

Vibrational Analysis of a Bat-Ball Collision

A Major Qualifying Project Report

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By

Brandon Boucher

Gregory Mannke

Mitch McClune

Date:

Approved:

Professor Mark W. Richman, Advisor

Abstract:

The purpose of this project is to develop a model of a bat-ball collision to predict the outgoing velocity of the ball and how it varies as the point of impact along the bat changes. The method we propose is to subtract the kinematic losses due to vibration from the existing rigid body model for bat ball collisions. Our model accounts for the mechanism by which kinetic energy is imparted to the ball (as if the bat were rigid) and the mechanism by which vibrational energy is imparted to the bat because naturally it deforms after impact. The problem is complicated by the fact that the impulsive force applied by the ball to the bat depends on the outgoing velocity of the ball, which in turn needs to be determined as part of the simultaneous solution procedure. We were able to determine that a “sweet spot” of a bat is the location where outgoing ball velocity is at a maximum due to minimal kinetic losses due to vibration. By combining the rigid body and vibrational models we were able to find the “sweet spot” of the bat taking into account energy losses due to vibration. This sweet spot is different than predicted by the rigid body model alone, but is not located at any of the nodes of the vibrational modes.

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Executive Summary:

From the point of view of a baseball player the “sweet spot” of the bat is the point along the barrel that causes the player to feel no vibration in his hands when striking a ball there. In scientific terms the “sweet spot” causes the ball to leave the bat with a higher exit velocity than it would if it had been struck at any other point. Although the “sweet spot” has a true scientific definition in terms of what it means for the baseball's exit velocity there have not been any conclusive answers for the reason for its location along the barrel. Some studies have identified it as the location of the bat's center of percussion, and other theories have said that it has to do with the bat's nodes of vibration. The reasoning for these theories is because of the fact that the batter feels virtually no vibration in his hands when a ball is hit on the “sweet spot”. Our study looks into the theory of the sweet spot having to do with the bat's modes of vibration. By simultaneously examining the kinetic energy imparted from the bat into the ball as well as the vibrational energy imparted to the bat from the ball's impact we were able to create a model for outgoing velocity which takes into account energy lost due to vibration.

Examining the vibrational characteristics of a constant profile dowel allowed us through the manipulating of equations the separation of variables to produce the mode shapes for the dowel. By examining these mode shapes after a simulated impact we were able to determine convergence after ten terms, meaning that after the first ten modes the displacement of the bat was identical to the tenth mode. We then could calculate velocity of the ten mode summation and use that to determine the total energy.

Upon comparison with the previous group's rigid body model of bat performance we noticed a twenty percent drop in energy transfer when vibration was factored in. Also there was a shift in the "sweet spot" from that of the rigid body model by about eleven percent closer to the end of the bat. This shift moved the "sweet spot" in between the nodes for the first and second modes of vibration of the bat showing that the location of the "sweet spot" was not directly on a node but rather in between the two.

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Variables and Values Used:

Rigid:

v_1' = Outgoing ball velocity

v_1 = Incoming ball velocity = 90mph

m_1 = Mass of the bat = 30 ounces

m_2 = Mass of the ball = 30 ounces

$$m = \frac{m_1}{m_2}$$

e = Bat/ball coefficient of restitution = 0.5

v_c = Velocity of the center of mass of the bat

I_c = Moment of inertia of the bat

y = Impact point to the center of mass

ΔT_r = Change of energy (rigid)

ω_{bat} = Angular velocity of the bat = 40 – 60 radians per second

Vibrations:

$w(x, t)$ = Displacement of the bat

E = Elastic Modulus = 2.02×10^6

I_{zz} = Moment area of inertia (based on radius – average real bat radius)

A = Area (based on radius – average real bat radius)

x = Distance along the bat

t = time

ρ = Mass density = 0.0257 pounds (mass) per square inch

J = Impulse (including vibrations)

a = Impact point

L = Length of the bat = 33 inches

ω_n = Frequencies of vibrations

$\beta_n L$ = Weighted frequencies

$W_n(x)$ = Mode shapes

I_{nn} = Orthogonal integral of W_n^2

$E_v(a)$ = Nondimensional energy vibration

Chapter 1: Introduction

Anyone that has ever played baseball knows what you mean when you refer to the sweet spot of the bat. The sweet spot is the part of the bat that causes the ball to exit the barrel at the fastest velocity after striking it. Most people would say hitting a ball on the sweet spot of the bat is almost indescribable. Others have claimed that when hitting a ball on the sweet spot you don't feel anything you just here the crack of the bat and you know that you hit the ball well. Although many people have experienced hitting a ball on the sweet spot, if you asked one hundred of them to explain to you in a scientific manner what is the sweet spot actually is you would get one hundred different answers none of which would have significant scientific merit.

Most people who describe the sweet spot of a bat will use the term vibration in their description. This is due to the fact that when a ball does not hit the sweet spot of a bat it imparts a tremendous amount of vibrations to the batters hands. So when hitting a ball on the sweet spot the first thing someone will notice is how little (if any) vibration that they feel in their hands. Many studies have been conducted to find out where this sweet spot is on a bat to give batters a better visual picture of the target that they want the ball to hit on the barrel. However the goal of this project is not to find out where the sweet spot is on a bat it is rather to identify what the sweet spot of a baseball bat actually is in terms of the nodes of vibration imparted onto the bat by the ball.

1.1 Background Research

A player identifies the “sweet spot” of a baseball bat as the point on the bat which they feel virtually no force when the ball is hit. It has been determined that there are two main reasons for this phenomenon. The first reason is that this point on the bat is a node of one of the main modes of vibration so that when it strikes the ball the mode of vibration is not excited and the player feels no vibration in his hands. The second explanation for the sweet spot is that it is the center of percussion on the bat meaning that when a player strikes the ball at this point no force is translated to the players hands. ^[7]

A study performed at the University of Sydney in 2006 theorizes that the “sweet spot” is not a single point on the bat but rather a small range which encompasses forces from the balls relation to the vibrational nodes as well as the center of percussion. However the center of percussion (COP) is not a single point on the barrel of the bat. When a bat is pivoted about an axis of rotation through the conjugate point and a ball strikes the COP there will in turn be no resulting force at the pivot point. The study theorizes that there is a range of COP on the bat which place the conjugate point under the players hands which is why they feel no resultant force when the ball is struck. ^[3]

The experiment was conducted using a Louisville Slugger model R161 wood bat with a length of 33 inches and a mass of 31 ounces. To obtain the experimental data piezo discs were taped to several locations along the bat and hooked up to an oscilloscope. The vibrations on the bat would result in a proportional voltage output from the discs which could be seen on the oscilloscope. The experimental procedure was conducted by dropping a ball suspended by a

one meter length of string onto the horizontal bat which was gripped firmly in a right handers grip. The string allowed the bat to be accurately struck with the same force at each predetermined point.

From the data gathered during the experiment a procedure it was determined that striking a bat in close proximity to the first and second fundamental nodes of vibration minimizes the force felt in the player's hands due to the significant damping force associated with holding a bat. The vibration amplitude cause by a ball striking outside of the approximately 3 cm long zone on the barrel would be enough to cause significant pain in the players hands and dramatically decrease the outgoing velocity of the ball.

A separate study conducted at Kettering University in Michigan looked into the vibrational characteristics of a 30 inch wooden little league bat. In this study it was determined that the "sweet spot" is not simply a single point on the bat but a range between the first and second nodes of vibration. In their experiments the researchers at Kettering used a bat suspended by rubber bands at each end to represent a "free/free" vibrational environment. It was noted that a players hands gripping the bat has a significant effect in dampening the overall vibrations of the bat but virtually no effect on the frequency or overall mode shapes of the vibrations which is what really matters in this study. ^[6]

In order to excite each of the modes of vibration for the bat it was struck with a force hammer along 30 separate points on the bat. The vibrational data was obtained using an accelerometer which recorded a frequency response function for the impact at each of the 30

locations. Then modal analysis software was used to curve fit the frequency response functions as well as determine mode shapes and frequencies.

The data obtained showed that striking the bat somewhere along the barrel between the nodes of the first and second modes of vibration also placed nodes at approximately where the player's hands would be when gripping the bat. This means that the player would feel little to no vibration in his hands. Also in terms of energy transfer striking the ball in this zone would generate the most speed and distance off of the bat because more energy is imparted directly into the ball rather than lost to the excitation of those modes of vibration.

1.2 Previous Work

To further help us with our understanding of baseball bat dynamics we first examined the previous project by Seth Chapman, Mark Dignum, and Lyle Miller. In their project titled "Calculating Wooden Baseball Bat Efficiency" they calculated the optimum point of contact on a bat based on outgoing ball velocity versus point of impact along the barrel. Their calculations were based on a rigid body dynamic model of a bat in which they kept the handle diameter constant at 1 inch and varied the profile of each bat resulting in a change in the moment of inertia. The change in the mass moment of inertia along the length of the bat resulted in drastically different bat performance. The purpose of finding the outgoing velocity in relation to the point of contact on the bat is that the point on the bat corresponding to the maximum outgoing velocity represents the sweet spot on the bat. This means that that a ball struck on this point of the bat will travel further than a ball struck at any other point, given the same swing plane and exit angles. [2]

To obtain their equation for maximum velocity the previous group first defined the velocity of the bat at a point just prior to impact with the equation given by

$$v_1 = v_c + y\omega_{bat} \quad (1-1)$$

v_c in the above equation represents the velocity of the bat's center of mass

$$v_c = (h + \bar{x})\omega_{bat} \quad (1-2)$$

After applying the conservation of linear and angular momentum to this equation you obtain:

$$m_2v_2 + m_1v_c = m_2v'_2 + m_1v'_c \rightarrow m_2v_2y + I_c\omega_{bat} = m_2v'_2y + I_c\omega'_{bat} \quad (1-3)$$

Where I_c represents the bats moment of inertia about its center of gravity. Next the bat-ball coefficient of restitution (bbcor) is defined by:

$$e = \frac{-(v'_1 - v'_2)}{(v_1 - v_2)} \quad (1-4)$$

From this equation they were able to derive an equation for the exit velocities of the bat and ball and in this case. This leaves us with the equation for outgoing ball velocity in terms of the angular velocity of the bat, the distance along the bat from the center of mass and the velocity of the center of mass after impact.

$$v'_1 = v'_c + \omega_{bat}'y \quad (1-5)$$

In this equation the velocity of the center of mass of the bat can simply be treated as the angular velocity of the bat times the distance from center of mass of the bat to the center of rotation:

$$v'_c = (h' + \bar{x})\omega_{bat}' \quad (1-6)$$

And by combining these two equations we obtain and rearranging the terms to make an equation in terms of the incoming velocity plus a value dependent on the bcor, mass ratio m, velocity of the center of mass of the bat, distance along the bat, and mass of the bat.

$$v'_2 = v_2 + \frac{m(1 + e)(v_c - v_2 + \omega y)}{[1 + m + (\frac{m_2}{I_c})y^2]} \quad (1-7)$$

This is the equation for calculating the outgoing ball velocity versus its point of contact on the bat and from this equation they were able to produce a graph of outgoing velocity versus position of impact along the barrel. Our group was able to replicate their results using these equations to obtain:

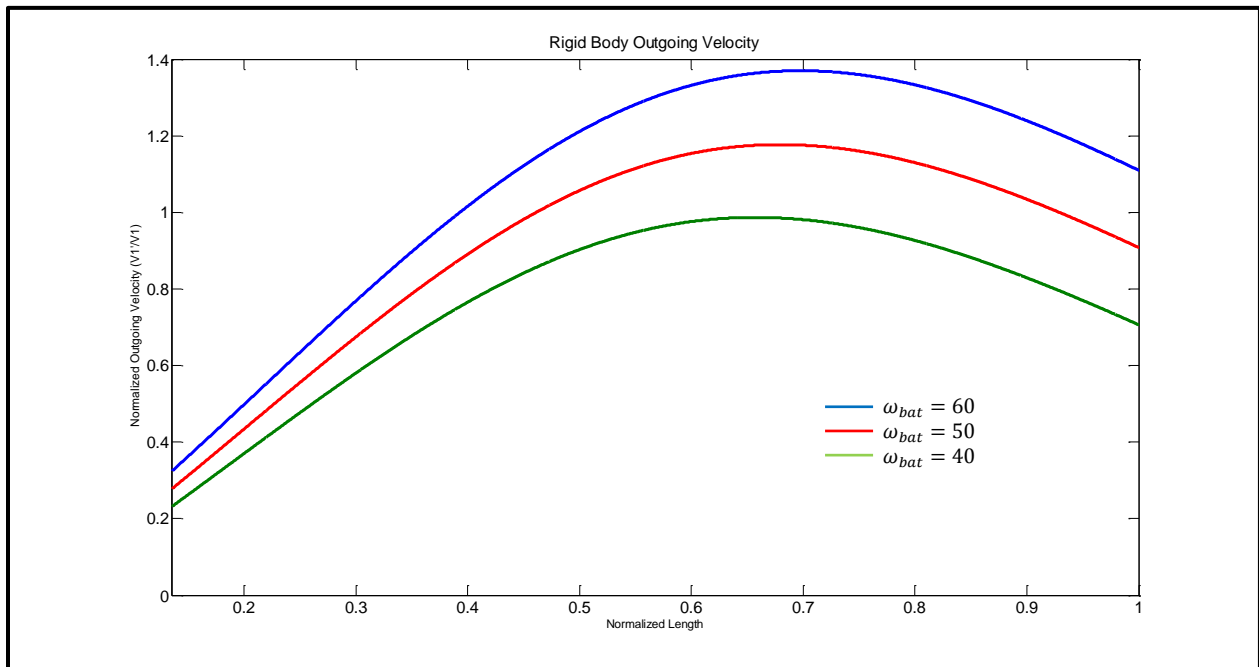


Figure 1 - Rigid Body Outgoing Velocity

Notice this assumes a uniform cross sectional area.

