Quasi-static Fracture Evolution with Cohesive Energy

by

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Abstract

The last fifteen years have seen much success in the analysis of quasi-static evolution for Griffith fracture, which is the mathematically natural starting point for studying fracture. At the same time, attempts have been made to show existence for similar models based on cohesive fracture rather than Griffith. These models are generally viewed as physically more realistic than Griffith, in that they are better models for crack nucleation. These attempts at existence proofs have been unsuccessful without very strong additional assumptions, for example, specifying the crack path a priori.

The main purpose of this thesis is to characterize as well as possible the mathematical difficulties in cohesive fracture, and to make progress toward an existence result without the prescribed crack path assumption. So far, the most powerful method for existence proofs is to build a sequence of approximate solutions, based on time discretization, and take the limit as the time steps go to zero. We show that there are mainly two complications on the cracks of these approximate solutions that we need to rule out in order to show existence. The first one is due to the potential oscillation of the crack path. The second is due to the potential splitting of a crack into two or more nearby cracks, with the same total jump in displacement.

We begin by first constructing an example illustrating how oscillations described above can affect the minimality of the limit. Then we prove that the splitting described above can be ruled out for any sequence of unilateral minimizers. With this result, we show how exactly oscillation affect the minimality on the limit of the sequence. We then move to the evolution problem and show the convergence of energy for almost every t. Based on this result we develop a method that allows us to analyze the problem using only a finite set of times. An application of this method is a proof of absolute continuity. Future work will be aimed at using the tools we developed to rule out oscillation and finally to prove existence results under more general assumptions.

Contents

Introduction

In 1920, Griffith laid the foundation for brittle fracture in [17]. He considered materials' toughness, denoted by G_c, and energy release rate, denoted by G, and regarded the propagation of fracture as the competition between G and G_c. G can be defined, in a two dimensional setting, as

$$G := -\frac{dW}{dl}$$

where W is the bulk energy and l is the length of a crack. Propagation can take place if $G = G_c$ and can not if $G < G_c$.

This idea is widely used to study crack propagation (e.g. [19]). However there are limitations on the model. It requires a pre-existing crack and it prescribes the crack path on the material. In 1998, Francfort A. G. and Marigo J.-J. addressed those issues by introducing a variational model of quasi-static growth for brittle cracks in [16]. They considered the formation and growth of a crack as the consequence of minimizing the sum of bulk energy and surface energy (fracture energy). Therefore the crack set is not prescribed, instead, the growth (direction and location) of cracks is determined by the minimization of the energy sum. To better illustrate the variational model introduced in [16], consider the quasi-static evolution dealt by Francfort G. A. and Larsen C. J. in [15], the first result for the Francfort-Marigo model without artificial restrictions on the crack set.

Let Ω be an open and bounded subset from $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary. Then consider the case of *generalized antiplanar shear* by assuming that the reference configuration is an infinite cylinder $\Omega \times \mathbb{R}$ and the displacement has the special form $(0, ..., 0, u(x_1, ..., x_N))$ where $u : \Omega \to \mathbb{R}$. The natural setting for u is to let $u \in SBV(\Omega)$ where SBV denotes the space of special functions of bounded variation. The space of SBV functions was introduced by De Giorgi E. and Ambrosio L. [11] and the definition of SBV functions is as follows

Definition 1. *$u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and the distributional derivative Du is a finite Radon measure. $u \in SBV(\Omega)$ if $u \in BV(\Omega)$ and Du can be split into two parts*

$$Du = \nabla u dx + [u]\nu\mathcal{H}^{N-1}\lfloor S_u$$

Here $\nabla u dx$ equals the part of the distributional derivative Du that is absolutely continuous with respect to Lebesgue measure. For SBV functions, the singular part of Du only contains the jump part $[u]\nu\mathcal{H}^{N-1}\lfloor S_u$. S_u denotes the jump set of u and $[u]$ denotes the size of the jump on S_u. It can be shown that S_u is countably $N-1$ rectifiable (see [14]). $\mathcal{H}^{N-1}$ is the $N-1$ dimensional Hausdorff measure and $\lfloor$ is the restriction of measures on sets. The vector ν is the unit normal to S_u. See [2], [13], [14] and [20] for more details on SBV functions.

Then consider the total energy

$$\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx + G_c \mathcal{H}^{N-1}(\Gamma)$$

where Γ is the crack set (which can be empty). For simplicity the bulk energy consists only of elastic energy $\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx$. The fracture energy here is the toughness G_c times the $\mathcal{H}^{N-1}$ measure or the size of crack set. Here can set G_c to be 1 as there's no difference in terms of proof. This kind of fracture energy is in general regarded as Griffith energy. Let $g \in W^{1,\infty}(\Omega)$ be the Dirichlet boundary condition and Γ be the pre-existing crack set, the displacement u will minimize the total energy in the following sense

$$\int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{N-1}(\Gamma \cup S_u) \leq \int_{\Omega} |\nabla v|^2 dx + \mathcal{H}^{N-1}(\Gamma \cup S_v) \tag{1}$$

for all $v = g$ on $\partial\Omega$.

The existence of a minimizer u can be shown using SBV compactness which is due to Ambrosio L. [1], also see [12] for more details. Before we give the theorem of SBV compactness, let's give the definition of SBV convergence.

Definition 2. *A sequence $u_n \in SBV(\Omega)$ converges in the sense of SBV to $u \in SBV(\Omega)$ if $u_n \to u$ in $L^1(\Omega)$ and*

$$\nabla u_n dx \overset{*}{\rightharpoonup} \nabla u dx$$
$$[u_n]\nu_n \mathcal{H}^{N-1} \lfloor S_{u_n} \overset{*}{\rightharpoonup} [u]\nu \mathcal{H}^{N-1} \lfloor S_u$$

where $\overset{*}{\rightharpoonup}$ denotes the weak convergence in measure.*

Let $\theta : [0,\infty) \to [0,\infty]$, $\varphi : (0,\infty) \to (0,\infty]$ be lower semi-continuous increasing functions and assume that

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty, \qquad \lim_{t\to 0} \frac{\varphi(t)}{t} = \infty.$$

Theorem 1. *(SBV compactness) Let $\{u_n\}_{n=1}^{\infty} \subset SBV(\Omega)$ such that*

$$\sup_n \left\{ \int_{\Omega} \theta(|\nabla u_n|) dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \right\} < \infty \tag{2}$$

and $\|u_n\|_{L^\infty}$ is uniformly bounded in n, then there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and an SBV function u such that

$$u_{n_k} \overset{SBV}{\rightharpoonup} u$$

To include the pre-existing crack set in the energy reflects one of the most important features a fracture problem have—irreversibility. It says the crack set can only grow bigger or stay the same, it can not be reversed. Then consider the continuous-time evolution where change of boundary condition is slow such that at each time t the material is able to reach an equilibrium. The detailed definition of quasi-static evolution with Griffith energy, borrowed from [15], is as follows

Definition 3. *(Quasi-static Evolution with Griffith Energy) Let* $g \in L^\infty([0,1], L^\infty(\mathbb{R}^N)) \cap W^{1,1}([0,1], H^1(\mathbb{R}^N))$ *be the Dirichlet boundary condition, a pair* $(u(t), \Gamma(t))$*, where* $u(t) \in SBV(\Omega)$ *for each* t *and* $\Gamma(t) := \cup_{\tau \leq t} S_{u(\tau)}$*, is said to be a quasi-static evolution if*

1. *Global stability.*
$$\int_\Omega |\nabla u(t)|^2 dx \leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v \backslash \Gamma(t)).$$
for all t *and all* $v \in SBV(\Omega)$ *s.t.* $v = g(t)$ *on* $\partial\Omega$

2. *Irreversibility.* $\Gamma(t_1) \subset \Gamma(t_2)$ *for all* $0 \leq t_1 \leq t_2 \leq 1$.

3. *Energy balance. Define* $E(t) := \int_\Omega |\nabla u(t)|^2 dx + \mathcal{H}^{N-1}(S_{u(t)} \cup \Gamma(t))$*, then*
$$E(t) = E(0) + 2\int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s) dx ds.$$

u represents the evolution of the displacement of the material. Γ represents the evolution of cracks. Notice here irreversibility is automatically satisfied due to the definition of $\Gamma(t)$. Global stability means the displacement of the material $u(t)$ at each time t is always globally stable in the sense that $u(t)$ minimizes the total energy with $\Gamma(t)$ as pre-existing crack. It reflects the quasi-static property that says at each time the material is in equilibrium. Energy is balanced in the sense that the energy at each time equals the energy at the beginning plus the work done up to time t.

Numerical implementations have been studied in [4], [5] and [6]. They were based on a finite time step approach. Meanwhile the continuous-time mathematical existence results were being studied. In [9] Dal Maso G. and Toader R. gave the first precise mathematical formulation of the model using a two-dimensional setting. They proved an existence result using the time discretization method introduced in [16] under the assumption that the bound on the number of connected components of cracks is set a priori. Later Chambolle A. [8] solved the planar elasticity setting under the same assumption on the number of connected components of crack sets. Time discretization, since its introduction, is widely used to approach quasi-static evolution problems. It provides an efficient way to approximate the continuous-time problem using discrete time solutions. One considers a sequence of minimizers as the discrete time step goes to 0. Then the problem becomes to show that the limit is also a minimizer.

A big success in showing the existence result for the continuous-time quasi-static evolution for Griffith energy is due to Francfort G. A. and Larsen C. J. in [15]. They showed the existence result with no other assumptions on the crack set than $N-1$ rectifiability. As mentioned above, the key step is to show the sequence of approximating solutions converges to a limit that is also a solution in some sense. First they consider a sequence that satisfies the following minimality for its own jump set

$$\int_\Omega |\nabla u_n|^2 dx \leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v \backslash S_{u_n})$$

for all $v = u_n$ on $\partial\Omega$ and for all n. Then they show that the limit is also a minimizer for its own jump set. Use the same technique (jump transfer) they show that the discrete time solutions converge to a global minimizer and thus show the global stability. Once the global stability is shown, the rest follows easily. Another contribution of [15] is the jump transfer technique that provides a method to alter crack set. This idea is widely used and extended in this thesis and in some of the proofs.

Meanwhile a cohesive model (see [3]) has been used to study crack propagation. For cohesive model the fracture energy also depends on the opening size of the crack. The fracture energy can be precisely written as

$$\int_\Gamma \varphi([u]) d\mathcal{H}^{N-1}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing, bounded and concave function with $\varphi(0) = 0$. Here $\varphi([u])$ is the energy density spent to create a crack with opening $[u]$ and $\varphi'([u])$ is the force density acting between the lips of crack. Typically the force decrease with distance and hence φ is concave. By letting $\varphi \equiv G_c$ one obtains Griffith energy. The total energy for a cohesive model can be written

$$E(u) := \int_\Omega |\nabla u|^2 dx + \int_\Gamma \varphi([u]) d\mathcal{H}^{N-1}.$$

Dal Maso G. and Zanini C. [10] and Cagnetti F. and Toader R. [7] showed existence results for quasi-static evolution with cohesive energy, for the case of prescribed crack path. When the crack path is fixed and crack set is regular enough, SBV functions are no longer needed to describe the displacement. One can define Sobolev functions on domain $\Omega \backslash \Gamma$ and study the behaviors of traces on two sides of the crack set. One of the biggest differences between [10] and [9] is the way they treat loading (increase) and unloading (decrease) of a crack. In [10], the unloading is constant and the problem is solved by using a special form of convergence. In [9], the unloading follows a convex function $\varphi(\cdot, z)_{z>0}$ where z is the size to which the crack has been opened up previously. The solution is based on the use of Young measures.

In this thesis we consider a model that is very similar to the one in [15] where the crack path is free. But instead of considering the Griffith energy we consider a cohesive energy for the fracture part. We use the same time discretization method to approach continuous-time evolution. We show that the method used to prove a existence result for Griffith energy does not apply to the cohesive energy. Then we consider a sequence of unilateral minimizers and show that the limit is a minimizer that picks up the oscillation on the sequence. Later we move to the evolution problem and first show the convergence of energy. Based on this result we develop a so called little o method that allows us to analyze the problem using only finite set of times.

Now we are in a position to give the details of the problem we are considering. As mentioned before for cohesive model the fracture energy depends on both the length and depth of a crack. The relation is described using a cohesive energy function $\varphi : [0, \infty) \to [0, \infty)$. For us, we consider a cohesive energy function that has the following properties.

1. $\varphi(0) = 0$.
2. φ is differentiable, $\varphi' > 0$ and $\varphi'(0) = \infty$.
3. φ is concave.
4. $\lim_{n\to\infty} \varphi(x) = M < \infty$.
5. $\frac{\varphi'(x)x}{\varphi(x)} \to C > 0$ as $x \to 0$.

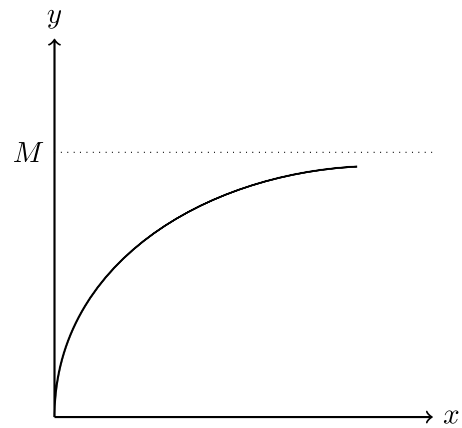

Figure 1: Cohesive energy function.

Here x represents the opening of a crack and $\varphi(x)$ represents the fracture energy density due to that opening. It is increasing because the bigger opening on the crack the more energy loss due to that opening. The reason why φ is concave has been mentioned before. $\varphi' > 0$ means all the fracture strictly stay in the cohesive zone. The properties of $\varphi'(0) = \infty$ and $\frac{\varphi'(x)x}{\varphi(x)} \to C$ are mathematically convenient. The first one allows us to use SBV compactness and the second one allows us to bound some functional of fracture defined on some small sets.

The energy form for cohesive model without pre-existing fracture is

$$E(u) = \int_{\Omega} |\nabla u|^2 dx + \int_{S_u} \varphi([u]) d\mathcal{H}^{N-1}.$$

A typical cohesive energy function may look like the graph in Figure 1. Next let's consider the energy when there's a pre-existing crack. For Griffith model we can measure the size of the union of current crack and pre-existing cracks by considering $\mathcal{H}^{N-1}(S_u \cup \Gamma)$. But for cohesive model it is more complicated. First let's give the cohesive energy function with history. Let $z > 0$, define cohesive energy function with history as follows

$$\tilde{\varphi}(x, z) = \begin{cases} \varphi(z) - \varphi'(z)(z - x) & 0 \leq x \leq z \\ \varphi(x) & z < x \end{cases} \tag{3}$$

where z represents the opening size of pre-existing crack. In other words, the same crack may have different fracture energy if there was another crack at the same location before. Define $\tilde{\varphi}(x, 0) = \varphi(x)$ when $z = 0$. The graph of the function may look like Figure 2.

This extra definition covers the case when the crack closes up. If a crack is always increasing in the evolution, it will follow $\varphi(x)$ straight. But if it starts to close up after the size reaches z it follows $\tilde{\varphi}(x, z)$. For an example and all the important properties of φ please refer to Appendix A.

Let $\Gamma \subset \Omega$ be $\mathcal{H}^{N-1}\sigma$-finite and let $\gamma \in L^1(\Gamma, \mathcal{H}^{N-1})$ denote the pre-existing cracks, our energy form will be

$$E(u, \gamma) \coloneqq \int_{\Omega} |\nabla u|^2 dx + \int_{S_u \cup \Gamma} \tilde{\varphi}([u], \gamma) d\mathcal{H}^{N-1}.$$

Figure 3 shows another example of cohesive energy function others consider.

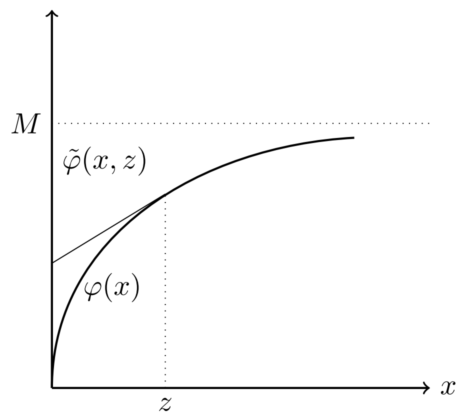

Figure 2: Graph of $\tilde{\varphi}(x, z)$.

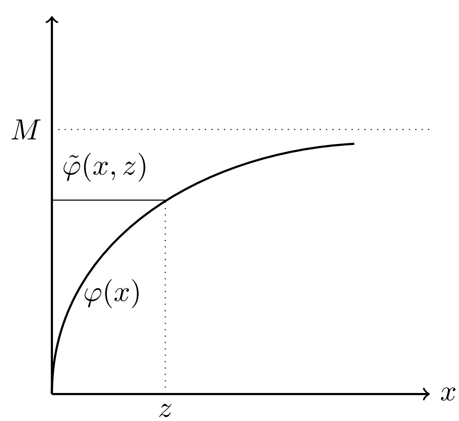

Figure 3: Another example.

It says you won't gain any energy back if you close up a crack, so the crack energy is fully irreversible. Another case is when you gain everything back, in which it follow φ entirely. From now on we will only consider φ as defined in (3).

Next let's have a look at the way we deal with boundary condition. Let Ω' be open, bounded and with Lipschitz boundary such that $\Omega \Subset \Omega'$. Let $g \in W^{1,\infty}(\Omega')$, we say an SBV function u satisfies the boundary condition g on $\partial\Omega$ or $u = g$ on $\partial\Omega$ if $u = g$ on $\Omega'\backslash\Omega$. By treating the boundary condition this way we allow the jump occur on the boundary $\partial\Omega$.

We describe the boundary condition, let $g(t) : [0,1] \to W^{1,\infty}(\Omega';\mathbb{R})$ s.t.

1. $\sup_t \|g(t)\|_{W^{1,\infty}} < \infty$.
2. $\nabla\dot{g}(t)$ exists and $\nabla\dot{g}(t) \in L^2(\Omega)$ $\forall t$, and $\sup_t \|\nabla\dot{g}(t)\|_{L^2} < \infty$.
3. $\nabla\dot{g}(t)$ is continuous on $[0,1]$.

Definition 4. *(Quasi-static Evolution with Cohesive Energy) We say a pair* $(u(t),\gamma(t)) : [0,1] \to SBV(\Omega) \times L^1(\Gamma(t),\mathcal{H}^{N-1})$*, where* $u(t) \in SBV(\Omega)$ *for each* t *and* $\gamma(t) := \sup_{\tau\le t}[u(t)]$ *defined on* $\Gamma(t)$*, is a quasi-static evolution that satisfies the boundary condition* $g(t)$ *if*

1. $\forall t \in [0,1]$

$$\int_\Omega |\nabla u(t)|^2 dx + \int_{S_{u(t)}\cup\Gamma(t)} \tilde{\varphi}([u(t)],\gamma(t))d\mathcal{H}^{N-1} \le \int_\Omega |\nabla v|^2 dx + \int_{S_v\cup\Gamma(t)} \tilde{\varphi}([v],\gamma(t))d\mathcal{H}^{N-1}$$

$\forall v \in SBV(\Omega)$ *s.t.* $v = g(t)$ *on* $\partial\Omega$*.*

2. $\Gamma(t_1) \subset \Gamma(t_2)$ *and* $\gamma(t_1) \le \gamma(t_2)$ *for all* $0 \le t_1 \le t_2 \le 1$*.*

3.

$$E(u(t),\gamma(t)) = E(u(0),\gamma(0)) + 2\int_0^t \int_\Omega \nabla u(s)\cdot\nabla\dot{g}(s)dxds$$

The existence of above evolution has been open for many years. The difficulty is to show that as the discrete time minimizers converge to a limit $u(t)$ in SBV, $u(t)$ also minimizes the energy. The way the cohesive energy is defined makes it sensitive to some of the complications in the fracture part of the sequence. We will illustrate exactly what those complications are in Chapter 1.

The following is the outline of the thesis.

Chapter 1

In Chapter 1, we first describes the procedure of time discretization. As the discrete time step goes to 0, we encounter a sequence of minimizers coupled with a sequence of pre-existing cracks and their limits. We then ask if the limit is also a minimizer. For Griffith model the answer is yes, in fact showing the limit is also a unilateral minimizer is the key part in showing the existence result for Griffith model. We show that for cohesive models the answer is no, by providing a counter example. We see that there are two potential complications on the fracture part of the sequence that will cause the limit not being a minimizer. One of them is the oscillation on the crack path, the other one is the splitting of one crack into two or more nearby cracks. We call the second one a "staircase". Then we construct a sequence of unilateral minimizers who have oscillations of the crack path. We show that the limit is not a unilateral minimizer. More interestingly, we show that the limit does have a minimality, but it is a minimality that encodes the oscillations.

Chapter 2

In Chapter 2, we continue our discussion by reconsidering the counter example from Chapter 1. We notice that the limit in the example picks up the oscillation of the sequence. We wonder if it is true for any other cases. We then consider a more general sequence and show that the limit will always pick up the oscillation, if there is oscillation of the sequence. To show this first we show the sequence does not have the second complication, the staircase situation, mentioned in Chapter 1. After excluding the staircase complication we show the oscillation will show up in the minimality of the limit. For the counter example we did not have this extra step of excluding staircase because we constructed the sequence to only have oscillation. Even though the limit of a sequence of a unilateral minimizers is not necessarily a unilateral minimizer, at least we show the staircase complication can be ruled out. This leaves only the oscillation to be the complication we need to rule out next. Further study indicates that it's impossible to exclude oscillation by only looking at a sequence of unilateral minimizers. So we move to the evolution problem to seek other ways to exclude oscillation.

Chapter 3

In Chapter 3, we consider the evolution problem. First we use time discretization to approximate the evolution. Then we formulate the energy and show the energy convergences for almost every t. Showing the convergence of energy is almost always the first step in showing anything else. We address the importance of convergence of energy by showing two applications of the result. First we show that in terms of energy the whole evolution in discrete time can be approximated by finitely many chosen times. It allows us to do analysis on finitely many minimizers. Using this idea we then show the absolute continuity result for the fracture energy of the sequence of minimizers at discrete time. As we can see in Chapter 2, the advantage of absolute continuity is to allow us apply advanced techniques like covering theorem.

Chapter 1

The Difficulties

1.1 Time discretization

As mentioned before, let $g(t) : [0,1] \to W^{1,\infty}(\Omega;\mathbb{R})$ be the boundary condition s.t.

1. $\sup_t \|g(t)\|_{W^{1,\infty}} < \infty$.
2. $\nabla \dot{g}(t)$ exists and $\nabla \dot{g}(t) \in L^2(\Omega)$ $\forall t$, and $\sup_t \|\nabla \dot{g}(t)\|_{L^2} < \infty$.
3. $\nabla \dot{g}(t)$ is continuous on $[0,1]$.

Then split the time line $[0,1]$ into 2^n pieces, denote $t_n^i = \frac{1}{2^n} i$. When $i = 0$, let u_n^0 be the solution to the following minimizing problem

$$u_n^0 = \min_{\substack{v \in SBV(\Omega) \\ v = g(t_n^0) \text{ on } \partial\Omega}} \left\{ \int_\Omega |\nabla v|^2 dx + \int_{S_v} \varphi([v]) d\mathcal{H}^{N-1} \right\}$$

When $0 \le i < 2^n$, let

$$u_n^{i+1} = \min_{\substack{v \in SBV(\Omega) \\ v = g(t_n^{i+1}) \text{ on } \partial\Omega}} \left\{ \int_\Omega |\nabla v|^2 dx + \int_{S_v \cup \Gamma_n^i} \tilde{\varphi}([v], \gamma_n^i) d\mathcal{H}^{N-1} \right\}$$

Where

$$\Gamma_n^i := \cup_{j=0}^i S_{u_n^j}$$
$$\gamma_n^i := \bigvee_{0 \le j \le i} [u_n^j]$$

We see γ_n^i is defined on Γ_n^i, and the existence of minimizers in each step can be derived from SBV compactness. Then define $u_n(t) = u_n^i$ for $t_n^i \le t < t_n^{i+1}$ and for $\forall 0 \le i < 2^n$. We see $u_n(t)$ is well defined on $[0,1]$ for $\forall n > 0$.

Define

$$E_n(t) := \int_\Omega |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1}$$

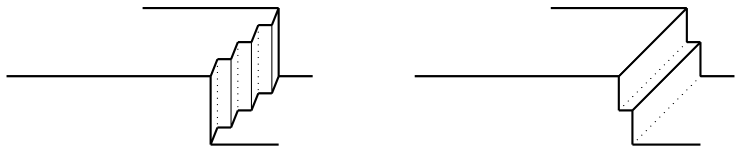

Figure 1.1: Two difficulties illustrated by pictures.

As approximation steps become more and more, one encounters a sequence of minimizers $u_n(t)$, a limit of the sequence $u(t)$, a sequence of pre-existing fracture or history $\gamma_n(t)$ and a limit of the history $\gamma(t)$ such that

$$\int_\Omega |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 dx + \int_{S_v \cup \Gamma_n(t)} \tilde{\varphi}([v], \gamma_n(t)) d\mathcal{H}^{N-1}$$

for each n and each $v = u_n(t)$ on $\Omega' \backslash \Omega$. The question is that is the limit a minimizer with $\gamma(t)$ as history? If we can show it then the global stability can be shown and the rest follows easily.

As we will see, in the next section, that the limit might not be a minimizer if there's oscillation on the sequence of history. In fact, there are two complications on the sequence that we need to rule out if we want to show the limit is also a minimizer. The first one is due to the potential oscillation of the crack path. The second is due to the potential of splitting a crack into two or more nearby cracks, with the same total jump in displacement. (see Figure 1.1)

In the next section we construct an example to show how oscillation affects the minimality of the limit. In the next chapter we show how to prove the second complication won't happen in a sequence of unilateral minimizers.

1.2 A Counter Example

1.2.1 1-D domain

In this section we construct a sequence of minimizers whose limit is not a minimizer. Though the sequence will be constructed in 2-D domain, we begin with examples in 1-D domain. Consider domain $I := [0, 1]$ and Dirichlet boundary condition $g(x) = x$ on $[0, 1]$. Let $\alpha \geq 1$ and φ be a cohesive energy function, let $u \in SBV(I)$ and define the energy

$$E_\alpha(u) := \int_0^1 |u'|^2 dx + \int_{S_u} \alpha \varphi([u]) d\mathcal{H}^0$$

Notice here $\mathcal{H}^0$ is the counting measure and $\int_{S_u} \alpha \varphi([u]) d\mathcal{H}^0 = \alpha \sum_{x_i \in S_u} \varphi([u](x_i))$. The existence of a minimizer of energy E_α can be shown using SBV compactness. Let u_α be a minimizer of the energy satisfying the Dirichlet condition g, i.e. $g(0) = 0$ and $g(1) = 1$. We can conclude the following

1. If u_α has a jump, it can only have one jump.
2. u'_α is constant on $[0, 1]$.
3. $2u'_\alpha = \alpha \varphi'([u_\alpha])$ if $[u_\alpha] > 0$.

