Analysis and homogenization of partial differential equations with discontinuous boundary conditions

by

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Abstract

Solutions of differential equations with discontinuous boundary conditions fail to belong in classical Sobolev spaces and hence presents a fundamental challenge in determining their effective behavior. In this dissertation, we consider three different boundary conditions for problems coming from the sciences and engineering, and use different approaches to circumvent this issue.

We first study a system of parabolic PDEs in moving domains modeling mass transfer in heterogeneous catalysis with a Robin boundary condition on the interface. The behavior of such systems becomes increasingly complex as the number of catalyst particles increases, which motivates the search for a homogenized model that would describe the asymptotic behavior of the solution to the problem. We transform the moving domain problem into a problem in a fixed domain by constructing a diffeomorphism out of the known solid particle velocities. We prove that solutions exist in any finite time and show that these solutions two-scale converge to solutions of a PDE/ODE system. We further prove corrector results for the solution and show strong convergence. Finally, we provide examples of solid velocities for which our result applies.

We then consider the elasticity problem for a homogeneous body with periodically distributed fractures. We first extend previous results on the dual formulations for an elastic body without fractures to a model of a homogeneous elastic body with fractures. In particular, in the framework of Legendre-Fenchel duality,

we were able to provide three equivalent formulations for the problem where the displacement, the stress, and the strain are the unknowns respectively. We also provide a characterization of the image of the convex cone of admissible displacements under the linearized strain tensor. Finally, we prove a homogenization result using Mosco convergence.

Lastly, we study the solvability of the Stokes equations in a bounded domain, describing the motion of a Newtonian fluid past moving rigid particles whose velocities are assumed to be known. We prescribe a Navier slip boundary condition on the fluid-solid interface. To solve the moving domain problem, we map the equations to a fixed domain using a diffeomorphism constructed from the solid particle velocities. The resulting equations can be thought of as a perturbation of the Stokes equations in a fixed domain. This motivates the use of a contraction mapping argument to show existence of solutions. We first construct weak solutions to the nonstationary Stokes equations in the fixed domain via Rothe's method. We then prove the higher regularity of the solution to the stationary Stokes equations in a bounded domain with slip boundary conditions and use this to show the existence of a strong solution for the nonstationary problem for any finite time interval using fixed-point methods. We leave the homogenization of this problem for future work.

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1 Introduction

The focus of this dissertation is the homogenization of partial differential equations whose solutions are discontinuous. The discontinuity comes from boundary conditions that reflect different physical phenomena. This creates a fundamental challenge when it comes to describing the macroscopic behavior of these heterogeneous systems, namely the lack of a sufficiently fine topology that contains all solutions at every sufficiently small scale. In this work, we consider different strategies to circumvent this obstacle, as well as analyses on the models we use and their solutions.

1.0.1 Well-posedness and homogenization of a coupled parabolic system modeling mass transfer in heterogeneous catalysis

Catalysts are substances that increase the rate of a chemical reaction. In industry, *heterogeneous* catalysts, which are in a different phase than the reactants and products, are widely used to enable faster large-scale production. These heterogeneous catalysts are small, ranging from micrometers to nanometers, and so modeling mass transfer can be computationally expensive. Homogenization theory gives us a way to approximate the model by looking at the limit behavior of such suspensions.

Our model is a system of parabolic equations coupled with a Robin boundary condition in a moving domain, which models the mass transport of a reactive solute in a bounded reactor with suspended catalysts:

$$\partial_t v_\varepsilon - D_F \Delta v_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla v_\varepsilon = 0, \quad \text{in } F_\varepsilon(t) \tag{1.1}$$

$$\partial_n v_\varepsilon = 0, \quad \text{on } \partial\Omega \tag{1.2}$$

$$D_F \partial_n v_\varepsilon = D_S \partial_n w_\varepsilon, \quad \text{on } \Gamma_\varepsilon(t) \tag{1.3}$$

$$D_F \partial_n v_\varepsilon + \alpha_\varepsilon (v_\varepsilon - w_\varepsilon) = 0, \quad \text{on } \Gamma_\varepsilon(t) \tag{1.4}$$

$$\partial_t w_\varepsilon - D_S \Delta w_\varepsilon + r w_\varepsilon = 0, \quad \text{in } S_\varepsilon(t) \tag{1.5}$$

$$v_\varepsilon(0) = v_{\varepsilon,0}, \quad \text{in } F_\varepsilon(0) \tag{1.6}$$

$$w_\varepsilon(0) = w_{\varepsilon,0}, \quad \text{in } S_\varepsilon(0). \tag{1.7}$$

Here, v_ε is the concentration of a solute that undergoes diffusion and advection in a bounded fluid domain $F_\varepsilon(t)$. The solute is adsorbed on the surface of suspended solid catalysts, described by the Robin boundary condition on $\Gamma_\varepsilon(t)$. It then diffuses into the solid catalysts $S_\varepsilon(t)$, where it is now denoted as w_ε, and reacts via linear kinetics. In our model, the catalysts are not necessarily fixed; they can move together with the fluid. Hence, the fluid domain $F_\varepsilon(t)$ and solid domain $S_\varepsilon(t)$ change in time and are described by the fluid and solid velocities that are known *a priori*.

Our approach was to map the moving domain to a periodic initial domain and use two-scale convergence to obtain an effective model. We define $v^\varepsilon := v_\varepsilon \circ \boldsymbol{X}_\varepsilon$ and $w^\varepsilon := w_\varepsilon \circ \boldsymbol{X}_\varepsilon$, where $\boldsymbol{X}_\varepsilon : \Omega \to \Omega$ is a diffeomorphism constructed from the known solid velocities. Thus, $(v^\varepsilon, w^\varepsilon)$ is the solution in the fixed domain.

Our main results were on the asymptotic behavior of these solutions. In particular, we determined their limits as $\varepsilon \to 0$ and obtained the equations that their limits satisfy. Physically, this corresponds to the suspension being homogenized, i.e., the size of the catalysts (which scale like ε) goes to zero and the number of catalyst particles (which scales like ε^{-1}) goes to infinity. Since $f_\varepsilon(t,\cdot) := v_\varepsilon(t,\cdot)\mathbb{1}_{F_\varepsilon(t)}(\cdot) + w_\varepsilon(t,\cdot)\mathbb{1}_{S_\varepsilon(t)}(\cdot)$ is not necessarily in $H^1(\Omega)$ because of the possible jump discontinuity on $\Gamma_\varepsilon(t)$, the homogenization is not straightforward.

To describe the limiting process, we made use of *two-scale convergence*:

Definition 1. *Let Ω and Y be bounded open sets in $\mathbb{R}^n$, and $T > 0$. A sequence $\{u_\varepsilon\}$ in $L^2\left((0,T)\times\Omega\right)$ is said to* ***two-scale converge*** *to a limit $u \in L^2\left((0,T)\times\Omega\times Y\right)$ if*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon(t,x)\phi\left(t,x,\frac{x}{\varepsilon}\right)dx\,dt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_Y u(t,x,y)\phi(t,x,y)\,dy\,dx\,dt, \tag{1.8}$$

for all $\phi \in L^2\left((0,T)\times\bar{\Omega};C_{per}\left(\bar{Y}\right)\right)$.

Instead of using more sophisticated extensions of v_ε and w_ε to the whole domain Ω, two-scale convergence allows us to use simpler ones, in particular extending these functions by zero outside F_ε and S_ε, respectively. Indeed, we proved:

Theorem 1. *Let $\overline{v^\varepsilon}$ be the zero extension of v^ε. Then, there exist $v^0 \in V$ and*

$v^1 \in L^2((0,T)\times\Omega; H^1_{per}(Y)/\mathbb{R})$ *such that, up to a subsequence, the following hold*

$$\overline{v^\varepsilon} \to v^0 \mathbb{1}_{Y_F} \qquad \textit{in the two-scale sense} \tag{1.9}$$

$$\overline{\nabla v^\varepsilon} \to \left(\nabla_x v^0 + \nabla_y v^1\right) \mathbb{1}_{Y_F} \qquad \textit{in the two-scale sense} \tag{1.10}$$

$$v^\varepsilon|_{\Gamma^\varepsilon} \to v^0 \qquad \textit{strongly in the two-scale sense on } \Gamma_\varepsilon \tag{1.11}$$

$$\overline{\partial_t v^\varepsilon} \rightharpoonup |Y_F|\partial_t v^0 \qquad \textit{weakly in } L^2\left((0,T)\times\Omega\right) \tag{1.12}$$

Theorem 2. *Let $\overline{w^\varepsilon}$ be the zero extension of w^ε. Then, there exists $w^0 \in V$ such that, up to a subsequence, the following hold*

$$\overline{w^\varepsilon} \to \chi_{Y_S} w^0 \qquad \textit{strongly in the two-scale sense} \tag{1.13}$$

$$\overline{\nabla w^\varepsilon} \to \mathbf{0} \qquad \textit{in the two-scale sense} \tag{1.14}$$

$$w^\varepsilon|_{\Gamma^\varepsilon} \to w^0 \qquad \textit{strongly in the two-scale sense on } \Gamma_\varepsilon \tag{1.15}$$

$$\overline{\partial_t w^\varepsilon} \rightharpoonup |Y_S|\partial_t w^0 \qquad \textit{weakly in } L^2\left((0,T)\times\Omega\right) \tag{1.16}$$

and that the limits v^0 and w^0 satisfy the following homogenized equations:

Theorem 3. *v^0, v^1, and w^0 are the the unique weak solutions of*

$$div_y\left(A_F^0(t,x,y)\left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y)\right) = 0, \quad \textit{in } (0,T)\times\Omega\times Y$$

$$|Y_F|\,\partial_t v^0 - div_x\left(\int_{Y_F} A_F^0(t,x,y)\left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y)\right) dy\right)$$
$$= |\Gamma|\,\alpha\left(v^0(t,x) - w^0(t,x)\right) \quad \textit{in } (0,T)\times\Omega$$

$$\partial_t w^0 + r w^0(t,x) = \frac{|\Gamma|}{|Y_S|}\alpha\left(w^0(t,x) - v^0(t,x)\right) \quad \textit{in } (0,T)\times\Omega$$

To demonstrate the utility of the limit equations as a useful proxy for the original model, we further proved:

Theorem 4.

$$
\begin{aligned}
&\|v^\varepsilon - v^0\|^2_{L^2((0,T)\times F_\varepsilon)} + \|w^\varepsilon - w^0\|^2_{L^2((0,T)\times S_\varepsilon)} \\
&+ \int_0^T \int_{F_\varepsilon} \left|\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right|^2 \to 0, \quad as\ \varepsilon \to 0.
\end{aligned} \tag{1.17}
$$

Lastly, we provide some examples of examples of solid velocities for which our result applies.

1.0.2 Elastic solids with fractures

Elastic solids with fractures that are in equilibrium can be described by an elliptic PDE with a nonlinear boundary condition on the fractures. The presence of these fractures influences how the solid responds to forces acting on it. We are interested in solids with periodically distributed fractures. The finer the heterogeneity of the system, i.e., the smaller and more numerous the fractures, the closer the solid will behave to a homogeneous solid with effective properties.

We make use of the model of the elasticity problem with fractures in [50]. Here, we assume that the elastic body, Ω having a fixed boundary $\partial\Omega$ is homogeneous and contains a single fracture inside its interior. The fracture is thought to be a smooth orientable surface which may or may not be connected, and is denoted by Σ_c. The extension to the case of periodically distributed fractures is standard. We write as Ω_F the set $\Omega\backslash\Sigma_c$. The model is as follows:

$$\text{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = 0 \;\text{ in } \Omega_F \tag{1.18}$$

$$\boldsymbol{\sigma} = \boldsymbol{A}\nabla_S(\boldsymbol{u}) \;\text{ in } \Omega_F \tag{1.19}$$

$$\boldsymbol{u} = \boldsymbol{0} \;\text{ on } \partial\Omega \tag{1.20}$$

$$[\boldsymbol{u}\cdot\boldsymbol{N}] \geq 0 \;\text{ on } \Sigma_c \tag{1.21}$$

$$\boldsymbol{\sigma n}|_1 = \sigma_{NN}\boldsymbol{N};\; \boldsymbol{\sigma n}|_2 = -\sigma_{NN}\boldsymbol{N};\; \sigma_{NN} \leq 0 \;\text{ on } \Sigma_c \tag{1.22}$$

$$\text{if } [\boldsymbol{u}\cdot\boldsymbol{N}] > 0 \,\text{on } F, \text{then } \sigma_{NN} = 0. \tag{1.23}$$

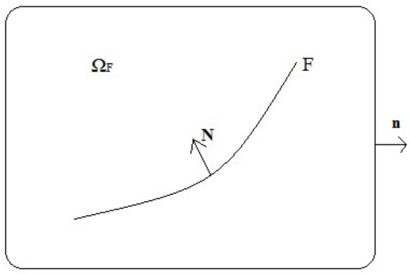

Figure 1.1: Elastic solid with fracture

Here, $\boldsymbol{N}$ refers to the unit normal on Σ_c, n is the outward unit normal on the boundary of Ω_F, $[\phi] = \phi_1 - \phi|_2$ refers to the jump of the field ϕ across the fracture Σ_c, where the subscripts 1 and 2 denote the faces of Σ_c in the direction of $\boldsymbol{N}$ and the opposite direction, respectively. $\sigma_{NN} = \boldsymbol{\sigma N}\cdot\boldsymbol{N}$. $A = [a_{ijkl}]$ is the elasticity tensor, assumed to have symmetry and positivity properties, i.e.,

$$AB\cdot B > 0, \;\text{for all } B \neq 0,\; B \in \mathbb{R}^{3\times 3}, \tag{1.24}$$

$$a_{ijkl} = a_{ijlk} = a_{jikl} = a_{jilk}, \tag{1.25}$$

$\boldsymbol{f}$ represents the body forces acting on the body. $\nabla_S(\cdot)$ is the linearized strain tensor.

We first prove some duality results. It was shown in [13] that the problem of finding a displacement vector that solves the elasticity problem is equivalent to finding a stress or strain tensor that solves a minimization problem. We extended their results, which dealt with the case of a homogeneous elastic solid without fractures, to the case with fractures. The new formulations are:

Problem 1 (Displacement Formulation)**.** *Find* $\boldsymbol{u} \in \boldsymbol{K}$ *such that*

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{K}} J(\boldsymbol{v}), \tag{1.26}$$

where $J(\boldsymbol{v}) := \dfrac{1}{2} \displaystyle\int_{\Omega_F} A \nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v})\, dx - \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, dx$ *for all* $\boldsymbol{v} \in \boldsymbol{V}$.

Problem 2 (Stress Formulation)**.** *Find* $\boldsymbol{\sigma} \in \mathbb{S}$ *such that*

$$g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\mu} \in \mathbb{S}} g(\boldsymbol{\mu}), \tag{1.27}$$

where $g(\boldsymbol{\mu}) = \dfrac{1}{2} \displaystyle\int_{\Omega_F} B\boldsymbol{\mu} : \boldsymbol{\mu}\, dx$ *for all* $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$*, and* $B = A^{-1}$.

Problem 3 (Strain Formulation)**.** *Find* $\boldsymbol{\pi} \in \mathbb{M}^+$ *such that*

$$\tilde{J}(\boldsymbol{\pi}) = \inf_{\boldsymbol{\mu} \in \mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}), \tag{1.28}$$

where $\tilde{J}(\boldsymbol{\mu}) := \dfrac{1}{2} \displaystyle\int_{\Omega_F} A\boldsymbol{\mu} : \boldsymbol{\mu}\, dx - \int_{\Omega_F} \boldsymbol{f} \cdot \mathcal{L}(\boldsymbol{\mu})\, dx$ *for all* $\boldsymbol{\mu} \in \mathbb{M}^+$.

Here, $\mathcal{L}(\boldsymbol{\mu})$ is the unique element in K such that $\nabla_S(\mathcal{L}(\boldsymbol{\mu})) = \boldsymbol{\mu}$. The spaces $\boldsymbol{K}, \mathbb{S}$, and $\mathbb{M}^+$ are the sets of admissible displacements, stresses, and strains. (See [53] for a precise definition of these spaces.)

We proved, up to a change of sign, that the displacement, stress, and strain formulations are dual problems:

Theorem 5.

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}) = -\inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}). \tag{1.29}$$

Moreover, we proved the following relationship among the minimizers of each problem:

Theorem 6. *Let $\bar{\boldsymbol{\sigma}} \in \mathbb{S}$, $\bar{\boldsymbol{v}} \in \boldsymbol{K}$, and $\bar{\boldsymbol{\mu}}$ such that*

$$g(\bar{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}), \quad J(\bar{\boldsymbol{v}}) = \inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}), \quad \tilde{J}(\bar{\boldsymbol{\mu}}) = \inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}). \tag{1.30}$$

Then

$$\bar{\boldsymbol{\sigma}} = A\nabla_S(\bar{\boldsymbol{v}}) = A\bar{\boldsymbol{\mu}}. \tag{1.31}$$

To describe the effective properties of the fractured material, we made use of Γ-convergence:

Definition 2. *We say that a sequence of functions $F_n : X \to \overline{\mathbb{R}}$, on a (first countable) topological space X, **$\boldsymbol{\Gamma}$-converges** to a function $F : X \to \overline{\mathbb{R}}$ if*

- *for every sequence $x_n \to x$ in X, we have that*

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n), \tag{1.32}$$

- *for each $x \in X$, there is a sequence x_n that converges to x in X such that*

$$F(x) \geq \limsup_{n\to\infty} F_n(x_n). \tag{1.33}$$

The starting point was to recast this problem as a minimization problem. We denote by Γ_ε the union of the fractures and $\Omega_\varepsilon := \Omega \setminus \Gamma_\varepsilon$, where Ω is some bounded subset of $\mathbb{R}^3$. We define the set of *admissible displacements*:

$$\boldsymbol{K}_\varepsilon = \{\boldsymbol{v} \in H^1(\Omega_\varepsilon) \mid \boldsymbol{v} = 0 \,\text{on}\, \partial\Omega \,\text{in}\, \partial\Omega,\, [\boldsymbol{v} \cdot \boldsymbol{N}] \geq 0 \,\text{in}\, \Gamma_\varepsilon\}. \tag{1.34}$$

Here n is the chosen normal of the fracture surface and $[\cdot]$ denotes the jump across the fractures. The set K_ε consists of displacement vector fields that are zero on the outer boundary and has positive jump on the fractures.

Solving the elliptic PDE that describes our problem is equivalent to finding a solution u_ε that solves the variational inequality:

$$\int_{\Omega_\varepsilon} \boldsymbol{A}\nabla_S(\boldsymbol{u}_\varepsilon) : \nabla_S(\boldsymbol{v} - \boldsymbol{u})\, dx \geq \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}_\varepsilon)\, dx \;\; \forall \boldsymbol{v} \in K_\varepsilon. \tag{1.35}$$

Here $\boldsymbol{A}$ is the elasticity tensor of the solid and $\boldsymbol{f}$ are the forces acting on the solid. This can then be written equivalently as the following minimization problem:

$$\min_{\boldsymbol{v}\in\boldsymbol{K}_\varepsilon} \left\{ \frac{1}{2} \int_{\Omega_\varepsilon} \boldsymbol{A}\nabla(\boldsymbol{v}) : \nabla(\boldsymbol{v}) - \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v} \right\}. \tag{1.36}$$

At this point, we run into a similar problem as before, in particular $\boldsymbol{u}_\varepsilon$ is not in $H^1(\Omega)$. We used a family of restriction-extension operators introduced in [9] to circumvent this problem. Indeed, we proved:

Theorem 7. *Define for* $\boldsymbol{v} \in L^2(\Omega)$,

$$J_\epsilon(\boldsymbol{v}) := \frac{1}{2}\int_{\Omega_\epsilon} \boldsymbol{A}\nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v}) - \int_{\Omega_\epsilon} \boldsymbol{f}\cdot\boldsymbol{v} + \chi_{K_\epsilon}(\boldsymbol{v}) \tag{1.37}$$

$$J_{hom}(\boldsymbol{v}) := \frac{1}{2}\int_{\Omega} \bar{\boldsymbol{\sigma}}^\circ(\nabla_S(\boldsymbol{v}))\nabla_S(\boldsymbol{v}) - \int_{\Omega} f\cdot\boldsymbol{v} + \chi_{H_0^1(\Omega)}(\boldsymbol{v}). \tag{1.38}$$

$$\tag{1.39}$$

Then

$$J_{hom} = \Gamma - \lim J_\epsilon, \tag{1.40}$$

in the strong $L^2(\Omega)$ *topology,*

where

$$\bar{\boldsymbol{\sigma}}^\circ(\nabla \boldsymbol{u}_0) := \int_Y \boldsymbol{A}(\nabla_x \boldsymbol{u}_0 + \nabla_y \boldsymbol{u}_1)\, dy, \tag{1.41}$$

and $\boldsymbol{u}_1 \in H_0^1(Y\setminus\Gamma)$ *solves the unit cell problem:*

$$\int_{Y\setminus\Gamma} \boldsymbol{A}(\nabla_x \boldsymbol{u}_0 + \nabla_y \boldsymbol{u}_1)\nabla_y(\boldsymbol{w} - \boldsymbol{u}_1)\, dy \geq 0, \tag{1.42}$$

for all $w \in H_0^1(Y\setminus\Gamma)$ *such that* $[\boldsymbol{w}\cdot\boldsymbol{N}] \geq 0$ *on* Γ*. Here,* Y *is the unit cell in* $\mathbb{R}^3$, $(0,1)^3$.

This result states that the response of an elastic solid (with periodically distributed fractures) to a distribution of forces is close to that of a solid described by the PDE:

$$\text{div } (\bar{\boldsymbol{\sigma}}^{\circ}(\nabla \boldsymbol{u}_0)) = \boldsymbol{f}, \quad \text{in } \Omega \tag{1.43}$$

$$\boldsymbol{u}_0 = \boldsymbol{0}, \quad \text{on } \partial\Omega, \tag{1.44}$$

1.0.3 Suspensions of rigid particles in a Newtonian fluid: well-posedness and regularity

Suspensions of rigid particles dispersed in an incompressible fluids are commonly found in industry. Initially, our goal in this project is to determine the effective viscosity of such suspensions. However, we were unable to finish the homogenization and have only proved well-posedness of the model and the regularity of its solution. The study of the emergent behavior of suspensions has a long history, dating back to Einstein's work on dilute suspensions [25]. The focus of this section is on the solvability of the Stokes equations with Navier-slip boundary conditions in a moving domain. Indeed, we consider the motion of an incompressible Newtonian fluid in a bounded domain with submerged rigid particles whose velocities are known. At the boundary of the fluid domain, we prescribe a Navier slip condition.

The goal is to find a velocity v and pressure q that satisfy

$$\partial_t v - \Delta v + \nabla q = f, \quad \text{in } \Omega(t),\ t \in (0,T) \tag{1.45}$$

$$\text{div } v = 0, \quad \text{in } \Omega(t),\ t \in (0,T) \tag{1.46}$$

$$v \cdot n = 0, \quad \text{in } \Gamma(t),\ t \in (0,T) \tag{1.47}$$

$$[\mathbb{D}(v)n]_\tau + \alpha(v - V)_\tau = 0, \quad \text{in } \Gamma(t),\ t \in (0,T) \tag{1.48}$$

$$v(0) = v_0, \ \text{ in } \Omega, \tag{1.49}$$

where the moving domain $\Omega(t)$ is defined by the velocities of the solid particles, given by

$$V_i(t,x) := h_i'(t) + M_i(t)\left(x - h_i(t)\right), \quad x \in \Gamma_i(t). \tag{1.50}$$

Here, h_i and M_i are in $C^\infty(0,T)$, $M_i(t)$ is skew-symmetric for all t, and $\Gamma(t)$ is the boundary of the solid particles at time t. Physically, the above equation says that V_i is a combination of a *translation* and a *rotation*.

Our approach is to use the diffeomorphism in [22] to write the problem in a fixed domain. This method was pioneered by Inoue and Wakimoto in their seminal paper for the Navier-Stokes equations in noncylindrical domains [37].

The problem can be solved using a fixed point argument. In [22], this was obtained using classical work by [54] using semigroup theory. In our case, this task is two-fold. We first show the H^2-regularity of solutions to the steady-state Stokes equations with slip boundary conditions. Indeed, we proved the following:

Theorem 8. *Suppose $f \in L^2(\Omega)$ and $\alpha > 0$. Then, the weak solution (u,p) to the stationary Stokes problem with Navier-slip boundary conditions belongs in*

$H^2(\Omega) \times H^1(\Omega)$.

We note that how we proved this, albeit similar to [1], differs in the way that the map we use to transform the local problem with a curved boundary to a domain with a straight boundary, preserves the normal boundary conditions, i.e., the jump of the normal component of the velocity across the interface fluid-solid interface is zero.

Rothe's method can then be used to show existence to the parabolic problem. We proved:

Theorem 9. *Suppose that the initial velocity u_0 belongs to $H^1(\Omega)$. Then the solution to the non-steady state Stokes equations with slip boundary conditions u is in $W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right)$.*

Finally, a fixed point argument gives us:

Theorem 10. *Let $F \in L^2\left((0,T)\times\Omega\right)$ and $u_0 \in H^1(\Omega)$. Then, there exist $(u,p) \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right) \times L^2\left(0,T;H^1(\Omega)\right)$ that solves the fixed domain problem.*

2 Well-posedness and Homogenization of a system of parabolic equations in moving domains modeling mass transfer in heterogeneous catalysis

2.1 Introduction

We consider a system of parabolic equations coupled with a Robin boundary condition in a moving domain:

$$\partial_t v_\varepsilon - D_F \Delta v_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla v_\varepsilon = 0, \quad \text{in } F_\varepsilon(t) \tag{2.1}$$

$$\partial_n v_\varepsilon = 0, \quad \text{on } \partial\Omega \tag{2.2}$$

$$D_F \partial_n v_\varepsilon = D_S \partial_n w_\varepsilon, \quad \text{on } \Gamma_\varepsilon(t) \tag{2.3}$$

$$D_F \partial_n v_\varepsilon + \alpha_\varepsilon \left(v_\varepsilon - w_\varepsilon\right) = 0, \quad \text{on } \Gamma_\varepsilon(t) \tag{2.4}$$

$$\partial_t w_\varepsilon - D_S \Delta w_\varepsilon + r w_\varepsilon = 0, \quad \text{in } S_\varepsilon(t) \tag{2.5}$$

$$v_\varepsilon(0) = v_{\varepsilon,0}, \quad \text{in } F_\varepsilon(0) \tag{2.6}$$

$$w_\varepsilon(0) = w_{\varepsilon,0}, \quad \text{in } S_\varepsilon(0). \tag{2.7}$$

These equations form a simple model for the mass transport of a reactive solute in a bounded reactor with suspended catalysts. Here, v_ε is the concentration of a solute that undergoes diffusion and advection in a bounded fluid domain $F_\varepsilon(t)$, that is adsorbed on the surface of suspended solid catalysts, described by the Robin boundary condition on $\Gamma_\varepsilon(t)$. It diffuses into the solid catalysts $S_\varepsilon(t)$, where it is now denoted as w_ε, and reacts with linear kinetics. It is assumed that the fluid and solid velocities are known. Given these velocities, we detail in the next section how the sets $F_\varepsilon(t)$ and $S_\varepsilon(t)$ are defined.

Heterogeneous catalysts are catalysts that are different in phase than reactants or products of chemical reactions that they catalyze. These are widely used in industry to make chemical processes cost effective by hastening the rate of chemical reactions. We are particularly interested in heterogeneous catalysts that are suspended in a liquid medium. These are often seen in water treatment applications, where a pollutant is degraded through photocatalytic reactions. See for example [24], [55], [41], [59] and the references therein.

Catalyst particle sizes are typically measured in micrometers and nanometers and are several orders of magnitude smaller than the scale of reactors. Numerical simulations that account for the contributions of each particle can be complex. This motivates the search for a simpler effective model that describes the mass transfer processes.

Some of the early work in this regard is [42] and [36]. In [42], they consider the mass transport of a reactive solute in a porous medium, where the catalysts are supported on the surface of the porous medium. The homogenization is carried out using formal asymptotic expansions. In [36], they consider a similar problem but couple the mass transfer equations with the Stokes equations. They assume that the fluid flow is independent of mass transfer and hence are able to use known homogenization results for the Stokes equations in porous media to rigorously obtain the homogenization result.

The kinetics and adsorption mechanics considered in [36] were linear and were far simpler than what are observed in practice. The authors in [17] extend this case to more realistic models. In particular, they both consider Langmuir and

Freundlich kinetics, which are nonlinear adsorption mechanisms. We also cite [5], where the authors consider a model where one has convection and diffusion in both the bulk fluid and the pore surface. With the assumption of periodicity of the velocity field that drives the advection, the authors were able to get a homogenization result using formal asymptotic expansions, and show that this is rigorous through two-scale convergence with drift.

In [29], they consider reaction-diffusion processes for multiple reacting species in a two-component porous medium with nonlinear flux conditions at the interface. In contrast with the earlier cited works, reactions occur throughout the two-component medium instead of only happening on pore surfaces. The authors use extension operators to extend solutions in the connected component of the domain into the whole domain. This requires some degree of regularity of the boundary of the connected component. We note that this was also used in [17]. The authors in [29] used the boundary unfolding operator and a compactness result for Banach-spaced valued functions to handle the convergence of the nonlinear terms.

For problems in an evolving domain, one usually assumes that the evolution is regular enough to map the moving domain into a fixed one. Homogenization problems in evolving domains commonly need a periodicity assumption on the fixed domain in order to use tools from homogenization theory, e.g. two-scale convergence, periodic unfolding methods. For example, in [47] the author considers the homogenization of a diffusion-reaction-advection problem in domains with evolving microstructure. The deformation of the domain is assumed to be regular enough that it can be mapped into a fixed periodic one. The author also assumes the strong two-scale convergence of the terms that arise from the change of variables.

This imposes a restriction on how the domain evolves. If the deformation veers away too much from a periodic structure, then classical homogenization methods might fail. In [23], the author considers a similar problem whose homogenization result is proved using the periodic unfolding method. A more recent work in this vein is [30], where the authors obtain the homogenization of a reaction-diffusion-advection problem in an evolving domain with nonlinear boundary conditions. The authors use similar tools as in [29] to handle the nonlinear terms. They also assume that the evolution guarantees the strong two-scale convergence of the terms arising from mapping to a fixed domain.

In this paper, we are interested in obtaining an effective model that describes the mass transport of a single chemical species in a reactor with suspended moving catalyst particles. To clarify the presentation, we assume linear kinetics and a constant diffusivity for the fluid and solid domains. Our approach is to map the moving domain into the periodic initial domain and use two-scale convergence to obtain an effective model. We define $v^\varepsilon := v_\varepsilon \circ \boldsymbol{X}_\varepsilon$ and $w^\varepsilon := w_\varepsilon \circ \boldsymbol{X}_\varepsilon$, where $\boldsymbol{X}_\varepsilon : \Omega \to \Omega$ is our constructed diffeomorphism. Thus, $(v^\varepsilon, w^\varepsilon)$ is the solution in the fixed domain. More details are provided in the latter sections. We show that the following convergences hold:

Theorem 3. *Let $\overline{v^\varepsilon}$ be the zero extension of v^ε. Then, there exist $v^0 \in L^2\left(0, T; H^1(\Omega)\right)$ and $v^1 \in L^2((0,T) \times \Omega; H^1_{per}(Y)/\mathbb{R})$ such that, up to a subsequence, the following* *hold*

$$\overline{v^\varepsilon} \to v^0 \mathbb{1}_{Y_F} \qquad \textit{in the two-scale sense} \tag{2.8}$$

$$\overline{\nabla v^\varepsilon} \to \left(\nabla_x v^0 + \nabla_y v^1\right) \mathbb{1}_{Y_F} \qquad \textit{in the two-scale sense} \tag{2.9}$$

$$v^\varepsilon|_{\Gamma^\varepsilon} \to v^0 \qquad \textit{strongly in the two-scale sense on } \Gamma_\varepsilon \tag{2.10}$$

$$\overline{\partial_t v^\varepsilon} \rightharpoonup |Y_F| \partial_t v^0 \qquad \textit{weakly in } L^2\left((0,T)\times\Omega\right) \tag{2.11}$$

Theorem 4. *Let $\overline{w^\varepsilon}$ be the zero extension of w^ε. Then, there exists $w^0 \in L^2\left((0,T)\times\Omega\right)$ such that, up to a subsequence, the following hold*

$$\overline{w^\varepsilon} \to \chi_{Y_S} w^0 \qquad \textit{strongly in the two-scale sense} \tag{2.12}$$

$$\overline{\nabla w^\varepsilon} \to \mathbf{0} \qquad \textit{in the two-scale sense} \tag{2.13}$$

$$w^\varepsilon|_{\Gamma^\varepsilon} \to w^0 \qquad \textit{strongly in the two-scale sense on } \Gamma_\varepsilon \tag{2.14}$$

$$\overline{\partial_t w^\varepsilon} \rightharpoonup |Y_S| \partial_t w^0 \qquad \textit{weakly in } L^2\left((0,T)\times\Omega\right) \tag{2.15}$$

and that the limits v^0 and w^0 satisfy the following homogenized equations:

Theorem 5. *v^0, v^1, and w^0 are the the unique weak solutions of*

$$div_y\left(A_F^0(t,x,y)\left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y)\right)\right) = 0, \quad \textit{in } (0,T)\times\Omega\times Y \tag{2.16}$$

$$|Y_F|\,\partial_t v^0 - div_x\left(\int_{Y_F} A_F^0(t,x,y)\left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y)\right) dy\right) \tag{2.17}$$

$$= |\Gamma|\,\alpha\left(v^0(t,x) - w^0(t,x)\right) \quad \textit{in } (0,T)\times\Omega$$

$$\partial_t w^0 + r w^0(t,x) = \frac{|\Gamma|}{|Y_S|}\alpha\left(w^0(t,x) - v^0(t,x)\right) \quad \textit{in } (0,T)\times\Omega \tag{2.18}$$

Because of the linearity of the reaction term, we did not need to use H^1 extensions of solutions onto the whole domains such as those used in [29]. We are still able to show a strong convergence result by using the fact that we are working with solutions to PDEs. Indeed, we have

Theorem 6.

$$\|v^\varepsilon - v^0\|^2_{L^2((0,T)\times F_\varepsilon)} + \|w^\varepsilon - w^0\|^2_{L^2((0,T)\times S_\varepsilon)} \tag{2.19}$$

$$+ \int_0^T \int_{F_\varepsilon} \left|\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t, x, \frac{x}{\varepsilon}\right)\right|^2 \to 0, \quad as\ \varepsilon \to 0. \tag{2.20}$$

The diffeomorphism we use to map the moving domain onto the initial domain comes from a standard construction found in works on fluid-solid interactions and the Navier-Stokes equations in moving domains originally introduced in [37]. We follow the construction in [22], where they considered the flow of a Newtonian fluid past moving rigid obstacles. The idea is to construct the diffeomorphism in such a way so that it is a rigid deformation for points initially on the solid domain, the identity map when sufficiently far away from the solid domain and a smooth transition in between these regions that is volume preserving. The volume preserving property simplifies the calculations in the homogenization but is not a necessary requirement. As long as the diffeomorphism is well behaved, i.e., its Jacobian is uniformly bounded away from zero, then the requirements we require from the solid velocities would allow one to handle the extra terms that come from the Jacobian.

The paper is organized as follows: in Section 2.2, we construct the diffeomorphism that allows us to map a moving domain problem into a fixed domain. We show that these problems are equivalent. In Section 17, we prove the existence of solutions via Rothe's method. We also obtain estimates on the solutions. In

Section 2.4, we prove our homogenization result via two-scale convergence and our strong convergence result. Finally, in Section 2.5, we provide some examples of solid motion for which our result applies.

2.2 Transformation to a fixed domain and weak solutions

Let Ω be a bounded subset of $\mathbb{R}^3$ with smooth boundary and Y_S be an open subset of $Y := (0,1)^3$ with smooth boundary such that $Y_S \subset\subset Y$. We denote $Y_F := Y \backslash \overline{Y_S}$. Let ε be a sequence of positive numbers that goes to zero. We define the following sets:

$$\Theta_\varepsilon := \left\{ \xi \in \mathbb{Z}^3 \mid \varepsilon\,(\xi + Y) \subset\subset \Omega \right\}$$
$$S_\varepsilon := \bigcup_{\xi \in \Theta_\varepsilon} \varepsilon\,(\xi + Y_S)$$
$$F_\varepsilon := \Omega \setminus \overline{S_\varepsilon}.$$

Observe that since Ω is bounded and has a smooth boundary, we have that

$$F_\varepsilon = \left(\bigcup_{\xi \in \Theta_\varepsilon} \varepsilon\,(\xi + Y_F) \right) \bigcup \Lambda_\varepsilon,$$

for some $\Lambda_\varepsilon \subset \Omega$ such that $|\Lambda_\varepsilon| \to 0$ as $\varepsilon \to 0$. As Θ_ε is finite, we can write

$$S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} \mathcal{O}_i.$$

The sets $\mathcal{O}_i$ represent the solid rigid particles at time zero, hence, S_ε are the solid catalysts and F_ε is the fluid domain at time zero. Below is an example of what the domain looks like at $t = 0$. Here, S_ε is the union of the purple circles and F_ε is the interior of the space in between these circles.

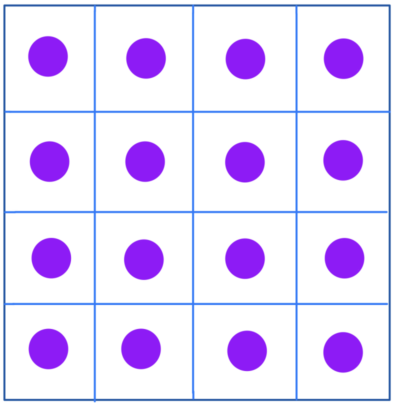

Figure 2.1: Domain at $t = 0$

From here onwards, we denote by y, the spatial variable in the domain at time zero and $x := x(t, y)$ to be spatial variable in the moving domain. To describe the moving domain, we obtain a transformation that maps $\overline{\mathcal{O}_i}$ to $\overline{\mathcal{O}_i(t)}$, i.e., a mapping between points from the solid at $t = 0$ to points in the solid at any time $t \in (0, T)$. This would come from the velocities of the solid particles that are assumed to be known *a priori*.