Chapter 2: Homogenous Vibrational Analysis

In this section we analyzed the vibration characteristics of a bat to determine its modes of vibration. First we started by looking into the vibrational characteristics of a beam with a given initial shape. From this model we could move on to a dowel with a given initial shape which gave us the model we used to represent our bat. The initial shape of the dowel represents its displacement immediately after the impact of a ball (which is the force that causes vibration).

Figures one and two show a free body diagram of a bent beam where $f(x,t)$ is the external forces, $M(x,t)$ is the bending moment within the beam, and $V(x,t)$ is the shear forces within the beam.

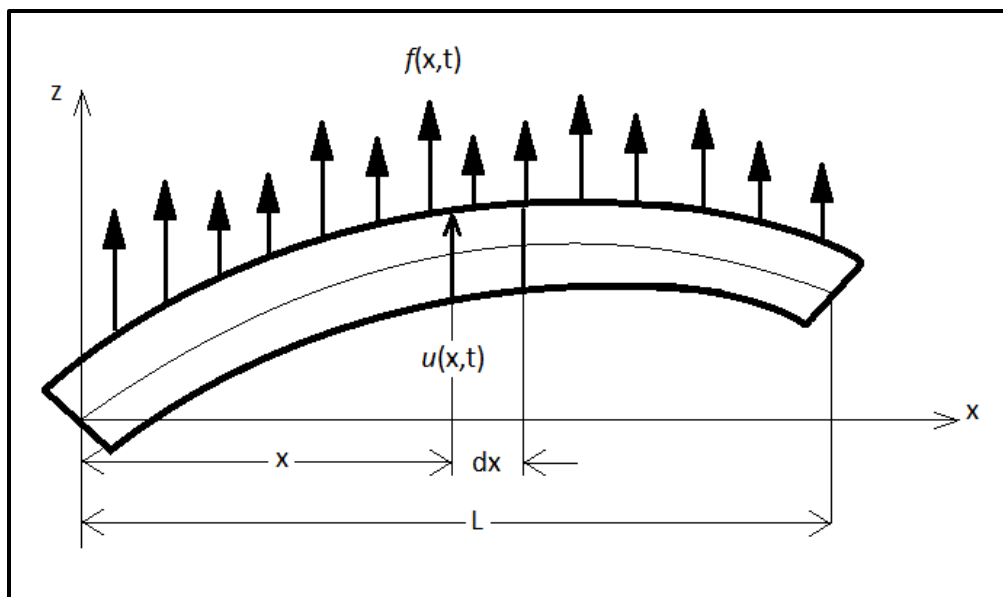


Figure 2 – Longitudinal Free Body Diagram

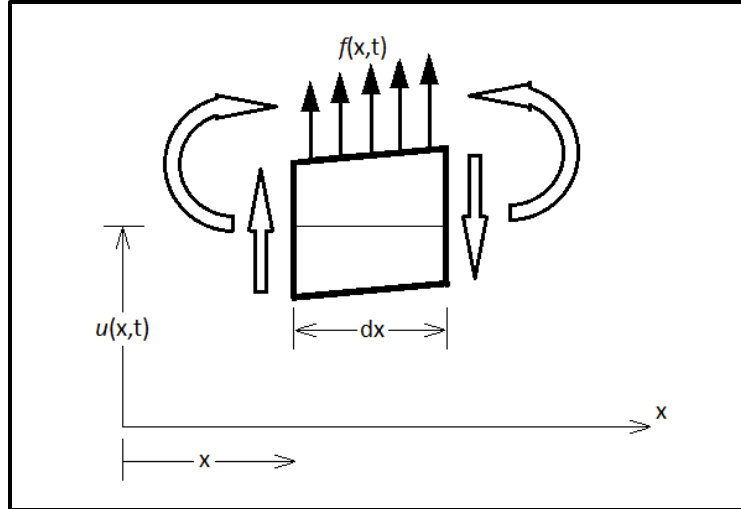


Figure 3 - Side View of Free Body Diagram

Summing the forces on the beam element in the z- direction results in the following equation:

$$-(V + dV) + f(x, t)dx + V = \rho A(x)dx \frac{\partial^2 w(x, t)}{\partial t^2} \quad (2-1)$$

where $\rho A(x)dx \frac{\partial^2 w(x, t)}{\partial t^2}$ is the inertia force acting on the element beam, ρ is the mass density, and $A(x)$ is the variable cross-sectional area of the beam. Summing the moments of force about the left most point on the centroid gives:

$$(M + dM) - (V + dV)dx + f(x, t)dx \frac{dx}{2} - M = 0 \quad (2-2)$$

Rewriting $dV = \frac{\partial V}{\partial x} dx$ and $dM = \frac{\partial M}{\partial x} dx$ and eliminating any terms involving a dx^2

(for the sake of our calculation $dx^2 \approx 0$) equations (2-1) and (2-2) can be rewritten as:

$$-\frac{\partial V(x, t)}{\partial x} + f(x, t) = \rho A(x) \frac{\partial^2 w(x, t)}{\partial t^2} \quad (2-3)$$

$$\frac{\partial M(x, t)}{\partial x} - V(x, t) = 0 \quad (2-4)$$

Using the relation of $V(x,t)$ to $M(x,t)$ in the previous equation to equation (2-3), the combination becomes:

$$-\frac{\partial^2 M(x, t)}{\partial x^2} + f(x, t) = \rho A(x) \frac{\partial^2 w(x, t)}{\partial t^2} \quad (2-5)$$

The moment with respect to the z-axis from the Euler-Bernoulli beam theory is

$$M_z(x, t) = EI_{zz}(x) \frac{\partial^2 w(x, t)}{\partial x^2} \quad (2-6)$$

While the shear force defined by the derivative of the moment with respect to x (relation of equation (2-6))

$$V_y(x, t) = \frac{\delta}{\delta x} \left[EI_{zz}(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right] \quad (2-7)$$

where E is the Young's Modulus of the beams material and $I_{zz}(x)$ is the moment of inertia of the beams cross section about the y-axis. Using the assumed Free- free boundary conditions:

$$M_z(x, t) = 0 \quad (2-8)$$

$$V_y(x, t) = 0 \quad (2-9)$$

The overall equation from Newton second law of motion ($F=ma$).

$$\frac{\partial^2}{\partial x^2} \left[EI_{zz}(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 w}{\partial t^2} = f(x, t) \quad (2-10)$$

A system assuming a uniform area, density, and Young's modulus was used to simplify equation (2-10). In later upcoming projects, the cross sectional area can be varied along the x-direction as a real world baseball bat does. By making these simple assumptions as well as assuming unforced vibrations ($f(x,t)=0$) the following partial differential equation was found:

$$c^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \quad (2-11)$$

Where $c^2 = \frac{EI_{zz}}{\rho A}$.

Separation of variables for the partial differential equation for the deflection s in the y direction:

$$w(x, t) = W(x)T(t) \quad (2-12)$$

With the separated solution the fourth derivative with respect to x and the second derivative with respect to t are taken. The following equation show the result of this procedure.

$$c^2 \frac{\partial^4 W}{\partial x^4} T + W \frac{\partial^2 T}{\partial t^2} = 0 \quad (2-13)$$

Then the variables are separated so only a function of x is on the left hand side and only a function of t is on the right hand side. Because these functions are equal to one another they must also be equal to the same constant. This provides the following equality:

$$\frac{c^2 \partial^4 W}{W \partial x^4} = -\frac{\partial^2 T}{\partial t^2} \frac{1}{T} = \omega^2 \quad (2-14)$$

The left hand side and the right hand side give separate ordinary differential equations (equations 15 and 16) that can be solved using simpler methods that would have previously been impossible with the partial differential equation.

$$\frac{\partial^2 T}{\partial t^2} + \omega^2 T = 0 \quad (2-15)$$

$$\frac{\partial^4 W}{\partial x^4} - \frac{\omega^2}{c^2} W = 0 \quad (2-16)$$

The solution to equation 15 is

$$T(t) = A \cos \omega t + B \sin \omega t \quad (2-17)$$

Where ω is known as the frequency of vibration. Allowing β^4 to equal $\frac{\omega^2}{c^2}$, the frequency is equal to

$$\omega = (\beta L)^2 \sqrt{\frac{EI_{zz}}{\rho AL^4}} \quad (2-18)$$

where L is the length of the baseball bat. The initial shape (w_0) and velocity (v_0) of the bat are defined by the following equations:

$$w(x, 0) = w_0(x) \quad (2-19)$$

$$\frac{dw}{dt}(x, 0) = v_0(x) \quad (2-20)$$

Where $u_0(x) = a_0 \sin \pi X$ (a sin function was used to mimic an initial shape) because of the nature of the bat where a_0 is the displacement at the midpoint of the beam. The initial velocity is defined by $v_0(x) = 0$ because the initial velocity of the bat will only affect the rigid body mode

of the bat (e.g. initial velocity will not affect the vibrations of the bat; it will only rotate and translate it).

The solution to equation 16 is

$$W(x) = D\cos \beta x + E\sin \beta x + F\cosh \beta x + G\sinh \beta x \quad (2-21)$$

The baseball bat is modeled as a free-free system because the force imparted by the baseball is far greater than the impact force of the ball. From the definition of a free-free system shown in equations (2-3) and (2-4) the following boundary conditions are realized:

$$\left. \frac{d^2W}{dx^2} \right]_{x=0} = 0 \quad \text{and} \quad \left. \frac{d^2W}{dx^2} \right]_{x=L} = 0 \quad (2-22)$$

$$(2-23)$$

$$\left. \frac{d^3W}{dx^3} \right]_{x=0} = 0 \quad \text{and} \quad \left. \frac{d^3W}{dx^3} \right]_{x=L} = 0 \quad (2-24)$$

$$(2-25)$$

Using boundary conditions for $x=0$ into equation (2-21), the constants were able to be related. Equations (2-26) and (2-27) gives the relations

$$F = D \quad \text{and} \quad E = G \quad (2-26)$$

$$F = D \quad \text{and} \quad E = G \quad (2-27)$$

Equations (2-23) and (2-25) gives two relations for the constant α defined as $\frac{F}{G}$:

$$\alpha = \frac{\sin \beta L - \sinh \beta L}{\cosh \beta L - \cos \beta L} = \frac{\cos \beta L - \cosh \beta L}{\sinh \beta L - \sin \beta L} \quad (2-28)$$

Simplifying equation (2-28) gives the expression

$$1 = \cos \beta L \cosh \beta L \quad (2-29)$$

which is impossible to solve for βL explicitly, but implicit solutions can be found graphically or by using a solver.

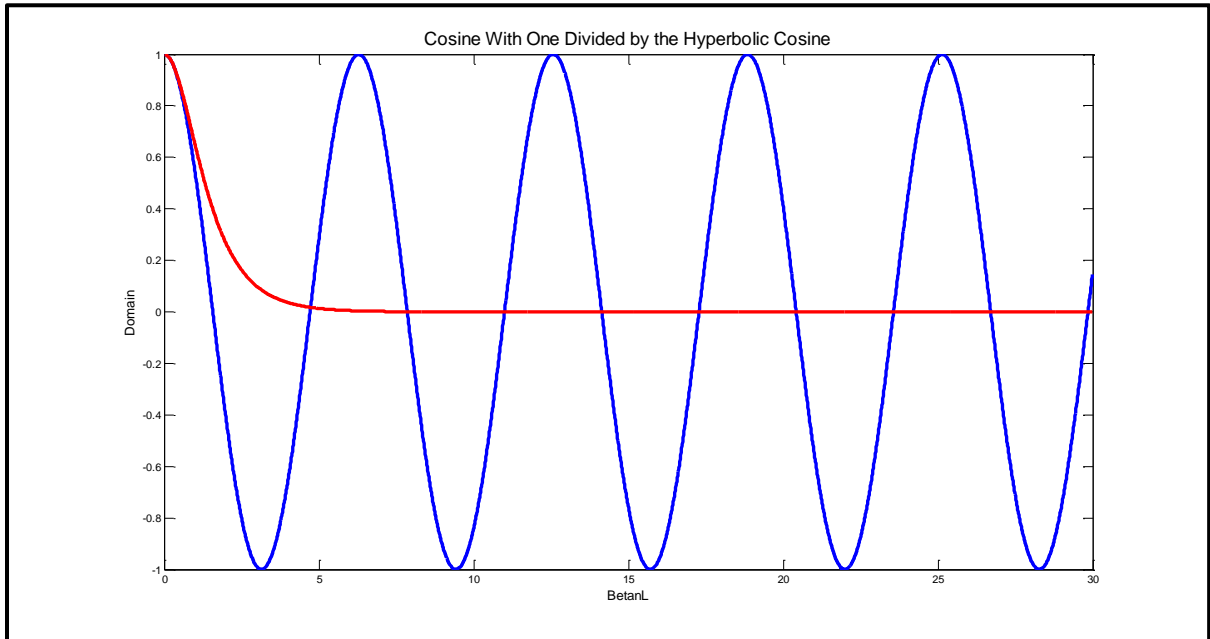


Figure 4 - Solving for $\beta_n L$

Figure 4 shows the cosine function plotted with the inverse of the hyperbolic cosine. The intersection of the two graphs gives the values for βL . The first few values for βL are solved by using the matlab solver, but for higher values of n the solution mimics the zeros of the cosine function. This gives an infinite amount of solutions for βL so it will be further denoted by $\beta_n L$. With $\beta_n L$ found the frequency ω_n is known. The x variables in equation (2-21) were normalized to make $X = \frac{x}{L}$. This allows $\beta_n L = B_n$ to be used seamlessly in (2-21). By rearranging the constants from equations (2-26) and (2-28) equation (2-30) becomes the infinite solutions to the new mode shape function:

$$W_n(x) = \sinh B_n X + \sin B_n X + \alpha_n (\cosh B_n X + G \sinh B_n X) \quad (2-30)$$

This also yields an infinite number of solutions for the function T(t) in equation (2-17)

generating equation (2-31):

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t \quad (2-31)$$

Figure 5 shows the first nine mode shapes (equation (2-30)) as a function of X.