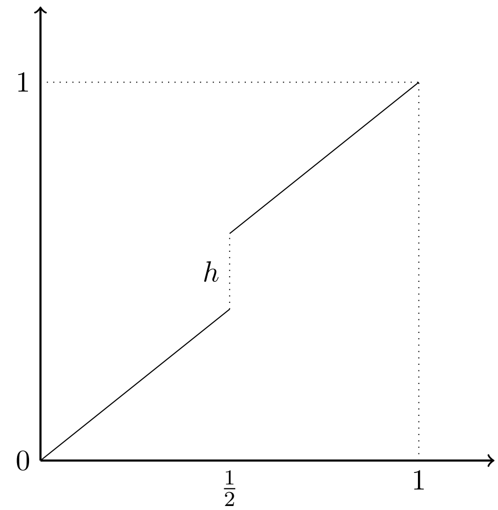

Figure 1.2: Graph of v.

If u_α has at least two jumps, we can move one jump to the other without changing the continuous part. By doing this we get a new SBV with smaller energy due to the strict concavity of φ. Since u_α has only one jump, let h_α denote the height of the jump. Notice that the uniqueness of the minimizer can not be guaranteed in the following sense. First of all, even though there is only one jump, the location of the jump can be anywhere on $[0, 1]$. Second, even if the location is fixed, the height might not be unique. Here the location is not important since we can always translate the location of the jump to one fixed point, say $\frac{1}{2}$.

Lemma 1. *There exists φ and α_0 such that for any $1 \leq \alpha \leq \alpha_0$, any minimizer of the energy $E_\alpha(v)$ satisfying $v = g$ on the boundary has a positive jump.*

Proof. First pick any cohesive energy function ϕ. Let v be a SBV function that satisfies the following conditions

1. v has only one jump at point $\{\frac{1}{2}\}$.
2. v' is constant on $[0, 1]$
3. $v(0) = 0$ and $v(1) = 1$.

Let h denote the height of the jump of v, the graph of v looks like Figure 1.2. Let $\beta \geq 0$ and due to the nature of v the energy $\int_0^1 |v'|^2 dx + \int_{S_v} \beta\phi([v]) d\mathcal{H}^0$ can also be written in terms of h as follows

$$f_\beta(h) = (h - 1)^2 + \beta\phi(h).$$

We see f_β is continuous and $f_\beta(0) = 1$. Choose $\beta_0 > 0$ and $\beta_1 > 0$ such that

1. $\beta_0 < \beta_1$
2. $f_\beta(\frac{1}{2}) < \frac{1}{2}$ for all β such that $\beta_0 \leq \beta \leq \beta_1$

Next define $\alpha_0 := \frac{\beta_0}{\beta_1} > 1$. We can see for any α such that $1 \le \alpha \le \alpha_0$, we have $\beta_1 \le \alpha\beta_1 \le \beta_0$, it follows

$$f_{\alpha\beta_1}(\frac{1}{2}) < \frac{1}{2}$$

Due to the definition of f_β, there exists some SBV function v that satisfies the Dirichlet condition g and

$$\int_0^1 |v'|^2 dx + \int_{S_v} \alpha\beta_1 \phi([v]) d\mathcal{H}^0 < \frac{1}{2} \tag{1.1}$$

Let $\varphi = \beta_1\phi$, we have

$$E_\alpha(v) < \frac{1}{2}$$

This shows each minimizer of the energy has positive jump, since if there's no jump the lowest energy it can get is 1. □

The following lemma can be seen as concavity of φ with α and history h.

Lemma 2. *$\forall$ φ and $0 < h \le 1$, $\exists \alpha_1 > 1$ s.t. for any $1 < \alpha < \alpha_1$ we have*

$$\alpha\tilde{\varphi}(a+b,h) < \alpha\tilde{\varphi}(b,h) + \varphi(a) \tag{1.2}$$

$\forall 0 \le b \le 1$ and $\forall 0 < a \le 1$.

Proof. First observe that the inequality (1.2) becomes equality when $a = 0$. Next we see

$$0 < \tilde{\varphi}(a+b,h) - \tilde{\varphi}(b,h) \le \tilde{\varphi}(a,h) - \tilde{\varphi}(0,h)$$

It follows

$$\frac{\varphi(a)}{\tilde{\varphi}(a+b,h) - \tilde{\varphi}(b,h)} \ge \frac{\varphi(a)}{\tilde{\varphi}(a,h) - \tilde{\varphi}(0,h)}$$

We see $\frac{\varphi(a)}{\tilde{\varphi}(a,h)-\tilde{\varphi}(0,h)} \to \infty$ as $a \to 0$ and $\frac{\varphi(a)}{\tilde{\varphi}(a,h)-\tilde{\varphi}(0,h)} > 1$ for all $0 < a \le 1$. Thus there exists $\delta > 1$ s.t. $\frac{\varphi(a)}{\tilde{\varphi}(a,h)-\tilde{\varphi}(0,h)} > \delta > 1$. Choose $\alpha_1 = \delta$ and let $1 < \alpha < \alpha_1$, we have $\frac{\varphi(a)}{\tilde{\varphi}(a+b,h)-\tilde{\varphi}(b,h)} > \alpha$ and thus (1.2) is proved. □

Let φ and α_0 be chosen the same as in lemma 1. Since $\varphi'(x) \to \infty$ as $x \to 0$, choose $0 < h < 1$ small such that $\varphi'(h) > 3$. According to lemma 2, there exists α_1 s.t. for any $1 < \alpha < \alpha_1$, (1.2) is satisfied. Then fix an α such that $1 < \alpha < \min\{\alpha_0, \alpha_1\}$. Let u_1 and u_α be minimizers of energy E_α that satisfies the boundary condition g, define $h_1 := [u_1]$ and $h_\alpha := [u_\alpha]$. First we see $2u'_\alpha = \alpha\varphi'(h_\alpha)$ and $2u'_1 = \varphi'(h_1)$. Since $u'_1 \le 1$ and $u'_\alpha \le 1$, we have $\varphi'(h_\alpha) \le 2$ and $\varphi'(h_1) \le 2$. It follows $h \le h_\alpha$ and $h \le h_1$ due to monotonicity of function φ'. We further conclude the following properties

$$\begin{gathered} h_1 > 0 \text{ and } h_\alpha > 0 \\ \alpha\tilde{\varphi}(a+b,h_\alpha) < \alpha\tilde{\varphi}(b,h_\alpha) + \varphi(a) \end{gathered} \tag{1.3}$$

We see $h_\alpha < 1$ and $h_1 < 1$. The second estimate is due to the fact

$$\alpha\tilde{\varphi}(a+b,h_\alpha) - \alpha\tilde{\varphi}(b,h_\alpha) \le \alpha\tilde{\varphi}(a+b,h) - \alpha\tilde{\varphi}(b,h)$$

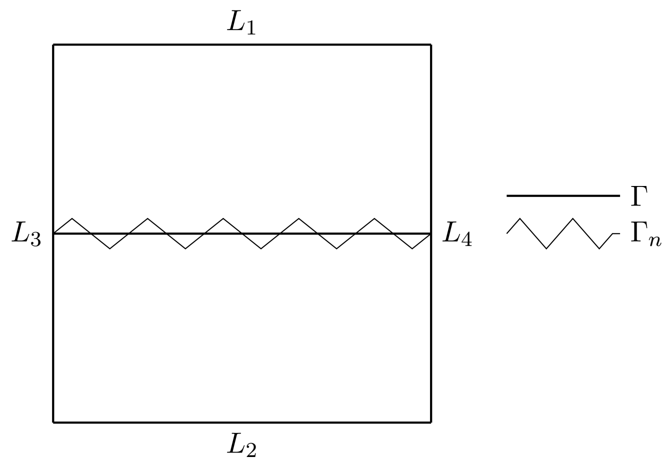

Figure 1.3: The 2-D domain.

for $h < h_\alpha$. From now on let's fix u_1 and u_α by fixing the location of the jumps at middle point $\frac{1}{2}$.

1.2.2 2-D domain

At this point we've specifically picked α, φ, u_1, u_α, h_1 and h_α. Next consider the 2-D square $[0,1]\times[0,1]$. Denote $L_1 = [0,1]\times\{1\}$, $L_2 = [0,1]\times\{0\}$, $L_3 = \{0\}\times[0,1]$ and $L_4 = \{1\}\times[0,1]$ as four boundaries of the square. Let Γ denote the set $[0,1]\times\{\frac{1}{2}\}$. Define $\gamma : \Gamma \to \mathbb{R}$ such that $\gamma = h_\alpha$ on Γ. For each n create a zigzag around Γ with amplitude equal to $\frac{1}{n}$. Adjust the angle of zigzag such that the total length equals α. Let Γ_n denote the zigzag. See Figure 1.3 for the graph. Define $\gamma_n = h_\alpha$ on Γ_n such that $\gamma_n = h_\alpha$ and let $\tilde{u}_n$ minimizes the following

$$\int_\Omega |\nabla \tilde{u}_n|^2 dx + \int_{S_{\tilde{u}_n}\cup\Gamma_n} \tilde{\varphi}([\tilde{u}_n],\gamma_n)d\mathcal{H}^1 \le \int_\Omega |\nabla v|^2 dx + \int_{S_v\cup\Gamma_n} \tilde{\varphi}([v],\gamma_n)d\mathcal{H}^1$$

over all v such that $v = 1$ on $[0,1]\times\{1\}$, $v = 0$ on $[0,1]\times\{0\}$ and $\frac{\partial v}{\partial n} = 0$ on the other two sides. So $\tilde{u}_n$ is a minimizer with history γ_n for each n. We see there exists a subsequence $\tilde{u}_n$(not relabeled) and a SBV limit $\tilde{u}_\infty$ s.t. $\tilde{u}_n \overset{SBV}{\rightharpoonup} \tilde{u}_\infty$.

Lemma 3.

$$\alpha\int_\Gamma \tilde{\varphi}([\tilde{u}_\infty],h_\alpha)d\mathcal{H}^1 + \int_{S_{\tilde{u}_\infty}\setminus\Gamma} \tilde{\varphi}([\tilde{u}_\infty])d\mathcal{H}^1 \le \liminf_{n\to\infty}\int_{S_{\tilde{u}_n}\cup\Gamma_n} \tilde{\varphi}([\tilde{u}_n],h_\alpha)d\mathcal{H}^1 \tag{1.4}$$

Proof. Fix $\epsilon > 0$, first choose a small rectangular region R_ϵ containing Γ s.t.

$$\begin{aligned}\int_{S_{\tilde{u}_\infty}\setminus\Gamma\cap R_\epsilon} \varphi([\tilde{u}_\infty])d\mathcal{H}^1 &\le \epsilon\\ \int_{\Omega\setminus R_\epsilon} \varphi([\tilde{u}_\infty])d\mathcal{H}^1 &\le \liminf_{n\to\infty}\int_{\Omega\setminus R_\epsilon}\varphi([\tilde{u}_n])d\mathcal{H}^1.\end{aligned} \tag{1.5}$$

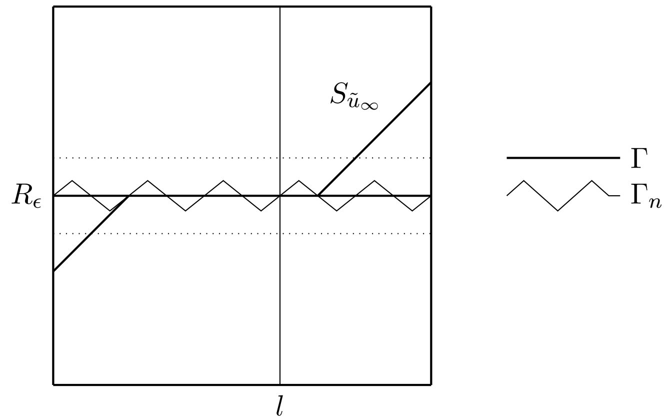

Figure 1.4: R_ϵ on the 2-D domain.

Next we see

$$\begin{aligned}\text{LHS of (1.4)} &= \alpha \int_{\Gamma} \tilde{\varphi}([\tilde{u}_\infty], h_\alpha) d\mathcal{H}^1 + \int_{S_{\tilde{u}_\infty} \setminus \Gamma \cap R_\epsilon} \varphi([\tilde{u}_\infty]) d\mathcal{H}^1 + \int_{\Omega \setminus R_\epsilon} \varphi([\tilde{u}_\infty]) d\mathcal{H}^1 \\ \text{RHS of (1.4)} &= \int_{\Gamma_n} \tilde{\varphi}([\tilde{u}_n], h_\alpha) d\mathcal{H}^1 + \int_{S_{\tilde{u}_n} \setminus \Gamma_n \cap R_\epsilon} \varphi([\tilde{u}_n]) d\mathcal{H}^1 + \int_{\Omega \setminus R_\epsilon} \varphi([\tilde{u}_n]) d\mathcal{H}^1\end{aligned} \tag{1.6}$$

We constrain our domain on R_ϵ and consider slices $\tilde{u}_n^l$, $\tilde{u}_\infty^l$ for $0 < l < 1$ along y axis(i.e. constrains of $\tilde{u}_n$ and $\tilde{u}_\infty$ on the set $\{l\} \times [0, 1]$). Let $x_i \in \{l\} \times [0, 1] \cap R_\epsilon$ be the locations where $\tilde{u}_n^l$ has jumps. Let Γ^l and Γ_n^l denote the intersection of Γ and Γ_n with $\{l\} \times [0, 1]$. We see for $\mathcal{L}^1$ a.e. l

$$[\tilde{u}_\infty^l(\Gamma^l)] \le \liminf_{n\to\infty} \sum_i [\tilde{u}_n^l(x_i)] \tag{1.7}$$

Combine with result from (1.3), we have

$$\begin{aligned}\alpha\tilde{\varphi}([\tilde{u}_\infty^l(\Gamma^l)], h_\alpha) &\le \liminf_{n\to\infty} \alpha\tilde{\varphi}(\sum_i [\tilde{u}_n^l(x_i)], h_\alpha) \\ &\le \liminf_{n\to\infty} \left[\alpha\tilde{\varphi}(\sum_{x_i \in \Gamma_n^l} [\tilde{u}_n^l(x_i)], h_\alpha) + \varphi(\sum_{x_i \notin \Gamma_n^l} [\tilde{u}_n^l(x_i)]) \right] \\ &\le \liminf_{n\to\infty} \left[\alpha \int_{R_\epsilon \cap \Gamma_n^l} \tilde{\varphi}([\tilde{u}_n^l], h_\alpha) d\mathcal{H}^0 + \int_{R_\epsilon \cap (S_{\tilde{u}_n^l} \setminus \Gamma_n^l)} \varphi([u_n^l]) d\mathcal{H}^0 \right]\end{aligned}$$

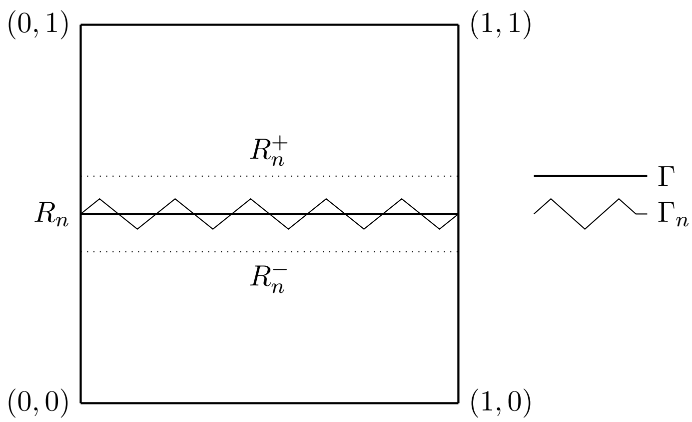

Figure 1.5: R_n on the 2-D domain.

According to Fatou's lemma, we have

$$\alpha \int_0^1 \tilde{\varphi}([\tilde{u}_\infty^l(\Gamma^l)], h_\alpha) dl \leq \liminf_{n\to\infty} \left[\alpha \int_0^1 \int_{R_\epsilon \cap \Gamma_n^l} \tilde{\varphi}([\tilde{u}_n^l], h_\alpha) d\mathcal{H}^0 dl + \int_0^1 \int_{R_\epsilon \cap (S_{\tilde{u}_n^l} \setminus \Gamma_n^l)} \varphi([u_n^l]) d\mathcal{H}^0 dl \right]$$
$$\leq \liminf_{n\to\infty} \left[\int_{R_\epsilon \cap \Gamma_n} \tilde{\varphi}([\tilde{u}_n], h_\alpha) d\mathcal{H}^1 + \int_{R_\epsilon \cap (S_{\tilde{u}_n} \setminus \Gamma_n)} \varphi([u_n]) d\mathcal{H}^1 \right]$$

And $\alpha \int_0^1 \tilde{\varphi}([\tilde{u}_\infty^l(\Gamma^l)], h_\alpha) dl = \alpha \int_\Gamma \tilde{\varphi}([\tilde{u}_\infty], h_\alpha) d\mathcal{H}^1$, next combine results from (1.5) and (1.6) we conclude the proof.

□

Lemma 4.

$$\limsup_{n\to\infty} \left[\int_\Omega |\nabla \tilde{u}_n|^2 dx + \int_{S_{\tilde{u}_n} \cup \Gamma_n} \tilde{\varphi}([\tilde{u}_n], h_\alpha) d\mathcal{H}^1 \right] \leq \int_\Omega |\nabla \bar{u}_\alpha| dx + \alpha \int_\Gamma \varphi([\bar{u}_\alpha]) d\mathcal{H}^1 \qquad (1.8)$$

where $\bar{u}_\alpha$ denotes the extension of u_α into 2-D domain, i.e. each slice of $\bar{u}_\alpha$ equals u_α.

Proof. For each n define a small region R_n that closely contain set Γ_n, as shown in Figure 1.5. Let u_α^+ denote the right limit of $u_\alpha(x)$ at $\frac{1}{2}$ and u_α^- the left limit. Let R_n^+ denote the horizontal boundary of R_n that is close to the set $[0,1] \times \{1\}$ and R_n^- the other horizontal boundary (see the above graph). Then define an SBV function w_n on Ω that satisfies the following conditions

1. $w_n = 1$ on $[0,1] \times \{1\}$ and $w_n = 0$ on $[0,1] \times \{0\}$.
2. $w_n = u_\alpha^+$ on R_n^+ and linear between R_n^+ and $[0,1] \times \{1\}$.
3. $w_n = u_\alpha^-$ on R_n^- and linear between R_n^- and $[0,1] \times \{0\}$.
4. For the values of w_n on R_n, define one side of Γ_n to be equal to u_α^+ and the other equal to u_α^- such that w_n only has jumps on Γ_n.

We can see that

$$\int_\Omega |\nabla w_n|^2 dx + \int_{S_{w_n}} \tilde{\varphi}([w_n], h_\alpha) d\mathcal{H}^1 \to \int_\Omega |\nabla \bar{u}_\alpha|^2 dx + \alpha \int_\Gamma \varphi([\bar{u}_\alpha]) d\mathcal{H}^1.$$

Since

$$\int_\Omega |\nabla \tilde{u}_n|^2 dx + \int_{S_{\tilde{u}_n} \cup \Gamma_n} \tilde{\varphi}([\tilde{u}_n], h_\alpha) d\mathcal{H}^1 \leq \int_\Omega |\nabla w_n|^2 dx + \int_{S_{w_n}} \tilde{\varphi}([w_n], h_\alpha) d\mathcal{H}^1$$

for each n, we conclude the proof. □

Let $\tilde{\gamma} : \Omega \to \mathbb{R}$ s.t. $\tilde{\Gamma} := supp(\tilde{\gamma})$ is $\mathcal{H}^1$ measurable and $\int_{\tilde{\Gamma}} \varphi(\tilde{\gamma}) d\mathcal{H}^1 < \infty$.

Lemma 5. *The limit $\tilde{u}_\infty$ has the following properties*

1. $\frac{\partial \tilde{u}_\infty}{\partial x} = 0$.
2. $\tilde{u}^l_\infty$ *is a minimizer of the energy* E_α.
3. $\exists \bar{w} : \Omega \to \mathbb{R}$ *that satisfies the boundary condition and*

$$\int_\Omega |\nabla \bar{w}|^2 dx + \int_{S_{\bar{w}} \cup \tilde{\Gamma}} \tilde{\varphi}([\bar{w}], \tilde{\gamma}) d\mathcal{H}^1 < \int_\Omega |\nabla \tilde{u}_\infty|^2 dx + \int_{S_{\tilde{u}_\infty} \cup \tilde{\Gamma}} \tilde{\varphi}([\tilde{u}_\infty], \tilde{\gamma}) d\mathcal{H}^1$$

for any $\tilde{\gamma}$.

Proof. Combine lemma 3 and lemma 4 we get

$$\int_\Omega |\nabla \tilde{u}_\infty|^2 dx + \alpha \int_\Gamma \tilde{\varphi}([\tilde{u}_\infty], h_\alpha) d\mathcal{H}^1 + \int_{S_{\tilde{u}_\infty} \setminus \Gamma} \varphi([\tilde{u}_\infty]) d\mathcal{H}^1 \leq \int_\Omega |\nabla \bar{u}_\alpha|^2 dx + \alpha \int_\Gamma \varphi([\bar{u}_\alpha]) d\mathcal{H}^1 \quad (1.9)$$

Let $v \in SBV([0,1])$ s.t. $v(0) = 0$ and $v(1) = 1$. Based on v let's construct $\hat{v}$ by moving all jumps of v to the point $\frac{1}{2}$ and keeping $\int_0^1 |v'|^2 dx = \int_0^1 |\hat{v}'|^2 dx$. Due to (1.3) we have

$$\int_0^1 |u'_\alpha|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \varphi([u_\alpha]) d\mathcal{H}^0 \leq \int_0^1 |\hat{v}'|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \tilde{\varphi}([\hat{v}], h_\alpha) d\mathcal{H}^0$$
$$\leq \int_0^1 |v'|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \tilde{\varphi}([v], h_\alpha) d\mathcal{H}^0 + \int_{S_v \setminus \{\frac{1}{2}\}} \varphi([v]) d\mathcal{H}^0$$

Next we see

$$\int_\Omega |\nabla \tilde{u}_\infty|^2 dx + \alpha \int_\Gamma \tilde{\varphi}([\tilde{u}_\infty], h_\alpha) d\mathcal{H}^1 + \int_{S_{\tilde{u}_\infty} \setminus \Gamma} \varphi([\tilde{u}_\infty]) d\mathcal{H}^1 \quad (1.10)$$
$$\leq \int_0^1 \int_0^1 |\nabla \tilde{u}^l_\infty|^2 dx dl + \alpha \int_0^1 \int_{\{\frac{1}{2}\}} \tilde{\varphi}([\tilde{u}^l_\infty], h_\alpha) d\mathcal{H}^0 dl + \int_0^1 \int_{S_{\tilde{u}^l_\infty} \setminus \{\frac{1}{2}\}} \varphi([\tilde{u}^l_\infty]) d\mathcal{H}^0 dl \quad (1.11)$$

This shows $\int_\Omega |\nabla \tilde{u}_\infty|^2 dx = \int_0^1 \int_0^1 |\nabla \tilde{u}^l_\infty|^2 dx dl$ and thus $\frac{\partial \tilde{u}_\infty}{\partial x} = 0$.
It follows that each slice $\tilde{u}^l_\infty$ is the same. According to (1.9) we have

$$\int_0^1 |\nabla \tilde{u}^l_\infty|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \tilde{\varphi}([\tilde{u}^l_\infty], h_\alpha) d\mathcal{H}^0 + \int_{S_{\tilde{u}_\infty} \setminus \{\frac{1}{2}\}} \varphi([\tilde{u}^l_\infty]) d\mathcal{H}^0 \leq \int_0^1 |\nabla u_\alpha|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \varphi([u_\alpha]) d\mathcal{H}^0$$

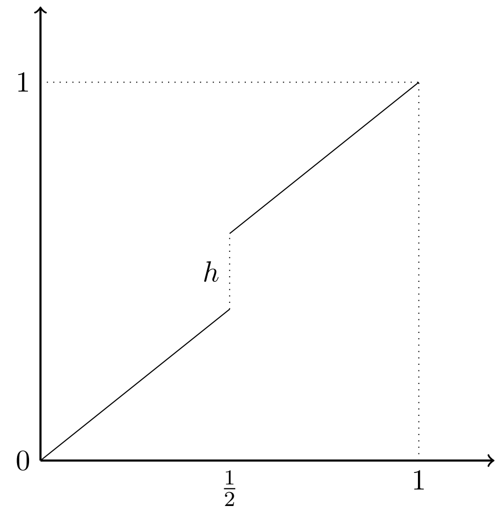

Figure 1.6: Graph of $\tilde{u}^l_\infty$.

We see $\int_{S_{\tilde{u}^l_\infty}\setminus\{\frac{1}{2}\}} \varphi([\tilde{u}^l_\infty])d\mathcal{H}^0 = 0$ because otherwise we can construct a function that gives smaller energy than u_α. Then we conclude that $\tilde{u}^l_\infty$ is also a minimizer to the energy E_α. It follows that $\tilde{u}^l_\infty$ has following properties

1. $\frac{d}{dx}\tilde{u}^l_\infty$ is constant.
2. $2\frac{d}{dx}\tilde{u}^l_\infty = \alpha\varphi'([\tilde{u}^l_\infty])$.
3. $S_{\tilde{u}^l_\infty} = \{\frac{1}{2}\}$.