Let $y \in \overline{\mathcal{O}_i}$, and consider the following ODE describing the trajectories of the solid catalysts:

$$\begin{aligned} &\boldsymbol{G}_i'(t, y) = \boldsymbol{h}_{\varepsilon,i}'(t) + \boldsymbol{M}_{\varepsilon,i}(t) \left(\boldsymbol{G}_i(t, y) - \boldsymbol{h}_{\varepsilon,i}(t)\right), \quad t > 0 \\ &\boldsymbol{G}_i(0, y) = y. \end{aligned}$$

Assumption 7. *We assume that* $\boldsymbol{M}_{\varepsilon,i} : [0, \infty) \to \mathbb{R}^{3\times 3}$ *is a skew-symmetric* *matrix and that both* $\boldsymbol{h}_{\varepsilon,i} : [0,\infty) \to \mathbb{R}^3$ *and* $\boldsymbol{M}_{\varepsilon,i}$ *satisfy*

$$\sup_{1\le i\le N(\varepsilon)} \|\boldsymbol{h}_{\varepsilon,i}\|_{L^\infty(0,\infty)} \le C\varepsilon^\gamma, \tag{2.21}$$

$$\sup_{1\le i\le N(\varepsilon)} \|\boldsymbol{M}_{\varepsilon,i}\|_{L^\infty(0,\infty)} \le C, \tag{2.22}$$

for some $\gamma \ge 1$ *and* $C > 0$ *that is independent of* ε*.*

This, then, defines an isomorphism $\boldsymbol{G}_i(t,\cdot) : \overline{\mathcal{O}}_i \mapsto \overline{\mathcal{O}_i(t)}$. In addition to (2.21), we further assume:

Assumption 8. *The solid velocities are slow enough such that*

$$\sup_{i\ne j} d\left(\mathcal{O}_i(t), \mathcal{O}_j(t)\right) \ge \delta\varepsilon,$$

for some $0 < \delta < 1$ *that is independent of* ε*. We assume that* δ *is small enough to guarantee that the solids do not intersect each other.*

We now define the fluid and solid domains, and the interface, respectively as:

$$\begin{aligned} F_\varepsilon(t) &:= \Omega \setminus \bigcup_{i=1}^m \overline{\mathcal{O}_i(t)} \\ S_\varepsilon(t) &:= \bigcup_{i=1}^m \mathcal{O}_i(t) \\ \Gamma_\varepsilon(t) &:= \partial S_\varepsilon(t). \end{aligned}$$

We now write our system as:

$$\partial_t v_\varepsilon - D_F \Delta v_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla v_\varepsilon = 0, \quad \text{in } F_\varepsilon(t) \tag{2.23}$$

$$\partial_n v_\varepsilon = 0, \quad \text{on } \partial\Omega \tag{2.24}$$

$$D_F \partial_n v_\varepsilon = D_S \partial_n w_\varepsilon, \quad \text{on } \Gamma_\varepsilon(t) \tag{2.25}$$

$$D_F \partial_n v_\varepsilon + \alpha_\varepsilon \left(v_\varepsilon - w_\varepsilon\right) = 0, \quad \text{on } \Gamma_\varepsilon(t) \tag{2.26}$$

$$\partial_t w_\varepsilon - D_S \Delta w_\varepsilon + r w_\varepsilon = 0, \quad \text{in } S_\varepsilon(t) \tag{2.27}$$

$$v_\varepsilon(0) = v_{\varepsilon,0}, \quad \text{in } F_\varepsilon(0) \tag{2.28}$$

$$w_\varepsilon(0) = w_{\varepsilon,0}, \quad \text{in } S_\varepsilon(0) \tag{2.29}$$

$$\tag{2.30}$$

Assumption 9. *We assume that* $\alpha_\varepsilon = \alpha\varepsilon$*, where* $\alpha > 0$*. Moreover, we assume that* $v_{\varepsilon,0} \in H^1\left(F_\varepsilon(0)\right)$*,* $w_{\varepsilon,0} \in H^1\left(S_\varepsilon(0)\right)$ *and that*

$$v_{\varepsilon,0} \xrightarrow{\varepsilon \to 0} v_0, \quad \text{in } L^2\left(\Omega\right), \tag{2.31}$$

$$w_{\varepsilon,0} \xrightarrow{\varepsilon \to 0} w_0, \quad \text{in } L^2\left(\Omega\right), \tag{2.32}$$

for some $v_0, w_0 \in L^2\left(\Omega\right)$*.*

A practical choice for the limit functions in the assumption above is $v_0 \equiv C$ and $w_0 \equiv 0$, which corresponds to starting with a uniform concentration of solute in the solution and using fresh catalysts.

Assumption 10. *We suppose that the map* $t \mapsto \boldsymbol{u}_\varepsilon(t, \cdot)$ *is smooth and that for* *each* $t \geq 0$*,* $\boldsymbol{u}_\varepsilon(t,\cdot) : F_\varepsilon(t) \to \mathbb{R}^3$ *is smooth as well. Moreover, we assume that*

$$\boldsymbol{u}_\varepsilon \xrightarrow{\varepsilon\to 0} \mathbf{0}, \quad \textit{in } L^2(\Omega).$$

Definition 11. *We define the spaces*

$$V_\varepsilon := \left\{ v : [0,T] \times F_\varepsilon(t) \to \mathbb{R} \mid \int_0^T \int_{F_\varepsilon(t)} |\partial_t v(t,x)|^2 + |v(t,x)|^2 + |\nabla v(t,x)|^2 \, dx\, dt < +\infty \right\}$$
$$W_\varepsilon := \left\{ w : [0,T] \times S_\varepsilon(t) \to \mathbb{R} \mid \int_0^T \int_{S_\varepsilon(t)} |\partial_t w(t,x)|^2 + |w(t,x)|^2 + |\nabla w(t,x)|^2 \, dx\, dt < +\infty \right\}$$

Definition 12. *We say that* $(v_\varepsilon, w_\varepsilon) \in V_\varepsilon \times W_\varepsilon$ *is a weak solution to the moving domain problem if for every* $(\varphi, \psi) \in V_\varepsilon \times W_\varepsilon$*, we have*

$$\begin{aligned}
&\int_0^T \int_{F_\varepsilon(t)} (\partial_t v_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla v_\varepsilon)\,\varphi + \int_0^T \int_{S_\varepsilon(t)} \partial_t w_\varepsilon \psi + \int_0^T \int_{F_\varepsilon(t)} D_F \nabla v_\varepsilon \cdot \nabla \varphi \\
&+ \int_0^T \int_{S_\varepsilon(t)} D_S \nabla w_\varepsilon \cdot \nabla \psi + \int_0^T \int_{S_\varepsilon(t)} r w_\varepsilon \psi \\
&= \int_0^T \int_{\Gamma_\varepsilon(t)} \alpha_\varepsilon (w_\varepsilon - v_\varepsilon)(\varphi - \psi)
\end{aligned}$$

The task now is to find a diffeomorphism between $F_\varepsilon(0)$ and $F_\varepsilon(t)$. We follow a classical construction used in fluid-solid interaction problems (see for instance [29]). We do this by defining a suitable *domain velocity* for Ω that will give the necessary diffeomorphism upon integration. Heuristically, we want this velocity to be the solid velocity inside the solid particles, zero when one is sufficiently far away from the solids, and to *glue* together these two velocites in between. For simplicity, we require it to be volume preserving.

To start, let B_{1_i}, B_{2_i} be open balls such that $\overline{\mathcal{O}}_i \subset B_{1_i} \subset \overline{B}_{1_i} \subset B_{2_i}$. We define for $k=1,2$:

$$B_{k_i}(t) := \{x = \boldsymbol{G}_i(t,y) |\, y \in B_{k_i}\}$$

Let $\eta \in C^\infty(\mathbb{R}^3 \times [0,T])$ be a cut-off function such that

- $0 \le \eta \le 1$,
- for $t \in [0,T]$, $\eta \equiv 1$ on $\cup B_{1_i}(t)$, $\eta \equiv 0$ on $\mathbb{R}^3 \setminus \cup B_{2_i}(t)$.

We let $K_i(t) :=$ support of $\nabla\eta(t,\cdot) \cap \overline{B}_{2_i}(t)$.

In order to get a volume preserving diffeomorphism, we need this domain velocity to have zero divergence everywhere. To do that, we subtract out the divergence of the terms where we expect the velocity to be nonzero. Indeed, we make the following calculations:

$$\begin{aligned}
\operatorname{div}_x\big(\eta(t,x)\boldsymbol{h}'_{\varepsilon,i}(t)\big) &= \nabla\eta(t,x)\cdot \boldsymbol{h}'_{\varepsilon,i}(t),\\
\operatorname{div}_x\left(\eta(t,x)\boldsymbol{M}_{\varepsilon,i}(t)\boldsymbol{h}_{\varepsilon,i}(t)\right) &= \nabla\eta(t,x)\cdot \boldsymbol{M}_{\varepsilon,i}(t)\boldsymbol{h}_{\varepsilon,i}(t),\\
\operatorname{div}_x\left(\eta(t,x)\boldsymbol{M}_{\varepsilon,i}(t)x\right) &= \nabla\eta(t,x)\cdot \boldsymbol{M}_{\varepsilon,i}(t)x + \eta(t,x)\operatorname{div}_x\left(\boldsymbol{M}_{\varepsilon,i}(t)x\right)\\
&= \nabla\eta(t,x)\cdot \boldsymbol{M}_{\varepsilon,i}(t)x,
\end{aligned}$$

since $\boldsymbol{M}_{\varepsilon,i}$ is skew-symmetric. These motivate us to define for $t \in [0,T]$ and $x \in \overline{\mathcal{O}_i(t)}$:

$$\begin{aligned}
\boldsymbol{b}_\varepsilon(t,x) := \eta(t,x)\sum_{i=1}^{N(\varepsilon)} \big(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\left(x - \boldsymbol{h}_{\varepsilon,i}(t)\right)\big)\, \mathbb{1}_{\overline{\mathcal{O}_i(t)}}(x)\\
- \sum_{i=1}^{N(\varepsilon)} \boldsymbol{B}_{K_i(t)}\big(\nabla\eta(t,\cdot)\cdot\big(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\cdot\big)\big)(x)\mathbb{1}_{\overline{\mathcal{O}_i(t)}}(x),
\end{aligned}$$

where $\boldsymbol{B}_{K_i(t)} : L^2(K_i(t)) \to H_0^1(K_i(t))$, is an operator such that,

$$\operatorname{div}\ \left(\boldsymbol{B}_{K_i(t)}(H)\right) = H,$$

and $\|\boldsymbol{B}_{K_i(t)}(H)\|_{H_0^1(K_i(t))} \leq C\,(K_i(t))\,\|H\|_{L^2(K_i(t)}$, see [58] for details. Based on our previous calculations, we have that

- $\boldsymbol{b}_\varepsilon(t,x) = \boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\,(x - \boldsymbol{h}_{\varepsilon,i}(t))$ for $x \in \overline{\mathcal{O}_i(t)}$,
- $\operatorname{div}\ \boldsymbol{b}_\varepsilon \equiv 0$ in Ω,
- $\boldsymbol{b}_\varepsilon \in C^\infty_{0,\sigma}\left(\mathbb{R}^r \times [0,T]; \mathbb{R}^3\right)$.

$\boldsymbol{b}_\varepsilon$ is the *domain velocity* that we need to define the necessary diffeomorphism. Indeed, we consider the following problem: for $y \in \mathbb{R}^3$,

$$\begin{aligned} \partial_t \boldsymbol{X}_\varepsilon(t,y) &= \boldsymbol{b}_\varepsilon\left(t, \boldsymbol{X}_\varepsilon(t,y)\right), \quad t > 0, \\ \boldsymbol{X}_\varepsilon(0,y) &= y. \end{aligned}$$

As $\boldsymbol{b}_\varepsilon$ is smooth, by Picard-Lindelöf, there exists a smooth function $\boldsymbol{X}_\varepsilon$ that solves the above ODE. Thus, restricting it to F_ε, we have that $\boldsymbol{X}_\varepsilon(t,\cdot) : F_\varepsilon \to F_\varepsilon(t)$ is the desired diffeomorphism. Similarly, restricting to S_ε, we have that $\boldsymbol{X}_\varepsilon(t,\cdot) : S_\varepsilon \to S_\varepsilon(t)$ is a diffeomorphism.

If the fluid and solid velocities are known and smooth enough, say u_F and u_S, respectively, and that across the fluid-solid interface $u_F = u_S$ at all times, then one can create a map from the initial domain to the domain at any time $t > 0$ by solving the problem:

$$\partial_t \phi(t,y) = u\left(t, \phi(t,y)\right), \quad t \in (0,T),$$
$$\phi(0,y) = y,$$

where u is equal to u_F in the fluid domain and u_S in the solid domain. We chose not to construct the diffeomorphism this way so that we do not have to deal with terms arising from the Jacobian of the diffeomorphism.

Definition 13. *We define the spaces*

$$V_{\varepsilon,0} := \left\{ v : [0,T] \times F_\varepsilon(0) \to \mathbb{R} \mid \int_0^T \int_{F_\varepsilon(0)} |\partial_t v(t,x)|^2 + |v(t,x)|^2 + |\nabla v(t,x)|^2 \, dx \, dt < +\infty \right\}$$
$$W_{\varepsilon,0} := \left\{ w : [0,T] \times S_\varepsilon(0) \to \mathbb{R} \mid \int_0^T \int_{S_\varepsilon(0)} |\partial_t w(t,x)|^2 + |w(t,x)|^2 + |\nabla w(t,x)|^2 \, dx \, dt < +\infty \right\}$$

For a function $v \in V_{\varepsilon,0}$ and $w \in W_{\varepsilon,0}$, we let $\tilde{v} := v \circ \boldsymbol{X}_\varepsilon$ and $\tilde{w} := w \circ \boldsymbol{X}_\varepsilon$. Because $\boldsymbol{X}_\varepsilon$ is a diffeomorphism, we have that $(v,w) \in V_{\varepsilon,0} \times W_{\varepsilon,0}$ if and only if $(\tilde{v}, \tilde{w}) \in V_\varepsilon \times W_\varepsilon$.

We have the following proposition:

Proposition 14. $(v_\varepsilon, w_\varepsilon)$ *is a weak solution to the moving domain problem if and only if* $\tilde{v}_\varepsilon \in V_{\varepsilon,0}$, $\tilde{w}_{\varepsilon,0} \in W_{\varepsilon,0}$ *and* $\tilde{v}_\varepsilon, \tilde{w}_\varepsilon$ *and that for every* $(\tilde{\varphi}, \tilde{\psi}) \in V_{\varepsilon,0} \times W_{\varepsilon,0}$ *we* *have:*

$$\int_0^T \int_{F_\varepsilon(0)} \left(\partial_t \tilde{v}_\varepsilon + (\boldsymbol{u}_\varepsilon \circ \boldsymbol{X}_\varepsilon - \partial_t \boldsymbol{X}_\varepsilon) \cdot (\nabla \boldsymbol{X}_\varepsilon)^{-T} \nabla \tilde{v}_\varepsilon\right) \tilde{\varphi} \tag{2.33}$$

$$+ \int_0^T \int_{S_\varepsilon(0)} \left(\partial_t \tilde{w}_\varepsilon - \partial_t \boldsymbol{X}_\varepsilon \cdot (\nabla \boldsymbol{X}_\varepsilon)^{-T} \nabla \tilde{w}_\varepsilon\right) \tilde{\psi}$$

$$+ \int_0^T \int_{F_\varepsilon(0)} A_F^\varepsilon \nabla \tilde{v}_\varepsilon \cdot \nabla \tilde{\varphi} + \int_0^T \int_{S_\varepsilon(0)} A_S^\varepsilon \nabla \tilde{w}_\varepsilon \cdot \nabla \tilde{\psi} + \int_0^T \int_{S_\varepsilon(0)} r \tilde{w}_\varepsilon \tilde{\psi} \tag{2.34}$$

$$= \int_0^T \int_{\Gamma_\varepsilon(0)} \alpha_\varepsilon \left(\tilde{w}_\varepsilon - \tilde{v}_\varepsilon\right) \left(\tilde{\varphi} - \tilde{\psi}\right),$$

where

$$A_F^\varepsilon(t,x) := \mathbb{1}_{F_\varepsilon(0)}(t) \left(\nabla \boldsymbol{X}_\varepsilon(t,x)\right)^{-1} \left(\nabla \boldsymbol{X}_\varepsilon(t,x)\right)^{-T}$$

$$A_S^\varepsilon(t,x) := \mathbb{1}_{S_\varepsilon(0)}(t) \left(\nabla \boldsymbol{X}_\varepsilon(t,x)\right)^{-1} \left(\nabla \boldsymbol{X}_\varepsilon(t,x)\right)^{-T}.$$

Proof. Let $t \in (0,T)$. Let $(\varphi, \psi) \in V_\varepsilon \times W_\varepsilon$. We begin with

$$\begin{aligned} \int_{F_\varepsilon(t)} D_F \nabla v_\varepsilon \cdot \nabla \phi &= \int_{F_\varepsilon(0)} D_F \nabla v_\varepsilon \circ \boldsymbol{X}_\varepsilon \cdot \nabla \phi \circ \boldsymbol{X}_\varepsilon \\ &= \int_{F_\varepsilon(0)} D_F \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-T} \nabla \left(v_\varepsilon \circ \boldsymbol{X}_\varepsilon(t)\right) \cdot \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-T} \nabla \left(\phi \circ \boldsymbol{X}_\varepsilon(t)\right) \\ &= \int_{F_\varepsilon(0)} A_F^\varepsilon \nabla \tilde{v}_\varepsilon \cdot \nabla \tilde{\phi}. \end{aligned}$$

Similarly,

$$\int_{S_\varepsilon(t)} D_S \nabla w_\varepsilon \cdot \nabla \psi = \int_{S_\varepsilon(0)} A_S^\varepsilon \nabla \tilde{w}_\varepsilon \cdot \nabla \tilde{\psi}.$$

As for the time derivative terms, we have

$$\begin{aligned}\int_{F_\varepsilon(t)} \left(\partial_t v_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla v_\varepsilon\right)\phi &= \int_{F_\varepsilon(0)} \left(\partial_t v_\varepsilon \circ \boldsymbol{X}_\varepsilon + \left(\boldsymbol{u} \circ \boldsymbol{X}_\varepsilon - \partial_t \boldsymbol{X}_\varepsilon\right) \cdot \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-T} \nabla \tilde{v}_\varepsilon\right) \varphi \circ \boldsymbol{X}_\varepsilon \\ &= \int_{F_\varepsilon(0)} \left(\partial_t \tilde{v}_\varepsilon + \left(\boldsymbol{u}_\varepsilon \circ \boldsymbol{X}_\varepsilon - \partial_t \boldsymbol{X}_\varepsilon\right) \cdot \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-T} \nabla \tilde{v}_\varepsilon\right) \tilde{\varphi},\end{aligned}$$

and

$$\int_{S_\varepsilon(t)} \partial_t w_\varepsilon \psi = \int_{S_\varepsilon(0)} \left(\partial_t \tilde{w}_\varepsilon - \partial_t \boldsymbol{X}_\varepsilon \cdot \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-T} \nabla \tilde{w}_\varepsilon\right) \tilde{\psi}$$

Since $\boldsymbol{X}_\varepsilon$ is a rigid displacement on $\Gamma_\varepsilon(t)$, we have

$$\int_{\Gamma_\varepsilon(t)} \alpha_\varepsilon \left(w_\varepsilon - v_\varepsilon\right)\left(\varphi - \psi\right) = \int_{\Gamma_\varepsilon(0)} \alpha_\varepsilon \left(\tilde{w}_\varepsilon - \tilde{v}_\varepsilon\right)\left(\tilde{\varphi} - \tilde{\psi}\right)$$

Lastly, as

$$\nabla \tilde{\varphi} = \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-T} \nabla \phi \circ \boldsymbol{X}_\varepsilon,$$

and

$$\nabla \varphi = \left(\nabla \boldsymbol{X}_\varepsilon^{-1}\right)^{-T} \nabla \tilde{\phi} \circ \boldsymbol{X}_\varepsilon^{-1},$$

the proposition follows. □

Assumption 15. *We assume that there exist* A_F^0, $A_S^0 \in L^2\left((0,T)\times\Omega; C_{per}(Y)\right)$ *such that*

$$\int_0^T \int_\Omega \left|A_F^\varepsilon(t,x) - A_F^0\left(t,x,\frac{x}{\varepsilon}\right)\right|^2 \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{2.35}$$

$$\int_0^T \int_\Omega \left|A_S^\varepsilon(t,x) - A_S^0\left(t,x,\frac{x}{\varepsilon}\right)\right|^2 \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{2.36}$$

i.e. we assume that A_F^ε *and* A_S^ε *are strongly two-scale convergent. This allows us* *later on to deal with their limits in the homogenization process. Similar assumptions have been made in [30] to deal with homogenization in evolving domains.*

2.3 Estimates and existence of solutions

From here onwards, we define $v^\varepsilon := \tilde{v}_\varepsilon$ and $w^\varepsilon := \tilde{w}_\varepsilon$, i.e., $(v^\varepsilon, w^\varepsilon)$ is fixed domain solution. Moreover, we have that $F_\varepsilon = F_\varepsilon(0)$, and $S_\varepsilon = S_\varepsilon(0)$. Observe that $(v^\varepsilon, w^\varepsilon)$ are the weak solutions to the following problem:

$$\partial_t v^\varepsilon - \text{div } (A_F^\varepsilon \nabla v^\varepsilon) + \boldsymbol{U}_\varepsilon^1 \cdot \nabla v^\varepsilon = 0, \qquad (0,T) \times F_\varepsilon \tag{2.37}$$

$$A_F^\varepsilon \nabla v^\varepsilon \cdot \boldsymbol{n} = 0, \qquad (0,T) \times \partial\Omega \tag{2.38}$$

$$A_F^\varepsilon \nabla v^\varepsilon \cdot \boldsymbol{n} = A_S^\varepsilon \nabla w^\varepsilon \cdot \boldsymbol{n}, \qquad (0,T) \times \Gamma_\varepsilon \tag{2.39}$$

$$A_F^\varepsilon \nabla v^\varepsilon \cdot \boldsymbol{n} + \alpha_\varepsilon (v^\varepsilon - w^\varepsilon) = 0, \qquad (0,T) \times \Gamma_\varepsilon \tag{2.40}$$

$$\partial_t w^\varepsilon - \text{div } (A_S^\varepsilon \nabla w^\varepsilon) - \boldsymbol{U}_\varepsilon^2 \cdot \nabla w^\varepsilon + r w^\varepsilon = 0, \qquad (0,T) \times S_\varepsilon \tag{2.41}$$

$$v^\varepsilon(0) = v_{\varepsilon,0}, \qquad \text{in } F_\varepsilon(0) \tag{2.42}$$

$$w^\varepsilon(0) = w_{\varepsilon,0}, \qquad \text{in } S_\varepsilon(0), \tag{2.43}$$

$$\tag{2.44}$$

where $\boldsymbol{U}_\varepsilon^1 := (\nabla \boldsymbol{X}_\varepsilon)^{-1} (\boldsymbol{u}_\varepsilon \circ \boldsymbol{X}_\varepsilon - \partial_t \boldsymbol{X}_\varepsilon)$ and $\boldsymbol{U}_\varepsilon^2 := (\nabla \boldsymbol{X}_\varepsilon)^{-1} \partial_t \boldsymbol{X}_\varepsilon$.

Because of the slowness Assumption 2.21 on the solid velocities, we have that the matrices A_F^ε and A_S^ε are coercive with a coercivity constant that is independent of ε. Indeed, we have

Lemma 16. *There exists $\beta > 0$ such that for all $\boldsymbol{v} \in \mathbb{R}^3$, we have*

$$\sup_{t,x,\varepsilon} \left(\boldsymbol{v}^T \boldsymbol{A}^\varepsilon(t,x)\boldsymbol{v}\right) \geq \beta \boldsymbol{v}^T \boldsymbol{v}, \tag{2.45}$$

where $\boldsymbol{A}^\varepsilon(t,x) := (\nabla \boldsymbol{X}_\varepsilon(t,x))^{-1} (\nabla \boldsymbol{X}_\varepsilon(t,x))^{-T}$.

Proof. Since $\boldsymbol{A}^\varepsilon$ is positive semidefinite, it is unitarily diagonalizable and all of its eigenvalues are nonnegative. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of $\boldsymbol{A}^\varepsilon(t,x)$ and $D :=$ diag $\{\lambda_1, \lambda_2, \lambda_3\}$. Let $\boldsymbol{v} \in \mathbb{R}^3$. Then, for some unitary matrix P, we have

$$\boldsymbol{v}^T \boldsymbol{A}^\varepsilon(t,x)\boldsymbol{v} = \boldsymbol{v}^T P^* D P \boldsymbol{v} = \boldsymbol{w} D \boldsymbol{w} = \sum_{i=1}^{3} \lambda_i w_i^2 \geq \lambda_1 \|\boldsymbol{w}\|_2^2 = \lambda_1 \|\boldsymbol{v}\|_2^2,$$

where the last equality follows from P being a unitary matrix. We now show that $\lambda_1 = \lambda_1(t,x,\varepsilon)$ can be uniformly estimated. First, we have

$$\det\left(\boldsymbol{A}^\varepsilon(t,x)\right) = \det\left(\nabla \boldsymbol{X}_\varepsilon(t,x)^{-1}\right)^2 = e^{-2\,tr\left(\int_0^t \nabla \boldsymbol{b}_\varepsilon(x, \boldsymbol{X}_\varepsilon(s,x)\,ds)\right)} \geq e^{-C(\|\nabla \boldsymbol{b}_\varepsilon\|_\infty)} \geq C_1,$$

for some $C_1 > 0$. The last inequality follows from Assumption 2.21 and

$$\nabla \boldsymbol{X}_\varepsilon(t,y) = e^{\int_0^t \nabla \boldsymbol{b}_\varepsilon(s, \boldsymbol{X}_\varepsilon(s,y))\,ds}.$$

See the last section on *Examples* for details. Next, we have that

$$\lambda_3 \leq \|\boldsymbol{A}^\varepsilon(t,x)\|_{L^2((0,T)\times\Omega)} \leq C\|\boldsymbol{A}^\varepsilon(t,x)\|_{L^\infty((0,T)\times\Omega)} \leq C\|\nabla \boldsymbol{b}_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leq C_2 < \infty.$$

Thus,

$$\lambda_1 = \frac{\det\left(\boldsymbol{A}^\varepsilon(t,x)\right)}{\lambda_3 \lambda_2} \geq \frac{\det\left(\boldsymbol{A}^\varepsilon(t,x)\right)}{\lambda_3^2} \geq \frac{C_1}{C_2} =: \beta > 0.$$

□

We have the following existence theorem:

Theorem 17. *There exists a unique weak solution* $(v_\varepsilon, w_\varepsilon)$ *to the fixed domain problem.*

We first construct approximate solutions by successively solving a sequence of steady-state problems. We proceed as follows. Let $N \in \mathbb{N}$ and define $k := \frac{T}{N}$. We set $v_\varepsilon^0 := v_{\varepsilon,0}$ and $w_\varepsilon^0 := w_{\varepsilon,0}$. For $m = 1, ..., N$, we have

Proposition 18. *Given* $v_\varepsilon^1, v_\varepsilon^2, ..., v_\varepsilon^{m-1} \in H^1(F_\varepsilon)$ *and* $w_\varepsilon^1, w_\varepsilon^2, ..., w_\varepsilon^{m-1} \in H^1(S_\varepsilon)$, *there exists a unique solution* $(v_\varepsilon^m, w_\varepsilon^m) \in H^1(F_\varepsilon) \times H^1(S_\varepsilon)$ *to the problem:*

$$\begin{aligned}
\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k} - div(A_F^{\varepsilon,m}\nabla v_\varepsilon^m) + \boldsymbol{U}_\varepsilon^{1,m} \cdot \nabla v_\varepsilon^m &= 0, && \textit{in } F_\varepsilon \\
A_F^{\varepsilon,m}\nabla v_\varepsilon^m \cdot \boldsymbol{n} &= 0, && \textit{on } \partial\Omega \\
A_F^{\varepsilon,m}\nabla v_\varepsilon^m \cdot \boldsymbol{n} &= A_S^{\varepsilon,m}\nabla w_\varepsilon^m \cdot \boldsymbol{n}, && \textit{on } \Gamma_\varepsilon \\
A_F^{\varepsilon,m}\nabla v_\varepsilon^m \cdot \boldsymbol{n} + \alpha_\varepsilon (v_\varepsilon^m - w_\varepsilon^m) &= 0, && \textit{on } \Gamma_\varepsilon \\
\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} - div(A_S^\varepsilon\nabla w_\varepsilon^m) - \boldsymbol{U}_\varepsilon^{2,m} \cdot \nabla w_\varepsilon^m + r w_\varepsilon^m &= 0, && \textit{in } S_\varepsilon,
\end{aligned}$$

where

$$A_F^{\varepsilon,m}(x) := A_F^\varepsilon(mk, x), \quad A_S^{\varepsilon,m}(x) := A_S^\varepsilon(mk, x),$$

and

$$\boldsymbol{U}_\varepsilon^{i,m} := \frac{1}{k}\int_{(m-1)k}^{mk} \boldsymbol{U}_\varepsilon^i \, dt, \quad i = 1, 2.$$

Proof. We define $a^m : [H^1(F_\varepsilon) \times H^1(S_\varepsilon)]^2 \to \mathbb{R}$ as

$$a^m((v,w),(\varphi,\psi)) := \int_{F_\varepsilon} \left(\frac{1}{k} v + \boldsymbol{U}_\varepsilon^{1,m} \cdot \nabla v \right) \varphi + \int_{S_\varepsilon} \left(\frac{1}{k} w + \boldsymbol{U}_\varepsilon^{2,m} \cdot \nabla w + rw \right) \psi \\ + \int_{F_\varepsilon} A_F^{\varepsilon,m} \nabla v \cdot \nabla \varphi + \int_{S_\varepsilon} A_S^{\varepsilon,m} \nabla w \cdot \nabla \psi + \int_{\Gamma_\varepsilon} \alpha_\varepsilon (v-w)(\varphi - \psi).$$

We also define $F^m : H^1(F_\varepsilon) \times H^1(S_\varepsilon) \to \mathbb{R}$ as

$$F^m(\varphi,\psi) := \int_{F_\varepsilon} \frac{1}{k} v^{m-1} \varphi + \int_{S_\varepsilon} \frac{1}{k} w^{m-1} \psi.$$

Then, we can write our problem as: find $(v_\varepsilon^m, w_\varepsilon^m) \in H^1(F_\varepsilon) \times H^1(S_\varepsilon)$ such that

$$a^m((v_\varepsilon^m, w_\varepsilon^m), (\varphi,\psi)) = F^m(\varphi,\psi),$$

for all $(\varphi,\psi) \in H^1(F_\varepsilon) \times H^1(S_\varepsilon)$.

We proceed to show that this is uniquely solvable by the Lax-Milgram lemma. Clearly, F^m is linear. Moreover, by induction, F^m is bounded on $H^1(F_\varepsilon) \times H^1(S_\varepsilon)$.

Now, a^m is clearly bilinear on $[H^1(F_\varepsilon) \times H^1(S_\varepsilon)]^2$. It remains to show that it is coercive. Indeed, let $(v,w) \in H^1(F_\varepsilon) \times H^1(S_\varepsilon)$. By Lemma 16, we have that

$$\int_{F_\varepsilon} A_F^{\varepsilon,m} \nabla v \cdot \nabla v + \int_{S_\varepsilon} A_S^{\varepsilon,m} \nabla w \cdot \nabla w \geq \beta \left(\|v\|_{L^2(F_\varepsilon)}^2 + \|w\|_{L^2(S_\varepsilon)}^2 \right). \tag{2.46}$$

Also,

$$\int_{\Gamma_\varepsilon} \alpha_\varepsilon (v-w)^2 \geq 0. \tag{2.47}$$

Now, by (2.21), we have that

$$\|\boldsymbol{U}_{\varepsilon}^{1,m}\|_{L^{\infty}((0,T)\times F_{\varepsilon})} + \|\boldsymbol{U}_{\varepsilon}^{2,m}\|_{L^{\infty}((0,T)\times S_{\varepsilon})} \leq C, \tag{2.48}$$

for some C that is independent of ε. Moreover,

$$\|\boldsymbol{U}_{\varepsilon}^{2,m}\|_{L^{\infty}((0,T)\times S_{\varepsilon})} \leq C\varepsilon^{\gamma}.$$

Suppose for now that v is $H^2(F_\varepsilon)$. Then, we have that

$$\begin{aligned}\int_{F_\varepsilon} v\boldsymbol{U}_{\varepsilon}^{1,m}\cdot\nabla v &= \int_{F_\varepsilon}\boldsymbol{U}_{\varepsilon}^{1,m}\cdot\nabla\left(\frac{v^2}{2}\right) \quad = \int_{F_\varepsilon}\text{div }\left(\frac{v^2}{2}\boldsymbol{U}_{\varepsilon}^{1,m}\right) - \frac{v^2}{2}\text{div }\boldsymbol{U}_{\varepsilon}^{1,m}\\ &= \int_{\Gamma_\varepsilon}\frac{v^2}{2}\boldsymbol{U}_{\varepsilon}^{1,m}\cdot n\\ &= 0,\end{aligned}$$

since $\boldsymbol{u}_\varepsilon = \partial_t\boldsymbol{X}_\varepsilon$ on $\Gamma_\varepsilon\cup\partial\Omega$ and

$$\begin{aligned}\text{div }\boldsymbol{U}_{\varepsilon}^{1,m} &= \text{div }\left(\frac{1}{k}\int_{(m-1)k}^{mk}(\nabla\boldsymbol{X}_\varepsilon)^{-1}\left(\boldsymbol{u}_\varepsilon\circ\boldsymbol{X}_\varepsilon - \partial_t\boldsymbol{X}_\varepsilon\right)\right)\\ &= \frac{1}{k}\int_{(m-1)k}^{mk}\text{div }\left((\nabla\boldsymbol{X}_\varepsilon)^{-1}\left(\boldsymbol{u}_\varepsilon\circ\boldsymbol{X}_\varepsilon\right)\right) - \partial_t\text{div }\boldsymbol{X}_\varepsilon = 0,\end{aligned}$$

since $\boldsymbol{X}_\varepsilon$ has divergence zero everywhere. Thus, by a density argument, it follows that if $v\in H^1(F_\varepsilon)$, we have

$$\text{div }\boldsymbol{U}_{\varepsilon}^{1,m} = 0.$$

Combining these estimates, we obtain that for small enough ε,

$$\begin{aligned} a^m\left((v,w),(v,w)\right) &\geq \frac{1}{k}\|v\|^2_{L^2(F_\varepsilon)} + \beta\|\nabla v\|^2_{L^2(F_\varepsilon)} \\ &+ \left(\frac{1}{k} + r - C\varepsilon^\alpha\right)\|w\|^2_{L^2(S_\varepsilon)} + \left(\beta - C\varepsilon^\gamma\right)\|\nabla w\|^2_{L^2(S_\varepsilon)} \\ &\geq C\left(\|v\|^2_{H^1(F_\varepsilon)} + \|w\|^2_{H^1(F_\varepsilon)}\right). \end{aligned}$$

Thus, a^m is coercive on $[H^1(F_\varepsilon) \times H^1(S_\varepsilon)]^2$. Hence, the proposition follows by the Lax-Milgram lemma. □

We now define the following approximate solutions:

$$\begin{aligned} v_{\varepsilon,k}(t,x) &:= \sum_{m=1}^{N} v^m_\varepsilon(x)\mathbb{1}_{[(m-1)k,mk)}(t) \\ w_{\varepsilon,k}(t,x) &:= \sum_{m=1}^{N} w^m_\varepsilon(x)\mathbb{1}_{[(m-1)k,mk)}(t) \\ \bar{v}_{\varepsilon,k}(t,x) &:= \sum_{m=1}^{N}\left(v^{m-1}_\varepsilon(x) + \frac{v^m_\varepsilon(x) - v^{m-1}_\varepsilon(x)}{k}(t-mk)\right)\mathbb{1}_{[(m-1)k,mk)}(t) \\ \bar{w}_{\varepsilon,k}(t,x) &:= \sum_{m=1}^{N}\left(w^{m-1}_\varepsilon(x) + \frac{w^m_\varepsilon(x) - w^{m-1}_\varepsilon(x)}{k}(t-mk)\right)\mathbb{1}_{[(m-1)k,mk)}(t) \end{aligned}$$

Observe that $\bar{v}_{\varepsilon,k}$ and $\bar{w}_{\varepsilon,k}$ are continuous in time. Their time derivatives are well-defined in a weak sense and approximate to the time derivatives of the fixed domain solutions. We now prove estimates for these functions.