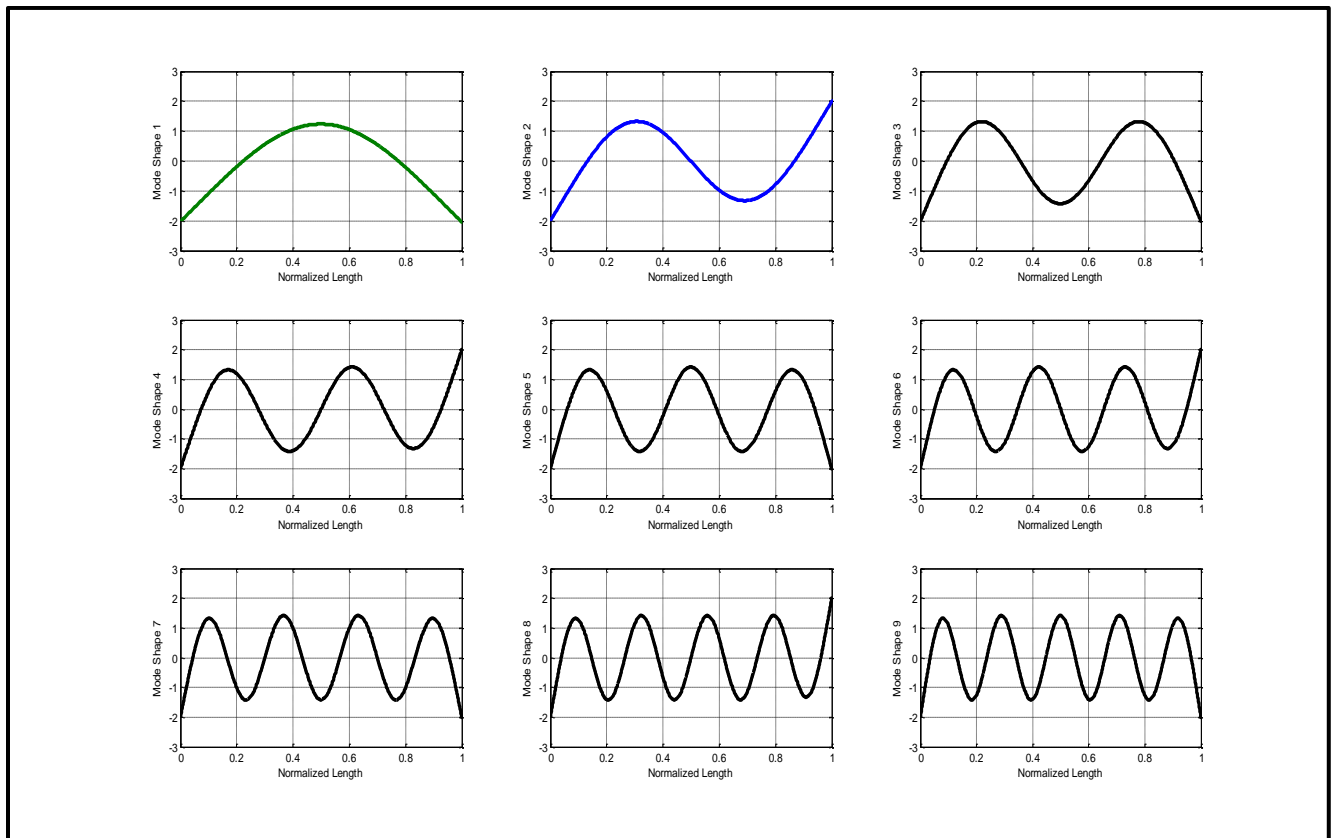


Figure 5 - Mode Shapes

Because of the infinite number of solutions for U(x) and T(t) the final solution can be expressed by the sum of all of the solutions. Combining equations (2-30) and (2-31) into the relation from equation (2-12) gives the general solution

$$w(x, t) = \sum_{n=1}^{\infty} W_n(x) T_n(t) \quad (2-32)$$

Where A_n and B_n from equation (2-31) are still unknown, but can be found by using the initial conditions stated in equations (2-19) and (2-20). Using these initial conditions the following equations were derived:

$$w(x, 0) = w_o(x) = \sum_{n=1}^{\infty} A_n W_n(x) \quad (2-33)$$

$$w_t(x, 0) = v_o(x) = \sum_{n=1}^{\infty} B_n \omega_n W_n(x) \quad (2-34)$$

Multiplying both sides of both equations by $W_n(x)$ and integrating over the length of the beam induces the orthogonality of the mode shapes, $W_n(x)$. The following integration of each of the $W_n(x)$'s multiplied together. Notice the following relation:

$$\int_0^L W_n(x) W_m(x) dx = \begin{cases} 0 & \text{when } n \neq m \\ I_{nn} & \text{when } n = m \end{cases} \quad (2-35)$$

Orthogonality Matrix									
	W1	W2	W3	W4	W5	W6	W7	W8	W9
W1	1.035942	3.71E-14	1.06E-05	-6.32E-12	1.49E-05	1.41E-10	1.91E-05	-1.30E-08	2.02E-05
W2	0	0.998458	3.31E-13	1.47E-05	-1.27E-10	1.88E-05	3.20E-08	2.30E-05	2.87E-06
W3	0	0	1.000082	-2.83E-12	1.89E-05	-1.65E-10	2.30E-05	1.39E-08	2.46E-05
W4	0	0	0	1.000016	-7.59E-11	2.30E-05	2.22E-08	2.72E-05	2.37E-06
W5	0	0	0	0	1.000023	-8.63E-10	2.72E-05	3.49E-08	2.93E-05
W6	0	0	0	0	0	1.000027	1.58E-08	3.14E-05	1.94E-06
W7	0	0	0	0	0	0	1.000031	6.00E-08	3.38E-05
W8	0	0	0	0	0	0	0	1.000035	1.63E-06
W9	0	0	0	0	0	0	0	0	1.000037

Figure 6 - Orthogonality Matrix

$B_n=0$ because $v_o=0$. Solving for the A_n 's gives the following solution:

$$A_n = \frac{1}{I_{nn}} \int_0^L w_o W_n(x) dx \quad (2-36)$$

The graph below shows an initial shape given to the homogenous solution. This shape is modeled by the equation $w_0 = 0.1 \sin\left(\frac{x}{2\pi}\right)$. This shape is similar to the first mode shape so the summation of the graph follows the shape of the first mode closely shown in figure 7.

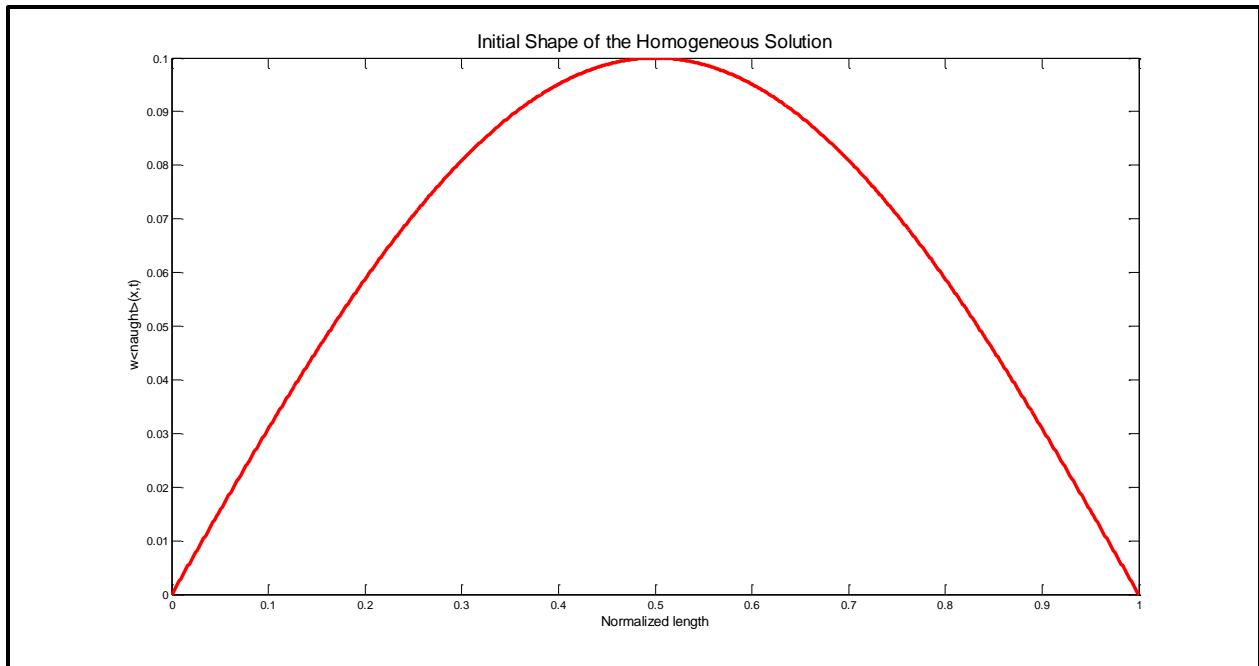


Figure 7 - Initial Shape of Homogeneous Solution

The graph bellow shows the nine separate plots at one time $t = \frac{\pi}{4\omega_1}$. The table following the graph show the error defined by the maximum distance between one plot and the next plot (i.e. two term summation compared to three term summation).

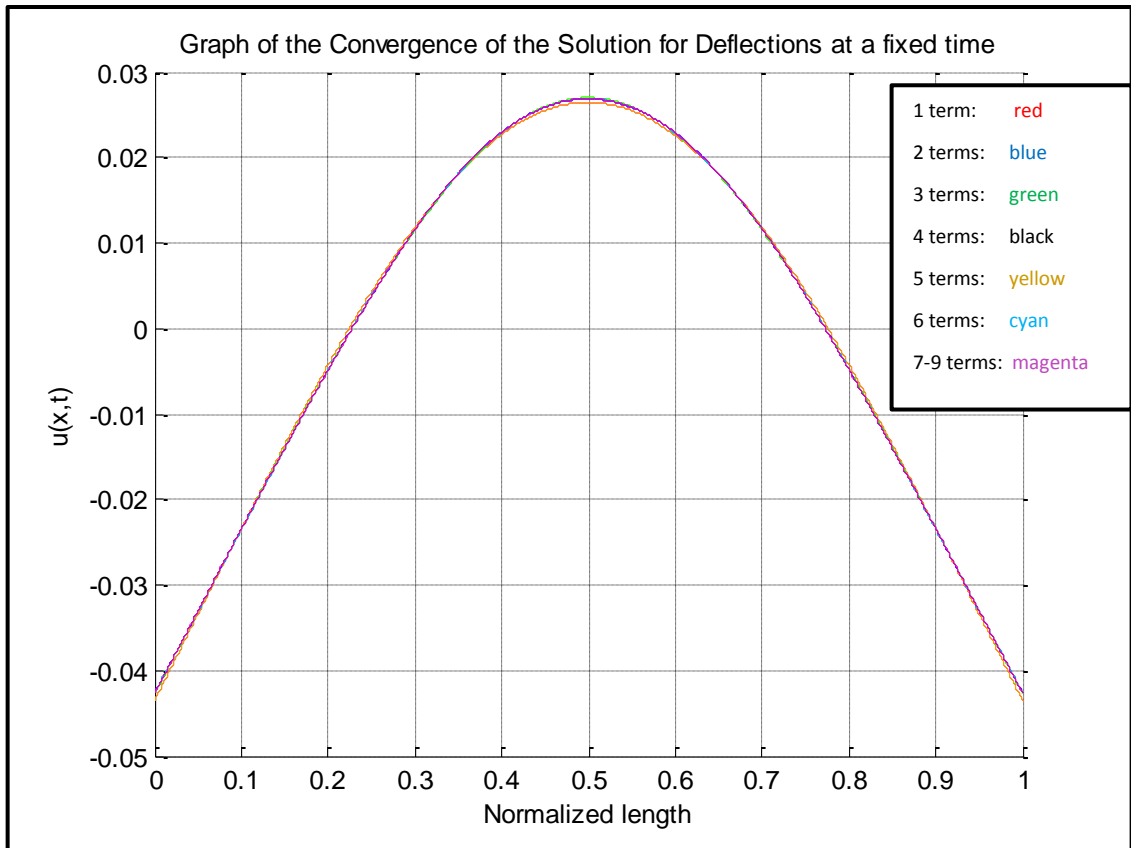


Figure 8 - Deflection Convergence

Error Difference	Maximum
Between:	Difference:
1 & 2	1.44E-15
2 & 3	0.000770952
3 & 4	1.53E-13
4 & 5	0.000137314
5 & 6	7.07E-12
6 & 7	6.48E-05
7 & 8	1.41E-09
8 & 9	3.12E-05

This convergence is also shown by the A values found numerically:

Numerical Values for A_n/L	
A1/L	0.030225263
A2/L	-1.29E-15
A3/L	0.000854218
A4/L	1.03E-13
A5/L	0.000139396
A6/L	-7.36E-12
A7/L	4.04E-05
A8/L	7.07E-10
A9/L	1.58E-05

