result states that for $0 \leq c = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$, the sequence of functionals

$$\mathcal{W}_\epsilon \ : \ V_{\epsilon,\delta} \subset V \to \mathbb{R}, \text{ with } \mathcal{W}_\epsilon(v) = \frac{1}{2} \parallel v \parallel_V + \int_\Sigma S_\perp H(|v_\tau|) - f(v)$$

Γ-converge with respect to the weak topology of V to, $\overline{\mathcal{W}} \ : \ V \to \mathbb{R}$ with

$$\overline{\mathcal{W}}(v) = \frac{1}{2} \parallel v \parallel_V^2 + \int_\Sigma S_\perp H(|v_\tau|) - f(v) + \frac{1}{2} c \sum_{i,j=1}^{3} \int_\Sigma C_{ij} v_i v_j$$

where V is the limit functional space, i.e., $V = \{v \in [H^1(\Omega_+)]^3 / v = 0 \quad \text{on} \quad \Gamma_d\}$ and C is a constant matrix computed in [47] with the help of a class of cell problems.

A brief physical interpretations of this result leads us to the following conclusions:
i) if the barriers are too large (i.e. $c = \infty$) then the fault is locked (no slip)
ii) if $c > 0$ then the fault behaves as a fault under a slip-dependent friction. The slip weakening rate of the equivalent fault is smaller then undisturbed fault. Since the limit slip weakening rate may be negative a slip-hardening effect can also be expected.
iii) if the barriers are too small (i.e. $c = 0$) then the presence of the barriers does not affect the friction law on the limit fault.

In the second part of this chapter we study the homogenization of the Steklov spectral problem associated to 1.0.23),(1.0.24), (1.0.25). The study of the firast eigenvalue of (1.0.28), (1.0.29), (1.0.30) provides information about the stability of the minimum points of $\mathcal{W}_\epsilon$ defined at (1.0.26). On the same functional setting as in Section 1 we considered the following Steklov type eigenvalue problem associated with (1.0.23), (1.0.24), (1.0.25),

Find $u^\epsilon : \Omega_+ \longrightarrow \mathbb{R}^3$, $u^\epsilon \neq 0$ and $\lambda^\epsilon \in \mathbf{R}$ such that

$$\sigma(u^\epsilon) = \mathcal{A}\epsilon(u^\epsilon), \qquad \text{div}\, \sigma(u^\epsilon) = 0, \quad \text{in } \Omega_+, \tag{1.0.28}$$

$$u^\epsilon = 0 \quad \text{on} \quad \Gamma_d, \qquad \sigma_{33}(u^\epsilon) = 0, \quad u^\epsilon_\tau = 0 \quad \text{on} \quad \Gamma^\epsilon_t, \tag{1.0.29}$$

$$\sigma_{33}(u^\epsilon) = 0, \qquad \sigma_\tau(u^\epsilon) = \lambda^\epsilon u^\epsilon_\tau \quad \text{on} \quad \Gamma^\epsilon_f, \tag{1.0.30}$$

which has the following variational formulation:

$$u^\epsilon \in V_{\epsilon,\delta}, \quad < u^\epsilon, v >_V = \lambda^\epsilon \int_{\Gamma^\epsilon_f} u^\epsilon_\tau \cdot v_\tau, \quad \forall v \in V_{\epsilon,\delta}. \tag{1.0.31}$$

where $V_{\epsilon,\delta}$ is the functional space defined at (1.0.27) in the Section **??**.

It is proved in [50] that the (eigenvalue, eigenvector) pairs for the above problem formes the spectrum of some suitable defined compact operators. In [50] the authors prove that if

$$\lambda^\epsilon_1 > c_{u^\epsilon} =: \text{ess}\sup_{x \in \Gamma^\epsilon_f} S(x)\gamma(|u^\epsilon_\tau(x)|), \tag{1.0.32}$$

where $u^\epsilon \in V_{\epsilon,\delta}$ is the solution of (1.0.23), (1.0.24), (1.0.25) λ_1^ϵ is the first eigenvalue of (1.0.31), γ (described in [47]) is a function depending on the friction coefficient μ, and $-S$ is the normal stress on Σ, then u^ϵ is an isolated local minimum for $\mathcal{W}_\epsilon$, i.e. there exists $\mu > 0$ such that

$$\mathcal{W}_\epsilon(u^\epsilon) < \mathcal{W}_\epsilon(v) \quad \forall\, v \in V_{\epsilon,\delta},\ v \neq u^\epsilon,\ \|v - u^\epsilon\|_V < \mu. \tag{1.0.33}$$

The result in Proposition (**??**) shows that the first eigenvalue of (1.0.31) provides information about the stability of the solution of the contact problem defined at (1.0.23), (1.0.24), (1.0.25). Therefore the limit analysis for the problem (1.0.31) is very important. Using the method developed by us in [66] we can pass to the limit in (1.0.31) and obtain the limit problem. We have that in the case when $0 \leq c = \lim_{\epsilon \to 0} \dfrac{\delta(\epsilon)}{\epsilon} < \infty$ then there is a decreasing sequence $\{\epsilon_j\}_{j\in\mathbf{N}}$ with $\epsilon_j \to 0$ such that $u_n^{\epsilon_j} \rightharpoonup u_n,\ \ \lambda_n^\epsilon \to \lambda_n$ where (λ_n, u_n) solves the limit problem, $\lambda_n \in \mathbb{R}$ and $u_n \in W$ such that:

$$\sigma(u_n) = \mathcal{A}\epsilon(u_n), \qquad \text{div}\, \sigma(u_n) = 0, \quad \text{in} \ \ \Omega_+, \tag{1.0.34}$$

$$u_n = 0 \ \text{ on } \Gamma_d \quad \sigma_{33}(u_n) = 0 \text{ on } \Sigma \tag{1.0.35}$$

$$\sigma_\tau(u_n) = u_{n\tau}(\lambda_n I_3 - cC) \quad \text{on} \quad \Sigma, \tag{1.0.36}$$

where W is the limit functional space defined in [47], I_3 is the unity matrix in $\mathcal{M}^{3\times3}$ and C is a constant matrix computed in [47] with the help of a class of cell problems.

Similarly as in the case of the Laplace operator, for the case of multiple eigenvalues a precise characterization of the limit behavior of the (1.0.31) is obtained using Mosco-convergence techniques.

The results obtained in this chapter have been published in [47].

Chapter 2

Boundary layer error estimates in homogenization

This chapter is dedicated to studying the error estimates for the classical problem in homogenization using suitable boundary layer correctors.

Let $\Omega \in \mathbb{R}^N$, denote a bounded convex polyhedron or a convex bounded domain with a sufficiently smooth boundary. Consider also the unit cube $Y = (0,1)^N$. It is well known that for $A \in L^\infty(Y)^{N\times N}$, Y-periodic with $m|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq M|\xi|^2$, $\forall \xi \in \mathbb{R}^N$ the solutions of

$$\begin{cases} -\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon(x)) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.0.1}$$

have the property that (see [73], [51], [8],[11]),

$$u_\epsilon \rightharpoonup u_0 \ \text{ in } H_0^1(\Omega)$$

where u_0 verifies

$$\begin{cases} -\nabla \cdot (\mathcal{A}^{hom}\nabla u_0(x)) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{2.0.2}$$

with $\mathcal{A}_{ij}^{hom} = M_Y(A_{ij}(y) + A_{ik}(y)\dfrac{\partial \chi_j}{\partial y_k})$ where $M_Y(\cdot) = \dfrac{1}{|Y|}\displaystyle\int_Y \cdot dy$ and $\chi_j \in W_{per}(Y)$ are the solutions of the local problem

$$-\nabla_y \cdot (A(y)(\nabla \chi_j + e_j)) = 0 \tag{2.0.3}$$

and

$$W_{per}(Y) = \{\chi \in H^1_{per}(Y) | M_Y(\chi) = 0\}.$$

where e_j for the canonical basis in $\mathbb{R}^N$.

We mention that, further in this chapter, ∇ and $(\nabla\cdot)$ will denote the full gradient and divergence operators respectively, and with $\nabla_x, (\nabla_x\cdot)$ and $\nabla_y, (\nabla_x\cdot)$ we will denote the gradient and the divergence in the slow and fast variable respectively.

Remark 2.0.1. *In the remainder of the chapter, we will denote by Φ the continuous extension of a given function $\Phi \in W^{p,m}(\Omega)$ with $p, m \in \mathbb{Z}$, to the space $W^{p,m}(\mathbb{R}^N)$. With minimal assumption on the smoothness of Ω this extension can be chosen independent of the domain, (see [77], Ch. VI, 3.1).*

The formal asymptotic expansion corresponding to the above results can be written as

$$u_\epsilon(x) = u_0(x) + \epsilon w_1(x, \frac{x}{\epsilon}) + ...$$

where

$$w_1(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_j} \tag{2.0.4}$$

We make the observation that Einstein summation convention will be used in the remainder of the chapter and that the letter C will denote a constant independent of any other parameter, otherwise specified.

A classical result (see [73], [51], [54],[8]), states that with additional regularity assumption on the local problem solutions χ_j or on u_0 one has

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon})||_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}} \tag{2.0.5}$$

Without any additional assumptions a similar result has been recently proved by G. Griso in [41], using the Periodic Unfolding method developed in [22], i.e.,

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j})||_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}||u_0||_{H^2(\Omega)} \tag{2.0.6}$$

with

$$x \in \tilde{\Omega}_\epsilon, \;\; Q_\epsilon(\phi)(x) = \sum_{i_1,..,i_N} M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon i)\bar{x}_{1,\xi}^{i_1} \cdot ...\bar{x}_{N,\xi}^{i_N}, \quad \xi = \left[\frac{x}{\epsilon}\right]$$

for $\phi \in L^2(\Omega)$, $i = (i_1, ..., i_N) \in \{0,1\}^N$ and

$$\bar{x}_{k,\xi}^{i_k} = \begin{cases} \dfrac{x_k - \epsilon\xi_k}{\epsilon} & \text{if } i_k = 1 \\ 1 - \dfrac{x_k - \epsilon\xi_k}{\epsilon} & \text{if } i_k = 0 \end{cases} \qquad x \in \epsilon(\xi + Y)$$

where $M_Y^\epsilon(\phi) = \dfrac{1}{\epsilon^N}\displaystyle\int_{\epsilon\xi+\epsilon Y}\phi(y)dy$ and $\tilde{\Omega}_\epsilon = \displaystyle\bigcup_\xi \{\epsilon\xi + \epsilon Y, \text{ with } (\epsilon\xi + \epsilon Y) \cap \Omega \neq \emptyset\}$

In order to improve the error estimates in (2.0.5) boundary layer terms have been introduced as solutions to

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\theta_\epsilon) = 0 \text{ in } \Omega \;\; , \;\; \theta_\epsilon = w_1(x, \frac{x}{\epsilon}) \text{ on } \partial\Omega \tag{2.0.7}$$

Assuming $A \in C^\infty(Y)$, Y-periodic matrix and a sufficiently smooth homogenized solution u_0 it has been proved in [11] (see also [54]) have shown that

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(\cdot)||_{H_0^1(\Omega)} \leq C\epsilon \tag{2.0.8}$$

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(\cdot)||_{L^2(\Omega)} \leq C\epsilon^2. \tag{2.0.9}$$

In [59], Moskow and Vogelius proved the above estimates assuming $A \in C^\infty(Y)$, Y-periodic matrix and $u_0 \in H^2(\Omega)$ or $u_0 \in H^3(\Omega)$ for (2.0.8) or (2.0.9) respectively.

Inequality (2.0.8) is proved in [1] for the case when $A \in L^\infty(Y)$ and $u_0 \in W^{2,\infty}(\Omega)$.

In [82], Sarkis and Versieux showed that the estimates (2.0.8) and respectively (2.0.9) still holds in a more general setting, when one has $u_0 \in W^{2,p}(\Omega)$, $\chi_j \in W^{1,q}_{per}(Y)$ for (2.0.8), and $u_0 \in W^{3,p}(\Omega)$, $\chi_j \in W^{1,q}_{per}(Y)$ for (2.0.9), where, in both cases, $p > N$ and $q > N$ satisfy $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. In [82] the constants in the right hand side of (2.0.8) and (2.0.9) are proportional to $||u_0||_{W^{2,p}(\Omega)}$ and $||u_0||_{W^{3,p}(\Omega)}$ respectively.

In order to improve the error estimate in (2.0.8) and (2.0.9) one needs to consider the second order boundary layer corrector, φ_ϵ defined as the solution of,

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\varphi_\epsilon) = 0 \text{ in } \Omega \ , \ \varphi_\epsilon(x) = \chi_{ij}(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j} \text{ on } \partial\Omega \tag{2.0.10}$$

where $\chi_{ij} \in W_{per}(Y)$ are solution of the following local problems,

$$\nabla_y \cdot (A\nabla_y \chi_{ij}) = b_{ij} + \mathcal{A}^{hom}_{ij} \tag{2.0.11}$$

with $\mathcal{A}^{hom}$ defined by (2.0.2), $M_Y(b_{ij}(y)) = -\mathcal{A}^{hom}_{ij}$, and $b_{ij} = -A_{ij} - A_{ik}\frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j)$.

For the case when $u_0 \in W^{3,\infty}(\Omega)$ and $\chi_{ij} \in W^{1,\infty}(Y)$, with the help of φ_ϵ defined in (2.0.10), Allaire and Amar proved in [1] the following result

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(\cdot) - \epsilon^2\chi_{ij}(\frac{\cdot}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}||u_0||_{W^{3,\infty}(\Omega)} \tag{2.0.12}$$

This result shows that with the help of the second order correctors one can essentialy improve the order of the estimate (2.0.8). In this chapter we will generalize the existing results and prove several error estimates results for (2.0.1), in the general case of bounded coefficients, i.e. $A \in L^\infty_{per}(Y)$. This is important as one can immediately see that regularity assumptions on the cell solutions χ_j, χ_{ij} imply extra smoothness of the coefficients matrix A and this is not the case in general (e.g., the case of composite materials), and on the other hand the homogenized solution u_0 is in general not smooth, for example in the case when Ω is neither convex nor smooth enough (see, [42]).

First, inspired by Griso's idea presented in [41], we use the periodic unfolding method developed in [22] and a general smoothing argument to replace $w_1(x, \frac{x}{\epsilon})$ defined at (2.0.4), by

$$u_1(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon}) Q_\epsilon(\frac{\partial u_0}{\partial x_j}) \tag{2.0.13}$$

in (2.0.8) and (2.0.9). For $u_0 \in H^2(\Omega)$ we prove

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon}) Q_\epsilon(\frac{\partial u_0}{\partial x_j}) + \epsilon\beta_\epsilon(\cdot)||_{H_0^1(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)} \tag{2.0.14}$$

where β_ϵ is defined by

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\beta_\epsilon) = 0 \text{ in } \Omega \ , \ \beta_\epsilon = u_1(x, \frac{x}{\epsilon}) \text{ on } \partial\Omega \tag{2.0.15}$$

Assuming $u_0 \in W^{3,p}(\Omega)$ with $p > N$ we obtain

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j} + \epsilon\theta_\epsilon(\cdot)||_{L^2(\Omega)} \leq C\epsilon^2 ||u_0||_{W^{3,p}(\Omega)}. \tag{2.0.16}$$

Next, we present a refinement of (2.0.12) for the case of nonsmooth coefficients and general data. To do this we start by describing the asymptotic behavior of φ_ϵ with respect to ϵ. The key difference between the case of smooth coefficients, and the nonsmooth case discussed in the present chapter is that in the former, by means of the maximum principle or Avellaneda's compactness results (see [7]), it can be proved that the second order boundary layer corrector φ_ϵ is bounded in $L^2(\Omega)$ and is of order $O(\frac{1}{\sqrt{\epsilon}})$ in $H^1(\Omega)$, while in the latter one cannot use the aforementioned techniques to describe the asymptotic behavior of φ_ϵ in $L^2(\Omega)$ or $H^1(\Omega)$. Moreover one can see that φ_ϵ is not bounded in $L^2(\Omega)$ in general (see [7]), and therefore one needs to address carrefuly the question of the asymptotic behavior of φ_ϵ with respect to ϵ. First, we can easily observe that $\epsilon\varphi_\epsilon$ can be interpreted as the solution of an elliptic problem with variable periodic coefficients and with weakly convergent data in $H^{-1}(\Omega)$. For this class of problems a result of Tartar, [79](see also [24]) implies

$$\epsilon\varphi_\epsilon \overset{\epsilon}{\rightharpoonup} 0 \text{ in } H^1(\Omega)$$

As a consequence of Lemma 2.2.4 we obtain that for $u_0 \in H^3(\Omega)$ and $\chi_j, \chi_{ij} \in W^{1,p}_{per}(Y)$, for some $p > N$, we have

$$||\epsilon\varphi_\epsilon||_{H^1(\Omega)} \leq C\epsilon^{\min\{\frac{1}{2}, 1-\frac{N}{p}\}} ||u_0||_{H^3(\Omega)} \tag{2.0.17}$$

Using (2.0.17) we are able to prove that for $u_0 \in H^3(\Omega)$ and $\chi_j, \chi_{ij} \in W^{1,p}_{per}$ with $p > N$ we have

$$||u^\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j(\frac{\cdot}{\epsilon})\frac{\partial u_0}{\partial x_j} + \epsilon\theta_\epsilon(\cdot) - \epsilon^2\chi_{ij}(\frac{\cdot}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\min\{\frac{3}{2}, 2-\frac{N}{p}\}} ||u_0||_{H^3(\Omega)}. \tag{2.0.18}$$

In Section 2.4 we use (2.0.18) to extend the results in [59] to the case of nonsmooth coefficients. Namely, in two dimensions Moskow and Vogelius (see [59]) considered the Dirichlet spectral problem associated to (2.0.1)

$$\begin{cases} -\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon(x)) = \lambda^\epsilon u_\epsilon & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.0.19}$$

The eigenvalues of (2.0.19) form an increasing sequence of positive numbers, i.e,

$$0 < \lambda_1^\epsilon \leq \lambda_2^\epsilon \leq ... \leq \lambda_j^\epsilon \leq ...$$

and it is well known that we have $\lambda_j^\epsilon \rightharpoonup \lambda_j$ as $\epsilon \to 0$ for any $j \geq 0$ where

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ...$$

are the Dirichlet eigenvalues of the homogenized operator, i.e.,

$$\begin{cases} -\nabla \cdot (\mathcal{A}^{hom}\nabla u(x)) = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.0.20}$$

For $A \in C^\infty(Y)$, Y-periodic, and assuming that the eigenfunctions of (2.0.20) belong to $H^{2+r}(\Omega)$, with $r > 0$, Moskow and Vogelius analyzed in [59], the first corrector of the homogenized eigenvalue of (2.0.20) and proved that (See Thm. 3.6), up to a subsequence,

$$\frac{\lambda^\epsilon - \lambda}{\epsilon} \to \lambda \int_\Omega \theta_* u dx \tag{2.0.21}$$

where θ_* is a weak limit of θ_ϵ in $L^2(\Omega)$, and u is the normal eigenvector associated to the eigenvalue λ.

Using (2.0.18) we show that the result obtained in [59] for the first corrector of the homogenized eigenvalue holds true in the general case of nonsmooth coefficients.

2.1 First order error estimates

The main result of this section is

Theorem 2.1.1. *Let u_ϵ, u_0, u_1, and β_ϵ be defined as in Section 1. Then we have*

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\beta_\epsilon(\cdot)||_{H_0^1(\Omega)} \leq C\epsilon||u_0||_{H^2(\Omega)}$$

Proof. The first step is to consider the mollified coefficient matrix $(A_{ij}^n)_{i,j=1}^N$, defined in the Appendix, with the properties $||A_{ij}^n||_{L^\infty} \leq ||A_{ij}||_{L^\infty}$, (A_{ij}^n) is a Y-periodic matrix, and

$$A_{ij}^n \to A_{ij} \text{ in } L^p(Y) \text{ for } 1 \leq p < \infty \tag{2.1.1}$$

For these coefficients the corresponding functions u_ϵ^n, χ_j^n, u_1^n, and β_ϵ^n defined similarly as in Section 1 , (2.0.1), (2.0.3), (2.0.13), and (2.0.15), respectively, satisfy (see Appendix):

$$\begin{aligned} \chi_j^n &\rightharpoonup \chi_j \quad \text{in } H^1_{per}(Y) \\ u_\epsilon^n &\overset{n}{\rightharpoonup} u_\epsilon \quad \text{in } H^1_0(\Omega) \\ u_1^n &\rightharpoonup u_1 \quad \text{in } H^1(\Omega) \\ \beta_\epsilon^n &\overset{n}{\rightharpoonup} \beta_\epsilon \quad \text{in } H^1(\Omega) \end{aligned} \tag{2.1.2}$$

We define

$$v_0^n(x,y) = A^n(y)Q_\epsilon(\nabla_x u_0) + A^n(y)\nabla_y u_1^n(x,y) \tag{2.1.3}$$

therefore

$$(v_0^n(x,y))_i = \left(A_{ij}^n(y) + A_{ik}^n(y)\frac{\partial \chi_j^n}{\partial y_k} \right) Q_\epsilon(\frac{\partial u_0}{\partial x_j}) \tag{2.1.4}$$

By using the definition of χ_j^n we have $\nabla_y \cdot v_0^n = 0$. Let us denote by

$$(C^n(y))_{ij} = A_{ij}^n(y) + A_{ik}^n(y)\frac{\partial \chi_j^n}{\partial y_k}$$

and $\mathcal{A}_n^{hom} = M_Y(C^n(y))$. It can be seen that

$$\nabla_y \cdot (v_0^n - \mathcal{A}_n^{hom} Q_\epsilon(\nabla_x u_0)) = 0 \tag{2.1.5}$$

Lemma 2.1.2. *There exists* $q^n(x,\cdot) \in [W_{per}(Y)]^N$ *such that* $curl_y q^n = v_0^n - \mathcal{A}_n^{hom} Q_\epsilon(\nabla_x u_0)$.

Proof. Let $B^n(y) = C^n(y) - \mathcal{A}_n^{hom}$. We then have

$$v_0^n - \mathcal{A}_n^{hom} Q_\epsilon(\nabla_x u_0) = B^n(y) Q_\epsilon(\nabla_x u_0) \tag{2.1.6}$$

We look for q^n of the form

$$q^n(x,y) = \phi^n(y) Q_\epsilon(\nabla_x u_0)$$

where $\phi^n(y) = (\phi_{ij}^n(y))_{ij}$ with $\phi_{ij}^n(y) \in W_{per}(Y)$.

If we denote by B_l^n the vector $B_l^n = (B_{il}^n)_i \in [L^2_{per}(Y)]^N$ we observe that $\nabla_y \cdot B_l^n = 0$. Hence from the Theorem 3.4 in Girault-Raviart [40] adapted to the periodic case, the vectors $\phi_l^n = (\phi_{il}^n)_i \in [W_{per}(Y)]^N$ are determined as the solutions to

$$curl_y \phi_l^n = B_l^n \text{ and } div_y \phi_l^n = 0; \tag{2.1.7}$$

Obviously we have

$$curl_y q^n(x,y) = v_0^n - \mathcal{A}_n^{hom} Q_\epsilon(\nabla_x u_0) \tag{2.1.8}$$

□

Remark 2.1.3. *From (2.1.2) it can be immediately seen that B^n is bounded independently of n in $[L^2(Y)]^{N\times N}$ and using the Appendix we have*

$$B^n \rightharpoonup B \ \ in\ [L^2(Y)]^{N\times N}$$

where, B has an identical form as B^n and it can be easily determined from the above limit. This together with (2.1.7) and Theorem 3.9 in [40] adapted for the periodic case implies that ϕ^n is bounded independently of n in $(W_{per}(Y))^{N\times N}$ and we have

$$\phi_l^n \rightharpoonup \phi_l \ \ in\ [W_{per}(Y)]^N \ \ where$$
$$curl_y\phi_l = B_l \ \ and\ div_y\phi_l = 0; \ \ for\ every\ l \in \{1,..,N\} \tag{2.1.9}$$

Next we define

$$v_1^n(x,y) = curl_x q^n(x,y)$$

and using Lemma 2.1.2 we have

$$\nabla_y \cdot v_1^n = -\nabla_x \cdot curl_y q^n = -\nabla_x \cdot v_0^n - f_\epsilon^n \tag{2.1.10}$$

where

$$f_\epsilon^n = -\nabla_x \cdot (\mathcal{A}_n^{hom} Q_\epsilon(\nabla_x u_0)).$$

We define

$$z_\epsilon^n(x) = u_\epsilon^n(x) - u_0(x) - \epsilon u_1^n(x, \frac{x}{\epsilon}) \tag{2.1.11}$$

$$\mu_\epsilon^n(x) = A^n(\frac{x}{\epsilon})\nabla u_\epsilon^n(x) - v_0^n(x, \frac{x}{\epsilon}) - \epsilon v_1^n(x, \frac{x}{\epsilon}) \tag{2.1.12}$$

From the above definitions, similarly as in [59] we obtain

$$A^n(\frac{x}{\epsilon})\nabla z_\epsilon^n(x) - \mu_\epsilon^n(x) = \epsilon(v_1^n(x, \frac{x}{\epsilon}) - A^n(\frac{x}{\epsilon})\nabla_x u_1^n(x, \frac{x}{\epsilon})) + A^n(\frac{x}{\epsilon})(Q_\epsilon(\nabla_x u_0) - \nabla_x u_0) \tag{2.1.13}$$

Next, we will prove that the L^2 norm of (2.1.13) is of order ϵ. In order to do this we will show that $v_1^n(x, \frac{x}{\epsilon})$ and $A^n(\frac{x}{\epsilon})\nabla_x u_1(x, \frac{x}{\epsilon})$ are bounded in L^2 independently of n and ϵ. We have the following estimate

Lemma 2.1.4. *Let $\Omega \subset \mathbb{R}^N$ as before. For any $\psi \in L^2(Y)$, Y-periodic, we have*

$$||\nabla_x Q_\epsilon(\frac{\partial u_0}{\partial x_j})\psi(\frac{x}{\epsilon})||_{L^2(\Omega)} \leq C||u_0||_{H^2(\Omega)}||\psi||_{L^2(Y)}$$

Proof. We recall the definition of Q_ϵ

$$Q_\epsilon(\phi)(x) = \sum_{i_1,..,i_N} M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon i)\bar{x}_{1,\xi}^{i_1} \cdot ...\bar{x}_{N,\xi}^{i_N}, \quad \xi = \left[\frac{x}{\epsilon}\right]$$

for any $x \in \tilde{\Omega}_\epsilon$, with $\tilde{\Omega}_\epsilon$ defined in the Appendix, and any $\phi \in L^2(\tilde{\Omega}_{\epsilon,2})$ with $\tilde{\Omega}_{\epsilon,2} = \{x \in \Omega; dist(x,\Omega) < 2\epsilon\}$, where $i = (i_1, ..., i_N) \in \{0,1\}^N$ and

$$\bar{x}_{k,\xi}^{i_k} = \begin{cases} \dfrac{x_k - \epsilon\xi_k}{\epsilon} & \text{if } i_k = 1 \\ 1 - \dfrac{x_k - \epsilon\xi_k}{\epsilon} & \text{if } i_k = 0 \end{cases} \qquad x \in \epsilon(\xi + Y).$$

The first order derivative Q_ϵ takes the form

$$\frac{\partial}{\partial x_1} Q_\epsilon(\phi)(x) = \sum_{i_1,\dots,i_N} \frac{M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon(1, i_2, \dots, i_n)) - M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon(0, i_2, \dots, i_n))}{\epsilon} \bar{x}_{2,\xi}^{i_2} \cdot \dots \cdot \bar{x}_{N,\xi}^{i_N}$$

and therefore

$$\int_{\epsilon\xi+\epsilon Y} \left| \frac{\partial}{\partial x_1} Q_\epsilon(\phi)(x) \right|^2 \left| \psi(\frac{x}{\epsilon}) \right|^2 \le 2^{N-1} \left| \frac{M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon(1, i_2, \dots, i_n)) - M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon(0, i_2, \dots, i_n))}{\epsilon} \right|^2 \cdot$$

$$\cdot \int_{\epsilon\xi+\epsilon Y} \left| \psi(\frac{x}{\epsilon}) \right|^2 dx =$$

$$= 2^{N-1} \left| \frac{M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon(1, i_2, \dots, i_n)) - M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon(0, i_2, \dots, i_n))}{\epsilon} \right|^2 \epsilon^N ||\psi||_{L^2(Y)}^2.$$

Using the definition of the mean M_Y^ϵ and the Schwartz inequality we get

$$\int_{\epsilon\xi+\epsilon Y} \left| \frac{\partial}{\partial x_1} Q_\epsilon(\phi)(x) \right|^2 \left| \psi(\frac{x}{\epsilon}) \right|^2 \le$$

$$\le C ||\psi||_{L^2(Y)}^2 \sum_{i_1,\dots,i_N} \int_{\epsilon\xi+\epsilon Y} \left| \frac{\phi(x + \epsilon(1, i_2, \dots, i_n)) - \phi(x + \epsilon(0, i_2, \dots, i_n))}{\epsilon} \right|^2 dx \le$$

$$\le C ||\psi||_{L^2(Y)}^2 \sum_{i_1,\dots,i_N} \int_{\epsilon\xi+\epsilon Y} \left(\left| \frac{\phi(x + \epsilon(1, i_2, \dots, i_n)) - \phi(x)}{\epsilon} \right|^2 + \left| \frac{\phi(x + \epsilon(0, i_2, \dots, i_n)) - \phi(x)}{\epsilon} \right|^2 \right) dx$$

After summing the above inequalities over $\xi \in \{\xi \in \mathbb{Z}^N;\ (\epsilon\xi + \epsilon Y) \cap \Omega \neq \emptyset\}$, and using the inequality between the differential quontiens and the gradient we obtain

$$\int_\Omega \left| \frac{\partial}{\partial x_1} Q_\epsilon(\phi)(x) \right|^2 \left| \psi(\frac{x}{\epsilon}) \right|^2 \le C ||\psi||_{L^2(Y)}^2 ||\nabla \phi||_{L^2(\tilde{\Omega}_{\epsilon,2})}^2$$

This yields

$$\int_\Omega |\nabla_x Q_\epsilon(\phi)|^2 \left| \psi(\frac{x}{\epsilon}) \right|^2 \le C ||\psi||_{L^2(Y)}^2 ||\nabla \phi||_{L^2(\tilde{\Omega}_{\epsilon,2})}^2.$$

Choosing ϕ to be the partial derivative of u_0 the conclusion of the Lemma follows. □

Applying Lemma 2.1.4 we can see that

$$||A^n(\frac{x}{\epsilon})\nabla_x u_1^n(x,\frac{x}{\epsilon})||_{L^2(\Omega)} \leq C||\chi_j^n||^2_{L^2(Y)}||u_0||_{H^2(\Omega)} \leq C||u_0||_{H^2(\Omega)} \tag{2.1.14}$$

and using Remark 2.1.3 we obtain

$$||v_1^n(x,\frac{x}{\epsilon})||_{L^2(\Omega)} \leq C\left(\sum_{i,j}||\phi_{ij}^n||^2_{L^2(Y)}\right)^{\frac{1}{2}}||u_0||_{H^2(\Omega)} \leq C||u_0||_{H^2(\Omega)} \tag{2.1.15}$$

Using (2.1.14), (2.1.15) and the properties of Q_ϵ we obtain the following estimate for the left hand side of (2.1.13):

$$||A^n(\frac{x}{\epsilon})\nabla z_\epsilon^n(x) - \mu_\epsilon^n(x)||_{L^2(\Omega)} \leq C\epsilon||u_0||_{H^2(\Omega)} \tag{2.1.16}$$

For $g \in L^2(\Omega)$ we define $w_\epsilon^n \in H_0^1(\Omega)$ solution of the following problem

$$-\nabla\cdot(A^n(\frac{x}{\epsilon})\nabla w_\epsilon^n) = g \ \text{ in } \Omega \ , \ w_\epsilon^n = 0 \ \text{ on } \partial\Omega \tag{2.1.17}$$

Obviously we have

$$||w_\epsilon^n||_{H_0^1(\Omega)} \leq ||g||_{H^{-1}(\Omega)} \tag{2.1.18}$$

Using $z_\epsilon^n + \epsilon\beta_\epsilon^n$ as a test function in (2.1.17), with β_ϵ defined by (2.0.15) we obtain

$$\int_\Omega (z_\epsilon^n + \epsilon\beta_\epsilon^n) g dx = \int_\Omega A^n(\frac{x}{\epsilon})\nabla z_\epsilon^n \cdot \nabla w_\epsilon^n dx \tag{2.1.19}$$

The right hand side can be estimated as follows

$$\int_\Omega A^n(\frac{x}{\epsilon})\nabla z_\epsilon^n\cdot\nabla w_\epsilon^n dx = \int_\Omega \left(A^n(\frac{x}{\epsilon})\nabla z_\epsilon^n - \mu_\epsilon^n\right)\cdot\nabla w_\epsilon^n dx - \int_\Omega \left(\nabla\cdot\mu_\epsilon^n\right) w_\epsilon^n dx \leq$$

$$\leq ||A^n(\frac{x}{\epsilon})\nabla z_\epsilon^n - \mu_\epsilon^n||_{L^2(\Omega)}||w_\epsilon^n||_{H_0^1(\Omega)} + ||\nabla\cdot\mu_\epsilon^n||_{H^{-1}(\Omega)}||w_\epsilon^n||_{H_0^1(\Omega)} \tag{2.1.20}$$

We note here that $\nabla\cdot\mu_\epsilon^n \in L^2(\Omega)$. Indeed:

$$\nabla\cdot\mu_\epsilon^n(x) = \nabla\cdot(A^n(\frac{x}{\epsilon})\nabla u_\epsilon^n(x)) - \nabla_x\cdot v_0^n(x,\frac{x}{\epsilon}) - \frac{1}{\epsilon}\nabla_y\cdot v_0^n(x,\frac{x}{\epsilon}) -$$

$$-\nabla_x\cdot v_1^n(x,\frac{x}{\epsilon}) - \nabla_y\cdot v_1^n(x,\frac{x}{\epsilon}) = -f(x) - \nabla_x\cdot(\mathcal{A}_n^{hom}Q_\epsilon(\nabla_x u_0))$$

To estimate the H^{-1} norm of $\nabla\cdot\mu_\epsilon^n$ we consider $\phi \in H_0^1(\Omega)$ and

$$\int_\Omega (\nabla\cdot\mu_\epsilon^n)\phi(x)dx = \int_\Omega (\mathcal{A}^{hom}Q_\epsilon(\nabla u_0) - \mathcal{A}^{hom}\nabla u_0)\nabla\phi dx + \int_\Omega (\mathcal{A}_n^{hom} - \mathcal{A}^{hom})Q_\epsilon(\nabla u_0)\nabla\phi dx \leq$$

$$\leq C||\nabla u_0 - Q_\epsilon(\nabla u_0)||_{L^2(\Omega)}||\phi||_{H_0^1(\Omega)} + ||\phi||_{H_0^1(\Omega)}||(\mathcal{A}_n^{hom} - \mathcal{A}^{hom})Q_\epsilon(\nabla u_0)||_{L^2(\Omega)} \leq$$

$$\leq C\epsilon ||u_0||_{H^2(\Omega)}||\phi||_{H^1_0(\Omega)} + K_n||\phi||_{H^1_0(\Omega)}||u_0||_{H^1(\Omega)} \tag{2.1.21}$$

where we used the properties of Q_ϵ and $K_n \doteq |\mathcal{A}^{hom} - \mathcal{A}^{hom}_n|$.
Therefore we proved that

$$||\nabla \cdot \mu^n_\epsilon||_{H^{-1}(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)} + K_n||u_0||_{H^1(\Omega)} \tag{2.1.22}$$

Thus (2.1.16) and (2.1.22) used in (2.1.21) imply

$$\left| \int_\Omega (z^n_\epsilon + \epsilon\beta^n_\epsilon) g dx \right| \leq C\epsilon ||u_0||_{H^2(\Omega)}||w^n_\epsilon||_{H^1_0(\Omega)} + CK_n||w^n_\epsilon||_{H^1_0(\Omega)} \leq$$
$$\leq C\epsilon ||u_0||_{H^2(\Omega)}||g||_{H^{-1}(\Omega)} + CK_n||g||_{H^{-1}(\Omega)}$$

where we used (2.1.18). From the above inequality we have

$$||z^n_\epsilon + \epsilon\beta^n_\epsilon||_{H^1_0(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)} + CK_n \tag{2.1.23}$$

From (2.1.1) and (2.1.2) we have that $K_n \to 0$ as $n \to \infty$. Using the Appendix we can pass to the limit when $n \to \infty$ in (2.1.23) and from (2.1.2) we get

$$||z_\epsilon + \epsilon\beta_\epsilon||_{H^1_0(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)}$$

which is exactly what needs to be proved. □

2.2 Second order error estimate

The L^2-norm of

$$u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(\cdot) \tag{2.2.1}$$

can be estimated with additional assumptions. Moscow and Vogelius obtain in [59], the ϵ^{1+r}, estimate for (2.2.1), for some $r > 0$, assuming that $u_0 \in H^{2+r}(\Omega)$ and $A \in C^\infty(Y)$. In this section we will improve this estimate and analyze the case of nonsmooth coefficents. Let $\chi^n_{ij} \in W_{per}(Y)$ solutions of

$$\nabla_y \cdot (A^n \nabla_y \chi^n_{ij}) = b^n_{ij} - M_Y(b^n_{ij}) \tag{2.2.2}$$

where

$$b^n_{ij} = -A^n_{ij} - A^n_{ik}\frac{\partial \chi^n_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A^n_{ik}\chi^n_j)$$

and $M_Y(.)$ is the average on Y. From Appendix, Corollary 6.2.8

$$|\nabla_y \chi^n_{ij}|_{L^2(Y)} < C \text{ and } \chi^n_{ij} \rightharpoonup \chi_{ij} \text{ in } W_{per}(Y), \ \forall i,j \in \{1,...,N\}$$

where

$$\int_Y A(y)\nabla_y \chi_{ij} \nabla_y \psi dy = (b_{ij} - M_Y(b_{ij}), \psi)_{(W_{per}(Y),(W_{per}(Y))')}$$

for any $\psi \in W_{per}(Y)$ and with

$$b_{ij} = -A_{ij} - A_{ik}\frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j).$$

In [82], an estimate of order ϵ^2 is proved, under the assumptions that $u_0 \in W^{3,p}(\Omega)$ and $\chi_j, \chi_{ij} \in W^{1,q}_{per}(Y)$ for $p, q > N$ where $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$.

Next, we will only assume that $u_0 \in W^{3,p}(\Omega)$ with $N < p \leq \infty$ to prove the estimate of order ϵ^2 for (2.2.1). Indeed we have,

Theorem 2.2.1. *Let u_ϵ, u_0, u_1 and θ_ϵ defined as in Section 2. If $u_0 \in W^{3,p}(\Omega)$, $N < p \leq \infty$ we have*

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(\cdot)||_{L^2(\Omega)} \leq C\epsilon^2 ||u_0||_{W^{3,p}(\Omega)} \tag{2.2.3}$$

Proof. For the sake of simplicity we will consider only the case when $N = 3$, the two dimensional case being similar. As in the previous section we can assume the smooth coefficients A^n (see (6.2.1)), and follow the same ideas as in [59] to define

$$u_2^n(x, y) = \chi_{ij}^n(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$$

For $p > N$ we have that

$$||\nabla_x u_2^n(., \frac{\cdot}{\epsilon})||_{L^2(\Omega)} \leq ||\chi_{ij}^n(\frac{\cdot}{\epsilon})||_{L^{\frac{2p}{p-2}}(\Omega)} ||\nabla_x \frac{\partial^2 u_0}{\partial x_j \partial x_i}||_{L^p(\Omega)}$$

and using a change in variables and the inequality (6.2.10) in the Appendix, we obtain

$$||\nabla_x u_2^n(., \frac{\cdot}{\epsilon})||^2_{L^2(\Omega)} \leq C\sum_{i,j} ||\nabla_x \frac{\partial^2 u_0}{\partial x_j \partial x_i}||^2_{L^p(\Omega)} \leq C||u_0||^2_{W^{3,p}(\Omega)} \tag{2.2.4}$$

As in [59] we will define

$$(v_*^n(x, y))_k = A_{ki}^n(y)\chi_j^n(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A_{kl}^n(y)\frac{\partial \chi_{ij}^n}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j \partial x_i} \tag{2.2.5}$$

Following similar arguments we can observe that $\nabla_x \cdot M_Y(v_*^n) = 0$. By introducing

$$R_{ki}^j = M_Y(A_{ki}^n \chi_j^n + A_{kl}^n \frac{\partial \chi_{ij}^n}{\partial y_l}).$$

Consider $\alpha_{ij}^n \in [L^2(Y)]^3$ defined by,

$$\alpha_{ij}^n = \begin{pmatrix} A_{1i}^n \chi_j^n + A_{1l}^n \frac{\partial \chi_{ij}^n}{\partial y_l} - R_{1i}^j \\ A_{2i}^n \chi_j^n + A_{2l}^n \frac{\partial \chi_{ij}^n}{\partial y_l} - R_{2i}^j \\ A_{3i}^n \chi_j^n + A_{3l}^n \frac{\partial \chi_{ij}^n}{\partial y_l} - R_{3i}^j \end{pmatrix} + \beta_{ij}^n$$

with

$$\begin{aligned}
\beta^n_{1j} &= (0, -\phi^n_{3j}, \phi^n_{2j})^T \\
\beta^n_{2j} &= (\phi^n_{3j}, 0, -\phi^n_{1j})^T \quad \text{for } j \in \{1,2,3\} \\
\beta^n_{3j} &= (-\phi^n_{2j}, \phi^n_{1j}, 0)^T
\end{aligned}$$

where T designates the transposition operation and ϕ^n_{ij} are defined at (2.1.7). Using the symmetry of the matrix A we observe that the vectors α^n_{ij} defined above, are divergence free with zero average over Y. This imply that there exists $\psi^n_{ij} \in [W_{per}(Y)]^3$, (see Theorem 3.4, [40] adapted for the periodic case) so that

$$curl_y \psi^n_{ij} = \alpha^n_{ij} \ \text{ and } div \psi^n_{ij} = 0 \ \text{ for any } i,j \in \{1,2,3\} \tag{2.2.6}$$

From Corollary 6.2.10 in Appendix and we observe that

$$\alpha^n_{ij} \rightharpoonup \alpha_{ij} \ \text{ in } [L^2(Y)]^3 \tag{2.2.7}$$

where the form of α_{ij} is identical with that of α^n_{ij} and can be obviously obtain from (2.2.7). Using the above convergence result and Theorem 3.9 from [40] adapted to the periodic case, we obtain that

$$\psi^n_{ij} \rightharpoonup \psi_{ij} \ , \ \text{ in } \ W_{per}(Y) \ \text{ for any } i,j \in \{1,2,3\}$$

and ψ_{ij} satisfy

$$curl_y \psi_{ij} = \alpha_{ij} \text{ and } div_y \psi_{ij} = 0 \ \text{ for } i,j \in \{1,2,3\} \tag{2.2.8}$$

Next define $p^n(x,y) = \psi^n_{ij}(y) \dfrac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$ and $v^n_2(x,y) = curl_x p^n(x,y)$. Obviously we have that $\nabla_x \cdot v^n_2 = 0$. It is also easy to check, that $\nabla_y \cdot v^n_2 = -\nabla_x \cdot v^n_*$, (see [59] for example). We set

$$\begin{aligned}
w^n_1(x,y) &= \chi^n_j(y) \frac{\partial u_0}{\partial x_j}(x) \\
r^n_0(x,y) &= A^n(y) \nabla_x u_0 + A^n(y) \nabla_y w^n_1(x,y)
\end{aligned}$$

$$\psi^n_\epsilon(x) = u^n_\epsilon(x) - u_0(x) - \epsilon w^n_1(x, \frac{x}{\epsilon}) - \epsilon^2 u^n_2(x, \frac{x}{\epsilon}) \tag{2.2.9}$$

$$\xi^n_\epsilon(x) = A^n(\frac{x}{\epsilon}) \nabla u^n_\epsilon - r^n_0(x, \frac{x}{\epsilon}) - \epsilon v^n_*(x, \frac{x}{\epsilon}) - \epsilon^2 v^n_2(x, \frac{x}{\epsilon}) \tag{2.2.10}$$

As in [59] we can write

$$A^n(\frac{x}{\epsilon}) \nabla \psi^n_\epsilon(x) - \xi^n_\epsilon(x) = \epsilon^2 (v^n_2(x, \frac{x}{\epsilon}) - A^n(\frac{x}{\epsilon}) \nabla_x u^n_2(x, \frac{x}{\epsilon})) \tag{2.2.11}$$

We use next, as in (2.2.4), the inequality (6.2.10) to obtain

$$||v^n_2(., \frac{.}{\epsilon})||_{L^2(\Omega)} \leq C ||u_0||_{W^{3,p}(\Omega)} \tag{2.2.12}$$

Using (2.2.4), (2.2.11) and (2.2.12) we get

$$||A^n(\frac{x}{\epsilon})\nabla\psi_\epsilon^n(x) - \xi_\epsilon^n(x)||_{L^2(\Omega)} \le C\epsilon^2||u_0||_{W^{3,p}(\Omega)} \tag{2.2.13}$$

Similarly as in [59] we have that $\nabla \cdot \xi_\epsilon^n(x) = 0$. Let us define φ_ϵ^n as solution of

$$\nabla \cdot (A^n(\frac{x}{\epsilon})\nabla\varphi_\epsilon^n) = 0 \text{ in } \Omega\ ,\ \varphi_\epsilon^n = u_2^n(x, \frac{x}{\epsilon}) \text{ on } \partial\Omega \tag{2.2.14}$$

Using again Corollary 6.2.10 in Appendix, we have that $\varphi_\epsilon^n \rightharpoonup \varphi_\epsilon$ in $H^1(\Omega)$ where φ_ϵ is the solution of

$$\nabla \cdot (A(\frac{x}{\epsilon})\nabla\varphi_\epsilon) = 0 \text{ in } \Omega\ ,\ \varphi_\epsilon = u_2(x, \frac{x}{\epsilon}) \text{ on } \partial\Omega \tag{2.2.15}$$

Then,

$$||\varphi_\epsilon||_{L^2(\Omega)} \le C||u_2(.,\frac{\cdot}{\epsilon})||_{L^\infty(\partial\Omega)} \le C||\chi_{ij}||_{L^\infty(Y)}||u_0||_{W^{3,p}(\Omega)} \le C||u_0||_{W^{3,p}(\Omega)} \tag{2.2.16}$$

where we used [53] for the L^∞ bound on χ_{ij}. Next, similarly as in [59] we have

$$||u_\epsilon^n(\cdot) - u_0(\cdot) - \epsilon w_1^n(\cdot,\frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon^n(\cdot) - \epsilon^2 u_2^n(\cdot,\frac{\cdot}{\epsilon}) + \epsilon^2\varphi_\epsilon^n||_{L^2(\Omega)} \le C\epsilon^2||u_0||_{W^{3,p}(\Omega)}$$

and passing to the limit when $n \to \infty$ using triangle inequality, (2.2.4) and (2.2.16) we get (2.2.3). □

Remark that the assumption that $u_0 \in W^{3,p}$, with $p > N$ was necessary for the estimate (2.2.16)

In the case of L^∞ coefficients, with the only assumptions that $\chi_j, \chi_{ij} \in W^{1,p}_{per}(Y)$ for some $p > N$ and $u_0 \in H^3(\Omega)$ the left hand side of (2.0.12) can be shown to be of order $\epsilon^{min\{\frac{3}{2}, 2-\frac{N}{p}\}}$. Indeed we have,

Theorem 2.2.2. *Let $u_0 \in H^3(\Omega)$. If there exists $p > N$ such that $\chi_j, \chi_{ij} \in W^{1,p}_{per}(Y)$ then we have*

$$||u_\epsilon(.) - u_0(.) - \epsilon w_1(.,\frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(.) - \epsilon^2\chi_{ij}(\frac{\cdot}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{H^1(\Omega)} \le C\epsilon^{\min\{\frac{3}{2}, 2-\frac{N}{p}\}}||u_0||_{H^3(\Omega)}$$

Proof. As we did before, for the sake of simplicity, we will assume $N = 3$ the two dimensional case beeing similar. For any $i, j \in \{1, 2, 3\}$ let $\psi_{ij} \in [W_{per}(Y)]^3$ be defined as in (2.2.8). The hypothesis on χ_j and χ_{ij} implies that α_{ij} defined at (2.2.7) belongs to the space $[L^p(Y)]^3$ and we have

$$||\alpha_{ij}||_{[L^p(Y)]^3} \le C(||\beta_{ij}||_{[L^p(Y)]^3} + ||\chi_j||_{L^p(Y)} + ||\chi_{ij}||_{W^{1,p}(Y)} \le C \text{ for } i, j \in \{1, 2, 3\} \tag{2.2.17}$$

Relation (2.2.17) and Remark 3.11 in [40] imply that

$$||\psi_{ij}||_{[W^{1,p}(Y)]^3} \le C \text{ for } i, j \in \{1, 2, 3\} \tag{2.2.18}$$

Define $p(x,y) = \psi_{ij}(y)\dfrac{\partial^2 u_0}{\partial x_i \partial x_j}(x)$ and $v_2(x,y) = curl_x p(x,y)$. We can see that $p \in H^1(\Omega, H^1_{per}(Y))$ and $v_2 \in L^2(\Omega, H^1_{per}(Y))$. Obviously we have that $\nabla_x \cdot v_2 = 0$ in the sense of distributions (see [59]). Next, using (2.2.5) we observe that $\nabla_x \cdot M_Y(v_*) = 0$ where v_* is such that

$$v_*^n \rightharpoonup v_* \ \text{ weakly in } L^2(\Omega, L^2_{per}(Y))$$

We have that

$$(v_*(x,y))_k = A_{ki}(y)\chi_j(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A_{kl}(y)\frac{\partial \chi_{ij}}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j \partial x_i}$$

Using this and the fact that

$$\int_{\Omega\times Y} (\nabla_y \cdot v_2)\Phi(x,y)dxdy = \int_{\Omega\times Y} (\nabla_y \cdot curl_x p(x,y))\Phi(x,y)dxdy =$$

$$= -\int_{\Omega\times Y} (\nabla_x \cdot curl_y p(x,y))\Phi(x,y)dxdy$$

for any smooth function $\Phi \in \mathcal{D}(\Omega; \mathcal{D}(Y))$, one can immediately see that

$$\nabla_y \cdot v_2 = -\nabla_x \cdot v_* \tag{2.2.19}$$

in the sense of distributions. Consider ψ_ϵ^n and ξ_ϵ^n defined at (2.2.9) and (2.2.10). We have

Lemma 2.2.3.

$$(i) \ \ ||\psi_\epsilon^n||_{W^{1,1}(\Omega)} < C \ \ \text{and} \ ||\xi_\epsilon^n||_{L^1(\Omega)} < C$$

and there exists $\psi_\epsilon \in W^{1,1}(\Omega)$ *and* $\xi_\epsilon \in L^1(\Omega)$ *such that*

$$\psi_\epsilon^n \overset{n}{\rightharpoonup} \psi_\epsilon \ , \ \nabla\psi_\epsilon^n \overset{n}{\rightharpoonup} \nabla\psi_\epsilon \ , \ \xi_\epsilon^n \overset{n}{\rightharpoonup} \xi_\epsilon \ , \ \ \textit{weakly-* in the sense of measures.}$$

Also we have

$$\psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon w_1(x, \frac{x}{\epsilon}) - \epsilon^2 u_2(x, \frac{x}{\epsilon})$$

$$\xi_\epsilon(x) = A(\frac{x}{\epsilon})\nabla u_\epsilon - r_0(x, \frac{x}{\epsilon}) - \epsilon v_*(x, \frac{x}{\epsilon}) - \epsilon^2 v_2(x, \frac{x}{\epsilon})$$

(ii) Moreover, $\xi_\epsilon \in L^2(\Omega)$, $\psi_\epsilon \in H^1(\Omega)$ *and we have*

$$A(\frac{x}{\epsilon})\nabla\psi_\epsilon(x) - \xi_\epsilon(x) = \epsilon^2(v_2(x, \frac{x}{\epsilon}) - A(\frac{x}{\epsilon})\nabla_x u_2(x, \frac{x}{\epsilon})) \tag{2.2.20}$$

with

$$\nabla \cdot \xi_\epsilon(x) = 0 \tag{2.2.21}$$

in the sense of distributions.