Lemma 19. *Given* $\varepsilon_0 > 0$*, for* $p = 1, ..., N$*, and* $\varepsilon \leq \varepsilon_0$*, the following holds:*

$$\|v_\varepsilon^p\|^2_{L^2(F_\varepsilon)} + \|w_\varepsilon^p\|^2_{L^2(S_\varepsilon)} + \sum_{m=1}^{p} \left(\|v_\varepsilon^m - v_\varepsilon^{m-1}\|^2_{L^2(F_\varepsilon)} + \|w_\varepsilon^m - w_\varepsilon^{m-1}\|^2_{L^2(S_\varepsilon)} \right)$$
$$+ k\beta \sum_{m=1}^{p} \|\nabla v_\varepsilon^m\|^2_{L^2(F_\varepsilon)} + k\left(\beta - C\varepsilon_0^\gamma\right) \sum_{m=1}^{p} \|\nabla w_\varepsilon^m\|^2_{L^2(S_\varepsilon)}$$
$$\leq \|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)}$$

Proof. By taking v_ε^m and w_ε^m to be test functions in the weak formulation of the steady-state problem for v_ε^m and w_ε^m, we obtain

$$\int_{F_\varepsilon} \left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k} v_\varepsilon^m \right) + \int_{S_\varepsilon} \left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} w_\varepsilon^m \right)$$
$$+ \int_{F_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{1,m} \cdot \nabla v_\varepsilon^m \right) v_\varepsilon^m - \int_{S_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{2,m} \cdot \nabla w_\varepsilon^m \right) w_\varepsilon^m$$
$$+ \int_{F_\varepsilon} A_F^{\varepsilon,m} \nabla v_\varepsilon^m \cdot \nabla v_\varepsilon^m + \int_{S_\varepsilon} A_S^{\varepsilon,m} \nabla w_\varepsilon^m \cdot \nabla w_\varepsilon^m + \int_{S_\varepsilon} r \left(w_\varepsilon^m\right)^2 + \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_\varepsilon^m - w_\varepsilon^m\right)^2$$
$$= 0.$$

Since,

$$\int_{F_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{1,m} \cdot \nabla v_\varepsilon^m \right) v_\varepsilon^m = 0,$$
$$- \int_{S_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{2,m} \cdot \nabla w_\varepsilon^m \right) w_\varepsilon^m \geq -C\varepsilon_0^\gamma \|w\|^2_{H^1(F_\varepsilon)},$$
$$\int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_\varepsilon^m - w_\varepsilon^m\right)^2 \geq 0,$$

and

$$2\left(v_\varepsilon^m - v_\varepsilon^{m-1}\right) v_\varepsilon^m = \left(v_\varepsilon^m\right)^2 - \left(v_\varepsilon^{m-1}\right)^2 + \left(v_\varepsilon^m - v_\varepsilon^{m-1}\right)^2,$$

we obtain

$$\begin{aligned}
&\|v_\varepsilon^m\|^2_{L^2(F_\varepsilon)} - \|v_\varepsilon^{m-1}\|^2_{L^2(F_\varepsilon)} + \|v_\varepsilon^m - v_\varepsilon^{m-1}\|^2_{L^2(F_\varepsilon)} \\
&+ \|w_\varepsilon^m\|^2_{L^2(S_\varepsilon)} - \|w_\varepsilon^{m-1}\|^2_{L^2(S_\varepsilon)} + \|w_\varepsilon^m - w_\varepsilon^{m-1}\|^2_{L^2(S_\varepsilon)} \\
&+ k\beta\|\nabla v_\varepsilon^m\|^2_{L^2(F_\varepsilon)} + k\left(\beta - C\varepsilon_0^\gamma\right)\|\nabla w_\varepsilon^m\|^2_{L^2(S_\varepsilon)} \\
&\le 0.
\end{aligned}$$

Summing from $m = 1, ..., p$, for $1 \le p \le N$, and noting that $\|v_\varepsilon^m - v_\varepsilon^{m-1}\|^2_{L^2(F_\varepsilon)}$, $\|w_\varepsilon^m - w_\varepsilon^{m-1}\|^2_{L^2(S_\varepsilon)} \ge 0$, the lemma follows. □

Lemma 20.

$$\|v_{\varepsilon,k}\|^2_{L^\infty(0,T;L^2(F_\varepsilon))} + \|w_{\varepsilon,k}\|^2_{L^\infty(0,T;L^2(S_\varepsilon))} \le \|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)}.$$

Proof. Let $t \in (0, T)$. Then,

$$v_{\varepsilon,k}(t,\cdot) = v_\varepsilon^m(\cdot), \quad w_{\varepsilon,k}(t,\cdot) = w_\varepsilon^m(\cdot),$$

for some $m \in \{1, ..., N\}$. But by the Lemma 19, we have that

$$\|v_{\varepsilon,k}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,k}\|^2_{L^2(S_\varepsilon)} \le \|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)}.$$

Taking the essential supremum over t in $(0, T)$ gives the lemma. □

Lemma 21.

$$\|v_{\varepsilon,k}\|^2_{L^2(0,T;H^1(F_\varepsilon))} + \|w_{\varepsilon,k}\|^2_{L^2(0,T;H^1(S_\varepsilon))} \le C\left(\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)}\right).$$

Proof. From Lemma 19, we have

$$\begin{aligned}\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)} &\geq k\beta \sum_{m=1}^{p} \|\nabla v_\varepsilon^m\|^2_{L^2(F_\varepsilon)} + k\left(\beta - C\varepsilon_0^\gamma\right) \sum_{m=1}^{p} \|\nabla w_\varepsilon^m\|^2_{L^2(S_\varepsilon)} \\ &= \beta \sum_{m=1}^{p} \int_{(m-1)k}^{mk} \|\nabla v_\varepsilon^m\|^2_{L^2(F_\varepsilon)} \\ &\quad + k\left(\beta - C\varepsilon_0^\alpha\right) \sum_{m=1}^{p} \int_{(m-1)k}^{mk} \|\nabla w_\varepsilon^m\|^2_{L^2(S_\varepsilon)} \\ &= \beta \|\nabla v_{\varepsilon,k}\|^2_{L^2((0,T)\times F_\varepsilon)} + \left(\beta - C\varepsilon_0^\alpha\right) \|\nabla w_{\varepsilon,k}\|^2_{L^2((0,T)\times S_\varepsilon)} \\ &\geq C\left(\beta, \varepsilon^\gamma\right) \left(\|\nabla v_{\varepsilon,k}\|^2_{L^2((0,T)\times F_\varepsilon)} + \|\nabla w_{\varepsilon,k}\|^2_{L^2((0,T)\times S_\varepsilon)} \right).\end{aligned}$$

Now, from Lemma 20, we have

$$\begin{aligned}\|v_{\varepsilon,k}\|^2_{L^2((0,T)\times F_\varepsilon)} + \|w_{\varepsilon,k}\|^2_{L^2((0,T)\times S_\varepsilon)} &\leq T\left(\|v_{\varepsilon,k}\|^2_{L^\infty(0,T;L^2(F_\varepsilon))} + \|w_{\varepsilon,k}\|^2_{L^\infty(0,T;L^2(S_\varepsilon))} \right) \\ &\leq T\left(\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)} \right).\end{aligned}$$

Combining these estimates proves the lemma. □

Lemma 22.

$$\|\partial_t \bar{v}_{\varepsilon,k}\|^2_{L^2(0,T;H^1(F_\varepsilon)^*)} + \|\partial_t \bar{w}_{\varepsilon,k}\|^2_{L^2(0,T;H^1(S_\varepsilon)^*)} \leq C\left(\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)} \right).$$

Proof. Let $(\varphi, \psi) \in L^2\left(0,T;H^1\left(F_\varepsilon\right)\right) \times L^2\left(0,T;H^1\left(S_\varepsilon\right)\right)$. Using these as test functions in the weak formulation of the PDE that v_ε^m and w_ε^m solve, we obtain

$$\begin{aligned}&\int_{F_\varepsilon}\left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k}\right)\varphi + \int_{S_\varepsilon}\left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}\right)\psi \\ &= -\int_{F_\varepsilon} A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\nabla\varphi - \int_{S_\varepsilon} A_S^{\varepsilon,m}\nabla w_\varepsilon^m\cdot\nabla\psi \\ &\quad - \int_{F_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{1,m}\cdot\nabla v_\varepsilon^m\right)\varphi + \int_{S_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{2,m}\cdot\nabla w_\varepsilon^m\right)\psi \\ &\quad - \int_{S_\varepsilon} r w_\varepsilon^m\psi - \int_{\Gamma_\varepsilon}\alpha_\varepsilon\left(v_\varepsilon^m - w_\varepsilon^m\right)(\varphi-\psi).\end{aligned}$$

Summing from $m = 1, ..., N$ and integrating in time, we obtain

$$\begin{aligned}&\int_0^T\int_{F_\varepsilon}(\partial_t\bar{v}_{\varepsilon,k})\varphi + \int_0^T\int_{S_\varepsilon}(\partial_t\bar{w}_{\varepsilon,k})\psi \\ &= \sum_{m=1}^N\left(\int_{(m-1)k}^{mk}\int_{F_\varepsilon}\left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k}\right)\varphi + \int_{(m-1)k}^{mk}\int_{S_\varepsilon}\left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}\right)\psi\right) \\ &= -\sum_{m=1}^N\int_{(m-1)k}^{mk}\left(\int_{F_\varepsilon} A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\nabla\varphi + \int_{S_\varepsilon} A_S^{\varepsilon,m}\nabla w_\varepsilon^m\cdot\nabla\psi + \int_{F_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{1,m}\cdot\nabla v_\varepsilon^m\right)\varphi\right. \\ &\qquad\qquad\left. - \int_{S_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{2,m}\cdot\nabla w_\varepsilon^m\right)\psi - \int_{S_\varepsilon} r w_\varepsilon^m\psi - \int_{\Gamma_\varepsilon}\alpha_\varepsilon\left(v_\varepsilon^m - w_\varepsilon^m\right)(\varphi-\psi)\right) \\ &= -\int_0^T\int_{F_\varepsilon} A_F^{\varepsilon,k}\nabla v_{\varepsilon,k}\cdot\nabla\varphi - \int_0^T\int_{S_\varepsilon} A_S^{\varepsilon,k}\nabla w_{\varepsilon,k}\cdot\nabla\psi - \int_0^T\int_{F_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{1,k}\cdot\nabla v_{\varepsilon,k}\right)\varphi \\ &+ \quad \int_0^T\int_{S_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{2,k}\cdot\nabla w_{\varepsilon,k}\right)\psi - \int_0^T\int_{S_\varepsilon} r w_{\varepsilon,k}\psi - \int_0^T\int_{\Gamma_\varepsilon}\alpha_\varepsilon\left(v_{\varepsilon,k} - w_{\varepsilon,k}\right)(\varphi-\psi),\end{aligned}$$

where

$$\begin{aligned}&A_F^{\varepsilon,k}(t,x) := \sum_{i=1}^N A_F^{\varepsilon,m}(x)\mathbb{1}_{[(m-1)k,mk)}(t), \quad A_S^{\varepsilon,k}(t,x) := \sum_{i=1}^N A_S^{\varepsilon,m}(x)\mathbb{1}_{[(m-1)k,mk)}(t) \\ &\boldsymbol{U}_\varepsilon^{i,k}(t,x) := \sum_{m=1}^N \boldsymbol{U}_\varepsilon^{i,m}(x)\mathbb{1}_{[(m-1)k,mk)}(t), \quad i = 1,2.\end{aligned}$$

We estimate the surface term as follows:

$$\int_0^T \int_{\Gamma_\varepsilon} |\alpha_\varepsilon (v_{\varepsilon,k} - w_{\varepsilon,k}) (\varphi - \psi)| \leq C \int_0^T \varepsilon \|v_{\varepsilon,k} - w_{\varepsilon,k}\|_{L^2(\Gamma_\varepsilon)} \|\varphi - \psi\|_{L^2(\Gamma_\varepsilon)}$$
$$\leq C \left(\|v_{\varepsilon,k}\|_{L^2(0,T;H^1(F_\varepsilon))} + \|w_{\varepsilon,k}\|_{L^2(0,T;H^1(S_\varepsilon))}\right) \left(\|\varphi\|_{L^2(0,T;H^1(F_\varepsilon))} + \|\psi\|_{L^2(0,T;H^1(F_\varepsilon))}\right),$$

where the last inequality follows from Lemma 30. Thus, we have

$$\left| \int_0^T \int_{F_\varepsilon} (\partial_t \bar{v}_{\varepsilon,k})\varphi + \int_0^T \int_{S_\varepsilon} (\partial_t \bar{w}_{\varepsilon,k})\psi \right|$$
$$\leq C \left(\|v_{\varepsilon,k}\|_{L^2(0,T;H^1(F_\varepsilon))} + \|w_{\varepsilon,k}\|_{L^2(0,T;H^1(S_\varepsilon))}\right) \left(\|\varphi\|_{L^2(0,T;H^1(F_\varepsilon))} + \|\psi\|_{L^2(0,T;H^1(F_\varepsilon))}\right)$$
$$\leq C \left(\|v_{\varepsilon,0}\|_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|_{L^2(S_\varepsilon)}\right) \left(\|\varphi\|_{L^2(0,T;H^1(F_\varepsilon))} + \|\psi\|_{L^2(0,T;H^1(F_\varepsilon))}\right).$$

Taking the supremum over all φ and ψ proves the lemma. □

Lemma 23. *The following convergences hold:*

$$v_{\varepsilon,k} - \bar{v}_{\varepsilon,k} \to 0, \quad \text{in } L^2 ((0,T) \times F_\varepsilon)$$
$$w_{\varepsilon,k} - \bar{w}_{\varepsilon,k} \to 0, \quad \text{in } L^2 ((0,T) \times S_\varepsilon)$$

Proof. Observe that

$$v_{\varepsilon,k}(t,x) - \bar{v}_{\varepsilon,k}(t,x)$$
$$= \sum_{m=1}^{N} \left(v_\varepsilon^m(x) - v_\varepsilon^{m-1}(x) - \frac{v_\varepsilon^m(x) - v_\varepsilon^{m-1}(x)}{k}(t - (m-1)k) \right) \mathbb{1}_{[(m-1)k, mk)}(t)$$
$$= \sum_{m=1}^{N} \left(\frac{v_\varepsilon^m(x) - v_\varepsilon^{m-1}(x)}{k}(mk - t) \right) \mathbb{1}_{[(m-1)k, mk)}(t).$$

Thus,

$$\begin{aligned}\|v_{\varepsilon,k}-\bar{v}_{\varepsilon,k}\|^2_{L^2((0,T)\times F_\varepsilon)} &= \frac{1}{k}\sum_{m=1}^{N}\|v_\varepsilon^m - v_\varepsilon^{m-1}\|^2_{L^2(F_\varepsilon)}\int_{(m-1)k}^{mk}(t-mk)^2\,dt \\ &= \frac{k^2}{3}\sum_{m=1}^{N}\|v_\varepsilon^m - v_\varepsilon^{m-1}\|^2_{L^2(F_\varepsilon)} \le \frac{k^2}{3}\Big(\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)}+\|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)}\Big) \\ &\to 0, \quad \text{as } k\to 0.\end{aligned}$$

where the last inequality follows from Lemma 19. Similarly, we have

$$\|v_{\varepsilon,k}-\bar{v}_{\varepsilon,k}\|^2_{L^2((0,T)\times F_\varepsilon)} \le \frac{k^2}{3}\Big(\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)}+\|w_{\varepsilon,0}\|^2_{L^2(S_\varepsilon)}\Big) \to 0, \quad \text{as } k\to 0.$$

□

We are now ready to prove our existence theorem.

Proof of Theorem 17. From the estimates, we have that, up to a subsequence, the following convergences hold:

$$\begin{aligned} v_{\varepsilon,k} &\rightharpoonup v^\varepsilon, && wk^* - L^\infty\left(0,T;L^2\left(F_\varepsilon\right)\right)\\ w_{\varepsilon,k} &\rightharpoonup w^\varepsilon, && wk^* - L^\infty\left(0,T;L^2\left(S_\varepsilon\right)\right)\\ v_{\varepsilon,k} &\rightharpoonup v^\varepsilon, && wk - L^2\left(0,T;H^1\left(F_\varepsilon\right)\right)\\ w_{\varepsilon,k} &\rightharpoonup w^\varepsilon, && wk - L^2\left(0,T;H^1\left(S_\varepsilon\right)\right)\\ \partial_t\bar{v}_{\varepsilon,k} &\rightharpoonup v^{\varepsilon*}, && wk - L^2\left(0,T;H^1\left(F_\varepsilon\right)^*\right)\\ \partial_t\bar{w}_{\varepsilon,k} &\rightharpoonup w^{\varepsilon*}, && wk - L^2\left(0,T;H^1\left(S_\varepsilon\right)^*\right)\end{aligned}$$

We also have

$$\bar{v}_{\varepsilon,k} \rightharpoonup \bar{v}^{\varepsilon}, \qquad wk - L^2\left(0,T;H^1\left(F_\varepsilon\right)\right)$$
$$\bar{w}_{\varepsilon,k} \rightharpoonup \bar{w}^{\varepsilon}, \qquad wk - L^2\left(0,T;H^1\left(S_\varepsilon\right)\right).$$

Thus, we have that $\partial_t \bar{v}_\varepsilon = v^{\varepsilon *}$ and $\partial_t \bar{w}_\varepsilon = w^{\varepsilon *}$. Moreover, by Lemma 23, we have that $v_\varepsilon = \bar{v}_\varepsilon$ and $w_\varepsilon = \bar{w}_\varepsilon$. Therefore, $\partial_t v_\varepsilon = \partial_t \bar{v}_\varepsilon$ and $\partial_t w_\varepsilon = \partial_t \bar{w}_\varepsilon$.

Now, let $(\varphi,\psi) \in L^2(0,T;H^1(F_\varepsilon)) \times L^2(0,T;H^1(S_\varepsilon))$. Then, we have

$$\begin{aligned}
&\int_0^T \int_{F_\varepsilon} (\partial_t \bar{v}_{\varepsilon,k})\varphi + \int_0^T \int_{S_\varepsilon} (\partial_t \bar{w}_{\varepsilon,k})\psi + \int_0^T \int_{F_\varepsilon} A_F^{\varepsilon,k} \nabla v_{\varepsilon,k} \cdot \nabla\varphi + \int_0^T \int_{S_\varepsilon} A_S^{\varepsilon,k} \nabla w_{\varepsilon,k} \cdot \nabla\psi \\
&+ \int_0^T \int_{F_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{1,k} \cdot \nabla v_{\varepsilon,k}\right)\varphi - \int_0^T \int_{S_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{2,k} \cdot \nabla w_{\varepsilon,k}\right)\psi \\
&- \int_0^T \int_{S_\varepsilon} r w_{\varepsilon,k}\psi - \int_0^T \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_{\varepsilon,k} - w_{\varepsilon,k}\right)(\varphi - \psi) \\
&= 0.
\end{aligned}$$

Since A_F^ε, A_S^ε, $\boldsymbol{U}_\varepsilon^1$, and $\boldsymbol{U}_\varepsilon^2$ are smooth, by the dominated convergence theorem, we have

$$\int_0^T \int_{F_\varepsilon} \left|A_F^\varepsilon - A_F^{\varepsilon,k}\right|^2 + \left|\boldsymbol{U}_\varepsilon^1 - \boldsymbol{U}_\varepsilon^{1,k}\right|^2 + \int_0^T \int_{S_\varepsilon} \left|A_S^\varepsilon - A_S^{\varepsilon,k}\right|^2 + \left|\boldsymbol{U}_\varepsilon^2 - \boldsymbol{U}_\varepsilon^{2,k}\right|^2 \to 0, \text{ as } k \to 0.$$

Also, by weak convergence in $L^2(0,T;H^1(F_\varepsilon))$ and $L^2(0,T;H^1(S_\varepsilon))$, we have that $v_{\varepsilon,k} - w_{\varepsilon,k} \rightharpoonup v^\varepsilon - w^\varepsilon$ weakly in $L^2((0,T) \times \Gamma_\varepsilon)$. Finally, taking the limit as $k \to 0$, we obtain

$$\begin{aligned}
&\int_0^T \int_{F_\varepsilon} \langle \partial_t v^\varepsilon, \varphi \rangle_{H^1(F_\varepsilon)^*, H^1(F_\varepsilon)} + \int_0^T \int_{S_\varepsilon} \langle \partial_t w^\varepsilon, \psi \rangle_{H^1(S_\varepsilon)^*, H^1(S_\varepsilon)} \\
&+ \int_0^T \int_{F_\varepsilon} A_F^\varepsilon \nabla v^\varepsilon \cdot \nabla \varphi + \int_0^T \int_{S_\varepsilon} A_S^\varepsilon \nabla w^\varepsilon \cdot \nabla \psi \\
&+ \int_0^T \int_{F_\varepsilon} \left(\boldsymbol{U}_\varepsilon^1 \cdot \nabla v^\varepsilon \right) \varphi - \quad \int_0^T \int_{S_\varepsilon} \left(\boldsymbol{U}_\varepsilon^2 \cdot \nabla w^\varepsilon \right) \psi \\
&- \int_0^T \int_{S_\varepsilon} r w^\varepsilon \psi - \int_0^T \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v^\varepsilon - w^\varepsilon \right) (\varphi - \psi) \\
&= 0,
\end{aligned}$$

for all $(\varphi, \psi) \in L^2\left(0, T; H^1\left(F_\varepsilon\right)\right) \times L^2\left(0, T; H^1\left(S_\varepsilon\right)\right)$. □

We now show that if the initial data has higher regularity, then we have that the time derivatives of the weak solutions are in L^2.

Theorem 24. *Suppose that* $(v_{\varepsilon,0}, w_{\varepsilon,0} \in H^1(F_\varepsilon) \times H^1(S_\varepsilon)$. *Then* $\partial_t v^\varepsilon \in L^2\left(0, T; H^1(F_\varepsilon)\right)$ *and* $\partial_t w \varepsilon \in L^2\left(0, T; H^1(S_\varepsilon)\right)$.

Proof. We choose $\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k}$ and $\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}$ be test functions in the weak formulations for the PDE that v_ε^m and w_ε^m satisfy. Thus, we obtain,

$$\begin{aligned}
&\int_{F_\varepsilon} \left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k} \right)^2 + \int_{S_\varepsilon} \left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} \right)^2 + \int_{F_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{1,m} \cdot \nabla v_\varepsilon^m \right) \left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k} \right) \\
&- \int_{S_\varepsilon} \left(\boldsymbol{U}_\varepsilon^{2,m} \cdot \nabla w_\varepsilon^m \right) \left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} \right) + \int_{F_\varepsilon} A_F^{\varepsilon,m} \nabla v_\varepsilon^m \cdot \left(\frac{\nabla v_\varepsilon^m - \nabla v_\varepsilon^{m-1}}{k} \right) \\
&+ \int_{S_\varepsilon} A_S^{\varepsilon,m} \nabla w_\varepsilon^m \cdot \left(\frac{\nabla w_\varepsilon^m - \nabla w_\varepsilon^{m-1}}{k} \right) + \int_{S_\varepsilon} r w_\varepsilon^m \frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} \\
&+ \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_\varepsilon^m - w_\varepsilon^m \right) \left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k} - \frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} \right) = 0.
\end{aligned}$$

Firstly, by Hölder's inequality and ((2.21), we have that

$$\int_{F_\varepsilon}\left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k}\right)^2 + \int_{F_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{1,m}\cdot\nabla v_\varepsilon^m\right)\left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k}\right)$$
$$\geq \frac{1}{2}\int_{F_\varepsilon}\left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k}\right)^2 - C\int_{F_\varepsilon}|\nabla v_\varepsilon^m|^2.$$

Similarly,

$$\int_{S_\varepsilon}\left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}\right)^2 - \int_{S_\varepsilon}\left(\boldsymbol{U}_\varepsilon^{2,m}\cdot\nabla w_\varepsilon^m\right)\left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}\right) + \int_{S_\varepsilon} r w_\varepsilon^m \frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}$$
$$\geq \frac{1}{2}\int_{S_\varepsilon}\left(\frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k}\right)^2 - C\|w_\varepsilon\|_{H^1(S_\varepsilon)}^2.$$

As for the diffusion terms, observe that we have

$$\begin{aligned}
A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\left(\nabla v_\varepsilon^m - v_\varepsilon^{m-1}\right) &= \frac{1}{2}\Big(A_F^{\varepsilon,m}\left(\nabla v_\varepsilon^m - v_\varepsilon^{m-1}\right)\cdot\left(\nabla v_\varepsilon^m - v_\varepsilon^{m-1}\right)\\
&\quad + A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\nabla v_\varepsilon^m - A_F^{\varepsilon,m}\nabla v_\varepsilon^{m-1}\cdot\nabla v_\varepsilon^{m-1}\Big)\\
&\geq \frac{1}{2}\left(A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\nabla v_\varepsilon^m - A_F^{\varepsilon,m}\nabla v_\varepsilon^{m-1}\cdot\nabla v_\varepsilon^{m-1}\right),
\end{aligned}$$

by Lemma 16. Thus, we have

$$\begin{aligned}
&\int_{F_\varepsilon} A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\left(\frac{\nabla v_\varepsilon^m - \nabla v_\varepsilon^{m-1}}{k}\right)\\
&\geq \frac{1}{2k}\int_{F_\varepsilon} A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\nabla v_\varepsilon^m - A_F^{\varepsilon,m}\nabla v_\varepsilon^{m-1}\cdot\nabla v_\varepsilon^{m-1}\\
&= \frac{1}{2k}\left(\int_{F_\varepsilon} A_F^{\varepsilon,m}\nabla v_\varepsilon^m\cdot\nabla v_\varepsilon^m - \int_{F_\varepsilon} A_F^{\varepsilon,m-1}\nabla v_\varepsilon^{m-1}\cdot\nabla v_\varepsilon^{m-1}\right)\\
&\quad + \frac{1}{2}\int_{F_\varepsilon}\left(\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right)\nabla v_\varepsilon^m\cdot\nabla v_\varepsilon^m.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{S_\varepsilon} A_S^{\varepsilon,m} \nabla w_\varepsilon^m \cdot \left(\frac{\nabla w_\varepsilon^m - \nabla w_\varepsilon^{m-1}}{k} \right) \\
&\geq \frac{1}{2k} \left(\int_{S_\varepsilon} A_S^{\varepsilon,m} \nabla w_\varepsilon^m \cdot \nabla w_\varepsilon^m - \int_{F_\varepsilon} A_S^{\varepsilon,m-1} \nabla w_\varepsilon^{m-1} \cdot \nabla w_\varepsilon^{m-1} \right) \\
&\quad + \frac{1}{2} \int_{S_\varepsilon} \left(\frac{A_S^{\varepsilon,m} - A_S^{\varepsilon,m-1}}{k} \right) \nabla w_\varepsilon^m \cdot \nabla w_\varepsilon^m.
\end{aligned}$$

As for the surface term, we have

$$\begin{aligned}
&\int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_\varepsilon^m - w_\varepsilon^m \right) \left(\frac{v_\varepsilon^m - v_\varepsilon^{m-1}}{k} - \frac{w_\varepsilon^m - w_\varepsilon^{m-1}}{k} \right) \\
&= \frac{1}{2k} \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left[\left[\left(v_\varepsilon^m - w_\varepsilon^m \right) - \left(v_\varepsilon^{m-1} - w_\varepsilon^{m-1} \right) \right]^2 + \left(v_\varepsilon^m - w_\varepsilon^m \right)^2 - \left(v_\varepsilon^m - w_\varepsilon^m \right)^2 \right] \\
&\geq \frac{1}{2k} \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(\left(v_\varepsilon^m - w_\varepsilon^m \right)^2 - \left(v_\varepsilon^m - w_\varepsilon^m \right)^2 \right).
\end{aligned}$$

We now integrate in time and sum over $m = 1, ..., N$ and use the above estimates to obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \int_{F_\varepsilon} (\partial_t \tilde{v}_{\varepsilon,k})^2 + \frac{1}{2}\int_0^T \int_{S_\varepsilon} (\partial_t \tilde{w}_{\varepsilon,k})^2 \\
&+ \frac{1}{2}\int_{F_\varepsilon} A_F^{\varepsilon,N} \nabla v_\varepsilon^N \cdot \nabla v_\varepsilon^N - \frac{1}{2}\int_{F_\varepsilon} A_F^{\varepsilon,0} \nabla v_{\varepsilon,0} \cdot \nabla v_{\varepsilon,0} \\
&+ \frac{1}{2}\int_{S_\varepsilon} A_S^{\varepsilon,N} \nabla w_\varepsilon^N \cdot \nabla w_\varepsilon^N - \frac{1}{2}\int_{S_\varepsilon} A_S^{\varepsilon,0} \nabla w_{\varepsilon,0} \cdot \nabla w_{\varepsilon,0} \\
&+ \frac{1}{2}\int_0^T \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_{\varepsilon,k} - w_{\varepsilon,k}\right)^2 - \frac{1}{2}\int_0^T \int_{\Gamma_\varepsilon \alpha_\varepsilon} \left(v_{\varepsilon,0} - w_{\varepsilon,0}\right)^2 \\
&\leq \frac{1}{2}\sum_{m=1}^N k \int_{F_\varepsilon} \left(\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right) \nabla v_\varepsilon^{m-1} \cdot \nabla v_\varepsilon^{m-1} \\
&+ \frac{1}{2}\sum_{m=1}^N k \int_{S_\varepsilon} \left(\frac{A_S^{\varepsilon,m} - A_S^{\varepsilon,m-1}}{k}\right) \nabla w_\varepsilon^{m-1} \cdot \nabla w_\varepsilon^{m-1} \\
&\quad + C\left(\|v_{\varepsilon,k}\|^2_{L^2(0,T;H^1(F_\varepsilon))} + \|w_{\varepsilon,k}\|^2_{L^2(0,T;H^1(F_\varepsilon))}\right) \\
&\leq \frac{1}{2}C\sum_{m=1}^N k \int_{F_\varepsilon} \left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 + \frac{1}{2}C\sum_{m=1}^N k \int_{S_\varepsilon} \left|\frac{A_S^{\varepsilon,m} - A_S^{\varepsilon,m-1}}{k}\right|^2 \\
&\quad + \frac{1}{2}C\sum_{m=1}^N k \int_{F_\varepsilon} \left|\nabla v_\varepsilon^{m-1}\right|^2 + \frac{1}{2}C\sum_{m=1}^N k \int_{S_\varepsilon} \left|\nabla w_\varepsilon^{m-1}\right|^2 \\
&\quad + C\left(\|v_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)}\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sum_{m=1}^N k \int_{F_\varepsilon} \left|\nabla v_\varepsilon^{m-1}\right|^2 &\leq C\left(\|\nabla v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|\nabla v_{\varepsilon,k}\|_{L^2((0,T)\times F_\varepsilon)}\right) \\
&\leq C\left(\|v_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)}\right).
\end{aligned}
$$

Similarly,

$$\sum_{m=1}^{N} k \int_{S_\varepsilon} \left|\nabla w_\varepsilon^{m-1}\right|^2 \leq C\left(\|v_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)}\right).$$

Also, for $x \in F_\varepsilon$, we have

$$\begin{aligned}\int_0^T |\partial_t A_F^\varepsilon(t,x)|^2 \, dt = \int_0^T &\left(|\partial_t A_F^\varepsilon(t,x)|^2 - \sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \mathbb{1}_{[(m-1)k,mk)}(t)\right) dt \\ &+ \int_0^T \sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \mathbb{1}_{[(m-1)k,mk)}(t)\, dt.\end{aligned}$$

Remember that $k = \Delta t$, so that

$$\begin{aligned}\int_0^T \sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \mathbb{1}_{[(m-1)k,mk)}(t)\, dt &= \sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \int_0^T \mathbb{1}_{[(m-1)k,mk)}(t)\, dt \\ &= \sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \Delta t.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|\sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \Delta t - \int_0^T |\partial_t A_F^\varepsilon(t,x)|^2 \, dt\right| \\ &\qquad \leq \int_0^T \left||\partial_t A_F^\varepsilon(t,x)|^2 - \sum_{m=1}^{N}\left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \mathbb{1}_{[(m-1)k,mk)}(t)\right| dt \\ &\qquad \to 0, \quad \text{as } N \to \infty,\end{aligned}$$

uniformly in x since A_F^ε is smooth. Thus, we have that

$$\sum_{m=1}^{N} k \int_{F_\varepsilon} \left|\frac{A_F^{\varepsilon,m} - A_F^{\varepsilon,m-1}}{k}\right|^2 \leq C,$$

for some constant C independent of ε and N. Similarly,

$$\sum_{m=1}^{N} k \int_{S_\varepsilon} \left| \frac{A_S^{\varepsilon,m} - A_S^{\varepsilon,m-1}}{k} \right|^2 \leq C.$$

We also have

$$\int_{F_\varepsilon} A_F^{\varepsilon,0} \nabla v_{\varepsilon,0} \cdot \nabla v_{\varepsilon,0} + \int_{S_\varepsilon} A_S^{\varepsilon,0} \nabla w_{\varepsilon,0} \cdot \nabla w_{\varepsilon,0} \leq C \left(\|v_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} \right),$$

and

$$\int_{F_\varepsilon} A_F^{\varepsilon,N} \nabla v_\varepsilon^N \cdot \nabla v_\varepsilon^N + \int_{S_\varepsilon} A_S^{\varepsilon,N} \nabla w_\varepsilon^N \cdot \nabla w_\varepsilon^N \geq 0.$$

Finally, the surface term can be estimated as

$$\int_{\Gamma_\varepsilon} \alpha_\varepsilon \left(v_{\varepsilon,0} - w_{\varepsilon,0}\right)^2 \leq C \left(\|v_{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w_{\varepsilon,0}\|_{L^2(S_\varepsilon)} + \varepsilon \left(\|\nabla v_{\varepsilon,0}\|_{L^2(F_\varepsilon)} + \|\nabla w_{\varepsilon,0}\|_{L^2(S_\varepsilon)} \right) \right),$$

for some constant C that is independent of ε. Combining these estimates, we obtain

$$\int_0^T \int_{F_\varepsilon} (\partial_t \tilde{v}_{\varepsilon,k})^2 + \int_0^T \int_{S_\varepsilon} (\partial_t \tilde{w}_{\varepsilon,k})^2 \leq C \left(\|v_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} \right) \leq C,$$

where the rightmost upper bound is independent of ε and k. □

Note that the same estimates hold for the solutions themselves. Indeed, we have

Corollary 25. *The following estimates hold:*

$$\|v^\varepsilon\|^2_{L^\infty(0,T;L^2(F_\varepsilon))} + \|w^\varepsilon\|^2_{L^\infty(0,T;L^2(S_\varepsilon))} + \|v^\varepsilon\|^2_{L^2(0,T;H^1(F_\varepsilon))} + \|w^\varepsilon\|^2_{L^2(0,T;H^1(S_\varepsilon))} \\ \le C\left(\|v^{\varepsilon,0}\|^2_{L^2(F_\varepsilon)} + \|w^{\varepsilon,0}\|^2_{L^2(S_\varepsilon)}\right),$$

$$\|\partial_t \bar{v}_\varepsilon\|^2_{L^2((0,T)\times F_\varepsilon)} + \|\partial_t \bar{w}_\varepsilon\|^2_{L^2((0,T)\times S_\varepsilon)} \le C\left(\|v_{\varepsilon,0}\|^2_{H^1(F_\varepsilon)} + \|w_{\varepsilon,0}\|^2_{H^1(S_\varepsilon)}\right),$$

for some constant C that is independent of ε.

2.4 Homogenization

For a function $f : U \to \mathbb{R}^d$, $d \ge 1$, where U is a subdomain of Ω, we denote by $\bar{f}$ its zero extension, i.e., $\bar{f} \equiv f$ in U and $\bar{f} \equiv 0$ in $\Omega \setminus U$. Using the estimates from the previous section, we prove the following convergences:

Theorem 26. *Let $\overline{v^\varepsilon}$ be the zero extension of v^ε. Then, there exist $v^0 \in L^2(0,T;H^1(\Omega))$ and $v^1 \in L^2((0,T)\times\Omega;H^1_{per}(Y)/\mathbb{R})$ such that, up to a subsequence, the following holds*

$$\overline{v^\varepsilon} \to v^0 \mathbb{1}_{Y_F} \quad \text{in the two-scale sense} \tag{2.49}$$

$$\overline{\nabla v^\varepsilon} \to \left(\nabla_x v^0 + \nabla_y v^1\right)\mathbb{1}_{Y_F} \quad \text{in the two-scale sense} \tag{2.50}$$

$$v^\varepsilon|_{\Gamma^\varepsilon} \to v^0 \quad \text{strongly in the two-scale sense on } \Gamma_\varepsilon \tag{2.51}$$

$$\overline{\partial_t v^\varepsilon} \rightharpoonup |Y_F|\partial_t v^0 \quad \text{weakly in } L^2((0,T)\times\Omega) \tag{2.52}$$

Theorem 27. *Let $\overline{w^\varepsilon}$ be the zero extension of w^ε. Then, there exists $w^0 \in$*

$L^2\left((0,T)\times\Omega\right)$ *such that, up to a subsequence, the following holds*

$$\overline{w^\varepsilon} \to \chi_{Y_S} w^0 \qquad \textit{strongly in the two-scale sense} \tag{2.53}$$

$$\overline{\nabla w^\varepsilon} \to \mathbf{0} \qquad \textit{in the two-scale sense} \tag{2.54}$$

$$w^\varepsilon|_{\Gamma^\varepsilon} \to w^0 \qquad \textit{strongly in the two-scale sense on } \Gamma_\varepsilon \tag{2.55}$$

$$\overline{\partial_t w^\varepsilon} \rightharpoonup |Y_S|\partial_t w^0 \qquad \textit{weakly in } L^2\left((0,T)\times\Omega\right) \tag{2.56}$$

Proof. The proofs of the convergences (2.49) and (2.50) follow exactly as in [3]. Note that it is necessary that Y_S be compactly contained in Y for the argument in [3] to work.

To prove the (2.53), we proceed similarly as in [3]. By estimates on w^ε, we have that $\overline{w^\varepsilon}$ and $\overline{\nabla w^\varepsilon}$ are bounded in $L^2(\Omega)$ and $L^2(\Omega)^N$, respectively. Then, up to a subsequence, we have that

$$\overline{w^\varepsilon} \to w^0 \quad \text{strongly in the two-scale sense}$$

$$\overline{\nabla w^\varepsilon} \to \boldsymbol{\xi}^0 \quad \text{in the two-scale sense}$$

for some $w^0 \in L^2((0,T)\times\Omega\times Y)$ and $\boldsymbol{\xi}^0 \in L^2((0,T)\times\Omega\times Y)^N$. As $\overline{w^\varepsilon}$ and $\overline{\nabla w^\varepsilon}$ are zero in $\Omega\setminus S^\varepsilon$, we have that $w^0(t,x,y)$ and $\boldsymbol{\xi}^0(t,x,y)$ are zero if $y\in Y\setminus Y_S$.

We show that w^0 is independent of $y\in Y_S$. Let $\boldsymbol{\psi}\in L^2(0,T;C_0^\infty(\Omega;C_{per}(Y)))^N$

such that it is equal to zero if $y \in Y \setminus Y_S$. Then,

$$\begin{aligned}
&\int_0^T \int_{S^\varepsilon} \nabla w^\varepsilon(t,x) \cdot \boldsymbol{\psi}\left(t,x,\frac{x}{\varepsilon}\right) \, dx\, dt \\
&= \int_0^T \int_{S^\varepsilon} \operatorname{div}\left(w^\varepsilon(t,x)\boldsymbol{\psi}\left(t,x,\frac{x}{\varepsilon}\right)\right) - w^\varepsilon(t,x)\operatorname{div}_x\left(\boldsymbol{\psi}\left(t,x,\frac{x}{\varepsilon}\right)\right) \, dx\, dt \\
&= -\int_0^T \int_{S^\varepsilon} w^\varepsilon(t,x)\left[\operatorname{div}_x\boldsymbol{\psi}\left(t,x,\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\operatorname{div}_y\boldsymbol{\psi}\left(t,x,\frac{x}{\varepsilon}\right)\right] \, dx\, dt.
\end{aligned}$$

Thus, multiplying both sided by ε and letting $\varepsilon \to 0$, we obtain that

$$\int_0^T \int_\Omega \int_{Y_S} w^0(t,x,y)\operatorname{div}_y\boldsymbol{\psi}\,(t,x,y) \; dy\, dx\, dt = 0,$$

i.e., w^0 is independent of $y \in Y_S$. Hence,

$$w^0(t,x,y) = w^0(t,x)\chi_{Y_S}(y). \tag{2.57}$$

We now show that $\boldsymbol{\xi}^0 = \mathbf{0}$. We proceed as in [29]. Indeed, let $\psi \in C_0^\infty\left((0,T)\times\Omega\right)$. Let $\boldsymbol{\psi} \in C_0^\infty(Y_S)^N$ such that div $\boldsymbol{\psi} = 0$. We extend $\boldsymbol{\psi}$ by zero to Y and $Y-$periodically to $\mathbb{R}^N$. Then,

$$\begin{aligned}
&\int_0^T \int_{S^\varepsilon} \nabla w^\varepsilon(t,x) \cdot \left(\psi(t,x)\boldsymbol{\psi}\left(\frac{x}{\varepsilon}\right)\right) \, dx\, dt \\
&= -\int_0^T \int_{S^\varepsilon} w^\varepsilon(t,x)\nabla\psi(t,x)\cdot\boldsymbol{\psi}\left(\frac{x}{\varepsilon}\right) \, dx\, dt \\
&\xrightarrow{\varepsilon\to 0} -\int_0^T \int_\Omega \int_{Y_S} w^0(t,x)\nabla\psi(t,x)\cdot\boldsymbol{\psi}(y)\, dy\, dx\, dt \\
&= -\int_0^T \int_\Omega w^0(t,x)\nabla\psi(t,x)\cdot\left(\int_{Y_S}\boldsymbol{\psi}(y)\, dy\right) dx\, dt \\
&= 0,
\end{aligned}$$

since $\boldsymbol{\psi}$ has compact support in Y_S and is divergence-free. As,

$$\begin{aligned}
&\int_0^T \int_{S^\varepsilon} \nabla w^\varepsilon(t,x) \cdot \left(\psi(t,x) \boldsymbol{\psi}\left(\frac{x}{\varepsilon}\right)\right) dx\, dt \\
&= \int_0^T \int_\Omega \overline{\nabla w^\varepsilon}(t,x) \cdot \left(\psi(t,x) \boldsymbol{\psi}\left(\frac{x}{\varepsilon}\right)\right) dx\, dt \\
&\xrightarrow{\varepsilon\to 0} \int_0^T \int_\Omega \int_{Y_S} \boldsymbol{\xi}^0(t,x,y) \cdot \psi(t,x) \boldsymbol{\psi}(y)\, dy\, dx\, dt,
\end{aligned}$$

we find that

$$\begin{aligned}
&\int_0^T \int_\Omega \psi(t,x) \left(\int_{Y_S} \boldsymbol{\xi}^0(t,x,y) \cdot \boldsymbol{\psi}(y)\, dy \right) dx\, dt \\
&= \int_0^T \int_\Omega \int_{Y_S} \boldsymbol{\xi}^0(t,x,y) \cdot \psi(t,x) \boldsymbol{\psi}(y)\, dy\, dx\, dt \\
&= 0,
\end{aligned}$$

for all $\psi \in C_0^\infty\left((0,T)\times\Omega\right)$. Thus, for *a.e.* (t,x) in $(0,T)\times\Omega$,

$$\int_{Y_S} \boldsymbol{\xi}^0(t,x,y) \cdot \boldsymbol{\psi}(y)\, dy = 0,$$

for all $\boldsymbol{\psi} \in C_0^\infty(Y_S)^N$ such that div $\boldsymbol{\psi} = 0$. Thus, there is a unique $p \in L^2\left((0,T)\times\Omega; H^1(Y_S)\right)$ such that

$$\boldsymbol{\xi}^0(t,x,y) = \nabla_y p(t,x,y), \quad (t,x,y) \in (0,T)\times\Omega\times Y_S.$$

Now, let $\psi \in C_0^\infty(0,T)$ such that $\psi \geq 0$. Let $\psi^1 \in C_0^\infty\left(\Omega; C^\infty(\overline{Y}_S)\right)$ and extend $\boldsymbol{\psi}$ by zero to Y and $Y-$periodically to $\mathbb{R}^N$. Testing the weak form of the solid problem with $\varepsilon\psi(t)\psi^1\left(x, \dfrac{x}{\varepsilon}\right)$, we have

$$\begin{aligned}
&\int_0^T \int_{S^\varepsilon} A_S^\varepsilon(t,x)\nabla w^\varepsilon(t,x)\cdot\psi(t)\nabla_y\psi^1\left(x,\frac{x}{\varepsilon}\right)\,dx\,dt\\
&= -\varepsilon\Bigg[\int_0^T \int_{S^\varepsilon} A_S^\varepsilon(t,x)\nabla w^\varepsilon(t,x)\cdot\psi(t)\nabla_x\psi^1\left(x,\frac{x}{\varepsilon}\right)\,dx\,dt\\
&\qquad + \int_0^T \left\langle \partial_t w^\varepsilon, \psi^1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\rangle_{H^1(S^\varepsilon)^*, H^1(S^\varepsilon)}\,dt\\
&\qquad + \int_0^T \int_{S^\varepsilon} \left(\partial_t\boldsymbol{\phi}^\varepsilon(t,x)\cdot\nabla w^\varepsilon(t,x)\right)\psi(t)\psi^1\left(x,\frac{x}{\varepsilon}\right)\,dx\,dt\\
&\qquad + \varepsilon\int_0^T \int_{\Gamma^\varepsilon} \left(\alpha v^\varepsilon - \beta w^\varepsilon\right)\psi(t)\psi^1\left(x,\frac{x}{\varepsilon}\right)\,dS_x\,dt\\
&\qquad + \int_0^T \int_{S^\varepsilon} r w^\varepsilon(t,x)\psi(t)\psi^1\left(x,\frac{x}{\varepsilon}\right)\,dx\,dt\Bigg].
\end{aligned}$$

Since the terms inside the brackets are bounded uniformly in ε, the right hand side of the above equation goes to zero as ε goes to zero.