Figure 9 - Values for A_n/L

Because the value for all even values of A (A_2/L ; A_4/l etc.) are so close to zero, the final homogeneous solution to describe the behavior of the beam can be simplified to the three term summation shown in equation (2-37):

$$w(x, t) = \sum_{n=1,3} W_n(x) T_n(t) \quad (2-37)$$

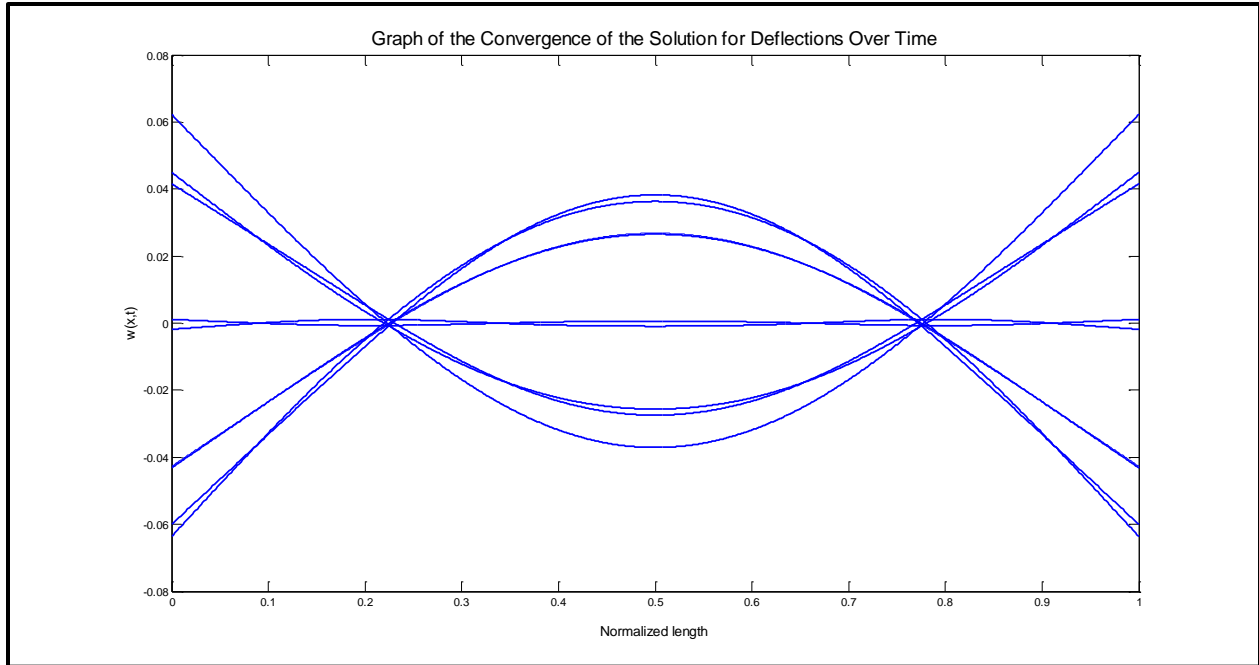


Figure 10 - Convergence over Time

Figure 10 shows the first and third term summation over a time period (the first frequency, $2\pi/\omega_1$, divided into eighths)

This solution only includes the first two nonzero terms ($n=1,3$) because of the fifth (third nonzero) term does not contribute enough to the equation effect the solution much. This can be seen in the previous figure that graphs the convergence of the term by term summation.

Chapter 3: Impact Vibrational Analysis

After we examined the vibration characteristics of a free/free beam given an initial shape we could then move on to the impact aspect of the problem. Once the vibrational modes of the beam were determined through our previous calculations we then had to examine how each mode reacted after an impact on the bat to simulate what would happen during a bat/ball collision. The general model for any force impact onto a beam is given by the Dirac delta function. This function represents a point impact at a specific point and a specific time on the beam. The graph of this function is represented by a line equal to zero over the entire integral except for an infinite spike at the point of impact. Another special characteristic is the value of the integral over the entire length is equal to one. This infinite spike at a specific point is what excites the vibration of the beam by modeling the impact and, depending on the location of the point impact, will cause different reactions from specific modes of vibration of the beam. How the delta function is defined in time is another interesting characteristic. Just before impact (at $t=0^-$) the shape of the bat is not affected. The bat's shape is then changed and at any time after there are no forces.

To examine the forces of vibration that take place on a bat being struck by a ball, we first examined the unit impulse on string fixed at both ends. The methods used to solve for a unit impulse on a string will then be applied and adjusted for the case of a baseball bat.

The impulse is applied at $l/4$ where l is the length of the string and the initial conditions are set to zero. This leaves us with the following equation:

$$\rho w_{tt}(x, t) + \gamma w_t(x, t) - \tau w_{xx}(x, t) = f(x, t) \quad (3-1)$$

In equation (3-1) $f(x, t)$ is taken to be $\delta(x - a)\delta(t)$. The function in this form represents where and what time the impact force is applied on the string. In the case study for equation (3-1) the impact, a , is equal to one quarter the distance from the end of the string or $l/4$.

Dirac Delta function for string at impact point of $l/4$:

$$f(x, t) = \delta\left(x - \frac{l}{4}\right)\delta(t) \quad (3-2)$$

The Eigen functions (mode shapes) of the un-damped, unforced clamped-clamped string are given in the form:

$$X_n(x) = a_n \sin\left(\frac{n\pi x}{l}\right) \quad (3-3)$$

The X_n represents the given vibrational mode shape for the string, each separate mode will have a different initial shape with different nodes which in turn will force the string to vibrate at a different frequency and amplitude.

The governing equation for the string will be equal to the following:

$$w_n(x, t) = T_n(t) * X_n(x) \quad (3-4)$$

Once again, in the governing equation, X_n represents the mode shape equation for the string. Equation (3-4) for the string can be plugged into equation (3-1). Once the governing equation is substituted into equation (3-1) we then multiply by $X_n(x)$, integrate over the length of the string and solve for $T_n(t)$.

This procedure results in equation (3-5):

$$\{\rho \ddot{T}_n(t) + \gamma \dot{T}_n(t) - \tau \left[-\left(\frac{n\pi}{l}\right)^2 \right] T_n(t)\} \sin\left(\frac{n\pi x}{l}\right) = \delta\left(x - \frac{l}{4}\right) \delta(t) \quad (3-5)$$

Next, we multiply equation (3-5) $\sin\left(\frac{n\pi x}{l}\right)$ and integrate which results in the following:

$$[\rho \ddot{T}_n(t) + \gamma \dot{T}_n(t) + \tau * \left(\frac{n\pi}{l}\right)^2 T_n(t)] \frac{l}{2} = \delta(t) \int_0^l \delta\left(x - \frac{l}{4}\right) \sin \frac{n\pi x}{l} dx = \delta(t) \left(\sin\left(\frac{n\pi}{4}\right)\right) \quad (3-6)$$

The response of an under-damped single degree of freedom system to an impulse can be modeled by the following equation:

$$m\ddot{x} + c\dot{x} + kx = \hat{F}\delta(t) \quad (3-7)$$

where $\delta(t)$ is a unit impulse

$$x(t) = \left(\frac{\hat{F}}{m w_d}\right) e^{-\zeta w_n t} \sin(w_d t) \quad (3-8)$$

where:

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2} m \omega$$

Therefore $w_d = w_n \sqrt{(1 - \zeta^2)}$ and $0 < \zeta < 1$ must be valid.

Our Equation for modeling the motion of a bat is similar to the one shown above except for our equation does not include a damping term. Another difference between the two equations is in the example of the string we are only required to take the second derivative of the mode shape. However, due to the complex structure of the bat in comparison with that of a string we are required to take the 4th derivative in order to retain the same mode shapes of vibration for the bat.

$$\frac{\partial^2}{\partial x^2} \left[EI_{zz}(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 w}{\partial t^2} = J\delta(x - a)\delta(t) \quad (3-9)$$

$$EI_{zz}w_{xxxx} + \rho Aw_{tt} = J\delta(x - a)\delta(t) \quad (3-10)$$

After closer examination the two above equations are equal for a uniform cross sectional area. In both equation (3-1) and (3-9) the Dirac delta function is defined by the following:

$$f(x, t) = J\delta(x - a)\delta(t) \quad (3-11)$$

The function in this form represents where and what time the impact force is applied on the bat. In the case study for equation (3-1) the impact, a , is equal to one quarter the distance from the end of the string. In the case of the uniform bat (equation (3-9)) the force is solved for any possible impact points along the bat defined as the variable (a).

The reasoning for representing the impact point with the symbol (a) is because this will allows us to evaluate the impulse on the bat for varying impact locations. In the case of the string, the impact point is taken to be a single location of $l/4$, but evaluating at a single impact point is not sufficient for our example because we wanted to be able to vary of impact point

over the length of the bat. This change can be seen by comparing Dirac delta function from (3-2) and (3-11). Equation (3-2) represents the Dirac delta function for the string with a single impact point of $l/4$ and equation (3-11) represents the Dirac delta function for the baseball bat with a varying impact point of (a).

The X_n in the string case represents the given vibrational mode shape for the string, each separate mode will have a different initial shape with different nodes which in turn will force the bat to vibrate at a different frequency and amplitude. In our derivation of this equation we use the term W_n to represent X_n in the following equation. W_n represents the vibration made shape for the baseball bat. The equation for $W_n(x)$ is obtained by using the method of separation of variables.

$$W_n(x) = \sinh B_n X + \sin B_n X + \alpha_n (\cosh B_n X + G \sinh B_n X) \quad (3-12)$$

Where α_n is equal to:

$$\alpha_n = \frac{\sin \beta_n L - \sinh \beta_n L}{\cosh \beta_n L - \cos \beta_n L} \quad (3-13)$$

And $B_n = \beta_n L$.

In the case of the string, the governing equation is equal to $w_n(x,t) = T_n(t)X_n(x)$, but in the case of the baseball bat the governing equation will take the form of $w_n(x,t) = T_n(t)W_n(x)$. This change in the governing equation is simple due to the change in the variables used to define the mode shape equation. In the case of the string, the mode shape equation is represented by X_n and therefore the governing equation for the string will be $w_n(x,t) = T_n(t)X_n(x)$. In the case of

the baseball bat, the mode shape equation is represented by W_n and therefore the governing equation for the string will be

$$w_n(x, t) = T_n(t) * W_n(x) \quad (3-14)$$

The governing equation for the bat can be plugged into equation (3-9). In the case of the bat we must take the 4th derivative of the modes of vibration for the baseball bat because equation (3-9) requires us to do so.

This procedure results equation (3-15) for the case of the bat:

$$[EI_{zz}(\beta_n^4)(W_n)]T_n(t) + \rho A \ddot{T}_n(t)W_n = f(x, t) \quad (3-15)$$

By taking the fourth derivative of equation (3-15) we are left with the same basic equation with an extra term of B_n^4 because taking the 4th derivative of a sine or cosine term leaves you with the original sine or cosine value. In the second portion of the equation the derivative is taken in terms of time (t), so the $W_n(x)$ value which depend up x, is left alone while the $T_n(t)$ value is taken to the second derivative.

In the case of the bat we multiplied equation (3-15) by $W_n(x)$ and integrated over length of the bat which leaves us with the following equation:

$$L[EI_{zz}(B_n^4)T_n(t) + \rho A \ddot{T}_n(t)] \int_0^L W_n^2 dx = \int_0^L f(x, t)W_n(x)dx \quad (3-16)$$

Our next step was to rewrite equation (3-16) in the form of a single degree of freedom oscillator. This resulted in the following equation:

$$\rho A L \ddot{T}_n(t) + [EI_{zz}(\beta_n^4 L)(W_n)]T_n(t) = J W_n(a) \delta(t) \quad (3-17)$$

In order to solve for the impulse, the next step was to model the response an under damped single degree of freedom system's response to an impulse. The following equation allows us to model such a response:

$$m\ddot{x} + c\dot{x} + kx = \hat{F} \delta(t) \quad (3-18)$$

where $\delta(t)$ is a unit impulse.

$$x(t) = \left(\frac{\hat{F}}{m w_d} \right) e^{-\zeta w_n t} \sin(w_d t) \quad (3-19)$$

where:

$$\zeta = c/2$$

Therefore $w_d = w_n \sqrt{1 - \zeta^2}$ and $0 < \zeta < 1$ must be valid.

The continuous system can be converted to take the form of a single degree of freedom system. This is done by taking the coefficients m , c and k of equation (3-16) and matching them up with the coefficients that result from taking the second derivative with respect to time.

In our case damping forces are not being considered because it does not need to be included when trying to find the energy loss in a system and therefore the damping coefficient c will be taken to be equal to 0.