Proof. Using the fact that, for any $i,j \in \{1,2,3\}$, $\chi_j^n, \chi_{ij}^n \in W_{per}(Y)$ and $\psi_{ij}^n \in [W_{per}(Y)]^3$ are bounded functions in this spaces, from the definition one can immediately see that

$$||\psi_\epsilon^n||_{W^{1,1}(\Omega)} < C \;\text{ and }\; ||\xi_\epsilon^n||_{L^1(\Omega)} < C.$$

Recall that

$$\chi_j^n \rightharpoonup \chi_j \;,\; \chi_{ij}^n \rightharpoonup \chi_{ij} \;\text{ in } W_{per}(Y) \;\text{ and }\; \psi_{ij}^n \rightharpoonup \psi_{ij} \;\text{ in } [W_{per}(Y)]^3.$$

Using the above convergence results and the Appendix the statement (i) in Lemma 2.2.3 follows immediately. Observe that $\chi_j, \chi_{ij} \in W_{per}^{1,p}(Y)$, with $p > 3$ imply

$$\psi_\epsilon \in H^1(\Omega) \tag{2.2.22}$$

To prove (2.2.22) it is enough to see that

$$||u_2(\cdot, \frac{\cdot}{\epsilon})||_{H^1(\Omega)} \leq \epsilon^2 ||\chi_{ij}||_{L^\infty(Y)} ||u_0||_{H^2(\Omega)} + \epsilon ||\chi_{ij}||_{W^{1,p}(Y)} ||u_0||_{H^3(\Omega)} + \epsilon^2 ||\chi_{ij}||_{L^\infty(Y)} ||u_0||_{H^3(\Omega)}$$

the rest of the necessary estimates being trivial. Similarly, from the definition of r_0, v_* and v_2 and the hypothesis $\chi_j, \chi_{ij} \in W_{per}^{1,p}(Y)$, with $p > 3$ we see that $\xi_\epsilon \in L^2(\Omega)$. Next note that we immediately have

$$A^n(\frac{x}{\epsilon}) \nabla \psi_\epsilon^n \overset{n}{\rightharpoonup} -A(\frac{x}{\epsilon}) \nabla \psi_\epsilon \;\text{ weakly-* in the sense of measures .} \tag{2.2.23}$$

Relation (2.2.20) follows immediately from (2.2.11), (2.2.23) the relations (6.2.1) in Appendix and a limit argument based on the convergence results obtained at (i). Recall that in the smooth case it is known from [59] that

$$\nabla \cdot \xi_\epsilon^n = 0$$

This is equivalent to

$$\int_\Omega \xi_\epsilon^n \nabla \Phi(x) dx = 0 \;\text{ for any } \Phi \in \mathcal{D}(\Omega)$$

Using the fact that $\xi_\epsilon \in L^2(\Omega)$, and that we have

$$\xi_\epsilon^n \overset{n}{\rightharpoonup} \xi_\epsilon \;\text{ weakly-* in the sense of measures}$$

we obtain (2.2.21). We make the remark that a different proof for (2.2.21) can be found in [82] □

Following the steps in the proof of Lemma 2.2.1 we observe that $\chi_j, \chi_{ij} \in W_{per}^{1,p}(Y)$, with $p > 3$, implies $\psi_{ij} \in W_{per}^{1,p}(Y)$. using this we obtain,

$$||\nabla_x u_2(\cdot, \frac{\cdot}{\epsilon})||_{L^2(\Omega)} \leq ||\chi_{ij}||_{L^\infty(Y)} ||\nabla_x \frac{\partial^2 u_0}{\partial x_j \partial x_i}||_{L^2(\Omega)} \leq ||\chi_{ij}||_{W^{1,p}(Y)} ||u_0||_{H^3(\Omega)} \leq C ||u_0||_{H^3(\Omega)} \tag{2.2.24}$$

$$||v_2(\cdot,\frac{\cdot}{\epsilon})||_{L^2(\Omega)} \leq C||\psi_{ij}||_{L^\infty(Y)}||\nabla_x \frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{L^2(\Omega)} \leq C\sum_{i,j}||\psi_{ij}||_{W^{1,p}(Y)}||u_0||_{H^3(\Omega)} \leq C||u_0||_{H^3(\Omega)} \tag{2.2.25}$$

where in (2.2.25) above we used (2.2.18). Similarly as in [59] using (2.2.24), (2.2.25) in (2.2.19), we arrive at

$$||A(\frac{x}{\epsilon})\nabla\psi_\epsilon(x) - \xi_\epsilon(x)||_{L^2(\Omega)} \leq C\epsilon^2||u_0||_{H^3(\Omega)}$$

In the general case when $A \in L^\infty(Y)$ and $u_0 \in H^3(\Omega)$, if we consider the second boundary layer φ_ϵ defined as in (2.2.14), using (2.2.22) and similar arguments as in [59] we obtain that

$$||u_\epsilon(.) - u_0(.) - \epsilon w_1(.,\frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(.) - \epsilon^2 u_2(x,\frac{x}{\epsilon}) + \epsilon^2\varphi_\epsilon||_{H_0^1(\Omega)} \leq C\epsilon^2||u_0||_{H^3(\Omega)} \tag{2.2.26}$$

Next we make the observation that without any further regularity assumption on u_0 or on the matrix of coefficients A one cannot make use of neither Avellaneda compactness result nor the maximum principle to obtain a bound and for φ_ϵ similar to (2.2.16). In fact in [7] it is presented an example where a solution of (2.2.14) would blow up in the L^2 norm. Although the unboundedness of φ_ϵ in L^2 we can still make the observation that using a result due to Luc Tartar [79] (see also [24], Section 8.5) concerning the limit analysis of the classical homogenization problem in the case of weakly convergent data in $H^{-1}(\Omega)$ together with a few elementary computations we can obtain that

$$\epsilon\varphi_\epsilon \overset{\epsilon}{\rightharpoonup} 0 \text{ in } H^1(\Omega)$$

Using the Periodic Unfolding Method in the spirit of Griso [41] we will be able to prove a very interesting Lemma, which would imply that, for $N \in \{2,3\}$, there exists $p > N$, such that

$$||\epsilon\varphi_\epsilon||_{H^1(\Omega)} \leq \epsilon^{\min\{\frac{1}{2},1-\frac{N}{p}\}} \tag{2.2.27}$$

Lemma 2.2.4. *Let $\Omega \subset \mathbb{R}^N$ be $C^{1,1}$ or convex. Consider the following problem,*

$$\begin{cases} -\nabla\cdot(A(\frac{x}{\epsilon})\nabla y_\epsilon) = h & \text{in } \Omega \\ y_\epsilon = g_\epsilon & \text{on } \partial\Omega \end{cases} \tag{2.2.28}$$

where $h \in L^2(\Omega)$, the coefficient matrix a satisfies the hypothesis of the first section, and we have that there exists $\phi_* \in W^{1,p}_{per}(Y)$ with $p > N$, and $z_\epsilon \in H^1(\Omega)$ such that

$$g_\epsilon(x) = \epsilon\phi_*(\frac{x}{\epsilon})z_\epsilon(x) \quad a.e. \quad \Omega. \tag{2.2.29}$$

Then if there exists $z \in H^1(\Omega)$ such that

$$z_\epsilon \rightharpoonup z \quad in \quad H^1(\Omega) \tag{2.2.30}$$

we have that there exists $y_* \in H^1_0(\Omega)$ such that

$$y_\epsilon \rightharpoonup y_* \quad in \quad H^1(\Omega) \tag{2.2.31}$$

and y_* satisfies

$$\begin{cases} \nabla \cdot (\mathcal{A}^{hom} \nabla y_*) = h & in \;\; \Omega \\ y_* = 0 & on \;\; \partial\Omega \end{cases} \tag{2.2.32}$$

and $\mathcal{A}^{hom}$ is the classical homogenized matrix defined at (2.0.2) in Section 1. Moreover if there exists $\alpha > 0$ such that

$$||z_\epsilon - z||_{L^2(\Omega)} \le C\epsilon^\alpha ||z||_{L^2(\Omega)} \tag{2.2.33}$$

then we have

$$||y_\epsilon - y_* - \epsilon\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j})||_{H^1(\Omega)} \le C\epsilon^{\min\{m, \beta(1-\frac{N}{p})\}} \left(||y_*||_{H^2(\Omega)} + ||z||_{H^1(\Omega)}\right) \tag{2.2.34}$$

where $\beta \doteq \min\{1, \alpha\}$, $m \doteq \min\{\alpha, \frac{1}{2}\}$, $\chi_j \in W_{per}(Y)$ are defined in (6.2.3) and Q_ϵ is defined at (2.0.6).

Proof. We could use Tartar's result concerning problems with weakly converging data in H^{-1} to prove (2.2.31) and (2.2.32), but we prefer to present here a different proof based on the Periodic unfolding Method developed in [22]. The method will give us the unfolded formulation for the limit problem and this in turn will help us, inspired by an idea of Griso (see [41]), to obtain the error estimate (2.2.34) for the solution y_ϵ. Homogenizing the data in problem (2.2.28) we obtain

$$\begin{cases} -\nabla \cdot \big(A(\frac{x}{\epsilon})\nabla(y_\epsilon - g_\epsilon)\big) = h + \nabla \cdot (A(\frac{x}{\epsilon})\nabla g_\epsilon) & \text{in } \;\Omega \\ y_\epsilon - g_\epsilon = 0 & \text{on } \;\partial\Omega \end{cases}$$

If we denote by $r_\epsilon = y_\epsilon - g_\epsilon$ we have

$$\begin{cases} -\nabla \cdot \big(A(\frac{x}{\epsilon})\nabla r_\epsilon\big) = h + \nabla \cdot (A(\frac{x}{\epsilon})\nabla g_\epsilon) & \text{in } \;\Omega \\ r_\epsilon = 0 & \text{on } \;\partial\Omega \end{cases} \tag{2.2.35}$$

Using r_ϵ as test function in (2.2.35) we obtain

$$\begin{aligned} ||r_\epsilon||_{H_0^1(\Omega)} &\le ||h||_{L^2(\Omega)} + ||A(\frac{x}{\epsilon})\nabla g_\epsilon||_{L^2(\Omega)} \le ||h||_{L^2(\Omega)} + ||\nabla g_\epsilon||_{L^2(\Omega)} \le \\ &\le ||h||_{L^2(\Omega)} + \epsilon||\phi_*(\frac{\cdot}{\epsilon})\nabla z_\epsilon(.)||_{L^2(\Omega)} + ||\nabla_y \phi_*(\frac{\cdot}{\epsilon}) z_\epsilon(.)||_{L^2(\Omega)} \le \\ &\le ||h||_{L^2(\Omega)} + C(||\phi_*||_{W^{1,p}(Y)} + 1)||z_\epsilon||_{H^1(\Omega)} \le C \end{aligned} \tag{2.2.36}$$

where for the last inequality above we used the assumptions on the matrix A and ϕ_*, Holder Inequality and (6.2.10) in the Appendix.

From (2.2.36) we have that there exists $y_* \in H_0^1(\Omega)$ such that on a subsequence still denoted by ϵ we have

$$r_\epsilon \rightharpoonup y_* \;\text{ weakly in }\; H_0^1(\Omega) \tag{2.2.37}$$

In fact we can see that

$$y_\epsilon \rightharpoonup y_* \text{ weakly in } H^1(\Omega) \tag{2.2.38}$$

Indeed we notice that

$$||g_\epsilon||_{L^2(\Omega)} = \epsilon||\phi_*(\frac{\cdot}{\epsilon})z_\epsilon(.)||_{L^2(\Omega)} \leq \epsilon||z_\epsilon||_{L^2(\Omega)} \tag{2.2.39}$$

and

$$\begin{aligned}||\nabla g_\epsilon||_{L^2(\Omega)} &\leq ||\nabla_y\phi_*(\frac{\cdot}{\epsilon})z_\epsilon(.)||_{L^2(\Omega)} + \epsilon||\phi_*(\frac{\cdot}{\epsilon})\nabla z_\epsilon(.)||_{L^2(\Omega)} \leq \\ &\leq C||\phi_*||_{W^{1,p}(Y)}||z_\epsilon||_{H^1(\Omega)} + \epsilon||\nabla z_\epsilon(.)||_{L^2(\Omega)} \leq C\end{aligned} \tag{2.2.40}$$

From (2.2.37), (2.2.39) and (2.2.40) we obtain (2.2.38). In order to prove that y_* is the solution of (2.2.32) we will consider first $v_\epsilon(x) = \psi(x)$, with $\psi \in \mathcal{D}(\Omega)$, as test functions in problem (2.2.35). We have,

$$\int_\Omega A(\frac{\cdot}{\epsilon})\nabla r_\epsilon \nabla\psi dx = \int_\Omega h\psi dx - \int_\Omega A(\frac{\cdot}{\epsilon})\nabla g_\epsilon \nabla\psi dx \tag{2.2.41}$$

We unfold (2.2.41) using Theorem 6.1.1 and we have

$$\begin{aligned}&\frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon\times Y} A(y)\mathcal{T}_\epsilon(\nabla r_\epsilon)\mathcal{T}_\epsilon(\nabla\psi)dxdy = \int_\Omega h\psi dx - \frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon\times Y} A(y)\mathcal{T}_\epsilon(\nabla g_\epsilon)\mathcal{T}_\epsilon(\nabla\psi)dxdy = \\ &= \int_\Omega h\psi dx - \frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon\times Y} A(y)\nabla_y\phi_*(y)\mathcal{T}_\epsilon(z_\epsilon)\mathcal{T}_\epsilon(\nabla\psi)dxdy - \frac{\epsilon}{|Y|}\int_{\tilde{\Omega}_\epsilon\times Y} A(y)\phi_*(y)\mathcal{T}_\epsilon(\nabla z_\epsilon)\mathcal{T}_\epsilon(\nabla\psi)dxdy\end{aligned} \tag{2.2.42}$$

Remark 2.2.5. *Using that $A \in L^\infty(Y)$, $\phi_* \in W^{1,p}(Y)$ and (2.2.30), it can be seen that the last term in the right hand side of (2.2.42) converges to zero when $\epsilon \to 0$.*

For the second integral in the right hand side of (2.2.42), Theorem 6.1.1 implies,

$$\frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon\times Y} A(y)\nabla_y\phi_*(y)\mathcal{T}_\epsilon(z_\epsilon)\mathcal{T}_\epsilon(\nabla\psi)dxdy \xrightarrow{\epsilon} \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)\nabla\psi dxdy \tag{2.2.43}$$

From (2.2.37), Remark 2.2.5), (2.2.43) and Theorem 6.1.1, we can pass at the limit when $\epsilon \to 0$ in (2.2.42), and obtain that there exists $\hat{y}_* \in L^2(\Omega, H^1_{per}(Y))$ such that

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(y)[\nabla_x y_* + \nabla_y\hat{y}_*]\nabla\psi dxdy = \int_\Omega h\psi dx - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)\nabla\psi dxdy \tag{2.2.44}$$

Next consider $v_\epsilon(x) = \epsilon\psi(x)\varphi(\frac{x}{\epsilon})$ with $\psi \in \mathcal{D}(\Omega)$ and $\varphi \in C^\infty_{per}(Y)$ as test functions in problem (2.2.35). Note that $v_\epsilon \rightharpoonup 0$ in $H^1(\Omega)$. We then have

$$\begin{aligned}&\int_\Omega A(\frac{x}{\epsilon})\nabla r_\epsilon\nabla v_\epsilon dx = \int_\Omega hv_\epsilon dx - \int_\Omega A(\frac{x}{\epsilon})\nabla g_\epsilon \nabla v_\epsilon dx \Leftrightarrow \\ &\Leftrightarrow \epsilon\int_\Omega A(\frac{x}{\epsilon})\nabla r_\epsilon\nabla\psi\varphi(\frac{x}{\epsilon})dx + \int_\Omega A(\frac{x}{\epsilon})\nabla r_\epsilon\psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx =\end{aligned}$$

$$= \int_{\Omega} hv_\epsilon dx - \epsilon \int_{\Omega} A(\frac{x}{\epsilon})\nabla g_\epsilon \nabla\psi\varphi(\frac{x}{\epsilon})dx - \int_{\Omega} A(\frac{x}{\epsilon})\nabla g_\epsilon \psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx. \quad (2.2.45)$$

We will first analyze the left hand side of (2.2.45). It is clear that

$$\epsilon \int_{\Omega} A(\frac{x}{\epsilon})\nabla r_\epsilon \nabla\psi\varphi(\frac{x}{\epsilon})dx \xrightarrow{\epsilon} 0 \quad (2.2.46)$$

For the second term of the left hand side in (2.2.45) we use property 3 in Theorem 6.1.1 and we have

$$\int_{\Omega} A(\frac{x}{\epsilon})\nabla r_\epsilon \psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx = \frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon \times Y} A(y)\mathcal{T}_\epsilon(\nabla r_\epsilon)\mathcal{T}_\epsilon(\psi)\nabla_y\varphi(y)dxdy. \quad (2.2.47)$$

Using property 5. in Theorem 6.1.1 we can pass to the limit in (2.2.47) and obtain,

$$\lim_{\epsilon\to 0}\int_{\Omega} A(\frac{x}{\epsilon})\nabla r_\epsilon \psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx = \frac{1}{|Y|}\int_{\Omega\times Y} A(y)[\nabla_x y_* + \nabla_y \hat{y}_*]\psi(x)\nabla_y\varphi(y)dxdy. \quad (2.2.48)$$

Next we will analyze the right hand side of (2.2.45). Easily can be proved that

$$\epsilon \int_{\Omega} A(\frac{x}{\epsilon})\nabla g_\epsilon \nabla\psi\varphi(\frac{x}{\epsilon})dx \xrightarrow{\epsilon} 0 \quad (2.2.49)$$

For the last integral in the right hand side of (2.2.45) we have

$$\int_{\Omega} A(\frac{x}{\epsilon})\nabla g_\epsilon \psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx = \epsilon \int_{\Omega} A(\frac{x}{\epsilon})\phi_*(\frac{x}{\epsilon})\nabla z_\epsilon \psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx +$$

$$+ \int_{\Omega} A(\frac{x}{\epsilon})\nabla_y\phi_*(\frac{x}{\epsilon})z_\epsilon(x)\psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx \quad (2.2.50)$$

Note that

$$\epsilon \int_{\Omega} A(\frac{x}{\epsilon})\phi_*(\frac{x}{\epsilon})\nabla z_\epsilon \psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx \xrightarrow{\epsilon} 0 \quad (2.2.51)$$

For the third integral in (2.2.50) we use property 3. in Theorem 6.1.1 and obtain

$$\int_{\Omega} A(\frac{x}{\epsilon})\nabla_y\phi_*(\frac{x}{\epsilon})z_\epsilon(x)\psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx = \frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon \times Y} A(y)\nabla_y\phi_*(y)\mathcal{T}_\epsilon(z_\epsilon)\mathcal{T}_\epsilon(\psi)\nabla_y\varphi(y)dxdy \quad (2.2.52)$$

Using Theorem 6.1.1 we can pass to the limit in (2.2.52)

$$\lim_{\epsilon\to 0}\int_{\Omega} A(\frac{x}{\epsilon})\nabla_y\phi_*(\frac{x}{\epsilon})z_\epsilon(x)\psi(x)\nabla_y\varphi(\frac{x}{\epsilon})dx = \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)\psi(x)\nabla_y\varphi(y)dxdy \quad (2.2.53)$$

From (2.2.46), (2.2.48), (2.2.49), (2.2.51) and (2.2.53) we can pass to the limit in (2.2.45) and obtain

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(y)[\nabla_x y_* + \nabla_y \hat{y}_*]\psi(x)\nabla_y\varphi(y)dxdy = -\frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)\psi(x)\nabla_y\varphi(y)dxdy \tag{2.2.54}$$

for all $\psi \in \mathcal{D}(\Omega)$ and $\varphi \in C^\infty_{per}(Y)$.

Summarizing, from (2.2.44) and (2.2.54) using the density of the tensor product $\mathcal{D}(\Omega) \times C^\infty_{per}(Y)$ in $L^2(\Omega, H^1_{per}(Y))$ we obtain the unfolded formulation of the limit problem, i.e.,

Find $(y_*, \hat{y}_*) \in H^1_0(\Omega) \times L^2(\Omega, H^1_{per}(Y))$ such that,

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(y)[\nabla_x y_* + \nabla_y \hat{y}_*](\nabla_x\psi(x) + \nabla_y\Phi(x,y))dxdy = \int_\Omega h\psi dx -$$

$$-\frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)((\nabla_x\psi(x) + \nabla_y\Phi(x,y))dxdy \tag{2.2.55}$$

for all$\psi \in \mathcal{D}(\Omega)$ and $\Phi \in L^2(\Omega, H^1_{per}(Y))$.

From (2.2.54) we will extract the limit problem verified by y_*. Indeed it can be easily checked that $\hat{y}_*(x,y)$ admits the following representation:

$$\hat{y}_*(x,y) = \chi_j(y)\frac{\partial y_*}{\partial x_j} - \phi_*(y)z(x) \tag{2.2.56}$$

where $\chi_j \in W_{per}(Y)$ are the corrector function defined in Section 1 at (2.0.3). Using (2.2.56) in (2.2.44) we obtain

$$\int_\Omega \mathcal{A}^{hom}\nabla y_*\nabla\psi dx = \int_\Omega h\psi dx \text{ for any } \psi \in \mathcal{D}(\Omega)$$

and this translates into:

$$\begin{cases} -\nabla\cdot(\mathcal{A}^{hom}\nabla y_*) = h & \text{in } \Omega \\ y_* \in H^1_0(\Omega) \end{cases} \tag{2.2.57}$$

where $\mathcal{A}^{hom}$ is defined at (2.0.2). Next, inspired by an idea of Griso [41], we will use the unfolded formulation (2.2.55) of the limit problem to obtain (2.2.34). We would like to make the observation that although a few of the steps we take in the proof are based on ideas of Griso, we decided to present the complete proof here, for the clarity of the exposition.

Let $\psi \in H^1_0(\Omega)$. From Theorem 6.1.2 in the Appendix we can see that there exists $\hat{\psi}_\epsilon \in L^2(\Omega, H^1_{per}(Y))$ such that the estimate (6.1.2) is satisfied. Therefore if we use the pair $(\psi, \hat{\psi}_\epsilon)$ as a test function in (2.2.55) we have

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(y)[\nabla_x y_* + \nabla_y \hat{y}_*](\nabla_x\psi(x) + \nabla_y\hat{\psi}_\epsilon)dxdy = \int_\Omega h\psi dx -$$

$$-\frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)(\nabla_x\psi(x)+\nabla_y\hat{\psi}_\epsilon)dxdy \tag{2.2.58}$$

Using (2.2.56), (2.2.58) becomes,

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(y)[\nabla_x y_* + \nabla_y\chi_j(y)\frac{\partial y_*}{\partial x_j} - \nabla_y\phi_*(y)z(x)](\nabla_x\psi(x)+\nabla_y\hat{\psi}_\epsilon)dxdy = \int_\Omega h\psi dx -$$

$$-\frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)((\nabla_x\psi(x)+\nabla_y\hat{\psi}_\epsilon)dxdy \tag{2.2.59}$$

Next consider ρ_ϵ the distance function defined in Proposition 6.1.3 in the Appendix. Using the fact that $y_* \in H^2(\Omega)$ and Proposition 6.1.3 from the Appendix we have,

$$|h_0 - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\rho_\epsilon[\nabla_x y_* + \nabla_y\chi_j(y)\frac{\partial y_*}{\partial x_j} - \nabla_y\phi_*(y)z(x)](\nabla_x\psi(x)+\nabla_y\hat{\psi}_\epsilon)dxdy| \leq$$

$$\leq C\epsilon^{\frac{1}{2}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.60}$$

with

$$h_0 = \int_\Omega h\psi dx - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)((\nabla_x\psi(x)+\nabla_y\hat{\psi}_\epsilon)dxdy \tag{2.2.61}$$

Note that

$$||\rho_\epsilon\nabla y_*||_{H^1(\Omega)} \leq C\epsilon^{-\frac{1}{2}}||y_*||_{H^2(\Omega)} \tag{2.2.62}$$

Indeed for $i \in \{1,..,N\}$ arbitrarily fixed we have

$$||\nabla_x\left(\rho_\epsilon\frac{\partial y_*}{\partial x_i}\right)||_{[L^2(\Omega)]^N} \leq ||\nabla_x\rho_\epsilon\frac{\partial y_*}{\partial x_i}||_{[L^2(\Omega)]^N} + ||\rho_\epsilon\nabla_x\left(\frac{\partial y_*}{\partial x_i}\right)||_{[L^2(\Omega)]^N} \leq$$

$$\leq ||\nabla_x\rho_\epsilon||_{[L^\infty(\hat{\Omega}_\epsilon)]^N}||\frac{\partial y_*}{\partial x_i}||_{L^2(\hat{\Omega}_\epsilon)} + ||\nabla_x\left(\frac{\partial y_*}{\partial x_i}\right)||_{[L^2(\Omega)]^N} \leq C\epsilon^{-\frac{1}{2}}||y_*||_{H^2(\Omega)}$$

where $\hat{\Omega}_\epsilon$ is defined in the Appendix before Proposition 6.1.3. Using Theorem 6.1.2) from the Appendix and (2.2.62) we can replace $\nabla_x\psi + \nabla_y\hat{\psi}_\epsilon$ by $\mathcal{T}_\epsilon$ in (2.2.60) and we obtain

$$|h_1 - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\rho_\epsilon[\nabla_x y_* + \nabla_y\chi_j(y)\frac{\partial y_*}{\partial x_j} - \nabla_y\phi_*(y)z(x)]\mathcal{T}_\epsilon(\nabla\psi)dxdy| \leq$$

$$\leq C\epsilon^{\frac{1}{2}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.63}$$

with

$$h_1 = \int_\Omega h\psi dx - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)z(x)\mathcal{T}_\epsilon(\nabla\psi)dxdy \tag{2.2.64}$$

Next, using inequalities 2 in Proposition 6.1.3 we can remove ρ_ϵ (2.2.63). Also, from property 6 in Proposition 6.1.3 we can replace $\nabla_x y_*$ with $M_Y^\epsilon(\nabla_x y_*)$, $\dfrac{\partial y_*}{\partial x_j}$ with $M_Y^\epsilon(\dfrac{\partial y_*}{\partial x_j})$ and $z(x)$ with $M_Y^\epsilon(z)$ in (2.2.63). Therefore we obtain

$$|h_1 - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)[M_Y^\epsilon(\nabla_x y_*) + \nabla_y\chi_j(y)M_Y^\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(y)M_Y^\epsilon(z)]\mathcal{T}_\epsilon(\nabla\psi)dxdy| \leq$$

$$\leq C\epsilon^{\frac{1}{2}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.65}$$

From (2.2.33) and Proposition 6.1.3 we can first replace z by z_ϵ and afterwards z_ϵ by $M_Y^\epsilon(z_\epsilon)$ in h_1 defined at (2.2.64) and (2.2.65) becomes

$$|h_2 - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)[M_Y^\epsilon(\nabla_x y_*) + \nabla_y\chi_j(y)M_Y^\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(y)M_Y^\epsilon(z)]\mathcal{T}_\epsilon(\nabla\psi)dxdy| \leq$$

$$\leq C\epsilon^{\min\{\alpha,\frac{1}{2}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) + C\epsilon||z_\epsilon||_{H^1(\omega)} \tag{2.2.66}$$

with

$$h_2 = \int_\Omega h\psi dx - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\nabla_y\phi_*(y)M_Y^\epsilon(z_\epsilon)\mathcal{T}_\epsilon(\nabla\psi)dxdy \tag{2.2.67}$$

Next note that

$$\epsilon||z_\epsilon||_{H^1(\omega)} \leq \epsilon^{\frac{1}{2}}||z||_{H^1(\omega)} \tag{2.2.68}$$

We also observe that

$$||M_Y^\epsilon(v)\psi(\frac{x}{\epsilon})||_{L^2(\hat{\Omega}_\epsilon)} \leq ||M_Y^\epsilon(v)||_{L^2(S_\epsilon)}||\psi(\frac{x}{\epsilon})||_{S_\epsilon} \leq \epsilon||v||_{L^2(\Omega)}||\psi||_{L^2(\Omega)} \tag{2.2.69}$$

for every $v \in L^2(\Omega)$ and $\psi \in L^2_{per}(Y)$, where

$$S_\epsilon = \bigcup_{\xi\in\Xi_\epsilon}(\epsilon\xi + \epsilon Y)$$

with $\Xi_\epsilon = \{(\epsilon\xi + \epsilon Y)\cap\hat{\Omega}_\epsilon\}$ and $\hat{\Omega}_\epsilon$ defined at Proposition 6.1.3 in the Appendix.

We used in (2.2.69) properties 4. and 5. of Proposition 6.1.3 in the Appendix.

From property 4. of Theorem 6.1.1 in the Appendix, together with (2.2.68) and (2.2.69, (2.2.66) becomes

$$|h_3 - \int_\Omega A(\frac{x}{\epsilon})[M_Y^\epsilon(\nabla_x y_*) + \nabla_y\chi_j(\frac{x}{\epsilon})M_Y^\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(\frac{x}{\epsilon})M_Y^\epsilon(z)]\nabla\psi dxdy| \leq$$

$$\leq C\epsilon^{\min\{\alpha,\frac{1}{2}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.70}$$

with

$$h_3 = \int_\Omega h\psi dx - \int_\Omega A(\frac{x}{\epsilon})\nabla_y\phi_*(\frac{x}{\epsilon})M_Y^\epsilon(z_\epsilon)\nabla\psi dxdy \tag{2.2.71}$$

Using property 6_1. of Proposition 6.1.3 we can replace $M_Y^\epsilon(\nabla_x y_*)$ with $\nabla_x y_*$ and from (2.2.69) we can introduce ρ_ϵ in front of $\nabla_y\chi_j(\frac{x}{\epsilon})M_Y^\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(\frac{x}{\epsilon})M_Y^\epsilon(z)$ in (2.2.70) and we have

$$|h_3 - \int_\Omega A(\frac{x}{\epsilon})\left[\nabla_x y_* + \rho_\epsilon\left(\nabla_y\chi_j(\frac{x}{\epsilon})M_Y^\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(\frac{x}{\epsilon})M_Y^\epsilon(z)\right)\right]\nabla\psi dxdy| \leq$$

$$\leq C\epsilon^{\min\{\alpha,\frac{1}{2}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.72}$$

Property 6_3 of Proposition 6.1.3 in the Appendix implies, (see [41]),

$$||(M_Y^\epsilon(v) - Q_\epsilon(v))\psi(\frac{x}{\epsilon})||_{L^2(\Omega)} \leq C\epsilon||v||_{H^1(\Omega)}||\psi||_{L^2(Y)} \tag{2.2.73}$$

for every $v \in H^1(\Omega)$ and $\psi \in L^2_{per}(Y)$.

Then we can replace $M_Y^\epsilon(z_\epsilon)$ with $Q_\epsilon(z_\epsilon)$ in h_3 defined at (2.2.71) and we can also replace $M_Y^\epsilon(\frac{\partial y_*}{\partial x_j})$ with $Q_\epsilon(\frac{\partial y_*}{\partial x_j})$ and $M_Y^\epsilon(z)$ with $Q_\epsilon(z)$ in (2.2.72). Therefore we obtain

$$|h_4 - \int_\Omega A(\frac{x}{\epsilon})\left[\nabla_x y_* + \rho_\epsilon\left(\nabla_y\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(\frac{x}{\epsilon})Q_\epsilon(z)\right)\right]\nabla\psi dxdy| \leq$$

$$\leq C\epsilon^{\min\{\alpha,\frac{1}{2}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.74}$$

with

$$h_4 = \int_\Omega h\psi dx - \int_\Omega A(\frac{x}{\epsilon})\nabla_y\phi_*(\frac{x}{\epsilon})Q_\epsilon(z_\epsilon)\nabla\psi dxdy \tag{2.2.75}$$

where we used (2.2.68) in (2.2.74) above.

Using Cauchy inequality, (6.2.10), the fact that $\phi_* \in W^{1,p}_{per}(Y)$ with $p > N$ and property 6 of Proposition 6.1.3 in the Appendix we can replace $Q_\epsilon(z_\epsilon)$ with z_ϵ in (2.2.75) and (2.2.74) becomes

$$|h_5 - \int_\Omega A(\frac{x}{\epsilon})\left[\nabla_x y_* + \rho_\epsilon\left(\nabla_y\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y\phi_*(\frac{x}{\epsilon})Q_\epsilon(z)\right)\right]\nabla\psi dxdy| \leq$$

$$\leq C\epsilon^{\min\{m,1-\frac{N}{p}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.76}$$

with $m = \min\{\alpha, \dfrac{1}{2}\}$ and

$$h_5 = \int_\Omega h\psi dx - \int_{\Omega\times Y} A(\frac{x}{\epsilon})\nabla_y\phi_*(\frac{x}{\epsilon})z_\epsilon\nabla\psi dxdy \tag{2.2.77}$$

Note that

$$\epsilon|\int_\Omega A(\frac{x}{\epsilon})\phi_*(\frac{x}{\epsilon})\nabla z_\epsilon\nabla\psi dx| \leq C\epsilon||\psi||_{H^1(\Omega)}||\phi_*||_{W^{1,p}(Y)}||z_\epsilon||H^1(\Omega) \tag{2.2.78}$$

Therefore we can introduce $\epsilon \int_\Omega A(\frac{x}{\epsilon})\phi_*(\frac{x}{\epsilon})\nabla z_\epsilon \nabla\psi dx$ in (2.2.77) and we have

$$|h_6 - \int_\Omega A(\frac{x}{\epsilon}) \left[\nabla_x y_* + \rho_\epsilon \left(\nabla_y \chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \nabla_y \phi_*(\frac{x}{\epsilon})Q_\epsilon(z)\right)\right] \nabla\psi dxdy| \leq$$

$$\leq C\epsilon^{\min\{m,1-\frac{N}{p}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.79}$$

with $m = \min\{\alpha, \frac{1}{2}\}$ and

$$h_6 = \int_\Omega h\psi dx - \int_{\Omega\times Y} A(\frac{x}{\epsilon})\nabla(\epsilon\phi_*(\frac{x}{\epsilon})z_\epsilon)\nabla\psi dxdy \tag{2.2.80}$$

Similarly as in Griso, [41], we can observe that

$$||\epsilon\frac{\partial\rho_\epsilon}{\partial x_j}Q_\epsilon(\frac{\partial v}{\partial x_i})\psi(\frac{\cdot}{\epsilon})||_{L^2(\Omega)} \leq ||\epsilon\frac{\partial\rho_\epsilon}{\partial x_j}||_{L^\infty(\hat{\Omega}_\epsilon)}||Q_\epsilon(\frac{\partial v}{\partial x_i})||_{L^2(\hat{\Omega}_\epsilon)}||\psi||_{L^2(Y)} \leq$$

$$\leq C\epsilon||v||_{H^2(\Omega)}$$

$$||\epsilon\rho_\epsilon\frac{\partial}{\partial x_j}Q_\epsilon(\frac{\partial v}{\partial x_i})\psi(\frac{\cdot}{\epsilon})||_{L^2(\Omega)} \leq \epsilon||\rho_\epsilon||_{L^\infty(\Omega)}||\frac{\partial}{\partial x_j}Q_\epsilon(\frac{\partial v}{\partial x_i})||_{L^2(\Omega)}||\psi||_{L^2(Y)} \leq$$

$$\leq C\epsilon||v||_{H^2(\Omega)} \tag{2.2.81}$$

for all $v \in H^2(\Omega)$ and $\psi \in L^2_{per}(Y)$.

Using the inequalities (2.2.81) in (2.2.79) we obtain

$$|h_7 - \int_\Omega A(\frac{x}{\epsilon}) \left[\nabla_x y_* + \nabla_x \left(\epsilon\rho_\epsilon \left(\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \phi_*(\frac{x}{\epsilon})Q_\epsilon(z)\right)\right)\right] \nabla\psi dxdy| \leq$$

$$\leq C\epsilon^{\min\{m,1-\frac{N}{p}\}}||\psi||_{H^1(\Omega)}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.82}$$

with $m = \min\{\alpha, \frac{1}{2}\}$ and h_7 is given by

$$h_7 = \int_\Omega h\psi dx - \int_{\Omega\times Y} A(\frac{x}{\epsilon})\nabla g_\epsilon \nabla\psi dxdy \tag{2.2.83}$$

where $g_\epsilon = \epsilon\phi_*(\frac{x}{\epsilon})z_\epsilon$.

If we consider $\psi = r_\epsilon - \left[y_* + \epsilon\rho_\epsilon\left(\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \phi_*(\frac{x}{\epsilon})Q_\epsilon(z)\right)\right]$ as a test function in the initial problem (2.2.35) from the elipticity of the matrix a we obtain

$$||\nabla r_\epsilon - \nabla_x \left[y_* + \epsilon\rho_\epsilon\left(\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \phi_*(\frac{x}{\epsilon})Q_\epsilon(z)\right)\right]||_{L^2(\Omega)} \leq$$

$$\leq C\epsilon^{\min\{m,1-\frac{N}{p}\}}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.84}$$

with $m = \min\{\alpha, \dfrac{1}{2}\}$ and where r_ϵ was defined at (2.2.35).

Using again (2.2.78) and property 2 of Proposition 6.1.3 in Appendix, we can remove ρ_ϵ from (2.2.84) and we have

$$||\nabla r_\epsilon - \nabla_x \left[y_* + \epsilon\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \epsilon\phi_*(\frac{x}{\epsilon})Q_\epsilon(z) \right] ||_{L^2(\Omega)} \leq$$

$$\leq C\epsilon^{\min\{m,1-\frac{N}{p}\}}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.85}$$

with $m = \min\{\alpha, \dfrac{1}{2}\}$. Note that

$$||\nabla g_\epsilon - \epsilon\nabla_x \left(\phi_*(\frac{\cdot}{\epsilon})Q_\epsilon(z) \right) ||_{L^2(\Omega)} \leq ||\epsilon\phi_*(\frac{\cdot}{\epsilon})(\nabla_x z_\epsilon - \nabla_x Q_\epsilon(z))||_{L^2(\Omega)} + ||\nabla_y \phi_*(\frac{\cdot}{\epsilon})(z_\epsilon - Q_\epsilon(z))||_{L^2(\Omega)}. \tag{2.2.86}$$

We ca easily see that (2.2.30), (2.2.33) and the properties of Q_ϵ (see Prop. 4.3 in the Appendix) imply that,

$$||z_\epsilon - Q_\epsilon(z)||_{L^2(\Omega)} \leq \epsilon^\beta ||z||_{H^1(\Omega)} \text{ and } ||\nabla_x z_\epsilon - \nabla_x Q_\epsilon(z)||_{L^2(\Omega)} \leq C \tag{2.2.87}$$

where $\beta \doteq \min\{1, \alpha\}$. Using (2.2.87) we obtain that,

$$||\epsilon\phi_*(\frac{\cdot}{\epsilon})\left(\nabla_x z_\epsilon - \nabla_x Q_\epsilon(z)\right)||_{L^2(\Omega)} \leq C\epsilon ||\phi_*||_{L^\infty(Y)}$$

and triangle inequality together with (6.2.10) give

$$||\nabla_y \phi_*(\frac{\cdot}{\epsilon})(z_\epsilon - Q_\epsilon(z))||_{L^2(\Omega)} \leq C\epsilon^{\beta(1-\frac{1}{N})} ||\phi_*||_{W^{1,p}(Y)} ||z||_{H^1(\Omega)}.$$

Using the properties of ϕ_* and the last two inequalities in (2.2.86) we obtain

$$||\nabla g_\epsilon - \epsilon\nabla_x \left(\phi_*(\frac{\cdot}{\epsilon})Q_\epsilon(z) \right) ||_{L^2(\Omega)} \leq C\epsilon^{\beta(1-\frac{1}{N})} ||z||_{H^1(\Omega)} \tag{2.2.88}$$

Inequality (2.2.88) used in (2.2.85) implies,

$$||\nabla y_\epsilon - \nabla_x \left[y_* + \epsilon\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) \right] ||_{L^2(\Omega)} \leq$$

$$\leq C\epsilon^{\min\{m,\beta(1-\frac{N}{p})\}}(||z||_{H^1(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.89}$$

where $\beta \doteq \min\{1, \alpha\}$ and $m = \min\{\alpha, \dfrac{1}{2}\}$. From (2.2.85) and (2.2.89) we obtain the statement of the Lemma. □

Then applying Lemma 2.2.4 with $h = 0$, $y_\epsilon = \epsilon\varphi_\epsilon$, $\phi_*(y) = \chi_{ij}(y)$, $z_\epsilon(x) = z(x) = \dfrac{\partial^2 u_0}{\partial x_i \partial x_j}$ we obtain that

$$||\epsilon\varphi_\epsilon||_{H^1(\Omega)} \leq C\epsilon^{\min\{\frac{1}{2},1-\frac{N}{p}\}}||u_0||_{H^3(\Omega)} \tag{2.2.90}$$

Using (2.2.90) in (2.2.26) we have

$$||u_\epsilon(.) - u_0(.) - \epsilon w_1(., \frac{.}{\epsilon}) + \epsilon\theta_\epsilon(.) - \epsilon^2\chi_{ij}(\frac{.}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{H^1(\Omega)} \leq C\epsilon^{\min\{\frac{3}{2},2-\frac{N}{p}\}}||u_0||_{H^3(\Omega)} \tag{2.2.91}$$

and this concludes the proof of Theorem 2.2.2 □

Remark 2.2.6. *It has been shown in [76] that the assumptions $\chi_j, \chi_{ij} \in W^{1,p}_{per}(Y)$ for some $p > N$ are implied by the conditions that the BMO semi-norm norm of the coefficients matrix a is small enough (see [76] for the precise statement). In a different work by M. Vogelius and Y.Y. Lin [52], it has been shown that one can have $\chi_j, \chi_{ij} \in W^{1,\infty}_{per}(Y)$ in the case of piecewise discontinuous matrix of coefficients when the discontinuities occur on certain smooth interfaces (see [52] for the precise statement). It is clear that the lack of smoothness in the matrix a and the fact that we only assume $u_0 \in H^3(\Omega)$ would not allow one to use neither Avellaneda compactness principle nor the maximum principle to bound φ_ϵ in the L^2.*

Corollary 2.2.7. *For $N = 2$ we could use a Meyers type regularity result and prove that there exists $p > 2$ such that $\chi_j, \chi_{ij} \in W^{1,p}_{per}(Y)$. Therefore Theorem 2.2.2 holds true in this case in the very general conditions that $u_0 \in H^3(\Omega)$ and $a \in L^\infty(Y)$.*

Remark 2.2.8. *We can see that, in the particular case when $p = +\infty$, the error estimate (2.2.90) will have order $O(\epsilon^{\frac{3}{2}})$ as is the case in [1], where they assume $u_0 \in W^{3,\infty}$ and $\chi_{ij} \in W^{1,\infty}(Y)$.*

Remark 2.2.9. *If one wants to remove the assumptions on χ_j, χ_{ij} and only assumes that the sequence $z_\epsilon, z \in W^{1,\infty}(\Omega)$ with $||z_\epsilon||_{W^{1,\infty}(\Omega)} \leq C||z||_{W^{1,\infty}(\Omega)}$, then using (2.2.33) we can first replace z by z_ϵ in h_1 defined at (2.2.64). Afterwards using property 5. of Theorem 6.1.1 in the Appendix we replace z_ϵ by $\mathcal{T}_\epsilon(z_\epsilon)$ in h_1 defined at (2.2.64) and (2.2.66). We continue the proof of Lemma 2.2.4 using similar arguments and if we have that α defined at (2.2.33) verifies $\alpha \geq \frac{1}{2}$ we obtain*

$$||\nabla y_\epsilon - \nabla g_\epsilon - \nabla_x \left[y_* + \epsilon\chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial y_*}{\partial x_j}) - \epsilon\phi_*(\frac{x}{\epsilon})Q_\epsilon(z) \right] ||_{L^2(\Omega)} \leq$$

$$\leq C\epsilon^{\frac{1}{2}}(||z||_{W^{1,\infty}(\Omega)} + ||y_*||_{H^2(\Omega)}) \tag{2.2.92}$$

Following similar steps as in the proof of Theorem 2.2.2 we obtain that the error estimate (2.2.90) will have order $O(\epsilon^{\frac{3}{2}})$.

2.3 A natural extra term in the first order corrector to the homogenized eigenvalue of a periodic composite medium

In this section we analyze the Dirichlet eigenvalues of an elliptic operator corresponding to a composite medium with periodic microstructure. This problem was initially studied in [59], for the case of C^∞ coefficients. We generalize their result to the case of L^∞ coefficients.

We will first state a simple consequence of Theorem 2.2.2 which will play a fundamental role further in our analysis.

Corollary 2.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex curvilinear polygon of class C^∞. Let $u_0 \in H^{2+r}(\Omega)$ with $r > 0$. In the conditions of Theorem 2.2.2, there exists a constant C_r independent of u_0 and ϵ such that*

$$||u_\epsilon(.) - u_0(.) - \epsilon w_1(., \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(.)||_{L^2(\Omega)} \leq C_r \epsilon^{1+r\min\{\frac{1}{2},1-\frac{N}{p}\}} ||u_0||_{H^{2+r}(\Omega)}$$

Proof. From Theorem 2.1.1, if $u_0 \in H^2(\Omega)$, we have

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(\cdot)||_{H_0^1(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)} \tag{2.3.1}$$

Indeed note that using the hypothesis on χ_j, χ_{ij} and the properties of Q_ϵ we have that

$$||\nabla w_1(\cdot, \frac{\cdot}{\epsilon}) - \nabla u_1(\cdot, \frac{\cdot}{\epsilon})||_{L^2(\Omega)} < \epsilon ||\chi_j||_{W^{1,p}(Y)} ||u_0||_{H^2(\Omega)} < C$$

where u_1 is defined by (2.0.13).

Also, using the definition of θ_ϵ and β_ϵ, we have

$$||\nabla\theta_\epsilon - \nabla\beta_\epsilon||_{L^2(\Omega)} < ||w_1(\cdot, \frac{\cdot}{\epsilon}) - u_1(\cdot, \frac{\cdot}{\epsilon})||_{H^1(\Omega)} < C$$

Using the last two inequalities in Theorem 2.1.1 we obtain (2.3.1). Next we may see that, for $u_0 \in H^3(\Omega)$, Theorem 2.2.2 immediately implies that

$$||u_\epsilon(.) - u_0(.) - \epsilon w_1(., \frac{\cdot}{\epsilon}) + \epsilon\theta_\epsilon(.)||_{L^2(\Omega)} \leq C\epsilon^{\min\{\frac{3}{2},2-\frac{N}{p}\}} ||u_0||_{H^3(\Omega)} \tag{2.3.2}$$

Using (2.3.1) and (2.3.2) together with a similar interpolation argument as in [59](see Theorem 2.4), we prove the statement of the Corollary. □

Next we will state the spectral problem and recall briefly the result obtained in [59]. On the domain $\Omega \subset \mathbb{R}^2$, we consider the spectral problem (2.0.19) associated with operator L_ϵ, i.e.,

$$\begin{cases} L_\epsilon v_\epsilon = -\nabla \cdot (A(\frac{x}{\epsilon})\nabla v_\epsilon(x)) = \lambda^\epsilon v_\epsilon & \text{in } \Omega \\ v_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3.3}$$

If we consider the eigenvalue problem for the operator $L \doteq -div(\mathcal{A}^{hom}\nabla)$ with $\mathcal{A}^{hom}$ defined at (2.0.2), i.e.,

$$\begin{cases} Lv = \lambda v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3.4}$$

then it is well known that for λ simple eigenvalue of (2.3.4), for each ϵ small enough, there exists λ^ϵ, an eigenvalue of (2.3.3) such that

$$\lambda^\epsilon \xrightarrow{\epsilon} \lambda$$

For any $f \in L^2(\Omega)$, we define $T_\epsilon f = u_\epsilon$ where $u_\epsilon \in H_0^1(\Omega)$ is the solution of $L_\epsilon u_\epsilon = f$ in Ω, and similarly $Tf = u_0$ with $u_0 \in H_0^1(\Omega)$ solution of $Lu_0 = f$. T_ϵ and T are compact and self adjoint operators from $L^2(\Omega)$ into $L^2(\Omega)$. Moreover $T_\epsilon \xrightarrow{\epsilon} T$ pointwise.

It can be seen that $\mu_k^\epsilon = \dfrac{1}{\lambda_k^\epsilon}$ are the eigenvalues of T_ϵ and $\mu_k = \dfrac{1}{\lambda_k}$ are the eigenvalues of T. From the definition of T_ϵ and T, the eigenvectors corresponding to μ_k^ϵ and respectively μ_k are the same as the eigenvectors of L_ϵ and L corresponding to λ_k^ϵ and respectively λ_k.

It is proved in [59] that if Ω is a bounded convex domain or bounded with a $C^{2,\beta}$ boundary we have that

$$|\lambda - \lambda_k^\epsilon| \leq C\epsilon \tag{2.3.5}$$

for ϵ sufficiently small. Moreover in the case of a smooth matrix of coefficients a, and for the eigenvectors of L in $H^{2+r}(\Omega)$, for some $r > 0$, using (2.0.8) and (2.0.9) and a result of Osborne [68], they obtain that

$$\lambda^{\epsilon_n} - \lambda = \epsilon_n \lambda \int_\Omega \bar{\theta}_{\epsilon_n} v dx + O(\epsilon_n^{1+r})$$

for any sequence $\epsilon_n \to 0$ and $\bar{\theta}_\epsilon$ defined by

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\bar{\theta}_\epsilon) = 0 \text{ in } \Omega \ , \ \bar{\theta}_\epsilon = \chi_j(\frac{x}{\epsilon})\frac{\partial v}{\partial x_j} \text{ on } \partial\Omega \tag{2.3.6}$$

The analysis in the case when the coefficients are only L^∞ employs a different argument and is presented in the following Lemma.