As for the left-hand side, due to Assumption 15, we have

$$\int_0^T \int_\Omega \left| A_S^\varepsilon(t,x) - A_S^0\left(t,x,\frac{x}{\varepsilon}\right)\right|^2 dx\,dt \to 0, \quad \text{as } \varepsilon\to 0,$$

we have

$$\begin{aligned}
&\int_0^T \int_{S^\varepsilon} A_S^\varepsilon(t,x)\nabla w^\varepsilon(t,x)\cdot\psi(t)\nabla_y\psi^1\left(x,\frac{x}{\varepsilon}\right)\,dx\,dt\\
&\xrightarrow{\varepsilon\to 0} \int_0^T \int_\Omega\int_{Y_S} A_S^0(t,x,y)\nabla_y p(t,x,y)\cdot\nabla_y\psi^1(t,x,y)\psi(t)\,dy\,dx\,dt.
\end{aligned}$$

Thus,

$$\int_0^T \int_\Omega\int_{Y_S} A_S^0(t,x,y)\nabla_y p(t,x,y)\cdot\nabla_y\psi^1(t,x,y)\psi(t)\,dy\,dx\,dt = 0.$$

Since Y_S is compactly contained in Y, we can take $\psi^1 \equiv p$. Thus, by ellipticity of A_S^0,

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \int_{Y_S} A_S^0(t,x,y)\nabla_y p(t,x,y) \cdot \nabla_y p(t,x,y)\psi(t)\, dy\, dx\, dt \\ &\geq \alpha_S^0 \int_0^T \int_\Omega \int_{Y_S} \psi(t) \left|\nabla_y p(t,x,y)\psi(t)\right|^2 \, dy\, dx\, dt, \end{aligned}$$

where $\alpha_S^0 > 0$. Then, if we take a sequence $\{\psi_n\}_n$ in $C_0^\infty(0,T)$ that converges to 1 in $L^2(0,T)$, we find that $\boldsymbol{\xi}^0 \equiv \nabla_y p \equiv \mathbf{0}$.

We prove the weak convergence of $\overline{\{\partial_t v^\epsilon\}}_{\varepsilon>0}$. Indeed, let $\varphi \in C_0^\infty(\Omega)$ and ζ be in $C_0^\infty(0,T)$. Since $\partial_t v^\epsilon$ is in $L^2\left((0,T)\times F^\epsilon(0)\right)$, by the the two-scale convergence of $\overline{v^\varepsilon}$ we have,

$$\begin{aligned} &\int_0^T \int_\Omega \overline{\partial_t v^\varepsilon}(t,x)\varphi(x)\zeta(t)\, dx\, dt \\ &= \int_0^T \int_{F^\varepsilon(0)} \partial_t v^\varepsilon(t,x)\varphi(x)\zeta(t)\, dx\, dt = -\int_0^T \int_{F^\varepsilon(0)} v^\varepsilon(t,x)\varphi(x)\zeta'(t)\, dx\, dt \\ &= -\int_0^T \int_\Omega \overline{v^\varepsilon}(t,x)\varphi(x)\zeta'(t)(x)\, dx\, dt \xrightarrow{\varepsilon\to 0} -|Y_F| \int_0^T \int_\Omega v^0(t,x)\varphi(x)\zeta'(t)\, dx\, dt \\ &= \int_0^T \langle |Y_F|\partial_t v^0, \varphi\rangle_{H^1(\Omega)^*, H^1(\Omega)}\zeta(t)\, dt, \end{aligned}$$

Since $\{\overline{\partial_t v^\varepsilon}\}_{\varepsilon>0}$ is bounded in $L^2\left((0,T)\times\Omega\right)$, we have that, up to a subsequence, it weakly converges to some g in $L^2\left((0,T)\times\Omega\right)$. Thus,

$$\int_0^T \int_\Omega \overline{\partial_t v^\varepsilon}(t,x)\varphi(x)\zeta(t)\, dx\, dt \xrightarrow{\varepsilon\to 0} \int_0^T \int_\Omega g(t,x)\varphi(x)\zeta(t)\, dx\, dt$$

Combining these convergences we have

$$\int_0^T \int_\Omega g(t,x)\varphi(x)\zeta(t)\,dx\,dt = \int_0^T \langle |Y_F|\partial_t v^0, \varphi\rangle_{H^1(\Omega)^*,H^1(\Omega)}\zeta(t)\,dt.$$

Since $C_0^\infty(\Omega) \times C_0^\infty(0,T)$ is dense in $L^2\left((0,T)\times\Omega\right)$, we have that $|Y_F|\partial_t v^0$ is in $L^2\left((0,T)\times\Omega\right)$ and (2.52) holds. Using similar arguments, we also have that (2.56) holds.

Finally, from the estimates on the traces of v_ε and w_ε on Γ_ε, (2.51) and (2.55) follow from Proposition 38.

□

2.4.1 Limit problem

We now show that the two-scale limits of the solutions of the microscopic problems solve the following two-scale homogenized problems.

Theorem 28. *The limits v^0, v^1, and w^0 in Theorems 26 and 27 are the the unique weak solutions of*

$$div_y\left(A_F^0(t,x,y)\left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y)\right)\right) = 0, \quad in\ (0,T)\times\Omega\times Y \tag{2.58}$$

$$|Y_F|\,\partial_t v^0 - div_x\left(\int_{Y_F} A_F^0(t,x,y)\big(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y)\big)\,dy\right) \tag{2.59}$$

$$= |\Gamma|\,\alpha\left(v^0(t,x) - w^0(t,x)\right) \quad in\ (0,T)\times\Omega$$

$$\partial_t w^0 + rw^0(t,x) = \frac{|\Gamma|}{|Y_S|}\alpha\left(w^0(t,x) - v^0(t,x)\right) \quad in\ (0,T)\times\Omega \tag{2.60}$$

Proof. Let $\zeta \in C_0^\infty(0,T)$, $\varphi \in C^\infty(\bar{\Omega})$, and $\varphi^1 \in C_0^\infty(\Omega; C_{per}^\infty(\bar{Y}))$. Then, using

$$\zeta(t)\left(\varphi(x) + \varepsilon\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right)$$

as a test function for the fluid problem, we have

$$\begin{aligned}
&\int_0^T \int_{F_\varepsilon} \partial_t v^\varepsilon(t,x)\zeta(t)\left(\varphi(x) + \varepsilon\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right)\, dx\, dt \\
&+ \int_0^T \int_{F_\varepsilon} \left(\left(\boldsymbol{u}^{\boldsymbol{\varepsilon}}(t,x) - \partial_t \boldsymbol{X}_\varepsilon(t,x)\right) \cdot \nabla v^\varepsilon\right)\zeta(t)\left(\varphi(x) + \varepsilon\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right)\, dx\, dt \\
&+ \int_0^T \int_{F^\varepsilon} \left(A_F^\varepsilon(t,x)\nabla v^\varepsilon(t,x) \cdot \nabla\left(\varphi(x) + \varepsilon\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right)\right)\zeta(t)\, dx\, dt \\
&= \varepsilon \int_0^T \int_{\Gamma^\varepsilon} \left(\alpha v^\varepsilon(t,x) - \beta w^\varepsilon(t,x)\right)\zeta(t)\left(\varphi(x) + \varepsilon\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right)\, dS_x\, dt
\end{aligned}$$

From the estimates on $\partial_t v^\varepsilon$ and the fact that φ^1 is smooth, we have

$$\begin{aligned}
\left|\int_0^T \int_\Omega \overline{\partial_t v^\varepsilon}(t,x)\zeta(t)\varphi^1\left(t, x, \frac{x}{\varepsilon}\right)\, dx\, dt\right| &\leq C\|\zeta\|_{L^\infty(0,T)}\left\|\varphi^1\left(\cdot, \cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2((0,T)\times\Omega)} \\
&\leq C\|\zeta\|_{L^\infty(0,T)}\|\varphi^1\|_{L^2((0,T)\times\Omega;C(Y))},
\end{aligned}$$

where the last inequality follows from Proposition 32 with the macroscopic domain being $(0,T)\times\Omega$ instead just Ω. Thus, we have

$$\begin{aligned}
&\int_0^T \int_{F_\varepsilon} \partial_t v^\varepsilon(t,x)\zeta(t)\left(\varphi(x) + \varepsilon\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right)\, dx\, dt \\
&= \int_0^T \int_\Omega \overline{\partial_t v^\varepsilon}(t,x)\zeta(t)\varphi(x) \\
&+ \varepsilon \int_0^T \int_\Omega \overline{\partial_t v^\varepsilon}(t,x)\zeta(t)\varphi^1\left(t, x, \frac{x}{\varepsilon}\right)\, dx\, dt \\
&\xrightarrow{\varepsilon\to 0} \int_0^T \int_\Omega |Y_F|\partial_t v^0(t,x)\zeta(t)\varphi(x)\, dx\, dt.
\end{aligned}$$

Using the estimates on the traces of v^ε and $^\varepsilon$, by Proposition 36, we have that

$$\left| \varepsilon \int_0^T \int_{\Gamma^\varepsilon} \alpha \left(v^\varepsilon(t,x) - w^\varepsilon(t,x) \right) \zeta(t) \left(\varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \, dS_x \, dt \right| \leq C \|\zeta\|_{L^\infty(0,T)} \|\varphi^1\|_{L^\infty(0,T)\times\Omega\times Y}$$

Hence,

$$\begin{aligned}
\varepsilon \int_0^T \int_{\Gamma^\varepsilon} \alpha(v^\varepsilon(t,x) - w^\varepsilon(t,x))\zeta(t) \left(\varphi(x) + \varepsilon \varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \, dS_x \, dt \\
&= \varepsilon \int_0^T \int_{\Gamma^\varepsilon} \alpha \left(v^\varepsilon(t,x) - w^\varepsilon(t,x) \right) \zeta(t) \left(\varphi(x) \right) \, dS_x \, dt \\
&\quad + \varepsilon^2 \int_0^T \int_{\Gamma^\varepsilon} \alpha \left(v^\varepsilon(t,x) - w^\varepsilon(t,x) \right) \zeta(t) \left(\varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \, dS_x \, dt \\
&\xrightarrow{\varepsilon \to 0} |\Gamma| \int_0^T \int_\Omega \alpha \left(v^0(t,x) - w^0(t,x) \right) \varphi(x) \zeta(t) \, dx \, dt.
\end{aligned}$$

We now turn to the diffusion term. From the estimates on $\overline{\nabla v^\varepsilon}$, we have

$$\begin{aligned}
&\left| \int_0^T \int_\Omega \overline{\nabla v^\varepsilon} \cdot \left(A_F^\varepsilon(t,x) - A_F^0 \left(t, x, \frac{x}{\varepsilon} \right) \right) \nabla \left(\varphi(x) + \varepsilon \varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \zeta(t) \chi_{Y_F} \left(\frac{x}{\varepsilon} \right) \, dx \, dt \right| \\
&\leq C \int_0^T \int_\Omega \left| A_F^\varepsilon(t,x) - A_F^0 \left(t, x, \frac{x}{\varepsilon} \right) \right|^2 \, dx \, dt \\
&\xrightarrow{\varepsilon \to 0} 0,
\end{aligned}$$

where the last line is due to Assumption 15. Therefore we have,

$$\int_0^T \int_{F^\varepsilon} \left(A_F^\varepsilon(t,x) \nabla v^\varepsilon(t,x) \cdot \nabla \left(\varphi(x) + \varepsilon \varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \right) \zeta(t)\, dx\, dt$$
$$= \int_0^T \int_\Omega \overline{\nabla v^\varepsilon} \cdot A_F^{0,T} \left(t, x, \frac{x}{\varepsilon} \right) \left(\nabla \varphi(x) + \varepsilon \nabla_x \varphi^1 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y \varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \zeta(t) \chi_{Y_F} \left(\frac{x}{\varepsilon} \right) dx\, dt$$
$$+ \int_0^T \int_\Omega \overline{\nabla v^\varepsilon} \cdot \left(A_F^{\varepsilon,T}(t,x) - A_F^{0,T} \left(t, x, \frac{x}{\varepsilon} \right) \right) \nabla \left(\varphi(x) + \varepsilon \varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) \zeta(t) \chi_{Y_F} \left(\frac{x}{\varepsilon} \right) dx\, dt$$
$$\xrightarrow{\varepsilon \to 0} \int_0^T \int_\Omega \int_Y \left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y) \right) \cdot A_F^{0,T}(t,x,y) \left(\nabla \varphi(x) + \nabla_y \varphi^1(x,y) \right) \zeta(t) \chi_{Y_F}(y)\, dy\, dx\, dt$$
$$= \int_0^T \int_\Omega \int_{Y_F} A_F^{0,T}(t,x,y) \left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y) \right) \cdot \left(\nabla \varphi(x) + \nabla_y \varphi^1(x,y) \right) \zeta(t)\, dy\, dx\, dt.$$

Lastly,

$$\int_0^T \int_{F_\varepsilon} \left(\left(\boldsymbol{u}^{\boldsymbol{\varepsilon}}(t,x) - \partial_t \boldsymbol{X}_\varepsilon(t,x) \right) \cdot \nabla v^\varepsilon \right) \zeta(t) \left(\varphi(x) + \varepsilon \varphi^1 \left(x, \frac{x}{\varepsilon} \right) \right) dx\, dt$$
$$\leq C \| \boldsymbol{u}^{\boldsymbol{\varepsilon}}(t,x) - \partial_t \boldsymbol{X}_\varepsilon \|_{L^\infty((0,T)\times\Omega)}$$
$$\xrightarrow{\varepsilon \to 0} 0.$$

Combining these calculations, we find that as $\varepsilon \to 0$,

$$\int_0^T \int_\Omega |Y_F| \partial_t v^0(t,x) \zeta(t) \varphi(x)\, dx\, dt$$
$$+ \int_0^T \int_\Omega \int_{Y_F} A_F^{0,T}(t,x,y) \left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y) \right) \cdot \left(\nabla \varphi(x) + \nabla_y \varphi^1(x,y) \right) \zeta(t)\, dy\, dx\, dt$$
$$= |\Gamma| \int_0^T \int_\Omega \alpha \left(v^0(t,x) - w^0(t,x) \right) \varphi(x) \zeta(t)\, dx\, dt,$$

i.e., v^0 and v^1 are weak solutions to the two-scale homogenized problem (2.58) and (2.59).

Similarly, we let $\zeta \in C_0^\infty(0,T)$ and $\psi \in C^\infty(\bar{\Omega})$. Then, using $\psi(t)\zeta(x)$ as a test function for the solid problem, we obtain:

$$\int_0^T \int_{S_\varepsilon} \partial_t w^\varepsilon(t,x)\zeta(t)\psi(x)\,dx\,dt + \int_0^T \int_{S^\varepsilon} A_S^\varepsilon(t,x)\nabla w^\varepsilon(t,x)\cdot\zeta(t)\nabla\psi(x)\,dx\,dt$$
$$+\int_0^T \int_{S^\varepsilon} \left(\partial_t \boldsymbol{X}_\varepsilon(t,x)\cdot\nabla w^\varepsilon(t,x)\right)\zeta(t)\psi(x)\,dx\,dt + \int_0^T \int_{S^\varepsilon} r w^\varepsilon(t,x)\zeta(t)\psi(x)\,dx\,dt$$
$$=\varepsilon\int_0^T \int_{\Gamma^\varepsilon} \alpha\left(w^\varepsilon(t,x)-v^\varepsilon(t,x)\right)\zeta(t)\psi(x)\,dS_x\,dt.$$

Using the convergences in Theorem 27 and the Assumption 15, we find that in the limit as $\varepsilon \to 0$, w^0 satisfies

$$\int_0^T \int_\Omega |Y_S|\partial_t w^0(t,x)\zeta(t)\psi(x)\,dx\,dt + |Y_S|\int_0^T \int_\Omega r w^0(t,x)\zeta(t)\psi(x)\,dy\,dx\,dt$$
$$= |\Gamma|\int_0^T \int_\Omega \alpha\left(w^0(t,x)-v^0(t,x)\right)\zeta(t)\psi(x)\,dx\,dt,$$

i.e., w^0 solves the ordinary differential equation (2.60). □

The well-posedness of the limit problem can be proven using standard methods. Thus, the convergences proven in this section hold for the whole sequences.

2.4.2 Corrector results and strong convergence

Observe that since $\overline{v}^\varepsilon(t)$ and $\overline{w}^\varepsilon(t)$ are not in $H^1(\Omega)$ for almost all t in $(0,T)$, we cannot conclude that they are strongly convergent in $L^2(\Omega)$ readily from Sobolev embedding theorems. In [29], the authors used instead an H^1- extension of v_ε that can be uniformly controlled in $H^1(\Omega)$. This is possible because of the regularity ∂S_ε. Since we are not using any H^1-extension, we will use the fact that v_ε and w_ε are solutions to PDEs to obtain strong convergence. In fact, we say something more

Theorem 29.

$$\|v^\varepsilon - v^0\|^2_{L^2((0,T)\times F_\varepsilon)} + \|w^\varepsilon - w^0\|^2_{L^2((0,T)\times S_\varepsilon)} \tag{2.61}$$
$$+ \int_0^T \int_{F_\varepsilon} \left|\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right|^2 \xrightarrow[\varepsilon\to 0]{} 0.$$

Proof. We have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v^\varepsilon(t) - v^0(t)\|^2_{L^2(F_\varepsilon)} + \|w^\varepsilon(t) - w^0(t)\|^2_{L^2(S_\varepsilon)}\right)\\
&+ \beta \int_{F_\varepsilon} \left|\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right|^2\\
&\leq \int_{F_\varepsilon} \left[(\partial_t v^\varepsilon)(t,x) - (\partial_t v^0)(t,x)\right]\left[v^\varepsilon(t,x) - v^0(t,x)\right]\\
&\quad + \int_{S_\varepsilon} \left[(\partial_t w^\varepsilon)(t,x) - (\partial_t w^0)(t,x)\right]\left[w^\varepsilon(t,x) - w^0(t,x)\right]\\
&\quad + \int_{F_\varepsilon} A_F^\varepsilon(t,x)\left[\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right]\\
&\qquad\qquad \cdot \left[\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right]\\
&= \int_{F_\varepsilon} (\partial_t v^\varepsilon)(t,x) v^\varepsilon(t,x) + \int_{S_\varepsilon} (\partial_t w^\varepsilon)(t,x) w^\varepsilon(t,x) + \int_{F_\varepsilon} A_F^\varepsilon(t,x)\nabla v^\varepsilon(t,x)\cdot\nabla v^\varepsilon(t,x)\\
&\quad + \int_{F_\varepsilon} \left(\partial_t v^0\right)(t,x) v^0(t,x) - \int_{F_\varepsilon} \left(\partial_t v^0\right)(t,x) v^\varepsilon(t,x) - \int_{F_\varepsilon} \left(\partial_t v^\varepsilon\right)(t,x) v^0(t,x)\\
&\quad + \int_{S_\varepsilon} \left(\partial_t w^0\right)(t,x) w^0(t,x) - \int_{S_\varepsilon} \left(\partial_t w^0\right)(t,x) w^\varepsilon(t,x) - \int_{S_\varepsilon} \left(\partial_t w^\varepsilon\right)(t,x)^0(t,x)\\
&\quad + \int_{F_\varepsilon} A_F^\varepsilon(t,x)\left[\nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right]\cdot\left[\nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right]\\
&\quad - 2\int_{F_\varepsilon} A_F^\varepsilon(t,x)\nabla v^\varepsilon(t,x)\cdot\left[\nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right].
\end{aligned}$$

Now, since v^ε and w^ε are solutions to PDEs, we get that

$$\begin{aligned}
&\int_{F_\varepsilon} (\partial_t v^\varepsilon)(t,x) v^\varepsilon(t,x) + \int_{S_\varepsilon} (\partial_t w^\varepsilon)(t,x) w^\varepsilon(t,x) + \int_{F_\varepsilon} A_F^\varepsilon(t,x) \nabla v^\varepsilon(t,x) \cdot \nabla v^\varepsilon(t,x) \\
&= -\int_{S_\varepsilon} A_S^\varepsilon(t,x) \nabla w^\varepsilon(t,x) \cdot \nabla w^\varepsilon(t,x) - \int_{S_\varepsilon} r\,(w^\varepsilon)^2 \\
&\quad - \int_{F_\varepsilon} \left((\boldsymbol{u}^\varepsilon(t,x) - \partial_t \boldsymbol{X}_{\boldsymbol{\varepsilon}}(t,x) \cdot \nabla v_\varepsilon(t,x))\, v^\varepsilon(t,x) \right) - \int_{\Gamma_\varepsilon} \alpha_{\varepsilon]} \left((v^\varepsilon - w^\varepsilon \right)^2 \\
&\leq \quad -\int_{F_\varepsilon} \left((\boldsymbol{u}^\varepsilon(t,x) - \partial_t \boldsymbol{X}_{\boldsymbol{\varepsilon}}(t,x) \cdot \nabla v^\varepsilon(t,x))\, v^\varepsilon(t,x) \right) - \int_{\Gamma_\varepsilon} \alpha_\varepsilon \left((v^\varepsilon(t,x) - w^\varepsilon(t,x) \right)^2,
\end{aligned}$$

we have that upon integrating, for almost all $t \in (0,T)$, we get

$$\begin{aligned}
&\frac{1}{2} \Big(\|v^\varepsilon(t) - v^0(t)\|^2_{L^2(F_\varepsilon)} + \|w^\varepsilon(t) - w^0(t)\|^2_{L^2(S_\varepsilon)} \Big) + \beta \int_0^t \int_{F_\varepsilon} \Big| \nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1 \Big(t, x, \frac{x}{\varepsilon} \Big) \Big| \\
&\leq -\Bigg(\int_0^t \int_{F_\varepsilon} (\partial_t v_\varepsilon)(t,x) v^0(t,x) + \int_0^t \int_{S_\varepsilon} (\partial_t w^\varepsilon)(t,x) w^0(t,x) \\
&\quad + \int_0^t \int_{F_\varepsilon} A_F^\varepsilon(t,x) \nabla v_\varepsilon(t,x) \cdot \Big[\nabla v^0(t,x) - \nabla_y v^1 \Big(t, x, \frac{x}{\varepsilon} \Big) \Big] \Bigg) \\
&\quad + \Bigg(\int_0^t \int_{F_\varepsilon} \left(\partial_t v^0 \right) \left(v^0 - v^\varepsilon \right) + \int_0^t \int_{S_\varepsilon} \left(\partial_t w^0 \right) \left(w^0 - w^\varepsilon \right) \\
&\quad + \int_0^t \int_{F_\varepsilon} A_F^\varepsilon(t,x) \Big(\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1 \Big(t, x, \frac{x}{\varepsilon} \Big) \Big) \cdot \Big(\nabla v^0(t,x) + \nabla_y v^1 \Big(t, x, \frac{x}{\varepsilon} \Big) \Big) \\
&\quad - \int_0^t \int_{F_\varepsilon} \left(\boldsymbol{u}^\varepsilon - \partial_t \boldsymbol{X}_\varepsilon \cdot \nabla v_\varepsilon \right) v^\varepsilon + \int_{F_\varepsilon} \left(v^0(0,x) - v_{\varepsilon,0}(x) \right)^2 + \int_{S_\varepsilon} \left(w^0(0,x) - w_{\varepsilon,0}(x) \right)^2 \Bigg) \\
&\quad - \int_0^t \int_{\Gamma_\varepsilon} \alpha (v^\varepsilon - w^\varepsilon)^2 \\
&=: I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon).
\end{aligned}$$

Now,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} I_1(\varepsilon)\\
&= -\lim_{\varepsilon\to 0}\left(\int_0^t\int_{F_\varepsilon}\left(\partial_t v^\varepsilon\right)(t,x)v^0(t,x) + \int_0^t\int_{S_\varepsilon}\left(\partial_t w^\varepsilon\right)(t,x)w^0(t,x)\right.\\
&\qquad\left. + \int_0^t\int_{F_\varepsilon} A_F^\varepsilon(t,x)\nabla v^\varepsilon(t,x)\cdot\left[\nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right]\right)\\
&= -\int_0^t\int_\Omega |Y_F|\partial_t v^0(t,x)v^0(t,x) + |Y_S|\partial_t w^0(t,x)w^0(t,x)\\
&\quad - \int_0^t\int_\Omega\int_{Y_F} A_F^0(t,x,y)\left(\nabla v^0(t,x) + \nabla_y v^1(t,x,y)\right)\cdot\left(\nabla v^0(t,x) + \nabla_y v^1(t,x,y)\right)\\
&= |\Gamma|\int_0^t\int_\Omega \alpha\left(v^0(t,x) - w^0(t,x)\right)^2\\
&= \int_0^t\int_\Omega\int_\Gamma \alpha\left(v^0(t,x) - w^0(t,x)\right)^2.
\end{aligned}
$$

Also, by the two-scale convergence of $v^\varepsilon, w^\varepsilon, \nabla v^\varepsilon$ and the strong convergence of $A_F^\varepsilon, \boldsymbol{u}^{\boldsymbol{\varepsilon}}, \partial_t \boldsymbol{X}^\varepsilon, v_{\varepsilon,0}, w_{\varepsilon,0}$, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} I_2(\varepsilon)\\
&= \lim_{\varepsilon\to 0}\left(\int_0^t\int_{F_\varepsilon}\left(\partial_t v^0\right)\left(v^0 - v^\varepsilon\right) + \int_0^t\int_{S_\varepsilon}\left(\partial_t w^0\right)\left(w^0 - w^\varepsilon\right)\right.\\
&\qquad + \int_0^t\int_{F_\varepsilon} A_F^\varepsilon(t,x)\left(\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right)\\
&\qquad\qquad\quad \cdot\left(\nabla v^0(t,x) + \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right)\\
&\qquad\left. - \int_0^t\int_{F_\varepsilon}\left(\boldsymbol{u}^\varepsilon - \partial_t \boldsymbol{X}_\varepsilon\cdot\nabla v^\varepsilon\right)v^\varepsilon + \int_{F_\varepsilon}\left(v^0(0,x) - v_{\varepsilon,0}(x)\right)^2 + \int_{S_\varepsilon}\left(w^0(0,x) - w_{\varepsilon,0}(x)\right)^2\right)\\
&= 0.
\end{aligned}
$$

Lastly,

$$\limsup_{\varepsilon\to 0} I_3(\varepsilon) = \limsup_{\varepsilon\to 0}\left(-\int_0^t\int_{\Gamma_\varepsilon}\alpha(v^\varepsilon - w^\varepsilon)^2\right) = -\liminf_{\varepsilon\to 0}\int_0^t\int_{\Gamma_\varepsilon}\alpha(v^\varepsilon - w^\varepsilon)^2.$$

Combining these limits, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\left(\frac{1}{2}\left(\|v^\varepsilon(t) - v^0(t)\|^2_{L^2(F_\varepsilon)} + \|w^\varepsilon(t) - w^0(t)\|^2_{L^2(S_\varepsilon)}\right)\right.\\ &\left. + \beta\int_0^t\int_{F_\varepsilon}\left|\nabla v^\varepsilon(t,x) - \nabla v^0(t,x) - \nabla_y v^1\left(t,x,\frac{x}{\varepsilon}\right)\right|^2\right)\\ &= \int_0^t\int_\Omega\int_\Gamma \alpha\left(v^0(t,x) - w^0(t,x)\right)^2 - \liminf_{\varepsilon\to 0}\int_0^t\int_{\Gamma_\varepsilon}\alpha\varepsilon(v^\varepsilon - w^\varepsilon)^2\\ &\le 0,\end{aligned}$$

since $v_\varepsilon - w_\varepsilon$ converges in the two-scale sense on the surface Γ_ε to $v^0 - w^0$. Since $T < \infty$, by the dominated convergence theorem, the theorem follows. □

2.5 Examples

We provide two examples of solid velocities for which our result applies. We find it interesting that these examples give different limit problems.

We consider the case when the solid velocities are *slow* in the sense that $\|\boldsymbol{b}_\varepsilon\| \sim \varepsilon^\alpha$ where $\alpha > 1$. We assume that

$$\begin{aligned}\boldsymbol{h}_{\varepsilon,i}(t) &= \varepsilon^\alpha \boldsymbol{h}_i(t)\\ \boldsymbol{M}_{\varepsilon,i}(t) &= \varepsilon^{\alpha-1}\boldsymbol{M_i}(t),\end{aligned}$$

and that $\|\boldsymbol{h}_i\|_\infty, \|\boldsymbol{M}_i\|_\infty \sim C$. This gives that $\|\boldsymbol{b}_\varepsilon\| \sim \varepsilon^\alpha$. We now calculate the gradient of $\boldsymbol{b}_\varepsilon$.

$$\nabla \boldsymbol{b}_\varepsilon(t,y) = \sum_{i=1}^{m_\varepsilon} \nabla \left(\eta_\varepsilon(t,y) \left(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\,(y - \boldsymbol{h}_{\varepsilon,i}(t)) \right) \right) \\ - \sum_{i=1}^{m_\varepsilon} \nabla \left(\boldsymbol{B}_{\varepsilon,K_i(t)} \left(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\,(y - \boldsymbol{h}_{\varepsilon,i}(t)) \right) \right).$$

We have that

$$\begin{aligned} \nabla(\eta_\varepsilon(t,y) h'_{\varepsilon,i}(t)) &= \boldsymbol{h}'_{\varepsilon,i}(t) \otimes \nabla \eta_\varepsilon(t,y) + \eta_\varepsilon(t,y) \nabla \boldsymbol{h}'_{\varepsilon,i}(t) \\ &= \boldsymbol{h}'_{\varepsilon,i}(t) \otimes \nabla \eta_\varepsilon(t,y) \\ &\sim \varepsilon^{\alpha-1}, \end{aligned}$$

since $\nabla \eta_\varepsilon(t,y) \sim \varepsilon^{-1}$. Similarly,

$$\begin{aligned} &\nabla \left(\eta_\varepsilon(t,y) \boldsymbol{M}_{\varepsilon,i}(t)\,(y - \boldsymbol{h}_{\varepsilon,i}(t)) \right) \\ &= \boldsymbol{M}_{\varepsilon,i}(t)\,(y - \boldsymbol{h}_{\varepsilon,i}(t)) \otimes \nabla \eta_\varepsilon(t,y) + \eta_\varepsilon(t,y) \nabla \left(\boldsymbol{M}_{\varepsilon,i}(t)\,(y - \boldsymbol{h}_{\varepsilon,i}(t)) \right) \\ &= \boldsymbol{M}_{\varepsilon,i}(t)\,(y - \boldsymbol{h}_{\varepsilon,i}(t)) \otimes \nabla \eta_\varepsilon(t,y) + \eta_\varepsilon(t,y) \boldsymbol{M}_{\varepsilon,i}(t) \\ &\sim \varepsilon^{\alpha-1}. \end{aligned}$$

For brevity, we let $\boldsymbol{B}^\varepsilon_{K_i(t)}(x) := \boldsymbol{B}_{K_i(t)} \left(\nabla \eta(t,\cdot) \cdot \varepsilon^{-1} \left(\boldsymbol{h}'_{\varepsilon,i}(t) + M_{\varepsilon,i}(t)\,(\varepsilon \cdot - \boldsymbol{h}_{\varepsilon,i}(t)) \right) \right)(x)$ and $\boldsymbol{B}_{\varepsilon,K_i(t)}(x) := r_\varepsilon \boldsymbol{B}^\varepsilon_{K_i(t)}\left(\frac{x}{\varepsilon}\right)$. We now estimate $\|\nabla \boldsymbol{B}_{\varepsilon,K_i}\|_\infty$. Indeed,

$$
\begin{aligned}
&\int_{B^{\varepsilon}_{2,i}(t)} \left|\boldsymbol{B}_{\varepsilon,K_i(t)}(x)\right|^p + \left|\nabla \boldsymbol{B}_{\varepsilon,K_i(t)}(x)\right|^p \, dx \\
&= \int_{B^{\varepsilon}_{2,i}(t)} \varepsilon^p \left|\boldsymbol{B}^{\varepsilon}_{K_i(t)} \left(\frac{x}{\varepsilon}\right)\right|^p + \left|\left(\nabla \boldsymbol{B}^{\varepsilon}_{K_i(t)}\right) \left(\frac{x}{\varepsilon}\right)\right|^p \, dx \\
&\leq \varepsilon^3 \left(1+\varepsilon^p\right) \int_{B^{\varepsilon}_{2,i}(t)} \varepsilon^p \left|\boldsymbol{B}^{\varepsilon}_{K_i(t)}\left(y\right)\right|^p + \left|\left(\nabla \boldsymbol{B}^{\varepsilon}_{K_i(t)}\right)(y)\right|^p \, dy \\
&\leq C\left(B_{2,i}(t)\right)^p \left(1+\varepsilon^p\right) \varepsilon^3 \int_{B_{2,i}(t)} \left|\nabla \eta(t,y) \cdot \varepsilon^{-1} \left(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\left(\varepsilon y - \boldsymbol{h}_{\varepsilon,i}(t)\right)\right)\right|^p \, dy \\
&= C\left(B_{2,i}(t)\right)^p \left(1+\varepsilon^p\right) \int_{B^{\varepsilon}_{2,i}(t)} \left|\nabla\left(\eta_\varepsilon\right)(t,x) \cdot \left(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)\left(x - \boldsymbol{h}_{\varepsilon,i}(t)\right)\right)\right|^p \, dx \\
&\leq C\left(B_{2,i}(t)\right)^p \left(1+\varepsilon^p\right) \varepsilon^{3+p(\alpha-1)}.
\end{aligned}
$$

Thus,

$$
\|\boldsymbol{B}_{\varepsilon,K_i(t)}\|_{L^p\left(B^{\varepsilon}_{2,i}(t)\right)} + \|\nabla \boldsymbol{B}_{\varepsilon,K_i(t)}\|_{L^p\left(B^{\varepsilon}_{2,i}(t)\right)} \leq C\left(B_{2,i}(t)\right)\left(1+\varepsilon^p\right)^{\frac{1}{p}} \varepsilon^{\frac{3}{p}+(\alpha-1)}.
$$

As $p \to \infty$,

$$
\|\boldsymbol{B}_{\varepsilon,K_i(t)}\|_{L^\infty\left(B^{\varepsilon}_{2,i}(t)\right)} + \|\nabla \boldsymbol{B}_{\varepsilon,K_i(t)}\|_{L^\infty\left(B^{\varepsilon}_{2,i}(t)\right)} \leq C\varepsilon^{\alpha-1}.
$$

Therefore, we obtain

$$
\|\nabla \boldsymbol{b}_\varepsilon\|_\infty \leq C\varepsilon^{\alpha-1},
$$

where C is independent of ε.

We now make estimates on the gradient of the diffeomorphism $\boldsymbol{X}_\varepsilon$.

$$\begin{aligned} \partial_t \boldsymbol{X}_\varepsilon(t,y) &= \boldsymbol{b}_\varepsilon\left(t, \boldsymbol{X}_\varepsilon(t,y)\right), \quad t \in (0,T) \\ \boldsymbol{X}_\varepsilon(0,y) &= y, \quad y \in \Omega(0). \end{aligned}$$

So that,

$$\begin{aligned} \partial_t \nabla \boldsymbol{X}_\varepsilon(t,y) &= \nabla \boldsymbol{b}_\varepsilon\left(t, \boldsymbol{X}_\varepsilon(t,y)\right) \nabla \boldsymbol{X}_\varepsilon(t,y), \quad t \in (0,T) \\ \nabla \boldsymbol{X}_\varepsilon(0,y) &= I, \quad y \in \Omega(0). \end{aligned}$$

Consider the following: fix $\boldsymbol{F} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and let $\boldsymbol{X}_\varepsilon(\boldsymbol{F})$ be the solution of the problem,

$$\begin{aligned} \partial_t \nabla \boldsymbol{X}_\varepsilon(\boldsymbol{F})(t,y) &= \nabla \boldsymbol{b}_\varepsilon\left(t, \boldsymbol{F}(t,y)\right) \nabla \boldsymbol{X}_\varepsilon(\boldsymbol{F})(t,y), \quad t \in (0,T) \\ \nabla \boldsymbol{X}_\varepsilon(0,y) &= I, \quad y \in \Omega(0). \end{aligned}$$

Then,

$$\nabla \boldsymbol{X}_\varepsilon(\boldsymbol{F})(t,y) = e^{\int_0^t \nabla \boldsymbol{b}_\varepsilon(s,F(s,y))\,ds}.$$

Thus, for $\boldsymbol{F} = \boldsymbol{X}_\varepsilon$, by uniqueness, we have

$$\nabla \boldsymbol{X}_\varepsilon(t,y) = e^{\int_0^t \nabla \boldsymbol{b}_\varepsilon(s,\boldsymbol{X}_\varepsilon(s,y))\,ds}.$$

Because $\nabla \boldsymbol{b}_\varepsilon \sim \varepsilon^{\alpha-1}$, we have that $\nabla \boldsymbol{b}_\varepsilon \to \mathbf{0}$ in L^∞, so that,

$$\nabla \boldsymbol{X}_\varepsilon \to \boldsymbol{I}, \quad \text{in } L^\infty\left((0,T)\times\Omega\right).$$

We now estimate $\nabla \boldsymbol{Y}_\varepsilon$. First, note that it is difficult to estimate $\nabla \boldsymbol{Y}_\varepsilon$ from

$$\begin{aligned} \partial_t \boldsymbol{Y}_\varepsilon(t,x) &= -\nabla \boldsymbol{Y}_\varepsilon(t,x) b_{r_\varepsilon}(t,x), \quad t \in (0,T) \\ \boldsymbol{Y}_\varepsilon(0,x) &= x, \quad x \in \Omega(0), \end{aligned}$$

since

$$\partial_t \nabla \boldsymbol{Y}_\varepsilon = -\nabla\nabla \boldsymbol{Y}_\varepsilon \boldsymbol{b}_\varepsilon - \nabla \boldsymbol{Y}_\varepsilon \nabla \boldsymbol{b}_\varepsilon,$$

and hence, estimating $\nabla \boldsymbol{Y}_\varepsilon$ depends on estimating second-order derivatives of $\boldsymbol{Y}_\varepsilon$. We estimate it from $\boldsymbol{X}_\varepsilon$ instead. Indeed, by the chain rule, we have

$$\nabla \boldsymbol{Y}_\varepsilon = \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-1} \circ \boldsymbol{Y}_\varepsilon.$$

Recall that $\nabla \boldsymbol{X}_\varepsilon(t,y) = e^{\int_0^t \nabla \boldsymbol{b}_\varepsilon(s,\boldsymbol{X}_\varepsilon(s,x))\,ds}$. Thus,

$$\left(\nabla \boldsymbol{X}_\varepsilon\right)^{-1}(t,y) = e^{-\int_0^t \nabla \boldsymbol{b}_\varepsilon(s,\boldsymbol{X}_\varepsilon(s,x))\,ds},$$

since $\int_0^t \nabla \boldsymbol{b}_\varepsilon\left(s, \boldsymbol{X}_\varepsilon(s,x)\right)\,ds$ and $-\int_0^t \nabla \boldsymbol{b}_\varepsilon\left(s, \boldsymbol{X}_\varepsilon(s,x)\right)\,ds$ commute. Therefore,

$$\nabla \boldsymbol{Y}_\varepsilon = \left(\nabla \boldsymbol{X}_\varepsilon\right)^{-1}\left(t, \boldsymbol{Y}_\varepsilon(t,x)\right) = e^{-\int_0^t \nabla \boldsymbol{b}_\varepsilon(s,x)\,ds}.$$

Thus,

$$\nabla \boldsymbol{Y}_\varepsilon \to \boldsymbol{I} \quad \text{in } L^\infty\left((0,T)\times\Omega\right).$$

Thus, we can take

$$A_F^0(t,x,y) := D_F \boldsymbol{I} \mathbb{1}_{Y_F}(y)$$
$$A_S^0(t,x,y) := D_S \boldsymbol{I} \mathbb{1}_{Y_S}(y),$$

so that the limit problem reads as

$$\begin{aligned}
\operatorname{div}_y \left(\left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y) \right) \right) &= 0, \quad \text{in } (0,T) \times \Omega \times Y_F \\
|Y_F| \, \partial_t v^0 - \operatorname{div}_x \left(\int_{Y_F} D_F \left(\nabla_x v^0(t,x) + \nabla_y v^1(t,x,y) \right) dy \right) & \\
&= |\Gamma| \left(\alpha v^0(t,x) - \beta w^0(t,x) \right) \quad \text{in } (0,T) \times \Omega \\
\partial_t w^0 + r w^0(t,x) &= \frac{|\Gamma|}{|Y_S|} \left(\beta w^0(t,x) - \alpha v^0(t,x) \right) \quad \text{in } (0,T) \times \Omega
\end{aligned}$$

We consider the case a similar case as previously but now with $\alpha = 1$ and the solid velocities are assumed to be periodic in space, i.e., the motion is the same for each cell but are not necessarily periodic in time.