Let $h = [\tilde{u}^l_\infty]$ and let Figure 1.6 be the graph of $\tilde{u}^l_\infty$. We see $\frac{d}{dx}\tilde{u}^l_\infty = 1-h$ and thus $2(1-h) = \alpha\varphi'(h)$ and further $2(1-h) > \varphi'(h)$. Let $\lambda > 0$ and increase the right limit of $\frac{d}{dx}\tilde{u}^l_\infty$ at $\{\frac{1}{2}\}$ by λ and decrease the left limit by λ. Let w_λ denote the new function. Next define the elastic energy $E^a(w_\lambda) := \int_0^1 |w'_\lambda|^2 dx$ and fracture energy $E^s(w_\lambda) := \varphi([w_\lambda])$. Further we see $E^a(w_\lambda)$ is decreased comparing to $E^a(\tilde{u}^l_\infty)$ and $E^s(w_\lambda)$ is increased comparing to $E^s(\tilde{u}^l_\infty)$. Next let $\Delta E^a(\lambda) := E^a(\tilde{u}^l_\infty) - E^a(w_\lambda)$ denote the elastic energy decreased and let $\Delta E^s(\lambda) := E^s(w_\lambda) - E^s(\tilde{u}^l_\infty)$ denote the fracture energy increased. Write down the details of $\Delta E^a(\lambda)$ and $\Delta E^s(\lambda)$

$$\Delta E^a(\lambda) = (1-h)^2 - (1-h-2\lambda)^2$$
$$\Delta E^s(\lambda) = \varphi(h+2\lambda) - \varphi(h)$$

We see both energies are differentiable with respect to λ and

$$\frac{\partial}{\partial\lambda}\Delta E^a(0) = 4(1-h)$$
$$\frac{\partial}{\partial\lambda}\Delta E^s(0) = 2\varphi'(h)$$

Since $4(1-h) > 2\varphi'(h)$, there exists positive small λ such that $\Delta E^a(\lambda) > \Delta E^s(\lambda)$, it follows $E(w_\lambda) < E(\tilde{u}^l_\infty)$. Next let $\bar{w}$ denote the extension of w_λ to 2D domain Ω, we see $\bar{w}$ satisfies the

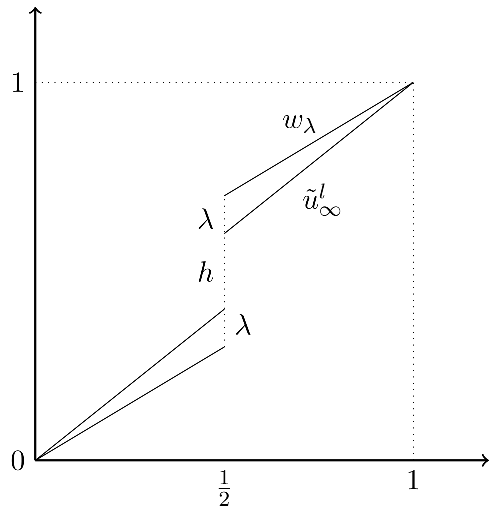

Figure 1.7: Graph of w_λ and $\tilde{u}^l_\infty$.

boundary conditions on $\partial\Omega$ and moreover $S_{\bar{w}} = S_{\tilde{u}_\infty} = \Gamma$, $[\bar{w}] > [\tilde{u}_\infty]$ on Γ and

$$\int_\Omega |\nabla \bar{w}|^2 dx + \int_{S_{\bar{w}}} \varphi([\bar{w}]) d\mathcal{H}^1 < \int_\Omega |\nabla \tilde{u}_\infty|^2 dx + \int_{S_{\tilde{u}_\infty}} \varphi([\tilde{u}_\infty]) d\mathcal{H}^1$$

It follows

$$\int_\Omega |\nabla \bar{w}|^2 dx + \int_{S_{\bar{w}} \cup \tilde{\Gamma}} \tilde{\varphi}([\bar{w}], \tilde{\gamma}) d\mathcal{H}^1 < \int_\Omega |\nabla \tilde{u}_\infty|^2 dx + \int_{S_{\tilde{u}_\infty} \cup \tilde{\Gamma}} \tilde{\varphi}([\tilde{u}_\infty], \tilde{\gamma}) d\mathcal{H}^1$$

for any $\tilde{\gamma}$. It is due to the fact that $\tilde{\varphi}(a, h) - \tilde{\varphi}(b, h) \leq \varphi(a) - \varphi(b)$ if $a \geq b$. □

Let $\eta : \Omega \to \mathbb{R}$ be defined

$$\eta = \begin{cases} \alpha & on\ \Gamma \\ 1 & elsewhere \end{cases}$$

It's not hard to show that the limit $\tilde{u}_\infty$ has the following minimality

$$\int_\Omega |\nabla \tilde{u}_\infty|^2 dx + \int_{S_{\tilde{u}_\infty}} \eta \varphi([\tilde{u}_\infty]) d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 dx + \int_{S_v} \eta \tilde{\varphi}([v], [\tilde{u}_\infty]) d\mathcal{H}^{N-1} \tag{1.12}$$

For Griffith model, the oscillation on the sequence won't affect the minimality of the limit. But this example shows that, for cohesive model, the oscillation on the sequence will destroy the minimality of the limit. More interestingly, as shown in this example, the limit seems to pick up the oscillation according to (1.12). In the next chapter we will discuss a more general case and show the limit will always pick up the oscillation, if there is one.

Chapter 2

Limits of Unilateral Minimizers

2.1 Introduction

In Chapter 1, we considered a counter example where a sequence of unilateral minimizers converges to a limit that is not a unilateral minimizer. Moreover we showed the limit does have a minimality, that picks up the oscillation, as shown in (1.12). We wonder if it is true for any other cases. In this chapter, we consider a sequence of unilateral minimizers with itself as history and show that the limit will always pick up the oscillation, if there is oscillation.

Let Ω' be an open and bounded subset in $\mathbb{R}^N$ such that $\Omega \Subset \Omega'$. Consider a sequence $\{g_n\}_{n=1}^{\infty} \subset W^{1,\infty}(\Omega')$ such that $g_n \to g$ in $W^{1,\infty}$.

Let u_n be a unilateral minimizer that minimizes the following

$$\int_\Omega |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 dx + \int_{S_v \cup S_{u_n}} \tilde{\varphi}([v],[u_n]) d\mathcal{H}^{N-1}$$

$\forall\ v = g_n$ on $\Omega' \backslash \Omega$. In addition, assume the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded from above in terms of the following energy

$$E(u_n) = \int_\Omega |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1}.$$

Due to SBV compactness we can extract a subsequence(not relabeled) and an SBV function u s.t. $u_n \overset{SBV}{\rightharpoonup} u$ in SBV. Then the following is true.

Theorem 2. *$\exists \alpha(x) : \Omega' \to \mathbb{R}$ and $\alpha \geq 1$ such that*

$$\int_\Omega |\nabla u|^2 dx + \int_{S_u} \alpha\varphi([u]) d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 dx + \int_{S_v \cup S_u} \alpha\tilde{\varphi}([v],[u]) d\mathcal{H}^{N-1}$$

$\forall\ v = g$ on $\Omega' \backslash \Omega$.

Remark 1. *Notice here $\exists b$ s.t. $\sup_n \|g_n\|_{W^{1,\infty}} \leq \frac{b}{2}$, by truncation argument the size of the jumps will stay below b, i.e.*

$$\sup_n [u_n] \leq b$$
$$\sup [u] \leq b.$$

Also notice, due to the fact that $\varphi'(x) \geq \varphi'(b) > 0$ for $x \leq b$, $\varphi'([u_n]) \geq \varphi'(b)$ and $\varphi'([u]) \geq \varphi'(b)$.

Remark 2. *The introduce of Ω' is to deal with the case where jumps of u_n converge to the boundary of Ω.*

2.2 Settings and Tools

Measure theory plays an important role here, let's recall some of notions. For more details refer to [13] and [2]. Let $U \subset \mathbb{R}^N$ be open. If μ is a real or vector valued measures on U, $|\mu|$ denotes its total variation. Unless otherwise stated, when we say a measure or Radon measure it usually means positive measure.

Definition 5. *Let μ be a positive measure and ν a real or vector valued measure on U. We say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$, if for every $B \subset U$ such that $\mu(B) = 0$ we have $|\nu|(B) = 0$.*

Definition 6. *Let μ and $\mu_n (n = 1, 2, ...)$ be vector Radon measures on U. We say μ_n converge weakly* to the measure μ, written $\mu_n \overset{*}{\rightharpoonup} \mu$, if*

$$\lim_{n\to\infty} \int_U f d\mu_n = \int_U f d\mu$$

for all $f \in C_c(U)$.

It's well known that if $\mu_n \overset{*}{\rightharpoonup} \mu$, the followings are true:

1. $|\mu|(K) \geq \limsup_{n\to\infty} |\mu_n|(K)$ for all compact set $K \subset U$.
2. $|\mu|(A) \leq \liminf_{n\to\infty} |\mu_n|(A)$ for all open set $A \subset U$.
3. if $|\mu_n| \overset{*}{\rightharpoonup} \lambda$, $\lambda \geq |\mu|$ and for all bounded Borel set B with $\lambda(\partial B) = 0$, we have $\mu(B) = \lim_{n\to\infty} \mu_n(B)$.

The following corollary of Besicovitch's Theorem is well known, but since it will be used frequently we will list it here. See [13] for the proof of the theorem.

Theorem 3. *Let μ be a Borel measure on $\mathbb{R}^N$, and $\mathcal{F}$ any collection of non-degenerate closed balls. Let A denote the set of centers of the balls in $\mathcal{F}$. Assume $\mu(A) < \infty$ and $\inf\{r : B(a,r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then for each open set $U \subset \mathbb{R}^N$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that*

$$\bigcup_{B\in\mathcal{G}} B \subset U$$

and

$$\mu\left((A \cap U) \setminus \bigcup_{B\in\mathcal{G}} B\right) = 0.$$

Let $u \in SBV(\Omega)$ and $t \in \mathbb{R}$, define

$$E_t := \{x \in \Omega : u > t\}.$$

And the reduced boundary of E_t is denoted by $\partial^* E_t$. Then we see E_t is a set of finite perimeter for $\mathcal{L}$-a.e. $t \in \mathbb{R}$. And the coarea formula for BV gives the following

$$|Du|(B) = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\partial^* E_t \cap B)dt$$

for any Borel B. It follows

$$|D^s u|(\Omega) = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\partial^* E_t \cap S_u)dt$$

and

$$\int_{\Omega} |\nabla u|dx = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\partial^* E_t \backslash S_u)dt$$

Another useful coarea formula

$$\int_{S_u} \varphi([u])d\mathcal{H}^{N-1} = \int_{-\infty}^{\infty} \int_{\partial^* E_t} \frac{\varphi([u])}{[u]} d\mathcal{H}^{N-1}dt$$

see [18] Lemma 6.3, pick $f(x,v) = |v|\frac{\varphi([u])}{[u]}\chi_{S_u}$.

2.3 Minimality with α

We delay the proof by introducing some of the preliminary results first. We split the proof of theorem 2 into three steps. First prove that the weak* limit of $\varphi([u_n])\mathcal{H}^{N-1}\lfloor S_{u_n}$ is absolutely continuous with respect to $\varphi([u])\mathcal{H}^{N-1}\lfloor S_u$. Then due to Radon-Nikodym theorem there exists a density function α' between the two measures, which gives us a candidate for α. Next we show that the cracks will eventually all combine to one reduced boundary, locally. This enables us to pass α to the limit to show the minimality with density α.

2.3.1 Absolute Continuity

If $u \in SBV$, let $\{[u] < h\}$ be the set $\{x \in S_u : [u](x) < h\}$. Define $\mu_n \coloneqq \varphi([u_n])\mathcal{H}^{N-1}\lfloor S_{u_n}$ and $\mu \coloneqq \varphi([u])\mathcal{H}^{N-1}\lfloor S_u$. Since μ_n is bounded, according to weak compactness in measure, there's a subsequence μ_n(not relabeled) and a Radon measure μ_∞ on Ω' such that $\mu_n \overset{*}{\rightharpoonup} \mu_\infty$. We see that in general $\mu \leq \mu_\infty$ and thus $\mu \ll \mu_\infty$, but here due to unilateral minimality we are going to show that the opposite is also true.

Remark 3. *Due to definition of μ and μ_n we see $\forall x \in \Omega'$*

$$\begin{aligned} \mu(\partial B(x,r)) &= 0 \\ \mu_n(\partial B(x,r)) &= 0, \forall\ n > 0 \end{aligned}$$

for $\mathcal{L}^1$-a.e. $r > 0$. Because $\mu(\partial B(x_1,r_1) \cap \partial B(x_2,r_2)) = 0$ for $(x_1,r_1) \neq (x_2,r_2)$, set $\{(x,r) \in (\Omega' \times \mathbb{R}) : \mu(\partial B(x,r)) > 0\}$ is at most countable. It follows set $\{x \in \Omega' : \exists r$ such that $\mu(\partial B(x,r)) > 0\}$ is at most countable. Similarly we can show $\{x \in \Omega' : \exists r$ and n such that $\mu_n(\partial B(x,r)) > 0\}$ is at most countable.

Immediately, from the first observation, we have $\forall x \in \Omega'$

$$\begin{aligned} \mathcal{H}^{N-1}(S_u \cap \partial B(x,r)) = 0 \\ \mathcal{H}^{N-1}(S_{u_n} \cap \partial B(x,r)) = 0, \forall n > 0 \end{aligned} \tag{2.1}$$

for $\mathcal{L}^1$*-a.e.* $r > 0$*. And from the second observation and the fact that* $N \geq 2$*, we have for* $\mathcal{H}^{N-1}$*-a.e.* $x \in S_u$

$$\begin{aligned} \mathcal{H}^{N-1}(S_u \cap \partial B(x,r)) = 0 \\ \mathcal{H}^{N-1}(S_{u_n} \cap \partial B(x,r)) = 0, \forall\ n > 0 \end{aligned} \tag{2.2}$$

for $\forall r > 0$*. This enables us to ignore the crack energy of* u *and* u_n *on boundary of* $B(x,r)$*.*

Let $B(x,r) \Subset \Omega'$, use T_u^+ to denote the trace of u inside the open ball. In other words we are regarding $B(x,r)$ as a domain for u. Similarly T_u^- denotes the trace of u on $\partial B(x,r)$ with $\Omega' \backslash \overline{B(x,r)}$ as its domain,

Lemma 6. *Fix* $x \in \Omega'$*, let* $\{u_n\}_{n=1}^\infty \subset BV(\Omega')$ *s.t.* $u_n \to u$ *in* $L^1(\Omega')$*. Then we can extract a subsequence* $\{n_k\}_{n=1}^\infty$ *s.t.*

$$\lim_{n\to\infty} \int_{\partial B(x,r)} |T^+_{u_{n_k}} - T^-_u| d\mathcal{H}^{N-1} = 0$$
$$\lim_{n\to\infty} \int_{\partial B(x,r)} |T^-_{u_{n_k}} - T^+_u| d\mathcal{H}^{N-1} = 0$$

for $\mathcal{L}^1$*-a.e.* $r > 0$*.*

Proof. Let $u \in BV$, we see

$$\int_{\partial B(x,r)} |T_u^+ - T_u^-| d\mathcal{H}^{N-1} = 0 \tag{2.3}$$

for $\mathcal{L}^1$-a.e. $r > 0$.

So

$$\begin{aligned} \int_{\Omega'} u dx &= \int_0^\infty \int_{\partial B(x,r)} u d\mathcal{H}^{N-1} dr = \int_0^\infty \int_{\partial B(x,r)} T_u^+ d\mathcal{H}^{N-1} dr \\ &= \int_0^\infty \int_{\partial B(x,r)} T_u^- d\mathcal{H}^{N-1} dr \end{aligned}$$

Let $\{v_n\}_{n=1}^\infty \subset BV$ s.t. $v_n \to 0$ in L^1. We see

$$\int_{\Omega'} |v_n| dx = \int_0^\infty \int_{\partial B(x,r)} T^+_{|v_n|} d\mathcal{H}^{N-1} dr \to 0.$$

It's well-known that we can extract a subsequence $\{n_k\}_{k=1}^\infty$ s.t.

$$\int_{\partial B(x,r)} T^+_{|v_{n_k}|} d\mathcal{H}^{N-1} \to 0$$

for $\mathcal{L}^1$-a.e. $r > 0$. Let $v_n = u_n - u$ we see

$$\int_{\partial B(x,r)} |T^+_{u_{n_k}} - T^+_u| d\mathcal{H}^{N-1} = \int_{\partial B(x,r)} T^+_{|u_{n_k}-u|} d\mathcal{H}^{N-1} \to 0.$$

Since $\int_{\partial B(x,r)} |T^+_u - T^-_u| d\mathcal{H}^{N-1} = 0$ for $\mathcal{L}^1$-a.e. $r > 0$, we see $\lim_{n\to\infty} \int_{\partial B(x,r)} |T^+_{u_{n_k}} - T^-_u| d\mathcal{H}^{N-1} = 0$ for $\mathcal{L}^1$-a.e. $r > 0$. The proof of the second half of the lemma is similar. □

Let $\{u_n\}_{n=1}^\infty$ be the sequence mentioned in the beginning of this section. Let $S_n \subset S_{u_n}$ be $\mathcal{H}^{N-1}$ measurable.

Lemma 7. *We have* $\forall \epsilon > 0$, $\exists \delta > 0$ *s.t.*

$$\int_{S_n \cap S_{u_n}} \varphi([u_n]) d\mathcal{H} \leq \epsilon$$

whenever $\int_{S_n \cap S_{u_n}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \leq \delta$.

Proof. Fix $\epsilon > 0$, as usual let's assume $S_n \subset S_{u_n}$. First we see $\frac{\varphi'(x)x}{\varphi(x)} \to C$ as $x \to 0$. So we can find $\tau > 0$ s.t. $\frac{\varphi'(x)x}{\varphi(x)} > \frac{C}{2}, \forall x \leq \tau$. Since $\varphi'(x)x$ is continuous, positive and $\varphi'(x)x = 0$ if and only if $x = 0$ for $0 \leq x \leq b$. Thus we have $m := \min_{\tau \leq x \leq b} \varphi'(x)x > 0$.

Then pick δ s.t.

$$(\frac{2\delta}{C} + \varphi(b)\frac{\delta}{m}) \leq \epsilon.$$

Then consider the following

$$\begin{aligned} \frac{C}{2}\int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} &\leq \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) \frac{\varphi'([u_n])[u_n]}{\varphi([u_n])} d\mathcal{H}^{N-1} \\ &\leq \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \\ &\leq \delta. \end{aligned}$$

It follows $\int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \frac{2\delta}{C}$.

Then

$$m\mathcal{H}^{N-1}(S_n \cap \{[u_n] > \tau\}) \leq \int_{S \cap Q(x,r) \cap \{[u_n] > \tau\}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \leq \delta$$

which gives $\mathcal{H}^{N-1}(S_n \cap \{[u_n] > \tau\}) \leq \frac{\delta}{m}$.

Finally

$$\begin{aligned} \int_{S_n} \varphi([u_n]) d\mathcal{H}^{N-1} &= \int_{S_n \cap \{[u_n] > \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} + \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} \\ &\leq \frac{2\delta}{C} + \varphi(b)\frac{\delta}{m} \\ &\leq \epsilon. \end{aligned}$$

□

We are in a position to show the following theorem.

Theorem 4. $\mu_\infty \ll \mu$

Proof. Fix $x \in \Omega'$, let $B(x,r) \Subset \Omega'$ where lemma (6) and remark (3) hold, which is true for $\mathcal{L}^1$-a.e. $r > 0$.

First, we claim that

$$\limsup_{n\to\infty} \int_{S_{u_n}\cap B(x,r)} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \leq \int_{S_u \cap B(x,r)} \varphi([u]) d\mathcal{H}^{N-1}. \tag{2.4}$$

For each n consider the test function v_n constructed as follows

$$v_n = \begin{cases} u & \text{on } B(x,r) \\ u_n & \text{on } \Omega' \backslash B(x,r). \end{cases}$$

Due to unilateral minimality

$$\int_{\Omega'} |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \int_{\Omega'} |\nabla v_n|^2 dx + \int_{S_{u_n} \cup S_{v_n}} \tilde{\varphi}([v_n],[u_n]) d\mathcal{H}^{N-1}.$$

Considering the construction of v_n, the above inequality can be written as

$$\begin{aligned} &\int_{B(x,r)} |\nabla u_n|^2 dx + \int_{B(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} + \int_{\Omega'\backslash B(x,r)} |\nabla u_n|^2 dx \\ &+ \int_{\Omega'\backslash B(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} + \int_{\partial B(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} \\ \leq &\int_{B(x,r)} |\nabla u|^2 dx + \int_{B(x,r)} \tilde{\varphi}([u],[u_n]) d\mathcal{H}^{N-1} + \int_{\Omega'\backslash B(x,r)} |\nabla u_n|^2 dx \\ &+ \int_{\Omega'\backslash B(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} + \int_{\partial B(x,r)} \tilde{\varphi}([v_n],[u_n]) d\mathcal{H}^{N-1}. \end{aligned}$$

It follows

$$\begin{aligned} &\int_{B(x,r)} |\nabla u_n|^2 dx + \int_{B(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} + \int_{\partial B(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} \\ \leq &\int_{B(x,r)} |\nabla u|^2 dx + \int_{B(x,r)} \tilde{\varphi}([u],[u_n]) d\mathcal{H}^{N-1} + \int_{\partial B(x,r)} \tilde{\varphi}([v_n],[u_n]) d\mathcal{H}^{N-1} \\ \leq &\int_{B(x,r)} |\nabla u|^2 dx + \int_{\partial B(x,r)} \tilde{\varphi}([v_n],[u_n]) d\mathcal{H}^{N-1} \\ &+ \int_{B(x,r)} \varphi([u]) d\mathcal{H}^{N-1} + \int_{B(x,r)} \tilde{\varphi}(0,[u_n]) d\mathcal{H}^{N-1}. \end{aligned}$$

The last inequality is due to the fact that $\varphi(x,z) \le \varphi(x) + \varphi(0,z)$. It follows

$$\begin{aligned}\int_{B(x,r)} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \le & \int_{B(x,r)} \varphi([u]) d\mathcal{H}^{N-1} + \int_{B(x,r)} |\nabla u|^2 dx - \int_{B(x,r)} |\nabla u_n|^2 dx \\ & + \int_{\partial B(x,r)} \tilde{\varphi}([v_n],[u_n]) d\mathcal{H}^{N-1}.\end{aligned}$$

Due to the fact $\int_{B(x,r)} |\nabla u|^2 dx \le \liminf_{n\to\infty} \int_{B(x,r)} |\nabla u_n|^2 dx$ and lemma(6) we conclude the first claim.

For the sake of contradiction assume there exists a Borel set $A \subset \Omega'$ s.t. $\mu(A) = 0$ but $\mu_\infty(A) = \delta > 0$. According to lemma (3) we can find $\bar{\delta} > 0$ s.t.

$$\int_{S_n \cap S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \le \frac{\delta}{2} \tag{2.5}$$

whenever

$$\int_{S_n \cap S_{u_n}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \le \bar{\delta}.$$

Let U be open s.t. $A \subset U \subset \Omega'$ and $\mu(U) \le \frac{1}{2}\bar{\delta}$. Then consider the collection $\mathcal{F}$ of balls $B(x,r)$ that satisfy the following conditions

$$\begin{aligned} x &\in A \\ B(x,r) &\subset U \\ \mu_\infty(\partial B(x,r)) &= 0 \\ (2.4) & \quad \text{holds .}\end{aligned}$$

According to Besicovitch covering theorem we can find a collection of countable disjoint closed balls $\{\overline{B(x_i,r_i)}\}_{i=1}^\infty$ s.t. $\mu_\infty(A \setminus \bigcup_{i=1}^\infty \overline{B(x_i,r_i)}) = 0$. Since $\mu_\infty(\partial B(x_i,r_i)) = 0\ \forall i$, we have $\mu_\infty(A \setminus \bigcup_{i=1}^\infty B(x_i,r_i)) = 0$. Then select a finite $N \in \mathbb{N}$ s.t. $\mu_\infty(A \setminus \bigcup_{i=1}^N B(x_i,r_i)) \le \frac{1}{8}\delta$. We conclude that

$$\mu(\bigcup_{i=1}^N B(x_i,r_i)) \le \frac{1}{2}\bar{\delta}$$
$$\mu_\infty(\bigcup_{i=1}^N B(x_i,r_i)) \ge \frac{7}{8}\delta.$$

Again since $\mu_\infty(\partial B(x_i,r_i)) = 0\ \forall i$, we have

$$\mu_\infty(\bigcup_{i=1}^N B(x_i,r_i)) = \lim_{n\to\infty} \int_{\bigcup_{i=1}^N B(x_i,r_i)} \varphi([u_n]) \ge \frac{7}{8}\delta. \tag{2.6}$$

Then consider (2.4) to get

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\bigcup_{i=1}^N B(x_i,r_i)}\varphi'([u_n])[u_n]d\mathcal{H}^{N-1}\\
\leq\ &\limsup_{n\to\infty}\int_{\bigcup_{i=1}^N B(x_i,r_i)}\varphi'([u_n])[u_n]d\mathcal{H}^{N-1}\\
\leq\ &\int_{\bigcup_{i=1}^N B(x_i,r_i)}\varphi([u])d\mathcal{H}^{N-1}\\
=\ &\mu(\bigcup_{i=1}^N B(x_i,r_i))\\
\leq\ &\frac{1}{2}\bar{\delta}.
\end{aligned}$$

Apply (2.5) to get

$$\lim_{n\to\infty}\int_{\bigcup_{i=1}^N B(x_i,r_i)}\varphi([u_n])d\mathcal{H}^{N-1}\leq\frac{\delta}{2}$$

which contradicts to (2.6). Thus the theorem has been proved. □

From the above theorem we see that μ_∞ also concentrates on S_u and $\mu_\infty \ll \mathcal{H}^{N-1}\lfloor S_u$ since $\mu \ll \mathcal{H}^{N-1}\lfloor S_u$. It follows that $D_\mu\mu_\infty$ exists and $0 < D_\mu\mu_\infty \leq \infty$ for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ and 0 else where. Also, regarding remark (3), we see for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$

$$D_\mu\mu_\infty(x) = \lim_{r\to 0}\frac{\mu_\infty(B(x,r))}{\mu(B(x,r))} = \lim_{r\to 0}\frac{\lim_{n\to\infty}\int_{S_{u_n}\cap B(x,r)}\varphi([u_n])}{\int_{S_u\cap B(x,r)}\varphi([u])}. \tag{2.7}$$

2.3.2 No-staircase Lemma

In this section both u_n and u will be the same as mentioned in the beginning of this section. We are going to show the most important theorem of this section. We show that the jumps of the sequence will eventually be combined to one reduced boundary of some set, basically it is due to unilateral minimality and concavity of the cohesive function $\varphi(x)$. The way we show it is something we call arguing from local. Fix $x \in S_u$ we shrink the cube $Q(x,r)$ to get nice results we want. Then we cover S_u with those carefully chosen cubes and sum over the errors to get global niceness.

We are switching from balls to cubes, technically there's no difference in terms of proof. First let's introduce some notations and show some results. Let $x \in S_u$, let $Q(x,r)$ be the cube with side length $2r$ and normal the same as $\nu(x)$, then define

$$\begin{aligned}
Q^-(x,r) &:= \{y \in Q(x,r) : (y-x)\cdot\nu(x) < 0\}\\
Q^+(x,r) &:= \{y \in Q(x,r) : (y-x)\cdot\nu(x) > 0\}\\
H(x,r,s) &:= \{y \in Q(x,r) : (y-x)\cdot\nu(x) = s\}\\
R_a^b(x,r) &:= \{y \in Q(x,r) : a < (y-x)\cdot\nu(x) < b\}.
\end{aligned}$$

The geometric meaning can be illustrated using Figure 2.1. Next we see $\mathcal{H}^{N-1}(H(x,r,s)) = (2r)^{N-1}$ for $-r \leq s \leq r$ and there exists $C(N,\delta)$ that does not depend on r s.t. $C(N,\delta) \to 0$ as

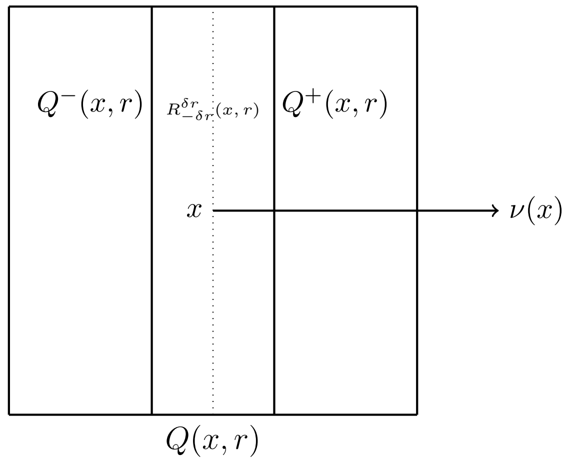

Figure 2.1: Illustration of $Q(x,r)$.