Lemma 2.3.2. *Let us assume the hypothesis of Corollary 2.3.1, λ a simple eigenvalue for L corresponding to the eigenfunction with $||v||_{L^2(\Omega)} = 1$, and $v \in H^{2+r}(\Omega)$ for some $r > 0$. Then,*

$$\lambda^{\epsilon_n} - \lambda = \epsilon_n \lambda \int_\Omega \bar{\theta}_{\epsilon_n} v dx + O(\epsilon_n^{1+r\min\{\frac{1}{2},1-\frac{N}{p}\}}) \tag{2.3.7}$$

for some subsequence $\epsilon_n \to 0$.

Proof. Using a result of Osborne [68], we can deduce as in [59] that

$$\frac{1}{\lambda} - \frac{1}{\lambda^{\epsilon_n}} = \langle (T - T_{\epsilon_n})v, v\rangle + O \tag{2.3.8}$$

Similarly as in [59] define $w_\epsilon = T_\epsilon v$ and using the definition of the operators T_ϵ, T observe that w_ϵ and $\frac{1}{\lambda}v$ solve the following boundary value problems in $H_0^1(\Omega)$,

$$L_\epsilon w_\epsilon = v \text{ and } L(\frac{1}{\lambda}v) = v.$$

In the conditions of lemma 2.3.2, using Corollary 2.3.3 we have that

$$||w_\epsilon(\cdot) - \frac{1}{\lambda}v(\cdot) - \frac{\epsilon}{\lambda}\chi_j(\frac{\cdot}{\epsilon})\frac{\partial v}{\partial x_j} + \frac{\epsilon}{\lambda}\bar{\theta}_\epsilon(\cdot)||_{L^2(\Omega)} = O(\epsilon^{1+\bar{r}})$$

where $\bar{r} = r\min\{\frac{1}{2}, 1 - \frac{N}{p}\} > 0$.

Following identical steps as in [59] (see Prop. 3.4) the statement of Lemma 2.3.2 follows. □

The fact that $\bar{\theta}_\epsilon$ is bounded in $L^2(\Omega)$ follows from $\chi_j \in W^{1,p}(Y)$, $p > 2$ and $v \in H^{2+r}(\Omega) \subset W^{1,\infty}(\Omega)$ for $r > 0$ and $\Omega \subset \mathbb{R}^2$. Using this and (2.3.5) from Lemma 2.3.2 we obtain that the result of Moskow and Vogelius (see Theorem 3.6) remains true in the general case of nonsmooth coefficients, i.e.,

Theorem 2.3.3. *In the hypothesis of Lemma 2.3.2 if λ_* is the limit of the sequence $\frac{(\lambda^{\epsilon_n} - \lambda)}{\epsilon_n}$ (as $\epsilon_n \to 0$) then there exists a function θ_*, weak limit point of the sequence $\bar{\theta}_\epsilon$ in $L^2(\Omega)$, so that*

$$\lambda_* = \lambda \int_\Omega \theta_* v dx$$

Conversely, if θ_* is a weak limit point of the sequence $\bar{\theta}_\epsilon$ in $L^2(\Omega)$ the there exists a sequence $\epsilon_n \to 0$ such that

$$\frac{(\lambda^{\epsilon_n} - \lambda)}{\epsilon_n} \to \lambda \int_\Omega \theta_* v dx$$

.

In the end we make the observation that the case when λ is a multiple eigenvalue can be treated similarly as in [59] (see Remark 3.7).

Chapter 3

Multiscale analysis of perforated materials

The periodic unfolding method (see [22]), as a simpler alternative to the two-scale convergence, was developed to study the limit behavior of periodic problems depending on a small parameter ε. As it turns out, the same philosophy applies to a whole range of periodic problems with small parameters, provided they have a specific period. The method is flexible enough to apply as well to almost any combinations of the preceding cases.

In this chapter, we present these various extensions and show how they apply to known results and allow for generalizations. This approach is significantly simpler than the original ones, both in spirit and in practice.

The plan of the chapter is as follows.

Section 3.1 is devoted to the presentation of various unfolding operators and their main properties. More precisely, in subsection 3.1.1, we recall the definition of the unfolding operator $\mathcal{T}_\varepsilon$ for the periodic case in fixed domains ([22] and [31]). In subsection 3.1.2, we present the unfolding operator adapted to the case of holes of size ε (with Neumann boundary condition) with period of same size (see [25] for details and applications). Subsection 3.1.3 introduces the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending of two small parameters ε and δ (corresponding to the scales ε and $\varepsilon\delta$) and was first introduced in a similar form in [19] and [20]. The following subsections deal again with an unfolding operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$ depending on the scales ε and $\varepsilon\delta$ when the latter occurs only on a layer. This approach never assumes the existence of an extension operator in the cells but is based on the Poincaré-Wirtinger inequality (subsections 3.1.1) and Sobolev-Poincaré-Wirtinger inequality (subsections 3.1.2 and 3.1.3). The latter requires that the dimension N be larger than 2.

The remainder of the chapter is devoted to the application to various linear problems in perforated domains and with oscillating coefficients. For simplicity, we assume a homogeneous Dirichlet boundary condition on the outer boundary of the domain, but more general boundary conditions can be handled provided the outer boundary is Lipschitz and the perforations do not intersect it. In each case, we obtain both the unfolded and the classical (standard) form for the limit problem. The operator $\mathcal{T}_\varepsilon$ allows to homogenize the coefficients of the differential operators, whereas the operators $\mathcal{T}_{\varepsilon,\delta}$ (or $\mathcal{T}^{bl}_{\varepsilon,\delta}$,...) generates the "strange terms" in the limit.

Section 3.2 concerns the homogenization of elliptic problems with oscillating coefficients, for volume ε-periodically distributed small holes of size $\varepsilon\delta$ with Dirichlet condition. These results are well known for the Laplace operator, with the appearance of the "strange term" (see [26] and references therein). For the case of oscillating coefficients, we refer to [29] where H-convergence is used. It should be noted that for technical reasons, our method fails to apply in dimension $N = 2$.

Section 3.3 considers small perforations of size $\varepsilon\delta$ which are distributed ε-periodically in a layer of thickness ε. It generalizes the results of [73], [61] and [26] to the case of oscillating coefficients.

Section 3.4 deals with the Neumann sieve problem with zero thickness and oscillating coefficients. For the case of constant coefficients, we refer the reader to [6], [30], [60], [72], [3] and [66]. In Section 3.5, the case of the thick sieve is treated (for which we refer to [35] for the case of the Laplace operator). The unfolding method was applied for the first time for sieve problems, in [63] also in the case of an operator with constant coefficients.

To conclude this section, we would like to point out that using the various unfolding operators introduced in this chapter, one can treat any combination of the previous problems, for instance, a medium with $\varepsilon-$size Neumann perforations and $\varepsilon\delta-$size Dirichlet holes in the bulk (see Figure 3.10), or even a thick sieve in such a medium. This will be presented in a forthcoming paper which will also include the proof of convergence for the energies.

3.1 The periodic unfolding operator

In this section we recall the general properties of the periodic unfolding operator introduced in [22] and include variants and generalizations, all based on the technique of unfolding. In particular, we introduce the notion of **unfolding criterion for integrals** (in short **u.c.i.**), in order to simplify the proofs where unfolding is used.

Let Y be the unit cube of $\mathbb{R}^N$ centered in the origin, $Y \doteq \left]-\frac{1}{2},\frac{1}{2}\right[^N$ (more general sets Y having the paving property in $\mathbb{R}^N$ can also be used, cf. [32]). We consider the periodical net on $\mathbb{R}^N$ (i.e. the subgroup $\mathbb{R}^N$) and all the corresponding translates of Y. By analogy with the one-dimensional case, to each $x \in \mathbb{R}^N$ we can associate its integer part, $[x]_Y$ belonging to the net, such that $x - [x]_Y \in Y$, the latter being its fractional part respectively, i.e, $\{x\}_Y = x - [x]_Y$ (see figure 3.1). These definitions are ambiguous, but only on a set of measure zero, which is enough for our purpose.

Therefore we have

$$x = \varepsilon \left\{\frac{x}{\varepsilon}\right\}_Y + \varepsilon \left[\frac{x}{\varepsilon}\right]_Y \quad \text{for any } x \in \mathbb{R}^N.$$

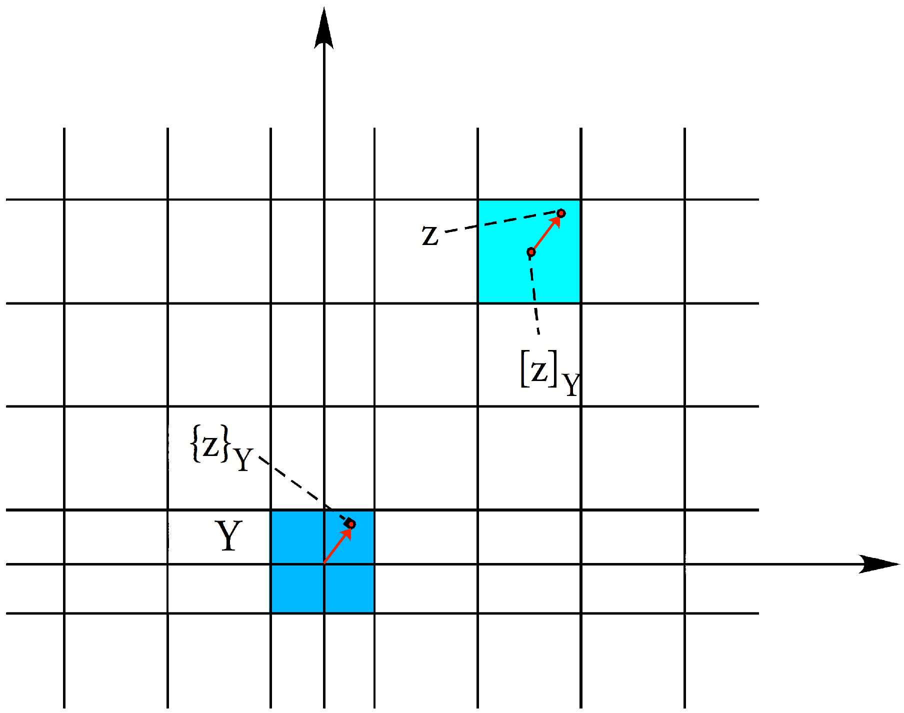


Figure 3.1: **The basic decomposition**

Let Ω be open and bounded in $\mathbb{R}^N$. We use the following notations

$$\widehat{\Omega}_\varepsilon = \left\{ x \in \Omega, \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon Y \right) \subset \Omega \right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon. \tag{3.1.1}$$

The set $\widehat{\Omega}_\varepsilon$ is the smallest finite union of εY cells contained in Ω, while Λ_ε is the subset of Ω containing the parts from εY cells intersecting the boundary $\partial\Omega$ (See Figure 3.2).

3.1.1 The case of fixed domains: the operator $\mathcal{T}_\varepsilon$

We recall here the definition of the unfolding operator and its main properties (for details and proofs we refer the reader to [22] anf [31]).

Definition 3.1.1. *For $\phi \in L^p(\Omega)$, the unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$ is defined as follows:*

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon \Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \text{if } (x,y) \in (\Omega \setminus \Lambda_\varepsilon) \times Y, \\ 0 & \text{if } (x,y) \in \ \Lambda_\varepsilon \times Y. \end{cases}$$

Theorem 3.1.2. (Properties of the operator $\mathcal{T}_\varepsilon$)
1. For any $v, w \in L^p(\Omega)$, $\mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\mathcal{T}_\varepsilon(w)$.

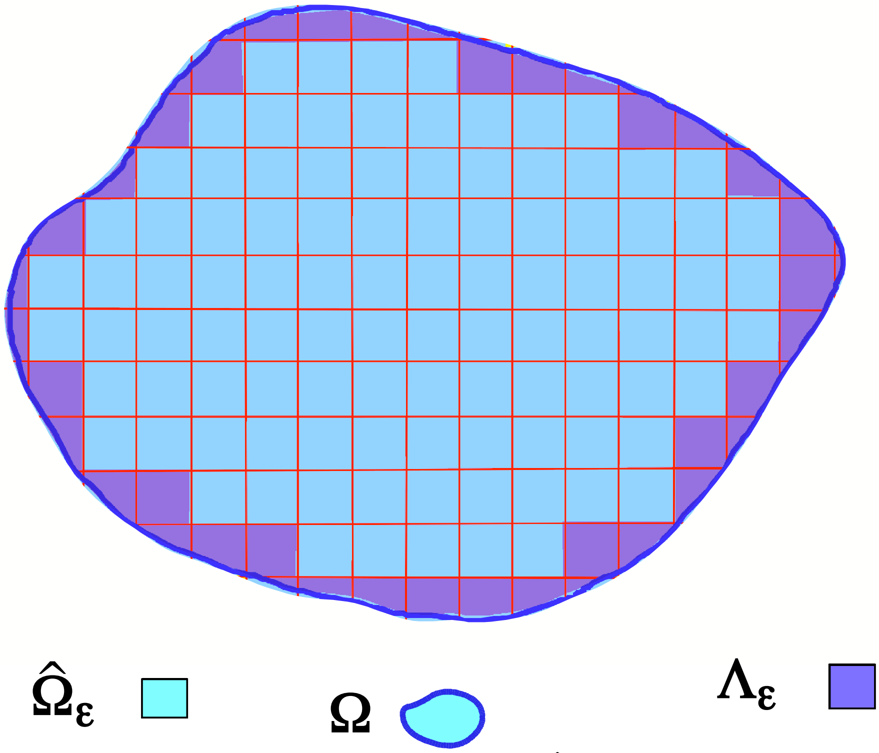


Figure 3.2: **The sets Ω, $\widehat{\Omega}_\varepsilon$ and Λ_ε**

2. *For any $w \in L^p(\Omega)$, one has the following "exact integration" formula:*

$$\int_{\Omega\times Y} \mathcal{T}_\varepsilon(w)(x,y)\, dx\, dy = \int_\Omega w(x)\, dx - \int_{\Lambda_\varepsilon} w(x)\, dx = \int_{\widehat{\Omega}_\varepsilon} w(x)\, dx.$$

3. *For any $u \in L^1(\Omega)$,*

$$\int_{\Omega\times Y} |\mathcal{T}_\varepsilon(u)|\, dxdy \leq \int_\Omega |u|\, dx.$$

4. *For any $u \in L^1(\Omega)$,*

$$\left| \int_\Omega u dx - \int_{\Omega\times Y} \mathcal{T}_\varepsilon(u)\, dxdy \right| \leq \int_{\Lambda_\varepsilon} |u| dx. \tag{3.1.2}$$

5. *Let $\{w_\varepsilon\} \subset L^2(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^2(\Omega)$. Then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w \text{ strongly in } L^2(\Omega \times Y).$$

6. *Let $w_\varepsilon \rightharpoonup w$ weakly in $H^1(\Omega)$. Then, there exists a subsequence and $\widehat{w} \in L^2\left(\Omega; H^1_{per}(Y)\right)$ such that*

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla_x w + \nabla_y \widehat{w} \text{ weakly in } L^2(\Omega \times Y).$$

Property 4 shows that any integral of a function w on Ω, is "almost equivalent" to the integral of its unfolded on $\Omega \times Y$, the "integration defect" arises only from the cells intersecting the boundary $\partial\Omega$ and is controlled by the right hand side integral in (3.1.2).

The following proposition, that we call **unfolding criterion for integrals (u.c.i.)**, is very useful tool when treating homogenization problems.

Proposition 3.1.3. (u.c.i.) *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |w_\varepsilon|\, dx \to 0,$$

then

$$\int_\Omega w_\varepsilon \, dx - \int_{\Omega\times Y} \mathcal{T}_\varepsilon(w_\varepsilon)\, dxdy \to 0.$$

Based on this result, in order to simplify the proofs in the sequel, we introduce the following notation:

Notation 3.1.4. *If $\{w_\varepsilon\}$ is a sequence satisfying u.c.i., we write*

$$\int_\Omega w_\varepsilon dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(w_\varepsilon)\, dxdy.$$

Corollary 3.1.5. *Let $\{u_\varepsilon\}$ be bounded in $L^2(\Omega)$ and $\{v_\varepsilon\}$ be bounded in $L^p(\Omega)$ with $p > 2$. Then we have*

$$\int_\Omega u_\varepsilon v_\varepsilon dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\, dxdy.$$

We end this subsection with the notion of local average of a function.

Definition 3.1.6. *The local average $M^\varepsilon_Y : L^p(\Omega) \mapsto L^p(\Omega)$, is defined for any ϕ in $L^p(\Omega)$, $1 \le p < \infty$, by*

$$M^\varepsilon_Y(\phi)(x) \doteq \int_Y \mathcal{T}_\varepsilon(\phi)(x,y)\, dy.$$

Remark 3.1.7. *The function $M^\varepsilon_Y(\phi)$ is indeed a local average, since*

$$M^\varepsilon_Y(\phi)(x) = \int_Y \mathcal{T}_\varepsilon(\phi)(x,y)\, dy = \begin{cases} \dfrac{1}{\varepsilon^N} \displaystyle\int_{\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon Y} \phi(\zeta)\, d\zeta, & \text{if } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{if } x \in \Lambda_\varepsilon. \end{cases}$$

Remark 3.1.8. *Note that $\mathcal{T}_\varepsilon(M^\varepsilon_Y(\phi)) = M^\varepsilon_Y(\phi)$ on the set $\Omega \times Y$.*

The next proposition, which will be frequently used as well, is classical:

Proposition 3.1.9. *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$ where $1 \le p < \infty$. Then we have*

$$M^\varepsilon_Y(w_\varepsilon) \to w \quad \text{strongly in } L^p(\Omega).$$

3.1.2 Unfolding in domains with volume-distributed "small" holes: the operator $\mathcal{T}_{\varepsilon,\delta}$

In Section 3.2, we will consider domains with holes of size $\varepsilon\delta$ (with $\delta \to 0$ with ε) and εY-periodically distributed. More precisely (see Figure 3.3), for a given open $B \subset\subset Y$ we denote $Y^*_\delta = Y \setminus \delta \overline{B}$ and define the perforated domain $\Omega^*_{\varepsilon,\delta}$ as

$$\Omega^*_{\varepsilon,\delta} = \Big\{ x \in \Omega, \text{ such that} \Big\{ \frac{x}{\varepsilon} \Big\} \in Y^*_\delta \Big\}. \tag{3.1.3}$$

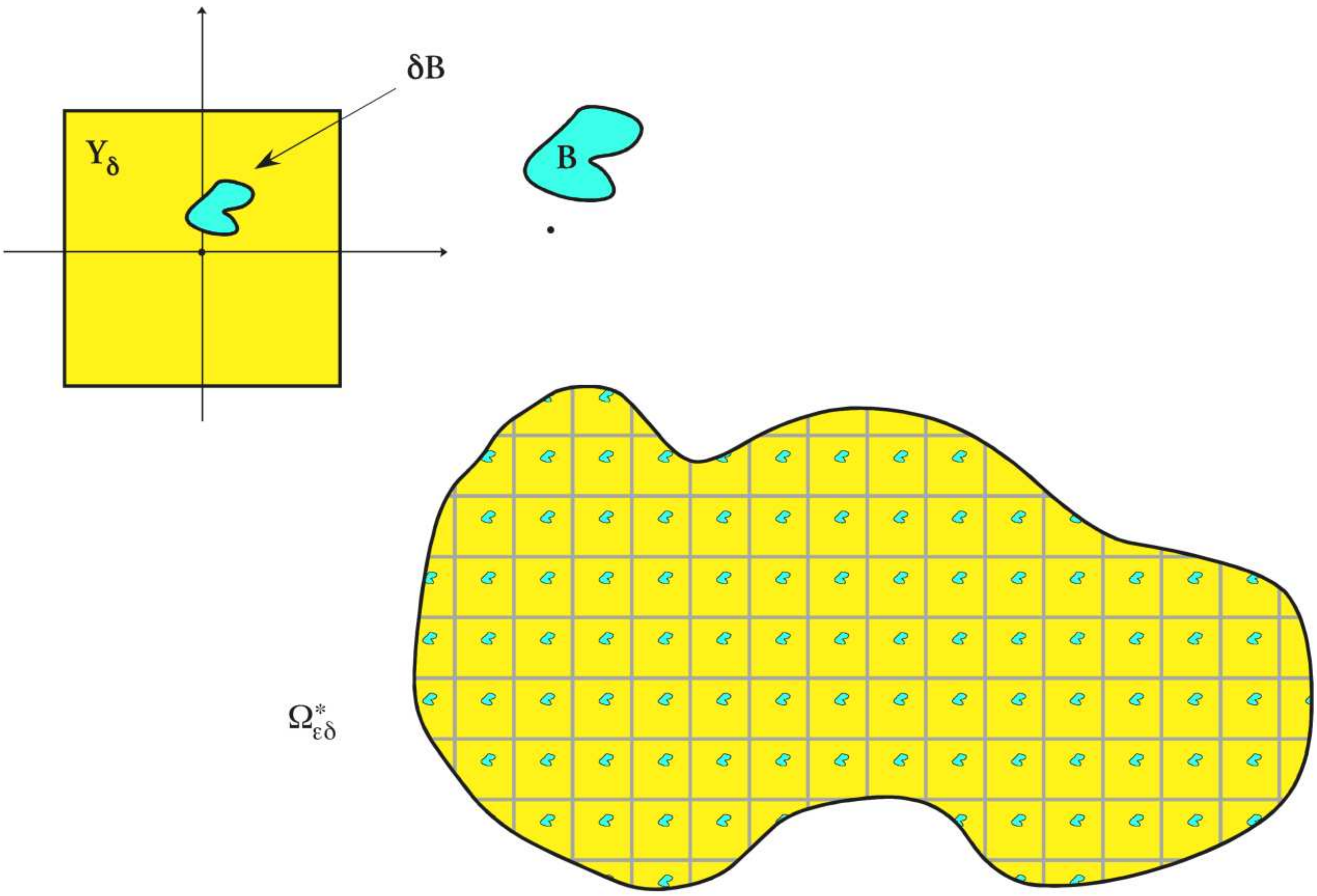


Figure 3.3: **The sets B and Y^*_δ and the corresponding $\Omega^*_{\varepsilon,\delta}$**

This geometry of domains with "small" holes requires another unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending on both parameters ε and δ. In the next sections, we will be concerned by functions $v_{\varepsilon,\delta}$ that vanish on the whole boundary of the perforated domain $\Omega^*_{\varepsilon,\delta}$, namely belonging to the space $H^1_0(\Omega^*_{\varepsilon,\delta})$. They are naturally extended by zero to the whole of Ω and these extensions denoted $\widetilde{v}_{\varepsilon,\delta}$ are functions in $H^1_0(\Omega)$. This justifies the introduction of $\mathcal{T}_{\varepsilon,\delta}$ on the fix domain Ω (but keeping in mind that our aim will be to apply it to $\widetilde{v}_{\varepsilon,\delta}$).

Definition 3.1.10. *For $\phi \in L^p(\Omega)$, $p \in [1, \infty[$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ is defined by*

$$\mathcal{T}_{\varepsilon,\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(x, \delta z) & \textit{if } (x,z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y, \\ 0 & \textit{otherwise.} \end{cases}$$

The following results follow directly from Theorem 2.2 by using the change of variable $z = (1/\delta)y$.

Theorem 3.1.11. (Properties of the operator $\mathcal{T}_{\varepsilon,\delta}$)
1. For any $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$.
2. For any $u \in L^1(\Omega)$, one has

$$\delta^N \int_{\Omega \times \mathbb{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)| \, dxdz \leq \int_\Omega |u| \, dx.$$

3. For any $u \in L^2(\Omega)$,

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega \times \mathbb{R}^N)} \leq \frac{1}{\delta^N} \|u\|^2_{L^2(\Omega)}.$$

4. For any $u \in L^1(\Omega)$,

$$\left| \int_\Omega u dx - \delta^N \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u) \, dxdz \right| \leq \int_{\Lambda_\varepsilon} |u| \, dx.$$

5. Let $u \in H^1(\Omega)$. Then

$$\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta} \nabla_z \big(\mathcal{T}_{\varepsilon,\delta}(u)\big) \quad \textit{in} \quad \Omega \times \frac{1}{\delta}Y.$$

Suppose $N \geq 3$, set $2^* = 2N/(N-2)$ and denote the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$ by C.
6. Let ω be open and bounded in $\mathbb{R}^N$. Then the following estimates hold:

$$\|\nabla_z \big(\mathcal{T}_{\varepsilon,\delta}(u)\big)\|^2_{L^2(\Omega \times \frac{1}{\delta}Y)} \leq \frac{\varepsilon^2}{\delta^{N-2}} \|\nabla u\|^2_{L^2(\Omega)}, \tag{3.1.4}$$

$$\|\mathcal{T}_{\varepsilon,\delta}\big(u - M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega; L^{2^*}(\mathbb{R}^N))} \leq \frac{C\varepsilon^2}{\delta^{N-2}} \|\nabla u\|^2_{L^2(\Omega)}, \tag{3.1.5}$$

and

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega \times \omega)} \leq \frac{2C\varepsilon^2}{\delta^{N-2}} \|\nabla u\|^2_{L^2(\Omega)} + 2|\,\omega\,| \; \|u\|^2_{L^2(\Omega)}.$$

7. Let $\{w_{\varepsilon,\delta}\}$ be a sequence of functions in $H^1(\Omega)$ which converges weakly to some w_0 when both ε and δ go to zero. Then, up to an subsequence, there is U in $L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$

and W *in* $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^2(\Omega \times \mathbb{R}^N)$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M_Y^\varepsilon(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big)\, 1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^N),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)).$$

Remark 3.1.12. *The use of the Poincaré-Wirtinger inequality in place of the Sobolev-Poincaré-Wirtinger inequality in estimate (3.1.5) gives*

$$\|\mathcal{T}_{\varepsilon,\delta}\big(u - M_Y^\varepsilon(u)\big)\|^2_{L^2(\Omega'\times\mathbb{R}^N)} \leq \frac{1}{\delta^2}\,\frac{C'\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

where C' *is the Poincaré-Wirtinger constant of* Y*. This estimate is not compatible with (3.1.4).*

Concerning the integral formulas, we have the following results, similar to those of the previous subsection.

Proposition 3.1.13. (u.c.i.) *If* $\{w_\varepsilon\}$ *is a sequence in* $L^1(\Omega)$ *satisfying*

$$\int_{\Lambda_\varepsilon} |w_\varepsilon|\, dx \to 0,$$

then

$$\int_\Omega w_\varepsilon\, dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(w_\varepsilon)\, dxdz.$$

Corollary 3.1.14. *Let* $\{u_\varepsilon\}$ *be bounded in* $L^2(\Omega)$ *and* $\{v_\varepsilon\}$ *be bounded in* $L^p(\Omega)$ *with* $p > 2$*. Then*

$$\int_\Omega u_\varepsilon v_\varepsilon dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u_\varepsilon)\mathcal{T}_{\varepsilon,\delta}(v_\varepsilon)dxdz.$$

3.1.3 The boundary-layer unfolding operator: the operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$

For sieve-type problems (Section 3.3 and 3.4 below), we consider the case of holes of size $\varepsilon\delta$, distributed in Σ'_ε, a layer of thickness ε parallel to a hyperplane in the open domain Ω in $\mathbb{R}^N$. We denote $x' \doteq (x_1, \cdots, x_{N-1})$, $\Pi \doteq \{x_N = 0\}$ and set $\Sigma = \Pi \cap \Omega$.

The layer Σ'_ε is defined as:

$$\Sigma'_\varepsilon = \Omega \cap \Big\{x; |x_N| < \frac{\varepsilon}{2}\Big\},$$

and by analogy with the (3.1.1), we introduce the corresponding sets

$$\widehat{\Sigma}'_\varepsilon = \Big\{ x \in \Sigma'_\varepsilon, \Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon Y\Big) \subset \Sigma'_\varepsilon \Big\}, \quad \Lambda'_\varepsilon = \Sigma'_\varepsilon, \backslash \widehat{\Sigma}'_\varepsilon,$$

and denote $\widehat{\Sigma}_\varepsilon = \widehat{\Sigma}'_\varepsilon \cap \Pi$.

The set $\widehat{\Sigma}'_\varepsilon$ is the smallest union of εY cells contained in Σ'_ε (see figure 3.4.)

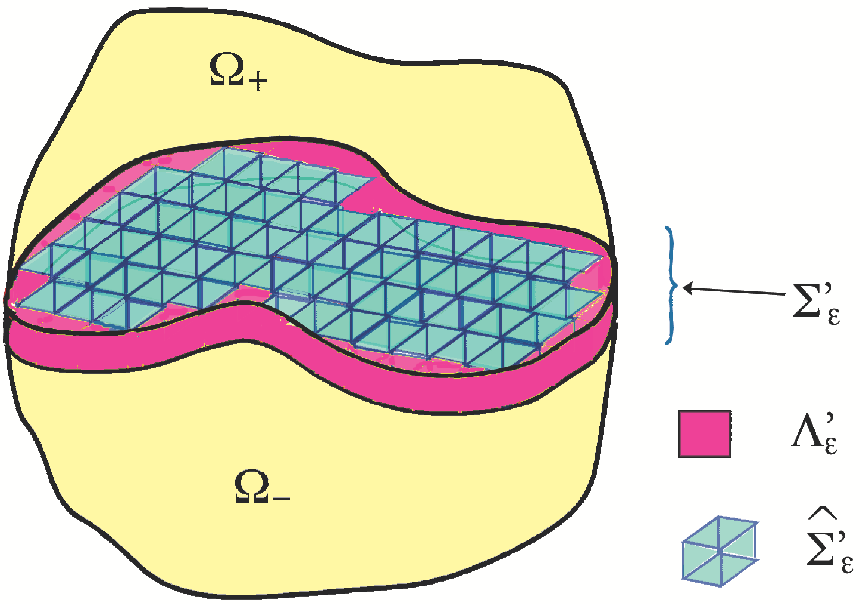


Figure 3.4: **The sets $\Sigma'_\varepsilon, \widehat{\Sigma}_\varepsilon$ and Λ'_ε**

Definition 3.1.15. *For $\phi \in L^p(\Sigma'_\varepsilon)$, $p \in [1, \infty[$ the unfolding operator $\mathcal{T}^{bl}_{\varepsilon,\delta} : L^p(\Sigma'_\varepsilon) \to L^p(\Sigma \times \mathbb{R}^N)$ is defined by*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)(x', z) = \begin{cases} \phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y + \varepsilon\delta z\Big) & \text{if } (x', z) \in \widehat{\Sigma}_\varepsilon \times \frac{1}{\delta}Y, \\ 0 & \text{otherwise .} \end{cases}$$

This operation, designed to capture the contribution of the barriers in the limit process, was originally used in [63].

We also introduce the notion of local average related to the hyperplane Σ.

Definition 3.1.16. *The local average $M^{\varepsilon,bl}_Y : L^p(\Sigma'_\varepsilon) \mapsto L^p(\Sigma)$, is defined for any ϕ in $L^p(\Sigma'_\varepsilon)$, $1 \le p < \infty$, by*

$$M^{\varepsilon,bl}_Y(\phi)(x') = \delta^N \int_{\frac{1}{\delta}Y} \mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)(x', z)\, dz = \begin{cases} \frac{1}{\varepsilon^N} \int_{\varepsilon[\frac{x'}{\varepsilon}] + \varepsilon Y} \phi(\zeta)\, d\zeta, & \text{if } x' \in \widehat{\Sigma}_\varepsilon, \\ 0 & \text{if } x' \in \Sigma \setminus \widehat{\Sigma}_\varepsilon. \end{cases}$$

Remark 3.1.17. *Since elements of $L^p(\Sigma)$ can be considered as functions of $L^p(\Sigma'_\varepsilon)$, $M_Y^{\varepsilon,bl}$ can be applied to them. With this convention, $\mathcal{T}^{bl}_{\varepsilon,\delta}(M_Y^{\varepsilon,bl}(\phi)) = M_Y^{\varepsilon,bl}(\phi)$ on the set Σ.*

We also have an equivalent of Proposition 3.1.9.

Proposition 3.1.18. *Let w_ε be a sequence such that $w_\varepsilon \rightharpoonup w$ weakly in $H^1(\Omega)$. Then*

$$M_Y^{\varepsilon,bl}(w_\varepsilon) \to w_{|\Sigma} \quad \textit{strongly in } L^2(\Sigma).$$

It is easy to check that most of the results stated in the previous subsection extend to $\mathcal{T}^{bl}_{\varepsilon,\delta}$.

Theorem 3.1.19. (Properties of the operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$)
1. For any $v, w \in L^p(\Sigma'_\varepsilon)$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(vw) = \mathcal{T}^{bl}_{\varepsilon,\delta}(v)\mathcal{T}^{bl}_{\varepsilon,\delta}(w).$$

2. For any $u \in L^1(\Sigma'_\varepsilon)$,

$$\varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(u)\; dxdz = \int_{\widehat{\Sigma}'_\varepsilon} u\; dx, \;\; \textit{and}$$

$$\varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} |\mathcal{T}^{bl}_{\varepsilon,\delta}(u)|\; dxdz \leq \int_{\Sigma'_\varepsilon} |u|\; dx.$$

3. For any $u \in L^2(\Sigma'_\varepsilon)$,

$$\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\|^2_{L^2(\Sigma\times\mathbb{R}^N)} \leq \frac{1}{\varepsilon\delta^N}\|u\|^2_{L^2(\Sigma'_\varepsilon)}.$$

4. For any $u \in L^1(\Sigma'_\varepsilon)$, one has

$$\left|\int_{\Sigma'_\varepsilon} u\; dx - \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(u)\; dxdz\right| \leq \int_{\Lambda'_\varepsilon} |u|\; dx.$$

5. Let u be in $H^1(\Sigma'_\varepsilon)$. Then,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big) \quad \textit{in} \quad \Sigma \times \frac{1}{\delta}Y.$$

Suppose $N \geq 3$, set $2^* = 2N/(N-2)$ and denote the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$ by C.
6. Let ω be open and bounded in $\mathbb{R}^N$. Then the following estimates hold:

$$\|\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big)\|^2_{L^2(\Sigma\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_\varepsilon)},$$

$$\|\mathcal{T}^{bl}_{\varepsilon,\delta}\big(u - M_Y^{\varepsilon,bl}(u)\big)\|^2_{L^2(\Sigma;L^{2^*}(\mathbb{R}^N))} \leq \frac{C\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_\varepsilon)},$$

and

$$\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\|^2_{L^2(\Sigma\times\omega)} \leq 2\frac{C\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_\varepsilon)} + 2|\,\omega\,|\,\|u\|^2_{L^2(\Sigma'_\varepsilon)}.$$

7. Let $w_{\varepsilon,\delta}$ be in $H^1(\Sigma'_\varepsilon)$ such that $\|\nabla w_{\varepsilon,\delta}\|_{L^2(\Sigma'_\varepsilon)}$ is bounded. Then, up to a subsequence, there exists U in $L^2(\Sigma; L^2_{loc}(\mathbb{R}^N))$ and W in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Sigma\times\mathbb{R}^N)$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M^\varepsilon_Y(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2(\Sigma; L^{2^*}(\mathbb{R}^N)),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big)\,1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in } L^2(\Sigma\times\mathbb{R}^N),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)).$$

Proposition 3.1.20. (u.c.i.) *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Sigma'_\varepsilon)$ satisfying*

$$\int_{\Lambda'_\varepsilon} |w_\varepsilon|\, dx \to 0,$$

then

$$\int_{\Sigma'_\varepsilon} w_\varepsilon\, dx \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\simeq} \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(w_\varepsilon)\, dxdz.$$

Corollary 3.1.21. *Let $\{u_\varepsilon\} \subset L^2(\Sigma'_\varepsilon)$ and $\{v_\varepsilon\} \subset L^p(\Sigma'_\varepsilon)$ with $p > 2$, such that $\|u_\varepsilon\|_{L^2(\Sigma'_\varepsilon)}$ and $\|v_\varepsilon\|_{L^p(\Sigma'_\varepsilon)}$ are bounded independently of ε. Then*

$$\int_{\Sigma'_\varepsilon} u_\varepsilon v_\varepsilon\, dx \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\simeq} \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(u_\varepsilon)\mathcal{T}^{bl}_{\varepsilon,\delta}(v_\varepsilon)\, dxdz.$$

For sieve problems, there is a need to distinguish between the subdomains above and below Σ. Set

$$\Omega_+ = \mathbb{R}^N_+ \cap \Omega, \quad \Omega_- = \mathbb{R}^N_- \cap \Omega, \quad Y_+ = \mathbb{R}^N_+ \cap Y, \quad Y_- = \mathbb{R}^N_- \cap Y.$$

We suppose that the two domains Ω_+ and Ω_- have a Lipschitz boundary.

For simplicity, we will make the convention that all the results stated for Ω_+, are true also for Ω_- unless specified otherwise. For any function u defined in Ω, we denote by u^+ its restriction to the domain Ω_+, i.e., $u^+ \equiv u|_{\Omega_+}$. Analogously, $u^- \equiv u|_{\Omega_-}$.

The corresponding definitions and propositions are the following:

Definition 3.1.22. *The local average $M^{\varepsilon,bl}_{Y_\pm} : L^p(\Sigma'_{\varepsilon\pm}) \mapsto L^p(\Sigma)$, is defined for any ϕ in $L^p(\Sigma'_{\varepsilon\pm})$, $1 \leq p < \infty$, by*

$$M^{\varepsilon,bl}_{Y_\pm}(\phi)(x') \doteq \frac{\delta^N}{|Y_\pm|}\int_{Y_\pm} \mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)(x',z)\, dz.$$

Proposition 3.1.23. *Let w_ε be a sequence such that $w_\varepsilon \rightharpoonup w^\pm$ weakly in $H^1(\Omega_\pm)$. Then*

$$M^{\varepsilon,bl}_{Y_\pm}(w_\varepsilon) \to w^\pm|_\Sigma \quad \textit{strongly in } L^2(\Sigma).$$

Theorem 3.1.24. *1. For all $\phi \in L^2(\Omega_\pm)$,*

$$\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\|^2_{L^2(\Sigma\times\mathbb{R}^N_\pm)} \leq \frac{1}{\varepsilon\delta^N}\|u\|^2_{L^2(\Sigma'_{\varepsilon\pm})}.$$

Suppose $N \geq 3$, set $2^* = 2N/(N-2)$ and denote the Sobolev-Poincaré-Wirtinger constant for $H^1(Y_\pm)$ by C.
2. Let $u \in H^1(\Omega_\pm)$. Let ω open and bounded in $\mathbb{R}^N_+$. Then the following estimates hold:

$$\|\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big)\|^2_{L^2(\Sigma\times\frac{1}{\delta}Y_\pm)} \leq \frac{\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_{\varepsilon\pm})},$$

$$\|\mathcal{T}^{bl}_{\varepsilon,\delta}\big(u - M^{\varepsilon,bl}_Y(u)\big)\|^2_{L^2(\Sigma;L^{2^*}(\mathbb{R}^N_\pm))} \leq \frac{C\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_{\varepsilon\pm})},$$

and

$$\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\|^2_{L^2(\Sigma\times\omega)} \leq 2\frac{C\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_{\varepsilon+})} + 2|\,\omega\,|\,\|u\|^2_{L^2(\Sigma'_{\varepsilon+})}.$$

A similar inequality is true for bounded open subsets of $\mathbb{R}^N_-$.
3. Let $w_{\varepsilon,\delta}$ be in $H^1(\Sigma'_{\varepsilon+})$ such that $\|\nabla w_{\varepsilon,\delta}\|_{L^2(\Sigma'_{\varepsilon+})}$ is bounded. Then, up to a subsequence there exists U^+ in $L^2(\Sigma; L^2_{loc}(\mathbb{R}^N_+))$ and W^+ in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N_\pm))$ with $\nabla_z W^+$ in $L^2(\Sigma\times\mathbb{R}^N_+)$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M^\varepsilon_{Y_+}(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y_+}\big) \rightharpoonup W^+ \quad \textit{weakly in } L^2(\Sigma; L^{2^*}(\mathbb{R}^N_+)),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big)\,1_{\frac{1}{\delta}Y_+} \rightharpoonup \nabla_z W^+ \quad \textit{weakly in } L^2(\Sigma\times\mathbb{R}^N_+),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\mathcal{T}^{bl}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U^+ \quad \textit{weakly in } L^2(\Sigma; L^2_{loc}(\mathbb{R}^N_+)).$$

The same result holds true for sequences in $H^1(\Sigma'_{\varepsilon-})$.

The equivalent of Proposition 3.1.20 (u.c.i.) also holds true in $\Omega_\pm$.

3.2 Homogenization in domains with small holes which are periodically distributed in volume

3.2.1 Functional setting

Let α and β be two real numbers such that $0 < \alpha < \beta$. For any open set $\mathcal{O}$ in $\mathbb{R}^N$, denote by $M(\alpha, \beta, \mathcal{O})$ the set of the $N \times N$ matrix-fields $A = (a_{ij})_{1\leq i,j\leq N} \in$

$(L^\infty(\mathcal{O}))^{N\times N}$ such that

$$\alpha|\lambda|^2 \leq (A(x)\lambda, \lambda), \quad |A(x)\lambda|^2 \leq \beta(A(x)\lambda, \lambda),$$

for any $\lambda \in \mathbb{R}^N$ and a.e. x in $\mathcal{O}$.

The perforated domain $\Omega^*_{\varepsilon,\delta}$ is defined by (3.1.3). Assume that the matrix field $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\leq i,j\leq N}$ belongs to $M(\alpha,\beta,\Omega)$. For $f \in L^2(\Omega)$, consider the following problem:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in H^1_0(\Omega^*_{\varepsilon,\delta}) \text{ satisfying} \\ \displaystyle\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega^*_{\varepsilon,\delta}} f\phi, \\ \forall \phi \in H^1_0(\Omega^*_{\varepsilon,\delta}). \end{cases} \qquad (\mathcal{P}_{\varepsilon,\delta})$$

In this section we suppose that $N \geq 3$ and study the asymptotic behavior of problem $(\mathcal{P}_{\varepsilon,\delta})$ as ε and $\delta = \delta(\varepsilon)$ are such that there exists a positive constant k_1 satisfying

$$k_1 = \lim_{\varepsilon\to 0} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon}, \quad \text{with } 0 \leq k_1 < \infty. \qquad (3.2.1)$$

3.2.2 Unfolded homogenization result

We now derive the unfolded formulation of the limit problem for $\mathcal{P}_{\varepsilon,\delta}$. In the limit we will observe the contribution of the periodic oscillations as well as the contribution of the perforations.

Here is the main theorem of this section.

Theorem 3.2.1. *Let A^ε belong to $M(\alpha,\beta,\Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon\big(A^\varepsilon\big)(x,y) \to A(x,y) \quad \text{a.e. in } \Omega \times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that as ε and $\delta \to 0$,

$$\mathcal{T}_{\varepsilon,\delta}\big(A^\varepsilon\big)(x,z) \to A_0(x,z) \quad \text{a.e. in } \Omega \times (\mathbb{R}^N \setminus B). \qquad (3.2.2)$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega), \qquad (3.2.3)$$

and there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$, and U satisfying (3.2.15) with $U - k_1 u_0$ in $L^2(\Omega; K_B)$, such that $(u_0, \widehat{u}, U)$ solves the equations

$$\int_Y A(x,y)\big(\nabla_x u_0(x) + \nabla_y \widehat{u}(x,y)\big)\nabla_y \phi(y)\, dy = 0, \qquad (3.2.4)$$

for a.e. x in Ω and all $\phi \in H^1_{per}(Y)$;

$$\int_{\mathbb{R}^N \setminus B} A_0(x,z)\, \nabla_z U(x,z)\, \nabla_z v(z)\, dz = 0, \tag{3.2.5}$$

for a.e. x in Ω and all $v \in K_B$ with $v(B) = 0$;

$$\int_{\Omega \times Y} A\big(\nabla_x u_0 + \nabla_y \widehat{u}\big)\nabla\psi\, dxdy - k_1 \int_{\Omega \times \partial B} A_0\, \nabla_z U\, \nu_B\, \psi\, d\sigma_z = \int_\Omega f\, \psi\, dx, \tag{3.2.6}$$

for all $\psi \in H^1_0(\Omega)$, where ν_B is the inward normal on ∂B and $d\sigma_z$ the surface measure.

For the proof of this theorem, we need the following two elementary results.

Lemma 3.2.2. *Let $\delta_0 > 0$. Then, for $N \geq 3$, the set*

$$\bigcup_{0<\delta<\delta_0} \{\phi \in H^1_{per}(Y); \phi = 0 \;\; on \;\; \delta B\}$$

is dense in $H^1_{per}(Y)$.

Proof. Let $\psi \in C^\infty_{per}(\overline{Y})$ be fixed. For $\delta_k \overset{k\to\infty}{\longrightarrow} 0$ consider $\phi_k \in H^1_{per}(Y)$ smooth with

$$\phi_k = \begin{cases} 0 & \text{on } \delta_k B, \\ 1 & \text{on } Y \backslash 2\delta_k B, \end{cases}$$

and such that $|\nabla \phi_k| \leq \frac{C}{\delta_k}$. Define $\Phi_k = \phi_k\, \psi$. We claim that Φ_k converges to ψ strongly in $H^1_{per}(Y)$. To do so, observe that

$$||\Phi_k - \psi||_{L^2(Y)} + ||\nabla \Phi_k - \nabla\psi||_{L^2(Y)} \leq \int_{2\delta_k B} |\psi|^2 dy + \int_{2\delta_k B} |\nabla\psi|^2 dy$$
$$+ \int_{2\delta_k B} |\nabla\phi_k|^2 |\psi|^2 dy.$$

For the last integral, using the definition of ϕ_k, one gets

$$\int_{2\delta_k B} |\nabla\phi_k|^2 |\psi|^2 dy \leq C^2 \delta_k^{N-2} \|\psi\|^2_{L^\infty(Y)}.$$

Hence,

$$\Phi_k \to \psi \quad \text{strongly in } H^1_{per}(Y).$$

Since $H^1_{per}(Y)$ is the closure of $C^\infty_{per}(\overline{Y})$ in the H^1-norm, a density argument completes the proof. □

Lemma 3.2.3. *Let v in $\mathcal{D}(\mathbb{R}^N) \cap K_B$ (i.e. $v = const. = v(B)$ on B), and set*

$$w_{\varepsilon,\delta}(x) = v(B) - v\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \;\; for \;\; x \in \mathbb{R}^N.$$

Then,

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \quad \text{weakly in } H^1(\Omega). \tag{3.2.7}$$

Proof. For δ small enough, the support of v is compact in $\frac{1}{\delta}Y$ and consequently,

$$\int_{\frac{1}{\delta}Y} |v(z)|^2 \, dz = \|v\|^2_{L^2(\mathbb{R}^n)}.$$

Clearly, $w_{\varepsilon,\delta}$ is uniformly bounded on $\mathbb{R}^N$. Observe that the set where $w_{\varepsilon,\delta}$ differs from $v(B)$ is $\bigcup_{\xi\in\mathbb{Z}^N} \varepsilon\xi + \varepsilon\delta\{\text{Support}(v)\}$, so that the measure of its intersection with Ω, is bounded by $C\delta^N$. Thus, $w_{\varepsilon,\delta}$ converges to $v(B)$ in every $L^q(\Omega)$ for finite q.

Since $\mathcal{T}_{\varepsilon,\delta}(w_\varepsilon)(x,z) = v(B) - v(z)$, property 4 from Theorem 3.1.11 gives

$$\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = -\frac{1}{\varepsilon\delta}\nabla_z v \quad \text{in} \quad \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \tag{3.2.8}$$

so that (see Theorem 3.1.2 (2)),

$$\|\nabla w_{\varepsilon,\delta}\|^2_{L^2(\widehat{\Omega}_\varepsilon)} \le \frac{\delta^{N-2}}{\varepsilon^2} |\,\Omega|\, \|\nabla_z\, v\|^2_{L^2(\mathbb{R}^N)}.$$

Due to (3.2.1), $\nabla w_{\varepsilon,\xi}$ is bounded in $L^2_{loc}(\Omega)$ which concludes the proof, since $w_{\varepsilon,\delta}$ is εY-periodic in $\mathbb{R}^N$. ☐

Proof of Theorem 3.1 (for the case $k_1 > 0$). Observe first that by the Lax-Milgram theorem, there exists a unique solution $u_{\varepsilon,\delta}$ of $(\mathcal{P}_{\varepsilon,\delta})$ and it satisfies

$$||u_{\varepsilon,\delta}||_{H^1_0(\Omega^*_{\varepsilon,\delta})} \le C||f||_{L^2(\Omega)}. \tag{3.2.9}$$

Still denoting $u_{\varepsilon,\delta}$ the extension by zero of $u_{\varepsilon,\delta}$ to the whole of Ω, (3.2.9) implies convergence (3.2.3), up to a subsequence. Next, by Theorem 3.1.2, there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$ such that

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \widehat{u} \;\text{ weakly in } L^2(\Omega \times Y). \tag{3.2.10}$$

By Theorem 3.1.11 (7), there exists some U in $L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{3.2.11}$$

By Proposition 3.1.9, one has

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon} M^\varepsilon_Y(u_{\varepsilon,\delta}) 1_{\frac{1}{\delta}Y} \to k_1 u_0 \;\text{ strongly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{3.2.12}$$

On the other hand, by Theorem 3.1.11 (5) there exists a W in $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Omega \times \mathbb{R}^N)$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\big(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) - M^\varepsilon_Y(u_{\varepsilon,\delta}) 1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)). \tag{3.2.13}$$

From (3.2.11), (3.2.12) and (3.2.13), one concludes

$$U = W + k_1 u_0, \quad \text{and } \nabla_z U = \nabla_z W,$$

and, by Theorem 3.1.11 (5) again

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon} \nabla_z \big(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\, 1_{\frac{1}{\delta}Y} = \delta^{\frac{N}{2}} \mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_z U \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^N). \tag{3.2.14}$$

From Definition 3.1.10, $\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Omega \times B$, so that by (3.2.11),

$$U = 0 \text{ on } \Omega \times B. \tag{3.2.15}$$

Now introduce the functional space K_B defined as follows:

$$K_B = \{\Phi \in L^{2^*}(\mathbb{R}^N)\ ;\ \nabla\Phi \in L^2(\mathbb{R}^N), \Phi = \mathit{const.} \text{ on } B\}. \tag{3.2.16}$$

Due to (3.2.15) , one actually has $W = U - k_1 u_0$ belongs to $L^2(\Omega; K_B)$.