In this example, we want to show that the resulting coefficient matrix A_F^ε and A_S^ε are periodic in space as well. Indeed, we first look at the extension of the solid velocity in the unit cell. We let

$$\begin{aligned}
\boldsymbol{b}(t,x) := \eta(t,x) \sum_i \left(\boldsymbol{h}_i'(t) + \boldsymbol{M}_i(t)(x - \boldsymbol{h}_i(t)) \right) \\
- \sum_i \boldsymbol{B}_{K_i(t)} \left(\nabla \eta(t,\cdot) \cdot \left(\boldsymbol{h}_i'(t) + \boldsymbol{M}_i(t) \left(\cdot - \boldsymbol{h}_i(t) \right) \right) \right)(x).
\end{aligned}$$

Now, the diffeomorphism that maps the unit cell at time zero to any positive time is obtained by solving

$$\partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}\left(t, \boldsymbol{X}(t,x)\right), \quad t > 0$$
$$\boldsymbol{X}(0,x) = x.$$

We want to show that $\boldsymbol{X}_\varepsilon(t,x) = \varepsilon \boldsymbol{X}\left(t, \frac{x}{\varepsilon}\right)$, where $\boldsymbol{X}$ is extended into the whole domain periodically. Indeed, note that

$$\begin{aligned}
\boldsymbol{b}_\varepsilon(t,x) &= \eta_\varepsilon(t,x) \sum_i \left(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t)(x - \boldsymbol{h}_{\varepsilon,i}(t)\right) \\
&\quad - \sum_i \boldsymbol{B}_{\varepsilon,K_i(t)} \left(\nabla \eta_\varepsilon(t,\cdot) \cdot \varepsilon^{-1} \left(\boldsymbol{h}'_{\varepsilon,i}(t) + \boldsymbol{M}_{\varepsilon,i}(t) \left(\varepsilon \cdot - \boldsymbol{h}_{\varepsilon,i}(t)\right)\right)\right)(x) \\
&= \eta\left(t, \frac{x}{\varepsilon}\right) \sum_i \varepsilon \left(\boldsymbol{h}'_i(t) + \boldsymbol{M}_i(t)\left(\frac{x}{\varepsilon} - \boldsymbol{h}_i(t)\right)\right) \\
&\quad - \sum_i \varepsilon \boldsymbol{B}_{K_i(t)} \left(\nabla \eta(t,\cdot) \cdot \left(\boldsymbol{h}'_i(t) + \boldsymbol{M}_i(t)\left(\cdot - \boldsymbol{h}_i(t)\right)\right)\right)\left(\frac{x}{\varepsilon}\right) \\
&= \varepsilon \boldsymbol{b}\left(t, \frac{x}{\varepsilon}\right).
\end{aligned}$$

Thus, $\boldsymbol{X}_\varepsilon$ satisfies

$$\partial_t \boldsymbol{X}_\varepsilon(t,x) = \boldsymbol{b}_\varepsilon\left(t, \boldsymbol{X}_\varepsilon(t,x)\right) = \varepsilon \boldsymbol{b}\left(t, \frac{1}{\varepsilon}\boldsymbol{X}_\varepsilon(t,x)\right).$$

So that,

$$\partial_t \left(\frac{1}{\varepsilon}\boldsymbol{X}_\varepsilon\right) = \boldsymbol{b}\left(t, \frac{1}{\varepsilon}\boldsymbol{X}_\varepsilon(t,x)\right),$$

for $t > 0$ and $x \in F_\varepsilon$. Since $x \in F_\varepsilon$ if and only if $x = \varepsilon y$ for some $y \in Y_F$, we have that for $y \in Y_F$, $\boldsymbol{X}_\varepsilon$ satifies

$$\begin{aligned} \partial_t \left(\frac{1}{\varepsilon} \boldsymbol{X}_\varepsilon \right) (t, \varepsilon y) &= \boldsymbol{b} \left(t, \frac{1}{\varepsilon} \boldsymbol{X}_\varepsilon (t, \varepsilon y) \right), \quad t > 0 \\ \frac{1}{\varepsilon} \boldsymbol{X}_\varepsilon (0, \varepsilon y) &= y. \end{aligned}$$

Thus, by uniqueness, we have $\frac{1}{\varepsilon} \boldsymbol{X}_\varepsilon (t, \varepsilon y) = \boldsymbol{X}(t, y)$, or that

$$\boldsymbol{X}_\varepsilon (t, x) = \varepsilon \boldsymbol{X} \left(t, \frac{x}{\varepsilon} \right), \quad (t, x) \in (0, \infty) \times F_\varepsilon.$$

We thus have

$$\begin{aligned} A_\varepsilon^F (t, x) = A_F^0 \left(t, \frac{x}{\varepsilon} \right) &:= D_F \mathbb{1}_{Y_F} \left(\frac{x}{\varepsilon} \right) (\nabla \boldsymbol{X})^{-T} \left(t, \frac{x}{\varepsilon} \right) (\nabla \boldsymbol{X})^{-1} \left(t, \frac{x}{\varepsilon} \right) \\ A_\varepsilon^S (t, x) = A_S^0 \left(t, \frac{x}{\varepsilon} \right) &:= D_S \mathbb{1}_{Y_S} \left(\frac{x}{\varepsilon} \right) (\nabla \boldsymbol{X})^{-T} \left(t, \frac{x}{\varepsilon} \right) (\nabla \boldsymbol{X})^{-1} \left(t, \frac{x}{\varepsilon} \right). \end{aligned}$$

In this case, the limit problem reads as

$$\begin{aligned} \operatorname{div}_y \left(A_F^0 (t, y) \left(\nabla_x v^0 (t, x) + \nabla_y v^1 (t, x, y) \right) \right) &= 0, \quad \text{in } (0, T) \times \Omega \times Y_F \\ |Y_F| \, \partial_t v^0 - \operatorname{div}_x \left(\int_{Y_F} A_F^0 (t, y) \big(\nabla_x v^0 (t, x) + \nabla_y v^1 (t, x, y) \big) \, dy \right) & \\ &= |\Gamma| \left(\alpha v^0 (t, x) - \beta w^0 (t, x) \right) \quad \text{in } (0, T) \times \Omega \\ \partial_t w^0 + r w^0 (t, x) &= \frac{|\Gamma|}{|Y_S|} \left(\beta w^0 (t, x) - \alpha v^0 (t, x) \right) \quad \text{in } (0, T) \times \Omega. \end{aligned}$$

Note that A_F^0 is not the identity matrix since $\nabla \boldsymbol{X}$ is not an orthogonal matrix nor the identity matrix.

2.6 Conclusions and future directions

We have shown that the solutions to (2.1)-(2.7) converge to the solutions of the effective model in Theorem 28. This allows us to study a more tractable model which is desirable since accounting for each catalyst particle in real-world set-ups is infeasible. However, our work only applies to cases where both the fluid and solid particles move slowly. For other cases, such as when there is vigorous mixing, more work is to be done to obtain a homogenization result similar to Theorem 28.

In theory, one can use the same diffeomorphism to map the problem onto a fixed domain, provided that the solid velocities are known beforehand. It is the homogenization that becomes difficult. Classical techniques in homogenization theory work well in regimes where the solid velocities are close to periodic motion. Outside these regimes, one needs different tools to describe the asymptotic behavior of the terms arising from the fluid and solid motion.

2.7 Appendix

The following lemma gives a weighted estimated on the L^2- norm of the traces of Sobolev functions [29].

Lemma 30.

1. *Let $\mathcal{O}$ be an arbitrary Lipschitz domain. Then, for any $\delta > 0$, there is some constant $C_\delta > 0$ such that for every $u \in H^1(\mathcal{O})$, we have*

$$\|u\|^2_{L^2(\partial\mathcal{O})} \leq C_\delta \|u\|^2_{L^2(\mathcal{O})} + \delta\varepsilon^2 \|\nabla u\|^2_{L^2(\mathcal{O})}.$$

2. *For every $\delta > 0$, there is some constant $C_\delta > 0$ such that for every $u^\varepsilon \in H^1(F_\varepsilon)$, we have*

$$\varepsilon \|u^\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \leq C_\delta \|u^\varepsilon\|^2_{L^2(F_\varepsilon)} + \delta\varepsilon^2 \|\nabla u^\varepsilon\|^2_{L^2(F_\varepsilon)}.$$

We recall the notion of *two-scale convergence* [3]. There is a notion for this convergence in the time-dependent case. We present two notions of this convergence as presented in [29].

Definition 31. *Let Ω and Y be bounded open sets in $\mathbb{R}^n$, and $T > 0$. A sequence $\{u_\varepsilon\}$ in $L^2\left((0,T) \times \Omega\right)$ is said to two-scale converge to a limit $u \in L^2\left((0,T) \times \Omega \times Y\right)$ if*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u_\varepsilon(t,x) \phi\left(t, x, \frac{x}{\varepsilon}\right)\, dx\, dt = \frac{1}{|Y|} \int_0^T \int_\Omega \int_Y u(t,x,y)\phi(t,x,y)\, dy\, dx\, dt,$$

for all $\phi \in L^2\left([0,T] \times \bar{\Omega}; C_{per}\left(\bar{Y}\right)\right)$.

We present a useful property of these special test functions [15].

Proposition 32. *Let φ be in $L^2\left(\Omega; C(Y)\right)$. Then $\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ is in $L^2\left(\Omega\right))$ with*

$$\left\|\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leq \|\varphi(\cdot,\cdot)\|_{L^2(\Omega; C_{per}(Y))}.$$

Definition 33. *We say that u_ε converges strongly in the two-scale sense to u if it two-scale converges to u and*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2((0,T)\times\Omega)} = \|u\|_{L^2((0,T)\times\Omega\times Y)}.$$

The following compactness result taken from [29] also holds for the above notion of two-scale convergence. The proof is essentially the same for the stationary case [3].

Proposition 34.

1. *Every bounded sequence* $\{u_\varepsilon\}$ *in* $L^2((0,T)\times\Omega)$ *has a two-scale convergent subsequence.*
2. *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $L^2((0,T);H^1(\Omega))$*. Then there exists* $u_0 \in L^2((0,T);H^1(\Omega))$ *and* $u_1 \in L^2\left((0,T)\times\Omega;H^1_{per}(Y)/\mathbb{R}\right)$ *and a subsequence, still denoted by* u_ε*, such that*

$$\begin{aligned} u_\varepsilon &\to u_0 && \text{in the two-scale sense,} \\ \nabla u_\varepsilon &\to \nabla_x u_0 + \nabla_y u_1 && \text{in the two-scale sense.} \end{aligned}$$

The notion of two-scale convergence can be extended to surfaces in $\mathbb{R}^N$ for the stationary case [4]. For the time-dependent case, we cite here a similar notion taken from [29]. We let Ω and Y bounded open sets of $\mathbb{R}^N$ and Γ an $(N-1)$-dimensional Lipschitz manifold compactly contained in Y. For $\varepsilon > 0$, we define Γ_ε to be the union of all $\varepsilon(\Gamma + k)$ for $k \in \mathbb{Z}$ that are contained in Ω.

Definition 35. *Let* $\{u_\varepsilon\}$ *be a sequence such that* $u_\varepsilon \in L^2((0,T)\times\Gamma_\varepsilon)$ *for each* $\varepsilon > 0$*. We say that* u_ε *converges in the two-scale sense on the surface* Γ_ε *to a limit*

$u_0 \in L^2\left((0,T) \times \Omega \times \Gamma\right)$ *if*

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} u_\varepsilon(t,x) \phi\left(t, x, \frac{x}{\varepsilon}\right) \, dS_x \, dt = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y) \phi(t,x,y) \, dS_y \, dx \, dt, \tag{2.62}$$

for all $\phi \in C\left([0,T] \times \bar{\Omega}; C_{per}(\Gamma)\right)$.

We present a useful property of these test functions [4].

Proposition 36. *Let* φ *be in* $C\left(\bar{\Omega}; C_{per}(Y)\right)$. *Then,* $\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ *is in* $L^2\left(\Gamma_\varepsilon\right)$ *and*

$$\varepsilon \left\| \varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\| \leq C \|\varphi\|^2_{C\left(\bar{\Omega}; C_{per}(Y)\right)},$$

for some constant $C > 0$ *that is independent of* ε.

Definition 37. *We say that the sequence* $\{u_\varepsilon\}$, *where* $u_\varepsilon \in L^2\left((0,T) \times \Gamma_\varepsilon\right)$ *for each* $\varepsilon > 0$ *converges strongly in the two-scale sense on* Γ_ε *if it converges in the two-scale sense on* Γ_ε *to* $u_0 \in L^2\left((0,T) \times \Omega \times \Gamma\right)$ *and*

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \|u_\varepsilon\|_{L^2((0,T) \times \Gamma_\varepsilon)} = \|u_0\|_{L^2((0,T) \times \Omega \times \Gamma)}. \tag{2.63}$$

We also have a similar compactness result for this notion of two-scale convergence on surfaces [4].

Proposition 38. *Let* $\{u_\varepsilon\}$ *be a sequence of functions such that* $u_\varepsilon \in L^2\left((0,T) \times \Gamma_\varepsilon\right)$ *for each* $\varepsilon > 0$. *Suppose* $\sqrt{\varepsilon} \|u_\varepsilon\|_{L^2((0,T) \times \Gamma_\varepsilon)} \leq C$ *for some constant* $C > 0$, *independent of* ε. *Then a subsequence exists that converges in the two-scale sense on* Γ_ε.

3 Dual formulations of the elasticity problem for a homogeneous elastic body with fractures

3.1 Introduction

The duality of displacement, stress, and strain formulations in elasticity without fractures was studied by Ciarlet et al., in [13]. In particular, they considered an homogeneous elastic body Ω in $\mathbb{R}^3$ with a body force $\boldsymbol{f}$ acting on it and surface traction $\boldsymbol{F}$ on a part of the boundary Γ_1. The three-dimensional linearized elasticity problem is then written as the following minimization problem:

Problem 4 (Displacement formulation)**.** *Find $\boldsymbol{u} \in \boldsymbol{V}$ such that*

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{V}} J(\boldsymbol{v}),$$

where, $J(\boldsymbol{v}) := \dfrac{1}{2} \displaystyle\int_\Omega A \nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v})\, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx - \int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{v}\, d\Gamma.$

Here, $J(\cdot)$ may be interpreted as the potential energy of the body and the above minimization problem can be thought of as a modern analog of the classical principle of minimum potential energy. $\boldsymbol{V}$ is the set of admissible displacements. Its definition is similar to the one presented in Section 3.2, but without any fractures.

The problem can also be formulated as a another minimization problem, for which the stress is the unknown:

Problem 5 (Stress formulation)**.** *Find* $\boldsymbol{\sigma} \in \mathbb{S}$ *such that*

$$g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\mu} \in \mathbb{S}} g(\boldsymbol{\mu}),$$

where $g(\boldsymbol{\mu}) = \dfrac{1}{2} \displaystyle\int_{\Omega} B\boldsymbol{\mu} : \boldsymbol{\mu}\, dx.$

Here B is the compliance tensor, i.e., $AB = I$. The function $g(\cdot)$ is the complimentary energy, and the above problem can be thought of as a modern version of the classical principle of minimum complimentary energy. Here $\mathbb{S}$ is the set of admissible stresses. We present a similar definition in Section 3.2.

Lastly, the authors present a different approach to the problem where the strain tensor field is the unknown. This is known as the intrinsic approach in some sources (see [13] and [14] and the references within):

Problem 6 (Strain formulation)**.** *Find* $\boldsymbol{\mu} \in \mathbb{M}^{\perp}$ *such that*

$$\tilde{J}(\boldsymbol{\mu}) = \inf_{\boldsymbol{\mu} \in \mathbb{M}^{*}} \tilde{J}(\boldsymbol{\mu}),$$

where $\tilde{J}(\boldsymbol{\mu}) := \dfrac{1}{2} \displaystyle\int_{\Omega} A\nabla_S(\boldsymbol{\mu}) : \nabla_S(\boldsymbol{\mu})\, dx - \int_{\Omega} \boldsymbol{f} \cdot \mathcal{L}(\boldsymbol{\mu})\, dx - \int_{\Gamma_1} \boldsymbol{F} \cdot \mathcal{L}(\boldsymbol{\mu}).$

Here, $\mathbb{M}$ is a set of tensors that has divergence $\mathbf{0}$ in $\boldsymbol{H}^{-1}(\Omega)$ such that the linear functionals acting on the trace space $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ defined by these tensors is zero. Here $\mathbb{M}^{\perp}$ refers to the orthogonal complement of $\mathbb{M}$ in $\mathbb{L}^2_S(\Omega_F)$. $\mathbb{L}^2_S(\Omega_F)$ is defined in Section 3.2. See [13] for more details.

Ciarlet et al. were able to show using Legendre-Fenchel duality that the displacement and stress formulations and the strain and stress formulations are dual formulations. The arguments for strong duality, i.e., the primal and dual problems attains the same objective value, relies on known results on elasticity. While it is already known that these are dual problems, the novelty lies in obtaining dual formulations through Legendre-Fenchel duality theory.

In this paper, we extend the results in [13] to the case of a fractured elastic body. It is a nonlinear extension of the results of that paper, since the space of admissible displacements for this case is not a linear space. A similar nonlinear extension can also be found in [32].

A model of the elasticity problem with fractures can be found in [50]. Here, the author assumed that the elastic body, Ω having a fixed boundary $\partial\Omega$ is homogeneous and contains a fracture inside its interior. The fracture is thought to be a smooth orientable surface which may or may not be connected, and is denoted by Σ_c. We write as Ω_F the set $\Omega\backslash\Sigma_c$. The formulation of the problem is written as,

Find $\boldsymbol{u}$ such that:

$$\text{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = 0 \;\text{ in } \Omega_F \tag{3.1}$$

$$\boldsymbol{\sigma} = \boldsymbol{A}\nabla_S(\boldsymbol{u}) \;\text{ in } \Omega_F \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \;\text{ on } \partial\Omega \tag{3.3}$$

$$[\boldsymbol{u}\cdot\boldsymbol{N}] \geq 0 \;\text{ on } \Sigma_c \tag{3.4}$$

$$\boldsymbol{\sigma n}|_1 = \sigma_{NN}\boldsymbol{N};\; \boldsymbol{\sigma n}|_2 = -\sigma_{NN}\boldsymbol{N};\; \sigma_{NN} \leq 0 \;\text{ on } \Sigma_c \tag{3.5}$$

$$\text{if } [\boldsymbol{u}\cdot\boldsymbol{N}] > 0 \,\text{on } F, \text{then } \sigma_{NN} = 0. \tag{3.6}$$

Here, $\boldsymbol{N}$ refers to the unit normal on Σ_c, n is the outward unit normal on the boundary of Ω_F, $[\phi] = \phi_1 - \phi|_2$ refers to the jump of the field ϕ across the fracture Σ_c, where the subscripts 1 and 2 denote the faces of Σ_c in the direction of $\boldsymbol{N}$ and

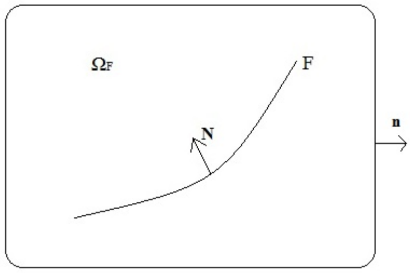

Figure 3.1: Elastic solid with fracture

the opposite direction, respectively. $\sigma_{NN} = \boldsymbol{\sigma}\boldsymbol{N}\cdot\boldsymbol{N}$. $A = [a_{ijkl}]$ is the elasticity tensor, assumed to have symmetry and positivity properties, i.e.,

$$AB\cdot B > 0, \text{ for all } B \neq 0,\ B \in \mathbb{R}^{3\times 3}, \tag{3.7}$$

$$a_{ijkl} = a_{ijlk} = a_{jikl} = a_{jilk}, \tag{3.8}$$

$\boldsymbol{f}$ represents the body forces acting on the body. $\nabla_S(\cdot)$ is the linearized strain tensor.

Equation (4.2) gives the constitutive relation for the elastic body, (4.3) says that on the outer boundary, the displacement is fixed, (4.4) implies that the body cannot penetrate itself on the crack, (4.5) shows that there is no friction on the crack and there is compression on it. Finally, (4.6) says that if the crack if open, there are no stresses on Σ_c.

Introducing the following spaces:

$$\boldsymbol{V}_F = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_F) \mid \boldsymbol{v} = 0 \,\text{on}\, \partial\Omega\},$$

$$\boldsymbol{K}_F = \{\boldsymbol{v} \in \boldsymbol{V} \mid [v_n] \geq 0 \,\text{on}\, \Sigma_c\}.$$

the problem is shown to be equivalent to the following variational formulation

Find $\boldsymbol{u} \in K_F$ such that

$$\int_{\Omega_F} A\nabla_S(\boldsymbol{u}) : \nabla_S(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{dx} \geq \int_{\Omega_F} \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{dx} \;\; \forall \boldsymbol{v} \in K_F. \tag{3.9}$$

Using classical results on variational inequalities, the author shows that the above problem has a unique solution. Kovtunenko in [39] studied the problem with the assumption that there is Coulomb friction on the fracture. The author introduced appropriate trace spaces for functions defined on the fracture as well as Green formulas. Using fixed point methods, the author was able to show existence of a solution to the variational formulation of the problem. We also mention [52] where the authors prove a homogenization result for an elastic solid with periodically distributed fractures using $\Gamma-$convergence and Mosco convergence.

We need a suitable characterization of symmetric tensors as strain of admissible displacements. The difficulty lies in the fact that the set of admissible displacements is a convex cone rather than a linear space as in [13]. Ciarlet et al. used the classical Banach closed range theorem to obtain this characterization. However, these are not directly applicable to our case.

Craven and Koliha in [19] obtained a generalization of Farkas' theorem. They provide necessary and sufficient conditions on the solvability of a linear problem posed in locally convex spaces. We use this characterization to write a suitable strain formulation to the elasticity problem with fractures.

In [13], Ciarlet et al. worked with a minimzation of the form

$$\inf_{x\in X} \left(f(v) + (g\circ h)(x)\right).$$

They were able to apply results from classical convex analysis. In particular, they considered the case where $f : X \to \mathbb{R} \cup \{+\infty\}$, $g : Y \to \mathbb{R} \cup \{+\infty\}$, $h : X \to Y$ is a continuous linear map, and X and Y are Banach spaces.

Under certain conditions, we have strong duality, i.e.,

$$\inf_{x \in X} \left(f(x) + (g \circ h)(x)\right) = \sup_{y^* \in Y^*} \left(-f^*\left(h^*(y^*)\right) - g^*(-y^*)\right).$$

Here f^* and g^* are the Fenchel conjugates of f and g, respectively, and Y^* is the dual space of Y.

However, in this paper, h is not linear. Moreover, the supremum problem is posed in a dual cone of a convex set. In order to obtain strong duality, we make use of [11], where the author considered dual formulations to minimzation problems of the same form but with h not necessarily linear, and X and Y are locally separated convex spaces. More details are provided in the following discussions.

In Section 3.2 we introduce some functional analytic preliminaries and the notation used in the paper. Section 3.3 gives the displacement, stress, and the strain formulations of the elasticity problem with fractures. These form analog formulations to those in [13]. Section 3.4 provides some preliminary results needed to obtain dual formulations of the stress problem in away similar to [13]. Section 3.5 talks about the characterization of the image of the strain operator under the set of admissible displacements. In Section 3.6, we obtain dual formulations to the stress formulation of the problem. We show that these dual problems are equivalent to the displacement and strain formulations. Furthermore, we prove strong duality. Lastly, we prove a relationship between the solutions of these problems.

3.2 Notation and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary denoted by Γ. Let Σ_c, the fracture, be an open oriented surface contained inside Ω, without self-intersection, and may not necessarily be connected, i.e., there may be several fractures. We denote $\Omega_F = \Omega \backslash \overline{\Sigma}_c$, $\overline{\Sigma}_c = \Sigma_c \cup \partial\Sigma_c$, where $\partial\Sigma_c$ is the boundary of the fracture. We assume that there is an extension Σ of Σ_c such that it divides the domain Ω into two subdomains Ω_1 and Ω_2 such that $\partial\Omega_1 = \Sigma^-$ and $\partial\Omega_2 = \Gamma \cup \Sigma^+$, where $\partial\Omega_1$ and $\partial\Omega_2$ are the boundaries of Ω_1 and Ω_2, respectively. We denote by N the unit normal vector on Σ and define by $\Sigma^\pm$ the opposite faces of Σ, with N pointing outwards of Σ^+. We let $\Sigma_c^\pm$ be the corresponding sections of $\Sigma^\pm$. We say that the boundary $\partial\Omega_F$ belongs to $C^{k,1}$ if $\partial\Omega_1$ and $\partial\Omega_2$ belong to $C^{k,1}$. Hereafter, we assume that $\partial\Omega_F$ belongs to $C^{1,1}$, so that the Green's formulas and trace theorems from [39] hold.

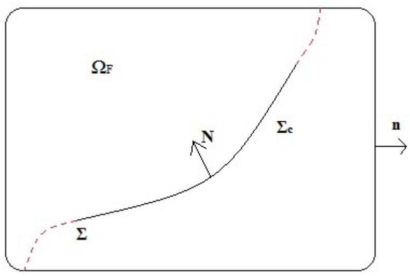

Figure 3.2: Extension of the fracture to Σ

We denote by $A = [a_{ijkh}]_{1 \leq i,j,k,h \leq 3}$ the fourth-order elasticity tensor. We assume that

$$a_{ijkh} \in L^\infty(\Omega_F) \tag{3.10}$$

$$a_{ijkh} = a_{jikh} = a_{khij}, \quad \forall i,j,k,h = 1,2,3 \tag{3.11}$$

$$\exists \alpha > 0 \text{ such that } \alpha|\boldsymbol{\sigma}|^2 \leq A\boldsymbol{\sigma} : \boldsymbol{\sigma} \quad \forall \boldsymbol{\sigma} \in \mathbb{R}^{3\times 3} \tag{3.12}$$

$$\exists \beta > 0 \text{ such that } |A\boldsymbol{\sigma}| \leq \beta|\boldsymbol{\sigma}| \quad \forall \boldsymbol{\sigma} \in \mathbb{R}^{3\times 3} \tag{3.13}$$

These imply the existence of the tensor $B = [b_{ijkh}]_{1\leq i,j,k,h\leq 3}$ such that $AB = I$ and having similar boundedness, coercivity, and symmetry properties.

Vector fields are denoted by bold lowercase Roman letters. Matrix fields are written using bold Greek letters. Sets and subsets of vector fields in $\mathbb{R}^3$ are denoted using capital, boldface Roman letters. Sets and subsets of matrix fields are written using special Roman capitals. We append a subscript S to denote spaces of symmetric matrix fields. We use the Einstein convention on repeated indices.

We use the conventional notations for the Sobolev spaces such as $H^1(\Omega)$, $H^{\frac{1}{2}}(\Sigma)$, and $H^{-\frac{1}{2}}(\Sigma)$. The set of infinitely differential functions with compact support defined on a set Ω is denoted by $D(\Omega)$. We apply the notation convention on sets of vector and tensor fields described above to these spaces whenever appropriate. We write the H^1-norm on a set Ω as $\|\cdot\|_{1,\Omega}$, where we omit Ω in the subscript when the context is clear. For the L^2-norm on a set Ω, we write the norm as $\|\cdot\|_{0,\Omega}$, where similarly, we omit Ω in the subscript when the context is clear.

We will make use of the following space to describe functions defined on the fracture:

$$\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c) := \{\boldsymbol{v} \in \boldsymbol{H}^{\frac{1}{2}}(\Sigma_c) \mid d^{-\frac{1}{2}}\boldsymbol{v} \in \boldsymbol{L}^2(\Sigma_c)\},$$

where $d \in C^{1,1}(\overline{\Sigma}_c)$, $d > 0$, $d = 0$ on $\partial\Sigma_c$, and $\lim_{x\to x_0} \frac{d(x)}{\operatorname{dist}(x,\partial\Sigma_c)} = \alpha \neq 0$ for every $x_0 \in \partial\Sigma_c$. $\operatorname{dist}(x, \partial\Sigma_c)$ refers to the distance from $x \in \Sigma_c$ to $\partial\Sigma_c$. This space is a Hilbert space with the norm

$$\|\boldsymbol{v}\|^2_{00,\Sigma_c} = \|\boldsymbol{v}\|^2_{\frac{1}{2},\Sigma_c} + \left\| d^{-\frac{1}{2}}\boldsymbol{v} \right\|^2_{0,\Sigma_c}.$$

We write the duality pairing on $\boldsymbol{H}^{\frac{1}{2}}(\Sigma)$ and its dual by $\langle\cdot,\cdot\rangle_{\frac{1}{2},\Sigma}$. Similarly, the duality pairing between $\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)$ and its dual is denoted by $\langle\cdot,\cdot\rangle_{00,\Sigma_c}$. We denote the jump across Σ_c by $[\boldsymbol{v}] = \boldsymbol{v}^+ - \boldsymbol{v}^-$, where $\boldsymbol{v}^\pm$ refers to the trace of $\boldsymbol{v}$ on corresponding faces of $\Sigma_c^\pm$.

We recall some trace theorems on these spaces. For details, see [39].

Proposition 1 (Trace Theorem 1)**.** *Let the boundary Γ belong to the class $C^{0,1}$, and let a function $\boldsymbol{u}$ belong to the space $\boldsymbol{H}^1(\Omega)$. Then there exists a linear continuous operator $\gamma : \boldsymbol{H}^1(\Omega) \to \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, which uniquely defines the trace $\gamma\boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ of $\boldsymbol{u}$ at Γ. Conversely, there exists a linear continuous operator $\boldsymbol{H}^{\frac{1}{2}} \to \boldsymbol{H}^1(\Omega)$ such that for any given $\boldsymbol{\varphi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, a function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ can be found such that $\gamma\boldsymbol{u} = \boldsymbol{\varphi}$ on Γ.*

Proposition 2 (Trace Theorem 2)**.** *Let the boundary $\partial\Omega_F$ belong to the class $C^{0,1}$, and let a function $\boldsymbol{u}$ belong to $\boldsymbol{H}^1(\Omega_F)$. Then there exists a linear continuous operator which uniquely defines at $\partial\Omega_F$ the values*

$$\boldsymbol{u}|_\Gamma \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \;\; \boldsymbol{u}^\pm \in \boldsymbol{H}^{\frac{1}{2}}(\Sigma_c), \;\; [\boldsymbol{u}] \in \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c).$$

Conversely, there exists a linear continuous operator such that for any given

$$\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \;\; \boldsymbol{\varphi}^{\pm} \in \boldsymbol{H}^{\frac{1}{2}}(\Sigma_c), \;\; [\boldsymbol{\varphi}] \in \boldsymbol{H}_{00}^{\frac{1}{2}}(\Sigma_c),$$

a function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_F)$ *can be found such that*

$$\boldsymbol{u} = \boldsymbol{\psi} \; on \; \Gamma, \;\; \boldsymbol{u}^{\pm} = \boldsymbol{\varphi}^{\pm} \; on \; \Sigma_c.$$

We can now define the following sets which will be important in the following discussions.

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_F) \mid \boldsymbol{v} = 0 \,\text{on}\, \boldsymbol{H}^{\frac{1}{2}}(\Gamma)\},$$

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V} \mid [v_n] \geq 0 \,\text{on}\, \boldsymbol{H}_{00}^{\frac{1}{2}}(\Sigma_c)\}.$$

For a subset U of a Banach space V, we define the polar cone of A as the following set

$$U^{-} := \{v^* \in V^* \mid \langle v^*, u\rangle_{V,V^*} \leq 0 \;\; \forall u \in U\}.$$

Similarly, the dual cone of A is defined to be the following set

$$U^{+} := \{v^* \in V^* \mid \langle v^*, u\rangle_{V,V^*} \geq 0 \;\; \forall u \in U\}.$$

We denote the duality pairing between V and its dual by $\langle \cdot, \cdot \rangle_{V,V^*}$.

Let X, Z be linear spaces, and C a nonempty convex subset of Z. Then C induces a partial order $\leq_C$ on Z. Indeed, we say $x \leq_C y$ if $y - x \in C$ for $x, y \in Z$.

We say that a function $g : X \to Z \cup \infty_C$ is C-convex if for all $x, y \in X$ and

$\lambda \in [0, 1]$,

$$g(\lambda x + (1-\lambda)y) \leq_C \lambda g(x) + (1-\lambda)g(y).$$

The linearized strain tensor $\nabla_S : \boldsymbol{D}(\Omega_F) \to \mathbb{D}_S(\Omega_F)$ is defined as

$$\nabla_S(\boldsymbol{v}) := \frac{1}{2}\left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right).$$

We also define the divergence operator div : $\mathbb{D}(\Omega_F) \to \boldsymbol{D}(\Omega_F)$:

$$(\text{div } \boldsymbol{\sigma})_i := \frac{\partial \sigma_{ij}}{\partial x_j}.$$

We now define the following space:

$$\mathbb{H}_S(\text{div}, \Omega_F) := \{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F) \mid \text{div } \boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega_F)\}.$$

We equip this space with the norm:

$$\|\boldsymbol{\sigma}\|^2_{\mathbb{H}_S(\text{div},\Omega_F)} := \|\boldsymbol{\sigma}\|^2_{\mathbb{L}^2_S(\Omega_F)} + \|\text{div } \boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2_S(\Omega_F)} \quad \forall \boldsymbol{\sigma} \in \mathbb{H}_S(\text{div}, \Omega_F).$$

We recall some Green's formulas relevant to our discussions, for details see [39].

Proposition 3 (Green Formula 1)**.** *Let the boundary* Γ *belong to the class* $C^{1,1}$ *and let a function* $\boldsymbol{\sigma}$ *belong to* $\mathbb{H}(div, \Omega_F)$*. There exists a linear continuous operator* $\mathbb{H}(div, \Omega_F) \to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ *which uniquely defines at the boundary* Γ *the values*

$$\sigma_n, \in H^{-\frac{1}{2}}(\Gamma), \;\; \boldsymbol{\sigma}_\tau \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \;\; \boldsymbol{\sigma}_\tau \cdot \boldsymbol{n} = 0,$$

and for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ *the generalized Green formula holds:*

$$\int_{\Omega} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, dx = -\int_{\Omega} div\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx + \langle \sigma_n, v_n \rangle_{\frac{1}{2},\Gamma} + \langle \boldsymbol{\sigma}_\tau, \boldsymbol{v}_\tau \rangle_{\frac{1}{2},\Gamma} \tag{3.14}$$

Proposition 4 (Green Formula 2)**.** *Let the boundary* $\partial\Omega_F$ *belong to the class* $C^{1,1}$*, let* $\boldsymbol{\sigma}$ *belong to* $\mathbb{H}(div, \Omega_F)$ *such that* $[\boldsymbol{\sigma n}] = 0$ *on* Σ*. Then there is a linear continuous operator* $\mathbb{H}(div, \Omega_F) \to \left(\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*$ *which uniquely defines at the crack* Σ_c *the values*

$$\sigma_n \in \left(H^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*, \;\; \boldsymbol{\sigma}_\tau \in \left(\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*, \;\; \boldsymbol{\sigma}_\tau \cdot \boldsymbol{n} = 0,$$

and for all $\boldsymbol{v} \in V$*, the generalized Green formula holds:*

$$\int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, dx = -\int_{\Omega_F} div\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx - \langle \sigma_n, [v_n] \rangle_{00,\Sigma_c} - \langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c} \tag{3.15}$$

We now define the set of admissible stresses as:

$$\mathbb{S} = \left\{ \boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F) \;\middle|\; \begin{array}{l} \mathrm{div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \text{ in } \boldsymbol{D}'(\Omega_F),\; \sigma_n \leq 0 \text{ on } H^{\frac{1}{2}}_{00}(\Sigma_c) \\ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \text{ on } \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c),\; [\sigma_n] = 0 \text{ on } H^{\frac{1}{2}}(\Sigma) \end{array} \right\}$$

In the coming discussion, it will be important to obtain a characterization of symmetric tensor fields as the strain of vectors coming from the set of admissible displacements. The following set plays an important role in this.

$$\mathbb{M} := \{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F) \mid \langle \nabla^*_S(\boldsymbol{\sigma}), \boldsymbol{v} \rangle_{V^*,V} \geq 0 \;\forall \boldsymbol{v} \in \boldsymbol{K}\}.$$

3.3 Formulations of the problem with fractures

We present three formulations of the elasticity problem with fractures. Analogs of these for the case without fractures present are in [13]. We begin with the displacement formulation which can be posed as a minimization problem.

Problem 7 (Displacement Formulation)**.** *Find* $\boldsymbol{u} \in \boldsymbol{K}$ *such that*

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{K}} J(\boldsymbol{v}),$$

where $J(\boldsymbol{v}) := \frac{1}{2}\int_{\Omega_F} A\nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v})\,dx - \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,dx$ *for all* $\boldsymbol{v} \in \boldsymbol{V}$*.*

There is an alternate formulation to this problem in which the stress is the unknown:

Problem 8 (Stress Formulation)**.** *Find* $\boldsymbol{\sigma} \in \mathbb{S}$ *such that*

$$g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\mu} \in \mathbb{S}} g(\boldsymbol{\mu}),$$

where $g(\boldsymbol{\mu}) = \frac{1}{2}\int_{\Omega_F} B\boldsymbol{\mu} : \boldsymbol{\mu}\,dx$ *for all* $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$*.*

Finally, we can recast the problem in which the strain is the unknown. This formulation is known in literature as the intrinsic formulation [14]. The use of the set $\mathbb{M}^+$ will be apparent in the coming sections.