Using equation (3-16) and relating it equation (3-18) it can be found that equation (3-16) takes the following form in our example:

$$m = \rho AL(I_{nn}) \quad (3-20)$$

$$c = 0$$

$$k = EI_{zz}\beta_n^4 LI_{nn} \quad (3-21)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{EI_{zz}(B_n^4)}{\rho A}} \quad (3-22)$$

The solution to the differential equation for (3-18) is the following:

$$T_n(t) = \frac{J W_n(a)}{\rho ALI_{nn}\omega_n} \sin\omega_n t \quad (3-23)$$

Now that we have solved for $T_n(t)$ we have a solution for $w(x,t) = T_n(t) * W_n(x)$ which takes the form

$$w(x, t) = J \left\{ \frac{L}{\sqrt{EI_{zz}\rho A}} \sum_{n=1}^{\infty} \frac{W_n(a)}{I_{nn}(\beta_n L)^2} \sin \left[\sqrt{\frac{EI_{zz}}{\rho AL^4}} (\beta_n L)^2 t \right] W_n(x) \right\} \quad (3-24)$$

3.1 Convergence of Infinite Solution

Because the $\beta_n L$ takes the form of $\frac{\pi(2n+1)}{2}$ the infinite series takes the form of

$$w_n = D \sum_{n=1}^{\infty} \left(\frac{2}{\pi(2n+1)} \right)^2 \quad (3-19)$$

Where D is on the same order of magnitude for each of the terms in the series so it can be treated as a constant outside of the series. This simplified series has a comparison series of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

By using Theorem 9.13 from Calculus with Analytic Geometry the simplified series converges because the comparison series also converges.^[5]

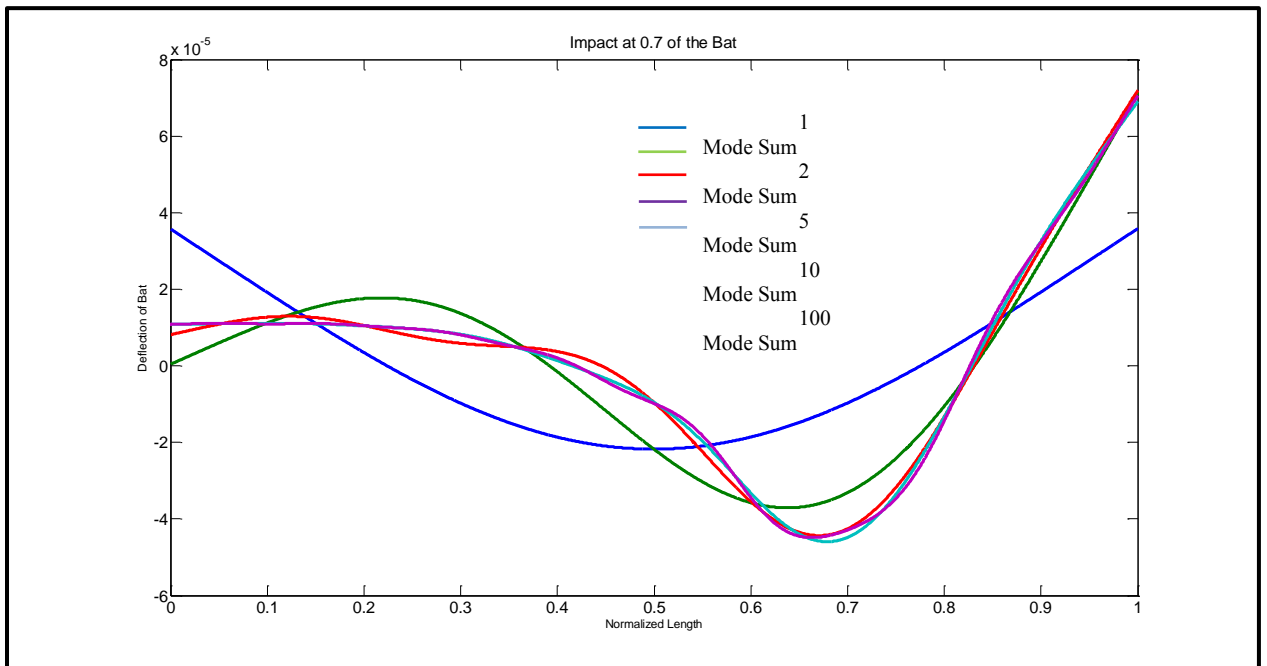


Figure 11 - Mode Shape Convergence

Figure 11 above shows the convergence of mode shapes for the bat. Any vibrating object has an infinite number of mode shapes which describe the shape in which the object will vibrate and the actual shape of the vibrating object is a summation of all the object's mode shapes. However, the higher the number mode shape the less influence it has on the vibrating displacement of the object. So in order to simplify our calculations we needed to determine the

number of modes which the series converges. By plotting the summation of 1, 2, 5, 10, and 100 modes we were able to see that the shape of the 10th and 100th modes was virtually identical which told us that the series converged after 10 terms. This meant that any summation of over 10 modes would give us the same result as the 10 term solution.

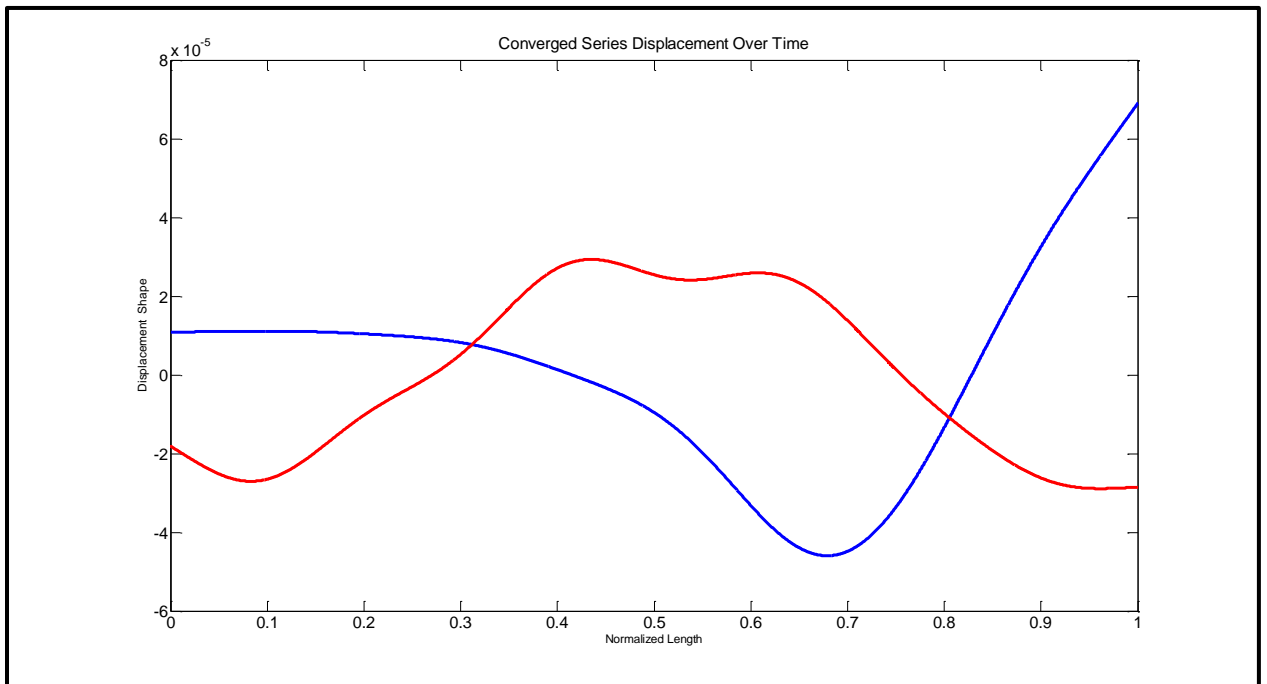


Figure 12 - Converged Series Displacement

Figure 12 above represents the general displacement shape of the bat over time. A video of this time changing graph can be seen at <http://youtu.be/XunyTrXgeNU>. The blue line is a summation of the first 10 modes of vibration immediately after the impact of a ball at 0.7 the normalized length of the bat. The red line on the graph shows the same 10 mode summation at a time half of the period of the first mode later.

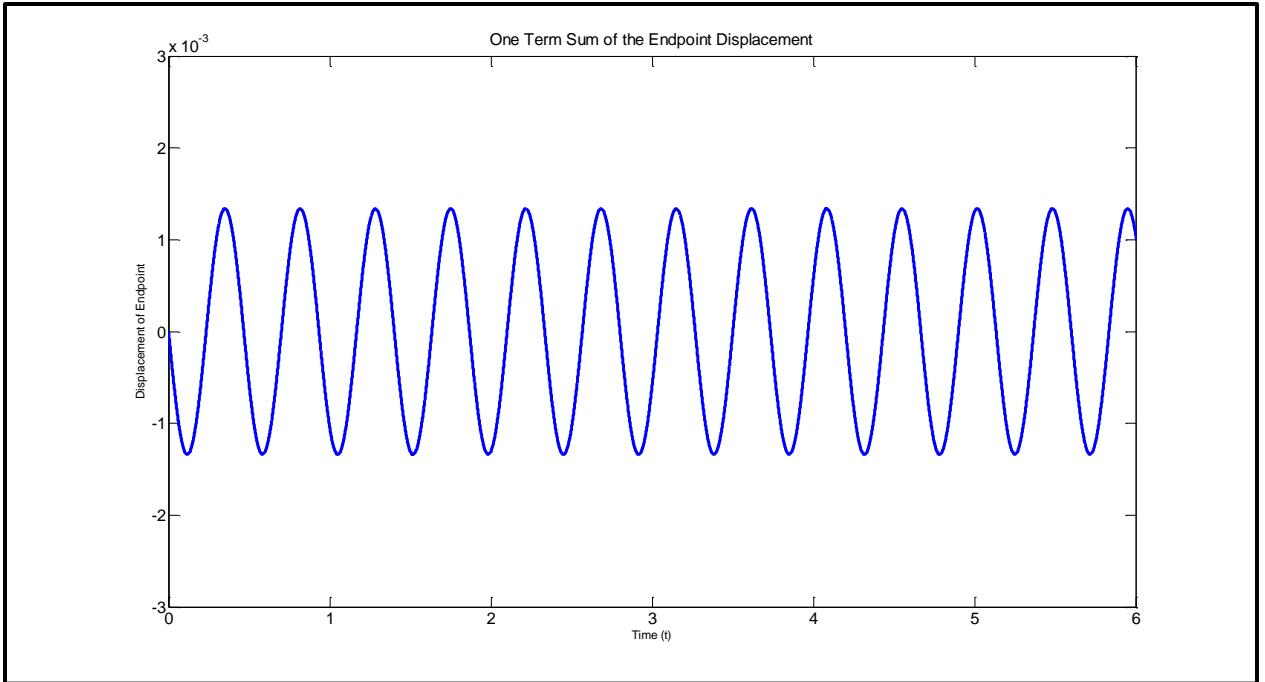


Figure 13 - One Term Sum of Endpoint Displacement

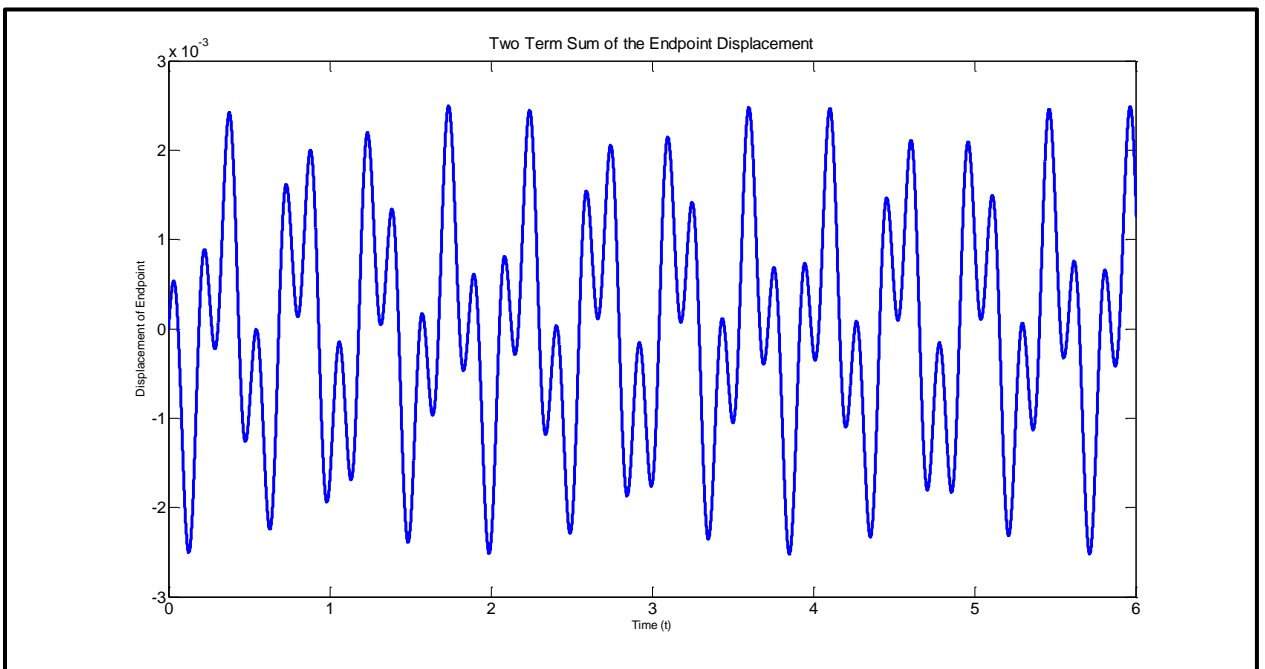


Figure 14 - Two Term Sum of Endpoint Displacement

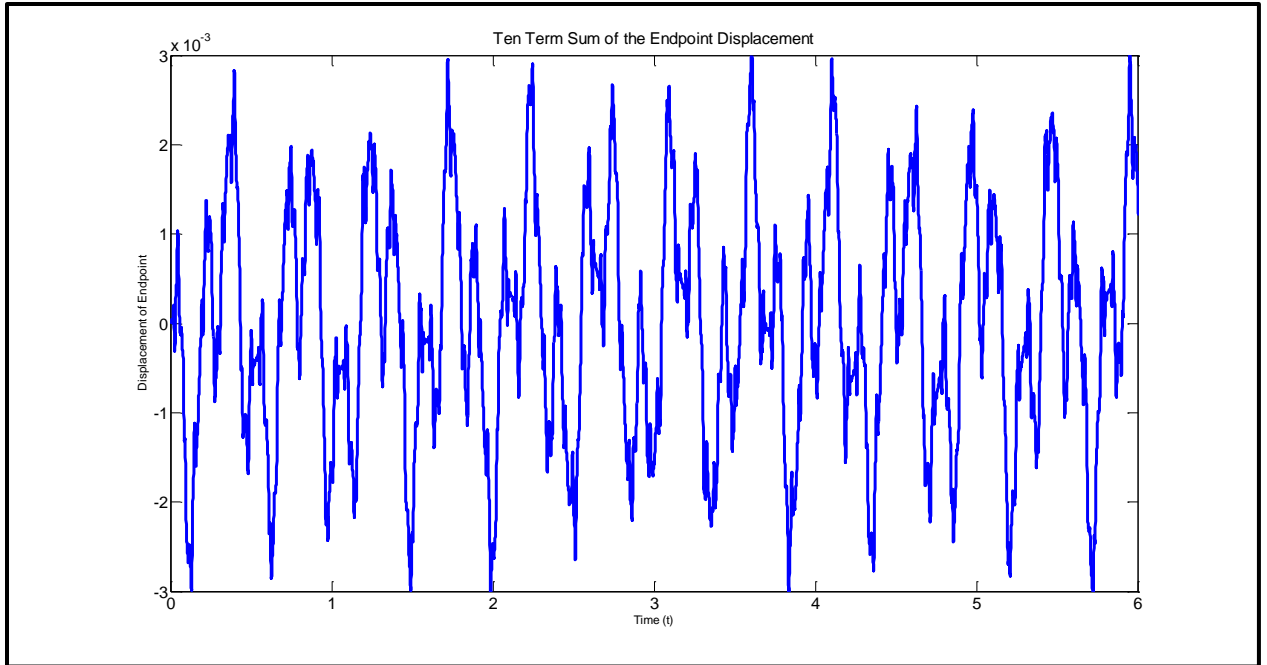


Figure 15 - Ten Term Sum of Endpoint Displacement

Figures 13, 14 and 15 above show the displacement of a single point on the bat over time. For this particular case it is the endpoint of the bat. Figure 13 only takes into account the first mode of vibration which is why the graph has a smooth oscillating shape. The next figure takes into account the first and second modes of vibration which cause the graph to become choppy because of the contributions from each mode. The last figure shows the endpoint displacement when all 10 terms are taken into account. The more terms that are added to the summation the more unpredictable the displacement will become due to all the different influences from the separate mode shapes.