$\delta \to 0$ and

$$\mathcal{H}^{N-1}(\partial R^{\delta r}_{-\delta r}(x,r) \backslash (H(x,r,-\delta r) \cup H(x,r,\delta r))) \leq C(N,\delta) r^{N-1} \quad \forall\ r. \tag{2.8}$$

Then we have

$$\int_{S_u} \varphi([u]) d\mathcal{H}^{N-1} \leq \varphi(b)\sqrt{\epsilon} r^{N-1} + \varphi(\sqrt{\epsilon})\mathcal{H}^{N-1}(S_u) \tag{2.9}$$

whenever $\int_{S_u} [u] d\mathcal{H}^{N-1} \leq \epsilon r^{N-1}$. Indeed, first consider

$$\mathcal{H}^{N-1}(\{[u > \delta]\})\delta \leq \int_{\{[u]>\delta\}} [u] d\mathcal{H}^{N-1} \leq \epsilon r^{N-1}.$$

It follows $\mathcal{H}^{N-1}(\{[u > \delta]\}) \leq \frac{\epsilon}{\delta} r^{N-1}$. Thus

$$\begin{aligned} \int_{S_u} \varphi([u]) d\mathcal{H}^{N-1} &= \int_{\{[u]>\delta\}} \varphi([u]) d\mathcal{H}^{N-1} + \int_{\{[u]\leq\delta\}} \varphi([u]) d\mathcal{H}^{N-1} \\ &\leq \varphi(b)\frac{\epsilon}{\delta} r^{N-1} + \varphi(\delta)\mathcal{H}^{N-1}(S_u). \end{aligned}$$

Pick $\delta = \sqrt{\epsilon}$ to conclude (2.9).

Let $t \in \mathbb{R}$, define

$$\begin{aligned} E_t &:= \{x \in \Omega : u > t\} \\ E_t^n &:= \{x \in \Omega : u_n > t\}. \end{aligned}$$

We see E_t and E_t^n are sets with finite perimeter for $\mathcal{L}^1$-a.e. t and $\forall\ n$. If a set E has finite perimeter, we use $\partial^* E$ and $\nu_E(x)$ to denote the reduced boundary of E and the generalized inner normal at $x \in \partial^* E$.

Lemma 8. *Let $\beta(x)$ be a $\mathcal{H}^{N-1}$ measurable function on S_u s.t. $\beta(x) > 0$ and $\int_{S_u} \beta(x) d\mathcal{H}^{N-1} < \infty$,* *then we have for all $\delta < 1$*

$$\lim_{r\to 0} \frac{\int_{S_u\cap Q(x,r)} \beta(y)d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x) \tag{2.10a}$$

$$\lim_{r\to 0} \frac{\int_{S_u\cap R^{\delta r}_{-\delta r}(x,r)} \beta(y)d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x) \tag{2.10b}$$

for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$. As usual $Q(x,r)$ and $R^{\delta r}_{-\delta r}(x,r)$ will be oriented by $\nu(x)$.

Proof. First let's write $S_u = \bigcup_{i=1}^{\infty} A_i$ where A_i is $\mathcal{H}^{N-1}$ rectifiable and $\mathcal{H}^{N-1}(A_i) < \infty$ for $\forall i$. It suffices to show (2.10a) and (2.10b) are true in each A_i. Define Radon measure $\mu := \beta\mathcal{H}^{N-1}\lfloor S_u$ and fix A_i we see for μ a.e. $x \in A_i$ (or $\mathcal{H}^{N-1}$-a.e. since $\beta(y) > 0$) $\lim_{r\to 0} \frac{\mu(A_i\cap Q(x,r))}{\mu(Q(x,r))} = 1$. That is

$$\lim_{r\to 0} \frac{\int_{A_i\cap Q(x,r)} \beta(y)d\mathcal{H}^{N-1}}{\int_{S_u\cap Q(x,r)} \beta(y)d\mathcal{H}^{N-1}} = 1.$$

Since $\mathcal{H}^{N-1}\lfloor A_i$ is Radon, we have for $\mathcal{H}^{N-1}\lfloor A_i$-a.e. $x \in A_i$, $\beta(x)$ is a Lebesgue point, i.e.

$$\lim_{r\to 0} \frac{\int_{A_i\cap Q(x,r)} \beta(y)d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i \cap Q(x,r))} = \beta(x)$$

for $\mathcal{H}^{N-1}\lfloor A_i$-a.e. $x \in A_i$. Then Besicovitch-Marstrand-Mattila theorem [2](page 83 theorem 2.63)says

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(A_i \cap Q(x,r))}{(2r)^{N-1}} = 1$$

for $\mathcal{H}^{N-1}$ a.e. $x \in A_i$ since A_i is $\mathcal{H}^{N-1}$ rectifiable and $\mathcal{H}^{N-1}(A_i) < \infty$. Thus we conclude

$$\lim_{r\to 0} \frac{\int_{S_u\cap Q(x,r)} \beta(y)d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x)$$

for $\mathcal{H}^{N-1}$ a.e. $x \in A_i$. And (2.10a) is proved.

Next consider (2.10b), the " $\leq$ " part is obvious. So let's show the " $\geq$ " part. Let D be a countable dense set in $\mathbb{R}$ s.t.

$$A_i = \bigcup_{t\in D} (A_i \cap \partial^* E_t)$$

and $\mathcal{H}^{N-1}(\partial^* E_t) < \infty$ for $\forall t$. We see for all $t \in D$, $\mathcal{H}^{N-1}$ a.e. $x \in A_i \cap \partial^* E_t$ is a Lebesgue point of $\chi_{A_i\cap\partial^* E_t}$ with Radon measure $\mathcal{H}^{N-1}\lfloor\partial^* E_t$ i.e.

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(A_i \cap \partial^* E_t \cap Q(x,r))}{\mathcal{H}^{N-1}(\partial^* E_t \cap Q(x,r))} = 1$$

for $\mathcal{H}^{N-1}$ a.e. $x \in A_i \cap \partial^* E_t$.
It is known that

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(\partial^* E_t \cap Q(x,r))}{(2r)^{N-1}} = 1,$$

thus

$$\lim_{r\to 0}\frac{\mathcal{H}^{N-1}(A_i\cap\partial^*E_t\cap Q(x,r))}{(2r)^{N-1}}=1.$$

Next we see

$$\lim_{r\to 0}\frac{\mathcal{H}^{N-1}(A_i\cap Q(x,r))}{(2r)^{N-1}}=1,$$

and therefore

$$\lim_{r\to 0}\frac{\mathcal{H}^{N-1}((A_i\backslash\partial^*E_t)\cap Q(x,r))}{(2r)^{N-1}}=0. \tag{2.11}$$

From [20](page 241 theorem 5.6.5) we see for $\forall\delta<1$

$$\frac{\mathcal{H}^{N-1}(\partial^*E_t\cap(Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)))}{(2r)^{N-1}}=0 \tag{2.12}$$

for $\mathcal{H}^{N-1}$ a.e. $x\in\partial^*E_t$. Then we get

$$\begin{aligned}
&\lim_{r\to 0}\frac{\mathcal{H}^{N-1}(A_i\cap(Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)))}{(2r)^{N-1}}\\
\le\ &\lim_{r\to 0}\left(\frac{\mathcal{H}^{N-1}(A_i\backslash\partial^*E_t\cap(Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)))}{(2r)^{N-1}}+\frac{\mathcal{H}^{N-1}(\partial^*E_t\cap(Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)))}{(2r)^{N-1}}\right)\\
\le\ &\lim_{r\to 0}\left(\frac{\mathcal{H}^{N-1}(A_i\backslash\partial^*E_t\cap Q(x,r))}{(2r)^{N-1}}+\frac{\mathcal{H}^{N-1}(\partial^*E_t\cap(Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)))}{(2r)^{N-1}}\right)\\
=\ &0
\end{aligned}$$

due to result (2.11) and (2.12). It follows

$$\lim_{r\to 0}\frac{\mathcal{H}^{N-1}(A_i\cap R^{\delta r}_{-\delta r}(x,r))}{\mathcal{H}^{N-1}(A_i\cap Q(x,r))}=1$$

for $\mathcal{H}^{N-1}$ a.e. $x\in A_i\cap\partial^*E_t$, and therefore for $\mathcal{H}^{N-1}$ a.e. $x\in A_i$.
Next consider

$$\lim_{r\to 0}\frac{\int_{A_i\cap Q(x,r)}|\beta(x)-\beta|d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i\cap Q(x,r))}=0$$

for $\mathcal{H}^{N-1}$ a.e. $x\in A_i$. Consider

$$\begin{aligned}
\left|\frac{\int_{A_i\cap R^{\delta r}_{-\delta r}(x,r)}\beta d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i\cap R^{\delta r}_{-\delta r}(x,r))}-\beta(x)\right| &\le \frac{\int_{A_i\cap R^{\delta r}_{-\delta r}(x,r)}|\beta(x)-\beta|d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i\cap R^{\delta r}_{-\delta r}(x,r))}\\
&= \frac{\int_{A_i\cap R^{\delta r}_{-\delta r}(x,r)}|\beta(x)-\beta|d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i\cap Q(x,r))}\frac{\mathcal{H}^{N-1}(A_i\cap Q(x,r))}{\mathcal{H}^{N-1}(A_i\cap R^{\delta r}_{-\delta r}(x,r))}.
\end{aligned}$$

So

$$\lim_{r\to 0}\frac{\int_{A_i\cap R^{\delta r}_{-\delta r}(x,r)}\beta(y)d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i\cap Q(x,r))}=\beta(x),$$

and thus

$$\lim_{r\to 0} \frac{\int_{A_i\cap R^{\delta r}_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x).$$

Then

$$\lim_{r\to 0} \frac{\int_{S_u\cap R^{\delta r}_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} \geq \lim_{r\to 0} \frac{\int_{A_i\cap R^{\delta r}_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x).$$

We conclude

$$\lim_{r\to 0} \frac{\int_{S_u\cap R^{\delta r}_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x)$$

for $\mathcal{H}^{N-1}$ a.e. $x \in A_i$. Therefore (2.10b) is proved. □

Remark 4. *The above lemma uses the fact that the fracture energy on S_u is mostly concentrated on some reduced boundary $\partial^* E_t$ locally in the sense of measure. That's why no matter how small δ is, the measure of $\partial^* E_t$ will stay mostly within $R^{\delta r}_{-\delta r}(x,r)$ as we shrink the cube $Q(x,r)$.*

During the following define

$$E(v,u) = \int_\Omega |\nabla v|^2 dx + \int_{S_u\cup S_v} \tilde{\varphi}([v],[u]) d\mathcal{H}^{N-1}$$

Lemma 9. *Let u_n and u be from section §2.1. Then we have for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$*

$$\lim_{r\to 0} \frac{\lim_{n\to\infty} \int_{Q(x,r)} |\nabla u_n| dx}{r^{N-1}} = 0.$$

Proof. From (2.2) in remark (3), we can choose r s.t. $\mathcal{H}^{N-1}(\partial Q(x,r)\cap S_{u_n}) = 0$, $\forall n$. Then define u'_n by

$$u'_n = \begin{cases} u_n & \Omega\backslash Q(x,r) \\ u^+(x) & Q^-(x,r) \\ u^-(x) & Q^+(x,r) \end{cases}.$$

So

$$\begin{aligned}
& E(u_n', u_n) \\
= & \int_{\Omega\setminus Q(x,r)} |\nabla u_n|^2 dx + \int_{S_{u_n}\cup S_{u_n'}} \tilde{\varphi}([u_n'], [u_n]) \\
= & \int_{\Omega\setminus Q(x,r)} |\nabla u_n|^2 dx + \int_{S_{u_n}\setminus Q(x,r)} \varphi([u_n]) \\
& + \int_{\partial Q(x,r)} \tilde{\varphi}([u_n'], [u_n]) + \int_{Q(x,r)\cap(H(x,r,0)\cup S_{u_n})} \tilde{\varphi}([u_n'], [u_n]) \\
\leq & \int_{\Omega\setminus Q(x,r)} |\nabla u_n|^2 dx + \int_{S_{u_n}\setminus Q(x,r)} \varphi([u_n]) + \mathcal{H}^{N-1}(\partial Q(x,r))\varphi(b) \\
& + \mathcal{H}^{N-1}(H(x,r,0))\varphi(b) + \int_{Q(x,r)\cap S_{u_n}} \varphi([u_n]) \\
\leq & \int_{\Omega\setminus Q(x,r)} |\nabla u_n|^2 dx + \int_{S_{u_n}\setminus Q(x,r)} \varphi([u_n]) + \int_{Q(x,r)\cap S_{u_n}} \varphi([u_n]) + Cr^{N-1}
\end{aligned}$$

for some constant $C < \infty$. It follows

$$\begin{aligned}
& E(u_n, u_n) - E(u_n', u_n) \\
\geq & \int_{\Omega\setminus Q(x,r)} |\nabla u_n|^2 dx + \int_{Q(x,r)} |\nabla u_n|^2 dx + \int_{S_{u_n}\setminus Q(x,r)} \varphi([u_n]) + \int_{S_{u_n}\cap Q(x,r)} \varphi([u_n]) \\
& - \int_{\Omega\setminus Q(x,r)} |\nabla u_n|^2 dx - \int_{S_{u_n}\setminus Q(x,r)} \varphi([u_n]) - \int_{Q(x,r)\cap S_{u_n}} \varphi([u_n]) - Cr^{N-1} \\
\geq & \int_{Q(x,r)} |\nabla u_n|^2 dx - Cr^{N-1}.
\end{aligned}$$

Due to unilateral minimality, we have $E(u_n, u_n) - E(u_n', u_n) \leq 0$. So

$$\int_{Q(x,r)} |\nabla u_n|^2 dx \leq Cr^{N-1}.$$

Then consider Cauchy-Schwartz inequality

$$\int_{Q(x,r)} |\nabla u_n| dx \leq |Q(x,r)|^{\frac{1}{2}} \left(\int_{Q(x,r)} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{2C} 2^{\frac{N}{2}} r^{N-\frac{1}{2}}.$$

So

$$\frac{\int_{Q(x,r)} |\nabla u_n| dx}{r^{N-1}} \leq \sqrt{2C} 2^{\frac{N}{2}} r^{\frac{1}{2}}.$$

Take the limit as $r \to 0$ we have proved the lemma. □

During the following, let $\{S_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{H}^{N-1}$ measurable sets.

Lemma 10. *Let u_n and u be from section §2.1. Let $0 < h \leq b$ and let $g(y) := \varphi'(y)(h-y) - (\varphi(h) - \varphi(y))$. For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ and $\forall \epsilon > 0$, exists $\delta > 0$ and R s.t. $\forall r < R$ there exists*

$N(r) \in \mathbb{N}$ *s.t.*

$$\sup_{n>N(r)} \int_{S_n \cap S_{u_n} \cap Q(x,r)} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1} < \epsilon r^{N-1}$$

for any sequence of $\mathcal{H}^{N-1}$ *measurable set* $\{S_n\}_{n=1}^{\infty}$ *s.t.*

$$\int_{S_n \cap S_{u_n} \cap Q(x,r)} g([u_n]) d\mathcal{H}^{N-1} < \delta r^{N-1}.$$

Proof. First we can assume $S_n \subset S_{u_n}$ always. We see $g(y) \geq 0$ and $g(y) = 0$ iff $y = h$. $g(y) \to \infty$ as $y \to 0$. And for $\mathcal{H}^{N-1}$ a.e. $x \in S_u$,

$$\begin{aligned} \lim_{r\to 0} \quad & \frac{\int_{S_u \cap Q(x,r)} \varphi([u])}{(2r)^{N-1}} = \varphi([u](x)) \\ D_\mu \mu_\infty(x) = \quad \lim_{r\to 0} \quad & \frac{\lim_{n\to\infty} \int_{S_{u_n} \cap Q(x,r)} \varphi([u_n])}{\int_{S_u \cap Q(x,r)} \varphi([u])} < \infty. \end{aligned}$$

Fix $\epsilon > 0$, let $\delta_0 < h$ s.t.

$$\max\{\varphi(h+\delta_0) - \varphi(h), \varphi(h) - \varphi(h-\delta_0)\} \frac{5 D_\mu \mu_\infty(x) \varphi(b) 2^{N-1}}{\varphi(h-\delta_0)} \leq \frac{\epsilon}{2}.$$

Then there exists $\delta > 0$ s.t.

$$2\varphi(b) \frac{\delta}{\min\{g(h-\delta_0), g(h+\delta_0)\}} \leq \frac{\epsilon}{2}.$$

Then there exists $R > 0$ s.t.

$$\begin{aligned} \lim_{n\to\infty} \int_{S_{u_n} \cap Q(x,r)} \varphi([u_n]) \quad & \leq \quad 2 D_\mu \mu_\infty(x) \int_{S_u \cap Q(x,r)} \varphi([u]) \\ \int_{S_u \cap Q(x,r)} \varphi([u]) \quad & \leq \quad 2\varphi([u](x))(2r)^{N-1} \end{aligned}$$

for $\forall r < R$. It follows

$$\lim_{n\to\infty} \int_{S_{u_n} \cap Q(x,r)} \varphi([u_n]) \leq 4 D_\mu \mu_\infty(x) \varphi([u]) r^{N-1} \leq 4 D_\mu \mu_\infty(x) \varphi(b) 2^{N-1} (2r)^{N-1}$$

for all $\forall r < R$.

Then $\forall r < R$, we can find a corresponding $N(r) \in \mathbb{N}$ s.t.

$$\int_{S_{u_n} \cap Q(x,r)} \varphi([u_n]) \leq 5 D_\mu \mu_\infty(x) \varphi(b) 2^{N-1} r^{N-1} \tag{2.13}$$

for all $n > N(r)$.

Next let's consider

$$\begin{aligned}
& \int_{S_n\cap Q(x,r)} |\varphi(h)-\varphi([u_n])|d\mathcal{H}^{N-1} \\
= & \int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|>\delta_0\}} |\varphi(h)-\varphi([u_n])|d\mathcal{H}^{N-1} \\
& + \int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}} |\varphi(h)-\varphi([u_n])|d\mathcal{H}^{N-1},
\end{aligned}$$

we see

$$\begin{gathered}
\textstyle\int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}} |\varphi(h)-\varphi([u_n])|d\mathcal{H}^{N-1} \\
\le \max\{\varphi(h+\delta_0)-\varphi(h), \varphi(h)-\varphi(h-\delta_0)\}\mathcal{H}^{N-1}(\{S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}).
\end{gathered}$$

Next consider

$$\begin{aligned}
& \varphi(h-\delta_0)\mathcal{H}^{N-1}(S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}) \\
\le & \int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}} \varphi([u_n])d\mathcal{H}^{N-1} \\
\le & C_0 r^{N-1}
\end{aligned}$$

where $C_0 = 5D_\mu\mu_\infty(x)\varphi(b)2^{N-1}$. Therefore $\mathcal{H}^{N-1}(S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}) \le \frac{C_0}{\varphi(h-\delta_0)} r^{N-1}$. It follows

$$\begin{gathered}
\int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}} |\varphi(h)-\varphi([u_n])|d\mathcal{H}^{N-1} \\
\le \max\{\varphi(h+\delta_0)-\varphi(h), \varphi(h)-\varphi(h-\delta_0)\}\frac{C_0}{\varphi(h-\delta_0)} r^{N-1}.
\end{gathered}$$

Next due to the choice of S_n we have

$$\int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|>\delta_0\}} g([u_n])d\mathcal{H}^{N-1} + \int_{S_n\cap Q(x,r)\cap\{|[u_n]-h|\le\delta_0\}} g([u_n])d\mathcal{H}^{N-1} \le \delta r^{N-1}.$$

It follows

$$\begin{aligned}
& \min\{g(h-\delta_0), g(h+\delta_0)\}\mathcal{H}^{N-1}(S_n\cap\{|[u_n]-h|>\delta_0\}) \\
& \qquad \le \int_{S_n\cap\{|[u_n]-h|>\delta_0\}} g([u_n])d\mathcal{H}^{N-1} \le \delta r^{N-1}.
\end{aligned}$$

So we have $\mathcal{H}^{N-1}(S_n \cap Q(x,r) \cap \{|[u_n]-h| > \delta_0\}) < \frac{\delta}{\min\{g(h-\delta_0), g(h+\delta_0)\}} r^{N-1}$. Finally we have

$$\begin{aligned}
& \int_{S_n \cap Q(x,r)} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1} \\
\leq\ & \Big[2\varphi(b) \frac{\delta}{\min\{g(h-\delta_0), g(h+\delta_0)\}} \\
& + \max\{\varphi(h+\delta_0) - \varphi(h), \varphi(h) - \varphi(h-\delta_0)\} \frac{C_0}{\varphi(h-\delta_0)} \Big] r^{N-1} \\
\leq\ & \epsilon r^{N-1}.
\end{aligned}$$

Take the sup to conclude the lemma. □

Using the same technique, we can also show the following lemma.

Lemma 11. *Let u_n and u be from section §2.1. Let $0 < h \leq b$. Then we have for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ and $\forall \epsilon > 0$, exists $\delta > 0$ and R s.t. $\forall r < R$ there exists $N(r) \in \mathbb{N}$ s.t.*

$$\sup_{n > N(r)} \int_{S_n \cap S_{u_n} \cap \{[u_n] \geq h\} \cap Q(x,r)} \left(\varphi'(h) - \varphi'([u_n])\right) d\mathcal{H}^{N-1} < \epsilon r^{N-1}$$

for any sequence of $\mathcal{H}^{N-1}$ measurable set $\{S_n\}_{n=1}^{\infty}$ s.t.

$$\int_{S_n \cap S_{u_n} \cap \{[u_n] \geq h\} \cap Q(x,r)} \left(\varphi([u_n]) - \varphi(h)\right) d\mathcal{H}^{N-1} < \delta r^{N-1}.$$

The following lemma can be proved using the exact technique in (7).

Lemma 12. *For $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.*

$$\int_{S_n \cap S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \epsilon r^{N-1}$$

whenever $\int_{S_n \cap S_{u_n}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \leq \delta r^{N-1}$.

Remark 5. *The point of the above two results is to show that if we have value of one functional of $[u_n]$ can be as small as we want, then we can infer some other functional of $[u_n]$ can be as small as we want.*

Notice that those results are based on the fact that u_n and u are from §2.1. They do not apply to general sequences of SBV functions.

The following theorem says, no matter how relatively small the region $R^{\delta r}_{-\delta r}(x,r)$ is, we can always shrink $Q(x,r)$ to some degree such that the fracture energy of u_n mostly concentrates on $R^{\delta r}_{-\delta r}(x,r)$. And the reason why we consider a small region $R^{\delta r}_{-\delta r}(x,r)$ rather than the whole cube $Q(x,r)$ is that later we are going to alter the values of u_n within the region $R^{\delta r}_{-\delta r}(x,r)$. Once we change the value of u_n on the region $R^{\delta r}_{-\delta r}(x,r)$, there could be new jumps created along the boundary $\partial R^{\delta r}_{-\delta r}(x,r)$. But we want the new jump created on the short side of $\partial R^{\delta r}_{-\delta r}(x,r)$ to be insignificant by reducing the 'length' of the short side.

Again, during the following we assume u_n and u are from section §2.1.

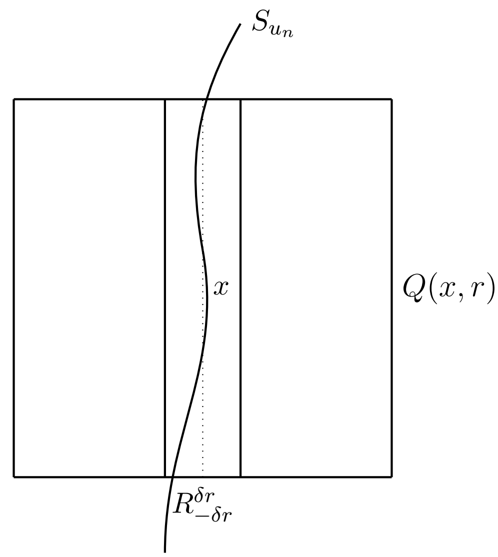

Figure 2.2: Illustration of no staircase.

Theorem 5. *For $\mathcal{H}^{N-1}$ a.e. $x \in S_u$, for $\forall \epsilon > 0$, $\exists \delta > 0$ and $R > 0$ s.t. $\forall r < R$ there $\exists \{t_n\}_{n=1}^{\infty} \subset (u^-, u^+)$ and $\{s_n\}_{n=1}^{\infty} \subset (\delta r, 2\delta r)$ s.t.*

$$\lim_{n\to\infty} \int_{S_{u_n} \cap R^{s_n}_{-s_n}(x,r) \cap \partial^* E^n_{t_n}} |\varphi([u_n]) - \varphi([u](x))| d\mathcal{H}^{N-1} < \epsilon r^{N-1}. \tag{2.14}$$

Moreover

$$\lim_{n\to\infty} \int_{(S_{u_n} \backslash \partial^* E^n_{t_n}) \cap R^{s_n}_{-s_n}(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} < \epsilon r^{N-1} \tag{2.15}$$

$$\lim_{n\to\infty} \mathcal{H}^{N-1}((\partial^* E^n_{t_n} \backslash S_{u_n}) \cap R^{s_n}_{-s_n}(x,r)) < \epsilon r^{N-1} \tag{2.16}$$

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(\partial R^{s_n}_{-s_n}(x,r) \backslash (H(x,r,-s_n) \cup H(x,r,s_n)) \leq \epsilon r^{N-1} \tag{2.17}$$

$$\lim_{n\to\infty} \left| \int_{S_{u_n} \cap R^{s_n}_{-s_n}(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} - \mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r)) \varphi([u](x)) \right| < \epsilon r^{N-1}. \tag{2.18}$$

Proof. For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ we have

$$\begin{aligned}
&\lim_{r\to 0} \fint_{Q^-(x,r)} |u^+(x) - u(y)|dy = 0 \\
&\lim_{r\to 0} \fint_{Q^+(x,r)} |u^-(x) - u(y)|dy = 0 \\
&\lim_{r\to 0} \frac{\int_{S_u\cap Q(x,r)} \varphi([u])}{(2r)^{N-1}} = \varphi([u](x)) \\
&\lim_{r\to 0} \frac{\lim_{n\to\infty} \int_{Q(x,r)} |\nabla u_n| dx}{r^{N-1}} = 0 \\
&\mathcal{H}^{N-1}(\partial Q(x,r)\cap S_u) = 0 \text{ and } \mathcal{H}^{N-1}(\partial Q(x,r)\cap S_{u_n}) = 0
\end{aligned} \tag{2.19}$$

and

$$D_\mu \mu_\infty(x) = \lim_{r\to 0} \frac{\lim_{n\to\infty} \int_{S_{u_n}\cap Q(x,r)} \varphi([u_n])}{\int_{S_u\cap Q(x,r)} \varphi([u])} < \infty.$$

Denote $h \coloneqq u^+(x) - u^-(x)$. Fix $\epsilon > 0$, regarding lemma (12), $\exists \delta_1 \le \epsilon$ s.t.

$$\int_{S_n\cap S_{u_n}} \varphi([u_n]) \le \epsilon r^{N-1} \tag{2.20}$$

whenever $\int_{S_n\cap S_{u_n}} \varphi'([u_n])[u_n] \le 3\delta_1 r^{N-1}$.

And for δ_1, regarding lemma (10), $\exists \delta_2 \le \delta_1$, R_2 and $N_2(r)$ s.t. $\forall r < R_2$

$$\sup_{n>N_2(r)} \int_{S_n\cap S_{u_n}\cap Q(x,r)} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1} < \delta_1 r^{N-1} \tag{2.21}$$

for any $\{S_n\}_{n=1}^\infty$ s.t. $\int_{S_n\cap S_{u_n}\cap Q(x,r)} \int g([u_n]) d\mathcal{H}^{N-1} < \delta_2 r^{N-1}$.