Using $\Phi(\cdot) = \varepsilon\psi(\cdot)\phi(\frac{\cdot}{\varepsilon})$ with $\psi \in \mathcal{D}(\Omega)$ and $\phi \in C^1_{per}(Y)$ vanishing in a neighborhood of the origin, as a test function in $(\mathcal{P}_{\varepsilon,\delta})$ we have

$$\varepsilon \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\psi\ \phi\Big(\frac{\cdot}{\varepsilon}\Big) + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\ \psi\ \nabla\phi\Big(\frac{\cdot}{\varepsilon}\Big) = \varepsilon \int_{\Omega^*_{\varepsilon,\delta}} f\psi\phi\Big(\frac{\cdot}{\varepsilon}\Big).$$

It is easy to see that the first integral, as well as the right hand side of the above equality, converges to zero. The second integral above is unfolded by $\mathcal{T}_\varepsilon$ to get

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \psi \nabla\phi(\frac{\cdot}{\varepsilon}) \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(A^\varepsilon)(x,y)\ \mathcal{T}_\varepsilon(\nabla_x u_{\varepsilon,\delta})(x,y) \nabla\phi(y)\ \mathcal{T}_\varepsilon(\psi)(x,y)dxdy \tag{3.2.17}$$

since the unfolding criterion of integrals (u.c.i.) is satisfied due to the choice of test functions. From (3.2.10), we can pass to the limit with respect to ε in (3.2.17). Then, by Lemma 3.2.2, we obtain (3.2.4), the first equation of the unfolded formulation for the limit problem. This equation describes the effect of the periodic oscillations of the coefficients in $(\mathcal{P}_{\varepsilon,\delta})$.

In order to describe the contribution of the perforations, we use the function $w_{\varepsilon,\delta}$ introduced in Lemma 3.2.3. For ψ in $\mathcal{D}(\Omega)$, use $w_{\varepsilon,\delta}\,\psi$ as a test function in $(\mathcal{P}_{\varepsilon,\delta})$ to obtain,

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\ \psi + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\ \nabla\psi\ w_{\varepsilon,\delta} = \int_{\Omega^*_{\varepsilon,\delta}} f\ w_{\varepsilon,\delta}\ \psi. \tag{3.2.18}$$

The first term in (3.2.18) is unfolded with $\mathcal{T}_{\varepsilon,\delta}$. Again, the choice of the test function (see (3.2.7)), implies that u.c.i. is satisfied, so by Corollary 3.1.14, we can write

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\ \psi \overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq} \delta^N \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon) \mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) \mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) \mathcal{T}_{\varepsilon,\delta}(\psi). \tag{3.2.19}$$

Therefore (3.2.19), together with (3.2.8), yields

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\, \psi \overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\delta^{\frac{N}{2}} \mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi). \quad (3.2.20)$$

From the following obvious inequality

$$||\mathcal{T}_{\varepsilon,\delta}(\psi) - \psi||_{L^\infty(\widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y)} \le C\,\varepsilon ||\nabla\psi||_{L^\infty(\Omega)},$$

we obtain

$$\mathcal{T}_{\varepsilon,\delta}(\psi)\nabla_z v \to \psi \nabla_z v \quad \text{strongly in } \; L^2(\Omega\times\mathbb{R}^N). \quad (3.2.21)$$

Convergences (3.2.14), (3.2.21), as well as hypothesis (3.2.2), allows us to pass to the limit in (3.2.20) to obtain

$$\lim_{\varepsilon\to 0} \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\, \psi \, dx = -k_1 \int_{\Omega\times(R^N\setminus B)} A_0(x,z)\, \nabla_z U(x,z) \nabla_z v(z)\, \psi(x)\, dxdz, \quad (3.2.22)$$

which by density, is true for any $v \in K_B$.

The second term in (3.2.18) is unfolded with $\mathcal{T}_\varepsilon$ and we have,

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\; w_{\varepsilon,\delta}\; \nabla\psi \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\; \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \mathcal{T}_\varepsilon(w_{\varepsilon,\delta}) \mathcal{T}_\varepsilon(\nabla\psi).$$

Using Theorem 3.1.2 and convergence (3.2.7), we can pass to the limit with respect to ε in the above equality to get

$$\lim_{\varepsilon\to 0} \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\; w_{\varepsilon,\delta}\; \nabla\psi = v(B) \int_{\Omega\times Y} A\, \big(\nabla_x u_0 + \nabla_y \widehat{u}\big)\, \nabla_x \psi, \quad (3.2.23)$$

where we also used the fact that $\mathcal{T}_\varepsilon(\nabla\psi)$ converges uniformly to $\nabla\psi$.

Passing to the limit with respect to ε in (3.2.18) and using (3.2.22) and (3.2.23), we obtain

$$v(B) \int_{\Omega\times Y} A\big(\nabla_x u_0 + \nabla_y \widehat{u}\big)\nabla\psi - k_1 \int_{\Omega\times(\mathbb{R}^N\setminus B)} A_0\, \nabla_z U\; \nabla v\; \psi \; = v(B) \int_\Omega f\; \psi, \quad (3.2.24)$$

which, by density, holds true for all $\psi \in H^1_0(\Omega)$ and $v \in K_B$. Choosing $v(B) = 0$ in (3.2.24) yields equation (3.2.5), whereupon the Stokes formula transforms (3.2.24) into (3.2.6). This concludes the proof of the theorem. □

3.2.3 Standard form for the limit problem

Here we show that the unfolded problem is well-posed and we give the formulation in terms of the macroscopic solution u_0 alone.

First, see [11], consider the classical correctors $\widehat{\chi}_j, j = 1, \dots, N$ defined by the cell problems

$$\begin{cases} \widehat{\chi}_j \in L^\infty(\Omega;\, H^1_{per}(Y)), \\ \displaystyle\int_Y A(x,y)\nabla(\widehat{\chi}_j - y_j)\ \nabla\phi\ dy = 0 \quad \text{a.e. } x \in \Omega, \\ \forall \phi \in H^1_{per}(Y). \end{cases} \tag{3.2.25}$$

Assuming u_0 is known and solving equation (3.2.4) for $\widehat{u}$ as a function of u_0, gives

$$\widehat{u}(x,y) = -\sum_{j=1}^{N} \frac{\partial u_0}{\partial x_j}(x)\ \widehat{\chi}_j(x,y),$$

which used in equation (3.2.6) from Theorem 3.2.1 yields

$$\int_\Omega \mathcal{A}^{\text{hom}}\ \nabla u_0 \nabla \psi\ dx - k_1 \int_{\Omega\times\partial B} A_0\ \nabla_z U\ \nu_B\ \psi\ d\sigma_z = \int_\Omega f\ \psi\ dx, \tag{3.2.26}$$

where, for a. e. x in Ω, $\mathcal{A}^{\text{hom}}(x)$ is the homogenized matrix

$$\mathcal{A}^{\text{hom}}_{ij}(x) = \int_Y \left(a_{ij}(x,y) - \sum_{k=1}^{N} a_{ik}(x,y) \frac{\partial \widehat{\chi}_j}{\partial y_k}(x,y) \right) dy. \tag{3.2.27}$$

Equation (3.2.26) is the variational formulation for

$$-\operatorname{div}(\mathcal{A}^{\text{hom}} \nabla u_0) - k_1 \int_{\partial B} A_0\ \nabla_z U\ \nu_B\ d\sigma_z = f. \tag{3.2.28}$$

It remains to clarify the connection between the second term in (3.2.28) and u_0. In order to do so, let θ be the solution of the corresponding "cell problem":

$$\begin{cases} \theta \in L^\infty(\Omega; K_B),\ \theta(x,B) \equiv 1, \\ \displaystyle\int_{\mathbb{R}^N\setminus B} {}^t\!A_0(x,z)\, \nabla_z \theta(x,z)\ \nabla_z \Psi(z)\ dz = 0 \quad \text{a.e. for } x \in \Omega, \\ \forall \Psi \in K_B \ \text{ with } \Psi(B) = 0. \end{cases} \tag{3.2.29}$$

From (3.2.29), (3.2.15) and Green's formula together with equation (3.2.5), we get

$$\int_{\partial B} A_0\, \nabla_z U\ \nu_B\ d\sigma_z = \int_{\partial B} A_0\, \nabla_z (U - k_1 u_0)\ \nu_B\ d\sigma_z = -k_1 u_0 \left(\int_{\partial B} {}^t\!A_0\, \nabla_z \theta\ \nu_B\ d\sigma_z \right),$$

so that equation (3.2.28) becomes

$$-\operatorname{div}(\mathcal{A}^{\text{hom}} \nabla u_0) + k_1^2\ \mathbf{\Theta} u_0 = f,$$

where

$$\mathbf{\Theta}(x) \doteq \int_{\partial B} {}^t\!A_0(x,z)\, \nabla_z \theta(x,z)\ \nu_B\ d\sigma_z. \tag{3.2.30}$$

Remark 3.2.4. *From definition (3.2.30) the function* $\mathbf{\Theta}(x)$ *equals*

$$\mathbf{\Theta}(x) = \int_{\mathbb{R}^N \setminus B} A_0(x,z)\, \nabla_z \theta(x,z)\, \nabla_z \theta(x,z)\; dz,$$

which is non-negative and can be interpreted as the local capacity of the set B.

In conclusion, by Lax-Milgram's theorem, we have

Theorem 3.2.5. *The limit function* u_0 *given by Theorem 3.2.1 is the unique solution of the homogenized equation*

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \mathcal{A}^{\text{hom}} \nabla u_0\, \nabla \psi + k_1^2 \int_\Omega \mathbf{\Theta}\; u_0\; \psi = \int_\Omega f\, \psi, \\ \forall \psi \in H_0^1(\Omega). \end{cases} \tag{3.2.31}$$

Remark 3.2.6. *The contribution of the oscillations of the matrix* A^ε *in the homogenized problem are reflected by the first term of the left hand side in (3.2.31). The contribution of the perforations is the zero order "strange term"* $k_1^2\, \mathbf{\Theta}(x)\, u_0$.

Remark 3.2.7.

1. *The proof is actually simpler for the case* $k_1 = 0$ *and the statement is included in Theorem 3.2.5: the small holes have no influence at the limit.*

2. *The case of* $\lim \dfrac{\delta^{\frac{N}{2}-1}}{\varepsilon} = \infty$ *is easy to analyze: from Theorem 3.1.11 (6),*

$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0 \text{ weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)).$$

On the other hand, since $\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \quad$ *in* $\Omega \times B$, *this implies that* $u_0 = 0$.

3.3 Homogenization in domains with small holes which are periodically distributed in a layer

3.3.1 Functional setting

As in the preceding section, we suppose that $N \geq 3$. We use the notations introduced in subsection 3.1.3 for domains with small holes contained in the layer Σ'_ε. The corresponding perforated layer $\Sigma'_{\varepsilon,\delta}$ is given by

$$\Sigma'_{\varepsilon,\delta} = \left\{ x \in \Sigma'_\varepsilon \text{ such that } \left\{ \frac{x}{\varepsilon} \right\}_Y \in Y^*_\delta \right\}.$$

The perforated domain is now (see Figure 3.5 for an example)

$$\Omega'_{\varepsilon,\delta} = \Omega \setminus \left\{ x \in \Sigma'_\varepsilon \text{ such that } \left\{ \frac{x}{\varepsilon} \right\}_Y \in \delta B \right\}.$$

The small perforations are of size $\varepsilon\delta$ with $\delta = \delta(\varepsilon)$ satisfying

$$k_2 = \lim_{\varepsilon \to 0} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}, \quad \text{where } 0 \le k_2 < \infty. \tag{3.3.1}$$

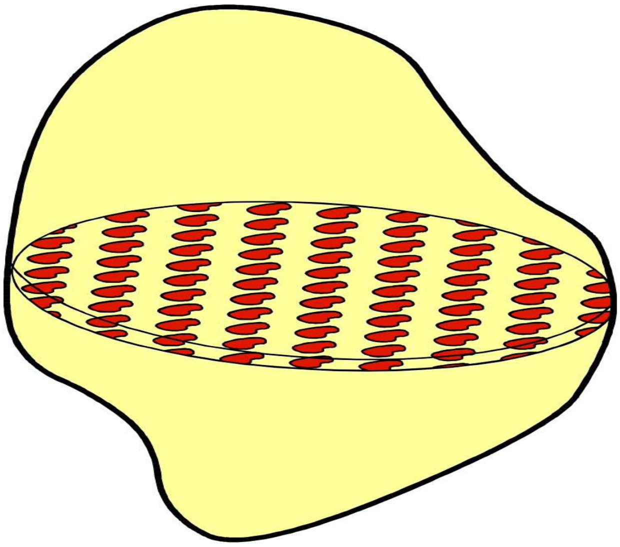


Figure 3.5: **An example of set $\Omega'_{\varepsilon,\delta}$: an electrostatic screen**

We consider the asymptotic behavior for the following problem:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in H^1_0(\Omega'_{\varepsilon,\delta}) \text{ satisfying} \\ \displaystyle\int_{\Omega'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega'_{\varepsilon,\delta}} f\phi, \qquad f \in L^2(\Omega), \\ \forall \phi \in H^1_0(\Omega'_{\varepsilon,\delta}). \end{cases} \tag{$\mathcal{P}'_{\varepsilon,\delta}$}$$

3.3.2 Unfolded homogenization result

Theorem 3.3.1. *Let A^ε belong to $M(\alpha, \beta, \Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon\big(A^\varepsilon\big)(x,y) \to A(x,y) \quad a.e. \ in \ \Omega \times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that, as ε and $\delta \to 0$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}\big(A^\varepsilon\big)(x', z) \to A_0(x', z) \quad a.e. \ in \ \Sigma \times (\mathbb{R}^N \setminus B). \tag{3.3.2}$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad weakly \ in \ H^1_0(\Omega),$$

and there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$, and U satisfying (3.3.11) with $U - k_2 u_0$ in $L^2(\Sigma; K_B)$, such that $(u_0, \widehat{u}, U)$ solves the equations

$$\int_Y A(x,y)\big(\nabla_x u_0(x) + \nabla_y \widehat{u}(x,y)\big)\nabla_y \phi(y)\, dy = 0, \tag{3.3.3}$$

for a.e. x in Ω and all $\phi \in H^1_{per}(Y)$;

$$\int_{\mathbb{R}^N \setminus B} A_0(x',z)\, \nabla_z U(x',z)\, \nabla_z v(z)\, dz = 0, \tag{3.3.4}$$

for a.e. x' in Σ and all $v \in K_B$ with $v(B) = 0$;

$$\int_{\Omega\times Y} A\, \big(\nabla_x u_0 + \nabla_y \widehat{u}\big)\nabla\psi - k_2 \int_{\Sigma\times\partial B} A_0\, \nabla_z U\, \nu_B\, \psi\, d\sigma_z = \int_\Omega f\, \psi, \tag{3.3.5}$$

for all $\psi \in H^1_0(\Omega)$, where ν_B and $d\sigma_z$ are the inward normal and the surface measure on ∂B.

For the proof of this theorem, we need the equivalent of Lemma 3.2.3 with a similar proof (with $\mathcal{T}_{\varepsilon,\delta}$ replaced by $\mathcal{T}^{bl}_{\varepsilon,\delta}$).

Lemma 3.3.2. *Let v in $\mathcal{D}(\mathbb{R}^N) \cap K_B$ and, for δ small enough, set*

$$w^{bl}_{\varepsilon,\delta}(x) = v(B) - v\Big(\frac{1}{\delta}\Big\{\frac{x'}{\varepsilon}\Big\}_Y, \frac{x_N}{\varepsilon\delta}\Big) \quad \text{for } x \in \mathbb{R}^N.$$

Then,

$$w^{bl}_{\varepsilon,\delta} \rightharpoonup v(B) \quad \text{weakly in } H^1(\Omega). \tag{3.3.6}$$

Proof of Theorem 3.3.1 (for the case $k_2 > 0$). We denote $u_{\varepsilon,\delta}$ the extension by zero to the whole of Ω of the solution of $(\mathcal{P}'_{\varepsilon,\delta})$. The reasoning is similar to that of the previous section. The following estimate is straightforward from $(\mathcal{P}'_{\varepsilon,\delta})$:

$$||u_{\varepsilon,\delta}||_{H^1_0(\Omega)} \leq C||f||_{L^2(\Omega)},$$

so that, up to a subsequence,

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega).$$

Equation (3.3.3) is obtained exactly as in the proof of Theorem 3.2.1.

By Theorem 3.1.19 (7), there exists some U in $L^2(\Sigma; L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2(\Sigma; L^2_{loc}(\mathbb{R}^N)). \tag{3.3.7}$$

Since $\mathcal{T}^{bl}_{\varepsilon,\delta}\big(M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})\big) = M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}$, Proposition 3.1.18 implies

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to k_2 u_{0|\Sigma} \;\text{ strongly in } L^2(\Sigma; L^2_{loc}(\mathbb{R}^N)). \tag{3.3.8}$$

On the other hand, Theorem 3.1.19 (7) gives a W in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Sigma \times \mathbb{R}^N)$, such that

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) - M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)). \tag{3.3.9}$$

From (3.3.7), (3.3.8) and (3.3.9), one concludes

$$U = W + k_2 u_0, \quad \text{and } \nabla_z U = \nabla_z W,$$

and, by Theorem 3.1.19 (7) again

$$\sqrt{\varepsilon}\,\delta^{\frac{N}{2}}\,\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) = \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\,1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad \text{weakly in } L^2(\Sigma \times \mathbb{R}^N). \tag{3.3.10}$$

From Definition 3.1.15, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Sigma \times B$, so (3.3.7) implies

$$U = 0 \ \text{ on } \ \Sigma \times B. \tag{3.3.11}$$

Therefore, $W = U - k_2 u_0$ belongs to $L^2(\Sigma; K_B)$.

In order to capture the contribution of the perforations to the limit problem, we adapt the proof of Theorem 3.2.1 and use Lemma 3.3.2. For $\psi \in \mathcal{D}(\Omega)$, let $\Phi \doteq \psi\, w^{bl}_{\varepsilon,\delta}$, be a test function in problem $(\mathcal{P}'_{\varepsilon,\delta})$. One obtains

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta}\,\psi + \int_{\Omega'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\,\nabla\psi\, w^{bl}_{\varepsilon,\delta} = \int_{\Omega'_{\varepsilon,\delta}} f\, w^{bl}_{\varepsilon,\delta}\,\psi. \tag{3.3.12}$$

Observe that since $w^{bl}_{\varepsilon,\delta}$ vanishes in the holes, one actually has

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta}\,\psi = \int_{\Sigma'_{\varepsilon}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta}\,\psi,$$

which unfolded with $\mathcal{T}^{bl}_{\varepsilon,\delta}$ gives

$$\int_{\Sigma'_{\varepsilon}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta}\,\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\simeq} \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla w^{bl}_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \tag{3.3.13}$$

Property 5 of Theorem 3.1.19 implies,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla w^{bl}_{\varepsilon,\delta}) = -\frac{1}{\varepsilon\delta}\nabla_z\, v,$$

so that (3.3.10) and (3.3.13) yield

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta}\,\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\simeq} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\big(-\nabla_z v\big)\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \tag{3.3.14}$$

From the compactness of the support of v and the straightforward inequality

$$||\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi)-\psi||_{L^\infty(\widehat{\Sigma}_\varepsilon\times\frac{1}{\delta}Y)}\leq c\varepsilon||\nabla_x\psi||_{L^\infty(\Omega)^N},$$

we obtain

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi)\nabla_z v\to\psi\nabla_z v\quad\text{strongly in }\ L^2(\Sigma\times\mathbb{R}^N). \tag{3.3.15}$$

This, together with convergences (3.3.1) and (3.3.10), as well as hypothesis (3.3.2), allows us to pass to the limit in (3.3.14) which now reads

$$\lim_{\varepsilon\to 0}\int_{\Sigma'_{\varepsilon,\delta}}A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\,\psi\,dx=-k_2\int_{\Sigma\times\mathbb{R}^N}A_0(x',z)\,\nabla_z U(x',z)\,\nabla_z v\,\psi\,dx'dz. \tag{3.3.16}$$

By a density argument, (3.3.15) is true for any v in K_B.

The second term in (3.3.12) is unfolded with $\mathcal{T}_\varepsilon$ and using Theorem 3.1.2, we get at the limit

$$\lim_{\varepsilon\to 0}\int_{\Omega'_{\varepsilon,\delta}}A^\varepsilon\nabla u_{\varepsilon,\delta}\,w^{bl}_{\varepsilon,\delta}\,\nabla\psi\,dx=v(B)\int_{\Omega\times Y}A(x,y)\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi\,dxdy,$$

which, with (3.3.16) gives equation (3.3.4). Equation (3.3.5) is obtained similarly. □

3.3.3 Standard form of the homogenized equation

Like in subsection 3.3.2, one can rewrite system (3.3.3)-(3.3.5) in the standard form. The result is stated in the next theorem, the proof of which follows the same lines as that of Theorem 3.2.5.

Theorem 3.3.3. *The limit function u_0 given by Theorem 3.3.1 is the solution of the homogenized equation*

$$\begin{cases} u_0\in H^1_0(\Omega),\\ \displaystyle\int_\Omega\mathcal{A}^{\text{hom}}\nabla u_0\,\nabla\psi+k_2^2\int_\Sigma\mathbf{\Theta}'u_0\,\psi=\int_\Omega f\,\psi,\\ \forall\psi\in H^1_0(\Omega),\end{cases} \tag{3.3.17}$$

where $\mathbf{\Theta}'$ is defined by (3.2.30) with x' in place of x.

Remark 3.3.4. *The strong formulation for (3.3.17) is the following:*

$$\begin{cases} -\text{div}\,\mathcal{A}^{\text{hom}}\nabla u_0 &=f &\text{in }\Omega\setminus\Sigma,\\ -\big[\mathcal{A}^{\text{hom}}\nabla u_0\big] &=(k_2)^2\mathbf{\Theta}'u_0 &\text{on }\ \Sigma,\\ u_0 &=0 &\text{on }\partial\Omega,\end{cases}$$

where $\big[\mathcal{A}^{\text{hom}}\,\nabla u_0\big]$ denotes the jump across Σ,

$$\big[\mathcal{A}^{\text{hom}}\nabla u_0\big]\doteq\mathcal{A}^{\text{hom}}\,\nabla u_0^-\,n^-+\mathcal{A}^{\text{hom}}\,\nabla u_0^+\,n^+\qquad\text{on }\Sigma,$$

n^+ and n^- denoting the respective exterior unit normal to Ω_+ and Ω_- on Σ.

Remark 3.3.5.

1. *The proof for the case* $k_2 = 0$ *is actually simpler, and the statement is included in Theorem 3.3.3: the small holes have no influence at the limit, i.e. the equation* $-\mathrm{div}\,\mathcal{A}^{\mathrm{hom}}\nabla u_0 = f$ *is satisfied in the whole of* Ω.

2. *As in Remark 3.2.7, for the case of* $\lim \dfrac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} = \infty$, *from Theorem 3.1.19 (6),*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0|_\Sigma \text{ weakly in } L^2(\Sigma; L^2_{loc}(\mathbb{R}^N)).$$

On the other hand, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \quad$ *in* $\Sigma \times B$ *implies that* $u_0|_\Sigma = 0$. *The limit problem splits into the two separate homogeneous Dirichlet problems in* Ω_+ *and* Ω_-

$$\begin{cases} -\mathrm{div}\,\mathcal{A}^{\mathrm{hom}}\nabla u_0 &= f \quad \text{in } \Omega_\pm \\ u_0 &= 0 \quad \text{on } \partial\Omega_\pm\ . \end{cases}$$

3.4 The thin Neumann sieve with variable coefficients

3.4.1 Functional setting

We use the same notations as in Section 3.2 and Section 3.3. For an open subset S of $Y \cap \Pi$ such that $\overline{S} \subset (Y \cap \Pi)$, set

$$Y_\delta = Y_+ \cup Y_- \cup \delta S,$$

and

$$S_{\varepsilon,\delta} = \Big\{ x \in \Sigma \text{ such that } \Big\{\frac{x}{\varepsilon}\Big\}_Y \in \delta S \Big\}.$$

For Ω open and bounded in $\mathbb{R}^N$ $(N \geq 3)$, define

$$\Omega^{bl}_{\varepsilon\delta} = \Omega_+ \cup \Omega_- \cup S_{\varepsilon,\delta} \quad \text{and} \quad \Sigma'_{\varepsilon,\delta} \doteq \Sigma'_\varepsilon \cap \Omega^{bl}_{\varepsilon\delta}.$$

The connection between Ω_+ and Ω_- occurs through the "sieve" consisting of the set $S_{\varepsilon,\delta}$ (see Figure 3.6). We assume that ε and δ satisfy assumption (3.3.1) of section 4:

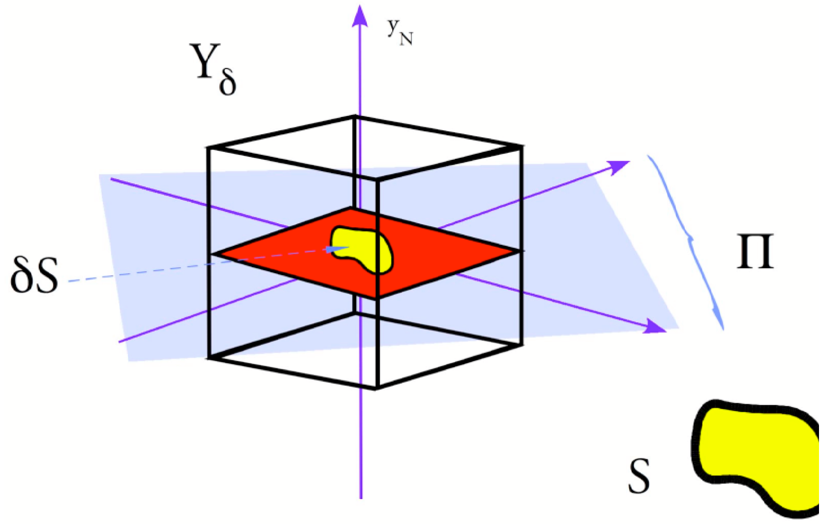
$$k_2 = \lim_{\varepsilon\to 0} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}, \quad \text{where } 0 \leq k_2 < \infty.$$

Consider the space

$$V = \{v \in H^1(\Omega_+ \cup \Omega_-);\ v = 0 \ \text{ on } \ \partial\Omega\}$$

which is a Hilbert space for the scalar product

$$< u, v >_V = \int_{\Omega_+ \cup \Omega_-} \nabla u\, \nabla v \quad \text{for all} \quad u, v \in V.$$



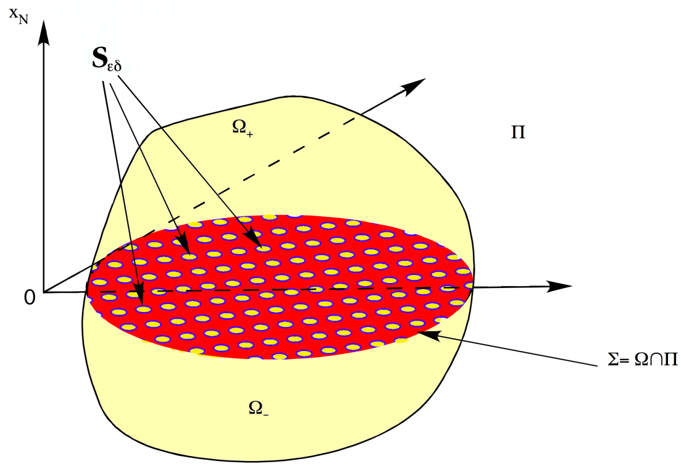


Figure 3.6: **The set Y_δ and the thin sieve $\Omega^{bl}_{\varepsilon\delta}$**

For simplicity, when v belongs to V, we denote ∇v the $L^2(\Omega)-$function which equals the gradient of v in $\Omega_+ \cup \Omega_-$ (this is the restriction to $\Omega_+ \cup \Omega_-$ of the distributional gradient of v). We also denote by $[v]$ the jump of v across Σ, $[v] \doteq v^+|_\Sigma - v^-|_\Sigma$, which belongs to $H^{1/2}(\Sigma)$. Finally set

$$V_{\varepsilon,\delta} = \{v \in V, [v] = 0 \ \text{ on } \ S_{\varepsilon,\delta}\}.$$

The thin Neumann sieve model is

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in V_{\varepsilon,\delta} \text{ satisfying} \\ \displaystyle\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega^{bl}_{\varepsilon,\delta}} f\phi, \quad f \in L^2(\Omega), \\ \forall \phi \in V_{\varepsilon,\delta}. \end{cases} \qquad (\mathcal{P}^{bl}_{\varepsilon,\delta})$$

3.4.2 Unfolded homogenization result

In this problem, the equivalent of the space K_B of Section 3.2 (see (3.2.16)), is

$$\widehat{K_S} = \Big\{\Phi \in H^1_{loc}(\mathbb{R}^N_+ \cup \mathbb{R}^N_-)\ ;\ \ \nabla\Phi \in L^2(\mathbb{R}^N_+ \cup \mathbb{R}^N_-),\, \big[\Phi\big] = 0 \text{ on } S\Big\}. \tag{3.4.1}$$

Proposition 3.4.1. *There exist two linear forms $l^\pm$ on $\widehat{K_S}$ such that for every Φ in $\widehat{K_S}$, the functions $\Phi^\pm - l^\pm(\Phi)$ belong to $L^{2^*}(\mathbb{R}^N_\pm)$.*

The space $\widehat{K_S}$ is Hilbert space for the norm

$$\|\Phi\|^2_{\widehat{K_S}} \doteq \|\nabla\Phi\|^2_{L^2(\mathbb{R}^N_+\cup\mathbb{R}^N_-)} + \Big(\frac{l^+(\Phi)+l^-(\Phi)}{2}\Big)^2. \tag{3.4.2}$$

Furthermore,

$$\widehat{K_S}^{\infty} \doteq \Big\{\Phi \in \widehat{K_S}, \Phi^\pm \in C^\infty(\mathbb{R}^N_\pm),\ \mathrm{supp}(\nabla\Phi^\pm)\ \textit{bounded in}\ \mathbb{R}^N_\pm\Big\},$$

is dense in $\widehat{K_S}$ for this norm.

Proof. Due to the Sobolev-Poincaré-Wirtinger inequality (applied in the sets $\frac{1}{\delta}Y_\pm$ with $\delta \to 0$), for every Φ in $\widehat{K_S}$, there exist two constants $l^\pm(\Phi)$ such that $\big(\Phi^\pm - l^\pm(\Phi)\big)$ belong to $L^{2^*}(\mathbb{R}^N_\pm)$.

It is well-known that the first term in (3.4.2) is a Hilbert semi-norm on the space $\widehat{K_S}$, so that, with the second term, it defines a norm. The density of $\widehat{K_S}^\infty$ in $\widehat{K_S}$ follows by a standard argument of truncation and regularization.

□

Theorem 3.4.2. *Let A^ε belong to $M(\alpha,\beta,\Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon\big(A^\varepsilon\big)(x,y) \to A(x,y) \quad \textit{a.e. in } \Omega\times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that, as ε and $\delta \to 0$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}\big(A^\varepsilon\big)(x',z) \to A_0(x',z) \quad \textit{a.e. in } \Sigma\times\mathbb{R}^N. \tag{3.4.3}$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}^{bl}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \textit{weakly in } V,$$

and there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$, $U \in L^2(\Sigma;\widehat{K}_S)$ satisfying

$$l^\pm(U) = k_2\, u^\pm_{0\,|\Sigma} \ \ \textit{for a.e.}\ \ x' \in \Sigma, \tag{3.4.4}$$

and such that $(u_0,\widehat{u},U)$ solves the following three equations:

$$\int_Y A(x,y)\big(\nabla_x u_0(x) + \nabla_y\widehat{u}(x,y)\big)\nabla_y\phi(y)\,dy = 0, \tag{3.4.5}$$

for a.e. x in Ω and all $\phi \in H^1_{per}(Y)$,

$$\int_{\mathbb{R}^N} A_0(x', z)\, \nabla_z U(x', z)\, \nabla_z v(z)\, dz = 0, \tag{3.4.6}$$

for a.e. x' in Σ and all $v \in \widehat{K}_S$ with $l^{\pm}(v) = 0$, and

$$\int_{\Omega\times Y} A\,\big(\nabla_x u_0 + \nabla_y \widehat{u}\big)\nabla\phi - k_2 \int_{\Sigma\times S} A_0\; \nabla_z U\, n^+\;\; [\phi]_\Sigma \;= \int_\Omega f\,\phi, \tag{3.4.7}$$

for all $\phi \in V$.

Proof (for the case $k_2 > 0$.) Let $u_{\varepsilon,\delta}$ be a test function in $(\mathcal{P}^{bl}_{\varepsilon,\delta})$. Using the Poincaré inequality on Ω_+ and Ω_-, there is a constant C (independent of ε, δ) such that,

$$||u_{\varepsilon,\delta}||_V \le C||f||_{L^2(\Omega)}.$$

Consequently, up to a subsequence, there exists $u_0 \in V$ such that

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } V.$$

By Theorem 3.1.2, one can also assume that there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$ with

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \widehat{u} \;\text{ weakly in }\; L^2(\Omega \times Y).$$

Using $\psi \in \mathcal{D}(\Omega)$ as a test function in $(\mathcal{P}^{bl}_{\varepsilon,\delta})$, and unfolding with operator $\mathcal{T}_\varepsilon$, we get

$$\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\phi\; dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\; dxdy.$$

Applying properties 5 and 6 of Theorem 3.1.2 we can pass to the limit to obtain

$$\int_{\Omega\times Y} A(x,y)[\nabla_x u_0 + \nabla_y \widehat{u}]\nabla_x \psi\; dxdy = \int_\Omega f\psi\; dx.$$

Next, consider $\phi \in H^1_{per}(Y)$ and $\psi \in \mathcal{D}(\Omega_+) \cup \mathcal{D}(\Omega_-)$. Using $\Phi(x) = \varepsilon\psi(x)\phi(\frac{x}{\varepsilon})$ as a test function in $(\mathcal{P}^{bl}_{\varepsilon,\delta})$ yields

$$\varepsilon \int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\psi\; \phi\Big(\frac{\cdot}{\varepsilon}\Big) + \int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\; \psi\; \nabla\phi\Big(\frac{\cdot}{\varepsilon}\Big) = \varepsilon \int_{\Omega^{bl}_{\varepsilon,\delta}} f\psi\phi\Big(\frac{\cdot}{\varepsilon}\Big).$$

As in subsection 3.3, passing to the limit gives (3.4.5).

By Theorem 3.1.24 (3), there exists $U \in L^2(\Sigma; L^2_{loc}(\mathbb{R}^N_\pm))$ such that (up to a subsequence)

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \mathcal{T}^{bl}_{\varepsilon,\delta}(u^\pm_{\varepsilon,\delta}) \rightharpoonup U^\pm \;\text{ weakly in }\;\; L^2(\Sigma; L^2_{loc}(\mathbb{R}^N_\pm)). \tag{3.4.8}$$

By construction $\mathcal{T}^{bl}_{\varepsilon,\delta}\big(M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta})\big) = M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta})1_{\frac{1}{\delta}Y_\pm}$. By Proposition 3.1.23, one has

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to k_2 u^\pm_{0\,|\Sigma} \quad \text{strongly in } L^2(\Sigma; L^2_{loc}(\mathbb{R}^N_\pm)). \tag{3.4.9}$$

By Theorem 3.1.24 (3) there exists a W in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W^\pm$ in $L^2(\Sigma\times\mathbb{R}^N_\pm)$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u^\pm_{\varepsilon,\delta}) - M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W^\pm \quad \text{weakly in } L^2(\Sigma; L^{2^*}(\mathbb{R}^N_\pm)). \tag{3.4.10}$$

From (3.4.8), (3.4.9) and (3.4.10), one concludes

$$U^\pm = W^\pm + k_2 u^\pm_{0\,|\Sigma}, \quad \text{and } \nabla_z U^\pm = \nabla_z W^\pm. \tag{3.4.11}$$

Again by Theorem 3.1.24 (3), one has the convergence

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u^\pm_{\varepsilon,\delta})\big) = \sqrt{\varepsilon}\delta^{\frac{N}{2}}\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u^\pm_{\varepsilon,\delta}) \rightharpoonup \nabla_z U^\pm \;\text{ weakly in }\; L^2(\Sigma\times\mathbb{R}^N_\pm). \tag{3.4.12}$$

From Definition 3.1.15, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u^+_{\varepsilon,\delta}) = \mathcal{T}^{bl}_{\varepsilon,\delta}(u^-_{\varepsilon,\delta})$ on $\Sigma\times S$, so that by convergences (3.4.8), (3.4.12) one has

$$[U(x',\cdot)] = 0 \;\text{ on }\; S \quad \text{for a.e. } x'\in\Sigma.$$

Therefore, $U \in L^2(\Sigma;\widehat{K_S})$, and (3.4.11) implies (3.4.4).

In order to obtain equations (3.4.6) and (3.4.7), choose a function v in $\widehat{K_S}^\infty$ and set

$$w_{\varepsilon,\delta}(x',x_N) = v\Big(\frac{1}{\delta}\Big\{\frac{x'}{\varepsilon}\Big\}_Y, \frac{x_N}{\varepsilon\delta}\Big).$$

Clearly, $[w_{\varepsilon,\delta}] = 0$ on $S_{\varepsilon,\delta}$ and $\nabla w^\pm_{\varepsilon,\delta}$ vanishes outside $\Sigma'_{\varepsilon,\delta}$ for δ small enough. One easily shows (as in Lemma 3.3.2) that

$$\begin{cases} \mathcal{T}_\varepsilon(w^\pm_{\varepsilon,\delta}) \to l^\pm(v) \quad \text{strongly in } L^2(\Omega_\pm), \\ w^\pm_{\varepsilon,\delta} \rightharpoonup l^\pm(v) \quad \text{weakly in } H^1(\Omega_\pm). \end{cases} \tag{3.4.13}$$

For $\psi\in\mathcal{D}(\Omega)$, using $\psi\, w_{\varepsilon\delta}$ as a test function in problem $(\mathcal{P}^{bl}_{\varepsilon,\delta})$ gives

$$\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\,\nabla\psi\; w_{\varepsilon,\delta} + \int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w_{\varepsilon,\delta}\;\psi = \int_{\Omega^{bl}_{\varepsilon,\delta}} f\; w_{\varepsilon,\delta}\;\psi. \tag{3.4.14}$$

The first term in (3.4.14) is unfolded with $\mathcal{T}_\varepsilon$ as usual. This yields

$$\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\,\nabla\psi\; w_{\varepsilon,\delta} \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\,dxdy.$$

Applying (3.4.13) and properties 5 and 6 of Theorem 3.1.2, one obtains

$$\lim_{\varepsilon\to 0}\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\,\nabla\psi\, w_{\varepsilon,\delta} = l^+(v)\int_{\Omega_+\times Y} A(x,y)\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi\,dxdy + l^-(v)\int_{\Omega_-\times Y} A(x,y)\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi\,dxdy.$$

The second term in (3.4.14) is unfolded with $\mathcal{T}^{bl}_{\varepsilon,\delta}$. The choice of the test function implies that u.c.i. is satisfied, so

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w_{\varepsilon,\delta}\,\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\simeq} \varepsilon\delta^N\int_{\Sigma\times\mathbb{R}^N}\mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \quad (3.4.15)$$

Property 4 from Theorem 3.1.19 gives

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = \frac{1}{\varepsilon\delta}\nabla_z v,$$

which, together with (3.4.15), yields

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w_{\varepsilon,\delta}\,\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\simeq} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\int_{\Sigma\times\mathbb{R}^N}\mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\sqrt{\varepsilon}\delta^{\frac{N}{2}}\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\,\nabla_z v\,\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \quad (3.4.16)$$

Convergences (3.4.3), (3.4.12), allow to pass to the limit in (3.4.16) to obtain

$$\lim_{\varepsilon\to 0}\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w_{\varepsilon,\delta}\,\psi = k_2\int_{\Sigma\times\mathbb{R}^N} A_0\,\nabla_z U(x',z)\,\nabla_z v\,\psi\,dx'dz.$$

Now, the limit in (3.4.14) becomes

$$\begin{aligned} &l^+(v)\int_{\Omega_+\times Y} A\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi + l^-(v)\int_{\Omega_-\times Y} A\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi+\\ &\quad+k_2\int_{\Sigma\times\mathbb{R}^N} A_0(x',z)\,\nabla_z U(x',z)\,\nabla_z v\,\psi dx'dz = l^+(v)\int_{\Omega_+} f\psi + l^-(v)\int_{\Omega_-} f\psi, \end{aligned} \quad (3.4.17)$$

which, by density, holds for every $v\in\widehat{K_S}$. Equation (3.4.6) is then simply obtained by choosing $l^+(v)=l^-(v)=0$ in (3.4.17).

Using (3.4.6) with an arbitrary v in $\widehat{K_S}^\infty$ one deduces by Green's formula that

$$\int_{\mathbb{R}^N_\pm} A_0\,\nabla_z U\,\nabla_z v\,dz = \int_S A_0\,\nabla_z U\,n^\pm\big(v(z')-l^\pm(v)\big)\,dz', \quad (3.4.18)$$

which still holds for every $v\in\widehat{K_S}$. Then, (3.4.18) together with (3.4.17) leads to

$$\begin{aligned} &l^+(v)\Big(\int_{\Omega_+\times Y} A\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi - k_2\int_{\Sigma\times S} A_0\,\nabla_z U\,n^+\psi\ -\int_{\Omega_+} f\psi\Big)\\ &+l^-(v)\Big(\int_{\Omega_-\times Y} A\big(\nabla_x u_0+\nabla_y\widehat{u}\big)\nabla_x\psi - k_2\int_{\Sigma\times S} A_0\,\nabla_z U\,n^-\psi\ -\int_{\Omega_-} f\psi\Big)\\ &+k_2\int_{\Sigma\times S}\big(A_0\,\nabla_z U\,n^+ + A_0\,\nabla_z U\,n^-\big)\,v\,\psi = 0. \end{aligned} \quad (3.4.19)$$

Taking $l^+(v) = l^-(v) = 0$ in (3.4.19), implies that

$$A_0 \nabla_z U \, n^+ + A_0 \nabla_z U \, n^- \doteq [A_0 \nabla_z U]_S = 0 \text{ a.e. on } \Sigma \times S. \tag{3.4.20}$$

Since $l^+(v)$ and $l^-(v)$ are independent, (3.4.19) now gives the following two formulas:

$$\begin{aligned} \int_{\Omega_+ \times Y} A\big(\nabla_x u_0 + \nabla_y \widehat{u}\big) \nabla_x \psi - k_2 \int_{\Sigma \times S} A_0 \nabla_z U \, n^+ \, \psi &= \int_{\Omega_+} f \psi, \\ \int_{\Omega_- \times Y} A\big(\nabla_x u_0 + \nabla_y \widehat{u}\big) \nabla_x \psi - k_2 \int_{\Sigma \times S} A_0 \nabla_z U \, n^- \, \psi &= \int_{\Omega_-} f \psi, \end{aligned} \tag{3.4.21}$$

which, by density, hold for every ψ in $H^1_0(\Omega)$. Let ϕ be arbitrary in V. Equation (3.4.7) is obtained by choosing $\psi = \phi^+$, respectively $\psi = \phi^-$ in (3.4.21), and adding the two corresponding equations. □

3.4.3 Standard form of the homogenized equation

As in subsection 3.4.2, one can write system (3.4.4), (3.4.5), (3.4.6), (3.4.7) in a standard form, with only u_0 as unknown.

First, from (3.4.6), the first term in the left-hand side of (3.4.7), can be written in terms of the standard homogenized operator

$$\int_{\Omega \times Y} A \left(\nabla_x u_0 + \nabla_y \widehat{u}\right) \nabla \phi = \int_{\Omega} \mathcal{A}^{\text{hom}} \nabla u_0 \, \nabla \phi,$$

for every ϕ in the space V, using the same cell-problems (3.2.25) and the same $\mathcal{A}^{\text{hom}}$ (as given by (3.2.27)).

Next, observe that for a given u_0, problem (3.4.4)-(3.4.6) for U, has a unique solution by the Lax-Milgram theorem (applied on a closed affine subspace of $\widehat{K_S}$).

Now, we show how equation (3.4.7) can be brought to the standard form. More precisely, it remains to clarify the connection between the term $-k_2 \int_S A_0 \; \nabla_z U \, n^+$ and $[u_0]_\Sigma$. In order to do so, let θ be the solution of the following "cell problem":

$$\begin{cases} \theta \in L^\infty(\Sigma; \widehat{K_S}), \; l^\pm(\theta) \equiv \pm 1, \\ \displaystyle\int_{\mathbb{R}^N} {}^t\!A_0(x', z) \, \nabla_z \theta(x', z) \; \nabla_z \Psi(z) \, dz = 0 \quad \text{for a.e. } x' \in \Sigma, \\ \forall \Psi \in \widehat{K_S} \; \text{ with } l^\pm(\Psi) = 0. \end{cases} \tag{3.4.22}$$

From (3.4.18) follows

$$\int_{\mathbb{R}^N_+ \cup \mathbb{R}^N_-} A_0 \nabla_z U \, \nabla_z v \, dz = \big(l^+(v) - l^-(v)\big) \int_S A_0 \nabla_z U \, n^- \, dz'. \tag{3.4.23}$$

Similarly, the solution of (3.4.22) is unique and satisfies for a.e. x' in Σ

$${}^t\!A_0 \, \nabla_z \theta \, n^+ + {}^t\!A_0 \, \nabla_z \theta \, n^- \doteq [\, {}^t\!A_0 \, \nabla_z \theta]_S = 0,$$

$$\int_{\mathbb{R}^N_+ \cup \mathbb{R}^N_-} {}^t\!A_0 \, \nabla_z \theta \, \nabla_z v \, dz = \big(l^+(v) - l^-(v) \big) \int_S {}^t\!A_0 \, \nabla_z \theta \, n^- \, dz'. \tag{3.4.24}$$

Formula (3.4.23) holds for $v = \theta$, whereas (3.4.24) does for $v = U$, so that combining the two yields

$$\int_S A_0 \, \nabla_z U \, n^- \, dz' = \frac{l^+(\theta) - l^-(\theta)}{2} \int_S A_0 \, \nabla_z U \, n^- \, dz' = \frac{l^+(U) - l^-(U)}{2} \int_S {}^t\!A_0 \, \nabla_z \theta \, n^- \, dz'.$$

Consequently, by (3.4.4)

$$k_2 \int_S A_0(x', z) \; \nabla_z U(x', z) \, n^- \, dz' = \frac{k_2^2}{2} \, \boldsymbol{\Theta}(x') \; [u_0]_\Sigma(x'),$$

where

$$\boldsymbol{\Theta}(x') \doteq \int_S {}^t\!A_0 \, \nabla_z \theta \, n^- \, dz' = - \int_S {}^t\!A_0 \, \nabla_z \theta \, n^+ \, dz',$$

the latter equality deriving from (3.4.23). Thus, equation (3.4.7) becomes

$$\int_\Omega \mathcal{A}^{\text{hom}} \nabla u_0 \, \nabla \phi \, dx + \frac{k_2^2}{2} \int_\Sigma \boldsymbol{\Theta}(x') \; [u_0]_\Sigma(x') \; [\phi]_\Sigma(x') \, dx' = \int_\Omega f \, \phi \, dx.$$

We have proved the following theorem:

Theorem 3.4.3. *The limit function u_0 given by Theorem 3.4.2 is the solution of the homogenized equation*

$$\begin{cases} u_0 \in V, \\ \displaystyle\int_\Omega \mathcal{A}^{\text{hom}} \nabla u_0 \, \nabla \phi + \frac{k_2^2}{2} \int_\Sigma \boldsymbol{\Theta} \; [u_0]_\Sigma \; [\phi]_\Sigma = \int_\Omega f \, \phi, \\ \forall \phi \in V. \end{cases} \tag{3.4.25}$$

Remark 3.4.4. *Taking $v = \theta$ in (3.4.24) shows that*

$$\boldsymbol{\Theta}(x') = \frac{1}{2} \int_{\mathbb{R}^N_+ \cup \mathbb{R}^N_-} A_0(x', z) \, \nabla_z \theta(x', z) \, \nabla_z \theta(x', z) \, dz,$$

which is non-negative. This implies existence and uniqueness of the solution u_0 of (3.4.25).

Remark 3.4.5. *The strong formulation for the solution u_0 of the limit problem is:*

$$\begin{cases} -\text{div} \mathcal{A}^{\text{hom}} \nabla u_0 &= f \quad \text{in } \Omega \setminus \Sigma, \\ \mathcal{A}^{\text{hom}} \nabla u_0 \, n^-|_\Sigma &= -\mathcal{A}^{\text{hom}} \nabla u_0 \, n^+|_\Sigma = \dfrac{k_2^2}{2} \, \boldsymbol{\Theta} \, [u_0]_\Sigma \, , \\ u_0 &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Remark 3.4.6. *In the case where A_0 is even with respect to z_N, θ vanishes on S. Then, $\mathbf{\Theta}(x')$ can be interpreted as the local capacity of the set S, the capacitary potential being $(1 \mp \theta^{\pm})$.*

Remark 3.4.7.

1. *The proof for the case $k_2 = 0$ is actually simpler and the statement is included in Theorem 3.4.3: the holes are too small to keep any connection between Ω_+ and Ω_-. The limit problem is split into two independent problems in each of these sets with mixed homogeneous boundary conditions,*

$$\begin{cases} -\mathrm{div}\mathcal{A}^{\mathrm{hom}}\nabla u_0 &= f \quad \text{in } \Omega_{\pm}, \\ \mathcal{A}^{\mathrm{hom}}\nabla u_0\, n^{\pm}|_{\Sigma} &= 0 \quad \text{on } \Sigma, \\ u_0 &= 0 \quad \text{on } \partial\Omega_{\pm} \setminus \Sigma. \end{cases}$$

2. *For the case of $\lim \dfrac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} = \infty$ Theorem 3.1.24 (2),*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0^{\pm}|_{\Sigma} \text{ weakly in } L^2(\Sigma; L^2_{loc}(\mathbb{R}^N_{\pm})).$$

On the other hand, $[\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})]_S = 0$ on $\Sigma \times S$ implies that $[u_0]|_{\Sigma} = 0$. Therefore, the limit problem is satisfied in the whole of Ω.

3.5 The thick Neumann sieve with variable coefficients

In this section we extend the results of Section 3.4 to the case of a thick Neumann sieve of thickness of order $\varepsilon > 0$. We will use the same notations, unless specified othewise, and we only sketch the main modifications of setting and of the proof.

For an open subset S of $Y \cap \Pi$ such that $S \subset\subset (Y \cap \Pi)$, we introduce the class $\mathcal{F}_{\mathcal{S}}$ of admissible sets, which we use to describe a thick sieve with holes shaped according to S.

Definition 3.5.1. *The subset set F of $\mathbb{R}^N$ is in $\mathcal{F}_{\mathcal{S}}$, if*

i) F is closed with connected complement in $\mathbb{R}^N$,

ii) F is symmetric with respect to all the hyperplanes of equations $\{z_j = 0,\ j \in \{1, \dots, N-1\}$ and $F = F_+ \cup F_- \cup \{\Pi \setminus S\}$,

iii) F is such that $F \cap \frac{1}{\delta}\overline{Y} \subset \Big\{|z_N| \leq \dfrac{1}{2\delta}\Big\}$ for any $0 < \delta << 1$,

iv) F_+ and F_- are unbounded with Lipschitz boundary,

v) there exists some positive R such that the boundaries ∂F_+ and ∂F_- outside the ball of radius R, are Lipschitz graphs over $\mathbb{R}^{N-1}$.