Chapter 4: Vibration Energy Losses

After solving for the displacement and proving the solution to be convergent, a factor of the impulse, J , remained in the solution. This factor is to be solved simultaneously by using both the energy of the rigid model and the energy taken from this system by the bat vibrations. This allowed us to produce graphs of the outgoing velocity with vibrations taken into consideration. The following equations lead to the solving of the impulse and finding the outgoing velocity depending on where on the ball contacted the bat.

By the definition of an impulse (mass time velocity) the following is true:

$$m_1 v_1' = m_1 v_1 + J \quad (4-1)$$

Where m_1 is the mass of the ball, v_1' is the outgoing velocity of the ball, v_1 is the incoming velocity (pitch speed) of the ball, and J is the total (combination of rigid and vibration) impulse of the bat-ball collision. Rearrangement of the equation gives:

$$v_1' = v_1 + \frac{J}{m_1} \quad (4-2)$$

To find the change in kinetic energy (in the form of $KE = \frac{1}{2} mass \times velocity^2$) the outgoing velocity of the ball needs to be squared. This gives the following:

$$v_1'^2 = v_1^2 + 2v_1 \frac{J}{m_1} + \left(\frac{J}{m_1}\right)^2 \quad (4-3)$$

This is used to find the total change in energy of the ball which is simplified to:

$$\Delta T = \frac{1}{2} m_1 (v_1'^2 - v_1^2) = v_1 J + \frac{J^2}{2m_1} \quad (4-4)$$

A rearrangement of equation (4-4) gives the following:

$$\frac{J^2}{2m_1} + v_1 J - \Delta T = 0 \quad (4-5)$$

This total change in energy is also defined as the change in the energy from the rigid body solution minus the energy from the velocity of the bat due to the vibrations.

$$\Delta T = (\Delta T)_r - (\Delta T)_v \quad (4-6)$$

The change in energy for the rigid body is defined by

$$(\Delta T)_r = \frac{1}{2} m_1 ((v'_1)_r^2 - v_1^2) \quad (4-7)$$

And can be found using the outgoing velocity found in the previous MQP's (chapter 1).

This outgoing velocity is defined by equation (1-7). Figure 16 shows the shapes of the change in energy found for the rigid body model.

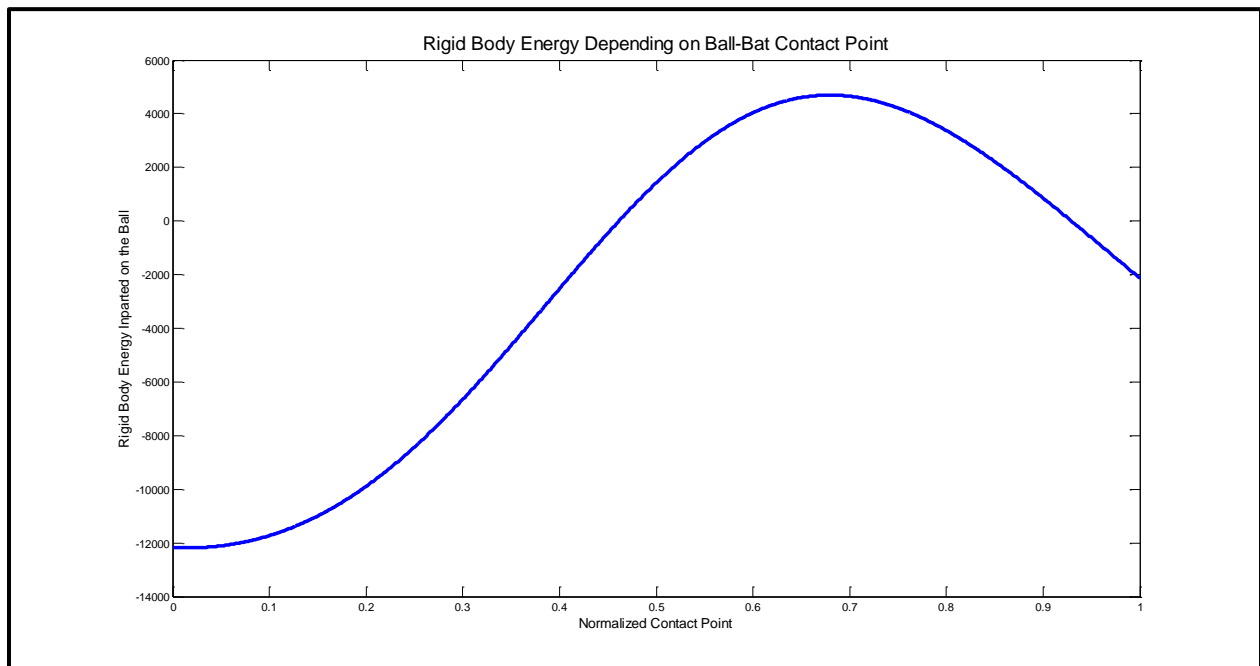


Figure 16 - Rigid Body Energy Depending on Contact Point

The energy absorbed by the vibrations is found in the form of $KE = \frac{1}{2} mass \times velocity^2$ as well. One half is multiplied by the integral of ρ times A over the length of the bat (mass) which is multiplied by the square of the velocity.

$$(\Delta T)_v = \frac{1}{2} \int_0^L \rho A v(x, t)^2 dx \quad (4-8)$$

The velocity of the bat is found by taking the time derivative of the displacement found using the vibrations equation from chapter 3 shown below:

$$w(x, t) = J \left\{ \frac{L}{\sqrt{EI_{zz}\rho A}} \sum_{n=1}^{\infty} \frac{W_n(a)}{I_{nn}(\beta_n L)^2} \sin \left[\sqrt{\frac{EI_{zz}}{\rho A L^4}} (\beta_n L)^2 t \right] W_n(x) \right\} \quad (4-9)$$

This series has been proven to be convergent using simply calculus theorems. However, when the time derivative of this equation is taken analytically or numerically (to find the velocity of the bat) the solution gives an infinite series that sums to infinity. Although this is true the derivative analysis gives the solution gives the form of:

$$v(x, t) = J \frac{d}{dt} \{w(x, t)\} \quad (4-10)$$

thus the energy takes the following form:

$$(\Delta T)_v = J^2 T \quad (4-11)$$

Combining equations (4-11) with (4-6) and inserting the result into equation (4-5) gives:

$$\left(\frac{1}{2m_1} + T \right) J^2 + v_1 J - (\Delta T)_r = 0 \quad (4-12)$$

Using the quadratic formula where $a = \left(\frac{1}{2m_1} + T\right)$, $b = v_1$, and $c = -(\Delta T)_r$ the

solution for J is the following:

$$J = \frac{-v_1 \pm \sqrt{v_1^2 + 4\left(\frac{1}{2m_1} + T\right)(\Delta T)_r}}{2\left(\frac{1}{2m_1} + T\right)} \quad (4-13)$$

Rearranging the variables in the solution for J and determining whether to use the plus or the minus sign¹ of the solution gives:

$$J = \frac{m_1}{2} \left\{ -v_1 + \sqrt{v_1^2 + 4\left(\frac{1}{2} + E_v(a)\right)\frac{(\Delta T)_r}{m_1}} \right\} \frac{1}{\left(\frac{1}{2} + E_v(a)\right)} \quad (4-14)$$

Where $E_v(a) = m_1 T$.

Based on equations (4-9) and (4-10) $E_v(a)$ takes the form of:

$$E_v(a) = \gamma \sum_{n=1}^{\infty} \frac{W_n(a)}{(2n + 1)^p} \quad (4-15)$$

Where γ is a magnitude of the vibrations, and p defines the speed of convergence.

Determination of these values were found by relating the incoming velocity to the outgoing velocity in a real world situation.

This can then be used to find the outgoing velocity of the baseball by the rearrangement (equation (4-17)) of the definition of the impulse (4-16) in a baseball bat shown in equations.

¹ Note the correct sign can be determined by making T (the contribution to impulse of vibrational energy) equal to zero in equation (4-13). The resulting solution when simplified and rearranged, it gives equation (4-2) for the rigid system only.

$$J = m_1(v'_1 - v_1) \quad (4-16)$$

$$v'_1 = v_1 + \frac{J}{m_1} \quad (4-17)$$

Figure 17 shows the Rigid body model plotted along with the 1, 2, and 10 mode summations for outgoing ball velocity using this energy method. The figure demonstrates the oscillating properties of the mode shapes because each mode shape will match the velocity of the pervious shape at its nodes of vibration. This is why the first mode touches the rigid body model at 2 points and the second mode touches the first at its 3 nodes of vibration. The 10 mode summation is considerable lower than the first and second because it takes into account all of the energy lost through the first 10 modes and gives us an accurate representation of the outgoing velocity.

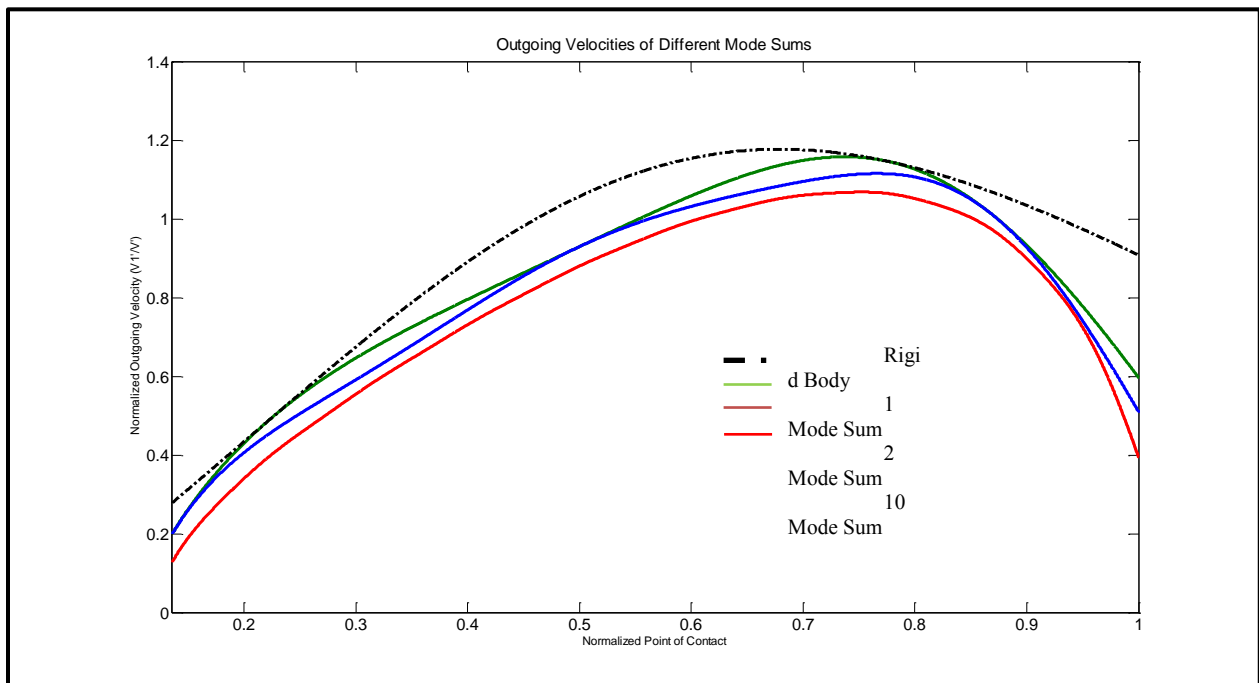


Figure 17 - Outgoing Velocity of Different Mode Sums

Figure 18 shows the 10 modes summations plotted against the rigid body model for 3 different swing speeds. The overlaid graphs show how the energy loss due to vibration affects the outgoing ball velocity along the barrel as well as a shift in the location of maximum outgoing or “sweet spot” to a location between the nodes of the first and second mode of vibration.