Problem 9 (Strain Formulation)**.** *Find* $\boldsymbol{\pi} \in \mathbb{M}^+$ *such that*

$$\tilde{J}(\boldsymbol{\pi}) = \inf_{\boldsymbol{\mu} \in \mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}),$$

where $\tilde{J}(\boldsymbol{\mu}) := \frac{1}{2}\int_{\Omega_F} A\boldsymbol{\mu} : \boldsymbol{\mu}\,dx - \int_{\Omega_F} \boldsymbol{f}\cdot\mathcal{L}(\boldsymbol{\mu})\,dx$ *for all* $\boldsymbol{\mu} \in \mathbb{M}^+$*.*

Here $\mathcal{L}(\boldsymbol{\mu})$ is the unique element in K such that $\nabla_S(\mathcal{L}(\boldsymbol{\mu})) = \boldsymbol{\mu}$. The existence of such an element in $\boldsymbol{K}$ will be discussed in Section 3.5.

It will be shown that indeed, up to a change of sign, the displacement and stress formulations are dual problems, and similarly the strain and stress formulations.

3.4 Auxilliary Results

We first show that the stress formulation of the problem can be written as a minimization over the entire space $\mathbb{L}^2_S(\Omega_F)$. To do this, we introduce the following functions. Let $h : \boldsymbol{V}^* \to \overline{\mathbb{R}}$ such that

$$h(\boldsymbol{v}^*) = \mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}^*). \tag{3.16}$$

We also define $\Lambda : \mathbb{L}^2_S(\Omega_F) :\to \boldsymbol{V}^*$ by

$$\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{V^*,V} = \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v}) \,\mathrm{dx} \ . \tag{3.17}$$

Proposition 5.

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} (g(\boldsymbol{\sigma}) + \mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma})) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} (g(\boldsymbol{\sigma}) + h(\Lambda\boldsymbol{\sigma}))\,.$$

Proof. Clearly,

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} (g(\boldsymbol{\sigma}) + \mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}))\,.$$

Thus, it suffices to show that

$$h(\Lambda\boldsymbol{\sigma}) = \mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}) \quad \forall\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F).$$

Indeed, let $\boldsymbol{\sigma} \in \mathbb{S}$. Then $\mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}) = 0$. Now, for $\boldsymbol{v} \in \boldsymbol{K}$, using Proposition 4

$$\begin{aligned}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{V^*,V} &= \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v}) \,\mathrm{dx} \\ &= \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} + \int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{dx} + \langle \sigma_n, [v_n]\rangle_{00,\Sigma_c} \leq 0.\end{aligned}$$

So that, $\Lambda\boldsymbol{\sigma} \in \boldsymbol{K}^-$ and hence, $h(\Lambda\boldsymbol{\sigma}) = 0$.

Now, let $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$ such that $h(\Lambda\boldsymbol{\sigma}) = 0$, then $\Lambda\boldsymbol{\sigma} \in \boldsymbol{K}^-$ and thus

$$\int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v}) \,\mathrm{dx} \geq \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{3.18}$$

Now, we let $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega_F)$. Then $\boldsymbol{v} := \pm\boldsymbol{\varphi} \in \boldsymbol{K}$. Substituting in (3.18), we obtain

$$\int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{\varphi}) \,\mathrm{dx} = \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{\varphi} \,\mathrm{dx}.$$

Thus,

$$\mathrm{div}\ \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \boldsymbol{D}'(\Omega_F). \tag{3.19}$$

As $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we have that $\boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F)$.

We now show that

$$\sigma_n \leq 0 \quad \text{in } \left(H^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*. \tag{3.20}$$

Let $\boldsymbol{v} \in \boldsymbol{K}$ such that $[\boldsymbol{v_\tau}] = 0$ in $\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)$. Then by (3.18) and the second Green's formula (4.9), we obtain

$$\begin{aligned}\int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} &\leq \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S \boldsymbol{v} \,\mathrm{dx} \\ &= -\int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{dx} - \langle \sigma_n, [v_n]\rangle_{00,\Sigma_c}\end{aligned}$$

So that by (3.19),

$$\langle \sigma_n, [v_n] \rangle_{00,\Sigma_c} \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

By the second trace theorem,

$$\langle \sigma_n, \psi \rangle_{00,\Sigma_c} \leq 0 \quad \forall \psi \in H_{00}^{\frac{1}{2}}(\Sigma_c),\ \psi \geq 0,$$

which proves (3.20).

Next, we prove that

$$[\sigma_n] = 0 \quad \text{in } H^{-\frac{1}{2}}(\Sigma). \tag{3.21}$$

Let $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ such that $\boldsymbol{\varphi}_\tau = 0$ in $H^{\frac{1}{2}}(\Sigma)$. Since $[\boldsymbol{\varphi}] = 0$ on Σ_c, we have that $\boldsymbol{\varphi} \in \boldsymbol{K}$. Thus, by (3.18) and the first Green's formula (3.14),

$$\begin{aligned}
\int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{\varphi} \,\mathrm{dx} \leq \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{\varphi}) \\
&= -\int_{\Omega_F} \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{\varphi} \,\mathrm{dx} - \left[\langle \sigma_n, \varphi_n \rangle_{\frac{1}{2},\Sigma} \right] - \left[\langle \boldsymbol{\sigma}_\tau, \boldsymbol{\varphi}_\tau \rangle_{\frac{1}{2},\Sigma} \right] \\
&= -\int_{\Omega_F} \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{\varphi} \,\mathrm{dx} - \left[\langle \sigma_n, \varphi_n \rangle_{\frac{1}{2},\Sigma} \right].
\end{aligned}$$

By (3.19),

$$\left[\langle \sigma_n, \varphi_n \rangle_{\frac{1}{2},\Sigma} \right] \leq 0.$$

Using $\pm\boldsymbol{\varphi}$ as test function, we have that

$$\left[\langle \sigma_n, \varphi_n \rangle_{\frac{1}{2},\Sigma} \right] = 0.$$

As $[\boldsymbol{\varphi}] = 0$ on Σ_c, we obtain

$$\langle [\sigma_n] , \varphi_n \rangle_{\frac{1}{2},\Sigma} = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{D}(\Omega),\ \boldsymbol{\varphi}_\tau = 0 \text{ in } \boldsymbol{H}^{\frac{1}{2}}(\Sigma),$$

which proves the claim. Lastly, we show that

$$\boldsymbol{\sigma}_\tau = 0 \quad \text{in } \left(\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*. \tag{3.22}$$

Indeed, let $\boldsymbol{v} \in \boldsymbol{V}$ such that $[v_n] = 0$ in $H^{\frac{1}{2}}_{00}(\Sigma_c)$. Then, $\boldsymbol{v} \in \boldsymbol{K}$. By (3.18) and Green's formula (4.9),

$$\begin{aligned}\int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{dx} &\leq \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, \mathrm{dx} \\ &= -\int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\, \mathrm{dx} - \langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c}.\end{aligned}$$

Using (3.19), we have that $\langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c} \leq 0$. So, using $\pm \boldsymbol{v}$ as a test function, we obtain that

$$\langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ [v_n] = 0.$$

Using the Trace Theorem as in (3.20), the claim follows. Hence, from (3.19), (3.20), (3.21), and (3.22), we have that $\boldsymbol{\sigma} \in \mathbb{S}$, i.e., $\mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}) = 0$. □

Next, we calculate h^* and h^{**}.

Proposition 6. $h^* = \mathbb{1}_{\boldsymbol{K}}$ *and* $h^{**} = h$

Proof. Firstly, since $\boldsymbol{K}$ is a convex set, h is convex on the reflexive space V. Thus, by the Fenchel-Moreau theorem, $h \equiv h^{**}$. We now show that $(\boldsymbol{K}^-)^- = \boldsymbol{K}$. As $\boldsymbol{V}$

is reflexive, we have that

$$(\boldsymbol{K}^-)^- = \left\{\boldsymbol{v} \in \boldsymbol{V} \mid \langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \leq 0 \quad \forall \boldsymbol{v}^* \in \boldsymbol{K}^-\right\}.$$

If $\boldsymbol{u} \in \boldsymbol{K}$, clearly $\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \leq 0$ for all $\boldsymbol{v}^* \in \boldsymbol{K}^-$. Thus, $\boldsymbol{K} \subset (\boldsymbol{K}^-)^-$.
Suppose that $\boldsymbol{u} \notin \boldsymbol{K}$. Since $\boldsymbol{K}$ is a closed and convex subset of $\boldsymbol{V}$, by the Hahn-Banach theorem, there exists a hyperplane which strictly separates $\{\boldsymbol{u}\}$ and $\boldsymbol{K}$, i.e., there is some $\boldsymbol{f} \in \boldsymbol{V}^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} < \alpha < \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Fix $\boldsymbol{v} \in \boldsymbol{K}$ and let $\lambda > 0$. Since $\boldsymbol{K}$ is a convex cone, $\lambda\boldsymbol{v} \in \boldsymbol{K}$. Then,

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} < \frac{1}{\lambda}\alpha \quad \forall \lambda > 0.$$

Letting $\lambda \to +\infty$,

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*,V} \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Thus, $\boldsymbol{f} \in \boldsymbol{K}^-$.
Similarly,

$$\lambda\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} < \alpha \quad \forall \lambda > 0.$$

As $\lambda \to 0^+$, we obtain that

$$0 \leq \alpha < \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}},$$

i.e., there is some $\boldsymbol{f} \in \boldsymbol{K}^-$ such that $\langle \boldsymbol{f}, \boldsymbol{u}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} > 0$. Hence, $\boldsymbol{u} \notin (\boldsymbol{K}^-)^-$ and the claim follows.

Next, we show that $(\mathbb{1}_{\boldsymbol{K}})^* = \mathbb{1}_{\boldsymbol{K}^-}$. As $\boldsymbol{V}$ is reflexive, we have that for $\boldsymbol{v}^* \in \boldsymbol{V}^*$,

$$(\mathbb{1}_{\boldsymbol{K}})^* = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \left(\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} - \mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v})\right) = \sup_{\boldsymbol{v}\in\boldsymbol{K}} \left(\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\right).$$

Suppose that $\boldsymbol{v}^* \in \boldsymbol{K}^-$. Then $\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \leq 0$ for all $\boldsymbol{v} \in \boldsymbol{K}$. Thus, $(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) \leq 0$. Since $\mathbf{0} \in \boldsymbol{K}$, we have that

$$(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) \geq \langle \boldsymbol{v}^*, \mathbf{0}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = 0.$$

Hence, $(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) = \mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}^*) = 0$.

Now, assume that $\boldsymbol{v}^* \notin \boldsymbol{K}^-$. Then, there exists some $\boldsymbol{v} \in \boldsymbol{K}$ such that $\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} > 0$. Hence,

$$(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) \geq \sup_{\lambda>0} \left(\langle \boldsymbol{v}^*, \lambda\boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\right) = +\infty.$$

Thus, $\mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}^*) = (\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) = +\infty$ and the claim is proved.

Since $\mathbf{0} \in \boldsymbol{K}^-$, $\boldsymbol{K}^-$ is a convex cone, and $\boldsymbol{V}^*$ is reflexive, arguing as previously, we obtain that $h^* = (\mathbb{1}_{\boldsymbol{K}^-})^* = \mathbb{1}_{(\boldsymbol{K}^-)^-} = \mathbb{1}_{\boldsymbol{K}}$. □

3.5 Characterization of the Range of the Strain Operator

The adjoint of the strain operator is the linear map $\nabla_S^* : \mathbb{L}^2_S(\Omega_F) \to \boldsymbol{V}^*$ such that for $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$, $\nabla_S^*(\boldsymbol{\sigma})$ is defined by,

$$\langle \nabla_S^*(\boldsymbol{\sigma}), \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\,\mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

It is important to obtain a suitable characterization of the image of the convex cone $\boldsymbol{K}$ under the strain operator. This will allow us to obtain a relationship between the stress and the strain formulations of the elasticity problem with fractures. We mention that the first result on the characterization of the space of admissible displacements can be found in [33]. For a characterization of matrix fields as linearized strain tensor fields, we refer to [7]. In their paper, since their space of admissible displacements is a linear space, the classical Banach Closed Range Theorem allows them this suitable characterization. In our case, we use results from [19], in particular Theorem 5, which is a generalization of Farkas' theorem that we tailor to our case in the following proposition.

Proposition 7. *If $\nabla_S : \boldsymbol{V} \to \mathbb{L}^2_S(\Omega_F)$ is strongly continuous and $\nabla_S(\boldsymbol{K})$ is strongly closed, then the following are equivalent for $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$:*

- *$\nabla_S(\boldsymbol{v}) = \boldsymbol{\mu}$ has a solution $\boldsymbol{v}$ in $\boldsymbol{K}$*
- *If $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$ such that $\nabla^*_S(\boldsymbol{\sigma}) \in \boldsymbol{K}^+$, then $\int_{\Omega_F} \boldsymbol{\mu} : \boldsymbol{\sigma}\, dx \geq 0$*

Remark 1. *The previous lemma implies that $\nabla_S(\boldsymbol{K}) = \mathbb{M}^+$.*

In order to utilize the previous remark, we have to show that the hypotheses of Proposition 7 hold true. Indeed we have the following result:

Proposition 8. *$\nabla_S : \boldsymbol{V} \to \mathbb{L}^2_S(\Omega_F)$ is strongly continuous and $\nabla_S(\boldsymbol{K})$ is strongly closed.*

Proof. There exists some $C > 0$ such that

$$\|\nabla_S(\boldsymbol{v})\|_0 \leq C\, \|\boldsymbol{v}\|_1 \;\; \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Thus, as ∇_S is a linear map, it is strongly continuous.

To show that $\nabla_S(\boldsymbol{K})$ is strongly closed, let $\{\nabla_S(\boldsymbol{v}_n)\}_{n=1}^{\infty}$ be a sequence of tensors in $\mathbb{L}^2_S(\Omega_F)$ such that $\boldsymbol{v}_n \in \boldsymbol{K}$ for each n and $\nabla_S(\boldsymbol{v}_n) \to \boldsymbol{D}$ for some $\boldsymbol{D} \in \mathbb{L}^2_S(\Omega_F)$. By Korn's inequality, we have that

$$\|\boldsymbol{v}_n - \boldsymbol{v}_m\|_1 \leq C\|\nabla_S(\boldsymbol{v}_n - \boldsymbol{v}_n)\|_0 \to 0,$$

as $n, m \to \infty$. Thus the sequence $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ is Cauchy in $\boldsymbol{V}$. As $\boldsymbol{K}$ is closed in $\boldsymbol{V}$, $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\boldsymbol{V}$ for some $\boldsymbol{v} \in \boldsymbol{K}$. By strong continuity of ∇_S, we have that $\boldsymbol{D} = \nabla_S(\boldsymbol{v})$. Hence, $\nabla_S(\boldsymbol{K})$ is strongly closed.

□

3.6 Duality

We present a treatment of primal-dual problems taken from [11]. Let X, Z be separated locally convex spaces and $F : X \to \bar{\mathbb{R}}$ be a proper function. To the primal problem

$$\inf_{x \in X} F(x),$$

we can assign a dual problem through the use of perturbation functions. Indeed, let $\Phi : X \times Y \to \bar{\mathbb{R}}$ such that $\Phi(x, 0) = F(x)$ for all $x \in X$. Then the primal problem may be written as

$$\inf_{x \in X} \Phi(x, 0).$$

The dual problem is

$$\sup_{z^* \in Z^*} (-\Phi^*(0, z^*)).$$

By specific choices of the perturbation function $\Phi(\cdot,\cdot)$, we arrive at different dual formulations of a given primal problem.

3.6.1 Stress-Displacement Duality

Let X, Z be separable locally convex spaces. Let $C \subset Z$ be a nonempty convex cone that partially orders Z, i.e., for $x, y \in Z$ such that $x \leq_C y$, we have that $y - x \in C$. We attach to Z a greatest element with respect to $\leq_C$, ∞_C, which is not in Z. We have that $x \leq_C \infty_C$ for all $x \in Z \cup \{\infty_C\}$. Let $f : X \to \bar{\mathbb{R}}$ be a proper function, $g : Z \to \bar{\mathbb{R}}$ be a proper, C-increasing function, i.e., $dom\, g := \{x \in Z | g(x) \in \mathbb{R}\} \neq \emptyset$, $g(x) > -\infty$ for all $x \in Z$, and for $x \leq_C y$, $g(x) \leq g(y)$. Let $h : X \to Z \cup \{\infty_C\}$ be a proper function such that $h(dom\, f \cap dom\, h) \cap dom\, g \neq \emptyset$.

We define the primal problem as

$$\inf_{x \in X} \{f(x) + (g \circ h)(x)\}. \tag{3.23}$$

We define $\Phi : X \times Z \to \bar{\mathbb{R}}$ such that $\Phi(x, z) := f(x) + g(h(x) + z)$ as the perturbation function. It can be shown (see [11]) that $\Phi^* : X^* \times Z^* \to \bar{\mathbb{R}}$ has for $(x^*, z^*) \in X^* \times Z^*$,

$$\Phi(x^*, z^*)^* = g^*(z^*) + (f + (z^* h))^*(x^*) + \mathbb{1}_{C^*}(z^*).$$

Hence, the corresponding dual problem is given by

$$\sup_{z \in C^+} \{-g^*(z^*) - (f + (z^* h))^* (0)\}, \tag{3.24}$$

Sufficient conditions that guarantee *strong duality*, i.e., the optimal values of the primal and dual problems coincide, are given in the following proposition taken from Chapter 1, Theorem 4.1 of [11].

Proposition 9. *Let X, Z be Fréchet spaces, C a nonempty convex cone contained in Z, $g : X \to \overline{\mathbb{R}}$ be proper and convex, $h : Z \to \overline{\mathbb{R}}$ be a $C-$increasing function such that if $z^* \notin C^*$, then $h^*(z^*) = +\infty$, $\Lambda : X \to Z \cup \{\infty_C\}$ be proper, $C-$convex such that $\Lambda(dom\, g \cap dom\, \Lambda) \cap\ dom\, h \neq \emptyset$. Suppose the following regularity conditions are satisfied:*

- *g and h are lower semicontinuous*
- *Λ is star-C lower semicontinuous*
- *$0 \in core\,(dom\, h - \Lambda(dom\, g \cap\ dom\, \Lambda))$.*

Then,

$$\inf_{x\in X} \left(g(x) + (h \circ \Lambda)\right)(x) = \sup_{z^*\in C^+} \left(-h^*(z^*) - (g + z^*\Lambda)^*(0)\right). \tag{3.25}$$

We go back to the stress formulation of the elasticity problem. Here, we set $X := \mathbb{L}^2_S(\Omega_F)$, $Z := \boldsymbol{V}^*$, $C := \boldsymbol{K}^+$, and g, h, and Λ to be the functions defined in the preliminaries. From Proposition 5, the stress formulation gives rise to the following primal problem

$$(P) \qquad \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} \{g(\boldsymbol{\sigma}) + (\mathbb{1}_{\boldsymbol{K}^-} \circ \Lambda)(\boldsymbol{\sigma})\}. \tag{3.26}$$

From Proposition 6 and (3.24), we have that the dual problem is:

$$(D1) \quad \sup_{\boldsymbol{v}\in\boldsymbol{K}}\{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g+\boldsymbol{v}\Lambda)^*(\mathbf{0})\}. \tag{3.27}$$

We show that the dual problem is equivalent to the displacement formulation of the elasticity problem.

Proposition 10. *For $\boldsymbol{v} \in \boldsymbol{K}$, we have that*

$$\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) + (g+\boldsymbol{v}\Lambda)^*(\mathbf{0}) = J(\boldsymbol{v}). \tag{3.28}$$

Moreover,

$$\sup_{\boldsymbol{v}\in\boldsymbol{K}}\{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g+\boldsymbol{v}\Lambda)^*(\mathbf{0})\} = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}). \tag{3.29}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{K}$. Then

$$\begin{aligned}
\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) + (g+\boldsymbol{v}\Lambda)^*(\mathbf{0}) &= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_{\boldsymbol{S}}(\boldsymbol{\Omega_F})}\{-g(\boldsymbol{\sigma}) - (\boldsymbol{v}\Lambda)(\boldsymbol{\sigma})\} \\
&= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_{\boldsymbol{S}}(\boldsymbol{\Omega_F})}\{-g(\boldsymbol{\sigma}) - \langle\Lambda\boldsymbol{\sigma},\boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\} \\
&= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_{\boldsymbol{S}}(\boldsymbol{\Omega_F})}\{-g(\boldsymbol{\sigma}) - \int_{\Omega_F}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} + \int_{\Omega_F}\boldsymbol{\sigma}:\nabla_S(\boldsymbol{v})\,\mathrm{dx}\} \\
&= g^*(\nabla_S(\boldsymbol{v})) - \int_{\Omega_F}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} \\
&= J(\boldsymbol{v}),
\end{aligned}$$

where the last equality due to $A^{-1} = B$. Lastly,

$$\sup_{\boldsymbol{v}\in\boldsymbol{K}}\{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g+\boldsymbol{v}\Lambda)^*(\mathbf{0})\} = -\inf_{\boldsymbol{v}\in\boldsymbol{K}}\{\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) + (g+\boldsymbol{v}\Lambda)^*(\mathbf{0})\} = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}).$$

□

Remark 2. *The previous proposition shows that, up to a change in sign, the (D1) and the displacement formulation are equivalent. In addition, the solution to (D1) is also a solution to the displacement problem.*

We now prove strong duality. We show that the hypotheses of Proposition 9 are satisfied.

$\mathbb{1}_{\boldsymbol{K}^-}$ is $\boldsymbol{K}^+$-increasing.

Proof. Let $\boldsymbol{v}_1^*, \boldsymbol{v}_2^* \in \boldsymbol{V}^*$ such that $\boldsymbol{v}_1^* \leq_{\boldsymbol{K}^+} \boldsymbol{v}_2^*$. To prove that $\mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}_1^*) \leq \mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}_2^*)$, it suffices to show that if $\boldsymbol{v}_2^* \in \boldsymbol{K}^-$, then $\boldsymbol{v}_1^* \in \boldsymbol{K}^-$. Indeed, suppose that $\boldsymbol{v}_2^* \in \boldsymbol{K}^-$. Then,

$$0 \geq \langle \boldsymbol{v}_2^*, \boldsymbol{v} \rangle_{\boldsymbol{V},\boldsymbol{V}^*} \geq \langle \boldsymbol{v}_1^*, \boldsymbol{v} \rangle_{\boldsymbol{V},\boldsymbol{V}^*}, \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

where the last inequality is because $\boldsymbol{v}_2^* - \boldsymbol{v}_1^* \in \boldsymbol{K}^+$. Thus, $\boldsymbol{v}_1^* \in \boldsymbol{K}^-$. □

Proposition 11. $\mathbf{0} \in \mathit{core}\,(\mathit{dom}\, \mathbb{1}_{\boldsymbol{K}^-} - \Lambda(\mathit{dom}\, g \cap \Lambda))$.

Proof. Since $\mathit{dom}\, g \cap \mathit{dom}\, \Lambda = \mathbb{L}^2_S(\Omega_F)$, we have that $W := \{\boldsymbol{v^*} - \lambda\boldsymbol{\sigma} \mid \boldsymbol{v^*} \in \boldsymbol{K}^-,\ \boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)\} = \mathit{dom}\, \mathbb{1}_{\boldsymbol{K}^-} - \Lambda(\mathit{dom}\, g \cap \mathit{dom}\, \Lambda)$. We show that $\mathbf{0} \in \mathrm{core}(W)$. Recall that for a given a linear space X and a nonempty subset A,

$$\mathrm{core}(A) := \{x_0 \in A \mid \forall x \in X,\, \exists t_x > 0 \text{ such that for all}\, t \in [0, t_x],\, x_0 + tx \in A\}.$$

Let $\boldsymbol{u}^* \in \boldsymbol{V}^*$. Set $t_{\boldsymbol{u^*}} = 1$. We claim that for each $t \in [0, 1]$, we can find some $\boldsymbol{v}_t^* \in \boldsymbol{K}^-$ and $\boldsymbol{\sigma}_t \in \mathbb{L}^2_S(\Omega_F)$ such that

$$\langle t\boldsymbol{u}^* - (\boldsymbol{v}_t^* - \lambda\boldsymbol{\sigma}_t), \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = 0\ \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{3.30}$$

We choose $\boldsymbol{v}_t^* = \mathbf{0} \in \boldsymbol{K}^-$. Consider the following problem:

Find $\boldsymbol{u}_t \in \boldsymbol{V}$ such that

$$\int_{\Omega_F} A\nabla_S(\boldsymbol{u}_t) : \nabla_S(\boldsymbol{v})\,\mathrm{dx} = \langle t\boldsymbol{u}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{3.31}$$

To show that this has a unique solution, first observe that the bilinear form defined by $a(\boldsymbol{u},\boldsymbol{v}) := \int_{\Omega_F} A\nabla_S(\boldsymbol{u}) : \nabla_S(\boldsymbol{v})\,\mathrm{dx}$ for all $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}$ is coercive and bounded. Now, $F(\boldsymbol{v}) := \langle t\boldsymbol{u}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx}\ \ \forall \boldsymbol{v} \in \boldsymbol{V}$ defines a linear function on $\boldsymbol{V}$. Moreover, it is bounded. Indeed,

$$|F(\boldsymbol{v})| \leq (t\,\|\boldsymbol{u}^*\|_V^* + \|\boldsymbol{f}\||_0)\,\|\boldsymbol{v}\|_1 \leq (\|\boldsymbol{u}^*\|_V^* + \|\boldsymbol{f}\||_0)\,\|\boldsymbol{v}\|_1 \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$

so that

$$\|F\|_{V^*} \leq \|\boldsymbol{u}^*\|_V^* + \|\boldsymbol{f}\||_0 < +\infty,$$

i.e., $F \in \boldsymbol{V}^*$.

Thus, by the Lax-Milgram theorem, a unique solution exists to (3.31). We now let $\boldsymbol{\sigma}_t := A\nabla_S(\boldsymbol{u}_t)$. Then we have that,

$$\langle t\boldsymbol{u}^*, \boldsymbol{v}\rangle_{V^*,V} = \int_{\Omega_F} \boldsymbol{\sigma}_t : \nabla_S(\boldsymbol{v})\,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} = \langle \mathbf{0} - \Lambda\boldsymbol{\sigma}_t, \boldsymbol{v}\rangle_{V^*,V} \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$

i.e., $t\boldsymbol{u}^* \in W$. Thus, $\mathbf{0} \in \mathrm{core}(W)$. □

Proposition 12. *Λ is $\boldsymbol{K}^+$ -convex and star-$\boldsymbol{K}^+$ lower semicontinuous.*

Proof. Let $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{L}_S^2(\Omega_F)$ and $t \in [0,1]$. To show that Λ is $\boldsymbol{K}^+$-convex, we must have that

$$\Lambda\,(t\boldsymbol{\sigma}_1 + (1-t)\boldsymbol{\sigma}_2) \leq_{\boldsymbol{K}^+} t\Lambda\boldsymbol{\sigma}_1 + (1-t)\Lambda\boldsymbol{\sigma}_2,$$

i.e., $t\Lambda\boldsymbol{\sigma}_1 + (1-t)\Lambda\boldsymbol{\sigma}_2 - \Lambda\left(t\boldsymbol{\sigma}_1 + (1-t)\boldsymbol{\sigma}_2\right) \in \boldsymbol{K}^+$.

But this follows immediately, as it is easy to verify that $t\Lambda\boldsymbol{\sigma}_1 + (1-t)\Lambda\boldsymbol{\sigma}_2 - \Lambda\left(t\boldsymbol{\sigma}_1 + (1-t)\boldsymbol{\sigma}_2\right) = \mathbf{0} \in \boldsymbol{K}^+$.

Now, since $(\boldsymbol{K}^+)^+ = \boldsymbol{K}$, to show that Λ is star-$\boldsymbol{K}^+$ lower semicontinuous, it suffices to show that for all $\boldsymbol{v} \in \boldsymbol{K}$, $\boldsymbol{v}\Lambda$ is lower semicontinuous.

Indeed, let $\{\boldsymbol{\sigma}_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{L}^2_S(\Omega_F)$ such that $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ in $\mathbb{L}^2_S(\Omega_F)$ for some $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$. Then,

$$\begin{aligned}
|(\boldsymbol{v}\Lambda)(\boldsymbol{\sigma}_n) - (\boldsymbol{v}\Lambda)(\boldsymbol{\sigma})| &= |\langle \Lambda\boldsymbol{\sigma}_n, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} - \langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}| \\
&= \left| \int_{\Omega_F} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_n) : \nabla_S(\boldsymbol{v})\, \mathrm{dx} \right| \\
&\leq C \left\|\boldsymbol{v}\right\|_1 \left\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\right\|_0 \to 0.
\end{aligned}$$

Thus, $\boldsymbol{v}\Lambda$ is continuous for each $\boldsymbol{v} \in \boldsymbol{K}$. Hence, Λ is star-$\boldsymbol{K}^+$ lower semicontinuous.

□

We now have the duality between the stress and the displacement formulations.

Theorem 11.

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}). \tag{3.32}$$

Proof. Propositions 3.6.1, 11, and 12 guarantee that the conditions of Proposition 9 are satisifed. Hence, we have that

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} \{g(\boldsymbol{\sigma}) + (\mathbb{1}_{\boldsymbol{K}^-} \circ \Lambda)(\boldsymbol{\sigma})\} = \sup_{\boldsymbol{v}\in\boldsymbol{K}} \{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g + \boldsymbol{v}\Lambda)^*(\mathbf{0})\}. \tag{3.33}$$

The result now follows from Propositions 5 and 10. □

Remark 3. *The above result shows that, up to a change in sign, the stress and displacement formulations are equivalent.*

3.6.2 Stress-Strain Duality

We follow a treatment on dual problems with cone constraints as presented in [11]. We adopt the definitions for the sets X, Z, C from the previous section. The primal problem we look at is

$$\inf_{x \in X} (f(x) + \mathbb{1}_{\mathcal{A}}(x)), \tag{3.34}$$

where $\mathcal{A} = \{x \in S | \, G(x) \in -C\}$. Here $S \subseteq X$ is a given nonempty set, $f : X \to \overline{\mathbb{R}}$ and $G : X \to Z \cup \{\infty_C\}$ are proper functions such that $dom\, f \cap S \cap G^{-1}(-C) \neq \emptyset$. We define the perturbation function $\Phi : X \times X \to \bar{\mathbb{R}}$ as

$$\Phi(x, y) := f(x + y) + \mathbb{1}_{\mathcal{A}}(x).$$

It can be shown that its Fenchel conjugate, $\Phi^* : X^* \times X^* \to \bar{\mathbb{R}}$, is

$$\Phi^*(x^*, y^*) = f^*(y^*) + \sup_{z \in \mathcal{A}} \langle x^* - y^*, z \rangle_{X,X^*}.$$

The *Fenchel* dual problem can then be written as

$$\sup_{y^* \in X^*} \{-f^*(y^*) - \sup_{x \in \mathcal{A}} \langle y^*, x \rangle\} \tag{3.35}$$

To obtain strong duality, we make use of the following result taken from Chapter 1, Theorem 3.5 of [11].

Proposition 13. *Let $S \subseteq X$ be a nonempty convex set. $f : X \to \overline{\mathbb{R}}$ be a proper and convex function and $g : X \to Z \cup \{\infty_C\}$ a proper and C-convex function such that dom $f \cap S \cap G^{-1}(-C) \neq \emptyset$. If there exists some $x' \in$ dom $f \cap \mathcal{A}$ such that f is continuous at x', then* (3.34) *and* (3.35) *agree and the dual has an optimal solution.*

We look at the stress formulation. Here, we take $X = Z := \mathbb{L}^2_S(\Omega_F)$, and $g : \mathbb{L}^2_S(\Omega_F) \to \mathbb{R}$ as defined in the preliminaries. We define the sets

$$S := \{\boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F) |\, \mathrm{div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \,\mathrm{in}\, \boldsymbol{D}'(\Omega_F)\},$$

$$C := \{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F) |\, \sigma_n \geq 0 \text{ on } H^{\frac{1}{2}}_{00}(\Sigma_c),\ \boldsymbol{\sigma}_\tau = \mathbf{0} \text{ on } \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c),\ [\sigma_n] = 0 \text{ on } H^{\frac{1}{2}}(\Sigma)\},$$

$$\mathcal{A} := \{\boldsymbol{\sigma} \in S |\, I\boldsymbol{\sigma} \in -C\},$$

where $I : \mathbb{L}^2_S(\Omega_F) \to \mathbb{L}^2_S(\Omega_F)$ is the identity map.

It is easy to see that C is a convex cone in Z and that $\mathcal{A} = \mathbb{S}$. The primal problem can be written as

$$(P2) \qquad \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)} (g(\boldsymbol{\sigma}) + \mathbb{1}_{\mathcal{A}}(\boldsymbol{\sigma})). \tag{3.36}$$

Following (3.35), the dual problem is

$$(D2) \qquad \sup_{\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)} \left(-g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma} \in \mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} \right) \tag{3.37}$$

We show that (*D2*) is equivalent to the strain formulation of the elasticity problem.

Proposition 14. *For $\boldsymbol{\mu} \in \mathbb{M}^-$, we have that*

$$g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\sigma} \in \mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\, dx = \tilde{J}(-\boldsymbol{\mu}). \tag{3.38}$$

Moreover,

$$\sup_{\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)} \left(-g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma} \in \mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\, dx \right) = - \inf_{\boldsymbol{\mu} \in \mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}). \tag{3.39}$$

Proof. Fix some $\boldsymbol{\pi} \in \mathbb{S}$ such that $\pi_n = 0$ in $H^{\frac{1}{2}}_{00}(\Sigma_c)$. We claim that for each $\boldsymbol{\sigma} \in \mathbb{S}$, there exists some $\boldsymbol{\tau} \in \mathbb{M}$ such that $\boldsymbol{\sigma} = \boldsymbol{\pi} + \boldsymbol{\tau}$. Indeed, it suffices to show that $\boldsymbol{\sigma} - \boldsymbol{\pi} \in \mathbb{M}$.

Let $\boldsymbol{v} \in \boldsymbol{K}$. Then,

$$\begin{aligned} \langle \nabla_S^*(\boldsymbol{\sigma} - \boldsymbol{\pi}), \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} &= \langle \boldsymbol{\sigma} - \boldsymbol{\pi}, \nabla_S(\boldsymbol{v}) \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \\ &= - \int_{\Omega_F} \operatorname{div}\, (\boldsymbol{\sigma} - \boldsymbol{\pi}) \cdot \boldsymbol{v}\, \mathrm{dx} - \langle \sigma_n - \pi_n, [v_n] \rangle_{00,\Sigma_c} \\ &= - \langle \sigma_n, [v_n] \rangle_{00,\Sigma_c} \geq 0. \end{aligned}$$

Hence, $\boldsymbol{\sigma} - \boldsymbol{\pi} \in \mathbb{M}$. Now, let $\boldsymbol{\mu} \in \mathbb{M}^-$. Thus, $-\boldsymbol{\mu} \in \mathbb{M}^+$. As $\nabla_S(\boldsymbol{K}) = \mathbb{M}^+$, there is some $\boldsymbol{v} \in \boldsymbol{K}$ such that $-\boldsymbol{\mu} = \nabla_S(\boldsymbol{v})$.

Observe that since $\mathbf{0} \in \mathbb{M}$, we have that

$$0 \leq \sup_{\boldsymbol{\tau} \in \mathbb{M}} \int_{\Omega_F} \boldsymbol{\mu} : \boldsymbol{\tau}\, \mathrm{dx} \leq 0.$$

Hence,

$$
\begin{aligned}
g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\,\mathrm{dx} &= g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\tau}\in\mathbb{M}} \int_{\Omega_F} \boldsymbol{\tau} : \boldsymbol{\mu}\,\mathrm{dx} + \int_{\Omega_F} \boldsymbol{\pi} : \boldsymbol{\mu}\,\mathrm{dx} \\
&= g^*(-\nabla_S(\boldsymbol{v})) - \int_{\Omega_F} \boldsymbol{\pi} : \nabla_S(\boldsymbol{v})\,\mathrm{dx} \\
&= g^*(\nabla_S(\boldsymbol{v})) + \int_{\Omega_F} \operatorname{div} \boldsymbol{\pi} \cdot \boldsymbol{v}\,\mathrm{dx} - \langle \pi_n, [v_n] \rangle_{00,\Sigma_c} \\
&= \frac{1}{2} \int_{\Omega_F} A\nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v})\,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{dx} \\
&= \tilde{J}(-\boldsymbol{\mu}).
\end{aligned}
$$

For the second assertion, we first claim that if $\boldsymbol{\mu} \notin \mathbb{M}^-$, then $\sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\,\mathrm{dx} = +\infty$. Indeed, suppose $\boldsymbol{\mu} \notin \mathbb{M}^-$. Then there is some $\boldsymbol{\omega} \in \mathbb{M}$ such that $\int_{\Omega_F} \boldsymbol{\mu} : \boldsymbol{\omega}\,\mathrm{dx} > 0$. As $\mathbb{M}$ is a convex cone, we have that

$$
\begin{aligned}
\sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\,\mathrm{dx} &= \sup_{\boldsymbol{\tau}\in\mathbb{M}} \left(\int_{\Omega_F} \boldsymbol{\tau} : \boldsymbol{\mu}\,\mathrm{dx} \right) + \int_{\Omega_F} \boldsymbol{\pi} : \boldsymbol{\mu}\,\mathrm{dx} \\
&\geq \sup_{t>0} \left(t \int_{\Omega_F} \boldsymbol{\omega} : \boldsymbol{\mu}\,\mathrm{dx} \right) + \int_{\Omega_F} \boldsymbol{\pi} : \boldsymbol{\mu}\,\mathrm{dx} \\
&= +\infty.
\end{aligned}
$$

From (3.38) and the above claim, we have that

$$\begin{aligned}
\sup_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)}\left(-g^*(\boldsymbol{\mu})-\sup_{\boldsymbol{\sigma}\in\mathbb{S}}\int_{\Omega_F}\boldsymbol{\sigma}:\boldsymbol{\mu}\,\mathrm{dx}\right) &= -\inf_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)}\left(g^*(\boldsymbol{\mu})+\sup_{\boldsymbol{\sigma}\in\mathbb{S}}\int_{\Omega_F}\boldsymbol{\sigma}:\boldsymbol{\mu}\,\mathrm{dx}\right)\\
&= -\inf_{\boldsymbol{\mu}\in\mathbb{M}^-}\left(g^*(\boldsymbol{\mu})+\sup_{\boldsymbol{\sigma}\in\mathbb{S}}\int_{\Omega_F}\boldsymbol{\sigma}:\boldsymbol{\mu}\,\mathrm{dx}\right)\\
&= -\inf_{\boldsymbol{\mu}\in\mathbb{M}^-}\tilde{J}(-\boldsymbol{\mu})\\
&= -\inf_{\boldsymbol{\mu}\in\mathbb{M}^+}\tilde{J}(\boldsymbol{\mu}).
\end{aligned}$$

□

We now have the strong duality result.

Theorem 12.

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = -\inf_{\boldsymbol{\mu}\in\mathbb{M}^+}\tilde{J}(\boldsymbol{\mu}). \tag{3.40}$$

Proof. Clearly, the identity operator, $I:\mathbb{L}^2_S(\Omega_F)\to\mathbb{L}^2_S(\Omega_F)$ is C-convex. Also, we have that $\mathrm{dom}\, g\cap S\cap I^{-1}(C)=\mathbb{L}^2_S(\Omega_F)\cap S\cap(-C)=\mathbb{S}\neq\emptyset$.