For $F \in \mathcal{F}_{\mathcal{S}}$, set

$$F_\delta = \delta F \cap Y, \quad \text{and} \quad F_{\varepsilon,\delta} = \Big\{x \in \Sigma'_\varepsilon \text{ such that } \Big\{\frac{x}{\varepsilon}\Big\}_Y \in F_\delta\Big\}.$$

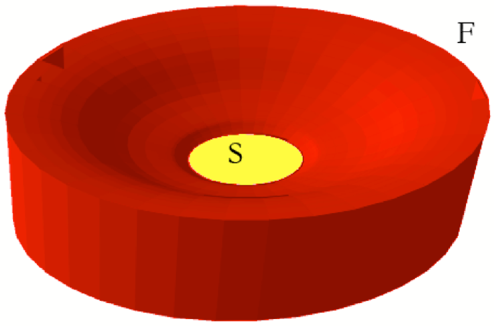


Figure 3.7: **An example of set F: the hole in the sieve**

Define

$$\Omega^{ns}_{\varepsilon\delta} = \Omega \setminus F_{\varepsilon\delta} \quad \text{and} \quad S_{\varepsilon,\delta} = \Omega^{ns}_{\varepsilon\delta} \cap \Pi.$$

Figure 3.7 present an example of admissible set F in dimensions 3. Figure 3.8 is the corresponding sieve. Figure 3.9 is a two dimensional cross-section.

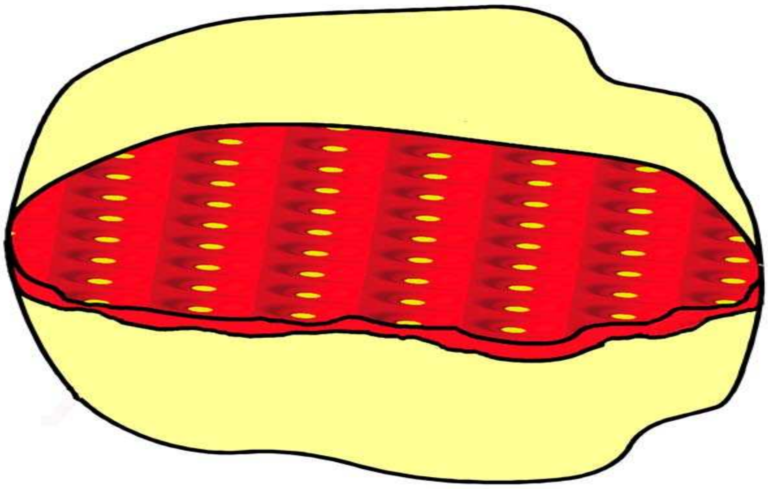


Figure 3.8: **The 3D geometry of the thick Neumann sieve**

We use the same space V as in Section 5, while the $V_{\varepsilon,\delta}$ is now

$$V_{\varepsilon,\delta} = \big\{ v \in H^1(\Omega^{ns}_{\varepsilon\delta\,+} \cup \Omega^{ns}_{\varepsilon\delta\,-}), v|_{\partial\Omega} = 0, [v]_{S_{\varepsilon,\delta}} = 0 \big\}.$$

The thick Neumann sieve problem can be stated as follows:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in V_{\varepsilon,\delta} \text{ satisfying} \\ \displaystyle\int_{\Omega^{ns}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega^{ns}_{\varepsilon,\delta}} f\phi, \quad f \in L^2(\Omega), \\ \forall \phi \in V_{\varepsilon,\delta}. \end{cases} \qquad (\mathcal{P}^{ns}_{\varepsilon,\delta})$$

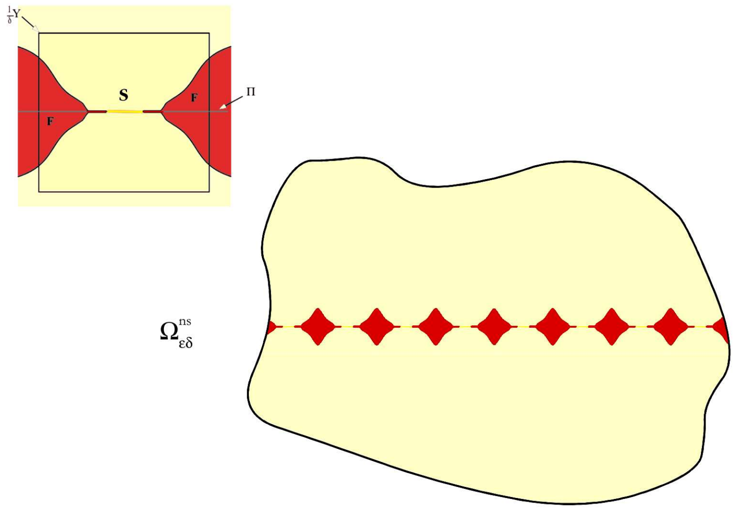


Figure 3.9: **A 2D cross-section of the set F and the domain $\Omega^{ns}_{\varepsilon,\delta}$**

The equivalent of the space $\widehat{K_S}$ (see (3.4.1)) is the following, where G denotes the complement of F:

$$\widetilde{K_G} = \Big\{ \Phi \in H^1_{loc}(G)\ ;\ \ \nabla\Phi \in L^2(G) \Big\}. \tag{3.5.1}$$

Proposition 3.5.2. *There exit two linear forms $l^{\pm}$ on $\widetilde{K_G}$ such that for every Φ in $\widetilde{K_G}$, the functions $\Phi^{\pm} - l^{\pm}(\Phi)$ belong to $L^{2^*}((\mathbb{R}^N \setminus F)_{\pm})$.*

The space $\widetilde{K_G}$ is a Hilbert space for the norm given by

$$\|\Phi\|^2_{\widetilde{K_G}} \doteq \|\nabla\Phi\|^2_{L^2((\mathbb{R}^N_+ \cup \mathbb{R}^N_-)\setminus F)} + l^+(\Phi)^2 + l^-(\Phi)^2.$$

Furthermore, for this norm, l^+ and l^- are continuous on $\widetilde{K_G}$ and

$$\widetilde{K_G}^{\infty} \doteq \Big\{ \Phi \in \widetilde{K_G}, \Phi \in C^\infty(G),\ \mathrm{supp}(\nabla\Phi)\ \textit{bounded in}\ G \Big\},$$

is dense in $\widetilde{K_G}$.

Proof. The proof is the same as that of Proposition 3.4.1. The only modification concerns the sequence of sets on which the Sobolev-Poincaré-Wirtinger inequality (with a uniform constant) is applied. In view of Definition 3.5.1(iv), this can be achieved on the sets $\frac{1}{\delta}Y_{\pm} \cap \{\pm z_N > R\} \cap G$ (making use of [77]). □

The unfolded limit problem and the standard homogenized equation are given in the next two theorems. Up to the modifications of notations indicated above, theirs proofs are the same as in Section 3.4.

Theorem 3.5.3. *Let Ω be open and bounded in $\mathbb{R}^N$, $N \geq 3$ and A^ε belong to $M(\alpha, \beta, \Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon\big(A^\varepsilon\big)(x, y) \to A(x, y) \quad \textit{a.e. in}\ \Omega \times Y.$$

Furthermore, suppose that there exists a matrix field A_0 *such that, as* ε *and* $\delta \to 0$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}\big(A^\varepsilon\big)(x',z) \to A_0(x',z) \quad a.e. \ in \ \Sigma \times (\mathbb{R}^N \setminus F).$$

Let $u_{\varepsilon,\delta}$ *be the solution of the problem* $(\mathcal{P}^{ns}_{\varepsilon,\delta})$. *Then*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad weakly \ in \ H^1_{loc}(\Omega \setminus \Sigma),$$

and there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$, $U \in L^2(\Sigma; \widetilde{K}_G)$ *satisfying*

$$l^\pm(U) = k_2(u_0^\pm)_{|\Sigma} \ \ for \ a.e. \ \ x' \in \Sigma,$$

and such that $(u_0, \widehat{u}, U)$ *solves the equations*

$$\int_Y A(x,y)\big(\nabla_x u_0(x) + \nabla_y \widehat{u}(x,y)\big)\nabla_y \phi(y)\, dy = 0,$$

for a.e. x *in* Ω *and all* $\phi \in H^1_{per}(Y)$ *;*

$$\int_G A_0(x',z)\, \nabla_z U(x',z)\, \nabla_z v(z)\, dz = 0,$$

for a.e. x' *in* Σ *and all* $v \in \widetilde{K}_G$ *with* $l^\pm(v) = 0$ *;*

$$\int_{\Omega\times Y} A\,\big(\nabla_x u_0 + \nabla_y \widehat{u}\big)\nabla\phi - k_2 \int_{\Sigma\times S} A_0\ \nabla_z U\, n^+\ [\phi]_\Sigma\ = \int_\Omega f\,\phi,$$

for all $\phi \in V$.

Theorem 3.5.4. *The limit function* u_0 *given by Theorem 3.5.3 is the solution of the homogenized equation*

$$\begin{cases} u_0 \in V, \\ \displaystyle\int_\Omega \mathcal{A}^{\rm hom}\nabla u_0\, \nabla\phi + \frac{k_2^2}{2}\int_\Sigma \mathbf{\Theta}\ [u_0]_\Sigma\ [\phi]_\Sigma = \int_\Omega f\,\phi, \\ \forall \phi \in V, \end{cases}$$

where

$$\mathbf{\Theta}(x') = \frac{1}{2}\int_G A_0(x',z)\, \nabla_z\theta(x',z)\, \nabla_z\theta(x',z)\, dz,$$

and θ *is the solution of the cell-problem*

$$\begin{cases} \theta \in L^\infty(\Sigma; \widetilde{K_G}),\ l^\pm(\theta(x',\cdot)) \equiv \pm 1, \\ \displaystyle\int_G {}^t\!A_0(x',z)\, \nabla_z\theta(x',\,z)\ \nabla_z\Psi(z)\, dz = 0, \qquad a.e.\ for\ x' \in \Sigma, \\ \forall \Psi \in \widetilde{K_G} \quad with \quad l^\pm(\Psi) = 0. \end{cases}$$

Remark 3.5.5. *The function* $\boldsymbol{\Theta}(x')$ *can be interpreted as the local relative capacity (in* G*) of the set* $\mathcal{C}(x')$ *defined as the set where* $\theta(x', \cdot)$ *vanishes, the capacitary potential being* $(1-\theta(x', \cdot))$ *"above* $\mathcal{C}(x')$*" and* $(1+\theta(x', \cdot))$ *"below* $\mathcal{C}(x')$*".*

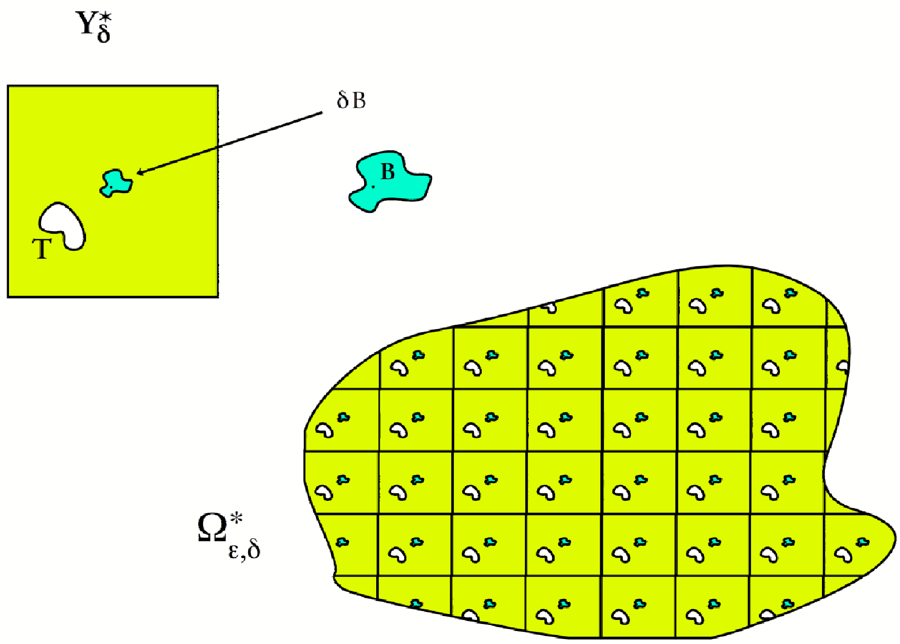


Figure 3.10: **The combination of a Neumann hole T and a Dirichlet hole** δ**B**

Chapter 4

A class of Steklov type problems associated to the Neumann sieve

In this chapter we study a spectral problem associated to the Neumann sieve (see Figure 4.1 for the geometry of the problem). Consider a plane Σ that separates a three dimensional domain Ω in two subdomains Ω_+ and Ω_- and distribute ϵ-periodically on Σ two dimensional small holes of diameter $\epsilon\delta(\epsilon) < \epsilon$ denoted by $S_{\epsilon,\delta}$.

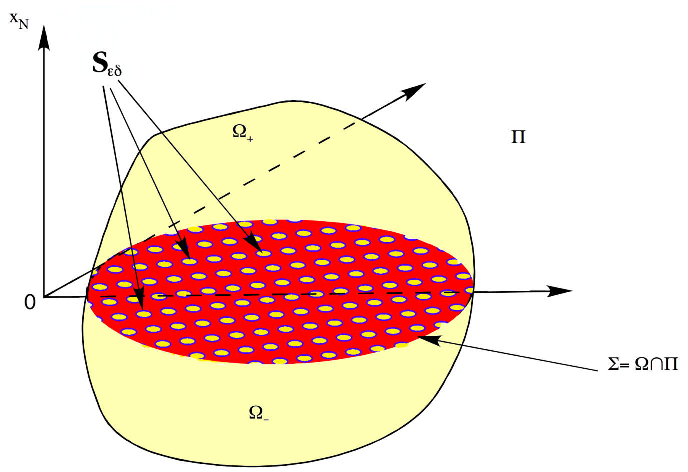


Figure 4.1: **The geometry of the Neumann Sieve**

Set

$$V = \{u \in H^1(\Omega_+) \cup H^1(\Omega_-) \mid u = 0 \ \text{ on } \ \partial\Omega\}$$

and

$$V_{\epsilon,\delta} = \{u \in V \mid [u] = 0 \ \text{ on } \ S_{\epsilon,\delta}\}$$

where $[u] = u^+ - u^-$ and $u^+ = u$ on Ω_+ and $u^- = u$ on Ω_-.

The Steklov-type spectral problem associated to the Neumann sieve problem, (see

Damlamian [30], Attouch [6]), is

$$\begin{cases} -\Delta u^\epsilon = 0 & \text{in} \quad \Omega_+ \cup \Omega_- \cup S_{\epsilon,\delta} \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = \lambda^\epsilon [u^\epsilon] & \text{on} \quad \Sigma - S_{\epsilon,\delta} \\ u^\epsilon = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.0.1}$$

An equivalent formulation can be expressed in terms of the Dirichlet to Neumann (DtN) map. For this, we consider for any z in $H^{\frac{1}{2}}(\Sigma)$ the solution v of the following problem

$$\inf\{\frac{1}{2}|\nabla v|^2_{L^2(\Omega_+\cup\Omega_-)} \mid v \in V \text{ with } [v]_\Sigma = z\}.$$

Let n be the unit normal to Σ towards Ω_+. Then denote by L the map from z to $Lz \doteq \frac{\partial v^+}{\partial n} = -\frac{\partial v^-}{\partial n}$. L is a well defined fixed operator from $H^{\frac{1}{2}}(\Sigma)$ to $H^{-\frac{1}{2}}(\Sigma)$. It is known that L^{-1} is onto V and is compact (see [78]).
Let $TV_{\epsilon,\delta}$ the subspace of $H^{\frac{1}{2}}(\Sigma)$ of elements which vanish on $S_{\epsilon,\delta}$ i.e, $TV_{\epsilon,\delta}$ is the trace subspace of $V_{\epsilon,\delta}$. An equivalent formulation to (4.0.1) is then, find $z_\epsilon \in TV_{\epsilon,\delta}$ and $\lambda^\epsilon \in \mathbf{R}$, such that

$$L(z_\epsilon) = \lambda^\epsilon z_\epsilon. \tag{4.0.2}$$

The spectral problem (4.0.1) is associated to the Neumann sieve problem:

$$\begin{cases} -\Delta u^\epsilon = f & \text{in} \quad \Omega_+ \cup \Omega_- \cup S_{\epsilon,\delta} \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = 0 & \text{on} \quad \Sigma - S_{\epsilon,\delta} \\ u^\epsilon = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.0.3}$$

Depending on the order of $\delta(\epsilon)$ with respect to ϵ it has been proved in Damlamian [30] that the homogenized problem is of the form

$$\begin{cases} -\Delta u = f & \text{on} \quad \Omega_+ \cup \Omega_- \\ \dfrac{\partial u^+}{\partial n} = -\dfrac{\partial u^-}{\partial n} = \dfrac{C}{4}[u] & \text{on} \quad \Sigma \end{cases}$$

where $C = 0$ if $\delta(\epsilon) \ll \epsilon$, C is the capacity in $\mathbf{R}^3$ of the holes if $\delta(\epsilon) \approx \epsilon$ or $C = \infty$ if $\delta(\epsilon) \gg \epsilon$ and $[u]$ was defined above.

This type of behavior was first observed in the work of Cioranescu and Murat [26] where the same problem but with three dimensional holes periodically distributed in the entire domain or on a hyperplane was studied.

Homogenization of a Stekloff type problem for perforated domains with three dimensional ϵ sized holes distributed in the entire domain has been studied in Vanninathan [81], using multiscale analysis and Tartar's method.

In Section 4.2 we set the functional framework and the problem to be analyzed. By using G-convergence techniques and the homogenization result of (4.0.3) obtained by Damlamian [30] we obtain in Section 4.3 the limit problem for (4.0.1),

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega_+ \cup \Omega_- \\ \dfrac{\partial u^+}{\partial n} = -\dfrac{\partial u^-}{\partial n} = \left(\lambda - \dfrac{C}{4}\right)[u] & \text{on} \quad \Sigma \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

or equivalently the limit problem for (4.0.2),

$$Lz = (\lambda - \frac{C}{4})z$$

where $L : H^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma)$ is the DtN operator defined above, and λ is a limit point of a sequence of eigenvalues $\{\lambda^\epsilon\}_{\epsilon>0}$ of (4.0.2) or (4.0.1).

We show that the entire sequence formed by the n-th eigenvalue of the ϵ-problem, i.e $\{\lambda_n^\epsilon\}_\epsilon$ converges to the n-th eigenvalue of the limit problem (4.0.4). When λ_n is a simple eigenvalue we can prove that the entire sequence of eigenvectors, u_n^ϵ associated to λ_n^ϵ for the problem (4.0.1) will converge to the eigenvector u_n associated to λ_n. Subsections 4.3.1 and 4.3.2 present the cases when $\lim_{\epsilon\to 0} \frac{\delta(\epsilon)}{\epsilon} = \infty$ and $\lim_{\epsilon\to 0} \frac{\delta(\epsilon)}{\epsilon} = 0$ respectively.

In the form (4.0.1) our problem is related to the modelling of earthquake initiation phase where one has a periodic system of faults on which slip-weakening friction is considered. The eigenvalues λ^ϵ provide stability properties of the solutions of the dynamic problem (see [45] and [47]). Also (4.0.1) can be considered as the spectral problem associated to a heat conduction problem where imperfectly conducting interfaces are present (see Sanchez-Palencia [62], Lipton and Vernescu [55] and Belyaev et al. [10]).

In the form (4.0.2) the problem is a spectral problem for DtN operator in domains perforated along a hyperplan. The asymptotic behavior of the spectrum is similar and is obtained as a consequence of the analysis for (4.0.1).

4.1 Problem Statement

Consider an open set $\Omega \subset \mathbf{R}^3$ and a plane Σ that separates Ω into two open subsets Ω_+, Ω_- such that

$$\Omega = \Omega_+ \cup \Omega_- \cup \Sigma$$

For simplicity we will consider in the sequel $\Sigma = \{z = 0\}$.

We define $Y = [0, 1]^2$ as the reference square and an open set $S \subset Y$. With $0 < \delta(\epsilon) \leq 1$ we construct on Σ ϵ-periodically distributed obstacles obtained by $\epsilon\delta(\epsilon)$-homothety from S and denote by $S_{\epsilon,\delta}$ its union:

$$S_{\epsilon,\delta} = \bigcup_{k\in\mathbb{Z}^2} (\epsilon\delta(\epsilon)S + k\epsilon)$$

We introduce the natural functional framework for our problem by defining

$$V = \{u \in H^1(\Omega_+) \cup H^1(\Omega_-) \mid u = 0 \text{ on } \partial\Omega\}, \quad V_{\epsilon,\delta} = \{u \in V \mid [u] = 0 \text{ on } S_{\epsilon,\delta}\}$$

where $[u]$ denotes the jump on Σ defined as above. V is a Hilbert space endowed with the following scalar product:

$$\langle u, v\rangle_V = \int_{\Omega_+\cup\Omega_-} \nabla u \nabla v$$

and $V_{\epsilon,\delta}$ is a subspace of V.

Let us remark that $H_0^1(\Omega)$ is a closed subspace of $V_{\epsilon,\delta}$ and denote by $W_{\epsilon,\delta} = (H_0^1(\Omega))^{\perp}$ its orthogonal in $V_{\epsilon,\delta}$ and by $W = (H_0^1(\Omega))^{\perp}$ its orthogonal in V. Thus $V_{\epsilon,\delta} = H_0^1(\Omega) \oplus W_{\epsilon,\delta}$ and , $V = H_0^1(\Omega) \oplus W$. Let us also define $P_{W_{\epsilon,\delta}} : V_{\epsilon,\delta} \to W_{\epsilon,\delta}$ the orthogonal projection onto $W_{\epsilon,\delta}$.

Also it is easy to see that the trace space of $V_{\epsilon,\delta}$, $TV_{\epsilon,\delta}$ is identical with the trace space of $W_{\epsilon,\delta}$, $TW_{\epsilon,\delta}$.

In this setting the problem (4.0.2) is equivalent with the following spectral problem: find $u^\epsilon \in V_{\epsilon,\delta}, \lambda^\epsilon \in \mathbf{R}_+$ such that

$$(\mathcal{P}_\epsilon) \quad \begin{cases} -\Delta u^\epsilon = 0 & \text{on } \ \Omega_+ \cup \Omega_- \cup S_{\epsilon,\delta} \\ \dfrac{\partial (u^\epsilon)^+}{\partial n} = -\dfrac{\partial (u^\epsilon)^-}{\partial n} = \lambda^\epsilon [u^\epsilon] & \text{on } \ \Sigma - S_{\epsilon,\delta} \end{cases} \tag{4.1.1}$$

The corresponding equivalent variational formulation is: find $u^\epsilon \in V_{\epsilon,\delta}, \lambda^\epsilon \in \mathbb{R}_+$ such that

$$\int_{\Omega_+ \cup \Omega_-} \nabla u^\epsilon \nabla w = \lambda^\epsilon \int_{\Sigma - S_{\epsilon,\delta}} [u^\epsilon][w], \ \text{ for any } \ w \in V_{\epsilon,\delta} \tag{4.1.2}$$

The equivalence between the problems (4.0.2) and (4.1.1) is understood in the sense that they have the same eigenvalues and related eigenvectors. Thus if $\{z_n^\epsilon\}_{n\in\mathbb{N}}$ is an orthonormal sequence of eigenvectors for (4.0.2), then the sequence $\{\frac{1}{\sqrt{\lambda_n^\epsilon}} u_n^\epsilon\}_{n\in\mathbb{N}}$, where u_n^ϵ is, for any $n \in \mathbb{N}$, the solution of

$$\inf\{\frac{1}{2}|\nabla v|^2_{L^2(\Omega_+\cup\Omega_-)} \mid v \in W \text{ with } [v]_\Sigma = z_n^\epsilon, \}. \tag{4.1.3}$$

is an orthonormal sequences of eigenvectors for problems (4.1.1). Conversely, if $\{u_n^\epsilon\}_{n\in\mathbb{N}}$ is an orthonormal sequence of eigenvectors for (4.1.1) then the sequence $\{z_n^\epsilon \sqrt{\lambda_n^\epsilon}\}_{n\in\mathbb{N}}$ with

$$z_n^\epsilon = [u_n^\epsilon] \text{ for } \ (n \in \mathbb{N}) \tag{4.1.4}$$

is an orthonormal sequence for (4.0.2).

From the compactness of L^{-1}, we have that L has an increasing sequence of eigenvalues $\{\lambda_n^\epsilon\}_{n\in\mathbf{N}}$ and an orthonormal sequence of corresponding eigenvectors $\{z_n^\epsilon\}_{n\in\mathbf{N}}$ in $L^2(\Sigma)$. Also from the Rayleigh's principle we have

$$\lambda_n^\epsilon = \inf_{\substack{z\in TW_{\epsilon,\delta}, z\perp z_i^\epsilon \\ i=\overline{1,n-1}}} \frac{< Lz, z >_{L^2(\Sigma)}}{\| z \|^2_{L^2(\Sigma)}} \tag{4.1.5}$$

4.2 Asymptotic analysis

Because of the equivalence relations (4.1.3), (4.1.4) we will study only the asymptotic behavior of (4.1.1). The similar results for the problem (4.0.2) will be stated as

corollaries.
Now it is easy to observe that from the equivalence relation, (4.1.3) and (4.1.4) we have

$$\inf_{\substack{z\in TW_{\epsilon,\delta}, z\perp z_i^\epsilon \\ i=\overline{1,n-1}}} \frac{< Lz, z >_{L^2(\Sigma)}}{\| z \|^2_{L^2(\Sigma)}} = \inf_{\substack{u\in W_{\epsilon,\delta}, u\perp u_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\| u \|_V^2}{\displaystyle\int_\Sigma [u]^2 d\sigma} \tag{4.2.1}$$

From (4.1.5) and (4.2.1) we get the following representation for λ_n^ϵ, i.e

$$\lambda_n^\epsilon = \inf_{\substack{u\in W_{\epsilon,\delta}, u\perp u_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\| u \|_V^2}{\displaystyle\int_\Sigma [u]^2 d\sigma} \tag{4.2.2}$$

Lemma 4.2.1. *If* $\lim\limits_{\epsilon\to 0} \dfrac{\delta(\epsilon)}{\epsilon} < \infty$ *then* $C_1 \leq \lambda_n^\epsilon$ *and* $\limsup\limits_{\epsilon} \lambda_n^\epsilon < \infty$ *where* C_1 *is a constant with respect to* ϵ *and* n.

Proof. Using the trace continuity and (4.2.2) we obtain

$$\lambda_n^\epsilon \geq C_1 \text{ for any } n \in \mathbf{N} \tag{4.2.3}$$

with C_1 not depending on ϵ, and therefore $\{\lambda_n^\epsilon\}$ is uniformly bounded from below.

We will prove next that all the limit points λ_n of $\{\lambda_n^\epsilon\}_\epsilon > 0$ are finite. We consider the following capacity potential:

$$\begin{cases} -\Delta w^\epsilon = 0 & \text{in} \quad B_\epsilon - \epsilon\delta(\epsilon)S \\ w^\epsilon = 1 & \text{on} \quad \epsilon\delta(\epsilon)S \\ w^\epsilon = 0 & \text{on} \quad \partial B_\epsilon \end{cases}$$

where B_ϵ is the ball of radius ϵ centered in the ϵY cube. The function w^ϵ is extended by periodicity on a layer of size ϵ around Σ and then by zero to $\mathbf{R}^3$. The sequence w^ϵ has the property that (see [6])

$$w^\epsilon \rightharpoonup 0 \quad \text{weakly in} \quad H^1(\Omega).$$

Consider $u \in V \cap [C^\infty(\Omega_+) \cup C^\infty(\Omega_-)]$, with the orthogonal decomposition $u = \bar{u}_1 + \bar{u}_2$, where $\bar{u}_1 \in W$ and $\bar{u}_2 \in H_0^1(\Omega)$. We suppose that $\bar{u}_1 \neq 0$ and $\bar{u}_2 \neq 0$. Define $z^\epsilon = (1 - w^\epsilon)u$, then z^ϵ satisfies

$$z^\epsilon \rightharpoonup u \ \text{ weakly in } V, \ \ [z^\epsilon] = (1 - w^\epsilon)[\bar{u}_1] \ \text{ on } \ \Sigma \ \text{ and } z^\epsilon \in V_{\epsilon,\delta}.$$

We make the observation that, for ϵ small enough, $z^\epsilon \notin H_0^1(\Omega)$ and $z^\epsilon \notin W_{\epsilon,\delta}$. Indeed, we have $[z^\epsilon] = (1 - w^\epsilon)[\bar{u}_1] \neq 0$ on Σ, since $\bar{u}_1 \in W$. On the other hand letting ϵ go to zero we obtain:

$$\lim_{\epsilon\to 0} < z^\epsilon, \bar{u}_2 >= \int_\Omega \nabla u \nabla \bar{u}_2 =\| \bar{u}_2 \|_V^2 > 0.$$

Therefore there exists $\epsilon_0 > 0$ such that $< z^\epsilon, \bar{u}_2 > \neq 0$ for any $\epsilon < \epsilon_0$, i.e $z^\epsilon \notin W_{\epsilon,\delta}$ for any $\epsilon < \epsilon_0$.

From (4.2.2) we have that

$$\lambda_1^\epsilon \leq \frac{\| P_{W_{\epsilon,\delta}} z^\epsilon \|_V^2}{\int_\Sigma [P_{W_{\epsilon,\delta}} z^\epsilon]^2} \leq \frac{\| z^\epsilon \|_V^2}{\int_\Sigma [z^\epsilon]^2} \leq \frac{C_1}{\int_\Sigma [z^\epsilon]^2},$$

where we used the orthogonal decomposition $V_{\epsilon,\delta} = W_{\epsilon,\delta} \oplus H_0^1(\Omega)$ in order to obtain

$$\int_\Sigma [P_{W_{\epsilon,\delta}} z^\epsilon]^2 = \int_\Sigma [z^\epsilon]^2.$$

Since $\{z^\epsilon\}$ is weakly convergent to u and using the continuity of the trace we get

$$\limsup_{\epsilon \to 0} \lambda_1^\epsilon \leq \frac{C_1}{\int_\Sigma [\bar{u}_1]^2} < \infty$$

where C_1 is a constant independent of ϵ.

Next we will use an induction argument to prove the statement for all $n \in \mathbf{N}$. Let's assume that

$$\limsup_{\epsilon \to 0} \lambda_k^\epsilon < \infty \qquad \text{for any } k \leq n-1. \tag{4.2.4}$$

We need to prove

$$\limsup_{\epsilon \to 0} \lambda_n^\epsilon < \infty$$

Let $\{\lambda_n^\epsilon\}_{\epsilon>0}$ be a subsequence of $\{\lambda_n^\epsilon\}_{\epsilon>0}$ still denoted by ϵ. Then, using the induction hypothesis (4.2.4), the orthonormality of the associated sequence of eigenvectors and a diagonalization argument we find a decreasing sequence $\{\epsilon_j\}_{j\in\mathbf{N}}$, such that $\epsilon_j \to 0$ and

$$u_k^{\epsilon_j} \stackrel{j}{\rightharpoonup} u_k \in \mathrm{W} \tag{4.2.5}$$

$$\lim_{j\to\infty} \lambda_k^{\epsilon_j} \doteq \lambda_k < \infty \tag{4.2.6}$$

for $k = \overline{1, n-1}$

Let z^ϵ be as in the proof of Lemma (4.2.1), with

$$\bar{u}_1 \notin \ span\{u_1, ..., u_{n-1}\} \tag{4.2.7}$$

We can do that because W has infinite dimension.

From (4.2.2) we obtain

$$\lambda_n^{\epsilon_j} = \inf_{\substack{u \in W_{\epsilon_j,\delta_j},\, u \perp u_i^{\epsilon_j} \\ i=\overline{1,n-1}}} \frac{\| u \|_V^2}{\int_\Sigma [u]^2 d\sigma} \tag{4.2.8}$$

Consider now

$$\bar{z}^{\epsilon_j} = z^{\epsilon_j} - \sum_{i=1}^{n-1} u_i^{\epsilon_j} < z^{\epsilon_j}, u_i^{\epsilon_j} >_V$$

First we can see that

$$< \bar{z}^{\epsilon_j}, u_i^{\epsilon_j} >_V = 0 \text{ for any } i = \overline{1, n-1} \tag{4.2.9}$$

Then $\bar{z}^{\epsilon_j} \in V_{\epsilon_j \delta_j}$ and $\bar{z}^{\epsilon_j} \notin H_0^1(\Omega)$ for j big enough. Indeed from (4.1.2) we have

$$< z^{\epsilon_j}, u_i^{\epsilon_j} >_V = \lambda_i^{\epsilon_j} \int_\Sigma [u_i^{\epsilon_j}][z^{\epsilon_j}]$$

and using the trace continuity, the definition of z^{ϵ_j}, (4.2.5) and (4.2.6) in the above relation implies

$$\bar{z}^{\epsilon_j} \rightharpoonup \bar{z} \doteq u - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i].$$

If we suppose

$$\left[u - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] \right] = 0$$

this is equivalent with

$$\left[\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] \right] = 0$$

which implies

$$\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] = 0 \tag{4.2.10}$$

because $\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_\Sigma [\bar{u}_1][u_i] \in W$ and W is orthogonal on $H_0^1(\Omega)$. But (4.2.10) leads to a contradiction with (4.2.7).

Therefore $[\bar{z}] \neq 0$ and this implies the statement, i.e $\bar{z}^{\epsilon_j} \notin H_0^1(\Omega)$ for j big enough.

Next using (4.2.9) and (4.2.8) we obtain

$$\lambda_n^{\epsilon_j} \leq \frac{\| P_{W_{\epsilon_j \delta_j}} \bar{z}^{\epsilon_j} \|_V^2}{\displaystyle\int_\Sigma [P_{W_{\epsilon_j, \delta_j}} \bar{z}^{\epsilon_j}]^2} \leq \frac{\| z^{\epsilon_j} \|_V^2}{\displaystyle\int_\Sigma [\bar{z}^{\epsilon_j}]^2} \leq \frac{C_1}{\displaystyle\int_\Sigma [\bar{z}^{\epsilon_j}]^2}$$

where C_1 is a constant independent of j. Passing to the limit when $j \to \infty$ we obtain

$$\limsup_{j \to \infty} \lambda_n^{\epsilon_j} \leq \frac{C_1}{\displaystyle\int_\Sigma [\bar{z}]^2} < \infty. \tag{4.2.11}$$

So we have proved that any subsequence of λ_n^ϵ has a subsequence $\{\lambda_n^{\epsilon_j}\}_{j \in \mathbf{N}}$ such that (4.2.11) is satisfied. Therefore we have that

$$\limsup_{\epsilon \to 0} \lambda_n^\epsilon < \infty$$

for any $n \in \mathbf{N}$ □

The next Corollary shows that the weak-limits u_n of the sequence $\{u_n^\epsilon\}_{\epsilon>0}$ of the normal eigenvectors associated to the eigenvalue λ_n^ϵ, cannot be zero.

Corollary 4.2.2. *Let $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ be the orthonormal sequence of eigenvectors associated to λ_n^ϵ for the problem $(\mathcal{P}_\epsilon)$. Then every weak-limit u_n of $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ (i.e., u_n such that on a subsequence $u_n^\epsilon \overset{\epsilon}{\rightharpoonup} u_n$), is nonzero.*

Proof. Because $\| u_n^\epsilon \|= 1$ a subsequence, still denoted by u_n^ϵ, will weakly converge to some u_n. Using the variational form of $(\mathcal{P}_\epsilon)$ we have

$$\lambda_n^\epsilon = \frac{1}{\int_\Sigma [u_n^\epsilon]^2}.$$

Letting ϵ go to zero above we obtain

$$\limsup \lambda_n^\epsilon = \frac{1}{\int_\Sigma [u_n]^2}.$$

Next using Lemma (4.2.1) we obtain that

$$\int_\Sigma [u_n]^2 \neq 0.$$

and this implies the statement. □

Remark 4.2.3. *Similar results hold for the problem (4.1.1),i.e, all the strong-$L^2(\Sigma)$ limit points of the sequence $\{z_n^\epsilon\}_\epsilon$ are nonzero.*

Let us now consider the duality operator $J^\epsilon \ : \ V_{\epsilon,\delta} \to (V_{\epsilon,\delta})'$

$$\langle J^\epsilon u, w\rangle_{(V_{\epsilon,\delta})', V_{\epsilon,\delta}} = \langle u, w\rangle_{V_{\epsilon,\delta}} \text{ for any } u, w \in V_{\epsilon,\delta}$$

J^ϵ is an operator of subdifferential type

$$J^\epsilon = \partial\varphi^\epsilon, \ \ \varphi^\epsilon \ : \ V_{\epsilon,\delta} \to R \tag{4.2.12}$$

$$\varphi^\epsilon(u) = \frac{1}{2} \| u \|^2_{V_{\epsilon,\delta}} \tag{4.2.13}$$

By using the results in Damlamian [30] and Attouch [6] we have the following lemma:

Lemma 4.2.4. *The sequence of functionals $\{\varphi^\epsilon\}$ is Γ-convergent weakly in V to φ given by*

$$\varphi(u) = \frac{1}{2}\left(\| u \|^2_V + \frac{C}{4}\int_\Sigma [u]^2\right)$$

where $C = \begin{cases} R \cdot cap\ S & \text{if } \lim\limits_{\epsilon\to 0} \dfrac{\delta(\epsilon)}{\epsilon} = R < \infty \\ \infty & \text{if } \lim\limits_{\epsilon\to 0} \dfrac{\delta(\epsilon)}{\epsilon} = \infty \end{cases}$

We have used $capS$ for the capacity of the set S in $\mathbf{R}^3$, i.e

$$capS = inf\{\int_{\mathbf{R}^3} | \nabla w |^2 dx \mid w \in H^1(\mathbf{R}^3), w \geq 1 \quad \text{a.e. on } S\}.$$

Corollary 4.2.5. *The sequence of operators J^ϵ is G convergent to $\partial\varphi$, with respect to the weak $\times$ strong topology in $V \times V'$.*

Proof. Using the G-convergence result for subdifferentials of Γ-convergent sequences (see Attouch [6]-Th.3.67) we have that the Γ-convergences of the sequence φ^ϵ to φ imply the G-convergence of the subdifferentials,

$$\partial\varphi^\epsilon \stackrel{G}{\rightarrow} \partial\varphi$$

□

Next, we state the first homogenization result for problem (4.1.1):

Theorem 4.2.6. *There is a decreasing sequence $\{\epsilon_j\}_j \in \mathbf{N}$ with $\epsilon_j \to 0$ such that $u_n^{\epsilon_j} \rightharpoonup u_n$, $\lambda_n^{\epsilon_j} \to \lambda_n$ where (λ_n, u_n) solves the limit problem $(\mathcal{P})$:*

$$(\mathcal{P}) \quad \begin{cases} -\Delta u_n = 0 \\ \dfrac{\partial (u_n)^+}{\partial n} = -\dfrac{\partial (u_n)^-}{\partial n} = \left(\lambda_n - \dfrac{C}{4}\right)[u_n] \end{cases}$$

where $C \neq \infty$ is as in Lemma 4.2.4.

Proof. Let an arbitrary fixed $n \in \mathbf{N}$. Let $\{\lambda_n^\epsilon\}_{\epsilon>0}$ be the sequence of eigenvalues for the problem $(\mathcal{P}_\epsilon)$ and u_n^ϵ the corresponding orthonormal sequence of eigenvectors. Then there is a subsequence $\{\epsilon_j\}_j \in \mathbf{N}$ such that:

$$u_n^{\epsilon_j} \rightharpoonup u_n \text{ and } \lambda_n^{\epsilon_j} \to \lambda_n$$

We have proved in Lemma 4.2.1 that $\lambda_n < \infty$.
Let $f_n^{\epsilon_j} \in V'$ be defined as

$$f_n^{\epsilon_j}(w) = \lambda_n^{\epsilon_j} \int_\Sigma [u_n^{\epsilon_j}][w] \quad \text{for all } w \in V.$$

Using the variational formulation (4.1.2) we have:

$$f_n^{\epsilon_j}(w) = < J^{\epsilon_j} u_n^{\epsilon_j}, w >_{(V_{\epsilon,\delta})', V_{\epsilon,\delta}} \quad \text{for all } w \in V_{\epsilon_j,\delta_j}.$$

This implies

$$f_n^{\epsilon_j} \in \partial\varphi^{\epsilon_j} \tag{4.2.14}$$

The next observation is that:

$$f_n^{\epsilon_j} \stackrel{j\to\infty}{\longrightarrow} f_n \quad \text{strongly in } V' \tag{4.2.15}$$

where

$$f_n(w) = \lambda_n \int_\Sigma [u_n][w] \quad \text{for all } w \in V$$

The proof of the above convergence is straightforward. Indeed,

$$\| f_n^{\epsilon_j} - f_n \|_{V'} = \sup_{\substack{w \in W \\ \|w\|_V \leq 1}} \left(\lambda_n^{\epsilon_j} \int_\Sigma [u_n^{\epsilon_j}][w] - \lambda_n \int_\Sigma [u_n][w] \right)$$

Now from the reflexivity of the space V we have that there exists $w_0^j \in V$ with $\| w_0^j \|_V \leq 1$ such that

$$\| f_n^{\epsilon_j} - f_n \|_{V'} = \left(\lambda_n^{\epsilon_j} \int_\Sigma [u_n^{\epsilon_j}][w_0^j] - \lambda_n \int_\Sigma [u_n][w_0^j] \right) =$$

$$= (\lambda_n^{\epsilon_j} - \lambda_n) \int_\Sigma [u_n^{\epsilon_j}][w_0^j] + \lambda_n \int_\Sigma [u_n^{\epsilon_j} - u_n][w_0^j]$$

Thus, from Cauchy-Schwartz inequality

$$\| f_n^{\epsilon_j} - f_n \|_{V'} \leq | \lambda_n^{\epsilon_j} - \lambda_n | \left(\int_\Sigma [u_n^{\epsilon_j}]^2 \right)^{1/2} \left(\int_\Sigma [w_0^j]^2 \right)^{1/2} +$$

$$+ \lambda_n \left(\int_\Sigma [u_n^{\epsilon_j} - u_n]^2 \right)^{1/2} \left(\int_\Sigma [w_0^j]^2 \right)^{1/2}.$$

Next we will use the following interpolation inequality (see[46]):

$$\| u \|_{L^2(\Sigma)}^2 \leq M \| u \|_{H^1(\Omega)} \| u \|_{L^2(\Omega)} \quad \forall u \in V \tag{4.2.16}$$

and the fact that $\| w_0^j \|_V \leq 1$ to obtain :

$$f_n^{\epsilon_j} \xrightarrow{j \to \infty} f_n \quad \text{strongly in } \left(V' \right).$$

Therefore from (4.2.14), (4.2.15) and using the Corollary (4.2.5) we obtain that:

$$f_n \in \partial \varphi(u_n). \tag{4.2.17}$$

But (4.2.17) is exactly the problem $(\mathcal{P})$. □

The limit problem $(\mathcal{P})$, is equivalent with the following spectral problem for the DtN operator defined above. Indeed problem $(\mathcal{P})$ is:
Find $\lambda \in \mathbf{R}$ and $z \in H^{\frac{1}{2}}(\Sigma)$ such that:

$$Lz = (\lambda - \frac{C}{4})z. \tag{4.2.18}$$

Using the equivalence relations (4.1.3) and (4.1.4) the next Corollary is a obvious consequence of the above discussions.

Corollary 4.2.7. *There is a decreasing sequence* $\{\epsilon_j\}_j \in \mathbf{N}$ *with* $\epsilon_j \to 0$ *such that* $z_n^{\epsilon_j} \to z_n$, *strongly in* $(L^2(\Sigma))$ *and* $\lambda_n^{\epsilon_j} \to \lambda_n$ *where* (λ_n, z_n) *solves the limit problem :*

$$Lz_n = (\lambda_n - \frac{C}{4})z_n$$

where $C \neq \infty$ *is as in Lemma 4.2.4.*

The main homogenization result is:

Theorem 4.2.8. *If* $\lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$ *then:*

i) $\lim_{\epsilon \to 0} \lambda_n^{\epsilon} = \lambda_n$ *on the entire sequence.*

ii) There is a decreasing sequence $\{\epsilon_j\}_{j \in \mathbf{N}}$ *with* $\epsilon_j \to 0$, *such that* $u_n^{\epsilon_j} \rightharpoonup u_n$ *weakly in* V *and* $z_n^{\epsilon_j} \to z_n$ *strongly in* $L^2(\Sigma)$,

iii) $\lambda_n = (\frac{C}{4} + \beta_n)$ *with* $(Lz_n = \beta_n z_n)$, *where* β_n *is the n-th eigenvalue of the DtN operator* L, *and* C *is as in Lemma 4.2.4.*

Proof. Suppose there is $\lambda \neq \lambda_n$ for any $(n \in \mathbf{N})$ eigenvalue for the limit problem. Let $u \in W$ be the associated normal eigenvector, i.e $\| u \|_V = 1$ and

$$< u, w >= (\lambda - \frac{C}{4}) \int_{\Sigma} [u][w] \quad \text{for all } w \in W.$$

There is $m \in \mathbb{N}$ such that

$$\lambda < \lambda_{m+1} \tag{4.2.19}$$

From the Lax Milgram lemma we have that there exists $w^{\epsilon} \in W_{\epsilon,\delta}$ such that

$$< J^{\epsilon} w^{\epsilon}, w >_{(V_{\epsilon,\delta}{}', V_{\epsilon,\delta})} = \lambda \int_{\Sigma} [u][w] \quad \text{for all } (w \in W_{\epsilon,\delta}).$$

Easily can be seen that w^{ϵ} is bounded in the norm of V.
Then on a subsequence still denoted by ϵ we have,

$$w^{\epsilon} \rightharpoonup \bar{w} \text{ as } \epsilon \to 0$$

for some $\bar{w} \in W$. But if we consider $f_\lambda \in V'$ with $f_\lambda(w) = \lambda \int_\Sigma [u][w]$ then clearly

$$f_\lambda(w) =< J^{\epsilon} w^{\epsilon}, w >_{(V_{\epsilon,\delta}{}', V_{\epsilon,\delta})} \Longrightarrow f_\lambda \in \partial\varphi^{\epsilon}(w^{\epsilon}).$$

So using the G-convergence result stated in (4.2.5) we obtain

$$f_\lambda \in \partial\varphi(\bar{w}) \Longleftrightarrow < \bar{w}, v > + \frac{C}{4} \int_{\Sigma} [u][\bar{w}] = \lambda \int_{\Sigma} [u][v]$$

for any $v \in W$.
Therefore, because of the definition of u we have that $u = \bar{w}$. Now by Uryson's property we can see that

$$w^{\epsilon} \rightharpoonup u \text{ when } (\epsilon \to 0)$$

Let

$$v^\epsilon = w^\epsilon - \sum_{i=1}^{m} u_i^\epsilon (w^\epsilon, u_i^\epsilon)_V$$

We can see that

$$(w^\epsilon, u_i^\epsilon)_V = \lambda_i^\epsilon \int_\Sigma [u_i^\epsilon][w^\epsilon] \xrightarrow{\epsilon} \lambda_i \int_\Sigma [u][u_i].$$

But using the variational form of problem $(\mathcal{P})$ the last integral in the above equality is zero by the assumption $\lambda \neq \lambda_n$ for any $(n \in \mathbf{N})$.

Thus $v^\epsilon \rightharpoonup u$ weakly in (V). Noticing that $v^\epsilon \in W_{\epsilon,\delta}$ and $v^\epsilon \perp u_i^\epsilon$ for all $\left(i = \overline{1,m}\right)$ from the Rayleigh's principle for (4.1.1) we have

$$\lambda_{m+1}^\epsilon \leq \frac{\| v^\epsilon \|_V^2}{\displaystyle\int_\Sigma [v^\epsilon]^2} \tag{4.2.20}$$

Now, from the definition of w^ϵ and the inequality (4.2.16) we have

$$\lim_{\epsilon \to 0} \| v^\epsilon \|_V^2 = \lim_{\epsilon \to 0} \| w^\epsilon \|_V^2 = \lambda \int_\Sigma [u]^2.$$

From the last relation and Theorem (4.2.6), passing to the limit when $\epsilon \to 0$ in (4.2.20) we obtain the contradiction.