Then pick $\delta > 0$ and $\delta_0 > 0$ s.t.

$$\begin{aligned}
\delta_0 &\le \delta_2 \\
C(N,\delta)\varphi(b) &\le \frac{\delta_2}{4} \\
\varphi(b)\delta_0 &\le \frac{\delta_2}{4}.
\end{aligned} \tag{2.22}$$

Then pick $\epsilon_0 > 0$ s.t.

$$\begin{aligned}
\varphi(b)\sqrt{3\frac{\epsilon_0}{\delta}} + 2^N \varphi(\sqrt{3\frac{\epsilon_0}{\delta}}) &\le \frac{\delta_2}{4} \\
C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} &\le \frac{\delta_2}{4} \\
\delta_0 h - \epsilon_0 &> 0
\end{aligned} \tag{2.23}$$

where $C_0 = 5 D_\mu \mu_\infty(x) \varphi(b) 2^{N-1}$.

Considering (2.19), we have for the chosen ϵ_0, $\exists R_1$ s.t.

$$\begin{aligned}
&\int_{Q^-(x,r)} |u^+(x) - u(y)|dy \le \epsilon_0 r^N \\
&\int_{Q^+(x,r)} |u^-(x) - u(y)|dy \le \epsilon_0 r^N \\
&\int_{S_u \cap Q(x,r)} \varphi([u]) \le 2\varphi([u](x))(2r)^{N-1} \\
&\lim_{n\to\infty} \int_{Q(x,r)} |\nabla u_n|dx \le \epsilon_0 r^{N-1} \\
&\lim_{n\to\infty} \int_{S_{u_n} \cap Q(x,r)} \varphi([u_n]) \le 2D_\mu \mu_\infty(x) \int_{S_u \cap Q(x,r)} \varphi([u])
\end{aligned} \tag{2.24}$$

for all $r \le R_1$.

Let $R = \min\{R_1, R_2\}$, we see $\forall r < R$, (2.20), (2.21) and (2.24) all hold. In particular $\forall r < R$, we can find $N_1(r) > N_2(r)$ s.t.

$$\int_{S_{u_n} \cap Q(x,r)} \varphi([u_n]) \le (5D_\mu \mu_\infty(x)\varphi(b)2^{N-1})r^{N-1} \quad \forall n > N_1(r). \tag{2.25}$$

Since we are fixing x and r during the rest of the proof, let R_b^a denote $R_b^a(x,r)$ and $H(s)$ denote $H(x,r,s)$. Because of L^1 convergence, we have

$$\int_{R_{-2\delta r}^{-\delta r} \cup R_{\delta r}^{2\delta r}} |u(y) - u_n(y)|dy \to 0$$

as $n \to \infty$. Then consider

$$\begin{aligned}
& \int_{R_{-2\delta r}^{-\delta r}} |u^+(x) - u_n(y)|dy + \int_{R_{\delta r}^{2\delta r}} |u^-(x) - u_n(y)|dy \\
\le\ & \int_{R_{-2\delta r}^{-\delta r}} |u^+(x) - u(y)|dy + \int_{R_{-2\delta r}^{-\delta r}} |u(y) - u_n(y)|dy \\
& + \int_{R_{\delta r}^{2\delta r}} |u^-(x) - u(y)|dy + \int_{R_{\delta r}^{2\delta r}} |u(y) - u_n(y)|dy \\
\le\ & \int_{R_{-2\delta r}^{-\delta r}} |u^+(x) - u(y)|dy + \int_{R_{\delta r}^{2\delta r}} |u^-(x) - u(y)|dy \\
& + \int_{R_{-2\delta r}^{-\delta r} \cup R_{\delta r}^{2\delta r}} |u(y) - u_n(y)|dy.
\end{aligned}$$

It follows that we can find $N(r) > N_1(r)$ s.t.

$$\int_{R_{-2\delta r}^{-\delta r}} |u^+(x) - u_n(y)|dy + \int_{R_{\delta r}^{2\delta r}} |u^-(x) - u_n(y)|dy \le 3\epsilon_0 r^N \qquad \forall n > N(r).$$

But

$$
\begin{aligned}
&\int_{R_{-2\delta r}^{-\delta r}} |u^+(x) - u_n(y)|dy + \int_{R_{\delta r}^{2\delta r}} |u^-(x) - u_n(y)|dy \\
=& \int_{\delta r}^{2\delta r} \left[\int_{H(s)} |T_{u_n}^- - u^-(x)| d\mathcal{H}^{N-1} + \int_{H(-s)} |T_{u_n}^- - u^+(x)| d\mathcal{H}^{N-1} \right] ds.
\end{aligned}
\tag{2.26}
$$

Pick $s_n \in (\delta r, 2\delta r)$ s.t. $\mathcal{H}^{N-1}(H(s_n) \cap S_{u_n}) + \mathcal{H}^{N-1}(H(-s_n) \cap S_{u_n}) = 0$ and

$$
\int_{H(s_n)} |T_{u_n}^- - u^-(x)| d\mathcal{H}^{N-1} + \int_{H(-s_n)} |T_{u_n}^- - u^+(x)| d\mathcal{H}^{N-1} \leq \frac{3\epsilon_0 r^N}{\delta r} = 3\frac{\epsilon_0}{\delta} r^{N-1}.
$$

Let $t \in (u^-(x), u^+(x))$, define the following u_t^n by

$$
u_t^n = \begin{cases} u_n & \Omega \backslash R_{-s_n}^{s_n} \\ u^+(x) & R_{-s_n}^{s_n} \cap E_t^n \\ u^-(x) & R_{-s_n}^{s_n} \backslash E_t^n \end{cases}. \tag{2.27}
$$

We see u_t^n is defined everywhere on Ω and $u_t^n \in SBV(\Omega)$, moreover we have

$$
\int_{S_{u_n} \cup S_{u_t^n}} \tilde{\varphi}([u_t^n], [u_n]) = \int_{\Omega \backslash \overline{R_{-s_n}^{s_n}}} \varphi([u_n]) + \int_{\partial R_{-s_n}^{s_n}} \varphi([u_t^n]) + \int_{R_{-s_n}^{s_n}} \tilde{\varphi}([u_t^n], [u_n]).
$$

Then we see the new crack energy on the boundary of $R_{-s_n}^{s_n}$ looks like following

$$
\begin{aligned}
&\int_{\partial R_{-s_n}^{s_n}} \varphi([u_t^n]) d\mathcal{H}^{N-1} \\
=& \int_{H(s_n)} \varphi(|T_{u_n}^- - u^-(x)|) d\mathcal{H}^{N-1} + \int_{H(-s_n)} \varphi(|T_{u_n}^- - u^+(x)|) d\mathcal{H}^{N-1} \\
&+ \int_{\partial R_{-s_n}^{s_n} \backslash (H(s_n) \cup H(-s_n))} \varphi([u_t^n]) d\mathcal{H}^{N-1} \\
\leq& \left[\varphi(b) \sqrt{3\frac{\epsilon_0}{\delta}} r^{N-1} + \varphi(\sqrt{3\frac{\epsilon_0}{\delta}}) \mathcal{H}^{N-1}(H(s_n) \cup H(-s_n)) + C(N,\delta)\varphi(b) \right] r^{N-1} \\
\leq& \left[\varphi(b) \sqrt{3\frac{\epsilon_0}{\delta}} r^{N-1} + 2^N \varphi(\sqrt{3\frac{\epsilon_0}{\delta}}) + C(N,\delta)\varphi(b) \right] r^{N-1} \\
\leq& \frac{1}{2}\delta_2.
\end{aligned}
\tag{2.28}
$$

The inequalities come from (2.8), (2.9), (2.22) and (2.23).

Next we see $R_{-s_n}^{s_n} \cap S_{u_t^n} = \partial E_t^n \cap R_{-s_n}^{s_n}$ except for a $\mathcal{H}^{N-1}$ measure 0 set and $[u_t^n] = u^+(x) -$

$u^-(x) = h$ on $\partial E_t^n \cap R_{-s_n}^{s_n}$, so

$$\begin{aligned}
&\int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cup\partial^* E_t^n)} \tilde{\varphi}([u_t^n],[u_n])d\mathcal{H}^{N-1} \\
= &\int_{R_{-s_n}^{s_n}\cap(S_{u_n}\setminus\partial^* E_t^n)} \tilde{\varphi}(0,[u_n])d\mathcal{H}^{N-1} + \int_{R_{-s_n}^{s_n}\cap(\partial^* E_t^n\setminus S_{u_n})} \varphi(h)d\mathcal{H}^{N-1} \\
&+ \int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cap\partial^* E_t^n)} \tilde{\varphi}(h,[u_n])d\mathcal{H}^{N-1}.
\end{aligned}$$

But we see

$$\begin{aligned}
&\int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cap\partial^* E_t^n)} \tilde{\varphi}(h,[u_n])d\mathcal{H}^{N-1} \\
= &\int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cap\partial^* E_t^n)\cap\{[u_n]>h\}} \varphi(h,[u_n])d\mathcal{H}^{N-1} \\
&+ \int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cap\partial^* E_t^n)\cap\{[u_n]\le h\}} \varphi(h,[u_n])d\mathcal{H}^{N-1} \\
= &\int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cap\partial^* E_t^n)\cap\{[u_n]>h\}} \left[\varphi([u_n]) - \varphi'([u_n])([u_n]-h)\right] d\mathcal{H}^{N-1} \\
&+ \int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cap\partial^* E_t^n)\cap\{[u_n]\le h\}} \varphi(h)d\mathcal{H}^{N-1}.
\end{aligned}$$

For cleanness let's drop domain $R_{-s_n}^{s_n}$,

$$\begin{aligned}
&\int_{R_{-s_n}^{s_n}\cap(S_{u_n}\cup\partial^* E_t^n)} \tilde{\varphi}([u_t^n],[u_n])d\mathcal{H}^{N-1} \\
= &\int_{S_{u_n}\setminus\partial^* E_t^n} \varphi([u_n])d\mathcal{H}^{N-1} - \int_{S_{u_n}\setminus\partial^* E_t^n} \varphi'([u_n])[u_n]d\mathcal{H}^{N-1} \\
&+ \int_{\partial^* E_t^n\setminus S_{u_n}} \varphi(h)d\mathcal{H}^{N-1} + \int_{S_{u_n}\cap\partial^* E_t^n\cap\{[u_n]>h\}} \varphi([u_n])d\mathcal{H}^{N-1} \\
&- \int_{S_{u_n}\cap\partial^* E_t^n\cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h)d\mathcal{H}^{N-1} \\
&+ \int_{S_{u_n}\cap\partial^* E_t^n\cap\{[u_n]\le h\}} \varphi(h)d\mathcal{H}^{N-1}.
\end{aligned}$$

Let $-\Delta := \int_{R^{s_n}_{-s_n}} \varphi([u_n]) - \int_{R^{s_n}_{-s_n}} \tilde{\varphi}([u^n_t], [u_n])$, we have

$$
\begin{aligned}
& -\Delta \\
= & \int_{S_{u_n}\cap\partial^* E^n_t} \varphi([u_n]) + \int_{S_{u_n}\backslash\partial^* E^n_t} \varphi([u_n]) - \int_{S_{u_n}\backslash\partial^* E^n_t} \varphi([u_n]) + \int_{S_{u_n}\backslash\partial^* E^n_t} \varphi'([u_n])[u_n] \\
& - \int_{\partial^* E^n_t\backslash S_{u_n}} \varphi(h) - \int_{S_{u_n}\cap\partial^* E^n_t\cap\{[u_n]>h\}} \varphi([u_n]) + \int_{S_{u_n}\cap\partial^* E^n_t\cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) \\
& - \int_{S_{u_n}\cap\partial^* E^n_t\cap\{[u_n]\le h\}} \varphi(h) + \int_{S_{u_n}\cap\partial^* E^n_t\cap\{[u_n]\le h\}} \varphi'([u_n])(h-[u_n]) \\
& - \int_{S_{u_n}\cap\partial^* E^n_t\cap\{[u_n]\le h\}} \varphi'([u_n])(h-[u_n]) \\
= & \int_{S_{u_n}} \varphi'([u_n])[u_n] - \int_{S_{u_n}\cap\partial^* E^n_t} \varphi'([u_n])h - \int_{\partial^* E^n_t\backslash S_{u_n}} \varphi(h) \\
& + \int_{S_{u_n}\cap\partial^* E^n_t\cap\{[u_n]\le h\}} \left[\varphi'([u_n])(h-[u_n]) - (\varphi(h)-\varphi([u_n]))\right].
\end{aligned}
$$

Let's fix $t^* \in (u^-(x), u^+(x))$, put back the integral domain $R^{s_n}_{-s_n}$ and split $-\Delta$ to two pieces, $-\Delta = A + B$ where

$$
A := \int_{R^{s_n}_{-s_n}\cap S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \left[\varphi'([u_n])(h-[u_n]) - (\varphi(h)-\varphi([u_n]))\right] d\mathcal{H}^{N-1} \tag{2.29}
$$

and

$$
\begin{aligned}
B \quad := \quad & \int_{R^{s_n}_{-s_n}\cap S_{u_n}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} - \int_{R^{s_n}_{-s_n}\cap S_{u_n}\cap\partial^* E^n_{t^*}} \varphi'([u_n]) h d\mathcal{H}^{N-1} \\
& - \int_{R^{s_n}_{-s_n}\cap(\partial^* E^n_{t^*}\backslash S_{u_n})} \varphi(h) d\mathcal{H}^{N-1}.
\end{aligned}
$$

Considering the co-area formula, B has the following form

$$
\begin{aligned}
B = & \int_{-\infty}^{+\infty} \int_{R^{s_n}_{-s_n}\cap S_{u_n}\cap\partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} dt \\
& - \int_{u_-}^{u^+} \int_{R^{s_n}_{-s_n}\cap S_{u_n}\cap\partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} dt - \int_{R^{s_n}_{-s_n}\cap(\partial^* E^n_{t^*}\backslash S_{u_n})} \varphi(h) d\mathcal{H}^{N-1}.
\end{aligned}
$$

Notice that $A \ge 0$ since $\varphi'([u_n])(h-[u_n]) - (\varphi(h)-\varphi([u_n])) \ge 0$ for all $[u_n] \le h$.

Then define

$$
\begin{aligned}
T^+_{\delta_0} & := \{u^- < t < u^+ : \mathcal{H}^{N-1}(R^{s_n}_{-s_n} \cap (\partial^* E^n_t \backslash S_{u_n})) \ge \delta_0 r^{N-1}\} \\
T^-_{\delta_0} & := \{u^- < t < u^+ : \mathcal{H}^{N-1}(R^{s_n}_{-s_n} \cap (\partial^* E^n_t \backslash S_{u_n})) < \delta_0 r^{N-1}\}.
\end{aligned}
$$

Notice $T^+_{\delta_0}$ is defined differently when δ_0 is a scaler. Since $\int_{R^{s_n}_{-s_n}} |\nabla u_n| dx \le \epsilon_0 r^{N-1}$, after applying co-area formula again we get

$$\begin{aligned}
&\int_{T^+_{\delta_0}} \mathcal{H}^{N-1}(R^{s_n}_{-s_n} \cap (\partial^* E^n_t \backslash S_{u_n}))dt + \int_{T^-_{\delta_0}} \mathcal{H}^{N-1}(R^{s_n}_{-s_n} \cap (\partial^* E^n_t \backslash S_{u_n}))dt \\
= & \int_{u^-}^{u^+} \mathcal{H}^{N-1}(R^{s_n}_{-s_n} \cap (\partial^* E^n_t \backslash S_{u_n}))dt \\
< & \ \epsilon_0 r^{N-1}.
\end{aligned}$$

It follows $|T^+_{\delta_0}| \leq \frac{\epsilon_0}{\delta_0}$, $|T^-_{\delta_0}| > h - \frac{\epsilon_0}{\delta_0}$. And since $h - \frac{\epsilon_0}{\delta_0} > 0$, we have $\frac{|T^+_{\delta_0}|}{|T^-_{\delta_0}|} \leq \frac{\epsilon_0}{\delta_0 h - \epsilon_0}$.

For fixed δ_0 and n we always let $t^*(\delta_0, n) \in T^-_{\delta_0}$ be specifically chosen s.t.

$$\int_{T^-_{\delta_0}} \left[\int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} - \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} \right] dt \geq 0.$$

We see that it's possible. From now on let t^* always be chosen that way.

Because of that, we have

$$\int_{T^-_{\delta_0}} \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} dt \geq |T^-_{\delta_0}| \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1}$$

and thus

$$\frac{\int_{T^-_{\delta_0}} \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} dt}{|T^-_{\delta_0}|} \geq \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1}.$$

It follows

$$\begin{aligned}
&|T^+_{\delta_0}| \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} \\
\leq & \ \frac{|T^+_{\delta_0}|}{|T^-_{\delta_0}|} \int_{T^-_{\delta_0}} \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} dt \\
\leq & \ \frac{|T^+_{\delta_0}|}{|T^-_{\delta_0}|} \int_{u^-}^{u^+} \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} dt \\
\leq & \ \frac{|T^+_{\delta_0}|}{|T^-_{\delta_0}|} \int_{R^{s_n}_{-s_n} \cap S_{u_n}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \\
\leq & \ \frac{|T^+_{\delta_0}|}{|T^-_{\delta_0}|} \int_{R^{s_n}_{-s_n} \cap S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \\
\leq & \ \frac{\epsilon_0}{\delta_0 h - \epsilon_0} \int_{S_{u_n} \cap Q(x,r)} \varphi([u_n]) \\
\leq & \ C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} r^{N-1}
\end{aligned} \tag{2.30}$$

where $C_0 = 5D_\mu \mu_\infty(x)\varphi(b)2^{N-1} > 0$, regarding (2.25). Next consider

$$\begin{aligned}
& B \\
\geq\ & \int_{u^-}^{u^+} \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} dt - \int_{u_-}^{u^+} \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} dt \\
& - \int_{R^{s_n}_{-s_n} \cap (\partial^* E^n_{t^*} \setminus S_{u_n})} \varphi(h) d\mathcal{H}^{N-1} \\
\geq\ & \int_{T^+_{\delta_0}} \left[\int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} - \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} \right] dt \\
& + \int_{T^-_{\delta_0}} \left[\int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} - \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} \right] dt \\
& - \int_{R^{s_n}_{-s_n} \cap (\partial^* E^n_{t^*} \setminus S_{u_n})} \varphi(h) d\mathcal{H}^{N-1} \\
\geq\ & \int_{T^+_{\delta_0}} \left[\int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_t} \varphi'([u_n]) d\mathcal{H}^{N-1} - \int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} \right] dt \\
& - \int_{R^{s_n}_{-s_n} \cap (\partial^* E^n_{t^*} \setminus S_{u_n})} \varphi(h) d\mathcal{H}^{N-1} \\
\geq\ & - \int_{T^+_{\delta_0}} \left[\int_{R^{s_n}_{-s_n} \cap S_{u_n} \cap \partial^* E^n_{t^*}} \varphi'([u_n]) d\mathcal{H}^{N-1} \right] dt - \int_{R^{s_n}_{-s_n} \cap (\partial^* E^n_{t^*} \setminus S_{u_n})} \varphi(h) d\mathcal{H}^{N-1}.
\end{aligned}$$

According to (2.30) and the fact that t^* is always chosen in $T^-_{\delta_0}$, we have

$$\begin{aligned}
B\ &\geq\ -C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} r^{N-1} - \varphi(h)\delta_0 r^{N-1} \geq (-C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} - \varphi(b)\delta_0) r^{N-1} \\
&\geq\ -\frac{\delta_2}{2} r^{N-1}.
\end{aligned} \tag{2.31}$$

Then consider the energy drop $E(u_n, u_n) - E(u^n_{t^*}, u_n)$ where

$$E(u_n, u_n) = \int_\Omega |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1}$$

and

$$\begin{aligned}
& E(u^n_{t^*}, u_n) \\
=\ & \int_\Omega |\nabla u^n_{t^*}|^2 dx + \int_{S_{u_n} \cup S_{u^n_{t^*}}} \tilde{\varphi}([u^n_{t^*}], [u_n]) d\mathcal{H}^{N-1} \\
=\ & \int_{\Omega \setminus R^{s_n}_{-s_n}} |\nabla u_n|^2 dx + \int_{\Omega \setminus \overline{R^{s_n}_{-s_n}}} \varphi([u_n]) + \int_{\partial R^{s_n}_{-s_n}} \varphi([u_n]) + \int_{R^{s_n}_{-s_n}} \tilde{\varphi}([u^n_{t^*}], [u_n]) \\
\leq\ & \int_{\Omega \setminus R^{s_n}_{-s_n}} |\nabla u_n|^2 dx + \int_{\Omega \setminus \overline{R^{s_n}_{-s_n}}} \varphi([u_n]) + \int_{R^{s_n}_{-s_n}} \tilde{\varphi}([u^n_{t^*}], [u_n]) + \frac{1}{2}\delta_2 r^{N-1}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&E(u_n,u_n)-E(u^n_{t^*},u_n)\\
\geq&\int_{R^{s_n}_{-s_n}}\varphi([u_n])-\int_{R^{s_n}_{-s_n}}\tilde{\varphi}([u^n_t],[u_n])-\frac{1}{2}\delta_2 r^{N-1}\\
=&-\Delta-\frac{1}{2}\delta_2 r^{N-1}\\
=&A+B-\frac{1}{2}\delta_2 r^{N-1}
\end{aligned}\tag{2.32}$$

considering the definition $-\Delta$. Then, according to unilateral minimality, we see $E(u_n,u_n)-E(u^n_{t^*},u_n)\leq 0$, and thus

$$A\leq -B+\frac{1}{2}\delta_2 r^{N-1}\leq \delta_2 r^{N-1}$$

considering (2.31). That is

$$\int_{R^{s_n}_{-s_n}\cap S_{u_n}\cap\partial^*E^n_{t^*}\cap\{[u_n]\leq h\}}\left[\varphi'([u_n])(h-[u_n])-(\varphi(h)-\varphi([u_n]))\right]d\mathcal{H}^{N-1}\leq\delta_2 r^{N-1}.$$

Due to (2.20) we have

$$\int_{R^{s_n}_{-s_n}\cap S_{u_n}\cap\partial^*E^n_{t^*}\cap\{[u_n]\leq h\}}(\varphi(h)-\varphi([u_n]))d\mathcal{H}^{N-1}\leq\delta_1 r^{N-1}\leq\epsilon r^{N-1}.$$

Therefore one direction has been proved.

Considering (2.32) and the fact that $A\geq 0$, we have

$$B\leq\frac{1}{2}\delta_2 r^{N-1}.$$

For clearness let's again drop the integral domain $R^{s_n}_{-s_n}$ for now. Consider the following

$$\begin{aligned}
B\ =&\ \int_{S_{u_n}}\varphi'([u_n])[u_n]-\int_{S_{u_n}\cap\partial^*E^n_{t^*}}\varphi'([u_n])h-\int_{\partial^*E^n_{t^*}\setminus S_{u_n}}\varphi(h)\\
=&\ \int_{S_{u_n}\cap\partial^*E^n_{t^*}}\varphi'([u_n])[u_n]+\int_{S_{u_n}\setminus\partial^*E^n_{t^*}}\varphi'([u_n])[u_n]-\int_{S_{u_n}\cap\partial^*E^n_{t^*}}\varphi'([u_n])h\\
&-\int_{\partial^*E^n_{t^*}\setminus S_{u_n}}\varphi(h)\\
=&\ \int_{S_{u_n}\cap\partial^*E^n_{t^*}\cap\{[u_n]>h\}}\varphi'([u_n])([u_n]-h)+\int_{S_{u_n}\cap\partial^*E^n_{t^*}\cap\{[u_n]\leq h\}}\varphi'([u_n])([u_n]-h)\\
&+\int_{S_{u_n}\setminus\partial^*E^n_{t^*}}\varphi'([u_n])[u_n]-\int_{\partial^*E^n_{t^*}\setminus S_{u_n}}\varphi(h)\\
\geq&\ \int_{S_{u_n}\cap\partial^*E^n_{t^*}\cap\{[u_n]\leq h\}}\varphi'([u_n])([u_n]-h)+\int_{S_{u_n}\setminus\partial^*E^n_{t^*}}\varphi'([u_n])[u_n]-\int_{\partial^*E^n_{t^*}\setminus S_{u_n}}\varphi(h).
\end{aligned}$$

First we see $\int_{\partial^* E^n_{t^*}\setminus S_{u_n}} \varphi(h) \le \varphi(b)\delta_0 r^{N-1} \le \frac{\delta_2}{4} r^{N-1}$. With a bit of manipulation we get

$$\begin{aligned}
&\int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \\
\le\ & \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \varphi'([u_n])(h-[u_n]) + \frac{3}{4}\delta_2 r^{N-1} \\
\le\ & \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \left[\varphi'([u_n])(h-[u_n]) - (\varphi(h)-\varphi([u_n]))\right] \\
& + \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} [\varphi(h)-\varphi([u_n])] + \frac{3}{4}\delta_2 r^{N-1} \\
\le\ & 3\delta_1 r^{N-1}.
\end{aligned}$$

(2.20) implies

$$\int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi([u_n]) \le \epsilon r^{N-1}.$$

Thus (2.15) has been proved. Next consider

$$\begin{aligned}
& -\Delta \\
=\ & \int_{S_{u_n}\cap\partial^* E^n_{t^*}} \varphi([u_n]) + \int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi([u_n]) - \int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi([u_n]) + \int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi'([u_n])[u_n] \\
& - \int_{\partial^* E^n_{t^*}\setminus S_{u_n}} \varphi(h) - \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi([u_n]) + \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) \\
& - \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \varphi(h) \\
=\ & \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi([u_n]) + \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \varphi([u_n]) + \int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi'([u_n])[u_n] \\
& - \int_{\partial^* E^n_{t}\setminus S_{u_n}} \varphi(h) - \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi([u_n]) + \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) \\
& - \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \varphi(h) \\
=\ & \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \varphi([u_n]) + \int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi'([u_n])[u_n] - \int_{\partial^* E^n_{t^*}\setminus S_{u_n}} \varphi(h) \\
& + \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) - \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} \varphi(h) \\
=\ & \int_{S_{u_n}\setminus\partial^* E^n_{t^*}} \varphi'([u_n])[u_n] + \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) \\
& - \int_{S_{u_n}\cap\partial^* E^n_{t^*}\cap\{[u_n]\le h\}} [\varphi(h)-\varphi([u_n])] - \int_{\partial^* E^n_{t^*}\setminus S_{u_n}} \varphi(h).
\end{aligned}$$

Again consider

$$0 \geq E(u_n, u_n) - E(u^n_{t^*}, u_n) \geq -\Delta - \frac{1}{2}\delta_2 r^{N-1}$$

we have

$$\begin{aligned} & \int_{S_{u_n}\cap\partial^* E^n_t \cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) \\ \leq & \int_{S_{u_n}\cap\partial^* E^n_t \cap\{[u_n]\leq h\}} [\varphi(h)-\varphi([u_n])] + \int_{\partial^* E^n_t \setminus S_{u_n}} \varphi(h) + \frac{1}{2}\delta_2 r^{N-1} \\ \leq & \frac{7}{4}\delta_1 r^{N-1}. \end{aligned} \tag{2.33}$$

Next consider

$$\begin{aligned} & \int_{S_{u_n}\cap\partial^* E^n_{t^*} \cap\{[u_n]>h\}} (\varphi([u_n])-\varphi(h)) \\ \leq & \int_{S_{u_n}\cap\partial^* E^n_{t^*} \cap\{[u_n]>h\}} (\frac{\varphi([u_n])}{[u_n]}[u_n]-\varphi(h)) \\ \leq & \int_{S_{u_n}\cap\partial^* E^n_{t^*} \cap\{[u_n]>h\}} (\frac{\varphi(h)}{h}[u_n]-\varphi(h)) \\ \leq & \frac{\varphi(h)}{h}\int_{S_{u_n}\cap\partial^* E^n_{t^*} \cap\{[u_n]>h\}} ([u_n]-h) \\ \leq & \frac{\varphi(h)}{\varphi'(M)h}\int_{S_{u_n}\cap\partial^* E^n_{t^*} \cap\{[u_n]>h\}} \varphi'(b)([u_n]-h) \\ \leq & \frac{\varphi(h)}{\varphi'(b)h}\int_{S_{u_n}\cap\partial^* E^n_{t^*} \cap\{[u_n]>h\}} \varphi'([u_n])([u_n]-h) \\ \leq & \frac{\varphi(h)}{\varphi'(b)h}\frac{7}{4}\delta_1 r^{N-1} \\ \leq & \frac{\varphi(h)}{\varphi'(b)h}\frac{7}{4}\epsilon r^{N-1}. \end{aligned}$$

The last estimate comes from (2.33). Thus the other direction has been proved. (2.15), (2.16), (2.17) and (2.18) should follow directly. □

2.3.3 Proof of Minimality with α

Define

$$\alpha(x) = \begin{cases} D_\mu \mu_\infty(x) & x \in S_u \\ 1 & \text{elsewhere} \end{cases}.$$

Lemma 13. *Let $v \in SBV(\Omega')$. For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ and $\forall \epsilon > 0$, $\exists \delta$ and R s.t. $\forall r < R$, $\exists \{v_n\}_{n=1}^\infty \subset SBV(\Omega')$ and $\{s_n\}_{n=1}^\infty \subset (\delta r, 2\delta r)$ s.t.*

$$\lim_{n\to\infty} \int_{(S_{v_n}\cup S_{u_n})R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) \leq \int_{(S_u\cup S_v)\cap R^{2\delta r}_{-2\delta r}(x,r)} \alpha\tilde{\varphi}([v],[u]) + \epsilon r^{N-1}. \tag{2.34}$$

Moreover

$$
\begin{aligned}
\nabla v_u &= 0 \qquad \text{on } R^{s_n}_{-s_n}(x,r) && (2.35)\\
v_n &= v \qquad \text{on } \Omega' \backslash R^{s_n}_{-s_n}(x,r) && (2.36)\\
\lim_{n\to\infty}\int_{\partial R^{s_n}_{-s_n}(x,r)} \varphi([v_n])d\mathcal{H}^{N-1} &\le \epsilon r^{N-1}. && (2.37)
\end{aligned}
$$

Proof. For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$

$$
\begin{aligned}
\alpha(x) &= \lim_{r\to 0}\frac{\int_{Q(x,r)}\alpha\varphi([u])}{\int_{Q(x,r)}\varphi([u])} = \lim_{r\to 0}\frac{\int_{Q(x,r)}\alpha\varphi([u])}{\varphi([u](x))(2r)^{N-1}} = \lim_{r\to 0}\frac{\int_{R^{\delta r}_{-\delta r}(x,r)}\alpha\varphi([u])}{\varphi([u](x))(2r)^{N-1}}\\
&= \lim_{r\to 0}\frac{\lim_{n\to\infty}\int_{R^{\delta r}_{-\delta r}(x,r)}\varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} \qquad (2.38)\\
\alpha(x) &= \lim_{r\to 0}\frac{\lim_{n\to\infty}\int_{R^{2\delta r}_{-2\delta r}(x,r)}\varphi([u_n])}{\varphi([u](x))(2r)^{N-1}}
\end{aligned}
$$

and

$$
\alpha(x)\tilde{\varphi}([v](x),[u](x)) = \lim_{r\to 0}\frac{\int_{R^{\delta r}_{-\delta r}(x,r)}\alpha\tilde{\varphi}([v],[u])d\mathcal{H}^{N-1}}{(2r)^{N-1}} \qquad (2.39)
$$

for $\forall\delta < \frac{1}{2}$.