We now show that $g:\mathbb{L}^2_S(\Omega)\to\mathbb{R}$ is continuous. Indeed, if $\boldsymbol{\sigma}_n\to\boldsymbol{\sigma}$ in $\mathbb{L}^2_S(\Omega_F)$, then

$$\begin{aligned}
|g(\boldsymbol{\sigma}_n)-g(\boldsymbol{\sigma})| &= \frac{1}{2}\left|\int_{\Omega_F}B\boldsymbol{\sigma}_n:\boldsymbol{\sigma}_n\,\mathrm{dx}-\int_{\Omega_F}B\boldsymbol{\sigma}:\boldsymbol{\sigma}\,\mathrm{dx}\right|\\
&= \frac{1}{2}\left|\int_{\Omega_F}B(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}):(\boldsymbol{\sigma}_n+\boldsymbol{\sigma})\,\mathrm{dx}\right|\\
&\leq \tilde{\beta}\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}\|_{\mathbb{L}^2_S(\Omega_F)}\to 0.
\end{aligned}$$

Thus, from Proposition 13, we have that

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma}\in\mathcal{A}} g(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)} \left(-g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} \right).$$

Finally, from (3.38), the result follows. □

3.6.3 Solutions of the primal and dual problems

In this subsection, we look at how the minimizers of the displacement, stress, and strain formulations are related to each other.

Theorem 13. *Let* $\bar{\boldsymbol{\sigma}} \in \mathbb{S}$*,* $\bar{\boldsymbol{v}} \in \boldsymbol{K}$*, and* $\bar{\boldsymbol{\mu}}$ *such that*

$$g(\bar{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}), \quad J(\bar{\boldsymbol{v}}) = \inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}), \quad \tilde{J}(\bar{\boldsymbol{\mu}}) = \inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}).$$

Then

$$\bar{\boldsymbol{\sigma}} = A\nabla_S(\bar{\boldsymbol{v}}) = A\bar{\boldsymbol{\mu}}. \tag{3.41}$$

Proof. The perturbation function used to obtain (*D1*) is $\Phi : \mathbb{L}^2_S(\Omega_F) \times \boldsymbol{V}^* \to \bar{\mathbb{R}}$,

$$\Phi(\boldsymbol{\sigma}, \boldsymbol{z}^*) := g(\boldsymbol{\sigma}) + \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\boldsymbol{\sigma} + \boldsymbol{z}^*).$$

From classical results in convex analysis *(see for instance,* [26] and [60]*)*, we have that $(\mathbf{0}, \bar{\boldsymbol{v}}) \in \partial\Phi(\bar{\boldsymbol{\sigma}}, \mathbf{0})$, i.e., for all $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$ and $\boldsymbol{u}^* \in \boldsymbol{V}^*$,

$$\langle (\boldsymbol{\mu}, \boldsymbol{u}^*) - (\bar{\boldsymbol{\sigma}}, \mathbf{0}), (\mathbf{0}, \bar{\boldsymbol{v}}) \rangle_{(\mathbb{L}^2_S(\Omega_F)\times\boldsymbol{V}^*)^*, \mathbb{L}^2_S(\Omega_F)\times\boldsymbol{V}^*} \leq \Phi(\boldsymbol{\mu}, \boldsymbol{u}^*) - \Phi(\bar{\boldsymbol{\sigma}}, \mathbf{0}).$$

This means that

$$\int_{\Omega_F} (\boldsymbol{\mu} - \bar{\boldsymbol{\sigma}}) : \mathbf{0} \,\mathrm{dx} + \langle \boldsymbol{u}^*, \bar{\boldsymbol{v}} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \leq g(\boldsymbol{\mu}) + \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\boldsymbol{\mu} + \boldsymbol{u}^*) - g(\bar{\boldsymbol{\sigma}}) - \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\bar{\boldsymbol{\sigma}}).$$

As $\mathbb{1}_{\boldsymbol{K}^-}(\Lambda\bar{\boldsymbol{\sigma}}) = \mathbb{1}_{\mathbb{S}}(\bar{\boldsymbol{\sigma}}) = 0$, we have that

$$\langle \boldsymbol{u}^*, \bar{\boldsymbol{v}} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \leq g(\boldsymbol{\mu}) + \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\boldsymbol{\mu} + \boldsymbol{u}^*) - g(\bar{\boldsymbol{\sigma}}).$$

If we set $\boldsymbol{\mu} := \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{u}^* := -\Lambda\bar{\boldsymbol{\sigma}}$, and noting that $\mathbf{0} \in \boldsymbol{K}^-$, then

$$\langle \Lambda\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{v}} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \geq 0.$$

As $\bar{\boldsymbol{v}} \in \boldsymbol{K}$ and $\Lambda\bar{\boldsymbol{\sigma}} \in \boldsymbol{K}^-$, we have that

$$0 = \langle \Lambda\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{v}} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = \int_{\Omega_F} \boldsymbol{f} \cdot \bar{\boldsymbol{v}} \,\mathrm{dx} - \int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}}) \,\mathrm{dx}.$$

Hence,

$$\int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}}) \,\mathrm{dx} = \int_{\Omega_F} \boldsymbol{f} \cdot \bar{\boldsymbol{v}} \,\mathrm{dx}. \tag{3.42}$$

From Theorem 11, we get

$$\begin{aligned} 0 &= g(\bar{\boldsymbol{\sigma}}) + J(\bar{\boldsymbol{v}}) \\ &= \frac{1}{2}\int_{\Omega_F} B\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} \,\mathrm{dx} + \frac{1}{2}\int_{\Omega_F} A\nabla_S(\bar{\boldsymbol{v}}) : \nabla_S(\bar{\boldsymbol{v}}) \,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{f} \cdot \bar{\boldsymbol{v}} \,\mathrm{dx} \\ &= \frac{1}{2}\int_{\Omega_F} B\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} \,\mathrm{dx} + \frac{1}{2}\int_{\Omega_F} A\nabla_S(\bar{\boldsymbol{v}}) : \nabla_S(\bar{\boldsymbol{v}}) \,\mathrm{dx} - \int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}}) \,\mathrm{dx}, \end{aligned}$$

where the last equality came from (3.42). Hence, we have that

$$g(\bar{\boldsymbol{\sigma}}) + g^*(\nabla_S(\bar{\boldsymbol{v}})) = \int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx}.$$

Looking at g as a convex, proper function on $\mathbb{L}^2_S(\Omega_F)$, this means that $\nabla_s(\bar{\boldsymbol{v}}) \in \partial g(\bar{\boldsymbol{\sigma}})$, i.e., for all $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$,

$$\int_{\Omega_F} (\boldsymbol{\mu} - \bar{\boldsymbol{\sigma}}) : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} \le g(\boldsymbol{\mu}) - g(\bar{\boldsymbol{\sigma}}).$$

If we set $\boldsymbol{\mu} := A\nabla_S(\bar{\boldsymbol{v}})$, then

$$\begin{aligned} g(A\nabla_S(\bar{\boldsymbol{v}})) - g(\bar{\boldsymbol{\sigma}}) &\ge \int_{\Omega_F} (A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} \\ &= \int_{\Omega_F} (A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : B(A\nabla_S(\bar{\boldsymbol{v}}))\,\mathrm{dx} \\ &= \int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : A\nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx}, \end{aligned}$$

where the last equality is due to B being symmetric.

Since

$$g(A\nabla_S(\bar{\boldsymbol{v}})) - g(\bar{\boldsymbol{\sigma}}) = \frac{1}{2}\int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : (A\nabla_S(\bar{\boldsymbol{v}}) + \bar{\boldsymbol{\sigma}})\,\mathrm{dx},$$

the above inequality becomes

$$\int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : A\nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} \le \int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : \bar{\boldsymbol{\sigma}}\,\mathrm{dx}. \tag{3.43}$$

The positive-definiteness of B and (3.43) imply

$$\alpha\|A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}\|^2_{\mathbb{L}^2_S(\Omega_F)} \leq \int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : (A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}})\,\mathrm{dx} \leq 0.$$

Hence, $\bar{\boldsymbol{\sigma}} = A\nabla_S(\bar{\boldsymbol{v}})$.

Now, as $\bar{\boldsymbol{\mu}} \in \mathbb{M}^+$, there exists some $\boldsymbol{u} \in \boldsymbol{K}$ such that $\bar{\boldsymbol{\mu}} = \nabla_S(\boldsymbol{u})$. Moreover as $\nabla_S(\boldsymbol{K}) = \mathbb{M}^+$, we have that $\boldsymbol{u}$ minimizes J on $\boldsymbol{K}$. Thus, from the previous arguments, we have that

$$A\nabla_S(\boldsymbol{u}) = \bar{\boldsymbol{\sigma}}.$$

Since ker $\nabla_S = \{\mathbf{0}\}$ in $\boldsymbol{V}$, we have that $\bar{\boldsymbol{v}} = \boldsymbol{u}$, so that

$$A\bar{\boldsymbol{\mu}} = A\nabla_S(\bar{\boldsymbol{v}}) = \bar{\boldsymbol{\sigma}}.$$

$\square$

4 Homogenization of the Elasticity problem for a material with fractures

4.1 Introduction

We consider a linear elasticity problem for a homogeneous solid with periodically distributed fractures. This problem was considered by Sanchez-Palencia in [50]. He proved that the problem, which we describe later, has a unique solution. Assuming that the solution has an asymptotic expansion, by formally taking limits, he obtained in the limit the homogenized problem *(without fractures)*. Properties of the homogenized stress is given in [50].

A few authors have considered proving rigorously this homogenization result. Attouch and Murat in [9] have considered a more general form of the problem but in the scalar case. Their stresses are assumed to be subgradients of a convex energy functional with other suitable properties, e.g., ellipticity, boundedness, etc. They proved that the energy functionals Γ-converge to the energy functional for the homogenized problem, in the strong $L^2(\Omega)$ topology. A major hurdle in this problem is to find a suitable space that will contain $H^1(\Omega_\epsilon)$ for all $\epsilon > 0$. $L^2(\Omega)$ does this but to accomplish Γ-convergence in this topology, one needs to prove that *(under certain conditions)* limit points of $\{u_\epsilon\}_\epsilon$, where $u_\epsilon \in H^1(\Omega_\epsilon)$, must be in $H^1(\Omega)$. To do this, the authors in [9] constructed suitable restriction-extension operator that allowed them to prove this under the condition that $\sup_\epsilon \|u_\epsilon\|_{H^1(\Omega_\epsilon)} < +\infty$. The construction of the restriction-extension operator required that the fracture in the unit cell, Y, has a neighborhood with smooth boundary that is compactly contained in the cell. An issue with their proof is justifying the limits of certain functionals. In particular, they were working with functions of the form,

$$\int_{\Omega_\epsilon} j(Z + \nabla w_Z(\frac{x}{\epsilon}))\, dx,$$

where w_Z is a periodic solution to a unit cell problem, given a constant tensor Z. One, however, cannot simply use weak convergence in $L^2(\Omega)$ to the average in the unit cell since the integrand is not necessarily in $L^2(\Omega)$, (w_Z *is only in* $H^1(Y \setminus \Gamma)$). We prove this assertion by partitioning the domain into cells compactly contained in Ω and those touching the boundary. We adapt standard arguments to proving the *limsup inequality* from [8].

Pastukhova in [46], with the assumption that $\limsup_\epsilon \|u_\epsilon\|_{H^1(\Omega_\epsilon)} < +\infty$, where u_ϵ is the solution for the periodic problem in Ω_ϵ, argued that $u_\epsilon \to u$ in $L^2(\Omega)$ and that the energies also converge. They assumed that the limit point of the any convergent subsequence must be in $H^1(\Omega)$. We argue that this is true using results from [9]. We also adjusted some of her arguments so that her arguments still hold without assuming weak convergence of certain functions in $H^1(\Omega)$. She also used periodicity of certain functionals to argue convergence. We justified such limits similarly as discussed before. Our proof of the liminf inequality is based on her arguments in [46].

We prove the homogenization result using Γ-convergence in the strong $L^2(\Omega)$ topology *(for the vector-valued case)*. We also prove that a Mosco-convergence result in the $L^2(\Omega)$ topology also holds.

4.2 Problem Formulation

Let Ω be an open bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The fracture is assumed to be a smooth surface which may or may not be connected, and is denoted by Γ. We define $\Omega_\Gamma := \Omega \setminus \Gamma$. The classical formulation of the problem in terms of displacements is

$$\text{div}\,\sigma + f = 0 \;\text{ in } \Omega_\Gamma \tag{4.1}$$

$$\sigma = A\nabla_S(u) \;\text{ in } \Omega_\Gamma \tag{4.2}$$

$$u = 0 \;\text{ on } \partial\Omega \tag{4.3}$$

$$[u \cdot n] \geq 0 \;\text{ on } \Gamma \tag{4.4}$$

$$\sigma n|_1 = \sigma_{nn} n;\; \sigma n|_2 = -\sigma_{nn} n;\; \sigma_{nn} \leq 0 \;\text{ on } \Gamma \tag{4.5}$$

$$\text{if } [u \cdot n] > 0 \,\text{on}\, \Gamma, \text{then } \sigma_{nn} = 0. \tag{4.6}$$

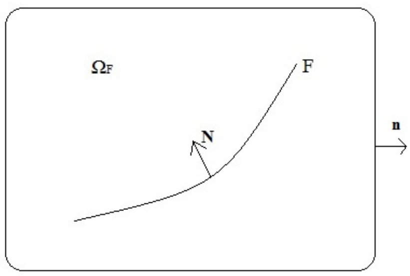

Figure 4.1: Elastic solid with fracture

Here, n refers to the unit normal on Γ, n is the outward unit normal on the boundary of Ω_Γ, $[\phi] = \phi|_1 - \phi|_2$ refers to the jump of the field ϕ across the fracture Γ, where the subscripts 1 and 2 denote the faces of F, in the direction of N and the opposite direction, respectively. $\sigma_{nn} = \sigma n \cdot n$. $A = [a_{ijkl}]$ is the elasticity tensor, assumed to have symmetry and positivity properties, i.e.,

$$AB \cdot B > 0, \quad B \neq 0,\, B \in \mathbb{R}^{3\times 3}, \tag{4.7}$$

$$a_{ijkl} = a_{ijlk} = a_{jikl} = a_{jilk}, \tag{4.8}$$

and f represents the body forces acting on the body. We denote by $\nabla_S(\cdot)$ the linearized strain tensor.

The constitutive relation for the elastic body is given by (4.2), (4.3) says that on the outer boundary, the displacement is fixed, (4.4) implies that the body cannot penetrate itself on the crack, (4.5) shows that there is no friction on the crack and there is compression on it. Finally, (4.6) says that if the crack is open, there are no stresses on Γ.

To define traces on the fracture, we follow the development in [39]. We assume that Γ can be extended into a smooth closed surface Σ that divides Ω into two disjoint sets and that Γ does not intersect itself. Observe that for a function $u \in H^1(\Omega \setminus \Gamma)$, we have that the jump of u, denoted by $[u]$, is zero in $\Sigma \setminus \Gamma$.

We introduce the following trace space:

$$H^{\frac{1}{2}}_{00}(\Gamma) := \{v \in H^{\frac{1}{2}}(\Gamma) \mid d^{-\frac{1}{2}} v \in L^2(\Gamma)\},$$

where $d \in C^{1,1}(\overline{\Sigma}_c)$, $d > 0$, $d = 0$ on $\partial\Gamma$, and $\lim_{x\to x_0} \frac{d(x)}{\mathrm{dist}(x,\partial\Gamma)} = \alpha \neq 0$ for every $x_0 \in \partial\Gamma$. $\mathrm{dist}(x, \partial\Gamma)$ refers to the distance from $x \in \Gamma$ to $\partial\Gamma$. This space is a

Hilbert space with the norm

$$\|v\|_{00,\Gamma}^2 = \|v\|_{\frac{1}{2},\Gamma}^2 + \left\| d^{-\frac{1}{2}} v \right\|_{0,\Gamma}^2 .$$

In [39], the author proves that $u \in H_{00}^{\frac{1}{2}}(\Gamma)$ if and only if the extension function

$$\bar{u} := \begin{cases} u & on\, \Gamma, \\ 0 & on\, \Sigma \setminus \Gamma, \end{cases}$$

belongs to $H^{\frac{1}{2}}(\Sigma)$. This characterization motivates the use of $H_{00}^{\frac{1}{2}}(\Gamma)$ to describe the jump $[u]$ on Γ for u in $H^1(\Omega_\Gamma)$.

We write the duality pairing on $H^{\frac{1}{2}}(\Sigma)$ and its dual by $\langle \cdot, \cdot \rangle_{\frac{1}{2},\Sigma}$. Similarly, the duality pairing between $H_{00}^{\frac{1}{2}}(\Gamma)$ and its dual is denoted by $\langle \cdot, \cdot \rangle_{00,\Gamma}$.

We define

$$\mathbb{H}(\mathrm{div}, \Omega_\Gamma) := \left\{ \sigma \in L^2(\Omega_\Gamma) | \mathrm{div}\ \sigma \in L^2(\Omega) \right\}.$$

We will use the following trace theorem and Green's theorem [39] to describe functions and functionals on the fracture.

Let the boundary Γ belong to the class $C^{0,1}$, and let a function u belong to $H^1(\Omega_\Gamma)$. Then there exists a linear continuous operator which uniquely defines at $\partial(\Omega_\Gamma)$ the values

$$u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega), \;\; u|_1, u|_2 \in H^{\frac{1}{2}}(\Gamma), \;\; [u] \in H_{00}^{\frac{1}{2}}(\Gamma).$$

Conversely, there exists a linear continuous operator such that for any given

$$\psi \in H^{\frac{1}{2}}(\partial\Omega),\;\; \varphi|_1, \varphi|_2 \in H^{\frac{1}{2}}(\Gamma),\;\; [\varphi] \in H_{00}^{\frac{1}{2}}(\Gamma),$$

a function $u \in H^1(\Omega_\Gamma)$ can be found such that

$$u = \psi \text{ on } \partial\Omega,\;\; u|_i = \varphi|_i \text{ on } \Gamma,\;\; i = 1, 2.$$

Let the boundary $\partial(\Omega_\Gamma)$ belong to the class $C^{1,1}$, let σ belong to $\mathbb{H}(\text{div}, \Omega_\Gamma)$. Then there is a linear continuous operator $\mathbb{H}(\text{div}, \Omega) \to \left(H_{00}^{\frac{1}{2}}(\Gamma)\right)^*$ which uniquely defines on the crack Γ the values

$$\sigma_n \in \left(H_{00}^{\frac{1}{2}}(\Gamma)\right)^*,\;\; \sigma_\tau \in \left(H_{00}^{\frac{1}{2}}(\Gamma)\right)^*,\;\; \sigma_\tau \cdot n = 0,$$

and for all $v \in V$, the generalized Green formula holds:

$$\int_\Omega \sigma : \nabla_S(v)\,\mathrm{dx} = -\int_\Omega \text{div }\sigma \cdot v\,\mathrm{dx} - \langle \sigma_n, [v_n] \rangle_{00,\Gamma} - \langle \sigma_\tau, [v_\tau] \rangle_{00,\Gamma} \tag{4.9}$$

We can now define the following spaces:

$$V := \{v \in H^1(\Omega_\Gamma) \mid v = 0 \,\text{in}\; H^{\frac{1}{2}}(\partial\Omega)\},$$

$$K := \{v \in V \mid [v_n] \geq 0 \,\text{on}\; H_{00}^{\frac{1}{2}}(\Gamma)).$$

The problem is then shown to be equivalent to the following variational formulation:

Find $u \in K$ such that

$$\int_{\Omega_\Gamma} A\nabla_S(u) : \nabla_S(v-u)\, dx \geq \int_{\Omega_\Gamma} f \cdot (v-u)\, dx \;\; \forall v \in K. \tag{4.10}$$

Sanchez-Palencia in [50] showed that a unique solution to problem (4.10) exists. It is standard to show that the above variational inequality is equivalent to the following minimization problem:

Problem 10 (Displacement formulation)**.** *Find $\bar{v} \in K$ such that*

$$j(\bar{v}) = \inf_{v \in K_\Gamma} j(v),$$

where $j(v) := \dfrac{1}{2} \int_{\Omega_\Gamma} A\nabla_S(v) : \nabla_S(v) - \int_{\Omega_\Gamma} f \cdot v.$

4.3 Periodic Problem

In this section we describe the periodic problem. Let $Y := (0,1)^3$ be the unit cell in $\mathbb{R}^3$. Let $\Gamma \subset Y$ be a smooth surface such that there exists an open neighborhood $\eta \subset\subset Y$ with smooth boundary containing Γ. We denote by $\mathbb{T}_\epsilon := \{z \in \mathbb{Z}^3 | \epsilon(Y + z) \subset\subset \Omega\}$. Let $\Gamma_\epsilon := \cup_{z \in \mathbb{T}_\epsilon} \epsilon(\Gamma + z)$ denote the periodically distributed fractures in Ω, and $\Omega_\epsilon := \Omega \setminus \Gamma_\epsilon$. The trace theorem can be extended naturally to this case, hence we can define similar function spaces:

$$V_\epsilon = \{v \in H^1(\Omega_\epsilon) \mid v = 0 \,\text{in}\; H^{\frac{1}{2}}(\partial\Omega)\},$$

$$K_\epsilon = \{v \in V_\epsilon \mid [v_n] \geq 0 \,\text{in}\; H^{\frac{1}{2}}_{00}(\Gamma_\epsilon)\}.$$

Using similar methods as those in the case for a single fracture, one can prove that a unique solution exists to the problem:

Find $u_\epsilon \in K_\epsilon$ such that

$$\int_{\Omega_\epsilon} A\nabla_S(u_\epsilon) : \nabla_S(v-u)\,dx \geq \int_{\Omega_\epsilon} f\cdot(v-u_\epsilon)\,dx \;\; \forall v \in K_\epsilon. \tag{4.11}$$

This can be written equivalently as the following minimization problem:

$$\min_{v\in K_\epsilon}\left\{\frac{1}{2}\int_{\Omega_\epsilon} A\nabla(v):\nabla(v) - \int_{\Omega_\epsilon} f\cdot v\right\}. \tag{4.12}$$

We are now interested in the limit as $\epsilon \to 0$. Sanchez-Palencia [50] used an asympotic expansion for u_ϵ of the form

$$u_\epsilon(x) = u_0(x,\frac{x}{\epsilon}) + \epsilon u_1(x,\frac{x}{\epsilon}) + \epsilon^2 u_2(x,\frac{x}{\epsilon}) + ...$$

and calculated formally the limit problem. The condition that $u_\epsilon \in K_\epsilon$ implies that $u_0(x,y) = u_0(x)$ and that $[u_1\cdot n] \geq 0$ on Γ. Moreover, formally letting $\epsilon \to 0$, he obtains that $u_0 \in H_0^1(\Omega)$ is a solution to the homogenized problem:

$$\text{div }\bar{\sigma}^\circ(\nabla u_0) + f = 0, \quad in\,\Omega,$$

where

$$\bar{\sigma}^\circ(\nabla u_0) := \int_Y A(\nabla_x u_0 + \nabla_y u_1)\,dy,$$

and $u_1 \in H_0^1(Y\setminus\Gamma)$ solves the unit cell problem:

$$\int_{Y\setminus\Gamma} A(\nabla_x u_0 + \nabla_y u_1)\nabla_y(w - u_1)\, dy \geq 0, \tag{4.13}$$

for all $w \in H_0^1(Y \setminus \Gamma)$ such that $[w \cdot n] \geq 0$ on Γ.

The homogenized problem is then equivalent to the following minimization problem:

$$\min_{v\in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega \bar{\sigma}^\circ(\nabla_S(v)) : \nabla_S(v) \right\} \tag{4.14}$$

Our goal now is to obtain a homogenization result by proving Mosco convergence [43] of the energy functionals found in (4.12) and (4.14).

4.4 Auxiliary lemmas

We list some lemmas which will be used to prove our main result.

Lemma 39. *[9] For any sequence $\{u_\epsilon\}_{\epsilon>0}$ satisfying $\sup_{\epsilon>0} \|u_\epsilon\|_{H^1(\Omega_\epsilon)} < \infty$, there exists a bounded sequence $\{Q_\epsilon(u_\epsilon)\}_{\epsilon>0}$ in $H^1(\Omega)$ such that*

$$\lim_{\epsilon\to 0} \|u_\epsilon - Q_\epsilon(u_\epsilon)\|_{L^2(\Omega)} = 0.$$

Remark 4. *The assumption that the Γ has a neighborhood η with smooth boundary that is compactly contained in Y allowed the authors in [9] to construct Q_ϵ. Their approach is to construct a restriction-extension operator that first restricts a function defined on $Y \setminus \Gamma$ on $Y \setminus \eta$ and extend it to the whole of Y. Doing the appropriate scaling and translations, one obtains the operator $Q_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega)$.*

Following [46], we define $C^\infty_{per}(Y_\Gamma)$ to be the set of smooth 1-periodic functions defined on $\mathbb{R}^N \setminus \Gamma_\epsilon$. We set $L^2_{per}(Y_\Gamma)$ to be the closure of this set in $L^2(Y)$. We then define $H^1_{per}(Y_\Gamma)$ to be the closure of the set of functions in $C^\infty_{per}(Y_\Gamma)$ that has support outside a neighborhood of Γ with respect to the $H^1(Y)$ norm. We then have the following lemmas from [46]:

Suppose that $a(y) \in L^2_{per}(Y \setminus \Gamma)$ and $\bar{a} = \int_{Y\setminus\Gamma} a\,dy = 0$. Then there exists $w \in H^1_{per}(Y,\Gamma)$ such that div $w(y) = a(y)$ and

$$\|w\|_{H^1(Y)} \leq C(\Gamma)\|a\|_{L^2(Y)}.$$

If $u_E(y)$ is a solution of the variational inequality (4.13) for a given tensor E with constant components, then the tensor function Z such that $Z = A(E + \nabla_S(u_E))$ has the following properties:

$$\text{div } Z = 0 \quad in\, Y \setminus \Gamma,$$

and the orthogonal decomposition of the vector $Z_n = Zn = Z_{nn}n + Z_\tau$ satisfies the following conditions on Γ:

$$Z_n|_1 = Z_{nn}|_1 n, \quad Z_n|_2 = -Z_{nn}|_1 n, \quad Z_{nn}|_1 \leq 0;$$
$$\int_{Y\setminus\Gamma} A(E + \nabla_S(u_E)) : (E + \nabla_S(u_E))\,dy = \bar{\sigma}^\circ(E)E = \bar{Z}E,$$

where $\bar{\sigma}^\circ(E) = \bar{Z} = \int_Y Z\,dy$.

4.5 Homogenization result

We extended the proof of the *limsup inequality* found in [9] to the vector-valued case, specialized to our particular case of linear elasticity. We provided necessary justifications for some of the convergences of the energy functionals. The proof of the *liminf inequality* is adapted from [46]. We adjusted her proof to our case and used Lemma 39 in some of the arguments. More details can be found below.

Define for $v \in L^2(\Omega)$,

$$J_\epsilon(v) := \frac{1}{2}\int_{\Omega_\epsilon} A\nabla_S(v) : \nabla_S(v) - \int_{\Omega_\epsilon} f \cdot v + \chi_{K_\epsilon}(v)$$
$$J_{hom}(v) := \frac{1}{2}\int_{\Omega} \bar{\sigma}^\circ(\nabla_S(v))\nabla_S(v) - \int_{\Omega} f \cdot v + \chi_{H_0^1(\Omega)}(v).$$

Then

$$J_{hom} = \Gamma - \lim J_\epsilon, \tag{4.15}$$

in the strong $L^2(\Omega)$ topology.

Proof. Observe that the map $v \mapsto \int_{\Omega_\epsilon} f \cdot v = \int_\Omega f \cdot v$ is continuous on $L^2(\Omega)$. Hence if we show that $F_{hom} = \Gamma - \lim F_\epsilon$ in the strong $L^2(\Omega)$ topology, where

$$F_\epsilon(v) := \frac{1}{2}\int_{\Omega_\epsilon} A\nabla_S(v) : \nabla_S(v) + \chi_{K_\epsilon}(v)$$
$$F_{hom}(v) := \frac{1}{2}\int_{\Omega} \bar{\sigma}^\circ(\nabla_S(v))\nabla_S(v) + \chi_{H_0^1(\Omega)}(v),$$

then the assertion of the theorem follows. We first prove the *limsup inequality*, i.e., for every $v \in L^2(\Omega)$, there is a sequence $\{v_\epsilon\}_{\epsilon>0}$ in $L^2(\Omega)$ such that

$$\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \le F_{hom}(v), \tag{4.16}$$

As $F_{hom}(v) = +\infty$ for $v \notin H_0^1(\Omega)$, it suffices to prove the inequality for $v \in H_0^1(\Omega)$. We proceed in several steps.

Step 1. Suppose v is affine, i.e., for some $Z \in \mathbb{R}^{N\times N}$ and $\alpha \in \mathbb{R}^N$,

$$v(x) = Zx + \alpha.$$

For $Z \in \mathbb{R}^{N\times N}$, let $w_Z \in H_0^1(Y \setminus \Gamma)$ be the unique solution of

$$\int_{Y\setminus\Gamma} A(Z + \nabla w_Z) : \nabla(w - w_Z)\, dy \geq 0,$$

for all $w \in H_0^1(Y \setminus \Gamma)$ such that $[w \cdot n] \geq 0$ on Γ.

We define $v_\epsilon(x) := v(x) + \epsilon w_Z(\frac{x}{\epsilon})$. As $[w_Z \cdot n] \geq 0$ on Γ and $v \in H_0^1(\Omega)$, we have that $v_\epsilon \in K_\epsilon$. Also, observe that $w_Z(\frac{x}{\epsilon}) \rightharpoonup \int_{Y\setminus\Gamma} w_Z(y)\, dy$ weakly in $L^2(\Omega)$ and hence is bounded in $L^2(\Omega)$. Thus, $v_\epsilon \to v$ in $L^2(\Omega)$.

Observe that we can find a finite number of translates of ϵY with disjoint interiors, $\{Y_i\}_{i=1}^{N(\epsilon)}$ and $\{Y_i'\}_{i=1}^{N'(\epsilon)}$ *(see Figure 2)* such that

$$\Omega \subset \left(\cup_{i=1}^{N(\epsilon)} Y_i\right) \cup \left(\cup_{i=1}^{N'(\epsilon)} Y_i'\right), \quad Y_i \subset\subset \Omega \quad i = 1, ..., N(\epsilon), \quad Y_i' \cap \partial\Omega \neq \emptyset \quad i = 1, ..., N'(\epsilon).$$

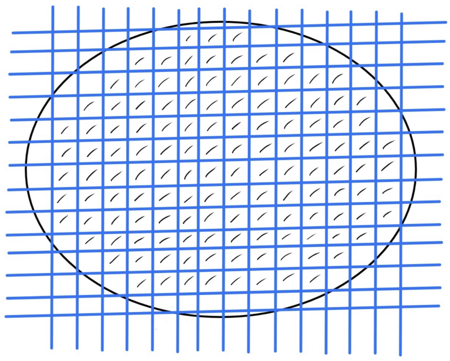

Figure 4.2: Domain with periodically distributed fractures

Denoting by $Y_{i\epsilon}$ and $Y_{i\epsilon}$ the ϵY translates without the fractures, we now calculate,

$$\begin{aligned}
F_\epsilon(v_\epsilon) &= \frac{1}{2}\int_{\Omega_\epsilon} A\nabla_S(v_\epsilon) : \nabla_S(v_\epsilon) = \frac{1}{2}\int_{\Omega_\epsilon} A\nabla(v_\epsilon) : \nabla(v_\epsilon) \\
&= \frac{1}{2}\int_{\Omega_\epsilon} A\left(Z + \nabla w_Z\left(\frac{x}{\epsilon}\right)\right) : \left(Z + \nabla w_Z\left(\frac{x}{\epsilon}\right)\right) \\
&\leq \frac{1}{2}\sum_i \left(\int_{Y_{i\epsilon}} A(Z + \nabla w_Z(\frac{x}{\epsilon})) : (Z + \nabla w_Z(\frac{x}{\epsilon}))\right) \\
&\quad + \frac{1}{2}\sum_i \left(\int_{Y'_{i\epsilon}} A(Z + \nabla w_Z(\frac{x}{\epsilon})) : (Z + \nabla w_Z(\frac{x}{\epsilon}))\right) \\
&\leq \frac{1}{2}\left(N(\epsilon)\epsilon^N + N'(\epsilon)\epsilon^N\right)\int_{Y\backslash\Gamma} A(Z + \nabla w_Z(\frac{x}{\epsilon})) : (Z + \nabla w_Z(\frac{x}{\epsilon})),
\end{aligned}$$

where we have used a change of variables to obtain the last inequality. Similar to arguments found in (Theorem 2.6, [15]), we have that

$$\begin{aligned}
N(\epsilon)\epsilon^N &\to \frac{|\Omega|}{|Y|} = |\Omega| \\
N'(\epsilon)\epsilon^{N-1} &\leq N\frac{|\Omega|}{|Y|} = N|\Omega|.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) &\le \frac{1}{2}|\Omega| \int_{Y\setminus\Gamma} A(Z+\nabla w_Z(\frac{x}{\epsilon})) : (Z+\nabla w_Z(\frac{x}{\epsilon})) \\ &= \frac{1}{2}\int_\Omega \int_{Y\setminus\Gamma} A(Z+\nabla w_Z(\frac{x}{\epsilon})) : (Z+\nabla w_Z(\frac{x}{\epsilon})) \\ &= F_{hom}(v).\end{aligned}$$

Step 2. Suppose v is a continuous piecewise affine function, i.e.,

$$v(x) = \sum_{i=1}^{l} (Z_i x + \alpha_i)\, \mathbb{1}_{\Omega_\epsilon^i}(x),$$

where Ω_ϵ^i forms a partition of Ω_ϵ.

We set $v_\epsilon^i := v(x) + \epsilon w_{Z_i}(\frac{x}{\epsilon})$ in Ω_ϵ^i. As $\nabla w_{Z_i}(\frac{\cdot}{\epsilon})$ is not necessarily equal to $\nabla w_{Z_j}(\frac{\cdot}{\epsilon})$ on the interface between Ω_ϵ^i and Ω_ϵ^j, we introduce smooth cut-off functions to obtain an appropriate sequence in $H^1(\Omega_\epsilon)$. We do this in the case $l = 2$. The general case follows similarly. Let $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$. For $\delta > 0$ small enough, we define $\Sigma_\delta := \{x \in \Omega | d(x, \Sigma) < \delta\}$. Let $\varphi_\delta \in C_0^\infty(\Omega)$ such that $0 \le \varphi_\delta \le 1$ and

$$\varphi_\delta = \begin{cases} 1 & \text{in } \Sigma_\delta, \\ 0 & \text{in } \Omega \setminus \Sigma_{2\delta}. \end{cases}$$

Set $v_\epsilon^\delta := (1-\varphi_\delta)v_\epsilon^i + \varphi_\delta v$. Observe that $v_\epsilon^\delta = v$ in Σ_δ and $[v_\epsilon^\delta \cdot n] = (1-\varphi_\delta)[w_{Z_i} \cdot n] \ge$

0. Hence, $v_\epsilon^\delta \in K_\epsilon$. Note that,

$$\nabla v_\epsilon^\delta = v_\epsilon^i \otimes \nabla(1-\varphi_\delta) + (1-\varphi_\delta)\nabla v_\epsilon^i + v \otimes \nabla\varphi_\delta + \varphi_\delta \nabla v.$$

Using the convexity of the map $Z \to AZ : Z$ for $Z \in \mathbb{R}^{N\times N}$, we have that for $0 < t < 1$,

$$\begin{aligned}
F_\epsilon(tv_\epsilon^\delta) &= \frac{1}{2}\sum_i \int_{\Omega_\epsilon^i} A\left(t(1-\varphi_\delta)\nabla v_\epsilon^i + t\varphi_\delta\nabla v + (1-t)\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right) : \\
&\qquad \left(t(1-\varphi_\delta)\nabla v_\epsilon^i + t\varphi_\delta\nabla v + (1-t)\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right) dx, \\
&\le \frac{1}{2}\sum_i \int_{\Omega_\epsilon^i} t(1-\varphi_\delta)A\nabla v_\epsilon^i : \nabla v_\epsilon^i + t\varphi_\delta A\nabla v : \nabla v \\
&\qquad + (1-t)A\left(\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right) : \left(\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right) dx, \\
&\le \frac{1}{2}\sum_i \int_{\Omega_\epsilon^i} A\nabla v_\epsilon^i : \nabla v_\epsilon^i + \int_{\Omega_\epsilon^i} A\nabla v : \nabla v \\
&\quad + (1-t)\int_{\Omega_\epsilon^i} A\left(\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right) : \left(\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right).
\end{aligned}$$

As $v_\epsilon^i \to v$ in $L^2(\Omega)$, we get

$$\begin{aligned}
\limsup_{\epsilon\to 0} F_\epsilon(tv^{\delta_\epsilon}) \le \frac{1}{2}\sum_i \int_\Omega \int_{Y\setminus\Gamma} A\left(Z_i + \nabla w_{Z_i}(y)\right) : \left(Z_i + \nabla w_{Z_i}(y)\right)\, dy \\
+ \frac{1}{2}\int_{\Sigma_{2\delta}} A\nabla v : \nabla v,
\end{aligned}$$

where we used *Step 1* to get the first term of the right-hand side of the inequality. As $v \in H^1(\Omega)$, we have that

$$\int_{\Sigma_{2\delta}} A\nabla v : \nabla v \to 0, \quad as\ \delta \to 0.$$

Thus,

$$\limsup_{\substack{\delta\to 0\\ t\to 1}}\limsup_{\epsilon\to 0} F_\epsilon(tv_\epsilon^\delta) \le \frac{1}{2}\sum_i \int_\Omega \int_{Y\setminus\Gamma} A\left(Z_i+\nabla w_{Z_i}(y)\right):\left(Z_i+\nabla w_{Z_i}(y)\right)\, dy$$
$$= F_{hom}(v).$$

By a standard diagonalization argument, we find a sequence $\delta(\epsilon)\to 0$ and $t(\epsilon)\to 1$ as $\epsilon\to 0$ such that

$$\limsup_{\epsilon\to 0} F(t(\epsilon)v_\epsilon^{\delta(\epsilon)}) \le \limsup_{\substack{\delta\to 0\\ t\to 1}}\limsup_{\epsilon\to 0} F_\epsilon(tv_\epsilon^\delta).$$

Hence, taking $v_\epsilon := t(\epsilon)v_\epsilon^{\delta(\epsilon)}$, we obtain

$$\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \le F_{hom}(v).$$

Step 3. We argue using density that (4.16) holds for $v\in H_0^1(\Omega)$. Indeed, let $v\in H_0^1(\Omega)$. Then there exists a sequence of continuous piecewise affine functions, $\{v_k\}_{k=1}^\infty$, that converges to v in $H^1(\Omega)$. From *Step 2*, for each k there is a sequence $\{v_{k,\epsilon}\}_{\epsilon>0}$ such that $v_{k,\epsilon}\to v_k$ in $L^2(\Omega)$ as $\epsilon\to 0$ and $\limsup_{\epsilon\to 0} F_\epsilon(v_{k,\epsilon})\le F_{hom}(v_k)$. Arguing similarly as in (Theorem 1.20, [8]), it can be shown that F_{hom} is convex and finitely valued on $H_0^1(\Omega)$ and hence is continuous. Thus, we obtain

$$F_{hom}(v) = \lim_{k\to\infty} F_{hom}(v_k) \ge \limsup_{k\to\infty}\limsup_{\epsilon\to 0} F_\epsilon(v_{k,\epsilon}) \ge \limsup_{\epsilon\to 0} F_\epsilon(v_{k(\epsilon),\epsilon}),$$

where the last inequality follows from a diagonalization argument. Taking $v_\epsilon := v_{k(\epsilon),\epsilon}$, we get the desired result.