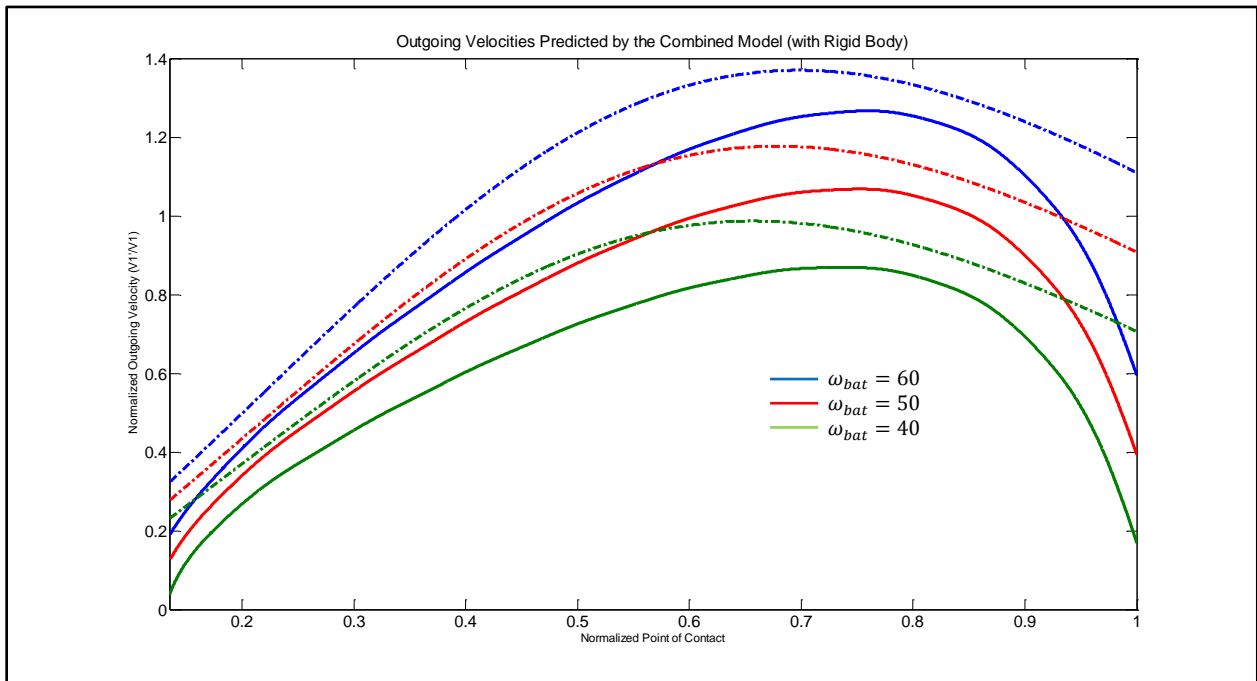


Figure 18 - Outgoing Velocity of 10 Mode Sums vs. Rigid Body

Conclusion

Our studies have shown that in terms of vibrational analysis the “sweet spot” of a bat lies somewhere between the nodes of the first and second modes of vibration, this means a ball struck at this point will exit the bat with the highest outgoing velocity. The reason for the peak in outgoing velocity is because when a ball is struck on the “sweet spot” of a bat the majority of energy is transferred from the bat to the ball. When a ball strikes outside of the “sweet spot” more energy will be lost during the collision due to the excitement of the modes of vibration within the bat, this phenomena is the reason a player sometimes feels a stinging vibration in his hands after hitting a ball. The vibration is caused by the excitement of the bats modes of vibration and the energy that goes into exciting these modes takes away from the energy transferred into the ball resulting in a lower outgoing velocity.

By comparing our findings to the “rigid body” model of a bat ball collision we were able to see the amount of energy lost due to vibration versus the energy transfer of an ideal elastic collision. On average for 3 different swing speeds we found that there was an energy drop of about 20 percent when vibration was factored into the collision. This was true for all points of impact along the bat even our newfound “sweet spot”. This is because even when the player strikes the sweet zone and does not excite the first and second modes of vibration upon impact there are still an infinite number of modes which are excited and although they are not large enough to feel in the players hands they still take away from the total energy transfer into the ball.

The models presented within our report are those representing the vibration of a constant profile wooden dowel. Our work would benefit from future analysis of how changing the profile of a bat will affect the location of the nodes of vibration and alter the size of the sweet spot.

Appendix A: References

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- [7] Sutton, Andrew D. "USING MODAL ANALYSIS TO INVESTIGATE BAT-BALL PERFORMANCE OF BASEBALL BATS." Thesis. University of Massachusetts Lowell, 2008 12 May 2010. Web. 7 Mar. 2012. <<http://m-5.uml.edu/umlbrc/Publications/Thesis-Andrew%20Sutton-2010.pdf>>.

Appendix B: Matlab Script

```
% Mode Shapes, Homogeneous Solution, Initial Shape, and Rigid Energy
% MQP
% Baseball Bat Vibration Analysis

%% Modes with Initial Shape
close all; clear all; clc;
%Non dimentinalized bat
x = 0:.033:33;
L = 33;
X = x/L;

%Values from properties of maple wood

E = 2.02e6; %psi
rhomass = 0.0257; %lbm/in^3
rhom = rhomass*32.2; %slug/in^3
radius = .83885; %radius of a bat based on density and
length)
A = (pi*radius^2); %Area
Izz = pi*(radius^4)/4; %moment area of inertia
omegabats = 50; %rad/sec
d = 2.5; %center of rotation from the bat handle
D = d/L; %normalized d
ao = 0.1; %intial displacment height (10% of the
length)
n = 9; %number of terms

P = omegabats*L^2*sqrt(rhom*A/(E*Izz));
%Solving for the frequency equation (solving for BetanL)
f = inline(' (cos(Beta).*cosh(Beta))-1', 'Beta');
BetanL=zeros(1,n);
for i = 7:n
    BetanL(i) = ((2*i)+1)*(pi/2);
end
BetanL(1) = fzero(f,4.73);
BetanL(2) = fzero(f,7.85);
BetanL(3) = fzero(f,10.995);
BetanL(4) = fzero(f,14.137);
BetanL(5) = fzero(f,17.27);
BetanL(6) = fzero(f,20.42);

%Preallocation
alpha = zeros(length(BetanL),1); Wn = zeros(length(BetanL),length(X));
Inn = zeros(length(BetanL),1); FunBn = zeros(length(BetanL),length(X));
IntBn = zeros(length(BetanL),1); BnoverL = zeros(length(BetanL),1);
omegan = zeros(length(BetanL),1); Funtest =
zeros(length(BetanL),length(X));
intttest = zeros(length(BetanL),length(length(BetanL)));
wn = zeros(length(BetanL),length(X)); w =
zeros(length(BetanL),length(X));
error = zeros(length(BetanL-1),1); FunAn =
zeros(length(BetanL),length(X));
```

```

IntAn = zeros(length(BetanL),1); AnoverL = zeros(length(BetanL),1);

w_naught = ao*sin(pi*X);

%Using each BetanL to find the mode shapes
for i = 1:length (BetanL)
    alpha(i) = (sin(BetanL(i))-sinh(BetanL(i)))./(cosh(BetanL(i))-
cos(BetanL(i)));
    Wn(i,:) =
(sinh(BetanL(i)*X)+(alpha(i)*cosh(BetanL(i)*X)))+(sin(BetanL(i)*X)+(alpha(i)*c
os(BetanL(i)*X)));
    FInn = Wn(i,:).*Wn(i,:);
    Inn(i) = trapz(X,FInn);
    %BC's (initial velocity)
    FunBn(i,:) = Wn(i,:).*(X+D);
    IntBn(i) = trapz(X,FunBn(i,:));
    BnoverL(i) = (P/(BetanL(i)*Inn(i)))*IntBn(i);
    %BC intial shape
    FunAn(i,:) = w_naught.*Wn(i,:);
    IntAn(i) = trapz(X,FunAn(i,:));
    AnoverL(i) = IntAn(i)./Inn(i);
    omegan(i) = (BetanL(i))^2*sqrt((E*Izz)/(rhom*A*L^4));
end

%Orthogonality Matrix
for i = 1:length (BetanL)
    for j = 1:length (BetanL)
        if i <= j
            Funtest (i,:) = Wn(i,:).*Wn(j,:);
            inttest (i,j) = trapz(X,Funtest(i,:));
        end
    end
end

%Plotting Mode shapes
figure(1)
for i = 1:length(BetanL)
    subplot(ceil((length(BetanL)/3)),3,i)
    plot(X,Wn(i,:));
    ylabel(['Mode ', num2str(i)])
    xlabel('Normalized length')
    axis([0 1 -3 3]);
end

%Plotting the Initial Shape
figure(2)
plot(X,w_naught,'red')
title('Initial Shape')
xlabel('Normalized length')
ylabel('u<naught>(x,t)')

%Plotting term by term sumation of the vibration solution @ a fixed
time
t = (pi/(4*omegan(1)));

```

```

for i = 1:length(BetanL)
    wn(i,:) = Wn(i,:)*AnoverL(i)*cos(omegan(i)*t);
end

%Summing Term by Term to Show Convergence
for i = 1:length(BetanL)
    j = i-1;
    if j < 1
        w(i,:) = wn(i,:);
    else
        w(i,:) = w(j,:)+wn(i,:);
    end
end

%Convergence Test
for i = 1:length(BetanL)-1
    error(i+1) = max(max(abs(w(i,:)-w(i+1,:))));
end

%Figure Showing Convergence of the Summation
figure (3)
title('Graph of the Convergence of the Solution for Deflections at
a Fixed Time')
xlabel('Normalized length')
ylabel('u(x,t)')
grid on
hold on
cc=hsv(length(BetanL));
for i = 1:length(BetanL)
    plot(X,w(i,:), 'color',cc(i,:));
end

%Figure Plotting the Previous Summation Over a Timespan
figure (4)
title('Graph of the Convergence of the Solution for Deflections Over
Time')
xlabel('Normalized length')
ylabel('u(x,t)')
hold on
t = 0:(pi/(4*omegan(1))):(2*pi/omegan(1));

for i = 1:length(t)
    w2(i,:) =
Wn(1,:)*AnoverL(1)*cos(omegan(1)*t(i))+Wn(2,:)*AnoverL(2)*cos(omegan(2)*t(i))
+Wn(3,:)*AnoverL(3)*cos(omegan(3)*t(i))+Wn(4,:)*AnoverL(4)*cos(omegan(4)*t(i))
+Wn(5,:)*AnoverL(5)*cos(omegan(5)*t(i))+Wn(6,:)*AnoverL(6)*cos(omegan(6)*t(i))
+Wn(7,:)*AnoverL(7)*cos(omegan(7)*t(i));%+Wn(8,:)*AnoverL(8)*cos(omegan(8)*
t(i))+Wn(9,:)*AnoverL(9)*cos(omegan(9)*t(i));
    plot(X,w2(i,:))
    pause
end

%% Impact
F = 1; %Impulse (force-time) 12.91 N-s
a = X;
t = (pi/2*(omegan(1)));

```

```

period = (2*pi)/omegan(1);

for k = 1:length(a)
    for i = 1:length(BetanL);

        Wna(i,:) =
(sinh(BetanL(i)*a(k))+sin(BetanL(i)*a(k)))+alpha(i).*(cosh(BetanL(i)*a(k))+co
s((BetanL(i)*a(k))));
        wpn(i,:) =
(F/(sqrt(rhom*A*E*Izz)))*((Wna(i,))/(Inn(i)*BetanL(i)^2))*sin(omegan(i)*t).*
Wn(i,:);

    end

    for i = 1:length(BetanL)
        j = i-1;
        if j < 1

            wp(i,:) = wpn(i,:);
        else
            wp(i,:) = wp(j,:)+wpn(i,:);
        end
    end

    w0 = 0;
    dt = .5;

    for i = 1:length(BetanL);

        Wna(i,:) =
(sinh(BetanL(i)*a(k))+sin(BetanL(i)*a(k)))+alpha(i).*(cosh(BetanL(i)*a(k))+co
s((BetanL(i)*a(k))));
        wpn(i,:) =
(F/(sqrt(rhom*A*E*Izz)))*((Wna(i,))/(Inn(i)*BetanL(i)^2))*sin(omegan(i)*dt).
*Wn(i,:);

    end

    for i = 1:length(BetanL)
        j = i-1;
        if j < 1

            wt1(i,:) = wpn(i,:);
        else
            wt1(i,:) = wt1(j,:)+wpn(i,:);
        end
    end

    for i = 1:length (BetanL)
        v(i,:) = (wt1(i,)-w0)/dt;
        %
        %
        figure (20)
        plot(X,v(i,:));
        FunV(i,:) = v(i,).*v(i,:);
        IntV(i) = trapz(X, FunV(i,:));
    end

```

```

        Ttimesm(i,k) = rhom*A*.5*IntV(i);
    end
    %T(i,j)--->i is # of terms;j is location of impact
end
figure (5)
for i = 1:length(BetanL)
    plot(X,wp(i,:))
    hold on
    pause
end

%% Rigid Body

omegabat = 50;

d = 2.5;
m1 = 0.00971279693*6;
m2 = 0.00971279693;
m = m1/m2;
xbar = L/2;
y = x-xbar;
Ic = (m1*(L^2))/12;

e = 0.5;
v2 = -1584;
vc = omegabat*(xbar+d);

v2primerid = v2+((m.*(1+e)*(vc-
v2+(omegabat.*y)))/(1+m+((m1*y.^2/Ic))));
V2prime = -v2primerid/v2;
figure (6)
plot(X,V2prime)

DeltaTr = m2*((v2primerid.^2)-(v2^2))/2;
figure (7)
plot(a,DeltaTr)