Thus $\forall\delta < \frac{1}{2}$ and $\forall\epsilon > 0$, $\exists R$ s.t.

$$
\begin{aligned}
\left|\alpha(x) - \frac{\lim_{n\to\infty}\int_{R^{\delta r}_{-\delta r}(x,r)}\varphi([u_n])}{\varphi([u](x))(2r)^{N-1}}\right| &\le \epsilon\\
\left|\alpha(x) - \frac{\lim_{n\to\infty}\int_{R^{2\delta r}_{-2\delta r}(x,r)}\varphi([u_n])}{\varphi([u](x))(2r)^{N-1}}\right| &\le \epsilon\\
\left|\alpha(x)\tilde{\varphi}([v](x),[u](x)) - \frac{\int_{R^{\delta r}_{-\delta r}(x,r)}\alpha\tilde{\varphi}([v],[u])d\mathcal{H}^{N-1}}{(2r)^{N-1}}\right| &\le \epsilon \qquad (2.40)\\
\left|\alpha(x)\tilde{\varphi}([v](x),[u](x)) - \frac{\int_{R^{2\delta r}_{-2\delta r}(x,r)}\alpha\tilde{\varphi}([v],[u])d\mathcal{H}^{N-1}}{(2r)^{N-1}}\right| &\le \epsilon
\end{aligned}
$$

for all $r < R$.

Then consider lemma (5), $\forall\epsilon > 0$ $\exists\delta > 0$ and $R > 0$ s.t. $\forall r < R$ there exists $\{t_n\}_{n=1}^{\infty} \subset (u^-, u^+)$

and $\{s_n\}_{n=1}^{\infty} \subset (\delta r, 2\delta r)$ s.t.

$$\lim_{n\to\infty} \int_{S_{u_n}\cap R^{s_n}_{-s_n}(x,r)\cap\partial^* E^n_{t_n}} |\varphi([u_n]) - \varphi([u](x))| d\mathcal{H}^{N-1} \le \epsilon r^{N-1} \tag{2.41}$$

$$\lim_{n\to\infty} \int_{(S_{u_n}\backslash\partial^* E^n_{t_n})\cap R^{s_n}_{-s_n}(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} \le \epsilon r^{N-1} \tag{2.42}$$

$$\lim_{n\to\infty} \mathcal{H}^{N-1}((\partial^* E^n_{t_n}\backslash S_{u_n}) \cap R^{s_n}_{-s_n}(x,r)) \le \epsilon r^{N-1} \tag{2.43}$$

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(\partial R^{s_n}_{-s_n}(x,r)\backslash(H(x,r,-s_n)\cup H(x,r,s_n)) \le \epsilon r^{N-1} \tag{2.44}$$

$$\lim_{n\to\infty} \left| \int_{S_{u_n}\cap R^{s_n}_{-s_n}(x,r)} \varphi([u_n]) d\mathcal{H}^{N-1} - \mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r))\varphi([u](x)) \right| \le \epsilon r^{N-1} \tag{2.45}$$

$$\lim_{n\to\infty} \int_{H(x,r,-s_n)} \varphi(|T^-_v - v^+(x)|) + \int_{H(x,r,s_n)} \varphi(|T^-_v - v^-(x)|) \le \epsilon r^{N-1}. \tag{2.46}$$

We see (2.46) can be shown using the same argument we used in (2.26). Regarding lemma (11) we also have

$$\lim_{n\to\infty} \int_{S_{u_n}\cap\{[u_n]\ge[u](x)\}\cap R^{s_n}_{-s_n}(x,r)\cap\partial^* E^n_{t_n}} \big(\varphi'([u](x)) - \varphi'([u_n])\big) \le \epsilon r^{N-1}. \tag{2.47}$$

Next consider

$$\begin{aligned}
& \frac{\mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r))}{(2r)^{N-1}} - \alpha(x) \\
= \quad & \frac{\mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r))}{(2r)^{N-1}} - \frac{\int_{R^{2\delta r}_{-2\delta r}(x,r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} \\
& + \frac{\int_{R^{2\delta r}_{-2\delta r}(x,r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} - \alpha(x) \\
\le \quad & \frac{\mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r))}{(2r)^{N-1}} - \frac{\int_{R^{s_n}_{-s_n}(x,r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} \\
& + \frac{\int_{R^{2\delta r}_{-2\delta r}(x,r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} - \alpha(x) \\
\le \quad & \frac{1}{\varphi([u](x))2^{N-1}} \epsilon + \epsilon.
\end{aligned} \tag{2.48}$$

The last inequality comes respectively from (2.45) and (2.40).

Next construct SBV function v_n as follows

$$v_n = \begin{cases} v & \text{on } \Omega\backslash R^{s_n}_{-s_n}(x,r) \\ v^+(x) & \text{on } R^{s_n}_{-s_n}(x,r) \cap E^n_{t_n} \\ v^-(x) & \text{on } R^{s_n}_{-s_n}(x,r)\backslash E^n_{t_n} \end{cases}.$$

First we see, according to (2.46) and (2.44), that

$$\int_{\partial R^{s_n}_{-s_n}(x,r)} \varphi([v_n]) \leq (\epsilon\varphi(b)+\epsilon)r^{N-1},$$

which proves (2.37).

Second we see

$$\begin{aligned}
& \int_{(S_{v_n}\cup S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) \\
= & \int_{(\partial^* E^n_{t_n}\cup S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \varphi(|v^+(x)-v^-(x)|,[u_n]) \\
= & \int_{(\partial^* E^n_{t_n}\cup S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v](x),[u_n]) \\
= & \int_{(\partial^* E^n_{t_n}\setminus S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \varphi([v](x)) + \int_{(S_{u_n}\setminus \partial^* E^n_{t_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}(0,[u_n]) \\
& + \int_{(\partial^* E^n_{t_n}\cap S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v](x),[u_n]) \\
\leq & (\varphi(b)\epsilon+\epsilon)r^{N-1} + \int_{(\partial^* E^n_{t_n}\cap S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v](x),[u_n]).
\end{aligned}$$

Denote $A_n = (\partial^* E^n_{t_n}\cap S_{u_n})\cap R^{s_n}_{-s_n}(x,r)$, let's consider

$$\begin{aligned}
\Delta \ := & \int_{A_n} \tilde{\varphi}([v](x),[u_n]) - \int_{A_n} \tilde{\varphi}([v](x),[u](x)) \\
\leq & \int_{A_n\cap\{[u_n]>[u](x)\}} \tilde{\varphi}([v](x),[u_n]) - \int_{A_n\cap\{[u_n]>[u](x)\}} \tilde{\varphi}([v](x),[u](x)).
\end{aligned}$$

If $[v](x)\leq [u](x)$

$$\begin{aligned}
\Delta \ \leq & \int_{A_n\cap\{[u_n]>[u](x)\}} \big(\varphi([u_n]) - \varphi'([u_n])([u_n]-[v](x)) - \varphi([u](x)) + \varphi'([u](x))([u](x)-[v](x))\big) \\
\leq & \int_{A_n\cap\{[u_n]>[u](x)\}} (\varphi([u_n])-\varphi([u](x))) + ([u](x)-[v](x)) \int_{A_n\cap\{[u_n]>[u](x)\}} \big[\varphi'([u](x))-\varphi'([u_n])\big] \\
\leq & (\epsilon+b\epsilon)r^{N-1},
\end{aligned}$$

the last inequality is due to (2.47).

If $[v](x) > [u](x)$,

$$\begin{aligned}\Delta &\leq \int_{A_n\cap\{[u_n]>[u](x)\}} \tilde{\varphi}([v](x),[u_n]) - \int_{A_n\cap\{[u_n]>[u](x)\}} \varphi([v](x)) \\ &\leq \int_{A_n\cap\{[u_n]>[v](x)\}} (\tilde{\varphi}([v](x),[u_n]) - \varphi([v](x))) \\ &\leq \int_{A_n\cap\{[u_n]>[v](x)\}} (\varphi([u_n]) - \varphi([v](x))) \\ &\leq \int_{A_n\cap\{[u_n]>[v](x)\}} (\varphi([u_n]) - \varphi([u](x))) \\ &\leq \epsilon r^{N-1}.\end{aligned}$$

It follows

$$\begin{aligned}&\int_{(S_{v_n}\cup S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) \\ &\quad - \int_{(\partial^* E^n_{t_n}\cap S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v](x),[u](x)) \\ &\leq \int_{(\partial^* E^n_{t_n}\cap S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) \\ &\quad - \int_{(\partial^* E^n_{t_n}\cap S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v](x),[u](x)) \\ &\quad + (\varphi(b)\epsilon + \epsilon) r^{N-1} \\ &\leq (2\epsilon + b\epsilon + \varphi(b)\epsilon + \epsilon) r^{N-1} \\ &= (3\epsilon + b\epsilon + \varphi(b)\epsilon) r^{N-1}.\end{aligned} \tag{2.49}$$

It follows

$$\begin{aligned}&\int_{(S_{v_n}\cup S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) - \int_{(S_u\cup S_v)\cap R^{s_n}_{-s_n}(x,r)} \alpha\tilde{\varphi}([v],[u]) \\ = &\int_{(S_{v_n}\cup S_{u_n})\cap R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) - \mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r))\tilde{\varphi}([v](x),[u](x)) \\ &+\mathcal{H}^{N-1}(\partial^* E^n_{t_n} \cap R^{s_n}_{-s_n}(x,r))\tilde{\varphi}([v](x),[u](x)) - \alpha(x)\tilde{\varphi}([v](x),[u](x))(2r)^{N-1} \\ &+\alpha(x)\tilde{\varphi}([v](x),[u](x))(2r)^{N-1} - \int_{(S_u\cup S_v)\cap R^{s_n}_{-s_n}(x,r)} \alpha\tilde{\varphi}([v],[u]) \\ \leq\ &(3\epsilon + b\epsilon + \varphi(b)\epsilon) r^{N-1} + \left[\frac{\tilde{\varphi}([v](x),[u](x))}{\varphi([u](x))}\epsilon + \epsilon\tilde{\varphi}([v](x),[u](x))2^{N-1}\right] r^{N-1} \\ &+2^{N-1}\epsilon r^{N-1} \\ \leq\ &\mathcal{O}(\epsilon) r^{N-1}\end{aligned}$$

regarding (2.49), (2.48) and (2.40). Thus we have proved the main result. (2.35) and (2.36) are due to construction of v_n. □

Once lemma (13) has been proved, it is straight forward to show the following result using Besicovitch's covering theorem.

Theorem 6.

$$\int_{\Omega'} |\nabla u|^2 dx + \int_{S_u} \alpha\varphi([u]) d\mathcal{H}^{N-1} \leq \int_{\Omega'} |\nabla v|^2 dx + \int_{S_u \cup S_v} \alpha\tilde{\varphi}([v],[u]) d\mathcal{H}^{N-1}$$

for all $v \in SBV(\Omega)$ *with* $v = u$ *on* $\Omega' \backslash \Omega$*.*

Proof. Recall that $\mu_n := \varphi([u_n])\mathcal{H}^{N-1}\lfloor S_{u_n} \overset{*}{\rightharpoonup} \mu_\infty$, the weak limit is absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor S_u$. Let $\mu_\infty = \xi\mathcal{H}^{N-1}\lfloor S_u$ where ξ is $\mathcal{H}^{N-1}$ measurable on S_u. Then define Radon measure

$$w := \alpha\tilde{\varphi}([v],[u])\mathcal{H}^{N-1}\lfloor S_u + \xi\mathcal{H}^{N-1}\lfloor S_u.$$

Fix $\epsilon > 0$, let U be open s.t. $S_u \subset U$ and $|U| << \epsilon$. Then consider the collection of closed cubes $\mathcal{F} : \{Q(x,r) : Q(x,r)$ satisfies the following $\}$

1. $x \in S_u$.
2. $Q(x,r)$ is oriented by $\nu(x)$.
3. $Q(x,r) \subset U$.
4. $\int_{Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)} \varphi([v]) \leq \epsilon\varphi([u](x))r^{N-1}$.
5. $\int_{Q(x,r)\backslash R^{\delta r}_{-\delta r}(x,r)} \xi \leq \epsilon\varphi([u](x))r^{N-1}$.

and

$$\begin{aligned} &\lim_{n\to\infty} \int_{(S_{v_n} \cup S_{u_n}) R^{s_n}_{-s_n}(x,r)} \tilde{\varphi}([v_n],[u_n]) \\ \leq &\int_{(S_u \cup S_v)\cap R^{2\delta r}_{-2\delta r}(x,r)} \alpha\tilde{\varphi}([v],[u]) + \epsilon\varphi([u](x))r^{N-1}. \end{aligned} \tag{2.50}$$

Here v_n, s_n and δ are the same as in lemma (13). Then according to Bescovitch covering theorem, we can find a countable family of disjoint closed cubes $\bigcup_{i=1}^\infty Q(x_i,r_i) \subset \mathcal{F}$ s.t. $w(S_u \backslash \bigcup_{i=1}^\infty Q(x_i,r_i)) = 0$. Pick an integer J s.t.

$$w(S_u \backslash \bigcup_{i=1}^J Q(x_i,r_i)) < \epsilon. \tag{2.51}$$

Since $Q(x_i,r_i)$ are disjoint, let's define

$$\tilde{v}_n = \begin{cases} v_n^i & \text{on each } Q(x_i,r_i) \\ v & \text{on } \Omega' \backslash \bigcup_{i=1}^J Q(x_i,r_i) \end{cases}.$$

During the following we let n be big enough. It follows

$$\begin{aligned} & \int_{\Omega'} |\nabla \tilde{v}_n|^2 dx + \int_{S_{\tilde{v}_n} \cup S_{u_n}} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{\Omega'} |\nabla v|^2 dx - \int_{S_v \cup S_u} \alpha \tilde{\varphi}([v], [u]) \\ \leq & \int_{S_{\tilde{v}_n} \cup S_{u_n}} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{S_v \cup S_u} \alpha \tilde{\varphi}([v], [u]) \\ \leq & \int_{S_{\tilde{v}_n} \cup S_{u_n} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \tilde{\varphi}([v], [u_n]) - \int_{S_v \cup S_u \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \alpha \tilde{\varphi}([v], [u]) \\ & + \int_{S_{\tilde{v}_n} \cup S_{u_n} \cap \bigcup_i Q(x_i, r_i)} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{S_v \cup S_u \cap \bigcup_i Q(x_i, r_i)} \alpha \tilde{\varphi}([v], [u]). \end{aligned}$$

We see

$$\begin{aligned} & \int_{S_v \cup S_{u_n} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \tilde{\varphi}([v], [u_n]) - \int_{S_v \cup S_u \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \alpha \tilde{\varphi}([v], [u]) \\ \leq & \int_{S_v \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([v]) + \int_{S_{u_n} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([u_n]) \\ & - \int_{S_v \cup S_u \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \alpha \tilde{\varphi}([v], [u]) \\ \leq & \int_{S_{u_n} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([u_n]). \end{aligned}$$

Due to (2.51), $\lim_{n \to \infty} \int_{S_{u_n} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([u_n]) \leq \epsilon$.

Then we see

$$\begin{aligned} & \int_{S_{\tilde{v}_n} \cup S_{u_n} \cap \bigcup_i Q(x_i, r_i)} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{S_v \cup S_u \cap \bigcup_i Q(x_i, r_i)} \alpha \tilde{\varphi}([v], [u]) \\ = & \int_{\bigcup_i Q(x_i, r_i) \setminus \bigcup R^{s_n^i}_{-s_n^i}} \tilde{\varphi}([v], [u_n]) - \int_{\bigcup_i Q(x_i, r_i) \setminus \bigcup R^{2\delta r_i}_{-2\delta r_i}} \alpha \tilde{\varphi}([v], [u]) \\ & + \int_{\bigcup R^{s_n^i}_{-s_n^i}} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{\bigcup R^{2\delta r_i}_{-2\delta r_i}} \alpha \tilde{\varphi}([v], [u]). \end{aligned}$$

It follows

$$\begin{aligned} & \int_{\bigcup R^{s_n^i}_{-s_n^i}} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{\bigcup R^{2\delta r_i}_{-2\delta r_i}} \alpha \tilde{\varphi}([v], [u]) \\ = & \sum_{i=1}^{J} \int_{R^{s_n^i}_{-s_n^i}} \tilde{\varphi}([v_n^i], [u_n]) - \sum_{i=1}^{J} \int_{R^{2\delta r_i}_{-2\delta r_i}} \alpha \tilde{\varphi}([v], [u]) \\ \leq & \ \epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1}. \end{aligned}$$

The inequality is due to (2.50). And

$$\begin{aligned}
& \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{s_n^i}_{-s_n^i}} \tilde{\varphi}([v],[u_n]) - \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{2\delta r_i}_{-2\delta r_i}} \alpha\tilde{\varphi}([v],[u]) \\
\leq\ & \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{s_n^i}_{-s_n^i}} \tilde{\varphi}([v],[u_n]) \\
\leq\ & \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{s_n^i}_{-s_n^i}} \varphi([v]) + \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{s_n^i}_{-s_n^i}} \varphi([u_n]) \\
\leq\ & \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{s_n^i}_{-s_n^i}} \varphi([v]) + \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{s_n^i}_{-s_n^i}} \varphi([u_n]) \\
\leq\ & \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{\delta r_i}_{-\delta r_i}} \varphi([v]) + \int_{\bigcup_i Q(x_i,r_i)\setminus\bigcup R^{\delta r_i}_{-\delta r_i}} \varphi([u_n]) \\
\leq\ & \sum_{i=1}^{J} \int_{Q(x_i,r_i)\setminus\bigcup R^{\delta r_i}_{-\delta r_i}} \varphi([v]) + \sum_{i=1}^{J} \int_{Q(x_i,r_i)\setminus\bigcup R^{\delta r_i}_{-\delta r_i}} \varphi([u_n]) \\
\leq\ & \epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1} + \epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1}.
\end{aligned}$$

The last inequality is due to condition (4) and (5) of the chosen cubes.

To sum up all the estimates, we see

$$\begin{aligned}
& \int_{\Omega'} |\nabla \tilde{v}_n|^2 dx + \int_{S_{\tilde{v}_n}\cup S_{u_n}} \tilde{\varphi}([\tilde{v}_n],[u_n]) - \int_{\Omega'} |\nabla v|^2 dx - \int_{S_v\cup S_u} \alpha\tilde{\varphi}([v],[u]) \\
\leq\ & \epsilon + 3\epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1} \\
\leq\ & \mathcal{O}(\epsilon).
\end{aligned}$$

We see $\sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1} \to \int_{S_u} \varphi([u]) d\mathcal{H}^{N-1}$ as we choose finer cover of S_u.

According to lower-semi-continuity and unilateral minimality

$$\begin{aligned}
& \int_{\Omega'} |\nabla u|^2 dx + \int_{S_u} \alpha\varphi([u]) d\mathcal{H}^{N-1} \\
\leq\ & \liminf_{n\to\infty} \left[\int_{\Omega'} |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \right] \\
\leq\ & \liminf_{n\to\infty} \left[\int_{\Omega'} |\nabla \tilde{v}_n|^2 dx + \int_{S_{\tilde{v}_n}\cup S_{u_n}} \varphi([\tilde{v}_n][u_n]) d\mathcal{H}^{N-1} \right] \\
\leq\ & \int_{\Omega'} |\nabla v|^2 dx + \int_{S_u\cup S_v} \alpha\tilde{\varphi}([v],[u]) d\mathcal{H}^{N-1} + \mathcal{O}(\epsilon).
\end{aligned}$$

□

This concludes our main result of the section. We show that the crack energy of u_n concentrates on reduced boundary of some level sets. And due to this condition, the density function α only depends on the oscillation, if there's any. Moreover we can show that the density α can be passed to the minimality independent of the choose of test function v.

Chapter 3

The Evolution Problem

3.1 Convergence of the Energy

Just by looking at a sequence of unilateral minimizers is not enough to exclude the oscillation complication. We then move on to the evolution problem because the history after all is union of all cracks from the previous minimizers at the discrete time. The problem is as approximating steps becomes more and more the minimizers become hard to control. So we hope to find finitely many minimizers that could represent the whole evolution to make our analysis easier.

Follow the time discretization procedure described in section §1.1, we see γ_n^i is defined on Γ_n^i, and the existence of minimizers in each step can be derived from SBV compactness. Then define $u_n(t) = u_n^i$ for $t_n^i \leq t < t_n^{i+1}$ and for $\forall 0 \leq i < 2^n$. We see $u_n(t)$ is well defined on $[0,1]$ for $\forall n > 0$.

Define

$$E_n(t) := \int_\Omega |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1}$$

At time t_n^{i+1}, pick $v = u_n(t_n^i) + g(t_n^{i+1}) - g(t_n^i)$ as a test function, we get

$$\begin{aligned}
E_n(t_n^{i+1}) \leq & \int_\Omega |\nabla u_n(t_n^i) + \nabla g(t_n^{i+1}) - \nabla g(t_n^i)|^2 dx + \int_{S_{u_n(t_n^i)} \cup \Gamma_n(t_n^i)} \tilde{\varphi}([u_n(t_n^i)], \gamma_n(t_n^i)) d\mathcal{H}^{N-1} \\
\leq & E_n(t_n^i) + \int_\Omega |\nabla g(t_n^{i+1}) - \nabla g(t_n^i)|^2 dx + 2 \int_\Omega \nabla u_n(t_n^i) \cdot (\nabla g(t_n^{i+1}) - \nabla g(t_n^i)) dx \\
\leq & E_n(t_n^i) + \int_\Omega \Big| \int_{t_n^i}^{t_n^{i+1}} \nabla \dot{g}(s) ds \Big|^2 dx + 2 \int_\Omega \nabla u_n(t_n^i) \cdot (\int_{t_n^i}^{t_n^{i+1}} \nabla \dot{g}(s) ds) dx \\
\leq & E_n(t_n^i) + \int_\Omega \Big(\int_{t_n^i}^{t_n^{i+1}} \big|\nabla \dot{g}(s)\big| ds \Big)^2 dx + 2 \int_{t_n^i}^{t_n^{i+1}} \int_\Omega \nabla u_n(t_n^i) \cdot \nabla \dot{g}(s) dx ds \\
\leq & E_n(t_n^i) + \int_\Omega \Big[\Big(\int_{t_n^i}^{t_n^{i+1}} \big|\nabla \dot{g}(s)\big|^2 ds \Big)^{\frac{1}{2}} (\Delta t_n)^{\frac{1}{2}} \Big]^2 dx + 2 \int_{t_n^i}^{t_n^{i+1}} \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds \\
\leq & E_n(t_n^i) + \Delta t_n \int_{t_n^i}^{t_n^{i+1}} \int_\Omega \big|\nabla \dot{g}(s)\big|^2 dx ds + 2 \int_{t_n^i}^{t_n^{i+1}} \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds
\end{aligned}$$

Then sum from i to j where $j > i$ we get

$$E_n(t_n^j) \leq E_n(t_n^i) + \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dxds$$

On the other hand for $t = t_n^i$, take $v = u_n^{i+1} + g(t_n^i) - g(t_n^{i+1})$ as test function to get

$$\begin{aligned} E_n(t_n^i) \leq& E_n(t_n^{i+1}) + \int_\Omega |\nabla g(t_n^i) - \nabla g(t_n^{i+1})|^2 dx + 2\int_\Omega \nabla u_n(t_n^{i+1}) \cdot (\nabla g(t_n^i) - \nabla g(t_n^{i+1}))dx \\ \leq& E_n(t_n^{i+1}) + \Delta t_n \int_{t_n^i}^{t_n^{i+1}} \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds - 2\int_{t_n^i}^{t_n^{i+1}} \int_\Omega \nabla u_n(s+\Delta t_n) \cdot \nabla \dot{g}(s) dxds \end{aligned}$$

So sum from i to j to get

$$E_n(t_n^i) \leq E_n(t_n^j) + \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds - 2\int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s+\Delta t_n) \cdot \nabla \dot{g}(s) dxds$$

It follows

$$\begin{aligned} E_n(t_n^i) - \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s+\Delta t_n) \cdot \nabla \dot{g}(s) dxds \\ \leq E_n(t_n^j) \leq \\ E_n(t_n^i) + \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dxds \end{aligned}$$

Set $i = 0$ and let $t \in I_\infty$, we see for sufficiently large n, $t = t_n^j$ for some n and j. Then we have

$$\begin{aligned} E(0) - \Delta t_n \int_0^t \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_0^t \int_\Omega \nabla u_n(s+\Delta t_n) \cdot \nabla \dot{g}(s) dxds \\ \leq E_n(t) \leq \\ E(0) + \Delta t_n \int_0^t \int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dxds \end{aligned} \tag{3.1}$$

Now we have bounds for the sequence $E_n(t)$, next is to show the bounds converge. The following results are pretty straight forward.