Using the equivalence relations (4.1.3), (4.1.4) and Corollary 4.2.7 we obtain ii) and iii). □

Next, following an idea in [6], we give a Mosco-convergence (see [6] for the definition of Mosco-convergence) result for the case $\lim\limits_{\epsilon \to 0} \dfrac{\delta(\epsilon)}{\epsilon} < \infty$:

Theorem 4.2.9. *Let* $\lim\limits_{\epsilon \to 0} \dfrac{\delta(\epsilon)}{\epsilon} < \infty$ *and* $i \in \mathbf{N}$ *arbitrary fixed.*
Then if m_i *is the order of multiplicity of* λ_i*, i.e*

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+m_i-1} < \lambda_{i+m_i}$$

then the sequence of subspaces generated by $\{u_i^\epsilon, \ldots, u_{i+m_i-1}^\epsilon\}$ *Mosco-converge in* $L^2(\Omega)$ *to the eigenspace* $\{u_i, \ldots, u_{i+m_i-1}\}$*, associated to* λ_i*.*

Proof. We remark that the multiplicity of λ_i^ϵ might be strictly smaller than that of λ_i. So if we denote

$$\{u_i^\epsilon, \ldots, u_{i+m_i-1}^\epsilon\} \doteq S_i^\epsilon \quad \text{and} \quad \{u_i, \ldots, u_{i+m_i-1}\} \doteq S_i$$

we can see that as in the above remark S_i^ϵ may be strictly larger than the eigenspace of λ_i^ϵ. Now from Theorem (4.2.8) we have that there is a subsequence $\{u_n^{\epsilon_j}\}_{j \in \mathbf{N}}$ such that

$$\lim_{\epsilon \to 0} \lambda_n^\epsilon = \lambda_n \quad \text{and} \quad u_n^{\epsilon_j} \rightharpoonup u_n,$$

where (u_n, λ_n) solve the spectral limit problem $\mathcal{P}$.
From the linearity of $\mathcal{P}^\epsilon$ and $\mathcal{P}$ we can say that

$$\limsup_{\epsilon \to 0} S_i^\epsilon \subset S_i.$$

We can easily see that for arbitrary fixed $i, j \in \mathbf{N}$, with $i \neq j$ and

$$u_i^\epsilon \rightharpoonup u_i \text{ and } u_j^\epsilon \rightharpoonup u_j$$

we have

$$< u_i, u_j >_V = 0.$$

Indeed from

$$0 = < u_i^\epsilon, u_j^\epsilon >_V = \lambda_i^\epsilon \int_\Sigma [u_i^\epsilon][u_j^\epsilon]$$

passing to the limit when $\epsilon \to 0$ we have

$$\lambda_i \int_\Sigma [u_i][u_j] = 0 \Longrightarrow < u_i, u_j >_V = 0$$

using the variational form of the limit problem. Next using the linear independence of $\{u_i, ..., u_{i+m_i-1}\}$ and the fact that the dimension of the eigenspace associated to λ_i is m_i we have in fact that

$$\limsup_{\epsilon \to 0} S_i^\epsilon = S_i.$$

Because of the compact imbeding of H^1 in L^2 we have that there is a subsequence ϵ_j such that

$$\liminf_{\epsilon \to 0} S_i^\epsilon = \limsup_{j \to \infty} S_i^{\epsilon_j}.$$

Now if there is v such that

$$v \notin \liminf_{\epsilon \to 0} S_i^\epsilon$$

then from the above relation we have

$$v \notin \limsup_{j \to \infty} S_i^{\epsilon_j} = S_i$$

which implies

$$S_i \subset \liminf_{\epsilon \to 0} S_i^\epsilon.$$

So we have proved the statement. $\square$

The next corollary is a consequence of the above results and states:

Corollary 4.2.10. *Let* $\lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$ *and* $i \in \mathbf{N}$ *arbitrary fixed.*
Then if m_i *is the order of multiplicity of* λ_i*, i.e*

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} = ... = \lambda_{i+m_i-1} < \lambda_{i+m_i}$$

then the sequence of subspaces generated by $\{z_i^\epsilon, ..., z_{i+m_i-1}^\epsilon\}$ *Mosco-converge in* $L^2(\Sigma)$ *to the eigenspace* $\{z_i, ..., z_{i+m_i-1}\}$*, associated to* λ_i *for the problem (4.2.18).*

Next we will analyze the case when λ_i is a simple eigenvalue of the limit problem. We have the following result:

Theorem 4.2.11. *Let* $\lim_{\epsilon\to 0}\dfrac{\delta(\epsilon)}{\epsilon}<\infty$*. If* $\lambda_n^\epsilon \to \lambda_n$ *and* λ_n *is a simple eigenvalue of the limit problem* $(\mathcal{P})$*, then the whole sequence* $\{u_n^\epsilon\}$ *is convergent,* $u_n^\epsilon \rightharpoonup u_n$*, where* u_n *is an eigenvector for* $(\mathcal{P})$ *associated to* λ_n *and* $\| u_n \|_V^2 = \left(\frac{1}{\beta_n}\frac{C}{4}+1\right)^{-1}$ *where* β_n *is as in Theorem 4.2.8 and* C *is defined in Lemma (4.2.4).*

Proof. Let $\tilde{u}_n$ be the orthonormal eigenvector associated to λ_n. Because λ_n is simple we will have that λ_n^ϵ will be simple for ϵ small enough. We can suppose

$$\langle u_n^\epsilon, \tilde{u}_n\rangle \geq 0, \tag{4.2.21}$$

for every $\epsilon>0$.
From the orthogonality of $(u_n^\epsilon)_{n\in\mathbf{N}}$ we have that their limits $(u_n)_{n\in\mathbf{N}}$ from a orthogonal subsequence . Indeed, using Theorem 4.2 and (4.2.16) we have that there is a subsequence still denoted by ϵ such that we can pass to the limit when $\epsilon\to 0$ in the next equality

$$0=\langle u_n^\epsilon, u_m^\epsilon\rangle_V = \lambda_n^\epsilon \int_\Sigma [u_n^\epsilon][u_m^\epsilon]$$

In the limit when $\epsilon\to 0$ in the last equality, we obtain

$$\lambda_n \int_\Sigma [u_n][u_m] = 0 \Leftrightarrow \langle u_n, u_m\rangle_V \cdot \frac{\lambda_n}{\lambda_n - \frac{C}{4}} = 0 \Leftrightarrow \langle u_n, u_m\rangle_V = 0,$$

and therefore the orthogonality of the limits eigenfunctions is proved.

On the other hand, using the orthonormality of $(u_n^\epsilon)_{n\in\mathbf{N}}$ we have that for any subsequence $(u_n^{\epsilon_j})_j$ there is a subsequence of it $(u_n^{\epsilon_{j_k}})_k$ such that $u_n^{\epsilon_{j_k}} \rightharpoonup u_n$.
Because λ_n is a simple eigenvalue we get that there is a constant r such that $u_n = r\cdot\tilde{u}_n$. Then from the orthonormality we get again

$$\| u_n^{\epsilon_{j_k}} \|_V = 1 \Leftrightarrow \langle u_n^{\epsilon_{j_k}}, u_n^{\epsilon_{j_k}}\rangle_V = 1 \Leftrightarrow \lambda_n^{\epsilon_{j_k}} \int_\Sigma [u_n^{\epsilon_{j_k}}][u_n^{\epsilon_{j_k}}] = 1$$

Passing to the limit when $k\to\infty$ in the above equality and using that $\lambda_n = \frac{C}{4}+\beta_n$ we obtain

$$(\frac{C}{4}+\beta_n)\int_\Sigma [u_n][u_n] = 1 \Leftrightarrow \langle u_n, u_n\rangle_V \cdot (\frac{1}{\beta_n}\frac{C}{4}+1) = 1$$

Therefore, because $u_n = r\cdot\tilde{u}_n \;\; \forall n\in\mathbf{N}$ we have that

$$\mid r \mid = \left(\frac{1}{\beta_n}\frac{C}{4}+1\right)^{-1/2}.$$

But because of (4.2.21) we get that $r>0$.Thus

$$r = \left(\frac{1}{\beta_n}\frac{C}{4}+1\right)^{-1/2}.$$

So we have proved that every subsequence $\{u_n^{\epsilon_{j_k}}\}_{k\in\mathbf{N}}$ of $\{u_n^\epsilon\}_{\epsilon>0}$ has a subsequence of it which converge in the weak topology of V, to $u_n = \tilde{u}_n \left(\frac{1}{\beta_n}\frac{C}{4}+1\right)^{-1/2}$. Therefore the conclusion follows immediately. $\square$

The next corollary follows from the equivalence relations (4.1.3), (4.1.4) and Corollary 4.2.7.

Corollary 4.2.12. *Let* $\lim_{\epsilon\to 0}\frac{\delta(\epsilon)}{\epsilon} < \infty$. *If* $\lambda_n^\epsilon \to \lambda_n$ *and* λ_n *is a simple eigenvalue of the limit problem* $(\mathcal{P})$ *then the entire sequence of eigenvectors for the problem (4.1.1),* $\{z_n^\epsilon\}_\epsilon$*, is convergent to* z_n *strongly in* $L^2(\Sigma)$*, where* z_n *is an eigenvector for (4.2.18) and* $\| z_n \|_{L^2(\Sigma)}^2 = \dfrac{1}{\frac{C}{4}+\beta_n}$.

4.2.1 Case $\lim_{\epsilon\to 0}\frac{\delta(\epsilon)}{\epsilon} = \infty$

In this case we can see that the sequence $\{\varphi^\epsilon\}_{\epsilon>0}$ defined in 4.2.4, Γ-converge to φ and we have

$$\varphi(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx & \text{if} \quad u \in H_0^1(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

Now suppose that there is $n \in \mathbf{N}$ such that $\lambda_n^\epsilon \xrightarrow{\epsilon} \lambda_n < \infty$.
Now using the same approach as before, we obtain from Theorem (4.2.6) and Corollary (4.2.5) that $y \in \partial\varphi(u_n)$.This means that

$$u_n \in Dom(\varphi) = H_0^1(\Omega).$$

But we know that $u_n^\epsilon \in W_{\epsilon,\delta} \subset W$ which means that

$$u_n \in W.$$

Using the fact that $W = (H_0^1(\Omega))^\perp$ in V we obtain $u_n = 0$, which contradicts Corollary 4.2.2. Therefore $\lambda_n^\epsilon \xrightarrow{\epsilon} \infty$. Now from the variational form of (4.1.1) if u_n^ϵ is the normal eigenvector associated to λ_n^ϵ we have

$$\frac{1}{\lambda_n^\epsilon} = \int_\Sigma [u_n^\epsilon]^2$$

Consider $u_n \in W$ to be the weak limit of u_n^ϵ when $\epsilon \to 0$. Passing to the limit for $\epsilon \to 0$ in the equality above we obtain

$$\int_\Sigma [u_n]^2 = 0$$

And this together with the fact that $u_n \in W$ and $W \perp H_0^1(\Omega)$ give us that $u_n = 0$. So in this case we have that all the eigenvectors of the $\mathcal{P}_\epsilon$ converges to zero and all the eigenvalues of the same problem converges to ∞.

4.2.2 Case $\lim\limits_{\epsilon\to 0}\dfrac{\delta(\epsilon)}{\epsilon}=0$

Although this can be seen as a particular case for all the results stated above, we will discus it separately due to the fact that we can obtain a stronger variant of Theorem (4.2.9).This case is very interesting because the holes $S_{\epsilon,\delta}$ "disappear" in the limit problem.

First, we can observe following ([6], Th.1.27) that in this case we have

$$w^\epsilon \to 0 \text{ strongly in } H_0^1(\Omega).$$

where w^ϵ is the capacity potential defined in Lemma 4.2.1.

It can be easily seen that for each $u \in W$, using w^ϵ, we can construct sequences, $\tilde{u}_\epsilon = u - w^\epsilon r_u$ with r_u smooth enough (for example in $W^{1,\infty}(\Omega_+)\cup W^{1,\infty}(\Omega_-)$) and $[r_u]=[u]$ on Σ, such that $\tilde{u}_\epsilon \in V_{\epsilon,\delta}$ and $P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon \to u$ strongly in W.
Using the strong convergence of the capacity potential we obtain:

$$\tilde{u}_\epsilon \to u \quad \text{strongly in } V \quad \text{when} \quad \epsilon \to 0. \tag{4.2.22}$$

Now we have

$$\tilde{u}_\epsilon = P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon + (\tilde{u}_\epsilon - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon).$$

Let $a=\lim\limits_{\epsilon\to 0}P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon$ and $b=\lim\limits_{\epsilon\to 0}(\tilde{u}_\epsilon - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon)$ be two (arbitrary chosen) weak limit points of $\{P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon\}_{\epsilon>0}$ and respectively $\{\tilde{u}_\epsilon - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon\}_{\epsilon>0}$.
It's easy to see that $a \in (H_0^1(\Omega))^\perp$ and $b \in H_0^1(\Omega)$. Therefore we obtain $u=a+b$. and thus $b=0$ and $a=u$. By the arbitrary choice of a and b, and the compactness of the above sequences we get that

$$\begin{aligned} P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon &\rightharpoonup u \\ \tilde{u}_\epsilon - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon &\rightharpoonup 0 \end{aligned} \tag{4.2.23}$$

Next we will use the following lemma in order to get the conclusion.

Lemma 4.2.13. *Let $\{a_n\}_{n\in\mathbf{N}}$ and $\{b_n\}_{n\in\mathbf{N}}$ be two sequences in V (where V can be a general Hilbert space, and $(.,.)$the scalar product in V) such that $a_n \perp b_n$ for every $n\in\mathbf{N}$.*
If $(a_n+b_n)\to L$ strongly in V and $a_n \rightharpoonup L$ then $a_n \to L$ and $b_n \to 0$ strongly in V as $n\to\infty$.

Proof. First we have that

$$\| a_n \|_V^2 = (a_n+b_n, a_n) \to \| L \|_V^2 .$$

Therefore we have that

$$\| a_n \|_V \to \| L \|_V$$

Now from reflexivity of V we get the result. □

Next, from the above lemma, (4.2.23), (4.2.22) and using the orthogonal decomposition of $\tilde{u}_\epsilon$ we obtain

$$\begin{aligned} P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon \rightarrow & \quad u \text{ strongly in } V \\ \tilde{u}_\epsilon - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon \rightarrow & \quad 0 \text{ strongly in } V \end{aligned} \tag{4.2.24}$$

Noticing that $W_{\epsilon,\delta}$ is a closed subspace of W, using (4.2.24) we can easily prove that

$$P_{W_{\epsilon,\delta}}u \rightarrow u \text{ for every } u \in W.$$

Indeed we have

$$\| P_{W_{\epsilon,\delta}}u - u \|_V \leq \| \tilde{u}_\epsilon - u \|_V + \| P_{W_{\epsilon,\delta}}u - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon \|_V + \| P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon - \tilde{u}_\epsilon \|_V \tag{4.2.25}$$

But for any $u \in V$ we have

$$\begin{aligned} \| P_{W_{\epsilon,\delta}}u - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon \|_V = \| P_{W_{\epsilon,\delta}}(P^\epsilon u) - P_{W_{\epsilon,\delta}}\tilde{u}_\epsilon \|_V \leq \| P^\epsilon u - \tilde{u}_\epsilon \|_V \quad \leq \\ \leq \| P^\epsilon u - u \|_V + \| \tilde{u}_\epsilon - u \|_V \leq 2 \cdot \| \tilde{u}_\epsilon - u \|_V \qquad . \end{aligned} \tag{4.2.26}$$

Thus, from (4.2.24), (4.2.22) and (4.2.26), the right hand member in (4.2.25) goes to 0 when $\epsilon \rightarrow 0$. This implies

$$P_{W_{\epsilon,\delta}}u \rightarrow u \text{ for every } u \in W. \tag{4.2.27}$$

Let $P_{W_{\epsilon,\delta}} \equiv \mathcal{R}_\epsilon$.
Now if for $u \in W_{\epsilon,\delta}$ we define $K^\epsilon : W_{\epsilon,\delta} \rightarrow W_{\epsilon,\delta}$ and $K : W \rightarrow W$ as

$$\langle K^\epsilon u, \ w\rangle_V = \int_{\Sigma - S_{\epsilon,\delta}} [u][w], \text{ for any } w \in W_{\epsilon,\delta}. \tag{4.2.28}$$

and

$$\langle Ku, \ w\rangle_V = \int_\Sigma [u][w], \text{ for any } w \in W. \tag{4.2.29}$$

we can see that K^ϵ and K,are compact and symmetric operators and they have the eigenvalues $\{\frac{1}{\lambda_n^\epsilon}\}_{n\in\mathbf{N}}$ and $\{\frac{1}{\beta_n}\}_{n\in\mathbf{N}}$ respectively and the associated eigenvectors sequence $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ and $\{u_n\}_{n\in\mathbf{N}}$ respectively. It is easy to check now that $\mathcal{R}_\epsilon$ verify the properties stated in ([51], section 11.1) In these conditions all the results obtained in ([51], chapter.11) are valid in our case too. Define

$$N(\beta_n, K) = \{u \in W, Ku = \frac{1}{\beta_n}u\}.$$

as in [51]. Now following the results in [51] we have $\lambda_n^\epsilon \rightarrow \beta_n$ and

$$| \frac{1}{\lambda_n^\epsilon} - \frac{1}{\beta_n} | \leq 2 \cdot \sup_{u\in N(\beta_n,K), \|u\|_W=1} \| K^\epsilon \mathcal{R}_\epsilon u - \mathcal{R}_\epsilon K u \|_W \tag{4.2.30}$$

and for the eigenvectors $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ the following stronger version of Theorem (4.2.9) holds.

Theorem 4.2.14. *Let $i \geq 1$ be an integer and*

$$\lambda_{i-1} < \lambda_i = ... = \lambda_{i+m_i-1} < \lambda_{m_i+i} \quad i.e,$$

the multiplicity of the eigenvalue λ_i is equal to m_i, then for any $w \in N(\beta_i, K)$, $\| w \|_V = 1$, there exists a linear combination $\bar{u}^\epsilon$ of eigenvectors u_i^ϵ, ..., $u_{i+m_i-1}^\epsilon$ of K^ϵ such that

$$\| \bar{u}^\epsilon - \mathcal{R}_\epsilon w \|_V \leq M_i \| K^\epsilon \mathcal{R}_\epsilon w - \mathcal{R}_\epsilon K w \|_V \tag{4.2.31}$$

where the constant M_i does not depend on ϵ

.

Remark 4.2.15. *The relation (4.2.31) in Theorem (4.2.14) can be rewritten as*

$$\| \bar{u}^\epsilon - w \|_V \leq M_i \| K^\epsilon \mathcal{R}_\epsilon w - \mathcal{R}_\epsilon K w \|_V + \| \mathcal{R}_\epsilon w - w \|_V .$$

Using (4.2.27) and the relation above we can see that Theorem (4.2.14) states in fact the Mosco-convergence in the strong topology of V of the sequence of the spaces generated by $\{u_i^\epsilon, ..., u_{i+m_i-1}^\epsilon\}$ to the eigenspace associated to λ_i, and this is stronger than Theorem (4.2.9).

Remark 4.2.16. *As a last observation we can see that using (4.1.3) and (4.1.4) similar results as those obtained in Section 3.1 and Section 3.2 can be stated and proved for the problem (4.0.2).*

Chapter 5

Homogenization results for a contact problem with friction arising in the modeling of the earthquake initiation phase

The origin of friction has to be found in the hard contacts between two rough surfaces and the the geometry of the contact, let us say the roughness, has been shown to be a decisive parameter for frictional behavior [75]. Since friction is a phenomenon that concerns both microscopic and macroscopic scales the contact on a geological fault is also modelled at the scale of the seismic waves (i.e. kilometric).

The friction properties are likely heterogeneous on the fault, particularly with the presence of barriers. By the term barrier, we denote here a patch on the fault plane where no slip occurs. This concept cannot be applied for the evolution of the fault at the geological time scale but it has been shown to be useful and relevant in the description of fault heterogeneity during an earthquake [69, 70].

The macroscopic behavior of a fault with small-scale heterogeneity of rupture resistance (small scale barriers) is difficult to relate to the local properties of the fault. A formal measure of the friction on the fault itself would just be a local particular law, that is varying with the position along the fault. In this chapter we focus on the following question : How can we obtain an effective (equivalent) friction law which, used on a homogeneous fault, leads to a slip evolution similar to the one produced on the heterogeneous fault ?

Mathematically the problem is related to the homogenization of the Newmann Sieve problem for the Laplacian studied by several authors [30, 26, 6, 21]. In the geophysical context the problem was studied (see [18, 17, 71]) in two dimensions (anti-plane geometry) to obtain the rescaling of the weakening rate through a spectral analysis.

The Newmann Sieve problem associated to the linear elasticity operator was studied by Lobo and Perez [56, **?**]. An extension to the non-linear case of the Neumann Sieve has been studied by Ansini in [3]. Our friction problem is similar to the previous, with the important difference that the tangential component of the displacement has zero jump on the barriers, and the limit analysis is therefore developed on a larger functional space.

Let us outline the content of the chapter.

In section 5.2 we consider the three dimensional shearing of an elastic domain which contains an internal boundary (the fault) located on a plane(the fault plane). The contact on the fault is described through a slip weakening friction (i.e. the decrease of the friction force with slip). This friction law is used in the geophysical context of earthquakes modeling and experimental studies [67] pointed out the good agreement of this model with experimental data. The symmetry of the displacement field with respect to the fault plane (see for instance [39] for the geophysical meaning) gives an important simplification of the problem: the normal over stress on the fault vanishes. The fact that the normal stress has a weak variation of during the dynamic rupture was already observed in direct computations [5, 57] as well as in the inversion of seismological data [27]. An important consequence of the above assumption is the fact that we can associate to the physical problem a minimization problem for the energy function. In modelling seismic phenomena, where at least two equilibria (before and after an earthquake) are involved, the energy function cannot be supposed convex.

In section 5.3 we obtain(as in [50] under slightly different assumptions) sufficient conditions of stability through the first eigenvalue of the tangent problem. Since this eigenvalue problem has an important significance in the description of the physical properties of the fault, we shall study it in the next section.

In section 5.4 we give the main results of the chapter. Firstly we set the perturbed (or heterogeneous) problem: a fault which has ϵ-periodically distributed barriers of radius r_ϵ. For $0 < c =: \lim_{\epsilon\to 0} \delta(\epsilon)/\epsilon < \infty$ we prove that the sequence of energy functionals Γ-converges to a limit energy functional. For the proof of liminf and the limsup inequalities we adapt an idea from [3].

The limit functional is associated to another slip weakening friction problem called the equivalent friction law. In the last part of this section we prove that the eigenvalues and eigenfunctions of the perturbed tangent problem converge to the eigenvalues and eigenfunctions of the equivalent (limit) tangent problem. For this we adapt G-convergence techniques, developed for the Newmann-Sieve problem in [66].

The slip weakening rate of the equivalent (or limit) fault is smaller then undisturbed fault. Since the limit slip weakening rate may be negative a slip-hardening effect can also be expected. Moreover, we have to point out that even if the small scale friction law is isotropic the equivalent one is not. This surprising fact is natural if we have in mind that the periodic distribution of the barriers is not isotropic, hence the limit problem will inherit this anisotropic geometrical perturbation. We have to mention here that this property was also obtained [**?**, 56] for an elastic body with a surface having small no-slip regions.We make the observation that the proof of the convergence is based there on the explicit computation of the solution for the cell problem, which in our case cannot be easily computed because of the general mixed type boundary conditions on parts of the boundary. This is the reason we chose the

Γ-convergence approach in our paper.

In the last section we give the physical interpretation of the previous theoretical results in the context of a barrier erosion process during the earthquake nucleation (or initiation) phase, which precedes the dynamic rupture. We point out the important role played by the process of erosion of the barriers in the effective properties of the homogenized fault. We deduce from our analysis that the nucleation phase can be divided in three time periods. Firstly we are dealing with a locking stage with no "macroscopic" slip. The second time period is characterized by a smaller, and even negative, weakening rate and by the loss of the isotropy of the friction law. The third time period corresponds to the last stage of (effective) initiation when the friction properties are the same with the undisturbed fault.

5.1 Statement of the physical problem

We consider the three dimensional shearing of an elastic domain $\mathcal{D} \subset \mathbb{R}^3$. If we denote by $u : \mathcal{D} \longrightarrow \mathbb{R}^3$ the displacement field, then the elastic constitutive equation and the equilibrium equation read

$$\sigma(u) = \mathcal{A}\epsilon(u), \quad \mathrm{div}(\mathcal{A}\epsilon(u)) = 0 \quad \text{in} \quad \mathcal{D}, \tag{5.1.1}$$

where $\mathcal{A}$ is the fourth order elastic tensor, $\sigma(u)$ is the over stress tensor and $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is the small strain tensor. $\mathcal{A}$ is a symmetric and positively defined fourth order tensor, i.e.

$$\mathcal{A}_{ijkl} \in L^\infty(\mathcal{D}), \quad \mathcal{A}(x)\epsilon \cdot \sigma = \mathcal{A}(x)\sigma \cdot \epsilon, \quad \text{a.e.} \quad x \in \mathcal{D}, \tag{5.1.2}$$

$$\text{such that} \ \ \mathcal{A}(x)\epsilon \cdot \epsilon \geq M_1|\epsilon|^2 \ \ \text{and} \ \ |\mathcal{A}(x)\epsilon| \leq M_2|\epsilon| \ \ \text{a.e} \ \ x \in \mathcal{D} \text{ with } \ M_1, M_2 > 0\ , \tag{5.1.3}$$

for all $i, j, k, l = \overline{1,3}$ and for all $\sigma, \epsilon \in \mathbf{R}^{3\times 3}_S$.

The smooth boundary $\Sigma = \partial\mathcal{D}$ is divided into two disjoint parts $\Sigma = \Sigma_d \cup \Gamma_f$: $\Sigma_d = \partial\bar{\mathcal{D}}$ the exterior boundary and Γ_f the interior one (i.e. it's a subset of the interior of $\bar{\mathcal{D}}$). For the sake of simplicity on the exterior boundary we shall suppose vanishing displacement conditions, i.e. $u = 0$ on Σ_d. The interior boundary is located in the plane $\Pi = \{x_3 = 0\}$, and will be called in the following the fault or fault region. We assume that the pre-stress $\sigma^\infty \in C^0(\bar{\mathcal{D}})$ is such that the fault does not open. Moreover the fault Γ_f is under a slip-dependent friction law:

$$[\sigma_{i3}(u)] = 0, \quad i = \overline{1,3}, \quad [u_3] = 0, \quad \text{on} \quad \Gamma_f, \tag{5.1.4}$$

$$\sigma_\tau(u) + \tau^\infty = -\mu(|[u_\tau]|)|\sigma_{33}(u) - S_\perp|\frac{[u_\tau]}{|[u_\tau]|} \quad \text{if} \quad [u_\tau] \neq 0 \ \text{ on } \ \Gamma_f, \tag{5.1.5}$$

$$|\sigma_\tau(u) + \tau^\infty| \leq \mu(0)|\sigma_{33}(u) - S_\perp| \quad \text{ if } \ [u_\tau] = 0 \ \text{ on } \ \Gamma_f, \tag{5.1.6}$$

where $[\ ]$ denotes the half of the jump across Γ_f, (i.e. $[w] = (w^+ - w^-)/2$), $\sigma_\tau(u) = -(\sigma_{13}(u), \sigma_{23}(u), 0)$ is the tangential over-stress, $\sigma_{33}(u)$ is the normal over-stress, $u_\tau =$

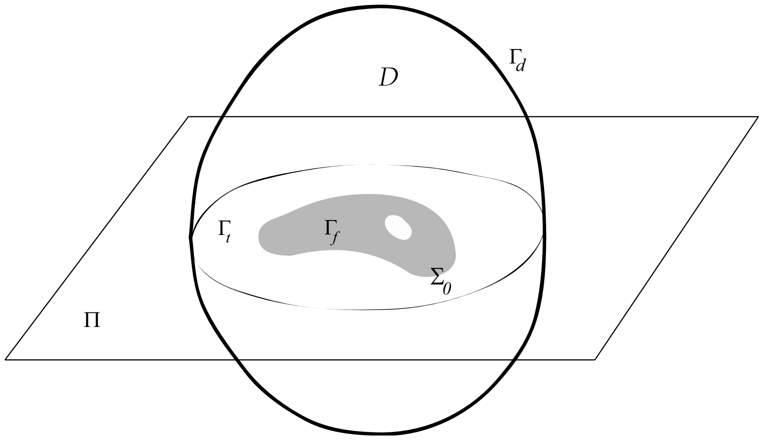


Figure 5.1: The geometry of the 3-D problem

$(u_1, u_2, 0)$ is the tangential displacement, and $\tau^\infty =: -(\sigma^\infty_{13}, \sigma^\infty_{23}, 0)$ and $-S_\perp =: \sigma^\infty_{33}$ are the tangential and the normal pre-stress acting on Γ_f. From the above assumptions on σ^∞ we have $S_\perp, \tau^\infty_i \in C^0(\Gamma_f)$. Equations (5.1.5)-(5.1.6) assert that the tangential (friction) stress is bounded by the normal stress multiplied by the value of the friction coefficient $\mu(0)$. If this limit is not attained sliding does not occur. Otherwise the friction stress is opposed to the slip $[u_\tau]$ and its absolute value depends on the slip modulus through $\mu(|[u_\tau]|)$. Concerning the regularity of $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ we suppose that the friction coefficient is a Lipschitz function with respect to the slip, i.e. there exists $L_\mu \geq 0$ such that

$$|\mu(s_1) - \mu(s_2)| \leq L_\mu |s_1 - s_2|, \tag{5.1.7}$$

and we denote by H its antiderivative

$$H(u) := \int_0^u \mu(s)\, ds.$$

We suppose that there exists $\gamma \in L^\infty(\mathbb{R}_+)$ and $a \geq 0$ such that

$$H(r) - H(s) \geq \mu(s)(r-s) - \gamma(s)(r-s)^2/2 - a|r-s|^3, \quad \forall r, s \geq 0. \tag{5.1.8}$$

Let us remark that if μ is two times differentiable with a bounded second derivative then (5.1.8) holds with $\gamma(s) = -\mu'(s)$. If μ is continuous but only piecewise differentiable then (5.1.8) holds with $\gamma(s) = -\min\{\mu'(s+), \mu'(s-)\}$.

A specific friction law with a linear piecewise slip weakening, which is a reasonable approximation of the experimental observations (see [67]), can be written as follows

$$\mu(s) = \begin{cases} \dfrac{\mu_d - \mu_s}{D_c} s + \mu_s & if\ s \leq D_c \\ \mu_d & if\ s \geq D_c \end{cases} \tag{5.1.9}$$

where $\mu_s > \mu_d$ are the static and, respectively, dynamic friction coefficients and D_c is the critical slip. In this case $\gamma(x) = (\mu_s - \mu_d)/D_c$.

We shall suppose in the following that $\mathcal{D}$ is symmetric with respect to the plane Π. As in [39] the following symmetries of the displacement field with respect to the plane Π will be considered:

$$u_1(x,-x_3) = -u_1(x,x_3), \quad u_2(x,-x_3) = -u_2(x,x_3), \quad u_3(x,-x_3) = u_3(x,x_3), \tag{5.1.10}$$

where $x = (x_1, x_2)$ and $(x, 0)$ belongs to Σ_0 the intersection of $\bar{\mathcal{D}}$ with the plane Π. In the case of an isotropic elastic material, i.e.

$$\mathcal{A}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + G\delta_{ik}\delta_{jl}, \tag{5.1.11}$$

with $\lambda, G > 0$ the Lamé coefficients, we deduce the following symmetries of the stress field σ_{33}:

$$\sigma_{33}(x,-x_3) = -\sigma_{33}(x,x_3).$$

The condition of continuity of the stress vector (5.1.4) on the fault plane Γ_f gives the fact that the normal over-stress σ_{33} does not present any variation during the slip

$$\sigma_{33}(x,0^+) = \sigma_{33}(x,0^-) = 0, \quad \text{ for any } (x,0) \in \Sigma_0. \tag{5.1.12}$$

Since the displacement field is continuous outside the faults, from the symmetry conditions (5.1.10) we get that the tangential displacement is vanishing outside Γ_f:

$$u_1(x,0^+) = u_1(x,0^-) = u_2(x,0^+) = u_2(x,0^-) = 0, \quad \text{for all } (x,0) \in \Sigma_0 \setminus \Gamma_f, \tag{5.1.13}$$

and the jump on Γ_f is the given by:

$$[u_i(x,0)] = u_i(x,0^+) = -u_i(x,0^-), \quad i = 1,2, \quad \text{for all } (x,0) \in \Gamma_f. \tag{5.1.14}$$

Let us denote by $\Omega := \mathcal{D} \cap \{x_3 > 0\}$ the upper half of the domain $\mathcal{D}$ and by $\Gamma_d := \Sigma_d \cap \{x_3 > 0\}$, $\Gamma_t := \Sigma_0 \setminus \Gamma_f$ which implies that $\partial\Omega = \Gamma_d \cup \Gamma_t \cup \Gamma_f$. From the above symmetry properties we can restrict ourselves to the upper half Ω of $\mathcal{D}$. We state the problem $(\mathcal{P})$: find the displacement field $u : \Omega \to \mathbb{R}^3$ solution of

$$\sigma(u) = \mathcal{A}\epsilon(u), \quad \text{div}(\mathcal{A}\epsilon(u)) = 0 \quad \text{in} \quad \Omega, \tag{5.1.15}$$

$$u = 0 \quad \text{on} \quad \Gamma_d, \qquad \sigma_{33}(u) = 0, \ u_\tau = 0 \quad \text{on} \quad \Gamma_t, \tag{5.1.16}$$

$$\sigma_{33}(u) = 0, \quad \begin{cases} \sigma_\tau(u) = -S_\perp\mu(|u_\tau|)\dfrac{u_\tau}{|u_\tau|} - \tau^\infty & \text{if} \quad u_\tau \neq 0 \\ |\sigma_\tau(u) + \tau^\infty| \leq S_\perp\mu(0) & \text{if} \quad u_\tau = 0. \end{cases} \quad \text{on } \Gamma_f, \tag{5.1.17}$$

5.2 Existence and stability

Let us denote by V the closed subspace of $[H^1(\Omega)]^3$ given by

$$V := \{v \in [H^1(\Omega)]^3 / v = 0 \quad \text{on} \quad \Gamma_d, \quad v_\tau = 0 \ \text{ on } \ \Gamma_t\}. \tag{5.2.1}$$

From Korn's inequality and Poincare's inequality one can easily deduce that the following inner product

$$< u, v >_V := \int_\Omega \mathcal{A}\epsilon(u) \cdot \epsilon(v), \quad \forall u, v \in V, \tag{5.2.2}$$

generates a norm, denoted by $\| \ \|_V$, which is equivalent with the natural norm on $[H^1(\Omega)]^3$ and

$$M_1 \| Du \|_{L^2}^2 \leq \| u \|_V^2 \leq M_2 \| Du \|_{L^2}^2 \quad \forall u \in V.$$

We have the following variational formulation of the physical problem (5.1.15)-(5.1.17), (see also [50])

$$u \in V, \qquad < u, u - v >_V + j(u, u) - j(u, v) \leq f(u - v) \qquad \forall v \in V, \tag{5.2.3}$$

where $j : V \times V \longrightarrow \mathbf{R}_+$ and $f : V \longrightarrow \mathbf{R}$ are given by

$$j(u, v) = \int_{\Sigma_0} S_\perp \mu(|u_\tau|)|v_\tau|, \qquad f(v) = -\int_{\Sigma_0} \tau^\infty \cdot v_\tau \quad \forall u, v \in V. \tag{5.2.4}$$

Let us introduce now the total energy functional $\mathcal{W} : V \longrightarrow \mathbf{R}$ given by

$$\mathcal{W}(v) = \frac{1}{2}\|v\|_V^2 + \int_{\Sigma_0} S_\perp H(|v_\tau|) - f(v), \quad \forall v \in V, \tag{5.2.5}$$

which characterizes the "physically acceptable" solutions. Indeed we have the following result.

Theorem 5.2.1. *If $u \in V$ is a local minimum for $\mathcal{W}$, then u is a solution of (5.2.3). Moreover there exists a global minimum for $\mathcal{W}$, i.e. there exists $u \in V$ such that*

$$\mathcal{W}(u) \leq \mathcal{W}(v), \qquad \forall v \in V. \tag{5.2.6}$$

Proof. Let u be a local minimum, i.e. there exists δ such that $\mathcal{W}(u) \leq \mathcal{W}(w)$ for all $w \in V$ with $\|w - u\|_V \leq \delta$. For all $v \in V$ we put $w = u + t(v - u)$, with $t > 0$ small enough, in the last inequality and we pass to the limit with $t \to 0$ to deduce (5.2.3).

In order to prove that $\mathcal{W}$ has a global minimum we remark that the trace map is compact from V to $L^2(\Gamma_f)$. Hence $v \to \int_{\Gamma_f} S_\perp H(|v_\tau|) - f(v)$ is weakly continuous on V, which implies that $\mathcal{W}$ is weakly lower semicontinuous. Bearing in mind that $\liminf \mathcal{W}(v) = \infty$ for $\|v\|_V \to \infty$, from a Weierstrass type theorem we deduce that $\mathcal{W}$ has at least one global minimum. $\square$

Let us consider now the following eigenvalue problem, which will be useful to characterize the stability of the local minima, $(\mathcal{E})$: find $u : \Omega \longrightarrow \mathbb{R}^3$, $u \neq 0$ and $\lambda \in \mathbf{R}$ such that

$$\sigma(u) = \mathcal{A}\epsilon(u), \quad \text{div}\, \sigma(u) = 0, \ \text{ in } \Omega, \tag{5.2.7}$$

$$u = 0 \ \text{ on } \ \Gamma_d, \qquad \sigma_{33}(u) = 0, \quad u_\tau = 0 \ \text{ on } \ \Gamma_t, \tag{5.2.8}$$

$$\sigma_{33}(u) = 0, \qquad \sigma_\tau(u) = \lambda u_\tau \ \text{ on } \ \Gamma_f, \tag{5.2.9}$$

which has the following variational formulation:

$$u \in V, \quad < u, v >_V = \lambda \int_{\Gamma_f} u_\tau \cdot v_\tau, \quad \forall v \in V. \tag{5.2.10}$$

The same technique as in [50] can be used to get the structure of the spectrum. For the convenience of the reader we shall give here the proof.

Theorem 5.2.2. *The eigenvalues and eigenfunctions of (5.2.10) form a sequence* $(\lambda_n, u_n)_{n\geq 1}$ *with* $0 < \lambda_1 \leq \lambda_2 \leq .....$ *and* $\lambda_n \longrightarrow +\infty$*. Moreover we have*

$$\frac{||u_1||_V^2}{\displaystyle\int_{\Gamma_f} |u_{1\tau}|^2 \, dx} = \lambda_1 = \min_{v \in V} \frac{||v||_V^2}{\displaystyle\int_{\Gamma_f} |v_\tau|^2 \, dx}. \tag{5.2.11}$$

Proof. Let $L = \{f = (f_1, f_2, 0)/f_1, f_2 \in L^2(\Sigma_0), f_1 = f_2 = 0 \quad \text{on} \quad \Gamma_t\}$ be a closed subspace of $[L^2(\Sigma_0)]^3$. Denote by $\gamma_\tau : V \to L$ the compact operator which associates to all $v \in V$ the tangential component of its trace on Γ_f, i.e.,

$$\gamma_\tau(v) \doteq v_\tau = v - (v \cdot n)n \quad \text{along } \Gamma_f$$

for any $v \in V$.
Let $V_1 = \ker \gamma_\tau$. Using the definition of V we can see that

$$V_1 = \{v \in V/ \ v_\tau = 0 \quad \text{on } \Sigma_0 \ \}.$$

Consider now

$$W = V_1^\perp = \{v \in V/ \quad < v, w >_V = 0 \ \ \forall \, w \in V_1\}.$$

Let $P_W : V \to W$ be the orthogonal projection onto W and define $T : L \to W$ to be the linear and bounded operator which associates to each $f \in L$ the unique solution $T(f) \in W$ of the following linear equation

$$\langle T(f), v \rangle_V = \int_{\Gamma_f} f \cdot v_\tau \, dx, \quad \forall v \in V. \tag{5.2.12}$$

We can define now the linear bounded operator $K : W \to W$ by $Kv \doteq T(v_\tau)$. From (5.2.12) we get

$$< Ku, v >_V = \int_{\Gamma_f} u_\tau v_\tau \tag{5.2.13}$$

for all $u, v \in W$, which implies that K is symmetric compact and strictly positive. Hence K has a positive and decreasing sequence of eigenvalues $(\beta_n)_{n\geq 1}$ with $\beta_n \to 0$ and an orthonormal sequence of corresponding eigenvectors, $(u_n)_{n\geq 1}$.
It's easy to observe that $\lambda_n \doteq \dfrac{1}{\beta_n}$ will be the eigenvalues of the problem $(\mathcal{E})$ and u_n will be the orthonormal eigenvectors corresponding to it.
Then Rayleigh's principle for K gives us the statement of the theorem. □

The following theorem makes use of the first eigenvalue of the above spectral problem to give sufficient conditions for a solution of (5.2.3) to be stable.

Theorem 5.2.3. *Let $u \in V$ be a solution of (5.2.3) and let λ_1 be the first eigenvalue of $\mathcal{E}$. If*

$$\lambda_1 > c_u =: \operatorname{ess\,sup}_{x\in\Gamma_f} S(x)\gamma(|u_\tau(x)|), \tag{5.2.14}$$

where γ has been defined in (5.1.8) and $-S$ is the normal stress on Γ_f, then u is an isolated local minimum for $\mathcal{W}$, i.e. there exists $\delta > 0$ such that

$$\mathcal{W}(u) < \mathcal{W}(v) \quad \forall\, v \in V,\ v \neq u,\ \|v-u\|_V < \delta. \tag{5.2.15}$$

Proof. Let us suppose that u is not a local minimum for $\mathcal{W}$, i.e. there exists $(v_m)_m \subset V$, $v_m \to u$ and $\mathcal{W}(v_m) \leq \mathcal{W}(u)$. If we put $v = v_m$ in (5.2.3) from the last inequality and from (5.1.8) we get:
$\|u-v_m\|_V^2 - \int_{\Gamma_f} S\gamma(|u_\tau|)(|u_\tau|-|v_{m\tau}|)^2 \leq 2a\int_{\Gamma_f} S\,|\,|v_{m\tau}|-|u_\tau|\,|^3$. Since the trace map is continuous from V to $L^3(\Gamma_f)$ the last inequality becomes

$$\|u-v_m\|_V^2 - c_u\int_{\Gamma_f}(|u_\tau|-|v_{m\tau})|)^2 \leq C\|u-v_m\|_V^3, \tag{5.2.16}$$

where C is a generic constant. If $c_u \leq 0$ then we obtain $1 \leq C\|u-v_m\|_V$, a contradiction. If $c_u > 0$ then from (5.2.11) and (5.2.16) we get $\dfrac{\lambda_1 - c_u}{\lambda_1}\|u-v_m\|_V^2 \leq C\|u-v_m\|_V^3$ which implies $\lambda_1 - c_u \leq C\lambda_1\|\bar{u}-v_m\|_V$. Since $v_m \to u$ we obtain $\lambda_1 - c_u \leq 0$, which contradicts (5.2.14). □

5.3 The perturbed problem

Denote by $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3; x_3 > 0\}$ and $\mathbb{R}^3_- = \{x \in \mathbb{R}^3; x_3 < 0\}$.

Throughout the chapter we will use M as an arbitrary constant independent of any parameter. Also by $B_1^2(0)$ we denote the two dimensional ball centered in 0 and with radius 1.
Let $\Gamma_f^0 \subset \Sigma_0$, with $dist(\Gamma_f^0, \Sigma_0) > 0$, be the unperturbed (or equivalent) fault and let $\epsilon > 0$ be a small parameter. Let S a fixed open set compactly inclosed in the 2-dimensional unit square and consider the lattice $\frac{\epsilon}{2}\mathbf{Z}^2$ on the plane Π. Let

$$Q^\epsilon_{i,2} = \left\{x_i^\epsilon + \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)^2\right\} \times \{0\} \tag{5.3.1}$$

denote the periodic cell on Π, centered at $x = (x_i^\epsilon, 0)$ where $x_i^\epsilon = i\epsilon$ for $i \in \mathbb{Z}^2$.
The perturbed fault $\Gamma_f^\epsilon \subset \Sigma_0$ has ϵ-periodically distributed holes, called (small scale) barriers. More precisely in each ϵ-square of the ϵ-lattice on the fault plane Π, the friction contact is considered outside an open set $S_{\epsilon,\delta}$ (small scale barrier) of size

$\epsilon\delta(\epsilon) < \epsilon$, (see Figure 5.2) with $S_{\epsilon,\delta} = \epsilon\delta(\epsilon)S + \frac{k}{2}\epsilon \;\; , \;\; k \in \mathbf{Z}^2$. For the simplicity of the exposition we will assume that $S_{\epsilon,\delta}$ is a 2-dimensional ball, denoted by $B^2(x_i^\epsilon, \epsilon\delta(\epsilon))$ centered in x_i^ϵ , $i \in \mathbf{Z}^2$ and of radius $\epsilon\delta(\epsilon)$. We shall denote by B_ϵ the set of all the microscopic barriers and let $\Gamma_f^\epsilon := \Gamma_f^0 \setminus \bar{B}_\epsilon$ be the perturbed fault. As before we define $\Gamma_t^\epsilon := \Sigma_0 \setminus \Gamma_f^\epsilon$.

Define now the spaces:

$$X = \{u \in [H^1(\Omega)]^3 / u = 0 \text{ on } \Gamma_d\}, \quad V = \{u \in X \;/\; u_\tau = 0 \;\text{ on } \Sigma \setminus \Gamma_f^0\}$$

and $W = V_1^\perp$ the orthogonal complement of V_1 in V.

We define the perturbed problem :

$(\mathcal{P}_\epsilon)$ find $u^\epsilon : \Omega \to \mathbb{R}^3$, solution of (5.1.15)-(5.1.17) with $\Gamma_f = \Gamma_f^\epsilon$ and $\Gamma_t = \Gamma_t^\epsilon$.

We consider

$$V_{\epsilon,\delta} := \{v \in [H^1(\Omega)]^3 / v = 0 \quad \text{on} \quad \Gamma_d, \quad v_\tau = 0 \quad \text{on} \quad \Gamma_t^\epsilon\}. \tag{5.3.2}$$

to formulate $(\mathcal{P}_\epsilon)$ in terms of the minimum of energy, $\mathcal{W}_\epsilon : V_{\epsilon,\delta} \to \mathbb{R}$, i.e.,

$$u_\epsilon \in V_{\epsilon,\delta} \qquad \mathcal{W}_\epsilon(u_\epsilon) \le \mathcal{W}_\epsilon(v) \qquad \forall v \in V_{\epsilon,\delta}. \tag{5.3.3}$$

We define the perturbed eigenvalue problem, associated to the above perturbed minimum problem, as

$(\mathcal{E}_\epsilon)$ find $u^\epsilon : \Omega \to \mathbb{R}^3$, and λ^ϵ solution of (5.2.7)-(5.2.9) with $\Gamma_f = \Gamma_f^\epsilon$ and $\Gamma_t = \Gamma_t^\epsilon$,

which has the variational formulation

$$u^\epsilon \in V_{\epsilon,\delta}, \qquad < u^\epsilon, v >_V = \lambda^\epsilon \int_{\Gamma_f^0} u_\tau^\epsilon \cdot v_\tau, \qquad \forall v \in V_{\epsilon,\delta}. \tag{5.3.4}$$

Let $L^\epsilon = \{f = (f_1, f_2, 0) / f_1, f_2 \in L^2(\Sigma_0), f_1 = f_2 = 0 \;\text{ on }\; \Gamma_t^\epsilon\}$ and let the tangential trace on Γ_f^ϵ be defined as before. Thus if we consider

$$V_1^\epsilon = \{v \in V_{\epsilon,\delta} \;;\; v_\tau = 0 \;\text{ on } \Gamma_f^\epsilon\}$$

we can see that

$$V_1^\epsilon = V_1 = \{v \in V; \;\; v_\tau = 0 \quad \text{on } \Sigma_0 \;\},$$

and V_1 is a subspace of $V_{\epsilon,\delta}$. Let's define $W_\epsilon \doteq V_1^\perp$ to be the orthogonal complement of V_1 in $V_{\epsilon,\delta}$, and $P_{W_\epsilon} : V_{\epsilon,\delta} \to W_\epsilon$ to be the orthogonal projection onto W_ϵ.
Then as is the proof of Theorem 5.2.2 we can write (5.3.4) as an eigenvalue problem for the operator $K^\epsilon : W_\epsilon \to W_\epsilon$, defined by

$$\langle K^\epsilon u, v \rangle = \int_{\Gamma_f^0} u_\tau v_\tau.$$

Thus the problem (5.3.4) will have an orthonormal sequence of eigenvectors $\{u_n^\epsilon\}_{n\ge1}$ and a sequence of corresponding eigenvalues $\{\lambda_n^\epsilon\}_{n\ge1}$, such that $0 < \lambda_1^\epsilon \le \lambda_2^\epsilon \le .....$, $\lambda_n^\epsilon \longrightarrow +\infty$, and

$$\frac{||u_1^\epsilon||_V^2}{\int_{\Gamma_f^\epsilon} |u_{1\tau}^\epsilon|^2 \, dx} = \lambda_1^\epsilon = \min_{v \in W_\epsilon} \frac{||v||_V^2}{\int_{\Gamma_f^\epsilon} |v_\tau|^2 \, dx}. \tag{5.3.5}$$

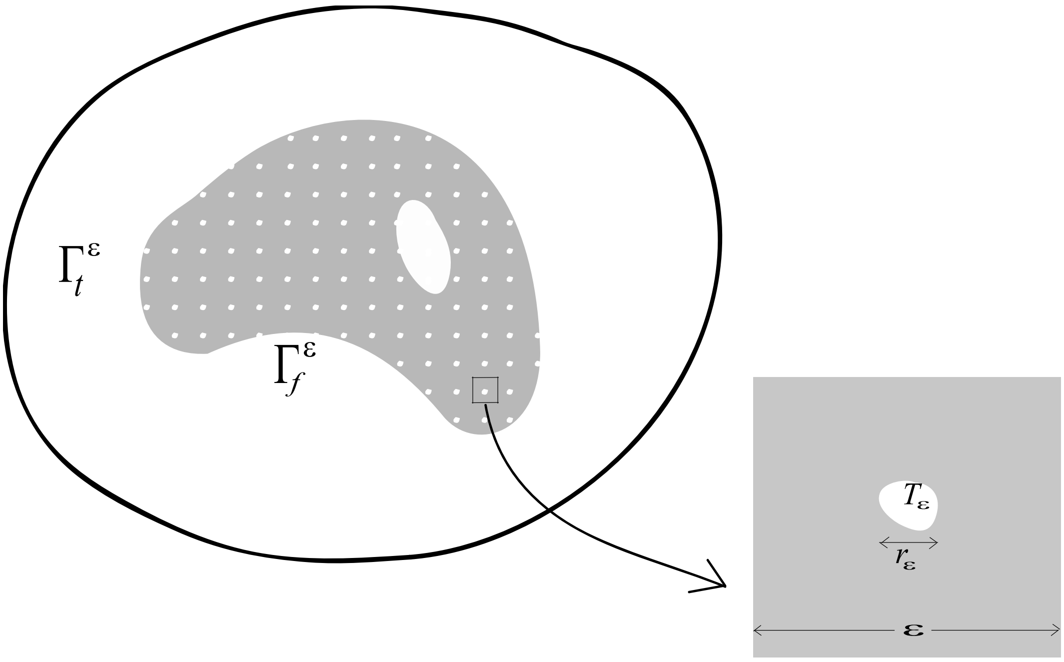


Figure 5.2: The splitting of Σ_0 into Γ_f^ϵ (the friction surface) in grey and Γ_t^ϵ (the barrier surface) in white for the the perturbed problem.

5.3.1 Asymptotic analysis of the problem $\mathcal{P}_\epsilon$

The main theorem concerning the homogenization of problem $\mathcal{P}_\epsilon$ is given next:

Theorem 5.3.1. *The sequence of functionals* $\mathcal{W}_\epsilon \;:\; V_{\epsilon,\delta} \to \mathbb{R}$

$$\mathcal{W}_\epsilon(v) = \frac{1}{2} \parallel v \parallel_V + \int_{\Gamma_f^0} SH(|v_\tau|) - f(v)$$

Γ*-converge with respect to the weak topology of* V *to,* $\overline{\mathcal{W}} \;:\; V \to \mathbb{R}$ *with*

$$\overline{\mathcal{W}}(v) = \frac{1}{2} \parallel v \parallel_V^2 + \int_{\Gamma_f^0} SH(|v_\tau|) - f(v) + \frac{1}{2} c \sum_{i,j=1}^{3} \int_{\Gamma_f^0} C_{ij} v_i v_j$$

where for $k, l = \overline{1,3}$

$$C_{kl} = \begin{cases} 0 & \textit{if } (k-3)(l-3) = 0 \\ \displaystyle\int_{\mathbb{R}^3_+} \mathcal{A}\epsilon(w^k)\epsilon(w^l)dx & \textit{otherwise,} \end{cases}$$

$0 < c = \lim\limits_{\epsilon \to 0} \dfrac{\delta(\epsilon)}{\epsilon} < \infty$ *and* w^k*, for* $k = \overline{1,2}$ *is the solution of the following local problem*

$$\begin{cases} \frac{\partial}{\partial y_j} \sigma_{ij}(w^k) = 0 & \textit{on } \mathbb{R}^3_+ \;\; \textit{for } i = 1,2,3 \\ \sigma_{33}(w^k) = 0, w^k_\tau = e^k & \textit{on } B_1^2(0) \\ \sigma_{i3}(w^k) = 0 & \textit{on } \mathbb{R}^2 \backslash B_1^2(0) \\ w^k(y) \to 0 & \textit{where } y_3 \geq 0 \textit{ and } |y| \to \infty \end{cases}$$

Before beginning the proof we make the following useful remark.