```

```

%% Three Values for different bat speeds (Rigid and Combination)
% MQP
% Baseball Bat Vibration Analysis

%% Modes with Initial Shape
close all; clear all; clc;
%Non dimentinalized bat
x = 0:.0033:33;
L = 33;
X = x/L;
a = X;

%Values from properties of maple wood

E = 2.02e6; %psi
rhomass = 0.0257; %lbm/in^3
rhom = rhomass*32.2; %slug/in^3
radius = .83885; %radius of a bat based on density and
length
A = (pi*radius^2); %Area
Izz = pi*(radius^4)/4; %moment area of inertia
omegabat1 = 40; %rad/sec
omegabat2 = 50;
omegabat3 = 60;
d = 2.5; %center of rotation from the bat handle
D = d/L; %normalized d
ao = 0.1; %intial displacement height (10% of the
length)
n = 10; %number of terms

P = omegabat1*L^2*sqrt(rhom*A/(E*Izz));
%Solving for the frequency equation (solving for BetanL)
f = inline(' (cos(Beta).*cosh(Beta))-1', 'Beta');
BetanL=zeros(1,n);
for i = 7:n
    BetanL(i) = ((2*i)+1)*(pi/2);
end
BetanL(1) = fzero(f,4.73);
BetanL(2) = fzero(f,7.85);
BetanL(3) = fzero(f,10.995);
BetanL(4) = fzero(f,14.137);
BetanL(5) = fzero(f,17.27);
BetanL(6) = fzero(f,20.42);

%Preallocation
alpha = zeros(length(BetanL),1); Wn = zeros(length(BetanL),length(X));
Inn = zeros(length(BetanL),1); FunBn = zeros(length(BetanL),length(X));
IntBn = zeros(length(BetanL),1); BnoverL = zeros(length(BetanL),1);
omegan = zeros(length(BetanL),1); Funtest =
zeros(length(BetanL),length(X));
intttest = zeros(length(BetanL),length(length(BetanL)));
wn = zeros(length(BetanL),length(X)); w =
zeros(length(BetanL),length(X));
error = zeros(length(BetanL-1),1); FunAn =
zeros(length(BetanL),length(X));
IntAn = zeros(length(BetanL),1); AnoverL = zeros(length(BetanL),1);

```

```

w_naught = ao*sin(pi*X);

%Using each BetanL to find the mode shapes
for i = 1:length (BetanL)
    alpha(i) = (sin(BetanL(i))-sinh(BetanL(i)))./(cosh(BetanL(i))-
cos(BetanL(i)));
    %Wn(i,:) =
(sinh(BetanL(i)*X)+sin(BetanL(i)*X))+alpha(i).*(cosh(BetanL(i)*X)+cos((BetanL
(i)*X)));

    Wn(i,:) =
(sinh(BetanL(i)*X)+(alpha(i)*cosh(BetanL(i)*X)))+(sin(BetanL(i)*X)+(alpha(i)*c
os(BetanL(i)*X)));
    FInn = Wn(i,:).*Wn(i,:);
    Inn(i) = trapz(X,FInn);
    %BC's (initial velocity)
    FunBn(i,:) = Wn(i,:).*(X+D);
    IntBn(i) = trapz(X, FunBn(i,:));
    BnoverL(i) = (P/(BetanL(i)*Inn(i)))*IntBn(i);
    %BC intial shape
    FunAn(i,:) = w_naught.*Wn(i,:);
    IntAn(i) = trapz(X, FunAn(i,:));
    AnoverL(i) = IntAn(i)./Inn(i);
    omegan(i) = (BetanL(i))^2*sqrt((E*Izz)/(rhom*A*L^4));
end

for i = 1:length(BetanL)
    Wna(i,:) =
(sinh(BetanL(i)*a)+(alpha(i)*cosh(BetanL(i)*a)))+(sin(BetanL(i)*a)+(alpha(i)*c
os(BetanL(i)*a)));
end

%% Rigid Body

d = 2.5;
m1 = 0.00971279693*6;
m2 = 0.00971279693;
m = m1/m2;
xbar = L/2;
y = x-xbar;
Ic = (m1*(L^2))/12;

e = 0.5;
v2 = -1584;
vc1 = omegabat1*(xbar+d);
vc2 = omegabat2*(xbar+d);
vc3 = omegabat3*(xbar+d);

v2primerid1 = v2+((m.*(1+e)*(vc1-
v2+(omegabat1.*y)))./(1+m+((m1*y.^2/Ic))));
v2primerid2 = v2+((m.*(1+e)*(vc2-
v2+(omegabat2.*y)))./(1+m+((m1*y.^2/Ic))));

```



```

v2primerid3 = v2+((m.*(1+e)*(vc3-
v2+(omegab3.*y))./(1+m+((m1*y.^2/Ic)))));
%V2prime = -v2primerid/v2;

DeltaTr1 = m2*((v2primerid1.^2)-(v2^2))/2;
DeltaTr2 = m2*((v2primerid2.^2)-(v2^2))/2;
DeltaTr3 = m2*((v2primerid3.^2)-(v2^2))/2;

T=0;

% Tm2=.1;

m = 2;
gamma = .2;

% Tm2 = 0;
for i = 1:length(BetanL)
    Tm2n(i,:) = gamma*((Wna(i,:)).^2)/((2*i)+1)^m;
end

for i = 1:length(BetanL)
    j = i-1;
    if j < 1
        Tm2(i,:) = Tm2n(i,:);
    else
        Tm2(i,:) = Tm2(j,:)+Tm2n(i,:);
    end
end

J1 = (m2/2).*(-
v2+sqrt((v2^2)+4.*((1/2)+Tm2(10,:)).*(DeltaTr1/(m2)))).*(1./((1/2)+Tm2(10,:))
);
J2 = (m2/2).*(-
v2+sqrt((v2^2)+4.*((1/2)+Tm2(10,:)).*(DeltaTr2/(m2)))).*(1./((1/2)+Tm2(10,:))
);
J3 = (m2/2).*(-
v2+sqrt((v2^2)+4.*((1/2)+Tm2(10,:)).*(DeltaTr3/(m2)))).*(1./((1/2)+Tm2(10,:))
);

v2primevib1 = -(v2+(J1/m2))/v2;
v2primevib2 = -(v2+(J2/m2))/v2;
v2primevib3 = -(v2+(J3/m2))/v2;

v2primevib1(1:1350) = 0;
v2primevib2(1:1350) = 0;
v2primevib3(1:1350) = 0;

figure(9)
%turn these on for the superimposed graph
% plot(a,v2primevib1)
hold on
% plot(a,v2primevib2)
% plot(a,v2primevib3)
plot(a,-v2primerid1/v2)
plot(a,-v2primerid2/v2)
plot(a,-v2primerid3/v2)

```

```

%% Brandon Boucher
% MQP
% Baseball Bat Vibration Analysis

%% Modes with Initial Shape
close all; clear all; clc;
%Non dimentinalized bat
x = 0:.0033:33;
L = 33;
X = x/L;

%Values from properties of maple wood

E = 2.02e6; %psi
rhomass = 0.0257; %lbm/in^3
rhom = rhomass*32.2; %slug/in^3
radius = .83885; %radius of a bat based on density and length
A = (pi*radius^2); %Area
Izz = pi*(radius^4)/4; %moment area of inertia
omegabat = 50; %rad/sec
d = 2.5; %center of rotation from the bat handle
D = d/L; %normalizedd d
ao = 0.1; %intial displacement height (10% of the
length)
n = 100; %number of terms

P = omegabat*L^2*sqrt(rhom*A/(E*Izz));
%Solving for the frequency equation (solving for BetanL)
f = inline('(cos(Beta).*cosh(Beta))-1', 'Beta');
BetanL=zeros(1,n);
for i = 7:n
    BetanL(i) = ((2*i)+1)*(pi/2);
end
BetanL(1) = fzero(f,4.73);
BetanL(2) = fzero(f,7.85);
BetanL(3) = fzero(f,10.995);
BetanL(4) = fzero(f,14.137);
BetanL(5) = fzero(f,17.27);
BetanL(6) = fzero(f,20.42);

%Preallocation
alpha = zeros(length(BetanL),1); Wn = zeros(length(BetanL),length(X));
Inn = zeros(length(BetanL),1); FunBn = zeros(length(BetanL),length(X));
IntBn = zeros(length(BetanL),1); BnoverL = zeros(length(BetanL),1);
omegan = zeros(length(BetanL),1); Funtest = zeros(length(BetanL),length(X));
intttest = zeros(length(BetanL),length(length(BetanL)));
wn = zeros(length(BetanL),length(X)); w = zeros(length(BetanL),length(X));
error = zeros(length(BetanL-1),1); FunAn = zeros(length(BetanL),length(X));
IntAn = zeros(length(BetanL),1); AnoverL = zeros(length(BetanL),1);

w_naught = ao*sin(pi*X);

%Using each BetanL to find the mode shapes
for i = 1:length (BetanL)

```

```

    alpha(i) = (sin(BetanL(i))-sinh(BetanL(i)))/(cosh(BetanL(i))-
cos(BetanL(i)));
    %Wn(i,:) =
(sinh(BetanL(i)*X)+sin(BetanL(i)*X))+alpha(i).*(cosh(BetanL(i)*X)+cos((BetanL
(i)*X)));

    Wn(i,:) =
(sinh(BetanL(i)*X)+(alpha(i)*cosh(BetanL(i)*X)))+(sin(BetanL(i)*X)+(alpha(i)*c
os(BetanL(i)*X)));
    FInn = Wn(i,:).*Wn(i,:);
    Inn(i) = trapz(X,FInn);
    %BC's (initial velocity)
    FunBn(i,:) = Wn(i,:).*(X+D);
    IntBn(i) = trapz(X,FunBn(i,:));
    BnoverL(i) = (P/(BetanL(i)*Inn(i)))*IntBn(i);
    %BC intial shape
    FunAn(i,:) = w_naught.*Wn(i,:);
    IntAn(i) = trapz(X,FunAn(i,:));
    AnoverL(i) = IntAn(i)./Inn(i);
    omegan(i) = (BetanL(i))^2*sqrt((E*Izz)/(rhom*A*L^4));
end
for i = 1:length (BetanL)
    Wnapp(i,:) = sin(BetanL(i)*X)-cos(BetanL(i)*X);
    FInnapp = Wnapp(i,:).*Wnapp(i,:);
    Innapp(i) = trapz(X,FInnapp);
end
%Orthogonality Matrix
for i = 1:length (BetanL)
    for j = 1:length (BetanL)
        if i <= j
            Funtest (i,:) = Wn(i,:).*Wn(j,:);
            intttest (i,j) = trapz(X,Funtest(i,:));
        end
    end
end
t = (pi/(4*omegan(1)));

for i = 1:length(BetanL)
    wn(i,:) = Wn(i,:)*AnoverL(i)*cos(omegan(i)*t);
end

%Summing Term by Term to Show Convergence
for i = 1:length(BetanL)
    j = i-1;
    if j < 1
        w(i,:) = wn(i,:);
    else
        w(i,:) = w(j,:)+wn(i,:);
    end
end

%Convergence Test
for i = 1:length(BetanL)-1
    error(i+1) = max(max(abs(w(i,:)-w(i+1,:))));
end

```

```

%% Impact
F = 1; %Impulse (force-time) 12.91 N-s
a = 0.7;
t = 0:(pi/(500*(omegan(1)))):(20*pi/((omegan(1))));
for k = 1:length(t)
for i = 1:length(BetanL);

    Wna(i,:) =
(sinh(BetanL(i)*a)+(alpha(i)*cosh(BetanL(i)*a))+(sin(BetanL(i)*a)+(alpha(i)*c
os(BetanL(i)*a))));
    S(i,:) = ((Wna(i,:))/(Inn(i)*BetanL(i)^2))*sin(omegan(i)*t).*Wn(i,:);

end

for i = 1:length(BetanL)
    j = i-1;
    if j < 1

        Sp(i,:) = S(i,:);
    else
        Sp(i,:) = Sp(j,:)+S(i,:);
    end
end

dt = 0.001;
dx = 0.0011;
vstar = Sp(length(BetanL),:)/dt;
vstarsq = (vstar.^2)*dx;
intvstarsq = sum(vstarsq);
m2 = 0.00971279693;

m1T = 12*((m2*(L^3))/(2*E*Izz))*intvstarsq;
%
figure (5)
% for i = 1:length(BetanL)
    plot(X,Sp(10,:))
    hold on
    pause
end

%% Rigid Body
a = X;
omegab = 50;

d = 2.5;
m1 = 0.00971279693*6;
m2 = 0.00971279693;
m = m1/m2;
xbar = L/2;
y = x-xbar;
Ic = (m1*(L^2))/12;

e = 0.5;

```

```

v2 = -1584;
vc = omegabat*(xbar+d);

v2primerid = v2+((m.*(1+e)*(vc-v2+(omegabat.*y))./(1+m+(m1*y.^2/Ic))));
V2prime = -v2primerid/v2;
figure (7)
plot(X,V2prime)
hold on

DeltaTrdm2 = ((v2primerid.^2)-(v2^2))/2;
figure (100)
plot(a,DeltaTrdm2)

m2T = 0.1;

gamma = .2;
for i = 1:length(BetanL)
    m2T(i) = gamma*(((Wna(i))^2)/((2*i)+1)^2);
end

% for i = length(BetanL)
%     for k = length(a)
%         Jm2(i,k) = (-
v2+sqrt((v2^2)+4*(m2*T(i,k)+(1/2))).*(DeltaTr(k)/(m2)))./(2*(m2*T(i,k)+(1/
2))));
%     end
% end
J = (m2/2)*(-v2+sqrt(((v2^2)+4*(.5+m2T))*DeltaTrdm2))*1/(.5+m2T);

v2prime = (v2+(J/m2))/v2;
figure (7)
plot (X,v2prime)

```