Lemma 14.

$$\sup_{t,n} \int_\Omega |\nabla u_n(t)|^2 dx < \infty$$

$$\sup_{t,n} E_n(t) < \infty$$

$$\sup_{t,n} \int_{\Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1} < \infty$$

Proof. At each time t, pick $g(t)$ as test function to get

$$\int_{\Omega} |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1}$$
$$\leq \int_{\Omega} |\nabla g(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1}$$

Since

$$\int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1} \geq \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1}$$

we get $\int_{\Omega} |\nabla u_n(t)|^2 dx \leq \int_{\Omega} |\nabla g(t)|^2 dx$, due to definition of $g(t)$ the first result is shown.

Considering (3.1),

$$E_n(t) \leq E(0) + \Delta t_n \int_0^t \int_{\Omega} \left|\nabla \dot{g}(s)\right|^2 dx ds + 2 \int_0^t \int_{\Omega} \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds$$

Due to Cauchy-Schwartz inequality and uniform boundedness of $\|\nabla \dot{g}(t)\|_{L^2}$, the second result is proved.

The third result is straight forward. $\square$

Let $I_n := \bigcup_{i=0}^{2^n} t_n^i$ and $I_\infty := \bigcup_{n=1}^{\infty} I_n$, we see I_∞ is a dense and countable subset of $[0, 1]$. For $a_1 \in I_\infty$ we have $\sup_n E_n(a_1) < \infty$, by SBV compactness we can extract a subsequence $\{u_n(a_1)\}_{n=1}^{\infty}$(not relabeled) and $u(a_1) \in SBV(\Omega)$ s.t. $u_n(a_1) \overset{SBV}{\rightharpoonup} u(a_1)$. Then apply the diagonal argument we can extract a subsequence $\{u_n(t)\}_{n=1}^{\infty}$(not relabeled) s.t.

$$u_n(t) \overset{SBV}{\rightharpoonup} u(t) \quad \forall\ t \in I_\infty$$

We see $u(t)$ is well defined on the dense and countable subset I_∞.
Define $\gamma_n(t) := \bigvee_{\tau<t} [u_n(\tau)]$ and $\Gamma_n(t) := \cup_{\tau<t} S_{u_n(\tau)}$. We have at each n the global minimality

$$\int_{\Omega} |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1} \tag{3.2}$$
$$\leq \int_{\Omega} |\nabla v|^2 dx + \int_{S_v \cup \Gamma_n(t)} \tilde{\varphi}([v], \gamma_n(t)) d\mathcal{H}^{N-1} \tag{3.3}$$

for $\forall v = g(t)$ on $\partial\Omega$, for all $t = t_n^i (0 \leq i \leq 2^n)$. Define $\gamma(t) := \bigvee_{\tau<t} [u(\tau)]$ and $\Gamma(t) := \bigcup_{\tau<t} S_{u(\tau)}$. Our first question is if it's true $\forall t \in I_\infty$

$$\int_{\Omega} |\nabla u(t)|^2 dx + \int_{S_{u(t)} \cup \Gamma(t)} \tilde{\varphi}([u(t)], \gamma(t)) d\mathcal{H}^{N-1} \tag{3.4}$$
$$\leq \int_{\Omega} |\nabla v|^2 dx + \int_{S_v \cup \Gamma(t)} \tilde{\varphi}([v], \gamma(t)) d\mathcal{H}^{N-1} \tag{3.5}$$

$\forall v \in SBV(\Omega)$ s.t. $v = g(t)$ on $\partial\Omega$. If the above minimality could be proved, the next step would be to extend the minimality to the whole time interval $[0, 1]$ and to show the global stability in quasi-static evolution. However, due to possible complications in the sequence of minimizers, it is almost impossible to obtain the minimality without excluding those complications mentioned in Chapter 1.

In this section we show that there exists a subsequence $E_n(t)$ (not relabeled) s.t. it converges for all $t \in [0,1]$. Before introducing the next lemma let's define the non-negative function,

$$m_n(t) := \int_{\Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1}$$

Since $\sup_n m_n(1) < \infty$, we have, according to lemma (24), there exists an non-decreasing function $m_\infty(t)$ and a subsequence $\{m_n(t)\}_{n=1}^\infty$(not relabeled) s.t. $m_n(t) \to m_\infty(t)$ $\forall t \in [0,1]$. Moreover $m_\infty(t)$ is a non-decreasing and bounded function defined on $[0,1]$, and thus continuous everywhere except for a countable subset. Denote $D := \{x \in [0,1] : m_\infty(x) \text{ is not continuous}\}$. We see D is at most countable.

Lemma 15. *If $t \in [0,1]\backslash D$, we have $\forall \epsilon > 0$, there $\exists \Delta t > 0$ and $N \in \mathbb{N}$ s.t.*

$$\|\nabla u_n(t_1) - \nabla u_n(t_2)\|_2 < \epsilon$$

$\forall t_1, t_2$ s.t. $t - \Delta t < t_1, t_2 < t + \Delta t$ and $\forall n > N$.

Proof. Let $\epsilon > 0$, first we see $\exists \Delta t > 0$ s.t. $m_\infty(t + \Delta t) - m_\infty(t - \Delta t) < \epsilon$. Then let $N \in \mathbb{N}$ be big s.t.

$$|m_\infty(t + \Delta t) - m_n(t + \Delta t)| < \epsilon$$
$$|m_\infty(t - \Delta t) - m_n(t - \Delta t)| < \epsilon$$

$\forall n > N$.
Let t_1, t_2 be s.t. $t - \Delta t < t_1 < t_2 < t + \Delta t$, we have $|m_n(t_2) - m_n(t_1)| < 3\epsilon$, $\forall n > N$, or equivalently

$$\int \tilde{\varphi}(0, \gamma_n(t_2)) d\mathcal{H}^{N-1} - \int \tilde{\varphi}(0, \gamma_n(t_1)) d\mathcal{H}^{N-1} < 3\epsilon$$

Next consider test function

$$u = \frac{1}{2}u_n(t_1) + \frac{1}{2}u_n(t_2) - \frac{1}{2}g(t_1) - \frac{1}{2}g(t_2) + g(t_1)$$

We see $u = g(t_1)$ on $\partial\Omega$ and according to minimality in discrete time we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n(t_1)|^2 dx + \int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_1)) d\mathcal{H}^{N-1} \\
\leq &\int_\Omega |\nabla u|^2 dx + \int \tilde{\varphi}([u], \gamma_n(t_1)) d\mathcal{H}^{N-1} \\
\leq &\int_\Omega |\nabla u|^2 dx + \int \tilde{\varphi}([u], \gamma_n(t_2)) d\mathcal{H}^{N-1} \\
\leq &\int_\Omega |\frac{1}{2}\nabla u_n(t_1) + \frac{1}{2}\nabla u_n(t_2)|^2 dx + \frac{1}{2}\int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_2)) d\mathcal{H}^{N-1} \\
&+ \frac{1}{2}\int \tilde{\varphi}([u_n(t_2)], \gamma_n(t_2)) d\mathcal{H}^{N-1} + O(|t_2 - t_1|)
\end{aligned}$$

Similarly we can deduce

$$
\begin{aligned}
&\int_\Omega |\nabla u_n(t_2)|^2 dx + \int \tilde{\varphi}([u_n(t_2)], \gamma_n(t_2)) d\mathcal{H}^{N-1} \\
\leq &\int_\Omega |\frac{1}{2}\nabla u_n(t_1) + \frac{1}{2}\nabla u_n(t_2)|^2 dx + \frac{1}{2}\int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_2)) d\mathcal{H}^{N-1} \\
&+ \frac{1}{2}\int \tilde{\varphi}([u_n(t_2)], \gamma_n(t_2)) d\mathcal{H}^{N-1} + O(|t_2 - t_1|)
\end{aligned}
$$

Summing the above two inequalities we get

$$
\begin{aligned}
&\int_\Omega |\nabla u_n(t_1)|^2 dx + \int_\Omega |\nabla u_n(t_2)|^2 dx + \int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_1)) d\mathcal{H}^{N-1} \\
&+ \int \tilde{\varphi}([u_n(t_2)], \gamma_n(t_2)) d\mathcal{H}^{N-1} \\
< &2\int_\Omega |\frac{1}{2}\nabla u_n(t_1) + \frac{1}{2}\nabla u_n(t_2)|^2 dx \\
&+ \int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_2)) d\mathcal{H}^{N-1} + \int \tilde{\varphi}([u_n(t_2)], \gamma_n(t_2)) d\mathcal{H}^{N-1} + O(|t_2 - t_1|)
\end{aligned}
$$

It follows

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u_n(t_2) - \nabla u_n(t_1)|^2 dx \\
= &\int_\Omega |\nabla u_n(t_1)|^2 dx + \int_\Omega |\nabla u_n(t_2)|^2 dx - 2\int_\Omega |\frac{1}{2}\nabla u_n(t_1) + \frac{1}{2}\nabla u_n(t_2)|^2 dx \\
\leq &\int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_2)) d\mathcal{H}^{N-1} - \int \tilde{\varphi}([u_n(t_1)], \gamma_n(t_1)) d\mathcal{H}^{N-1} + O(|t_2 - t_1|) \\
\leq &\int \tilde{\varphi}(0, \gamma_n(t_2)) d\mathcal{H}^{N-1} - \int \tilde{\varphi}(0, \gamma_n(t_1)) d\mathcal{H}^{N-1} + O(|t_2 - t_1|) \\
\leq &3\epsilon + O(|t_2 - t_1|)
\end{aligned}
$$

This concludes the proof. □

For $t \in [0,1]$, let's define

$$\theta_n(t) = \int_\Omega \nabla u_n(t) \cdot \nabla \dot{g}(t) dx$$

and the first obvious result we can see is

$$\sup_{n,t} \theta_n(t) < \infty.$$

According to weak convergence in L^2, $\theta_n(t) \to \int_\Omega \nabla u(t) \cdot \nabla \dot{g}(t) dx$ for $\forall t \in I_\infty$. Then we show the sequence $\theta_n(t)$ is Cauchy a.e. on $[0,1]$.

Lemma 16. *For $\forall t \in [0,1] \backslash D$, $\{\theta_n(t)\}_{n=1}^\infty$ is Cauchy.*

Proof. Let $\epsilon > 0$, according to lemma (15) we have there exists $\Delta t > 0$ and N s.t.

$$\|\nabla u_n(t_1) - \nabla u_n(t_2)\|_2 < \epsilon$$

for $\forall t_1, t_2 \in [t - \Delta t, t + \Delta t]$ and $\forall n > N$. Let $\tau \in [t - \Delta t, t + \Delta t] \cap I_\infty$, we can find $N_1 > N$ s.t.

$$|\theta_k(\tau) - \theta_l(\tau)| < \epsilon$$

for $\forall k, l > N_1$. According to the definition of $\nabla u_n(t)$ on $[0,1]$, $\nabla u_n(t) = \nabla u_n(t_n^{i(n,t)})$ where $i(n,t)$, depending on n and t, is the largest integer s.t. $t_n^{i(n,t)} < t$. Then let $N_2 > N_1$ be s.t. $t_n^{i(n,t)} > t - \Delta t$ for $\forall n > N_2$. It follows

$$\begin{aligned}
|\theta_k(t) - \theta_l(t)| =& \Big| \int_\Omega \nabla u_k(t_k^{i(k,t)}) \cdot \nabla \dot{g}(t) dx - \int_\Omega \nabla u_l(t_l^{i(l,t)}) \cdot \nabla \dot{g}(t) dx \Big| \\
\leq& \Big| \int_\Omega \nabla u_k(t_k^{i(k,t)}) \cdot \nabla \dot{g}(t) dx - \int_\Omega \nabla u_k(t_k^{i(k,t)}) \cdot \nabla \dot{g}(\tau) dx \Big| \\
&+ \Big| \int_\Omega \nabla u_k(t_k^{i(k,t)}) \cdot \nabla \dot{g}(\tau) dx - \int_\Omega \nabla u_k(\tau) \cdot \nabla \dot{g}(\tau) dx \Big| \\
&+ \Big| \int_\Omega \nabla u_k(\tau) \cdot \nabla \dot{g}(\tau) dx - \int_\Omega \nabla u_l(\tau) \cdot \nabla \dot{g}(\tau) dx \Big| \\
&+ \Big| \int_\Omega \nabla u_l(\tau) \cdot \nabla \dot{g}(\tau) dx - \int_\Omega \nabla u_l(t_l^{i(l,t)}) \cdot \nabla \dot{g}(\tau) dx \Big| \\
&+ \Big| \int_\Omega \nabla u_l(t_l^{i(l,t)}) \cdot \nabla \dot{g}(\tau) dx - \int_\Omega \nabla u_l(t_l^{i(l,t)}) \cdot \nabla \dot{g}(t) dx \Big| \\
\leq& O(|t - \tau|) + O(\epsilon) + \epsilon + O(\epsilon) + O(|\tau - t|)
\end{aligned}$$

for $\forall k, l > N_2$. This concludes the proof. □

Remark 6. *We see there exists $\theta(t) : [0,1] \backslash D \to \mathbb{R}$ s.t. $\theta_n(t) \to \theta(t)$ for all $t \in [0,1] \backslash D$. Since D is at most countable and apply the compactness in $\mathbb{R}$ and diagonal argument we can find a subsequence (not labeled) and $\theta(t)$ s.t. $\theta_n(t) \to \theta(t)$ for $\forall t \in [0,1]$. From there we can show the convergence of $E_n(t)$.*

Lemma 17. *Let $t \in I_\infty$, the following is true*

$$\int_0^t \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \dot{g}(s) dx ds \to \int_0^t \theta(s) ds$$

Proof. It suffices to show

$$\Big| \int_0^t \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \dot{g}(s) dx ds - \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds \Big| \to 0$$

First of all we see

$$
\begin{aligned}
&\Big|\int_0^t\int_\Omega \nabla u_n(s+\Delta t_n)\cdot\nabla\dot g(s+\Delta t_n)dxds-\int_0^t\int_\Omega \nabla u_n(s)\cdot\nabla\dot g(s)dxds\Big|\\
=&\Big|\int_{\Delta t_n}^{t+\Delta t_n}\int_\Omega \nabla u_n(s)\cdot\nabla\dot g(s)dxds-\int_0^t\int_\Omega \nabla u_n(s)\cdot\nabla\dot g(s)dxds\Big|\\
\le&\Big|\int_0^{\Delta t_n}\theta_n(s)ds-\int_t^{t+\Delta t_n}\theta_n(s)ds\Big|\\
\le&\int_0^{\Delta t_n}\big|\theta_n(s)\big|ds+\int_t^{t+\Delta t_n}\big|\theta_n(s)\big|ds\to 0\ as\ n\to\infty
\end{aligned}
$$

The convergence is due to uniform boundedness of $\theta_n(t)$. Then consider

$$
\begin{aligned}
&\Big|\int_0^t\int_\Omega \nabla u_n(s+\Delta t_n)\cdot\nabla\dot g(s+\Delta t_n)dxds-\int_0^t\int_\Omega \nabla u_n(s+\Delta t_n)\cdot\nabla\dot g(s)dxds\Big|\\
\le&\int_0^t\Big|\int_\Omega \nabla u_n(s+\Delta t_n)\cdot(\nabla\dot g(s+\Delta t_n)-\nabla\dot g(s))dx\Big|ds\\
\le&\int_0^t\Big(\int_\Omega|\nabla u_n(s+\Delta t_n)|^2dx\Big)^{\frac12}\Big(\int_\Omega|\nabla\dot g(s+\Delta t_n)-\nabla\dot g(s)|^2dx\Big)^{\frac12}ds
\end{aligned}
$$

The above estimate goes to 0 as $n\to\infty$ because $\Big(\int_\Omega|\nabla u_n(s+\Delta t_n)|^2dx\Big)^{\frac12}$ is bounded over s and $\Big(\int_\Omega|\nabla\dot g(s+\Delta t_n)-\nabla\dot g(s)|^2dx\Big)^{\frac12}$ goes to 0 for $\forall s$, apply again D.C.T. Thus the lemma has been proved. □

Remark 7. *Thus we have* $E_n(t)\to E(0)+2\int_0^t\theta(s)ds$, $\forall t\in[0,1]$.

3.2 Little o Method and Its Applications

The idea here is to see if we can find finitely many fixed times that can roughly represent the whole process. The convergence of energy shows that maybe we have a way.

Fix t, let's look at the energy at a small time step Δt further. Let $\Delta t>\Delta t_n$, we have

$$
E_n(t+\Delta t)\le E_n(t)+\Delta t_n\int_t^{t+\Delta t}\int_\Omega\big|\nabla\dot g(s)\big|^2dxds+2\int_t^{t+\Delta t}\int_\Omega\nabla u_n(s)\cdot\nabla\dot g(s)dxds
$$

Then let $v(t+\Delta t)$ be any SBV function s.t. $v(t+\Delta t)=g(t+\Delta t)$ on $\partial\Omega$, we have

$$
\begin{aligned}
E_n(t)\le&\int_\Omega|\nabla v(t+\Delta t)-\nabla g(t+\Delta t)+\nabla g(t)|^2dx+\int\tilde\varphi([v(t+\Delta t)],\gamma_n(t))\\
\le&\int_\Omega|\nabla v(t+\Delta t)|^2dx+\int\tilde\varphi([v(t+\Delta t)],\gamma_n(t))+\Delta t\int_t^{t+\Delta t}\int_\Omega\big|\nabla\dot g(s)\big|^2dxds\\
&-2\int_t^{t+\Delta t}\int_\Omega(\nabla v(t+\Delta t))\cdot\nabla\dot g(s)dxds
\end{aligned}
$$

Combine the above estimate, we get

$$\begin{aligned}&\int \tilde{\varphi}([u_n(t+\Delta t)],\gamma_n(t+\Delta t))d\mathcal{H}^{N-1} - \int \tilde{\varphi}([v(t+\Delta t)],\gamma_n(t))d\mathcal{H}^{N-1}\\ \leq&(\Delta t_n+\Delta t)\int_t^{t+\Delta t}\int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_t^{t+\Delta t}\int_\Omega (\nabla u_n(s)-\nabla v(t+\Delta t))\cdot\nabla\dot{g}(s)dxds\\ &+\int_\Omega |\nabla v(t+\Delta t)|^2dx - \int_\Omega |\nabla u_n(t+\Delta t)|^2 dx\end{aligned}$$

If we let $v(t+\Delta t) = u_n(t+\Delta t)$, the above estimate gives us

$$\begin{aligned}&\int \tilde{\varphi}([u_n(t+\Delta t)],\gamma_n(t+\Delta t))d\mathcal{H}^{N-1} - \int \tilde{\varphi}([u_n(t+\Delta t)],\gamma_n(t))d\mathcal{H}^{N-1}\\ \leq&(\Delta t_n+\Delta t)\int_t^{t+\Delta t}\int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_t^{t+\Delta t}\int_\Omega (\nabla u_n(s)-\nabla u_n(t+\Delta t))\cdot\nabla\dot{g}(s)dxds\end{aligned}$$

3.2.1 Finitely many minimizers

Let $p\in\mathbb{N}$, split time interval $[0,1]$ evenly into 2^p pieces and we have 2^p+1 times. To choose 2^p because after we fix p those finitely chosen times will be a subset of I_n as $n>p$. During the following we always assume $n>p$. It follows, from above estimate, that

$$\begin{aligned}&\int_{S_{u_n(t_p^{i+1})}\cup\Gamma_n(t_p^{i+1})} \tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1} - \int_{S_{v(t_p^{i+1})}\cup\Gamma_n(t_p^{i})} \tilde{\varphi}([v(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\ \leq&(\Delta t_n+\Delta t_p)\int_{t_p^i}^{t_p^{i+1}}\int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_{t_p^i}^{t_p^{i+1}}\int_\Omega (\nabla u_n(s)-\nabla v(t_p^{i+1}))\cdot\nabla\dot{g}(s)dxds\\ &+\int_\Omega |\nabla v(t_p^{i+1})|^2dx - \int_\Omega |\nabla u_n(t_p^{i+1})|^2 dx\end{aligned} \tag{3.6}$$

for any SBV function $v(t_p^{i+1}) = g(t_p^{i+1})$ on $\partial\Omega$. We see immediately, by letting $v(t_p^{i+1}) = u_n(t_p^{i+1})$, that

$$\int_{S_{u_n(t_p^{i+1})}\cup\Gamma_n(t_p^{i+1})} \tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1} - \int_{S_{u_n(t_p^{i+1})}\cup\Gamma_n(t_p^{i})} \tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1} \tag{3.7}$$

$$\leq(\Delta t_n+\Delta t_p)\int_{t_p^i}^{t_p^{i+1}}\int_\Omega \left|\nabla \dot{g}(s)\right|^2 dxds + 2\int_{t_p^i}^{t_p^{i+1}}\int_\Omega (\nabla u_n(s)-\nabla u_n(t_p^{i+1}))\cdot\nabla\dot{g}(s)dxds \tag{3.8}$$

Let $t \in I_\infty$, we are going to sum the above estimate over all t_p^i s.t. $t_p^i \leq t$.

$$\sum_{i=0}^{t_p^{i+1}\leq t} \Big[\int \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int \tilde{\varphi}([v(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \Big]$$

$$\leq (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dxds + 2 \sum_{i=0}^{t_p^{i+1}\leq t} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(s) - \nabla v_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dxds$$

$$+ \sum_{i=0}^{t_p^{i+1}\leq t} \int_\Omega |\nabla v_n(t_p^{i+1})|^2 dx - \int_\Omega |\nabla u_n(t_p^{i+1})|^2 dx$$

$$\leq (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dxds + 2 \sum_{i=0}^{t_p^{i+1}\leq t} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(s) - \nabla u_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dxds$$

$$+2 \sum_{i=0}^{t_p^{i+1}\leq t} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(t_p^{i+1}) - \nabla v_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dxds$$

$$+ \sum_{i=0}^{t_p^{i+1}\leq t} \int_\Omega |\nabla v_n(t_p^{i+1})|^2 dx - \int_\Omega |\nabla u_n(t_p^{i+1})|^2 dx$$

Next define

$$G_n^p(t) = (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dxds + 2 \sum_{i=0}^{t_p^{i+1}\leq t} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(s) - \nabla u_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dxds$$

Before introducing the next lemma let $f : [0,1] \to \mathbb{R}$ s.t. $\sup_{0\leq x\leq 1} |f(x)| < \infty$. Define it's step function with 2^n equi-length partitions as follows

$$f^{(n)}(x) = \begin{cases} f(0) & x = 0 \\ f(\frac{i+1}{2^n}) & \frac{i}{2^n} < x \leq \frac{i+1}{2^n},\ 0 \leq i \leq 2^n - 1 \end{cases}$$

Lemma 18. *Let $t \in I_\infty$, we have*

$$\lim_{p\to\infty} \lim_{n\to\infty} \int_0^t \int_\Omega (\nabla u_n(s) - \nabla u_n^{(p)}(s)) \cdot \nabla \dot{g}(s) dxds = 0$$

Proof. First we see $\theta_n(s) = \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx \to \theta(s)$ for $\forall 0 < s < 1$ and $\int_0^t |\theta_n(s) - \theta(s)| ds \to 0$ $\forall t \in I_\infty$. Then show $\theta(s)$ is continuous except for a at most countable subset in $[0,1]$. Let $s \in \{\tau \in [0,1] : m_\infty(\tau)$ is continuous$\}$. From previous work, we see $\{\tau \in [0,1] : m_\infty(\tau)$ is continuous$\}^c$ is at most countable and $\forall \epsilon > 0$, $\exists \Delta s > 0$ s.t.

$$\|\nabla u_n(s_1) - \nabla u_n(s_2)\|_{L^2} < \epsilon, \forall s_1, s_2 \in (s - \Delta s, s + \Delta s)$$

Let $y, z \in (s - \Delta s, s + \Delta s)$, we see

$$\begin{aligned}|\theta(y) - \theta(z)| &\le |\theta(y) - \theta_n(y)| + |\theta_n(y) - \theta_n(z)| + |\theta_n(z) - \theta(z)| \\ &\le O(\epsilon)\end{aligned}$$

Thus we have shown that $\theta(s)$ is continuous except for a countable subset on $[0, 1]$. Let $\epsilon > 0$, according to lemma 23 we see

$$\int_0^t |\theta(s) - \theta^{(p)}(s)| ds \to 0$$

as $p \to \infty$. Then

$$\begin{aligned}&\Big| \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}(s) dx ds \Big| \\ \le & \Big| \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds - \int_0^t \theta(s) ds \Big| + \Big| \int_0^t \theta(s) ds - \int_0^t \theta^{(p)}(s) ds \Big| \\ &+ \Big| \int_0^t \theta^{(p)}(s) ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}^{(p)}(s) dx ds \Big| \\ &+ \Big| \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}^{(p)}(s) dx ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}(s) dx ds \Big|\end{aligned}$$

We see

$$\Big| \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds - \int_0^t \theta(s) ds \Big| + \Big| \int_0^t \theta^{(p)}(s) ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}^{(p)}(s) dx ds \Big| \to 0$$

as $n \to \infty$. And

$$\begin{aligned}&\Big| \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}^{(p)}(s) dx ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \dot{g}(s) dx ds \Big| \\ \le & \int_0^t \|\nabla u_n^{(p)}(s)\|_{L^2} \|\nabla \dot{g}^{(p)}(s) - \nabla \dot{g}(s)\|_{L^2} ds \\ \le & \sup_s \|\nabla u_n^{(p)}(s)\|_{L^2} \int_0^t \|\nabla \dot{g}^{(p)}(s) - \nabla \dot{g}(s)\|_{L^2} ds\end{aligned}$$

We see $\sup_s \|\nabla u_n^{(p)}(s)\|_{L^2} < \infty$ and $\|\nabla \dot{g}^{(p)}(s) - \nabla \dot{g}(s)\|_{L^2} \to 0$ for $\forall s$ as $p \to \infty$. Apply D.C.T. to conclude the lemma. □

Then it's not hard to see that

$$\lim_{p \to \infty} \lim_{n \to \infty} G_n^p(t) = 0 \quad \forall t \in I_\infty \tag{3.9}$$

Next we show two applications based on above result.