Remark 5.3.2. *The result above can be rewritten in the following way:* $\Gamma-\lim_{\epsilon\to 0}\mathcal{W}_\epsilon = \overline{\mathcal{W}}$ *where* $\overline{\mathcal{W}}: V \to \mathbb{R}$ *is defined as*

$$\overline{\mathcal{W}}(v) = \frac{1}{2} \parallel v \parallel_V^2 + \int_{\Gamma_f^0} S_\perp H(|v_\tau|) - f(v) + \frac{1}{2} c \int_{\Gamma_f^0} v_\tau C v_\tau$$

with c *and the matrix* C *defined as above.*

Proof. We mention that the asymptotic analysis of this problem uses similar techniques as those developed in [3] and [4]. For the convenience of the reader we present the proof of our results, referring to the above mentioned papers when needed. The next lemma can be easily adapted from [3] using Korn's inequality.

Lemma 5.3.3. *Let* (u_j) *be bounded in* V *and let* $N, k \in \mathbb{N}$. *Let* ϵ_j *be a decreasing sequence of positive numbers converging to* 0 *and let*

$$\mathbb{Z}_{f,1}^j = \{i \in \mathbb{Z}^2 / B^2(x_i^{\epsilon_j}, \epsilon_j \delta(\epsilon_j)) \cap \Gamma_f^0 \neq \emptyset\} \text{ and } \mathbb{Z}_f^j = \{i \in \mathbb{Z}^2 / Q_{i,2}^{\epsilon_j} \subset \Gamma_f^0\}$$

Let (ρ_{ϵ_j}) *be a sequence of positive numbers, such that* $N\rho_{\epsilon_j} < \frac{1}{2}\epsilon_j$. *For all* $i \in \mathbb{Z}_f^j$ *there exists* $k_i \in \{0, \ldots, k-1\}$ *such that, having set*

$$C_i^j = \{x \in \mathbb{R}_+^3 \ / \ 2^{-k_i-1} N\rho_{\epsilon_j} < |x - (x_i^\epsilon, 0)| < 2^{-k_i} N\rho_{\epsilon_j}\}$$

$u_j^i = \dfrac{1}{|C_i^j|}\displaystyle\int_{C_i^j} u_j dx$ *(the mean value of* u_j *on* C_i^j*) and* $\rho_i^j = \frac{3}{4} 2^{-k_i} N\rho_{\epsilon_j}$ *(the middle radius of* C_i^j*) there exists a sequence* (w_j) *such that*

$$w_j = u_j \quad on \ \Omega \setminus \bigcup_{i \in \mathbb{Z}_f^j} C_i^j$$

$w_j(x) = u_j^i$ *if* $|x - (x_i^\epsilon, 0)| = \rho_j^i$ *and* $x_3 > 0$, *for* $i \in \mathbb{Z}_f^j$, *and*

$$\sum_{i \in \mathbb{Z}_f^j} \int_{C_i^j} (\mathcal{A}\epsilon(w_j)\epsilon(w_j) + \mathcal{A}\epsilon(u_j)\epsilon(u_j)dx) \leq \frac{M}{k},$$

where M *is independent of* j.
Moreover, if $\rho_{\epsilon_j}^3 = o(\epsilon_j^2)$, *and the sequence* $(|Du_j|^2)$ *is equi-integrable in* Ω, *then we can choose* $k_i = 0$ *for all* $i \in \mathbb{Z}_f^j$ *and*

$$\lim_{j\to+\infty} \sum_{i \in \mathbb{Z}_f^j} \int_{C_i^j} (\mathcal{A}\epsilon(w_j)\epsilon(w_j) + \mathcal{A}\epsilon(u_j)\epsilon(u_j)) = 0$$

As in [3] we can make the following remark.

Remark 5.3.4. *If $u_j \to u$ strongly in $L^2(\Omega)$ and $\sup_j \mathcal{W}_{\epsilon_j}(u_j) < +\infty$ then $u_j \rightharpoonup u$ weakly in V. Moreover if (w_j) is defined as in the above lemma then $w_j \to u$ strongly in $L^2(\Omega)$ and since (w_j) is bounded in V we get that also (w_j) converges weakly to u in V. If $(|Du_j|^2)$ is equi-integrable then $(|Dw_j|^2)$ is also equi-integrable.*

For any $R \in \mathbb{R}_+$ we will denote by $B_R(x) \subset \mathbb{R}^3$ be the ball centered in $x \in \mathbb{R}^3$, and radius R, and $B_R^+(x) = B_R(x) \cap \mathbb{R}^3_+$.

Let

$$\phi_N(z) = \inf \left\{ \int_{B_N^+(0)} \mathcal{A}\epsilon(v)\epsilon(v) / \ v \in H^1(B_N^+(0), \mathbb{R}^3), v_\tau = 0 \text{ on } B_1^2(0), v = z \text{ on } \partial B_N^+(0) \setminus \Sigma_0 \right\} \tag{5.3.6}$$

We will now make another useful remark:

Remark 5.3.5. *If f is a convex function and $0 \leq f(A) \leq M(1 + |A|^2)$ then f is locally Lipschitz i.e.,*

$$|f(A) - f(B)| \leq M(1 + |A| + |B|)|A - B|, \quad \textit{for all } A, B \in \mathcal{M}^{3\times3}. \tag{5.3.7}$$

In addition if f is also homogeneous of order 2, then

$$|f(A) - f(B)| \leq M(|A| + |B|)|A - B| \quad \textit{for all } A, B \in \mathcal{M}^{3\times3}. \tag{5.3.8}$$

Proof. Indeed from (5.3.7) if we consider a sequence ϵ_j, such that $\epsilon_j \to 0$ we have

$$|f(A) - f(B)| = \epsilon_j^2 \left| f\left(\frac{1}{\epsilon_j}A\right) - f\left(\frac{1}{\epsilon_j}B\right)\right| \leq \epsilon_j^2 M \left(1 + \frac{1}{\epsilon_j}|A| + \frac{1}{\epsilon_j}|B|\right) \frac{1}{\epsilon_j}|A - B|.$$

Thus

$$|f(A) - f(B)| \leq M\epsilon_j|A - B| + M(|A| + |B|))|A - B|$$

for all $A, B \in \mathcal{M}^{3\times3}$ and for any $j \in \mathbb{N}$.Then passing to the limit where $j \to \infty$ we have (5.3.8). □

Now we have

Lemma 5.3.6. *For all $N \in \mathbb{N}$ with $N > 2$, ϕ_N defined above verifies*

$$| \phi_N(z) - \phi_N(w) | \leq M \mid w - z \mid (\mid z \mid + \mid w \mid) \quad \textit{for all } z, w \in \mathbb{R}^3 \tag{5.3.9}$$

Proof. Fix $1 > \nu > 0$. Using the definition of $\phi_N(z)$ we find $\bar{w} \in H^1(B_N^+(0); \mathbb{R}^3)$, with
$\bar{w} = 0$ on $\partial B_N^+(0) \setminus \Sigma_0$ and $\bar{w}_\tau = -z_\tau$ on $B_1^2(0)$ such that

$$\int_{B_N^+(0)} \mathcal{A}\epsilon(\bar{w})\epsilon(\bar{w}) \leq \phi_N(z) + \nu \tag{5.3.10}$$

Let $\varphi \in C_0^\infty(B_2(0))$ be a cutoff function such that $\phi = 1$ on $B_1(0)$ and $| D\varphi | \leq M$. For $w \in \mathbb{R}^3$ define $\psi = \bar{w} + (1-\varphi)(w-z)$ on $B_N^+(0)$. So we can see that

$$\begin{cases} \psi_\tau = -z_\tau & \text{on } B_1^2(0) \\ \psi = w - z & \text{on } \partial B_N^+(0) \setminus \Sigma_0. \end{cases}$$

So ψ is a test function for $\phi_N(w)$. Thus using (5.1.3), (5.3.8) and (5.3.6) we have

$$\begin{aligned} \phi_N(w) - \phi_N(z) \ &\leq \int_{B_N^+(0)} \mathcal{A}\epsilon(\psi)\epsilon(\psi)dx - \int_{B_N^+(0)} \mathcal{A}\epsilon(\bar{w})\epsilon(\bar{w}) + \nu \\ &\leq M \int_{B_N^+(0)} (| \epsilon(\psi) | + | \epsilon(\bar{w}) |) \cdot (| \epsilon(\psi) - \epsilon(\bar{w}) |)dx + \nu \quad (5.3.11) \end{aligned}$$

Next from the definition of the test function ψ in (5.3.11) we obtain

$$\begin{aligned} \phi_N(w) - \phi_N(z) \ &\leq \ M \int_{B_N^+(0)} (2 | \epsilon(\bar{w}) | + | w - z | | D\varphi |) | w - z | | D\varphi | + \nu \\ &\leq \ M | w - z | \left(\int_{B_N^+(0)} | \epsilon(\bar{w}) |^2 dx \right)^{1/2} \cdot \left(\int_{B_N^+(0)} | D\varphi |^2 dx \right)^{1/2} \\ &+ \ M | w - z |^2 \int_{B_N^+(0)} | D\varphi |^2 dx + \nu. \qquad (5.3.12) \end{aligned}$$

Since $N > 2$ we have that $\int_{B_N^+(0)} | D\varphi | \ dx$ and $\int_{B_N^+(0)} | D\varphi |^2 \ dx$ are constants independent of N and by condition (5.1.3) and the definition of ϕ_N we get

$$M \int_{B_N^+(0)} | \epsilon(\bar{w}) |^2 dx \leq \int_{B_N^+(0)} \mathcal{A}\epsilon(\bar{w})\epsilon(\bar{w}) \leq \phi_N(z) + \nu \qquad (5.3.13)$$

and from Korn's inequality and monotonicity of ϕ_N

$$\begin{aligned} \phi_N(z) \ &\leq \ \phi_2(z) \leq M | z |^2 \inf \left\{ \int_{B_2^+(0)} | Dv |^2 \ dx \ \ v \in H^1(B_2^+(0); \mathbb{R}^3) \right., \\ &\left. v_\tau = 0 \ \text{ on } \ B_1^2(0), \ v = \frac{z}{| z |} \ \text{ on } \ \partial B_2^+(0) \setminus \Sigma_0 \right\}. \qquad (5.3.14) \end{aligned}$$

But

$$\left\{ v \ / \ v \in H^1(B_2^+(0); \mathbb{R}^3), \ v_\tau = 0 \ \text{ on } \ B_1^2(0), v = \frac{z}{| z |} \ \text{ on } \ \partial B_2^+(0) \setminus \Sigma_0 \right\}$$

contains

$$\left\{ v \ / \ v \in H^1(B_2(0) \setminus C_{1,2}; \mathbb{R}^3), \ v = \frac{z}{| z |} \ \text{ on } \ \partial B_2^+(0) \setminus \Sigma_0, \ v = 0 \ \text{ on } \ B_2(0) \cap \mathbb{R}^3_- \right\}$$

as a subset, where

$$C_{1,2} = \left\{(x',0) \in \mathbb{R}^3 \;:\; 1 \le |x'| < 2\right\}.$$

From (5.3.14) and the above inclusion we obtain

$$\phi_N(z) \;\le\; \phi_2(z) \le M \mid z \mid^2 \inf \left\{ \int_{B_2^+(0)} \mid Dv \mid^2 dx \;\; /v \in H^1(B_2(0) \setminus C_{1,2}, \mathbb{R}^3), \right.$$

$$\left. v = \frac{z}{\mid z \mid} \;\text{ on }\; \partial B_2^+(0) \setminus \Sigma_0,\; v = 0 \;\text{ on }\; B_2(0) \cap \mathbb{R}^3_- \right\}.$$

Now following the ideas in [3] (see (4.9)), we obtain

$$\phi_N(z) \le M \frac{\mid z \mid^2}{2} \mathrm{Cap}\ (B_1^2(0))$$

where $\mathrm{Cap}(B_1^2(0))$ is the usual capacity, i.e.,

$$\mathrm{Cap}(B_1^2(0)) = \inf \left\{ \int_{\mathbb{R}^3} \mid D\phi \mid^2 dx, \phi \in H^1(\mathbb{R}^3), \phi = 1 \;\text{ on }\; B_1^2(0) \right\}$$

By (5.3.13) we get

$$\int_{B_N^+(0)} \mid \epsilon(\bar{w}) \mid^2 dx \le M(\frac{\mid z \mid^2}{2} + \nu) \tag{5.3.15}$$

Now, from the results obtained so far, and (5.3.15) and (5.3.12) we have

$$\phi_N(w) - \phi_N(z) \le M \mid w - z \mid \left(\frac{\mid z \mid^2}{2} + \nu \right)^{1/2} + M \mid w - z \mid^2 + \nu \le$$

$$\le M \mid w - z \mid ((\nu + 1) \mid z \mid + \mid w \mid + \sqrt{\nu}) + \nu.$$

Now by the arbitrariness of ν we get that

$$\phi_N(w) - \phi_N(z) \le M \mid w - z \mid (\mid z \mid + \mid w \mid).$$

□

Lemma 5.3.7. *$\phi_N \to \phi$ uniformly where $\phi(z) = \sum_{k,l=1}^{3} C_{kl} z_k z_l$, and C_{kl} is given by (5.3.16) and the local problem (LP).*

Proof. From Ascoli-Arzela's Theorem we have that $\phi_N \to \phi$ uniformly on compact sets of $\mathbb{R}^3$.

For any $N \in \mathbb{N}$ the problem (5.3.9) has a unique solution $\tilde{w} + z$ for fixed $z \in \mathbb{R}^3$. The Euler-Lagrange equation for $\tilde{w}$ is

$$\begin{cases} \sigma(\tilde{w}) = \mathcal{A}\epsilon(\tilde{w}),\ -\mathrm{div}\sigma(\tilde{w}) = 0 & \text{on } B_N^+(0) \\ \sigma_{33}(\tilde{w}) = 0, \tilde{w}_\tau = -z_\tau & \text{on } B_1^2(0) \\ \sigma_{ij}(\tilde{w}) n_j = 0 & \text{on } C_{1,N} \\ \tilde{w} = 0 & \text{on } \partial B_N^+(0) \backslash \Sigma_0 \end{cases}$$

So $\phi_N = \int_{B_N^+(0)} \mathcal{A}\epsilon(\tilde{w})\epsilon(\tilde{w})dx = \sum_{k,l=1}^{3} C_{kl}^N z_k z_l$ where

$$C_{kl}^N = \begin{cases} 0 & (k-3)(l-3)=0 \\ \int_{B_N^+(0)} \mathcal{A}\epsilon(w_N^k)\epsilon(w_N^l) & \text{otherwise,} \end{cases} \qquad \text{for} \quad k,l=\overline{1,3}$$

where w_N^k for $k=\overline{1,2}$ is the solution of the following local problem,

$$\begin{cases} \frac{\partial}{\partial y_j}\sigma_{ij}(w_N^k) = 0 & \text{on } B_N^+(0) \qquad \text{for } i=\overline{1,3} \\ \sigma_{33}(w_N^k)=0,\ w_{N\tau}^k = e^k & \text{on } B_1^2(0) \\ \sigma_{i3}(w_N^k)=0 & \text{on } C_{1,N} \\ w_N^k = 0 & \text{on } \partial B_N^+(0)\setminus\Sigma \end{cases}$$

where $\{e^k\}_{k=\overline{1,3}}$ is the canonical base of $\mathbb{R}^3$. So $\phi_N \to \phi$ uniformly on compacts subsets of $\mathbb{R}^3$ where

$\phi(z) = \sum_{k,l=1}^{3} C_{kl} z_k z_l$ and

$$C_{kl} = \begin{cases} 0 & (k-3)(l-3)=0 \\ \int_{\mathbb{R}^3_+} \mathcal{A}\epsilon(w^k)\epsilon(w^l)dx & \text{otherwise} \end{cases}, \quad \text{for } k,l=\overline{1,3} \tag{5.3.16}$$

and w^k for $k=\overline{1,2}$ is the solution of the following local problem,

$$(LP)\begin{cases} \frac{\partial}{\partial y_j}\sigma_{ij}(w^k)=0 & \text{on } \mathbb{R}^3_+ \quad \text{for } i=\overline{1,3} \\ \sigma_{33}(w^k)=0,\ w_\tau^k = e^k & \text{on } B_1^2(0) \\ \sigma_{i3}(w^k)=0 & \text{on } \mathbb{R}^2 - B_1^2(0) \\ w^k(y)\to 0 & \text{when } \mid y \mid \to \infty \end{cases}$$

□

Remark 5.3.8. *From Lemma 5.3.7 we can see that*

$$\phi_N(z) = \phi_N(z_\tau) \ \textit{ and } \ \phi(z)=\phi(z_\tau).$$

Now using Remark 5.3.8, by similar techniques as in ([3], see Prop 4.4) we have.

Lemma 5.3.9. *Let $u_j \to u$ weakly in V and bounded in $L^\infty(\Omega;\mathbb{R}^3)$. Consider ψ_j to be defined as*

$$\psi_j = \sum_{i\in\mathbb{Z}_f^j} \phi_N(u_{j\tau}^i)\chi_{Q_{i,2}^{\epsilon_j}}$$

where

$$Q_{i,2}^{\epsilon_j} = (x_i^{\epsilon_j},0) + \left(-\frac{\epsilon_j}{2},\frac{\epsilon_j}{2}\right)^2,$$

and u_j^i and $\mathbb{Z}_f^j$ are defined in Lemma 5.3.3.

Then we have

$$\lim_{j\to\infty}\int_{\Gamma_f^0} \mid \psi_j - \phi_N(u_\tau) \mid ds = 0$$

Proof. First we will show that

$$|\Gamma_f^0 \setminus \bigcup_{i\in\mathbb{Z}_f^j} Q_{i,2}^{\epsilon_j}| \overset{j\to\infty}{\longrightarrow} 0.$$

Indeed, let

$$w_j^{''} = \bigcup_{\substack{i\in\mathbb{Z}^2\setminus\mathbb{Z}_f^j \\ Q_{i,2}^{\epsilon_j}\cap\Gamma_f^0\neq\emptyset}} Q_{i,2}^{\epsilon_j}.$$

Then easily can be seen that

$$\left(\Gamma_f^0 \setminus \bigcup_{i\in\mathbb{Z}_f^j} Q_{i,2}^{\epsilon_j}\right) \subset w_j^{''}$$

and therefore we obtain the limit when $j \to \infty$ that

$$\lim_{j\to\infty} |\Gamma_f^0 \setminus \bigcup_{i\in\mathbb{Z}_f^j} Q_{i,2}^{\epsilon_j}| \leq \lim_{j\to\infty} \mathcal{H}^2(w_j^{''}) \leq \mathcal{H}^2(\partial\Gamma_f^0) = 0 \tag{5.3.17}$$

Using Remark 5.3.8 we get

$$\lim_{j\to\infty} \int_{\Gamma_f^0} \mid \psi_j - \phi_N(u_\tau) \mid ds = \lim_{j\to\infty} \int_{\Gamma_f^0} \left| \sum_{i\in\mathbb{Z}_f^j} \phi_N(u_j^i)\chi_{Q_{i,2}^{\epsilon_j}} - \phi_N(u) \right| \leq$$

$$\leq \lim_{j\to\infty} \left(\sum_{i\in\mathbb{Z}_f^j} \int_{Q_{i,2}^{\epsilon_j}} \mid \phi_N(u_j^i) - \phi_N(u) \mid ds \right) + \lim_{j\to\infty} \int_{\Gamma_f^0\setminus\bigcup_{i\in\mathbb{Z}_f^j} Q_{i,2}^{\epsilon_j}} |\phi_N(u)|$$

Now from Lemma 5.3.6, and (5.3.17),the uniform boundedness of ϕ_N, Lemma 5.3.6 and the boundedness of $(u_j)_j$ in $L^\infty(\Omega, \mathbb{R}^3)$ we get

$$\lim_{j\to\infty} \int_{\Gamma_f^0} \mid \psi_j - \phi_N(u_\tau) \mid ds \leq M \lim_{j\to\infty} \sum_{i\in\mathbb{Z}_f^j} \int_{Q_{i,2}^{\epsilon_j}} \mid u_j^i - u \mid ds$$

By similar arguments as in ([3], Prop 4.4.) we can prove that

$$\lim_{j\to\infty} \sum_{i\in\mathbb{Z}_f^j} \int_{Q_{i,2}^{\epsilon_j}} \mid u_j^i - u \mid ds = 0$$

and this proves the statement of the Lemma. □

Using Remark 5.3.8, Lemma 5.3.7, Lemma 5.3.9, by similar arguments as in ([3], Sec. 5) we obtain the liminf inequality

Lemma 5.3.10. *Consider $\rho_{\epsilon_j} \doteq \epsilon_j \delta(\epsilon_j)$ defined as in Lemma 1 such that*

$$0 < \lim_{j\to\infty} \frac{\delta(\epsilon_j)}{\epsilon_j} = c < +\infty. \tag{5.3.18}$$

Let $k, N \in M$ fixed, and $N > 2^k$. Then for any sequence $(u_j)_j$, such that $u_j \in V_{\epsilon_j}$ and $u_j \rightharpoonup u$ weakly in V, we have

$$\liminf_{j\to\infty} \| u_j \|_V^2 \geq \| u \|_V^2 + c \sum_{k,l=1}^{3} \int_{\Gamma_f^0} C_{kl} u_k u_l ds$$

where the matrix $(C_{kl})_{k,l=\overline{1,3}}$ is defined in Theorem 5.3.1.

Proof. Remark 5.3.8 is very important. Because of the property mentioned in Remark 5.3.8, we can follow the proof in ([3], Sec. 5), although in our case the space $V_{\epsilon,\delta}$ is not the same as in [3] and therefore the functions ϕ_N and ϕ respectively are not the same as in [3].
Let $u \in V$ and consider $\{u_j\}_j$ such that $u_j \in V_{\epsilon_j}$ and $u_j \rightharpoonup u$. Let w_j and ρ_j^i as in Lemma 5.3.3 and

$$E_j = \bigcup_{i\in\mathbb{Z}_f^j} B_i^j \quad \text{with} \quad B_i^j = B_{\rho_j^i}(x_i^{\epsilon_j}, 0) \cap \{x \in \mathbb{R}^3 \mid x_3 > 0\} \quad \text{for all} \quad i \in \mathbb{Z}_f^j$$

We have

$$\liminf_{j\to\infty} \| u_j \|_V^2 \geq \liminf_{j\to\infty} \int_{\Omega\setminus E_j} \mathcal{A}\epsilon(u_j)\epsilon(u_j)dx + \liminf_{j\to\infty} \int_{E_j} \mathcal{A}\epsilon(u_j)\epsilon(u_j)dx \tag{5.3.19}$$

Next we can see that

$$\frac{M}{k} + \liminf_{j\to\infty} \int_{\Omega\setminus E_j} \mathcal{A}\epsilon(u_j)\epsilon(u_j)dx \geq \| u \|_V^2 \tag{5.3.20}$$

The proof of (5.3.20) is identical with the proof in ([3], Prop 5.1)) and therefore we won't present here.

Now similarly as in [3] let's define, for fixed $j \in \mathbb{N}$ and $i \in \mathbb{Z}_f^j$

$$\varphi(x) = \begin{cases} w_j((x_i^{\epsilon_j}, 0) + \epsilon_j \delta(\epsilon_j) x) & \text{if } x \in B^+_{\frac{3}{4}2^{-k_i}N}(0) \\ u_j^i & \text{if } x \in B_N^+(0) \setminus B^+_{\frac{3}{4}2^{-k_i}N}(0), \end{cases} \tag{5.3.21}$$

where $k_i \in \overline{1, k-1}$ and u_j^i are as in Lemma 5.3.3.

As in [3] we have, by Lemma 5.3.3

$$\frac{M}{k} + \liminf_{j\to\infty} \int_{E_j} \mathcal{A}\epsilon(u_j)\epsilon(u_j)dx \geq \liminf_{j\to\infty} \int_{E_j} \mathcal{A}\epsilon(w_j)\epsilon(w_j)dx \tag{5.3.22}$$

and using φ, defined in (5.3.21), as a test function in the definition of ϕ_N (see [3],(5.31)) we get

$$\liminf_{j\to\infty}\int_{E_j}\mathcal{A}\epsilon(w_j)\epsilon(w_j)dx \geq \liminf_{j\to\infty}\frac{\delta(\epsilon_j)}{\epsilon_j}\sum_{i\in\mathbb{Z}_f^j}\epsilon_j^2\phi_N(u_j^i)$$

By Remark 5.3.8 and the hypothesis (5.3.18) we obtain

$$\liminf_{j\to\infty}\int_{E_j}\mathcal{A}\epsilon(w_j)\epsilon(w_j)dx \geq c\cdot\liminf_{j\to\infty}\sum_{i\in\mathbb{Z}_f^j}\epsilon_j^2\phi_N(u_{j\tau}^i). \tag{5.3.23}$$

Note now that for any $j\in\mathbf{N}$ we have

$$\sum_{i\in\mathbb{Z}_f^j}\epsilon_j^2\phi_N(u_{j\tau}^i) = \int_{\Gamma_f^0}\psi_j ds, \tag{5.3.24}$$

where ψ_j, has been defined in Lemma 5.3.9. Combining (5.3.22), (5.3.23), (5.3.24) we obtain

$$\frac{M}{k}+\liminf_{j\to\infty}\int_{E_j}\mathcal{A}\epsilon(u_j)\epsilon(u_j)dx \geq c\liminf_{j\to\infty}\int_{\Gamma_f^0}\psi_j ds \tag{5.3.25}$$

and from (5.3.20) and (5.3.25) we obtain

$$\frac{M}{k}+\liminf_{j\to\infty}\int_{\Omega}\mathcal{A}\epsilon(u_j)\epsilon(u_j)dx \geq \| u \|_V^2 + c\liminf_{j\to\infty}\int_{\Gamma_f^0}\psi_j ds \tag{5.3.26}$$

From Lemma 5.3.7, Lemma 5.3.9, (5.3.26) and the arbitrariness of $k\in\mathbb{N}$, using Lemma 3.5 from [13] as in ([3], sec. Prop 5.2), we can "remove" the $L^\infty(\Omega,\mathbb{R}^3)$ boundedness hypothesis from Lemma 5.3.9, and obtain the liminf inequality in a similar manner. □

Next, we will prove the limsup inequality.

Lemma 5.3.11. *Let $\delta(\epsilon_j)$ be such that*

$$0 < \lim_{j\to\infty}\frac{\delta(\epsilon_j)}{\epsilon_j} = c < +\infty.$$

Then for all $u\in V$ and for all $\delta>0$ there exists a sequence $u_j\in V_{\epsilon_j,\delta_j}$ converging to u, in the weak topology of V, such that

$$\limsup_{j\to\infty}\| u_j \|_V^2 - \delta \leq \| u \|_V^2 + c\int_{\Gamma_f^0}\sum_{k,l=1}^{3}C_{kl}u_k u_l ds,$$

where $(C_{kl})_{k,l=\overline{1,3}}$ has been defined in Lemma 5.3.7.

Proof. Without loss of generality we will assume δ small enough. Again Remark 5.3.8 allows us to follow the same arguments as in ([3], Sec. 6).

Indeed, suppose first that $u \in L^\infty(\Omega, \mathbb{R}^3)$. Recall that $B_{N\rho_{\epsilon_j}} \equiv B_{N\rho_{\epsilon_j}}(x_i^{\epsilon_j}, 0)$ and $B^+_{N\rho_{\epsilon_j}} = B_{N\rho_{\epsilon_j}} \cap \{x_3 > 0\}$. From Lemma 5.3.3 for $u_j \equiv u$ and $\rho_{\epsilon_j} = \frac{4}{3}\epsilon_j \delta(\epsilon_j)$ and from the equi-integrability condition we obtain a sequence $(w_j)_j$ such that

$$w_j = u_j^i = \frac{1}{|C_i^j|}\int_{C_i^j} u \ \text{ on } \ \partial B^+_{N\epsilon_j\delta(\epsilon_j)} \setminus \Sigma_0.$$

where by $|A|$ we denoted the usual superficial measure supported by A. Define

$$v_j = w_j \ \text{ on } \ \Omega \setminus \bigcup_{i\in\mathbb{Z}^j_{f,1}} B^+_{N\epsilon_j\delta(\epsilon_j)} \tag{5.3.27}$$

Then because $|\bigcup_{i\in\mathbb{Z}^j_{f,1}} B^+_{N\epsilon_j\delta(\epsilon_j)}| \sim \epsilon_j\delta^3(\epsilon_j)$, and $w_j \rightharpoonup u$ weakly in V, we obtain that $v_j \rightharpoonup u$ weakly in V. We will define v_j on $\bigcup_i B^+_{N\epsilon_j\delta(\epsilon_j)}$ below.

Next, using similar arguments as in ([3], Sec. 6) we get

$$\limsup_{j\to\infty} \| v_j \|_V^2 \leq \| u \|_V^2 + \limsup_{j\to\infty} \int_{\bigcup_{i\in\mathbb{Z}^j_{f,1}} B_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \ . \tag{5.3.28}$$

From Lemma 5.3.7, we have that for any $\delta > 0$, there is an $N_0 \in \mathbb{N}$ such that

$$\phi(z) - \frac{\delta}{2} \leq \phi_N(z) \leq \phi(z) + \frac{\delta}{2} \quad \text{for any } z \ \text{ with } \ | z | \leq m, \tag{5.3.29}$$

and for any $N \geq N_0$, where $m = \| u \|_{L^\infty(\Omega,\mathbb{R}^3)}$.

By the definition of ϕ_N there is $w_j^i \in H^1(B_N^+(0); \mathbb{R}^3)$, $w^i_{j\tau} = 0$ on $B_1^2(0)$ and $w_j^i = u_j^i$ on $\partial B_N^+(0) \setminus \Sigma_0$, such that

$$\int_{B_N^+(0)} \mathcal{A}\epsilon(w_j^i)\epsilon(w_j^i)dx \leq \phi_N(u_j^i) + \frac{\delta}{2} \leq \phi(u_j^i) + \delta = \phi(u^i_{j\tau}) + \delta \tag{5.3.30}$$

where we used Remark 5.3.8 for the last equality above. Next, similar as in ([3], Sec. 6) we define v_j on $\bigcup_{i\in\mathbb{Z}^j_{f,1}} B_{N\rho_{\epsilon_j}}$ to be

$$v_j = w_j^i\left(\frac{x - (x_i^{\epsilon_i}, 0)}{\epsilon_j\delta(\epsilon_j)}\right) \ \text{ on } B_{N\epsilon_j\delta(\epsilon_j)} \quad \text{for } \ i \in \mathbb{Z}_f^j \tag{5.3.31}$$

and

$$v_j = h \cdot \left(\frac{x - (x_i^{\epsilon_i}, 0)}{\epsilon_j\delta(\epsilon_j)}\right) w_j(x) \text{ on } B^+_{N\epsilon_j\delta(\epsilon_j)} \quad \text{for } \ i \in \mathbb{Z}^j_{f,1} \setminus \mathbb{Z}^j_f \tag{5.3.32}$$

where $0 \leq h \leq 1$ is the same scalar function used in ([3]), i.e., $h = 1$ on $\partial B_N^+(0) \setminus \Sigma_0$ an $h = 0$ on $B_1^2(0)$.

From (5.3.27), (5.3.31) and (5.3.32) we can see that $v_j \in V_{\epsilon_j}$ and from (5.3.30) we have that ,

$$\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx = \epsilon_j\delta(\epsilon_j)\int_{B_N^+(0)} \mathcal{A}\epsilon(w_j^i)\epsilon(w_j^i)dx \leq \frac{\delta(\epsilon_j)}{\epsilon_j}(\epsilon_j^2\phi(u^i_{j\tau}) + \epsilon_j^2\delta) \tag{5.3.33}$$

for any $i \in \mathbb{Z}_f^j$.

Obviously we have

$$\limsup_{j\to\infty}\sum_{i\in\mathbb{Z}^j_{f,1}}\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \leq \limsup_{j\to\infty}\sum_{i\in\mathbb{Z}^j_{f,1}\setminus\mathbb{Z}^j_f}\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \tag{5.3.34}$$

$$+ \limsup_{j\to\infty}\sum_{i\in\mathbb{Z}^j_f}\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \tag{5.3.35}$$

Now let $w_j' = \bigcup_{i\in\mathbb{Z}^j_{f,1}\setminus\mathbb{Z}^j_f} Q^{\epsilon_j}_{i,2}$. For any $i \in \mathbb{Z}^j_{f,1} \setminus \mathbb{Z}^j_f$ we have

$$\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \leq M(N)\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mid Dv_j \mid^2 dx \leq$$

$$\leq \frac{1}{\epsilon_j^2\delta^2(\epsilon_j)}M(N)\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} |Dh|^2|w_j|^2dx + \int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} |Dw_j|^2dx. \tag{5.3.36}$$

Then using (5.3.36) and the equi-integrability and L^∞ bound of w_j we obtain

$$\limsup_{j\to\infty}\sum_{i\in\mathbb{Z}^j_{f,1}\setminus\mathbb{Z}^j_f} \cdot \int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \leq M(N)\lim_{j\to\infty}\frac{\delta(\epsilon_j)}{\epsilon_j}\lim_{j\to\infty}\mathcal{H}^2(w_j') \leq M(N)\cdot c\cdot\mathcal{H}^2(\partial\Gamma_f^0) = 0. \tag{5.3.37}$$

Next summing in (5.3.33) for all $i \in \mathbb{Z}^j_f$ and passing to the limit when $j \to \infty$ we get

$$\limsup_{j\to\infty}\sum_{i\in\mathbb{Z}^j_f}\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}} \mathcal{A}\epsilon(v_j)\epsilon(v_j)dx \leq \lim_{j\to\infty}\frac{\delta(\epsilon_j)}{\epsilon_j}\limsup_{j\to\infty}\sum_{i\in\mathbb{Z}^j_f}(\epsilon_j^2\phi(u^i_{j\tau}) + \epsilon_j^2\delta). \tag{5.3.38}$$

From (5.3.29) we have that

$$\sum_{i\in\mathbb{Z}^j_f}\epsilon_j^2\phi(u^i_{j\tau}) \leq \sum_{i\in\mathbb{Z}^j_f}\left(\epsilon_j^2\phi_N(u^i_{j\tau}) + \epsilon_j^2\frac{\delta}{2}\right) \leq M\delta + \sum_{i\in\mathbb{Z}^j_f}\epsilon_j^2\phi_N(u^i_{j\tau}). \tag{5.3.39}$$

From (5.3.39) and Lemma 5.3.9, we obtain

$$\limsup_{j\to\infty}\sum_{i\in\mathbb{Z}_f^j}\epsilon_j^2\phi(u_{j\tau}^i)\leq M\delta+\limsup_{j\to\infty}\int_{\Gamma_f^0}\psi_j ds=M\delta+\int_{\Gamma_f^0}\phi_N(u_\tau)ds. \tag{5.3.40}$$

From (5.3.34), (5.3.37) and (5.3.38), we obtain that there exists a positive constant $M=M\left(\| u \|_{L^\infty},c,M_1,M_2,|\Gamma_f^0|\right)$

$$\limsup_{j\to\infty}\sum_{i\in\mathbb{Z}_{f,1}^j}\int_{B^+_{N\epsilon_j\delta(\epsilon_j)}}\mathcal{A}\epsilon(v_j)\epsilon(v_j)dx\leq c\cdot\sum_{k,l=1}^{3}\int_{\Gamma_f^0}C_{kl}u_k u_l ds+M\delta. \tag{5.3.41}$$

From (5.3.41) and (5.3.28) we obtain

$$\limsup_{j\to\infty}\| v_j \|_V^2\leq\| u \|_V^2+c\sum_{k,l=1}^{3}\int_{\Gamma_f^0}C_{kl}u_k u_l ds+M\delta.$$

Because of the fact that M is a constant independent of δ the statement follows taking for example $\delta\doteq\frac{\delta}{M}$ in (5.3.33) and (5.3.30). Next the boundedness assumption for $u\in L^\infty(\Omega;\mathbb{R}^3)$ can be removed exactly by the same arguments in [3]. □

Next we make the simple observation that the functional $v\rightsquigarrow\int_{\Gamma_f^0}S_\perp H(|v_\tau|)-f(v)$ is continuous with respect to the weak topology on V. This can be seen by the trace continuity and the definition of the function H. Also easily we can observe that the limit functional does not depend on the particular subsequence ϵ_j and therefore by Uryson's property for the Γ-limits, using Lemma 5.3.10 and Lemma 5.3.11 and the above observations we proved Theorem 5.3.1. □

The cases $c=0$ and $c=\infty$ are discussed in the following Remark:

Remark 5.3.12. *We can see that when $c=0$ the influence of the barriers disappear in the limit problem. Indeed in this case we obtain*

$$\Gamma-\lim_{\epsilon\to0}\mathcal{W}_\epsilon=\mathcal{W}$$

where $\mathcal{W}:V\to\mathbb{R}^3$

$$\mathcal{W}(v)=\frac{1}{2}\| v \|_V^2+\int_{\Gamma_f^0}S_\perp H(|v_\tau|)-f(v).$$

In the other case $c=\infty$ we obtain that

$$\Gamma-\lim_{\epsilon\to0}\mathcal{W}_\epsilon=\mathcal{W}$$

with $\mathcal{W}:V\to\mathbb{R}^3$ and

$$\mathcal{W}(u)=\begin{cases}\frac{1}{2}\| u \|_V^2 & \text{if } u\in V_1\\ \infty & \text{otherwise}\end{cases}$$

5.3.2 Asymptotic analysis of the spectral problem $\mathcal{E}_\epsilon$

Rayleigh's principle for the operator K^ϵ gives us

$$\lambda_n^\epsilon = \inf_{\substack{u\in W_\epsilon, u\perp u_i^\epsilon \\ i=\overline{1,n-1}}} \frac{\| u \|_V^2}{\displaystyle\int_{\Gamma_f^0} u_\tau^2}. \tag{5.3.42}$$

where $\{u_i^\epsilon\}_i$ form the orthonormal sequence of eigenvectors for K^ϵ corresponding to the sequence of eigenvalues $\{\lambda_i^\epsilon\}_i$. Using trace inequality and (5.1.3) we obtain

$$\lambda_n^\epsilon \geq M \quad \text{for any } n \in \mathbf{N} \tag{5.3.43}$$

with M not depending on ϵ, and therefore $\{\lambda_n^\epsilon\}$ is uniformly bounded from below. Next we will prove that all the limit points λ_n of $\{\lambda_n^\epsilon\}_\epsilon > 0$ are finite.

Lemma 5.3.13. *If* $\lim_{\epsilon\to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$ *then we have* $\limsup_{\epsilon\to 0} \lambda_n^\epsilon < \infty$ *for any* $n \in \mathbf{N}$.

Proof. Let $u \in V$ such that $u = \bar{u}_1 + \bar{u}_2$ where $0 \neq \bar{u}_1 \in W$ and $0 \neq \bar{u}_2 \in V_1$.
Next consider the recovering sequence for u, i.e., $\bar{u}_\epsilon$ defined in the proof of Theorem 5.3.1,(5.3.27) and (5.3.31). We have that $\bar{u}_\epsilon \in V_{\epsilon,\delta}$ and $\bar{u}_\epsilon \rightharpoonup u$ weakly in V. Obviously from the definition of $u \in V$ we can see that there is an $\epsilon_0 > 0$ such that

$$\bar{u}_\epsilon \notin V_1 \ \text{ and } \bar{u}_\epsilon \notin W_\epsilon \tag{5.3.44}$$

for $\epsilon < \epsilon_0$.
Indeed if $\bar{u}_\epsilon \in V_1$ for a subsequence still denoted by ϵ,with $\epsilon \to 0$,then

$$0 =< \bar{u}_\epsilon, \bar{u}_1 >_V \to < \bar{u}_1, \bar{u}_1 >_V = \| \bar{u}_1 \|_V^2 > 0$$

and therefore the contradiction. Similarly it can be seen that $\bar{u}_\epsilon \notin W_\epsilon$ for all $\epsilon < \epsilon_0$.
From (5.3.42) we have that

$$\lambda_1^\epsilon \leq \frac{\| P_{W_\epsilon}\bar{u}_\epsilon \|_V^2}{\displaystyle\int_{\Gamma_f^0} (P_{W_\epsilon}\bar{u}_\epsilon)_\tau^2} \leq \frac{\| \bar{u}_\epsilon \|_V^2}{\displaystyle\int_{\Gamma_f^0} \bar{u}_{\epsilon\tau}^2} \leq \frac{M}{\displaystyle\int_{\Gamma_f^0} \bar{u}_{\epsilon\tau}^2}.$$

Since $\bar{u}_\epsilon$ is weakly convergent to u and using the continuity of the trace we get

$$\limsup_{\epsilon\to 0} \lambda_1^\epsilon \leq \frac{M}{\displaystyle\int_{\Gamma_f^0} \bar{u}_{1\tau}^2} < \infty$$

where we used the orthogonal decomposition

$$V_{\epsilon,\delta} = W_\epsilon \oplus V_1$$

in order to obtain

$$\int_{\Gamma_f^0} (P_{W_\epsilon} \bar{u}_\epsilon)_\tau^2 = \int_{\Gamma_f^0} \bar{u}_{\epsilon\tau}^2.$$

Next we will use an induction argument to prove the statement for all $n \in \mathbf{N}$. Let's assume that

$$\limsup_{\epsilon \to 0} \lambda_k^\epsilon < \infty \qquad \text{for any } k \leq n-1. \tag{5.3.45}$$

We need to prove

$$\limsup_{\epsilon \to 0} \lambda_n^\epsilon < \infty$$

Let $\{\lambda_n^\epsilon\}_{\epsilon>0}$ be a subsequence of $\{\lambda_n^\epsilon\}_{\epsilon>0}$ still denoted by ϵ. Then, using the induction hypothesis (5.3.45), the orthonormality of the associated sequence of eigenvectors and a diagonalization argument we find a decreasing sequence $\{\epsilon_j\}_{j\in\mathbf{N}}$,such that $\epsilon_j \to 0$ and

$$u_k^{\epsilon_j} \overset{j}{\rightharpoonup} u_k \in \mathrm{W} \tag{5.3.46}$$

$$\lim_{j\to\infty} \lambda_k^{\epsilon_j} \doteq \lambda_k < \infty \tag{5.3.47}$$

for $k = \overline{1, n-1}$

Let $u \in V$, $u = \bar{u}_1 + \bar{u}_2$ where $0 \neq \bar{u}_1 \in W$ and $0 \neq \bar{u}_2 \in V_1$, with

$$\bar{u}_1 \notin \ span\{u_1, ..., u_{n-1}\} \tag{5.3.48}$$

We can do that because W has infinite dimension.

Let $\bar{u}_\epsilon$ be the recovering sequence defined before such that $\bar{u}_\epsilon \in V_{\epsilon,\delta}$ and $\bar{u}_\epsilon \rightharpoonup u$.

From (5.3.42) we obtain

$$\lambda_n^{\epsilon_j} = \inf_{\substack{u \in W_{\epsilon_j}, u \perp u_i^{\epsilon_j} \\ i=\overline{1,n-1}}} \frac{\| u \|_V^2}{\displaystyle\int_{\Gamma_f^0} u_\tau^2 d\sigma} \tag{5.3.49}$$

Consider now

$$\bar{z}^{\epsilon_j} = \bar{u}_{\epsilon_j} - \sum_{i=1}^{n-1} u_i^{\epsilon_j} < \bar{u}_{\epsilon_j}, u_i^{\epsilon_j} >_V \tag{5.3.50}$$

First we can see that

$$< \bar{z}^{\epsilon_j}, u_i^{\epsilon_j} >_V = 0 \text{ for any } i = \overline{1, n-1} \tag{5.3.51}$$

Then $\bar{z}^{\epsilon_j} \in V_{\epsilon_j}$ and $\bar{z}^{\epsilon_j} \notin V_1$ for j big enough.

Indeed from (5.3.4) we have

$$< \bar{u}_{\epsilon_j}, u_i^{\epsilon_j} >_V = \lambda_i^{\epsilon_j} \int_{\Gamma_f^0} u_{i\tau}^{\epsilon_j} \, \bar{u}_{\epsilon_j\tau} \quad \text{for} \ \ i = \overline{1, n-1} \tag{5.3.52}$$

and from the trace continuity, the definition of $\bar{u}^{\epsilon_j}$, (5.3.46) and (5.3.47) letting j go to the ∞ in (5.3.52) and using the result in (5.3.50) we have

$$\bar{z}^{\epsilon_j} \rightharpoonup \bar{z} \doteq u - \sum_{i=1}^{n-1} u_i \lambda_i \int_{\Gamma_f^0} u_{i\tau} \bar{u}_{1\tau}.$$

If we suppose $z_\tau = 0$ on Γ_f^0 this is equivalent to

$$\left(u - \sum_{i=1}^{n-1} u_i \lambda_i \int_{\Sigma_0} u_{i\tau} \bar{u}_{1\tau} \right)_\tau = 0 \quad \text{on } \Sigma_0$$

and this is equivalent to

$$\left(\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_{\Gamma_f^0} u_{i\tau} \bar{u}_{1\tau} \right)_\tau = 0 \quad \text{on } \Sigma_0$$

which implies

$$\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_{\Gamma_f^0} u_{i\tau} \bar{u}_{1\tau} = 0 \tag{5.3.53}$$

because $\bar{u}_1 - \sum_{i=1}^{n-1} u_i \lambda_i \int_{\Gamma_f^0} u_{i\tau} \bar{u}_{1\tau} \in W$ and $W \perp V_1$.

But (5.3.53) leads to a contradiction with (5.3.48).
Therefore $\bar{z}_\tau \neq 0$ and this implies the statement, i.e $\bar{z}^{\epsilon_j} \notin V_1$ for j big enough.
Next using (5.3.51) and (5.3.49) we obtain

$$\lambda_n^{\epsilon_j} \leq \frac{\| P_{W_{\epsilon_j}} \bar{z}^{\epsilon_j} \|_V^2}{\int_{\Gamma_f^0} (P_{W_{\epsilon_j}} \bar{z}^{\epsilon_j})_\tau^2} \leq \frac{\| z^{\epsilon_j} \|_V^2}{\int_{\Gamma_f^0} (\bar{z}^{\epsilon_j})_\tau^2} \leq \frac{M}{\int_{\Gamma_f^0} (\bar{z}^{\epsilon_j})_\tau^2}.$$

Passing to the limit when $j \to \infty$ we obtain

$$\limsup_{j \to \infty} \lambda_n^{\epsilon_j} \leq \frac{M}{\int_{\Gamma_f^0} \bar{z}_\tau^2} < \infty. \tag{5.3.54}$$

So we have proved that any subsequence of λ_n^ϵ has a subsequence $\{\lambda_n^{\epsilon_j}\}_{j \in \mathbf{N}}$ such that (5.3.54) is satisfied. Therefore we have that

$$\limsup_{\epsilon \to 0} \lambda_n^\epsilon < \infty$$

for any $n \in \mathbf{N}$ □

The next corollary shows that the weak-limits u_n of the sequence $\{u_n^\epsilon\}_{\epsilon > 0}$ of the normal eigenvectors associated to the eigenvalue λ_n^ϵ, cannot be zero.

Corollary 5.3.14. *Let $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ is the orthonormal sequence of eigenvectors associated to λ_n^ϵ for the problem $(\mathcal{E}_\epsilon)$.*
Then for any $n \in \mathbb{N}$ we have that every weak-limit u_n of $\{u_n^\epsilon\}_{n\in\mathbf{N}}$ (i.e., u_n such that on a subsequence $u_n^\epsilon \overset{\epsilon}{\rightharpoonup} u_n$), is nonzero.

Proof. Let $n \in \mathbb{N}$ be arbitrary fixed. Let u_n be a weak limit of $\{u_n^\epsilon\}$.Thus there exists a subsequence of $\{u_n^\epsilon\}_\epsilon$ still denoted by ϵ such that $u_n^\epsilon \rightharpoonup u_n$.
Using the variational form of $\mathcal{E}_\epsilon$ and the normality of $\{u_n^\epsilon\}$ we have

$$\lambda_n^\epsilon = \frac{1}{\int_{\Gamma_f^0} {u_{n\tau}^\epsilon}^2}.$$

Letting ϵ go to zero above we obtain

$$\lambda_n = \frac{1}{\int_{\Gamma_f^0} u_{n\tau}^2}.$$

Next using Lemma 5.3.13 we obtain that

$$\int_{\Gamma_f^0} u_{n\tau}^2 \neq 0.$$

and this together with the arbitrariness of n implies the statement. □

Let us now consider the duality operator $J^\epsilon : V_{\epsilon,\delta} \to (V_{\epsilon,\delta})'$

$$\langle J^\epsilon u, w\rangle_{(V_{\epsilon,\delta})', V_{\epsilon,\delta}} = \langle u, w\rangle_V \quad \text{for any} \quad u, w \in V_{\epsilon,\delta}$$

J^ϵ is an operator of subdifferential type

$$J^\epsilon = \partial\varphi^\epsilon, \ \ \varphi^\epsilon : V_{\epsilon,\delta} \to \mathbb{R} \tag{5.3.55}$$

$$\varphi^\epsilon(u) = \frac{1}{2} \parallel u \parallel_V^2 \tag{5.3.56}$$

Lemma 5.3.15. *The sequence of operators J^ϵ is G convergent to $\partial\varphi$, with respect to the weak $\times$ strong topology in $V \times V'$.*

Proof. From the proof of Theorem 5.3.1 the sequence of functionals $\{\varphi^\epsilon\}$ is Γ-convergent weakly in V to φ given by

$$\varphi(v) = \frac{1}{2} \parallel v \parallel_V^2 + \frac{1}{2} c \int_{\Gamma_f^0} C_{ij} v_i v_j = \frac{1}{2} \parallel v \parallel_V^2 + \frac{1}{2} c \int_{\Gamma_f^0} v_\tau C v_\tau$$

where c and the matrix $(C_{ij})_{i,j=\overline{1,3}}$ are defined in Theorem 5.3.1. Using the G-convergence result for subdifferentials of Γ-convergent sequences (see Attouch [6]-Th.3.67) we have that the Γ-convergences of the sequence φ^ϵ to φ imply the G-convergence of the subdifferentials,

$$\partial\varphi^\epsilon \overset{G}{\to} \partial\varphi$$

□

Theorem 5.3.16. *There is a decreasing sequence* $\{\epsilon_j\}_j \in \mathbf{N}$ *with* $\epsilon_j \to 0$ *such that* $u_n^{\epsilon_j} \rightharpoonup u_n$, $\lambda_n^{\epsilon_j} \to \lambda_n$ *where* (λ_n, u_n) *solves the limit problem,* $\lambda_n \in \mathbb{R}$ *and* $u_n \in W$ *such that:*

$$\sigma(u_n) = \mathcal{A}\epsilon(u_n), \qquad div\, \sigma(u_n) = 0, \quad in\ \Omega, \tag{5.3.57}$$

$$u_n = 0 \ \ on\ \Gamma_d \quad \sigma_{33}(u_n) = 0\ on\ \Sigma_0 \tag{5.3.58}$$

$$\sigma_\tau(u_n) = u_{n\tau}(\lambda_n I_3 - cC) \quad on \quad \Gamma_f^0, \tag{5.3.59}$$

where I_3 *is the unity matrix in* $\mathcal{M}^{3\times 3}$ *and* c *and the matrix* C *have been defined in Theorem 5.3.1.*

Proof. Let an arbitrary fixed $n \in \mathbf{N}$. Let $\{\lambda_n^\epsilon\}_{\epsilon>0}$ be the sequence of eigenvalues for the problem $(\mathcal{E}_\epsilon)$ and u_n^ϵ the corresponding orthonormal sequence of eigenvectors. Then there is a subsequence $\{\epsilon_j\}_j \in \mathbf{N}$ such that:

$$u_n^{\epsilon_j} \rightharpoonup u_n \ \text{ and } \ \lambda_n^{\epsilon_j} \to \lambda_n$$

We have proved in Lemma 5.3.13 that $\lambda_n < \infty$ for all $n \in \mathbb{N}$.

Let $f_n^{\epsilon_j} \in V'$ be defined as

$$f_n^{\epsilon_j}(w) = \lambda_n^{\epsilon_j} \int_{\Gamma_f^0} u_{n\tau}^{\epsilon_j} w_\tau \quad \text{for all } w \in V.$$

Using the variational formulation (5.3.4) we have:

$$f_n^{\epsilon_j}(w) =< J^{\epsilon_j} u_n^{\epsilon_j}, w >_{(V_{\epsilon_j})', V_{\epsilon_j}} \quad \text{for all } w \in V_{\epsilon_j}.$$

This implies

$$f_n^{\epsilon_j} \in \partial\varphi^{\epsilon_j}(u_n^{\epsilon_j}) \tag{5.3.60}$$

The next observation is that:

$$f_n^{\epsilon_j} \overset{j\to\infty}{\longrightarrow} f_n \quad \text{strongly in } V' \tag{5.3.61}$$

where

$$f_n(w) = \lambda_n \int_{\Gamma_f^0} u_{n\tau} w_\tau \quad \text{for all } w \in V$$

The proof of the above convergence is straightforward. Indeed,

$$\| f_n^{\epsilon_j} - f_n \|_{V'} = \sup_{\substack{w\in V \\ \|w\|_V \le 1}} \left| \lambda_n^{\epsilon_j} \int_{\Gamma_f^0} u_{n\tau}^{\epsilon_j} w_\tau - \lambda_n \int_{\Gamma_f^0} u_{n\tau} w_\tau \right|$$

Now from the reflexivity of the space V we have that there exists $w_0^j \in V$ with $\| w_0^j \|_V \le 1$ such that

$$\| f_n^{\epsilon_j} - f_n \|_{V'} = \left| \lambda_n^{\epsilon_j} \int_{\Gamma_f^0} u_{n\tau}^{\epsilon_j} w_{0\tau}^j - \lambda_n \int_{\Gamma_f^0} u_{n\tau} w_{0\tau}^j \right| =$$

$$= \left| (\lambda_n^{\epsilon_j} - \lambda_n) \int_{\Gamma_f^0} u_{n\tau}^{\epsilon_j} w_{0\tau}^j + \lambda_n \int_{\Gamma_f^0} (u_{n\tau}^{\epsilon_j} - u_{n\tau}) w_{0\tau}^j \right|$$

Thus, from Cauchy-Schwartz inequality

$$\| f_n^{\epsilon_j} - f_n \|_{V'} \leq |\lambda_n^{\epsilon_j} - \lambda_n| \left(\int_{\Gamma_f^0} |u_{n\tau}^{\epsilon_j}|^2 \right)^{1/2} \left(\int_{\Gamma_f^0} |w_{0\tau}^j|^2 \right)^{1/2} +$$

$$+ |\lambda_n| \left(\int_{\Gamma_f^0} |u_{n\tau}^{\epsilon_j} - u_{n\tau}|^2 \right)^{1/2} \left(\int_{\Gamma_f^0} |w_{0\tau}^j|^2 \right)^{1/2}.$$

Next we will use the following interpolation inequality (see [48]):

Lemma 5.3.17. *Let $\Omega \subset \mathbf{R}^d$ be as above and let $\alpha \in [2, \dfrac{2(d-1)}{d-2}]$ if $d \geq 3$ and $\alpha \geq 2$ if $d = 2$. Then, for $\beta = \dfrac{d(\alpha - 2) + 2}{2\alpha}$ if $d \geq 3$ or if $d = 2$ and $\alpha = 2$, and for all $\beta \in]\dfrac{\alpha - 1}{\alpha}, 1[$ if $d = 2$ and $\alpha > 2$, there exists a constant $C = C(\beta)$ such that:*

$$||v||_{L^\alpha(\partial\Omega)} \leq C||v||_{L^2(\Omega)}^{1-\beta} ||v||_{H^1(\Omega)}^{\beta}, \quad \forall v \in H^1(\Omega). \tag{5.3.62}$$

In our case $d = 3$, $\alpha = 2$ and $\beta = \frac{1}{2}$ and thus the inequality becomes,

$$\| u_\tau \|_{L^2(\Sigma)}^2 \leq M \| u \|_{H^1(\Omega)} \| u \|_{L^2(\Omega)} \quad \forall u \in V. \tag{5.3.63}$$

Using the trace inequality, (5.3.63) and the fact that $\| w_0^j \|_V \leq 1$ we obtain :

$$f_n^{\epsilon_j} \overset{j \to \infty}{\longrightarrow} f_n \quad \text{strongly in } V'.$$

Therefore from (5.3.60), (5.3.61) and using the Lemma 5.3.15 we obtain that:

$$f_n \in \partial \varphi(u_n). \tag{5.3.64}$$

But (5.3.64) is equivalent with:

$$\langle u_n, w \rangle_V = \int_{\Gamma_f^0} u_{n\tau} (\lambda_n I_3 - cC) w_\tau \quad \text{for any} \quad w \in W. \tag{5.3.65}$$

which is the variational formulation for the problem (5.3.57), (5.3.58), (5.3.59).

From the arbitrariness of $n \in \mathbf{N}$ we have that the Theorem 5.3.16 is proved for all positive integers n. □

The main homogenization result of this section is:

Theorem 5.3.18. *If $c = \lim\limits_{\epsilon \to 0} \dfrac{\delta(\epsilon)}{\epsilon} < \infty$ then for any $n \in \mathbb{N}$ we have*

i) $\lim\limits_{\epsilon \to 0} \lambda_n^\epsilon = \lambda_n$ on the entire sequence,and λ_n is the n-th eigenvalue of the limit problem,

ii) There is a decreasing sequence $\{\epsilon_j\}_{j \in \mathbf{N}}$ with $\epsilon_j \to 0$ such that $u_n^{\epsilon_j} \rightharpoonup u_n$, where u_n is the normal eigenvector for the limit problem associated to λ_n.

Proof. Suppose there is λ eigenvalue of the limit problem such that $\lambda \neq \lambda_n$ for any $n \in \mathbf{N}$.
Let $u \in W$ be the normal eigenvector associated to λ, i.e, $\| u \|_V = 1$ and

$$\langle u, w\rangle_V = \int_{\Gamma_f^0} u_\tau(\lambda I_3 - cC)w_\tau \quad \text{for any} \quad w \in W. \tag{5.3.66}$$

Now obviously there is $m \in \mathbf{N}$ such that

$$\lambda < \lambda_{m+1}. \tag{5.3.67}$$

From the Lax-Milgram lemma we have that there exists $w^\epsilon \in W_\epsilon$ such that

$$< J^\epsilon w^\epsilon, w >_{(V_{\epsilon,\delta}{}', V_{\epsilon,\delta})} = \lambda \int_{\Gamma_f^0} u_\tau w_\tau \quad \text{for all } w \in W_\epsilon.$$

It can be seen easily that w^ϵ is bounded in the norm of V.
Then on a subsequence still denoted by ϵ we have,

$$w^\epsilon \rightharpoonup \bar{w} \text{ as } \epsilon \to 0$$

for some $\bar{w} \in W$. But if we consider $f_\lambda \in V'$ with $f_\lambda(w) = \lambda \int_{\Gamma_f^0} u_\tau w_\tau$ then clearly from the definition of w_ϵ an J^ϵ we have

$$f_\lambda(w) =< J^\epsilon w^\epsilon, w >_{(V_{\epsilon,\delta}{}', V_{\epsilon,\delta})} \Longrightarrow f_\lambda \in \partial\varphi^\epsilon(w^\epsilon).$$

So using the G-convergence result stated in Lemma 5.3.15 we obtain

$$f_\lambda \in \partial\varphi(\bar{w}) \Longleftrightarrow < \bar{w}, v >_V + c\int_{\Gamma_f^0} v_\tau C\bar{w}_\tau = \lambda \int_{\Gamma_f^0} u_\tau v_\tau$$

for any $v \in W$.

Therefore, from (5.3.66) we have that $u = \bar{w}$. Now by Uryson's property we can see that

$$w^\epsilon \rightharpoonup u \text{ when } \epsilon \to 0.$$

Let

$$v^\epsilon = w^\epsilon - \sum_{i=1}^{m} u_i^\epsilon \langle w^\epsilon, u_i^\epsilon\rangle_V$$

Using the interpolation inequality (5.3.63) and (5.3.4) we obtain

$$\langle w^\epsilon, u_i^\epsilon\rangle_V = \lambda_i^\epsilon \int_{\Gamma_f^0} u_{i\tau}^\epsilon w_\tau^\epsilon \xrightarrow{\epsilon} \lambda_i \int_{\Gamma_f^0} u_{i\tau} u_\tau \quad \text{for } i = \overline{1, m}.$$

On the other hand using the definition of w^ϵ we can see that

$$\langle w^\epsilon, u_i^\epsilon\rangle_V = \lambda \int_{\Gamma_f^0} u_\tau u_{i\tau}^\epsilon \xrightarrow{\epsilon} \lambda \int_{\Gamma_f^0} u_{i\tau} u_\tau \quad \text{for } i = \overline{1, m}.$$

Now because $\lambda \neq \lambda_i$ for all $i = \overline{1,m}$ from the last two relations we have that

$$\int_{\Gamma_f^0} u_{i\tau} u_\tau = 0 \;\text{ for all }\; i = \overline{1,m}.$$

Thus $\langle w^\epsilon, u_i^\epsilon \rangle_V \xrightarrow{\epsilon} 0$ and therefore $v^\epsilon \rightharpoonup u$ weakly in V. Noticing that $v^\epsilon \in W_\epsilon$ and $v^\epsilon \perp u_i^\epsilon$ for all $i = \overline{1,m}$ from the Rayleigh's principle for $(\mathcal{E}_\epsilon)$ we have

$$\lambda_{m+1}^\epsilon \leq \frac{\| v^\epsilon \|_V^2}{\displaystyle\int_{\Gamma_f^0} (v_\tau^\epsilon)^2}. \tag{5.3.68}$$

Now, from the definition of w^ϵ and the trace continuity we have

$$\lim_{\epsilon \to 0} \| v^\epsilon \|_V^2 = \lim_{\epsilon \to 0} \| w^\epsilon \|_V^2 = \lambda \int_{\Gamma_f^0} (u_\tau)^2.$$

From the last relation, the inequality (5.3.63) and Theorem 5.3.16, passing to the limit when $\epsilon \to 0$ in (5.3.68) we obtain the contradiction. So i) above has been proved and ii) is exactly the same as in Theorem 5.3.16. □

Next, following an idea in ([6]), we give a Mosco-convergence (see [6] for the definition of Mosco-convergence) result for the case $c = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$:

Theorem 5.3.19. *Let $c = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} < \infty$ and $i \in \mathbf{N}$ arbitrary fixed and let $\{\lambda_n^\epsilon, u_n^\epsilon\}_n$ be the couple of eigenvalues and normal eigenfunctions for $\mathcal{E}_\epsilon$.*
Then if m_i is the order of multiplicity of λ_i,i.e

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} = ... = \lambda_{i+m_i-1} < \lambda_{i+m_i} \tag{5.3.69}$$

then the sequence of subspaces generated by $\{u_i^\epsilon, ..., u_{i+m_i-1}^\epsilon\}$ Mosco-converge in $L^2(\Omega)$ to the eigenspace $\{\tilde{u}_i, ..., \tilde{u}_{i+m_i-1}\}$, associated to λ_i.

Proof. We remark that the multiplicity of λ_i^ϵ might be strictly smaller than that of λ_i. So if we denote

$$span\{u_i^\epsilon, ..., u_{i+m_i-1}^\epsilon\} \doteq S_i^\epsilon \;\text{ and }\; span\{\tilde{u}_i, ..., \tilde{u}_{i+m_i-1}\} \doteq S_i$$

we can see that as in the above remark S_i^ϵ may be strictly larger than the eigenspace of λ_i^ϵ. Now from Theorem 5.3.18 we have that, for any $n \in \mathbb{N}$, there is a subsequence still denoted by ϵ such that

$$\lim_{\epsilon \to 0} \lambda_n^\epsilon = \lambda_n \;\text{ and }\; u_n^{\epsilon_j} \rightharpoonup u_n \quad \text{weakly in } V,$$

where (u_n, λ_n) solve the spectral limit problem (5.3.57), (5.3.58) and (5.3.59).
From the linearity of $\mathcal{E}_\epsilon$ and $\mathcal{E}$ we can say that

$$\limsup_{\epsilon \to 0} S_i^\epsilon \subset S_i.$$

We can easily see that for arbitrary fixed $l, j \in \{i, ..., i + m_i - 1\}$, with $l \neq j$ and

$$u_l^\epsilon \rightharpoonup u_l \;\text{ and }\; u_j^\epsilon \rightharpoonup u_j$$

we have

$$u_l \neq u_j \tag{5.3.70}$$

Indeed suppose that there are $l, j \in \{i, ..., i + m_i - 1\}$, with $l \neq j$ such that $u_l = u_j$.Then from

$$2 = \| u_l^\epsilon - u_j^\epsilon \|_V^2 = \int_{\Sigma_0} (\lambda_l^\epsilon u_{l\tau}^\epsilon - \lambda_j^\epsilon u_{j\tau}^\epsilon)(u_{l\tau}^\epsilon - u_{j\tau}^\epsilon)$$

passing to the limit when $\epsilon \to 0$ using the inequality (5.3.63) we obtain the contradiction.

Next we will prove that set $\{u_i, ..., u_{i+m_i-1}\}$ is linear independent.

Indeed let

$$\sum_{k=i}^{i+m_i-1} c_k u_k = 0 \tag{5.3.71}$$

We have for any $j \in \{i, ..., i + m_i - 1\}$ that

$$c_j = \langle \sum_{k=i}^{i+m_i-1} c_k u_k^\epsilon, u_j^\epsilon \rangle_V = \lambda_j^\epsilon \int_{\Sigma_0} \left(\sum_{k=i}^{i+m_i-1} c_k u_{k\tau}^\epsilon \right) u_{j\tau}^\epsilon \xrightarrow{\epsilon} \lambda_j \int_{\Sigma_0} \left(\sum_{k=i}^{i+m_i-1} c_k u_{k\tau} \right) u_{j\tau} = 0$$

where the last equality above comes from (5.3.71).

Using the linear independence of $\{u_i, ..., u_{i+m_i-1}\}$, (5.3.70) and the fact that the dimension of the eigenspace associated to λ_i is m_i we have in fact that

$$S_i = span\{u_i, ..., u_{i+m_i-1}\}$$

and therefore

$$\limsup_{\epsilon \to 0} S_i^\epsilon = S_i.$$

Because of the compact imbedding of V in $[L^2]^3$ we have that there is a subsequence ϵ_j such that

$$\liminf_{\epsilon \to 0} S_i^\epsilon = \limsup_{j \to \infty} S_i^{\epsilon_j}.$$

Now if there is v such that

$$v \notin \liminf_{\epsilon \to 0} S_i^\epsilon$$

then from the above relation we have

$$v \notin \limsup_{j \to \infty} S_i^{\epsilon_j} = S_i$$

which implies

$$S_i \subset \liminf_{\epsilon \to 0} S_i^\epsilon.$$

So we have proved the statement. □

In the next Remark we will briefly discuss the cases $c = 0$ and $c = \infty$.

Remark 5.3.20. *The case $c = 0$ can be seen as a particular case of the previous theorems. The limit problem for the problem $\mathcal{E}_\epsilon$ is*

$$\sigma(u_n) = \mathcal{A}\epsilon(u_n), \qquad div\, \sigma(u_n) = 0, \quad in\ \Omega, \tag{5.3.72}$$

$$u_n = 0 \ \ on\ \Gamma_d \quad \sigma_{33}(u_n) = 0\ on\ \Sigma_0 \tag{5.3.73}$$

$$\sigma_\tau(u_n) = \lambda_n u_{n\tau} \quad on \quad \Gamma_f^0. \tag{5.3.74}$$

In the other case $c = \infty$ we have seen that the sequence $\{\varphi^\epsilon\}_{\epsilon>0}$ defined in Lemma 5.3.15, Γ-converge to φ and we have

$$\varphi(u) = \begin{cases} \| u \|_V^2\, dx & if \quad u \in V_1 \\ \infty & otherwise \end{cases}$$

Now suppose that there is $n \in \mathbf{N}$ such that $\lambda_n^\epsilon \xrightarrow{\epsilon} \lambda_n < \infty$.
Now using the same approach as before, from Theorem 5.3.16 and Lemma 5.3.15 we obtain that $f_\lambda \in \partial\varphi(u_n)$ where f_λ has been defined above. This means that

$$u_n \in Dom(\varphi) = V_1.$$

But we know that $u_n^\epsilon \in W_{\epsilon,\delta} \subset W$ which means that

$$u_n \in W.$$

Using the fact that $W = V_1{}^\perp$ in V we obtain $u_n = 0$,which contradicts Corollary 5.3.14. Then our assumption that $\lambda_n < \infty$ is false. Now from the variational form of (5.3.4) if u_n^ϵ is the normal eigenvector associated to λ_n^ϵ we have

$$\frac{1}{\lambda_n^\epsilon} = \int_{\Sigma_0} (u_{n\tau}^\epsilon)^2$$

Consider $u_n \in W$ to be the weak limit of u_n^ϵ when $\epsilon \to 0$. Passing to the limit for $\epsilon \to 0$ in the equality above we obtain

$$\int_{\Sigma_0} (u_{n\tau})^2 = 0$$

And this together with the fact that $u_n \in W$ and $W \perp V_1$ give us that $u_n = 0$. So in this case we have that all the eigenvectors of the $\mathcal{E}_\epsilon$ converges to zero and all the eigenvalues of the same problem converges to ∞.

5.4 Physical Interpretation

We give here the physical interpretation of the previous theoretical results concerning the macroscopic behavior of a fault with small-scale heterogeneity of rupture resistance (small scale barriers). Through theorems 4.2 and 4.16 we have obtained an effective (or equivalent) friction law which, used on a homogeneous fault, leads to a slip evolution similar to the one produced on the heterogeneous fault. More precisely, for a fault which has ϵ-periodically distributed barriers of radius r_ϵ we have proved that for $0 < c =: \lim_{\epsilon \to 0} \delta(\epsilon)/\epsilon < \infty$ the sequence of energy functionals Γ-converges to a limit energy functional. This limit functional is associated to another slip weakening friction problem called the equivalent friction law. These results can be interpreted in the context of a barrier erosion process during the nucleation phase of an earthquake.

The earthquake nucleation (or initiation) phase, preceding the dynamic rupture, has been pointed out by detailed seismological observations (e.g. [44, 37]) and it has been recognized in laboratory experiments (e.g. [36, 67]) to be related to the slip-weakening friction. This physical model was thereafter used in the qualitative description of the initiation phase in unbounded (e.g. [16, 2]) and bounded (e.g. [33, 80]) fault models. Important physical properties of the nucleation phase (characteristic time, critical fault length, etc.) were obtained in [16, 33, 34] through simple mathematical properties of the unstable evolution.

During the nucleation phase, the stress concentration and at the boundary between the barriers and the slipping zone exceeds the the barriers' strength and a part of the barrier is broken (i.e. it is transformed in a slipping zone). The evolution of the shape and of the distribution of the barriers can change the effective frictional properties of the fault and can explain the qualitatively different behaviors with the same local friction law.

In order to see how the barriers evolution change the effective friction properties during the initiation phase let us imagine that we deal with a external loading process on the time interval $[0, T]$. Since the loading rate of the tectonic plates is very slow we can suppose that the process is quasi-static. In this context $[0, T]$, the nucleation (or initiation) phase of an earthquake, turns out to be the transition between the quasi-static and the dynamic slip. The fault will be supposed to have periodically distributed barriers of period ϵ (small non-dimensional distance with respect to the fault length) and of a variable diameter $\epsilon\delta(\epsilon, t)$ (non-dimensional length) with $t \in [0, T]$. The erosion of the barriers is described by the fact that the function $t \to \delta(\epsilon, t)$ is non-increasing. Regarding the evolution of the parameter

$$A_\epsilon(t) =: \frac{\delta(\epsilon, t)}{\epsilon}$$

we can distinguish three periods of time. At the beginning of the process, $[0, T_1]$, the diameter of the barriers is large (i.e. $A_\epsilon(t)$ is very large). The second period of time $[T_1, T_2]$ the parameter $A_\epsilon(t)$ is of the order of unity and the last period $[T_2, T]$ the parameter $A_\epsilon(t)$ is very small.

1) In the first period of time $[0, T_1]$ the barriers are too large with respect to the distance between them, (i.e. $c(t) =: \lim_{\epsilon \to 0} \delta(\epsilon, t)/\epsilon = \infty$) and the equivalent fault is locked (i.e. no large scale slip even if we can have a small scale slip). This means that the presence of the "large" barriers (i.e. with diameters of the same order of the distance between them) will imply that the effective static friction force is larger

than the local one. Such a fault can stand "large-scale" locked without slipping even if the loading is greater than the local friction resistance.

2) In the second period of time $[T_1, T_2]$ the ratio between the barrier radius and the inter-barrier distance is of order of the ratio between the the inter-barrier distance and the fault length (i.e. $0 < c(t) =: \lim_{\epsilon \to 0} \delta(\epsilon, t)/\epsilon < \infty$). In this case on the equivalent fault is acting a slip weakening friction law with a smaller weakening rate. That means that during this period of time the equivalent fault has a larger critical slip D_c. The presence of barriers that slow down the growth of the instability is accounted for in the effective law by an initial weakening rate that is much smaller than that for the local laws. Since the initial weakening of a friction law determines the initiation duration, as discussed in [49], the initiation time associated with a large earthquake which develops on a large area of an heterogeneous fault can be important. The equivalent slip weakening rate may be also negative, hence a slip-hardening effect can be expected. This type of friction properties were used in [83] in describing the dynamic rupture arrest. Moreover, the large scale (equivalent) friction law is not isotropic (i.e. the tangential stress and the slip are not collinear). This can be explained by the fact that the periodic distribution of the barriers is not isotropic, hence the limit problem will heritage this anisotropic geometrical perturbation.

3) In the third period of time $[T_2, T]$ the barriers are too small with respect to the distance between them, (i.e. $c(t) =: \lim_{\epsilon \to 0} \delta(\epsilon, t)/\epsilon = 0$) and the presence of the barriers does not affect the friction law on the equivalent fault. That means that the effective friction law is the same as the local one only in the last stage of nucleation phase. Moreover the slip weakening rate at the end of the initiation is larger than the rate of the initial stage of nucleation.

Let us summarize now the role played by the process of erosion of the barriers in the effective properties of the homogenized fault. In this context the time period $[0, T_1]$ turns to be the "(effective) locking period", the second one $[T_1, T_2]$ is the "first stage of (effective) initiation" and the last one $[T_2, T]$ becomes the "last stage of (effective) initiation".

i) The effective friction resistance (static friction) is greater than the local one.

ii) The slip weakening rate is smaller at the beginning of initiation phase than at the end. This imply a concave shape of the friction distribution with respect to the slip of the effective friction law. From the concavity of the friction law we can expect a long initiation phase.

iii)A negative weakening rate (i.e. hardening of the friction force) can be present in some cases at the beginning of the initiation phase.

iv) A loss of the isotropicity of the friction force can be remarked during the first stage of the nucleation phase.

We have to mention that the partition of the initiation phase into two stages with two weakening rates was also pointed out in [17] into a different context. Indeed, in [17] they analyze a dynamic two dimensional (anti-plane) process, and the separation between the two stages is given by the fact that barriers are (almost) instantaneously broken. In contrast in the present analysis this separation is given by a quasi-static erosion of the barriers.

Chapter 6

Appendix

In this Section we will present the proofs for some of the results used in the previous Sections and which were not included in the main body of the chapter for the sake of clarity of the exposition.

6.1 Definition and Properties of the Unfolding Operator

Let $\Xi_\epsilon = \{\xi \in \mathbb{Z}^N ; (\epsilon\xi + \epsilon Y) \cap \Omega \neq \emptyset\}$ and define

$$\tilde{\Omega}_\epsilon = \bigcup_{\xi \in \Xi_\epsilon} (\epsilon\xi + \epsilon Y) \tag{6.1.1}$$

Let us also consider $H^1_{per}(Y)$ to be the closure of $C^\infty_{per}(Y)$ in the H^1 norm, where $C^\infty_{per}(Y)$ is the subset of $C^\infty(\mathbb{R}^N)$ of Y-periodic functions, and

$$W_{per}(Y) \doteq \left\{ v \in H^1_{per}(Y)/\mathbb{R} \ , \ \frac{1}{|Y|} \int_Y v dy = 0 \right\}$$

(see [24] for properties).

Next, similarly as in [22],[31], if we have a periodical net on $\mathbb{R}^N$ with period Y, by analogy with the one-dimensional case, to each $x \in \mathbb{R}^N$ we can associate its integer part, $[x]_Y$, such that $x - [x]_Y \in Y$ and its fractional part respectively, i.e, $\{x\}_Y = x - [x]_Y$. Therefore we have:

$$x = \epsilon \left\{ \frac{x}{\epsilon} \right\}_Y + \epsilon \left[\frac{x}{\epsilon} \right]_Y \quad \text{for any } x \in \mathbb{R}^N.$$

We will recall in the following the definition of the Unfolding Operator as it have been introduced in [22](see also [31]), and review a few of its principal properties. Let the unfolding operator be defined as $\mathcal{T}_\epsilon : L^2(\tilde{\Omega}_\epsilon) \to L^2(\tilde{\Omega}_\epsilon \times Y)$ with

$$\mathcal{T}_\epsilon(\phi)(x, y) = \phi(\epsilon \left[\frac{x}{\epsilon} \right]_Y + \epsilon y) \quad \text{for all} \quad \phi \in L^2(\tilde{\Omega}_\epsilon)$$

We have (see [22]):

Theorem 6.1.1. *For any $v, w \in L^2(\Omega)$ we have*

1.

$$\mathcal{T}_\epsilon(vw) = \mathcal{T}_\epsilon(v)\mathcal{T}_\epsilon(w)$$

2.

$$\nabla_y(\mathcal{T}_\epsilon(u)) = \epsilon\mathcal{T}_\epsilon(\nabla_x u) \quad \text{where} \quad u \in H^1(\Omega)$$

3.

$$\int_\Omega u dx = \frac{1}{|Y|}\int_{\tilde{\Omega}_\epsilon \times Y} \mathcal{T}_\epsilon(u) dx dy$$

4.

$$\left|\int_\Omega u dx - \int_{\Omega\times Y} \mathcal{T}_\epsilon(u) dx dy\right| < |u|_{L^1(\{x\in\tilde{\Omega}_\epsilon;\ dist(x,\partial\Omega)<\sqrt{n}\epsilon\})}$$

5.

$$\mathcal{T}_\epsilon(\psi) \to \psi \quad \text{uniformly on } \Omega\times Y \text{ for any } \psi \in \mathcal{D}(\Omega)$$

6.

$$\mathcal{T}_\epsilon(w) \to w \quad \text{strongly in} \quad L^2(\Omega\times Y)$$

7. Let $\{w_\epsilon\} \subset L^2(\Omega\times Y)$ such that $w_\epsilon \to w$ in $L^2(\Omega)$. Then

$$\mathcal{T}_\epsilon(w_\epsilon) \to w \text{ in } L^2(\Omega\times Y)$$

8. Let $w_\epsilon \rightharpoonup w$ in $H^1(\Omega)$. Then there exists a subsequence and $\hat{w} \in L^2\left(\Omega; H^1_{per}(Y)\right)$ such that:

$$a)\ \mathcal{T}_\epsilon(w_\epsilon) \rightharpoonup w \text{ in } L^2(\Omega; H^1(Y))$$

$$b)\ \mathcal{T}_\epsilon(\nabla w_\epsilon) \rightharpoonup \nabla_x w + \nabla_y \hat{w} \text{ in } L^2(\Omega\times Y)$$

Another important property of the Unfolding Operator it is presented in the next Theorem due to Damlamian and Griso, see [41].

Theorem 6.1.2. *For any $w \in H^1(\Omega)$ there exists $\hat{w}_\epsilon \in L^2(\Omega, H^1_{per}(Y))$ such that*

$$\begin{cases} ||\hat{w}_\epsilon||_{L^2(\Omega, H^1_{per}(Y))} \leq C||\nabla_x w||_{[L^2(\Omega)]^N} \\ ||\mathcal{T}_\epsilon(\nabla_x w) - \nabla_x w - \nabla_y \hat{w}_\epsilon||_{L^2(Y, H^{-1}(\Omega))} \leq C\epsilon||\nabla_x w||_{[L^2(\Omega)]^N} \end{cases} \tag{6.1.2}$$

where C only depends on N and Ω.

Next present some interesting technical results obtained in [41] which are used in Section 4. Define $\rho_\epsilon(.) = inf\{\frac{\rho(.)}{\epsilon}, 1\}$ where $\rho(x) = dist(x, \partial\Omega)$. Define also $\hat{\Omega}_\epsilon = \{x \in \Omega\ ;\ \rho(x) < \epsilon\}$ and for any $\phi \in L^2(\Omega)$ consider $M^\epsilon_Y(\phi)(x) = \frac{1}{|Y|}\int_Y \mathcal{T}_\epsilon(\phi)(x,y)dy$. Let $v \in H^2(\Omega)$ be arbitrarily fixed, and the regularization Q_ϵ defined at (2.0.6). Then (see Griso [41], for the proofs)

Proposition 6.1.3. *We have*

$$1.\ \ ||\nabla_x \rho_\epsilon||_{L^\infty(\Omega)} = ||\nabla_x \rho_\epsilon||_{L^\infty(\hat{\Omega}_\epsilon)} = \epsilon^{-1}$$

$$2.\ \ ||(1-\rho_\epsilon)v||_{[L^2(\omega)]^N} \leq ||v||_{[L^2(\hat{\Omega}_\epsilon)]^N} \leq C\epsilon^{\frac{1}{2}}||v||_{H^1(\Omega)} \ \ \textit{for any} \ \ v \in H^1(\Omega)$$

$$3.||\nabla_x v||_{L^2(\hat{\Omega}_\epsilon)} \leq C\epsilon^{\frac{1}{2}}||v||_{H^2(\Omega)} \Rightarrow ||Q_\epsilon(\nabla_x v)||_{L^2(\hat{\Omega}_\epsilon)} + ||M_Y^\epsilon(\nabla_x v)||_{L^2(\hat{\Omega}_\epsilon)} \leq C\epsilon^{\frac{1}{2}}||v||_{H^2(\Omega)}$$

for any $v \in H^2(\Omega)$.

$$4.\ ||\psi(\frac{\cdot}{\epsilon})||_{L^2(\hat{\Omega}_\epsilon)} + ||\nabla_y \psi(\frac{\cdot}{\epsilon})||_{L^2(\hat{\Omega}_\epsilon)} \leq C\epsilon^{\frac{1}{2}}||\psi||_{H^1(Y)} \ \textit{for every} \ \psi \in H^1_{per}(Y)$$

$$5.\ \ ||M_Y^\epsilon(v)||_{L^2(\Omega)} \leq ||v||_{L^2(\tilde{\Omega}_\epsilon)} \ \ \textit{for any} \ \ v \in L^2(\tilde{\Omega}_\epsilon)$$

$$6.\ \begin{cases} ||v - M_Y^\epsilon(v)||_{L^2(\Omega)} \leq C\epsilon||\nabla v||_{[L^2(\Omega)]^N} \\ ||v - \mathcal{T}_\epsilon(v)||_{L^2(\Omega\times Y)} \leq C\epsilon||\nabla v||_{[L^2(\Omega)]^N} \\ ||Q_\epsilon(v) - M_Y^\epsilon(v)||_{L^2(\Omega)} \leq C\epsilon||\nabla v||_{[L^2(\Omega)]^N} \qquad \textit{for any} \ \ v \in H^1(\Omega) \end{cases}$$

$$7.\ \ ||Q_\epsilon(v)\psi(\frac{\cdot}{\epsilon})||_{L^2(\Omega)} \leq C||v||_{L^2(\tilde{\Omega}_{\epsilon,2})}||\psi||_{L^2(Y)} \ \ \textit{for any} \ \ v \in L^2(\tilde{\Omega}_{\epsilon,2}) \ \textit{and} \ \psi \in L^2(Y)$$

6.2 Convergence results a the smoothing argument

Let $m_n \in C^\infty$ be the standard mollifying sequence, i.e., $0 < m_n \leq 1$, $\int_{\mathbb{R}^N} m_n dz = 1$, $sppt(m_n) \subset B(0, \frac{1}{n})$. Define $A^n(y) = (m_n * A)(y)$, where a has been defined in the Introduction (see (2.0.1)). We have:

$$\begin{gathered} 1.A^n - Y \ \ \text{periodic matrix} \\ 2.|A^n|_{L^\infty} < |A|_{L^\infty} \\ 3.A^n \to A \ \ \text{in} \ \ L^p \ \ \text{for any} \ \ p \in (1,\infty) \end{gathered} \tag{6.2.1}$$

From (6.2.1) we have that $c|\xi|^2 \leq A^n_{ij}(y)\xi_i\xi_j \leq C|\xi|^2 \ \forall \xi \in \mathbb{R}^N$. Define

$$(\mathcal{A}_n^{hom})_{ij} = M_Y(A^n_{ij}(y) + A^n_{ik}(y)\frac{\partial \chi_j^n}{\partial y_k}) \tag{6.2.2}$$

where $M_Y(\cdot) = \dfrac{1}{|Y|}\displaystyle\int_Y \cdot dy$ and $\chi_j^n \in W_{per}(Y)$ are the solutions of the local problem

$$-\nabla_y \cdot (A(y)(\nabla \chi_j^n + e_j)) = 0 \tag{6.2.3}$$

Next we present a few important convergence results needed in the smoothing argument developed in the previous Sections.

Lemma 6.2.1. *Let $f_n, f \in H^{-1}(\Omega)$ with $f_n \rightharpoonup f$ in $H^{-1}(\Omega)$ and let $b^n, b \in L^2(\Omega)$, with*

$$c|\xi|^2 \leq b^n_{ij}(y)\xi_i\xi_j \leq C|\xi|^2$$
$$c|\xi|^2 \leq b_{ij}(y)\xi_i\xi_j \leq C|\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and

$$b^n \to b \;\; in \;\; L^2(\Omega)$$

Consider $\zeta_n \in H^1_0(\Omega)$ the solution of

$$\int_\Omega b^n(x)\nabla\zeta_n\nabla\psi dx = \int_\Omega f_n\psi dx$$

for any $\psi \in H^1_0(\Omega)$. Then we have

$$\zeta_n \rightharpoonup \zeta \;\; in \;\; H^1_0(\Omega)$$

and ζ verifies

$$\int_\Omega b(x)\nabla\zeta\nabla\psi dx = \int_\Omega f\psi dx \;\; for\ any \;\; \psi \in H^1_0(\Omega).$$

Proof. Immediately can be observed that

$$||\zeta_n||_{H^1_0(\Omega)} \leq C$$

and therefore there exists ζ such that on a subsequence still denoted by n we have

$$\zeta_n \rightharpoonup \zeta \;\; \text{in} \;\; H^1_0(\Omega) \tag{6.2.4}$$

For any smooth $\psi \in H^1_0(\Omega)$ easily it can be seen that

$$\int_\Omega b^n(x)\nabla\zeta_n\nabla\psi dx \to \int_\Omega b(x)\nabla\zeta\nabla\psi dx$$

and this implies the statement of the Lemma. Due to the uniqueness of φ one can see that the limit (6.2.4) holds on the entire sequence. □

Remark 6.2.2. *Using similar arguments it can be proved that the results of Lemma 6.2.1 hold true if we replace the Dirichlet boundary conditions with periodic boundary conditions.*

Corollary 6.2.3. *Let $u^n_\epsilon \in H^1_0(\Omega)$ be the solution of*

$$\begin{cases} -\nabla \cdot (A^n(\frac{x}{\epsilon})\nabla u^n_\epsilon) = f & in\ \Omega \\ u^n_\epsilon = 0 & on\ \partial\Omega \end{cases}$$

We then have

$$u^n_\epsilon \overset{n}{\rightharpoonup} u_\epsilon \;\; in \;\; H^1_0(\Omega)$$

where u_ϵ verifies

$$\begin{cases} -\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = f & in\ \Omega \\ u_\epsilon = 0 & on\ \partial\Omega \end{cases}$$

Proof. Using (6.2.1) we have that

$$A^n(\frac{x}{\epsilon}) \xrightarrow{n} A(\frac{x}{\epsilon}) \text{ in } L^2(\Omega)$$

and the statement follows immediately from Remark 6.2.2. □

Corollary 6.2.4. *for $j \in \{1, ..., N\}$, let $\chi_j^n \in W_{per}(Y)$ be the solution of*

$$-\nabla_y \cdot (A^n(y)(\nabla \chi_j^n + e_j)) = 0 \tag{6.2.5}$$

where $\{e_j\}_j$ denotes the canonical basis of $\mathbb{R}^N$. Then we have

$$\chi_j^n \rightharpoonup \chi_j \text{ in } W_{per}(Y)$$

where $\chi_j \in W_{per}(Y)$ verifies

$$-\nabla_y \cdot (A(y)(\nabla \chi_j + e_j)) = 0$$

Proof. From (6.2.1) we obtain

$$\frac{\partial}{\partial y_i} A_{ij}^n(y) \rightharpoonup \frac{\partial}{\partial y_i} A_{ij}(y) \text{ in } (W_{per}(Y))'$$

The statement of the Remark follows then immediately from Remark 6.2.2. □

Proposition 6.2.5. *Let $v \in [H^1(\Omega)]^N$ be arbitrarily fixed and for every $j \in \{1, .., N\}$, let $\chi_j \in W_{per}(Y)$ be defined as in (6.2.3), and $\chi_j^n \in W_{per}(Y)$, for $j \in \{1, .., N\}$, to be the solutions of (6.2.5.*

Define $h^n(x, \frac{x}{\epsilon}) = \chi_j^n(\frac{x}{\epsilon})v_j$, $h(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon})v_j$, $g^n(x, \frac{x}{\epsilon}) = \chi_j^n(\frac{x}{\epsilon})Q_\epsilon(v_j)$, $g(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon})Q_\epsilon(v_j)$. We have that

1. $g^n \overset{n}{\rightharpoonup} g$ *in* $H^1(\Omega)$

2. *If* $v \in [W^{1,p}(\Omega)]^N$, $p > N$, *then* $, h^n \overset{n}{\rightharpoonup} h$ *in* $H^1(\Omega)$

Proof. First note that applying Corollary 6.2.4 to the sequence $\{\chi_j^n\}_n$ we have

$$\chi_j^n \overset{n}{\rightharpoonup} \chi_j \text{ in } W_{per}(Y) \tag{6.2.6}$$

Next we have

$$||g^n(x, \frac{x}{\epsilon})||_{H^1(\Omega)}^2 = \int_\Omega (\chi_j^n(\frac{x}{\epsilon})Q_\epsilon(v_j))^2 dx + \frac{1}{\epsilon^2}\int_\Omega (\nabla_y \chi_j^n(\frac{x}{\epsilon})Q_\epsilon(v_j))^2 dx + \int_\Omega (\chi_j^n(\frac{x}{\epsilon})\nabla_x Q_\epsilon(v_j))^2 dx \tag{6.2.7}$$

and

$$||h^n(x,\frac{x}{\epsilon})||^2_{H^1(\Omega)} = \int_\Omega (\chi_j^n(\frac{x}{\epsilon})v_j)^2dx + \frac{1}{\epsilon^2}\int_\Omega (\nabla_y \chi_j^n(\frac{x}{\epsilon})v_j)^2dx + \int_\Omega (\chi_j^n(\frac{x}{\epsilon})\nabla_x v_j)^2dx \tag{6.2.8}$$

For the first convergence in Theorem 6.2.5 we use that

$$||\chi_j^n(\frac{x}{\epsilon})Q_\epsilon(v_j)||_{H^1(\Omega)} \leq C||\chi_j^n||_{W_{per}(Y)} \tag{6.2.9}$$

Next we can see that (6.2.7) imply that

$$||g^n(x,\frac{x}{\epsilon}) - g(x,\frac{x}{\epsilon})||^2_{L^2(\Omega)} = \int_\Omega \left(\chi_j^n(\frac{x}{\epsilon}) - \chi_j(\frac{x}{\epsilon})\right)^2 (Q_\epsilon(v_j))^2dx$$

and using (6.2.9) we obtain the desired result.

For the second convergence result in Theorem 6.2.5 we will recall now a very important inequality (see [53], Chp. 2) to be used for our estimates.

For any $p > N$ we have

$$||\phi||_{L^{\frac{2p}{p-2}}(\Omega)} \leq c(p)(||\phi||_{L^2(\Omega)} + ||\nabla\phi||^{\frac{N}{p}}_{L^2(\Omega)}||\phi||^{1-\frac{N}{p}}_{L^2(\Omega)}) \tag{6.2.10}$$

for any $\phi \in H^1(\Omega)$ and where $c(p)$ is a constant which depends only on q, N, Ω.

For $v \in [W^{1,p}(\Omega)]^N$ with $p > N$, using (6.2.6), the Sobolev imbedding $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ and (6.2.10) in (6.2.8) we obtain

$$||h^n(x,\frac{x}{\epsilon})||^2_{H^1(\Omega)} < C$$

where the constant C above does not depend on n.

Next we can easily observe that

$$||h^n(x,\frac{x}{\epsilon}) - h(x,\frac{x}{\epsilon})||^2_{L^2(\Omega)} = \int_\Omega \left(\chi_j^n(\frac{x}{\epsilon}) - \chi_j(\frac{x}{\epsilon})\right)^2 (v_j)^2dx$$

and in either of the above cases, (6.2.6) and a few simple manipulations imply that

$$h^n(x,\frac{x}{\epsilon}) \xrightarrow{n} h(x,\frac{x}{\epsilon}) \text{ in } L^2(\Omega)$$

This together with the bound on the sequence $\{h^n(x,\frac{x}{\epsilon})\}_n$ implies the statement of the Corollary. □

The two convergence results in the next Corollary will follow immediately from Proposition 6.2.4.

Corollary 6.2.6. *Let* $w_1^n(x,\frac{x}{\epsilon}) = \chi_j^n(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_j}$ *and* $u_1^n(x,\frac{x}{\epsilon}) = \chi_j^n(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j})$. *Then we have*

1. If $u_0 \in W^{3,p}(\Omega)$ for $p > N$,

$$w_1^n \overset{n}{\rightharpoonup} w_1 \quad in \quad H^1(\Omega)$$

2. If $u_0 \in H^2(\Omega)$,

$$u_1^n \overset{n}{\rightharpoonup} u_1 \quad in \quad H^1(\Omega)$$

Corollary 6.2.7. *Let θ_ϵ^n be the solution of*

$$-\nabla \cdot (A^n(\frac{x}{\epsilon})\nabla\theta_\epsilon^n) = 0 \quad in\ \Omega\ ,\ \theta_\epsilon^n = w_1^n(x, \frac{x}{\epsilon}) \quad on\ \partial\Omega \tag{6.2.11}$$

and β_ϵ^n be the solution of

$$-\nabla \cdot (A^n(\frac{x}{\epsilon})\nabla\beta_\epsilon^n) = 0 \quad in\ \Omega\ ,\ \beta_\epsilon^n = u_1^n(x, \frac{x}{\epsilon}) \quad on\ \partial\Omega \tag{6.2.12}$$

We have that

(i) if $u_0 \in W^{3,p}(\Omega)$, $p > N$, then

$$\theta_\epsilon^n \overset{n}{\rightharpoonup} \theta_\epsilon \quad in \quad H^1(\Omega)$$

(ii) if $u_0 \in H^2(\Omega)$, then

$$\beta_\epsilon^n \overset{n}{\rightharpoonup} \beta_\epsilon \quad in \quad H^1(\Omega)$$

where θ_ϵ and β_ϵ satisfies

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\theta_\epsilon) = 0 \quad in\ \Omega\ ,\ \theta_\epsilon = w_1(x, \frac{x}{\epsilon}) \quad on\ \partial\Omega \tag{6.2.13}$$

and

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla\beta_\epsilon) = 0 \quad in\ \Omega\ ,\ \beta_\epsilon = u_1(x, \frac{x}{\epsilon}) \quad on\ \partial\Omega \tag{6.2.14}$$

Proof. Using Corollary 6.2.6 and a few simple arguments one can simply show that

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla w_1^n(x, \frac{x}{\epsilon})) \overset{n}{\rightharpoonup} -\nabla \cdot (A(\frac{x}{\epsilon})\nabla w_1(x, \frac{x}{\epsilon})) \text{ in } H^{-1}(\Omega)$$

and

$$-\nabla \cdot (A(\frac{x}{\epsilon})\nabla w_1^n(x, \frac{x}{\epsilon})) \overset{n}{\rightharpoonup} -\nabla \cdot (A(\frac{x}{\epsilon})\nabla w_1(x, \frac{x}{\epsilon})) \text{ in } H^{-1}(\Omega)$$

Homogenizing the data in the problems (6.2.11) and (6.2.12) and using Corollary 6.2.6 and Lemma 6.2.1 the statement follows immediately. □

Corollary 6.2.8. *For any $i, j \in \{1, .., N\}$ let $\chi_{ij}^n \in W_{per}(Y)$ be the solutions of:*

$$\nabla_y \cdot (A^n \nabla_y \chi_{ij}^n) = b_{ij}^n - M_Y(b_{ij}^n) \tag{6.2.15}$$

where

$$b_{ij}^n = -A_{ij}^n - A_{ik}^n \frac{\partial \chi_j^n}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}^n \chi_j^n)$$

and $M_Y(.)$ is the average on Y.

Then we have

$$\chi_{ij}^n \rightharpoonup \chi_{ij} \ \ in \ \ W_{per}(Y) \ \ for\ any \ \ i,j \in \{1,..,N\}$$

where χ_{ij} *satisfies*

$$\int_Y A(y)\nabla_y \chi_{ij} \nabla_y \psi dy = (b_{ij} - M_Y(b_{ij}), \psi)_{((W_{per}(Y))', W_{per}(Y))} \tag{6.2.16}$$

for any $\psi \in W_{per}(Y)$ *and with*

$$b_{ij} = -A_{ij} - A_{ik}\frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j).$$

Proof. For any $\psi \in W_{per}(Y)$, we have that,

$$\int_Y (b_{ij}^n - M_Y(b_{ij}^n)\psi dy = \int_Y (-A_{ij}^n - A_{ik}^n \frac{\partial \chi_j^n}{\partial y_k})\psi dy + (\mathcal{A}_n^{hom})_{ij}\int_Y \psi dy + \int_Y A_{ki}^n \chi_j^n \frac{\partial \psi}{\partial y_k} dy \tag{6.2.17}$$

where we have used that $M_Y(b_{ij}^n) = -(\mathcal{A}_n^{hom})_{ij}$ (see [59]).

Using (6.2.1), (6.2.6), and simple manipulations we can prove that

$$A_{ik}^n \frac{\partial \chi_j^n}{\partial y_k} \rightharpoonup A_{ik}\frac{\partial \chi_j}{\partial y_k} \ \ \text{in} \ \ L^2(Y) \tag{6.2.18}$$

and

$$A_{ik}^n \chi_j^n \rightharpoonup A_{ik}\chi_j \ \ \text{in} \ \ L^2(Y) \tag{6.2.19}$$

From (6.2.18), (6.2.1) and (6.2.2) we have that

$$(\mathcal{A}_n^{hom})_{ij} \to \mathcal{A}_{ij}^{hom} \tag{6.2.20}$$

Finally using (6.2.1), (6.2.6), (6.2.18) and (6.2.19) in (6.2.17) we obtain that

$$b_{ij}^n - M_Y(b_{ij}^n) \rightharpoonup b_{ij} - M_Y(b_{ij}) \ \ \text{in} \ \ (W_{per}(Y))'$$

This and Remark 6.2.2 complete the proof of the statement. □

Remark 6.2.9. *We can easily observe that we have*

$$A_{ij}^n \chi_{ij}^n \overset{n}{\rightharpoonup} A_{ij}\chi_{ij}, \ A_{ij}^n \frac{\partial \chi_{ij}^n}{\partial y_k} \overset{n}{\rightharpoonup} A_{ij}\frac{\partial \chi_{ij}}{\partial y_k} \ \ weakly\ in\ W_{per}(Y)$$

Corollary 6.2.10. *Let* $u_0 \in H^2(\Omega)$ *be the solution of the homogenized problem (2.0.2) and* $\chi_{ij}^n, \chi_{ij} \in W_{per}(Y)$ *be defined by (6.2.15) and (6.2.16). Suppose that there exists* $p > N$ *such that* $u_0 \in W^{3,p}(\Omega)$. *Define* $u_2^n(x,y) = \chi_{ij}^n(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$ *and* $u_2(x,y) = \chi_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$. *Consider* φ_ϵ^n *the solution of*

$$\nabla \cdot (A^n(\frac{x}{\epsilon})\nabla \varphi_\epsilon^n) = 0 \ \ in \ \ \Omega \ , \ \varphi_\epsilon^n = u_2^n(x, \frac{x}{\epsilon}) \ \ on \ \ \partial\Omega \tag{6.2.21}$$

Then we have that

$$u_2^n(x, \frac{x}{\epsilon}) \stackrel{n}{\rightharpoonup} u_2(x, \frac{x}{\epsilon}) \;\; and \;\; \varphi_\epsilon^n \stackrel{n}{\rightharpoonup} \varphi_\epsilon \;\; in \;\; H^1(\Omega)$$

where φ_ϵ *satisfies*

$$\nabla \cdot (A(\frac{x}{\epsilon})\nabla\varphi_\epsilon) = 0 \;\; in \;\; \Omega \; , \; \varphi_\epsilon = u_2(x, \frac{x}{\epsilon}) \;\; on \;\; \partial\Omega \tag{6.2.22}$$

Proof. Following similar arguments as those used in Corollary 6.2.5 we can prove that

$$u_2^n(x, \frac{x}{\epsilon}) \stackrel{n}{\rightharpoonup} \chi_{ij}(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) \;\; \text{in} \;\; H^1(\Omega)$$

Using the above convergence result and similar ideas as in Corollary 6.2.7 we complete the proof of the statement. □

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