3.2.2 Application of little o analysis: approximation with finite minimizers

Define the following

$$\gamma_n^p(t) = \bigvee_{i=0}^{t_p^i \le t} [u_n(t_p^i)]$$

$$\Gamma_n^p(t) = \bigcup_{i=0}^{t_p^i \le t} S_{u_n(t_p^i)}$$

We see $\gamma_n^p(t)$ is defined on $\Gamma_n^p(t)$.

Lemma 19.

$$\lim_{p\to\infty} \lim_{n\to\infty} \Big(\int_{\Gamma_n(t)} \tilde{\varphi}(\gamma_n^p(t), \gamma_n(t)) d\mathcal{H}^{N-1} - \int_{\Gamma_n^p(t)} \varphi(\gamma_n^p(t)) d\mathcal{H}^{N-1} \Big) = 0$$

Proof. Due to (3.7), we have

$$\begin{aligned} \sum_{i=0}^{t_p^{i+1} \le t} \Big(&\int_{S_{u_n(t_p^{i+1})} \cap \Gamma_n(t_p^{i+1})} \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} \\ &- \int_{S_{u_n(t_p^{i+1})} \cap \Gamma_n(t_p^{i})} \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i})) d\mathcal{H}^{N-1} \Big) \\ \le G_n^p(t) \end{aligned}$$

Let $y > z$, we see in general the value of $\tilde{\varphi}(x, y) - \tilde{\varphi}(x, z)$ decreases as x increases. So

$$\begin{aligned} &\sum_{i=0}^{t_p^{i+1} \le t} \Big(\int_{\Gamma_n(t_p^{i+1})} \tilde{\varphi}(\gamma_n^p(t), \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\Gamma_n(t_p^{i})} \tilde{\varphi}(\gamma_n^p(t), \gamma_n(t_p^{i})) d\mathcal{H}^{N-1} \Big) \\ \le &\sum_{i=0}^{t_p^{i+1} \le t} \Big(\int_{S_{u_n(t_p^{i+1})} \cap \Gamma_n(t_p^{i+1})} \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} \\ &\qquad - \int_{S_{u_n(t_p^{i+1})} \cap \Gamma_n(t_p^{i})} \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i})) d\mathcal{H}^{N-1} \Big) \\ \le &G_n^p(t) \end{aligned}$$

But we see the l.h.s of above inequality can be simplified to

$$\int_{\Gamma_n(t)} \tilde{\varphi}(\gamma_n^p(t), \gamma_n(t)) d\mathcal{H}^{N-1} - \int_{\Gamma_n^p(t)} \varphi(\gamma_n^p(t)) d\mathcal{H}^{N-1}$$

Thus we prove the lemma. □

The result is pretty straight forward once we have (3.9). It shows that we can approximate the whole history evolution by choosing finitely many minimizers.

3.2.3 Application of little o analysis: absolute continuity

As we see in Chapter 2, absolute continuity is very important in that it allows us to use advance tools like covering theorem. But it is very hard to show the result if we take into account the history. As $n \to \infty$, the number of minimizers in the history goes to infinity and it is hard to predict what the they will behave as a whole. But little method allows us to fix finitely many times and do analysis on those finitely many chosen times. This gives us a way to prove the absolute continuity for the evolution problem. First let $h(x) : \Gamma(t) \to (0, \infty)$ and $h(x) \in L^1(\Gamma(t); \mathcal{H}^{N-1})$.

Lemma 20. *Let $t \in I_\infty$, define $\mu_n := \tilde{\varphi}([u_n(t)], \gamma_n(t))\mathcal{H}^{N-1}\lfloor S_{u_n(t)} \cup \Gamma_n(t)$. We see there exists a Radon measure μ_∞ and a subsequence s.t.*

$$\mu_n \overset{*}{\rightharpoonup} \mu_\infty$$

Let $\mu := h\mathcal{H}^{N-1}\lfloor\Gamma(t)$. We have $\mu_\infty \ll \mu$.

Proof. The way we prove the lemma is very similar to the one we use to prove absolutely continuity in section Chapter 2. Except here we combine the little o method. Assume it is not true, we can find a set $A \subset \Omega$ s.t. $\mu(A) = 0$ but $\mu_\infty(A) = \delta > 0$.
First we see, according to (3.9)

$$\lim_{p\to\infty} \lim_{n\to\infty} G_n^p(t) = 0$$

where

$$G_n^p(t) = (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dxds + 2\int_0^t \int_\Omega (\nabla u_n(s) - \nabla u_n^{(p)}(s)) \cdot \nabla \dot{g}(s) dxds$$

Fix p to be big s.t. $\lim_{n\to\infty} G(n,p) \leq \frac{1}{8}\delta$ and let P be s.t. $t_p^P = t$.
Since $\mu(A) = 0$, we have $|A| = 0$ too. Let U be open and $A \subset U \subset \Omega$ s.t.

$$\sup_s \|\nabla \dot{g}(s)\|_{L^\infty} \limsup_{n\to\infty} \int_U |\nabla u_n(t_p^i)| dx \leq \frac{1}{8}\frac{\delta}{P} \quad \forall 1 \leq i \leq P \tag{3.10}$$

$$\sup_s \|\nabla \dot{g}(s)\|_{L^\infty} \int_U |\nabla u(t_p^i)| dx \leq \frac{1}{8}\frac{\delta}{P} \quad \forall 1 \leq i \leq P \tag{3.11}$$

$$\int_U \varphi([u(t_p^i)]) d\mathcal{H}^{N-1} \leq \frac{1}{8}\frac{\delta}{P} \quad \forall 1 \leq i \leq P \tag{3.12}$$

The last inequality comes from the fact that $\mu(A) = 0$ implies $\int_A \varphi([u(t_p^i)]) = 0 \ \forall i$.
Then finely cover A with each ball chosen s.t.

$$\begin{aligned} x &\in A \\ B(x,r) &\subset U \\ \mu_n(\partial B(x,r)) &= 0 \quad for \ \forall \ n > 0 \\ \mu_\infty(\partial B(x,r)) &= 0 \end{aligned}$$

According to Besicovitch covering theorem we can find a countable disjoint family of closed balls

$\{B(x_j, r_j)\}_{j=1}^{\infty}$ s.t.

$$\mu_\infty(A \backslash \bigcup_j^{\infty} B(x_j, r_j)) = 0 \tag{3.13}$$

Since $\mu_\infty(\partial B(x,r)) = 0$ we can let all balls be open. Then select a finite $N \in \mathbb{N}$ s.t. $\mu_\infty(A \backslash \bigcup_{i=1}^N B(x_i, r_i)) < \frac{1}{8}\delta$. We see

$$\mu(\bigcup_{i=1}^N B(x_i, r_i)) < \epsilon$$
$$\mu_\infty(\bigcup_{i=1}^N B(x_i, r_i)) \geq \frac{7}{8}\delta$$

Denote $B := \bigcup_{i=1}^N B(x_i, r_i)$ and consider the following test functions

$$v_n(t_p^i) = \begin{cases} u(t_p^i) & B \\ u_n(t_p^i) & \Omega \backslash B \end{cases}$$

Previously we showed that

$$\begin{aligned}
&\sum_{i=0}^{P-1} \int \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int \tilde{\varphi}([v_n(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \\
\leq &(\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dx ds + 2 \sum_{i=0}^{P-1} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(s) - \nabla v_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dx ds \\
&+ \sum_{i=0}^{P-1} \int_\Omega |\nabla v_n(t_p^{i+1})|^2 dx - \int_\Omega |\nabla u_n(t_p^{i+1})|^2 dx \\
\leq &(\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dx ds + 2 \sum_{i=0}^{P-1} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(s) - \nabla u_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dx ds \\
&+ 2 \sum_{i=0}^{P-1} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(t_p^{i+1}) - \nabla v_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dx ds + \sum_{i=0}^{P-1} \int_\Omega |\nabla v_n(t_p^{i+1})|^2 dx - \int_\Omega |\nabla u_n(t_p^{i+1})|^2 dx \\
\leq &G(n,p) + 2 \sum_{i=0}^{P-1} \int_{t_p^i}^{t_p^{i+1}} \int_\Omega (\nabla u_n(t_p^{i+1}) - \nabla v_n(t_p^{i+1})) \cdot \nabla \dot{g}(s) dx ds \\
&+ \sum_{i=0}^{P-1} \int_\Omega |\nabla v_n(t_p^{i+1})|^2 dx - \int_\Omega |\nabla u_n(t_p^{i+1})|^2 dx
\end{aligned}$$

According to the definition of $v_n(t_p^i)$, we have

$$\begin{aligned}
&2\sum_{i=0}^{P-1}\int_{t_p^i}^{t_p^{i+1}}\int_\Omega (\nabla u_n(t_p^{i+1})-\nabla v_n(t_p^{i+1}))\cdot\nabla\dot{g}(s)dxds\\
=&2\sum_{i=0}^{P-1}\int_{t_p^i}^{t_p^{i+1}}\int_B (\nabla u_n(t_p^{i+1})-\nabla u(t_p^{i+1}))\cdot\nabla\dot{g}(s)dxds\\
\leq&2\sum_{i=0}^{P-1}\Delta t_p\sup_s\|\nabla\dot{g}(s)\|_{L^\infty}\int_B|\nabla u_n(t_p^{i+1})-\nabla u(t_p^{i+1})|dx\\
\leq&2\Delta t_p\sup_s\|\nabla\dot{g}(s)\|_{L^\infty}\sum_{i=0}^{P-1}\Big[\int_B|\nabla u_n(t_p^{i+1})|dx+\int_B|\nabla u(t_p^{i+1})|dx\Big]
\end{aligned}$$

Again according to the definition of $v_n(t_p^i)$

$$\sum_{i=0}^{P-1}\int_\Omega|\nabla v_n(t_p^{i+1})|^2dx-\int_\Omega|\nabla u_n(t_p^{i+1})|^2dx=\sum_{i=0}^{P-1}\int_B|\nabla u(t_p^{i+1})|^2dx-\int_B|\nabla u_n(t_p^{i+1})|^2dx$$

According to lower-semi-continuity we have

$$\int_B|\nabla u(t_p^{i+1})|^2dx\leq\liminf_{n\to\infty}\int_B|\nabla u_n(t_p^{i+1})|^2dx$$

Combine to estimates from above we can get

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=0}^{P-1}\int\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int\tilde{\varphi}([v_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\
\leq&\frac{1}{8}\delta+2\Delta t_p\frac{2}{8}\delta\leq\frac{5}{8}\delta
\end{aligned}$$

Again in light of the definition of $v_n(t_p^i)$ we have

$$\begin{aligned}
&\sum_{i=0}^{P-1}\int\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int\tilde{\varphi}([v_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\
=&\sum_{i=0}^{P-1}\int_{\Omega\setminus B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{\Omega\setminus B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\
&+\sum_{i=0}^{P-1}\int_{\partial B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{\partial B}\tilde{\varphi}([v_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\
&+\sum_{i=0}^{P-1}\int_{B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{B}\tilde{\varphi}([u(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}
\end{aligned}$$

It is clear that

$$\sum_{i=0}^{P-1}\int_{\Omega\setminus B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{\Omega\setminus B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\geq 0$$

And according to Lemma (6)

$$\lim_{n\to\infty}\int_{\partial B}\tilde{\varphi}([v_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}=\lim_{n\to\infty}\int_{\partial B}\tilde{\varphi}(0,\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}$$

So

$$\lim_{n\to\infty}\sum_{i=0}^{P-1}\int_{\partial B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{\partial B}\tilde{\varphi}([v_n(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\geq 0$$

This shows

$$\lim_{n\to\infty}\sum_{i=0}^{P-1}\int_{B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{B}\tilde{\varphi}([u(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\leq\frac{5}{8}\delta$$

Then we see

$$\begin{aligned}&\sum_{i=0}^{P-1}\int_{B}\tilde{\varphi}([u_n(t_p^{i+1})],\gamma_n(t_p^{i+1}))d\mathcal{H}^{N-1}-\int_{B}\tilde{\varphi}([u(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\ =&\int_{B}\tilde{\varphi}([u_n(t_p^{P})],\gamma_n(t_p^{P}))d\mathcal{H}^{N-1}+\sum_{i=1}^{P-1}\Big[\int_{B}\tilde{\varphi}([u_n(t_p^{i})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\\ &-\int_{B}\tilde{\varphi}([u(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\Big]-\int_{B}\varphi([u(t_p^{1})])d\mathcal{H}^{N-1}\end{aligned}$$

Moreover

$$\begin{aligned}&\sum_{i=1}^{P-1}\Big[\int_{B}\tilde{\varphi}([u_n(t_p^{i})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}-\int_{B}\tilde{\varphi}([u(t_p^{i+1})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}\Big]\\ \geq&\sum_{i=1}^{P-1}\Big[\int_{B}\tilde{\varphi}([u_n(t_p^{i})],\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}-\int_{B}\tilde{\varphi}(0,\gamma_n(t_p^{i}))d\mathcal{H}^{N-1}-\int_{B}\varphi([u(t_p^{i+1})])d\mathcal{H}^{N-1}\Big]\\ \geq&\sum_{i=1}^{P-1}-\int_{B}\varphi([u(t_p^{i+1})])d\mathcal{H}^{N-1}\end{aligned}$$

Thus we have

$$\int_B \tilde{\varphi}([u_n(t_p^P)], \gamma_n(t_p^P)) d\mathcal{H}^{N-1} - \sum_{i=0}^{P-1} \int_B \varphi([u(t_p^{i+1})]) d\mathcal{H}^{N-1}$$
$$\leq \sum_{i=0}^{P-1} \int_B \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_B \tilde{\varphi}([u(t_p^{i+1})], \gamma_n(t_p^{i})) d\mathcal{H}^{N-1}$$

It follows

$$\lim_{n\to\infty} \int_B \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1} \leq \frac{5}{8}\delta + \frac{1}{8}\delta = \frac{6}{8}\delta$$

It contradicts to the fact that $\mu_\infty(\bigcup_{i=1}^{N} B(x_i, r_i)) \geq \frac{7}{8}\delta$. □

Appendix

A. Properties of cohesive energy function φ

An example of φ that fits the description in Chapter 1. Let

$$\varphi(x) = x + x\log(\frac{1}{x})$$

We see $\varphi'(x) = \log(\frac{1}{x})$ and $\varphi''(x) = -\frac{1}{x}$. So $\varphi'(0) = \infty$ and

$$\frac{\varphi'(x)x}{\varphi(x)} = \frac{\log(\frac{1}{x})x}{x + x\log(\frac{1}{x})} \to 1 \text{ as } x \to 0$$

Some properties of φ:

1. Let $x, \lambda > 0$, we have

$$\begin{aligned} \varphi(\lambda x) &< \lambda\varphi(x) \qquad if \lambda > 1 \\ \varphi(\lambda x) &> \lambda\varphi(x) \qquad if \lambda < 1 \end{aligned}$$

2. Subadditivity

$$\varphi\left(\sum_{i=1}^{n} x_i\right) < \sum_{i=1}^{n} \varphi(x_i) \qquad \text{where } x_i > 0 \ for \ all \ i$$

3. $\frac{\varphi(x)}{x} > \varphi'(x)$ for $x > 0$.

4. Subadditivity of $\tilde{\varphi}(\cdot, \lambda)$ for $\lambda > 0$. i.e.

$$\tilde{\varphi}(x + y, \lambda) \le \tilde{\varphi}(x, \lambda) + \tilde{\varphi}(y, \lambda) \quad \forall \ x, \ y \ge 0$$

moreover

$$\begin{aligned} \tilde{\varphi}(x + y, \lambda) &\le \tilde{\tilde{\varphi}}(x, \lambda) + \varphi(y) \\ & or \\ \tilde{\varphi}(x + y, \lambda) &\le \varphi(x) + \tilde{\varphi}(y, \lambda) \end{aligned}$$

5.

$$\tilde{\varphi}(x, h) - \tilde{\varphi}(y, h) \le \varphi(x) - \varphi(y)$$

for all $h \geq 0$ and $x \geq y$.

6.
$$\tilde{\varphi}(a+b,h) - \tilde{\varphi}(b,h) \leq \tilde{\varphi}(a,h) - \tilde{\varphi}(0,h)$$
for all $a \geq 0$, $b \geq 0$ and $h \geq 0$.

B. Useful lemmas

Lemma 21. *Let $f \in L^1(\Omega, \mu)$ where Ω is open. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets s.t. $A_n \subset \Omega$ for $\forall n$. If $\mu(A_n) \to 0$, then we have*

$$\int_{A_n} f dx \to 0$$

Proof. It suffices to show $\int_\Omega |f|\chi_{A_n} d\mu \to 0$. Let's argue by contradiction and assume $\lim_{n\to\infty} \int_\Omega |f|\chi_{A_n} d\mu \neq 0$. Then we can find a subsequence $\{A_{n_k}\}$ and $\delta > 0$ s.t. $\int_\Omega |f|\chi_{A_{n_k}} d\mu > \delta$ for all $k = 1, 2, ...$ First we see $\int_\Omega |\chi_{A_{n_k}} - 0| d\mu = \int_\Omega \chi_{A_{n_k}} d\mu = \mu(A_{n_k}) \to 0$. So $\chi_{A_{n_k}} \to 0$ in $L^1(\Omega, \mu)$. Thus we can extract a subsequence $\{n_{k_i}\}_{i=1}^{\infty}$ s.t. $\chi_{A_{n_{k_i}}}(x) \to 0$ μ-a.e. $x \in \Omega$. It follows $|f|\chi_{A_{n_{k_i}}} \to 0$ μ a.e. Then we see $|f|\chi_{A_{n_{k_i}}} \leq |f| \in L^1(\Omega, \mu)$ for $\forall i$, and due to D.C.T. we have $\int_\Omega |f|\chi_{A_{n_{k_i}}} d\mu \to 0$, which contradicts to the fact that $\int_\Omega |f|\chi_{A_{n_k}} d\mu > \delta$ for all $k = 1, 2, ...$ Therefore the lemma has been shown. □

Lemma 22. *Let $f \in L^1(0, 1)$, then*

$$\max_{0 \leq i < n} \int_{\frac{i}{n}}^{\frac{i+1}{n}} f dx \to 0 \text{ as } n \to \infty$$

Proof. Assume it's not true, we can find a subsequence $\{n_k\}_{k=1}^{\infty}$ and for each k an index $i(n_k)$ that depends on k and $\delta > 0$ s.t.

$$\int_{\frac{i(n_k)}{n_k}}^{\frac{i(n_k)+1}{n_k}} f dx > \delta \ \forall \ k$$

We see $\left|(\frac{i(n_k)}{n_k}, \frac{i(n_k)+1}{n_k})\right| = \frac{1}{n_k} \to 0$, and according to lemma (21) we see $\int_{\frac{i(n_k)}{n_k}}^{\frac{i(n_k)+1}{n_k}} f dx \to 0$. A contradiction. □

Lemma 23. *Let $f \in L^\infty([0, 1])$ s.t. f is continuous a.e. Let $f_n(x) := f(\frac{i}{n})$ when $\frac{i-1}{n} \leq x \leq \frac{i}{n}$, for $\forall$ $1 \leq i \leq n$. Then $\int_0^1 |f - f_n| dx \to 0$.*

Proof. Let $E = \{x \in [0, 1] : f(x)$ is not continuous$\}$. Let $\epsilon > 0$ and $U_E \subset (0, 1)$ be open s.t. $E \subset U_E$ and $|U_E| \leq \frac{\epsilon}{4\|f\|_{L^\infty}}$.
First we claim $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$if \left| \sup_{\frac{i-1}{n} \leq x \leq \frac{i}{n}} f(x) - \inf_{\frac{i-1}{n} \leq x \leq \frac{i}{n}} f(x) \right| > \frac{\epsilon}{2}, \quad then \ [\frac{i-1}{n}, \frac{i}{n}] \subset U_E$$

We argue by contradiction, assume the claim is not true, we can find a subsequence $\{n_j\}_{j=1}^{\infty}$ s.t. for each n_j, we can find an index $i(n_j)$ that depends on n_j s.t.

$$\Big| \sup_{\frac{i(n_j)-1}{n_j} \leq x \leq \frac{i(n_j)}{n_j}} f(x) - \inf_{\frac{i(n_j)-1}{n_j} \leq x \leq \frac{i(n_j)}{n_j}} f(x) \Big| > \frac{\epsilon}{2}$$

and $[\frac{i(n_j)-1}{n_j}, \frac{i(n_j)}{n_j}] \cap U_E^c \neq \emptyset$. Meanwhile we can find $x_{n_j}, y_{n_j}, z_{n_j} \in [\frac{i(n_j)-1}{n_j}, \frac{i(n_j)}{n_j}]$ s.t. $|f(x_{n_j}) - f(y_{n_j})| > \frac{\epsilon}{4}$ and $z_{n_j} \in U_E^c$. We see there's a further subsequence and $x \in [0,1]$ s.t. $x = \lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} y_{n_j} = \lim_{j\to\infty} z_{n_j}$. It follows that f is not continuous at x since $|f(x_{n_j}) - f(y_{n_j})| > \frac{\epsilon}{4}$ for $\forall j$, so $x \in U_E$. But since U_E^c is closed we also deduce $x \in U_E$, a contradiction. Thus the claim has been shown.
Next we see

$$\begin{aligned}
\int_0^1 |f - f_n| dx = & \sum_{|\sup f(x) - \inf f(x)| \leq \frac{\epsilon}{2}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |f - f_n| dx + \sum_{|\sup f(x) - \inf f(x)| > \frac{\epsilon}{2}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |f - f_n| dx \\
\leq & \sum_{|\sup f(x) - \inf f(x)| \leq \epsilon} \frac{1}{n}\frac{\epsilon}{2} + \frac{\epsilon}{4\|f\|_{L^\infty}} 2\|f\|_{L^\infty} \\
\leq & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

□

Lemma 24. *Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of non-decreasing functions defined on $[0,1]$. Assume $\sup_{n,x} |f_n(x)| < \infty$, then there's a subsequence $\{n_j\}_{j=1}^{\infty}$ and a non-decreasing bounded f defined on $[0,1]$ s.t.*

$$f_{n_j}(x) \to f(x) \ for \ \forall x \in [0,1]$$

Proof. Let $D \subset [0,1]$ be dense and countable. Then by compactness in $\mathbb{R}$ and diagonal argument we can extract a subsequence $\{n_j\}_{j=1}^{\infty}$ and f defined on D s.t. $f_{n_j}(x) \to f(x)$ for $\forall x \in D$. We see $f(x)$ is non-decreasing and bounded on D, i.e. $f(x) \leq f(y)$ whenever $x, y \in D$ and $x \leq y$. Next let $x \in [0,1] \backslash D$, define

$$f^+(x) = \inf_{y > x, y \in D} f(y) \qquad f^-(x) = \sup_{y < x, y \in D} f(y)$$

We see $f^-(x) \leq f^+(x)$ and define $f(x) = \frac{f^+(x) - f^-(x)}{2}$ for $x \in [0,1] \backslash D$. We see $f(x)$ is non-decreasing and bounded on $[0,1]$. Then we see $f(x)$ is continuous except on a set E that is at most countable.
Next let $x \in [0,1] \backslash E$, let $\epsilon > 0$ we can find $\Delta x > 0$ s.t. $|f(y) - f(x)| < \epsilon$ whenever $|x - y| < \Delta x$.

Now restrict $y, z \in D$, s.t. $x - \Delta x < y < x < z < x + \Delta x$. Then we have

$$\begin{aligned}
|f_{n_j}(x) - f(x)| \leq& |f_{n_j}(x) - f_{n_j}(y)| + |f_{n_j}(y) - f(y)| + |f(y) - f(x)| \\
\leq& |f_{n_j}(z) - f_{n_j}(y)| + |f_{n_j}(y) - f(y)| + |f(y) - f(x)| \\
\leq& |f_{n_j}(z) - f(z)| + |f(z) - f(y)| + |f(y) - f_{n_j}(y)| + |f_{n_j}(y) - f(y)| + |f(y) - f(x)| \\
\leq& O(\epsilon)
\end{aligned}$$

Thus we see $f_{n_j}(x) \to f(x)$ for $x \in [0,1] \backslash E$. But since E is at most countable, we can use compactness and diagonal argument to extract a further subsequence s.t. f_{n_j}(not relabeled) converges to $f(x)$ on $[0,1]$. And it's not hard to show f is non-decreasing. □

Let $\Omega \subset \mathbb{R}^n$ be open. Let μ be Radon on Ω and $\{f_n\}_{n=1}^{\infty} \subset L^1(\Omega; \mu)$ s.t. $f_n \overset{L^1}{\to} 0$. Then

Lemma 25. *$\exists$ a subsequence $\{n_j\}_{j=1}^{\infty}$ s.t.*

$$\lim_{j \to \infty} \lim_{r \to 0} \frac{\int_{B(x,r)} f_{n_j} d\mu}{\mu(B(x,r))} = 0$$

for μ-a.e. $x \in \Omega$.

Proof. First we see $\exists$ a subsequence s.t. $f_{n_j} \to 0$ for μ-a.e. Let $D_0 \subset \Omega$ be the set s.t. $f_{n_j} \to 0$ for $\forall x \in D_0$. We see $\mu(\Omega \backslash D_0) = 0$.
Then for each j, according to Lebesgue-Besicovitch differentiation theorem,

$$\lim_{r \to 0} \frac{\int_{B(x,r)} f_{n_j} d\mu}{\mu(B(x,r))} = f_{n_j}(x)$$

for $\mu - a.e. x \in \Omega$. Let $D_j \subset \Omega$ be the set s.t. the above is true for $x \in D_j$. Then consider the set $D := \cap_{j=0}^{\infty} D_j$. We see

$$\lim_{j \to \infty} \lim_{r \to 0} \frac{\int_{B(x,r)} f_{n_j} d\mu}{\mu(B(x,r))} = \lim_{j \to \infty} f_{n_j}(x) = 0$$

for all $x \in D$. And $\mu(\Omega \backslash D) \leq \sum_{j=0}^{\infty} \mu(\Omega \backslash D_j) = 0$. Thus the lemma has been proved. □

Notation

$a \vee b, a \wedge b$	maximum and minimum of a and b
$B(x, r)$	open ball centered at x with radius r
$Q(x, r)$	open cube centered at x with side r
χ_E	indicator function of set E
$\overline{E}$	closure of set E
∂E	topological boundary of set E
$V \Subset U$	V is compactly contained in U
$\mathcal{L}^N$	Lebesgue measure on $\mathbb{R}^N$
$\mathcal{H}^{N-1}$	$N-1$ dimensional Hausdorff measure
$\mu \lfloor A$	μ restricted to the set A
$D_\nu \mu$	derivative of μ with respect to ν
$\mu \ll \nu$	μ is absolutely continuous with respect to ν
BV	functions of bounded variation
SBV	special functions of bounded variation
S_u	jump set of u
$[u]$	size of the jump on S_u
ν	unit normal to the jump set
Du	distributional derivative of u
$D^a u$	absolutely continuous part of Du with respect to $\mathcal{L}^N$
$\partial^* E$	reduced boundary of E
E_t	$\{x \in \Omega : u > t\}$
$u_n \overset{SBV}{\rightharpoonup} u$	u_n converges to u in the sense of SBV
$\mu_n \overset{*}{\rightharpoonup} \mu$	μ_n converges weak* to μ

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