We now prove the *liminf inequality*, i.e., if $v \in L^2(\Omega)$ and $\{v_\epsilon\}_{\epsilon>0}$ is a sequence in $L^2(\Omega)$ such that $v_\epsilon \to v$ in $L^2(\Omega)$, then

$$F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon). \tag{4.17}$$

If $\liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon) = +\infty$, we are done. Suppose now that $\liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq M$ for some $M > 0$. We first need to show that $F_{hom}(v) < \infty$, i.e., $v \in H_0^1(\Omega)$. By the assumption, there is a subsequence such that $F_{\epsilon'}(v_{\epsilon'}) \leq M$ for all ϵ'. Necessarily, $v_{\epsilon'}$ belongs to $K_{\epsilon'}$ for each ϵ'. By coercivity of $F_{\epsilon'}$, we obtain

$$\sup_{\epsilon'} \|v_{\epsilon'}\|_{H^1(\Omega_\epsilon)} < \infty.$$

Using Lemma 39, we obtain a bounded sequence $\{Q_{\epsilon'}(v_{\epsilon'})\}_{\epsilon'}$ in $H^1(\Omega)$ such that

$$\|v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'})\|_{L^2(\Omega)} \to 0, \quad as\ \epsilon' \to 0.$$

As $H^1(\Omega) \subset\subset L^2(\Omega)$, up to a subsequence, for some $u \in H^1(\Omega)$, $Q_{\epsilon'}(v_{\epsilon'}) \to u$ in $L^2(\Omega)$. Then,

$$\|v - u\|_{L^2(\Omega)} \leq \|v - v_{\epsilon'}\|_{L^2(\Omega)} + \|v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'})\|_{L^2(\Omega)} + \|Q_{\epsilon'}(v_{\epsilon'}) - u\|_{L^2(\Omega)} \to 0,$$

Hence, $v \in H^1(\Omega)$. As v_ϵ has trace zero on $\partial\Omega$ for every ϵ, we have that $v \in H_0^1(\Omega)$. This guarantees that $F_{hom}(v) < \infty$. We now show that $F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon)$. Indeed, let E be an arbitrary symmetric tensor in $\mathbb{R}^{N\times N}$. Let Z be as in Lemma

4.4. We define $Z_\epsilon(x) := Z(\frac{x}{\epsilon})$. By Young's inequality, we obtain

$$\frac{1}{2}A\nabla v_\epsilon : \nabla_\epsilon \geq Z_\epsilon : \nabla v_\epsilon - \frac{1}{2}BZ_\epsilon : Z_\epsilon.$$

Let $\omega \subset\subset \Omega$ be a subdomain and $\varphi \in C_0^\infty(\omega)$, $0 \leq \varphi \leq 1$. Set $\omega_\epsilon := \omega \setminus \Gamma_\epsilon$. Then,

$$\frac{1}{2}\int_{\omega_\epsilon} \varphi A\nabla v_\epsilon : \nabla_\epsilon \geq \int_{\omega_\epsilon} \varphi Z_\epsilon : \nabla v_\epsilon - \frac{1}{2}\int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon. \tag{4.18}$$

Now, using Lemma 4.4 and Lemma 4.2 *(Green's formula)*, we obtain

$$\begin{aligned}
\int_{\omega_\epsilon} \varphi Z_\epsilon : \nabla v_\epsilon &= \int_{\omega_\epsilon} Z_\epsilon : (\nabla(\varphi v_\epsilon) - v_\epsilon \otimes \nabla\varphi) \\
&= -\int_{\omega_\epsilon} \text{div } Z_\epsilon \cdot (\varphi v_\epsilon) - \langle (Z_\epsilon)_n, [\varphi v_\epsilon]_n \rangle_{\frac{1}{2},\Gamma_\epsilon} \\
&\quad - \langle (Z_\epsilon)_\tau, [\varphi v_\epsilon]_\tau \rangle_{\frac{1}{2},\Gamma_\epsilon} - \int_{\omega_\epsilon} Z_\epsilon : (v_\epsilon \otimes \nabla\varphi) \\
&\geq -\int_{\omega_\epsilon} \nabla\varphi Z_\epsilon v_\epsilon \, dx \\
&= -\int_{\omega_\epsilon} \nabla\varphi \left(Z_\epsilon - \bar{Z}\right) v_\epsilon - \bar{Z}\int_{\omega_\epsilon} \nabla\varphi v_\epsilon.
\end{aligned}$$

Using Lemma 4.4, we obtain $W \in H^1_{per}(Y, \Gamma)$ such that $\|W\|_{H^1(Y)} \leq C(\Gamma)$ and

$$\text{div}_y W(y) = Z(y) - \bar{Z}.$$

Thus,

$$Z_\epsilon(x) - \bar{Z} = \text{div}_y W(y)|_{y=\frac{x}{\epsilon}} = \text{div}_x W(\frac{x}{\epsilon}).$$

Since,

$$\int_{\omega_\epsilon} \left(\mathrm{div}_x W\left(\frac{x}{\epsilon}\right)\right)^2 \leq C \int_{\omega_\epsilon} \left|\nabla W(\frac{x}{\epsilon})\right|^2 = C\epsilon^N \int_Y |\nabla W(y)|^2 \, dy \leq C \int_Y |\nabla W(y)|^2 \, dy,$$

it follows that

$$\int_{\omega_\epsilon} \nabla\varphi \left(Z_\epsilon - \bar{Z}\right) v_\epsilon = \int_{\omega_\epsilon} \nabla\varphi \left(\epsilon \mathrm{div}_x W(\frac{x}{\epsilon})\right) v_\epsilon \to 0.$$

As $v_\epsilon \to v$ in $L^2(\Omega)$,

$$\bar{Z} \int_{\omega_\epsilon} \nabla\varphi v_\epsilon = \bar{Z} \int_\omega \nabla\varphi v_\epsilon \to \bar{Z} \int_\omega \nabla\varphi v = -\bar{Z} \int_\omega \varphi \nabla v. \tag{4.19}$$

We thus obtain,

$$\liminf_{\epsilon\to 0} \int_{\omega_\epsilon} \varphi Z_\epsilon : \nabla v_\epsilon \geq \bar{Z} \int_\omega \varphi \nabla v. \tag{4.20}$$

Using the definition of Z and the technique of partitioning the cells into those inside the domain and those intersecting the boundary, we get

$$\begin{aligned}
\int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon = & \int_{\omega_\epsilon} A\left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) : \left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) \\
& \leq \sum_i \int_{Y_{i_\epsilon}} A\left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) : \left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) \\
& \quad + \sum_i \int_{Y'_{i_\epsilon}} A\left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) : \left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) \\
& \leq \epsilon^N \left(N(\epsilon) + N'(\epsilon)\right) \int_{Y\setminus\Gamma} A\left(E + \nabla u_E(y)\right) : \left(E + \nabla u_E(y)\right) \, dy,
\end{aligned}$$

where $N(\epsilon)$ and $N'(\epsilon)$ counts the cells inside ω and those intersecting $\partial\omega$, respectively. Thus, we have

$$\begin{aligned}\limsup_{\epsilon\to 0}\int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon &\leq |\omega| \int_{Y\setminus\Gamma} A\left(E+\nabla u_E\left(y\right)\right) : \left(E+\nabla u_E\left(y\right)\right)\, dy \\ &= \int_\omega \int_{Y\setminus\Gamma} A\left(E+\nabla u_E\left(y\right)\right) : \left(E+\nabla u_E\left(y\right)\right)\, dy\, dx \\ &= \int_\omega \bar{Z}E\, dx.\end{aligned}$$

Combining this with (4.18) and (4.20), we obtain

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{1}{2}\int_{\omega_\epsilon}\varphi A\nabla v_\epsilon : \nabla v_\epsilon &\geq \int_\omega \varphi\bar{Z}\nabla v - \liminf_{\epsilon\to 0}\frac{1}{2}\int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon \\ &\geq \int_\omega \varphi\bar{Z}\nabla v - \frac{1}{2}\int_\omega \bar{Z}E.\end{aligned}$$

As φ is arbitrary, it holds that for any subdomain ω,

$$\liminf_{\epsilon\to 0}\frac{1}{2}\int_{\omega_\epsilon} A\nabla v_\epsilon : \nabla v_\epsilon \geq \int_\omega \bar{Z}\nabla v - \frac{1}{2}\int_\omega \bar{Z}E = \int_\omega \bar{\sigma}^\circ(E)\nabla v - \frac{1}{2}\int_\omega \bar{\sigma}^\circ(E)E.$$

If g is a continuous piecewise affine function, we can extend the above estimate to

$$\liminf_{\epsilon\to 0}\frac{1}{2}\int_{\omega_\epsilon} A\nabla v_\epsilon : \nabla v_\epsilon \geq \int_\omega \bar{\sigma}^\circ(\nabla_S(g))\nabla v - \frac{1}{2}\int_\omega \bar{\sigma}^\circ(\nabla_S(g))\nabla_S(g). \tag{4.21}$$

Since $v \in H^1_0(\Omega)$, we can choose a sequence of continuous piecewise affine functions $\{g^\delta\}_{\delta>0}$ such that $g^\delta \to v$ in $H^1(\Omega)$. The map $E \mapsto \bar{\sigma}^\circ(E)$ is Lipschitz continuous on $\mathbb{R}^{N\times N}$ (Theorem 7.2, [50]). Thus, we have that $\bar{\sigma}^\circ(\nabla_S(g^\delta)) \to \bar{\sigma}^\circ(\nabla_S(v))$ in $L^2(\Omega)$. This, together with (4.21) gives,

$$\liminf_{\epsilon\to 0}\frac{1}{2}\int_{\omega_\epsilon} A\nabla v_\epsilon : \nabla v_\epsilon \geq \frac{1}{2}\int_\omega \bar{\sigma}^\circ(\nabla_S(v))\nabla_S(v),$$

i.e., $F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon)$. Hence,

$$F_{hom} = \Gamma - \lim\ F_\epsilon,$$

in the strong $L^2(\Omega)$ topology, and thus

$$J_{hom} = \Gamma - \lim\ J_\epsilon,$$

in the strong $L^2(\Omega)$ topology. □

Remark 5. *In [46], to show* (4.20)*, the author used weak convergence of u_ϵ in $H^1(\Omega)$ which does hold since u_ϵ is not in $H^1(\Omega)$. We used a slightly different argument to show that* (4.20) *holds.*

Moreover, we have the following result J_ϵ Mosco converges to J_{hom} in the $L^2(\Omega)$ topology.

Proof. As Mosco convergence is stable under continuous perturbations, it suffices to prove the following:

limsup inequality: For each v in $L^2(\Omega)$, there exists a sequence $\{v_\epsilon\}_{\epsilon>0}$ belonging to $L^2(\Omega)$ such that $v_\epsilon \to v$ in the strong topology of $L^2(\Omega)$ and

$$\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq F_{hom}(v), \tag{4.22}$$

liminf inequality: For each v in $L^2(\Omega)$ and sequence $\{v_\epsilon\}_{\epsilon>0}$ in $L^2(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in the weak topology of $L^2(\Omega)$, it holds that

$$F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon). \tag{4.23}$$

The proof of the *limsup inequality* was discussed in Theorem 4.5. The proof of the *liminf inequality* also holds even if we only assume that $v_\epsilon \rightharpoonup v$ weakly in $L^2(\Omega)$. Indeed, (4.19) still holds. Moreover, we can still show that $v = u$ a.e., so that $v \in H^1(\Omega)$. To see this, observe that

$$v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'}) \rightharpoonup v - u, \quad \text{weakly in } L^2(\Omega).$$

By the uniform boundedness principle,

$$\|v - u\|_{L^2(\Omega)} \leq \liminf_{\epsilon' \to 0} \|v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'})\|_{L^2(\Omega)} = 0.$$

The rest of the proof then follows similarly to that of Theorem 4.5. □

5 Stokes flow past moving rigid obstacles with slip boundary conditions

5.1 Introduction

We consider the motion of an incompressible Newtonian fluid in a bounded domain with submerged rigid particles whose velocities are known. At the boundary of the fluid domain, we prescribe a Navier slip condition. The goal is to find a velocity v and pressure q that satisfies

$$\begin{aligned} \partial_t v - \Delta v + \nabla q = f, &\quad \text{in } \Omega(t),\, t \in (0,T) \\ \operatorname{div} v = 0, &\quad \text{in } \Omega(t),\, t \in (0,T) \\ v \cdot n = 0, &\quad \text{in } \Gamma(t),\, t \in (0,T) \\ [\mathbb{D}(v)n]_\tau + \alpha(v - V)_\tau = 0, &\quad \text{in } \Gamma(t),\, t \in (0,T) \\ v(0) = v_0, &\quad \text{in } \Omega, \end{aligned}$$

where the moving domain $\Omega(t)$ is defined through the motion of the solid particles given by

$$V(t,x) := h_i'(t) + M_i(t)\,(x - h_i(t))\,, \quad x \in \Gamma_i(t),$$

h_i and M_i are in $C^\infty(0,T)$ and $M_i(t)$ is skew-symmetric for all t. Here $\Gamma(t)$ is the boundary of the solid particles at time t. More details are given in the next section.

Such a system of equations can be used as a simple model of suspensions of rigid solids in a Newtonian fluid undergoing Brownian motion. Although Brownian motion is known to have rough trajectories, in this model we are making the greatly simplifying assumption that the solid velocity is smooth.

There have been plenty of work done on both the Navier-Stokes and Stokes equations with slip boundary conditions. The slip condition used in this paper dates back to Navier [45]. It is more recently used in modeling fluid-solid interactions, most especially to resolve the no-collision paradoxes that have been known to exist for both the Stokes [18] and Navier-Stokes [35] equations under the usual no-slip boundary conditions.

Plenty of work has been done on both the well-posedness and regularity of solutions to both the Navier-Stokes and Stokes equations. A recent comprehensive paper that goes through the theory is [1] *(See also the references therein)*. In this paper, the authors look into the L^p-theory of the stationary Stokes and Navier-Stokes equations. For a recent treatment of the non stationary theory, we refer to [7] and their references. In this paper, they used the semigroup theory for the Stokes operator with slip conditions having a non constant friction coefficient to obtain strong solutions to the Navier-Stokes equations.

With regards to work on the Stokes and Navier-Stokes equations in moving domains, a lot has been done in the past several decades. One of the earliest mathematical treatments on this is in [28] where the domain is prescribed for every time. We mention first some of the work done in the case where the solid motion is coupled with the fluid velocity. In the case of no-slip boundary conditions, early works such as [21] and [27] prove the existence of weak solutions. In [56], the author proves the existence of strong solutions whereas in two dimensions they prove global solvability, and in three dimensions a local in-time existence and global solvability for small data. For the case of slip boundary conditions, we have [31] and [12] that prove the existence of weak solutions up to collisions.

In the case where the solid moves with some known velocity or when the evolution of the domain is known a priori, the general treatment is to map the problem in a cylindrical domain. There are numerous works in this regard and we mention [51], [20], and the references therein as some examples.

One of the transformations used comes from [37], where under some smoothness assumptions on the evolution of the domain, one has a divergence preserving transformation that maps the problem into a fixed domain. In our work, we do the same and map the moving domain problem into a fixed one. To transform the problem, we proceeded similarly as in [48]. The equations are the same except for the boundary conditions. We then provide an elementary proof of the H^2-regularity of solutions to the Stationary Stokes problem with slip boundary conditions and use Rothe's method to obtain a strong non stationary solution. To solve the full problem, we proceeded similarly as in [22] by solving the problem using a fixed-point argument.

5.2 Transformation to a fixed domain problem

Let U be a bounded subset of $\mathbb{R}^3$ and $\{\mathcal{O}_i\}_{i=1}^m$ be bounded, pairwise disjoint subsets of U such that $\partial U, \partial\mathcal{O}_i \in C^3$ for all i. The sets $\mathcal{O}_i$ represent the solid rigid particles at time zero. We let $\Omega := U \setminus \cup_{i=1}^m \mathcal{O}_i$ be the initial fluid domain.

From here onwards, we denote by y, the spatial variable in the domain at time zero and $x := x(t, y)$ to be spatial variable in the moving domain.

In order to describe the moving domain, we first need to obtain a transformation that maps $\overline{\mathcal{O}}_i$ to $\overline{\mathcal{O}_i(t)}$, i.e., a mapping between points from the solid at $t = 0$ to points in the solid at any time $t \in (0, T)$. This would come from the known velocity of the solid particles. Indeed, let $y \in \overline{\mathcal{O}_i}$, and consider the following ODE:

$$\begin{aligned} G_i'(t, y) &= h_i'(t) + M_i(t) \left(G_i(t, y) - h_i(t) \right), \quad t \in (0, T) \\ G_i(0, y) &= y. \end{aligned}$$

This, then, defines an isomorphism $G_i(t, \cdot) : \overline{\mathcal{O}}_i \mapsto \overline{\mathcal{O}(t)}_i$. With this, we can now define the domain at time $t \in [0, T]$ as $\Omega(t) := U \setminus \cup_{i=1}^m \mathcal{O}_i(t)$. We make the important assumption that h_i and M_i guarantee that the solids remain at least a positive distance $d > 0$ away from each other at all times.

The task now is to find a diffeomorphism between Ω and $\Omega(t)$. We do this by defining a suitable *domain velocity* for U that will give the necessary diffeomorphism upon integrating. Heuristically, we want this velocity to be the solid velocity inside the solid particles, zero when one is sufficiently far away from the solids, and *glues* together these two velocites in between. We also need it to be volume preserving.

To start, let B_{1_i}, B_{2_i} be open balls such that $\overline{\mathcal{O}}_i \subset B_{1_i} \subset \overline{B}_{1_i} \subset B_{2_i}$. We define for $k = 1, 2$:

$$B_{k_i}(t) := \{x = G_i(t, y) | \, y \in B_{k_i}\}$$

Let $\eta \in C^\infty \left(\mathbb{R}^3 \times [0, T] \right)$ be a cut-off function such that

- $0 \leq \eta \leq 1$,
- for $t \in [0, T]$, $\eta \equiv 1$ on $\cup B_{1_i}(t)$, $\eta \equiv 0$ on $\mathbb{R}^3 \setminus \cup B_{2_i}(t)$.

We let $K_i(t) :=$ support of $\nabla \eta(t, \cdot) \cap \overline{B}_{2_i}(t)$. We introduce this cut-off function to achieve our goal of having a domain velocity that matches the solid velocities in the solids, zero far away from them, and glues them together in between.

In order to get a volume preserving diffeomorphism, we need this domain velocity to be divergence free. To do that, we subtract out the divergence of the terms where we expect the velocity to be nonzero. Indeed, we make the following calculations:

$$\begin{aligned}\operatorname{div}_x\left(\eta(t,x)h_i'(t)\right) &= \nabla\eta(t,x)\cdot h_i'(t) + \eta(t,x)\operatorname{div}_x\left(h_i'(t)\right)\\ &= \nabla\eta(t,x)\cdot h_i'(t).\end{aligned}$$

Also,

$$\operatorname{div}_x\left(\eta(t,x)M_i(t)h_i(t)\right) = \nabla\eta(t,x)\cdot M_i(t)h_i(t).$$

Lastly,

$$\begin{aligned}\operatorname{div}_x\left(\eta(t,x)M_i(t)x\right) &= \nabla\eta(t,x)\cdot M_i(t)x + \eta(t,x)\operatorname{div}_x\left(M_i(t)x\right)\\ &= \nabla\eta(t,x)\cdot M_i(t)x,\end{aligned}$$

since

$$\begin{aligned}\operatorname{div}_x\left(M_i(t)x\right) &= \sum_j \partial_{x_j}\left(M_i(t)x\right)_j = \sum_j \partial_{x_j}\left(\sum_k \left(M_i(t)\right)_{jk} x_k\right)\\ &= \sum_{j,k}\left(M_i(t)\right)_{jk}\delta_{kj} = \sum_j \left(M_i(t)\right)_{jj}\\ &= \operatorname{tr}\left(M_i(t)\right)\\ &= 0.\end{aligned}$$

These motivate us to define for $t \in [0,T]$ and $x \in \overline{\mathcal{O}}_i(t)$:

$$b(t,x) := \eta(t,x) \sum_{i=1}^{m} \left(h_i'(t) + M_i(t)\,(x - h_i(t))\right) - \sum_{i=1}^{m} B_{K_i(t)} \left(\nabla \eta(t,\cdot) \cdot (h_i'(t) + M_i(t)\cdot)\right)(x),$$

where $B_{K_i(t)} : L^2(K_i(t)) \to H_0^1(K_i(t))$, is the operator such that,

$$\text{div } \left(B_{K_i(t)}(H)\right) = H,$$

and $\|B_{K_i(t)}(H)\|_{H_0^1(K_i(t))} \leq C\,(K_i(t))\,\|H\|_{L^2(K_i(t)}$. See [58] for details. Based on our previous calculations, we have that

- $b(t,x) = h_i'(t) + M_i(t)\,(x - h_i(t))$ for $x \in \overline{\mathcal{O}}_i(t)$,
- div $b \equiv 0$,
- $b \in C_{0,\sigma}^{\infty}\left(\mathbb{R}^r \times [0,T]; \mathbb{R}^3\right)$.

b is the *domain velocity* that we need to define the necessary diffeomorphism. Indeed, we consider the following problem: for $y \in \mathbb{R}^3$,

$$\begin{aligned} \partial_t \phi(t,y) &= b\left(t, \phi(t,y)\right), \quad t \in (0,T), \\ \phi(0,y) &= y. \end{aligned}$$

As b is smooth, by Picard-Lindelof, there exists a smooth function ϕ that solves the above ODE. Thus, restricting it to Ω, we have that $\phi(t,\cdot) : \Omega \to \Omega(t)$ is the desired diffeomorphism.

Roughly what ϕ is, is that outside $B_{2_i}(t)$, it is the identity map; inside $\overline{\mathcal{O}}_i(t)$, ϕ is the rigid displacement $G_i(t)$; and in between $\partial B_{2_i}(t)$ and $\overline{\mathcal{O}}_i(t)$, ϕ can be thought of as a *glue* between these two maps.

Now that we have the transformation that maps points in the fixed domain to points in the moving domain, we move on to defining the transformation that maps functions defined on the moving domain to ones defined on the fixed domain.

We introduce the following transformations: for $t \in [0,T]$ and $y \in \Omega$,

$$\begin{aligned} U(t,y) &:= (\Phi v)(t,y) := (\nabla \phi)^{-1}(t,y)\, v\,(t,\phi(t,y)) \\ p(t,y) &:= q\,(t,\phi(t,y)) \end{aligned}$$

These were first introduced in [37]. The reason that the velocity is mapped differently than the pressure is because we want the transformed velocity to also be solenoidal. This map guarantees that div $U \equiv 0$ in Ω. See [37] for details.

The resulting PDE that these transformed functions solve have been calculated in [37]. Our task now is to look into how the slip boundary condition is changed under this transformation. This is given by the following lemma:

The velocity v satisfies

$$[\mathbb{D}(v)n]_\tau + \alpha(v - V)_\tau = 0, \quad \text{in } \Gamma(t),\ t \in (0,T)$$

if and only if U satisfies

$$[\mathbb{D}(U)\mu]_{\tau_\mu} + \alpha\,(U - \Phi(V)) = 0 \quad \text{on } (0,T) \times \Gamma,$$

where μ is the outer normal to Γ and $[w]_{\tau_\mu} := w - (w \cdot \mu)\mu$ for any vector field w defined on Γ.

Proof. First, we have that $v = \Phi^{-1}U$. We then apply Φ to the slip condition on $\Gamma(t)$. Indeed, for $t \in (0,T)$ and $y \in \Gamma$:

$$\begin{aligned}
\Phi\left(\nabla_x\left(\Phi^{-1}U\right)\right)(t,y) &= \left(\nabla_y\phi\right)^{-1}(t,y)\left(\nabla_x\left(\Phi^{-1}U\right)\right)(t,\phi(t,y)) \\
&= \left(\nabla_y\phi\right)^{-1}(t,y)\nabla_y\left[\left(\Phi^{-1}U\right)(t,\phi(t,y))\right]\left(\nabla_y\phi\right)^{-1}(t,y) \\
&= \left(\nabla_y\phi\right)^{-1}(t,y)\nabla_y\left[\left(\nabla_y\phi\right)(t,y)\left(\nabla_yU\right)(t,y)\right]\left(\nabla_y\phi\right)^{-1}(t,y) \\
&= \left(\nabla_y\phi\right)^{-1}(t,y)\left(\nabla_y\phi\right)(t,y)\left(\nabla_yU\right)(t,y)\left(\nabla_y\phi\right)^{-1}(t,y) \\
&= \left(\nabla_yU\right)(t,y)\left(\nabla_y\phi\right)^{T}(t,y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Phi\left(\left[\nabla_x\left(\Phi^{-1}U\right)\right]^T\right)(t,y) &= \left(\nabla_y\phi\right)^{-1}(t,y)\left[\left(\nabla_y\phi\right)(t,y)\left(\nabla_yU\right)(t,y)\left(\nabla_y\phi\right)^{-1}(t,y)\right]^T \\
&= \left(\nabla_y\phi\right)^{-1}(t,y)\left(\nabla_y\phi\right)^{-T}(t,y)\left(\nabla_yU\right)^T(t,y)\left(\nabla_y\phi\right)^T(t,y) \\
&= \left(\nabla_yU\right)^T(t,y)\left(\nabla_y\phi\right)^T(t,y),
\end{aligned}$$

since $\nabla_y\phi$ is an orthogonal matrix on Γ. Setting,

$$\mu(t,y) := \left(\nabla\phi\right)^T(t,y)n\left(t,\phi(t,y)\right),$$

we have that μ is the unit outward normal on Γ. We now calculate:

$$\begin{aligned}
&\left[\mathbb{D}\left(\Phi^{-1}U\right)n\right](t,y) \\
&= \left(\nabla U\right)(t,y)\left(\nabla\phi\right)(t,y)n\left(t,\phi(t,y)\right) + \left(\nabla U\right)^T(t,y)\left(\nabla\phi\right)^T(t,y)n\left(t,\phi(t,y)\right) \\
&= \left(\mathbb{D}(U)\mu\right)(t,y).
\end{aligned}$$

Also,

$$
\begin{aligned}
&\left[\mathbb{D}\left(\Phi^{-1}U\right)n\cdot n\right](t,\phi(t,y))\\
&=(\nabla\phi)(t,y)\,(\nabla\phi)^{-1}(t,y)\left(\mathbb{D}\left(\Phi^{-1}U\right)\right)(t,\phi(t,y))\cdot n\,(t,\phi(t,y))\\
&=(\nabla\phi)^{-1}(t,y)\left(\mathbb{D}\left(\Phi^{-1}U\right)\right)(t,\phi(t,y))\cdot(\nabla\phi)^{T}(t,y)n\,(t,\phi(t,y))\\
&=(\mathbb{D}(U)\mu\cdot\mu)\,(t,y).
\end{aligned}
$$

Thus, we have,

$$
\begin{aligned}
&\Phi\left[\left(\mathbb{D}\left(\Phi^{-1}U\right)n\cdot n\right)n\right](t,y)\\
&=(\nabla\phi)^{-1}(t,y)\left[\left(\mathbb{D}\left(\Phi^{-1}U\right)n\cdot n\right)(t,\phi(t,y))\right]n\,(t,\phi(t,y))\\
&=\left[\left(\mathbb{D}\left(\Phi^{-1}U\right)n\cdot n\right)(t,\phi(t,y))\right](\nabla\phi)^{T}(t,y)n\,(t,\phi(t,y))\\
&=\left[(\mathbb{D}(U)\mu\cdot\mu)\,\mu\right](t,y).
\end{aligned}
$$

Combining these calculations, we obtain:

$$
\begin{aligned}
\Phi\left(\left[\mathbb{D}\left(\Phi^{-1}U\right)n\right]_{\tau}\right)(t,y)&=\Phi\left(\mathbb{D}\left(\Phi^{-1}U\right)n-\left(\mathbb{D}\left(\Phi^{-1}U\right)n\cdot n\right)n\right)(t,y)\\
&=(\mathbb{D}(U)\mu)\,(t,y)-\left[(\mathbb{D}(U)\mu\cdot\mu)\,\mu\right](t,y)\\
&:=[\mathbb{D}(U)\mu]_{\tau_\mu}(t,y).
\end{aligned}
$$

The slip boundary condition on $\Gamma(t)$ then becomes:

$$
[\mathbb{D}(U)\mu]_{\tau_\mu}+\alpha\,(U-\Phi(V))=0\quad\text{on }(0,T)\times\Gamma.
$$

As Φ is an isomorphism, one can do similar calculations for the reverse implication.

□

Thus, as was proven in [37], together with the calculations above, we have that (v,q) is a strong solution to the Stokes problem in the moving domain if and only if (U,p) is a strong solution to the following:

$$\begin{aligned}
\partial_t U + (\mathcal{M} - \mathcal{L})U &= f - \mathcal{G}p, \quad \text{in } (0,T)\times\Omega \\
\operatorname{div} U &= 0, \quad \text{in } (0,T)\times\Omega \\
U\cdot\mu &= 0, \quad \text{on } (0,T)\times\Gamma \\
[\mathbb{D}(U)\mu]_{\tau_\mu} + \alpha\,(U - \Phi(V)) &= 0 \quad \text{on } (0,T)\times\Gamma \\
U(0) &= U_0, \quad \text{in } \Omega,
\end{aligned}$$

where $U_0 := \Phi(v_0)$. Here, $\mathcal{L}$ corresponds to the transformed Stokes operator, $\mathcal{M}$ came from transforming the time derivative, and $\mathcal{G}$ is due to transforming the pressure term. These operators are defined as follows *(see* [37] *for details)*:

$$\begin{aligned}
[\mathcal{L}U]_i &:= \sum_{j,k=1}^{3} \partial_j\left(g^{jk}\partial_k U_i\right) + 2\sum_{j,k,l=1}^{3} g^{kl}\Gamma^i_{jk}\partial_l U_j \\
&\quad + \sum_{j,k,l=1}^{3}\left[\left(\partial_k\left(g^{kl}\Gamma^i_{jl}\right) + \sum_{m=1}^{n} g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right)U_j\right] \\
[\mathcal{M}U]_i &= \sum_{j=1}^{n}\partial_t\big(\nabla\phi^{-1}\big)_j\partial_j U_i + \sum_{j,k=1}^{3}\left[\left(\Gamma^i_{jk}\partial_t\big(\nabla\phi^{-1}\big)_k + (\partial_k\big(\nabla\phi^{-1}\big)_i)(\partial_j\partial_t(\nabla\phi)_k)\right)U_j\right] \\
[\mathcal{G}p]_i &= \sum_{j=1}^{3} g^{ij}\partial_j q,
\end{aligned}$$

where

$$g^{ij} := \nabla\phi^{-1}\left(\nabla\phi^{-1}\right)^T,$$

is the metric contravariant tensor,

$$g_{ij} := (\nabla\phi)^T \nabla\phi,$$

is the metric covariant tensor, and Christoffel's symbol

$$\Gamma_{ij}^k := \frac{1}{2}\sum_{l=1}^{3} g^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right).$$

Finally, we wish to quotient out the solid velocity in the boundary condition on Γ. To do this, note first that for $t \in (0,T)$ and $y \in \Gamma$, we have

$$\begin{aligned}
&\mathbb{D}\left(\Phi V\right)(t,y)\\
&= \nabla\left((\nabla\phi)^{-1}(t,y)V\left(t,\phi(t,y)\right)\right) + \left[\nabla\left((\nabla\phi)^{-1}(t,y)V\left(t,\phi(t,y)\right)\right)\right]^T\\
&= (\nabla\phi)^{-1}(t,y)\nabla\left(V\left(t,\phi(t,y)\right)\right) + \left[(\nabla\phi)^{-1}(t,y)\nabla\left(V\left(t,\phi(t,y)\right)\right)\right]^T\\
&= (\nabla\phi)^{-1}(t,y)\nabla V\left(t,\phi(t,y)\right)\nabla\phi(t,y)\\
&\quad + (\nabla\phi)^T(t,y)\left(\nabla V\right)^T\left(t,\phi(t,y)\right)(\nabla\phi)^{-T}(t,y)\\
&= (\nabla\phi)^T(t,y)\left(\nabla V\right)\left(t,\phi(t,y)\right)\left(\nabla\phi\right)(t,y) - (\nabla\phi)^T(t,y)\left(\nabla V\right)\left(t,\phi(t,y)\right)\left(\nabla\phi\right)(t,y)\\
&= 0.
\end{aligned}$$

Thus, setting $u(t,y) := U(t,y) - (\Phi b)(t,y)$ and noting that $b \equiv V$ on $\Gamma(t)$, we obtain that (v,q) is a strong solution to the moving domain problem if and only if (U, p) is a strong solution to:

$$
\begin{aligned}
\partial_t u - \Delta u + \nabla p = F + (\mathcal{L} - \Delta)u - \mathcal{M}u + (\nabla - \mathcal{G})p, \quad &\text{in } (0,T)\times\Omega \\
\text{div } u = 0, \quad &\text{in } (0,T)\times\Omega \\
u \cdot \mu = 0, \quad &\text{on } (0,T)\times\Gamma \\
[\mathbb{D}(u)\mu]_{\tau_\mu} + \alpha u = 0, \quad &\text{on } (0,T)\times\Gamma \\
u(0) = u_0, &
\end{aligned}
$$

where $F := f - \partial_t(\Phi b) - (\mathcal{M} - \mathcal{L})(\Phi b)$, $u_0 := U_0 - (\Phi b)(0)$. Note that since Φ is divergence preserving, we recover that u is solenoidal if and only if v is solenoidal.

5.3 Stationary problem

We aim to prove the well-posedness of the following problem:

$$
\begin{aligned}
-\Delta u + \nabla p = f, \quad &\text{in } \Omega \\
\text{div } u = 0, \quad &\text{in } \Omega \\
u \cdot \mu = 0, \quad &\text{on } \Gamma \\
[\mathbb{D}(u)\mu]_\tau + \alpha u = 0, \quad &\text{on } \Gamma.
\end{aligned}
$$

We first look into the existence of a weak solution, i.e., there exist $u \in H^1_{\sigma,\tau}(\Omega)$ and $p \in L^2_{loc}(\Omega)$ such that

$$
2\int_\Omega \mathbb{D}(u) : \mathbb{D}(\varphi) + 2\int_\Gamma \alpha u_\tau \cdot \varphi_\tau - \int_\Omega p \text{ div } \varphi = \int_\Omega f \cdot \varphi, \quad \forall \varphi \in H^1_\tau(\Omega).
$$

Equivalently, we can look for $u \in H^1_{\sigma,\tau}(\Omega)$ such that

$$2\int_\Omega \mathbb{D}(u) : \mathbb{D}(\varphi) + 2\int_\Gamma \alpha u_\tau \cdot \varphi_\tau = \int_\Omega f \cdot \varphi, \quad \forall \varphi \in H^1_{\sigma,\tau}(\Omega).$$

Korn's inequality applied to Ω guarantees the well-posedness of the above problem in $H^1_{\sigma,\tau}(\Omega) \times L^2_{loc}(\Omega)$, for instance see [1].

5.3.1 Regularity

The goal of this section is to prove that for sufficiently regular data to the stationary problem, we have that both the velocity and pressure have higher regularity. In particular, we have

Theorem 14. *Suppose $f \in L^2(\Omega)$ and $\alpha > 0$. Then, the weak solution (u,p) to the stationary Stokes problem belongs in $H^2(\Omega) \times H^1(\Omega)$.*

Proof. We break down the proof in several stages. First, note that the interior regularity is standard. We focus on the regularity up to the boundary. In this line, we introduce a change of coordinates that transforms portion of the domains into a domain with a flat boundary. Indeed, let $x_0 \in \Gamma$ and without loss of generality, we assume that $x_0 = 0$. By regularity of Γ, upon relabeling of axes, we may assume that

$$\Omega \cap B(x_0, r) = \{x = (x_1, x_2, x_3) \,|\, x_3 < H(x'),\, x' := (x_1, x_2)\},$$

for some $H \in C^3(\mathbb{R}^2; \mathbb{R})$. We let

$$\psi(x) := (x_1, x_2, x_3 - H(x')).$$

Thus,

$$\psi^{-1}(y) = (y_1, y_2, y_3 + H(y')).$$

We let $s > 0$ be small enough such that $B(0, s) \cap \mathbb{R}^3_+ \subset \Omega' := \psi\left(\Omega \cap B(0, r)\right)$. We let $V' := B\left(0, \frac{s}{2}\right) \cap \mathbb{R}^3_+$. We define:

$$\tilde{u}(y) := (\Psi u)(y) := \left(\nabla\psi^{-1}(y)\right)^{-1} u\left(\psi^{-1}(y)\right), \quad y \in \Omega'.$$

Note that:

$$\left(\Psi^{-1}\tilde{u}\right)(x) = \left(\nabla\psi^{-1}\right)(\psi(x))\,\tilde{u}\left(\psi(x)\right).$$

With these, we have that $u \cdot \mu = 0$ on Γ if and only if $\tilde{u} \cdot n = 0$ on $\Gamma' := \psi(\Gamma)$, where n is the outward unit normal on Γ'. Moreover, since $\det\nabla\psi^{-1} = 1$ in Ω', we have that $\mathrm{div}_y\tilde{u} = (\mathrm{div}_x u)\left(\psi^{-1}(y)\right) = 0$ in Ω'.

Now, let $(u, p) \in H^1_{\sigma,\tau}(\Omega) \times L^2_{loc}(\Omega)$ be the weak solution to the stationary Stokes problem, i.e., for all $\varphi \in H^1_\tau(\Omega)$:

$$\int_\Omega \mathbb{D}(u) : \mathbb{D}(\varphi) + \int_\Gamma \alpha u_\tau \cdot \varphi_\tau - \int_\Omega p \text{ div } \varphi = \int_\Omega F \cdot \varphi,$$

where $F := \frac{1}{2}f$.

5.3.1.1 Tangential regularity of the velocity

We apply a change of variables:

$$\int_\Omega \mathbb{D}(u) : \mathbb{D}(\varphi) = \int_{\psi^{-1}(\Omega')} \mathbb{D}(u) : \mathbb{D}(\varphi) = \int_{\Omega'} \mathbb{D}(u) \circ \psi^{-1} : \mathbb{D}(\varphi) \circ \psi^{-1}.$$

Let $\varphi' \in H^1_{\sigma,\tau}(\Omega')$ and define $\varphi := \Psi^{-1}\varphi'$. Then, $\varphi \in L^2(\Omega)$. Indeed,

$$\varphi = \psi^{-1}\varphi' = \left(\nabla\psi^{-1} \circ \psi\right)(\varphi' \circ \psi) = (\nabla\psi)^{-1}(\varphi' \circ \psi) \in L^2(\Omega),$$

since $\varphi' \in L^2(\Omega)$ and $(\nabla\psi)^{-1} \in L^\infty(\Omega)$. Moreover $\nabla\varphi \in L^2(\Omega)$ since,

$$\nabla\varphi = \nabla\left((\nabla\psi)^{-1}\right) \cdot (\varphi' \circ \psi) + (\nabla\psi)^{-1}\left((\nabla\varphi') \circ \psi\right)\nabla\psi \in L^2(\Omega),$$

since $\nabla\varphi' \in L^2(\Omega)$ and $\psi, \psi^{-1} \in C^3$. Thus, $\varphi \in H^1(\Omega)$.

Now, as $\Psi\varphi = \varphi'$, $\varphi' \cdot n = 0$ on Γ', and div $\varphi' = 0$ in Ω', we have that $\varphi \cdot n = 0$ on Γ and div $\varphi = 0$ in Ω. Hence, $\varphi \in H^1_{\sigma,\tau}(\Omega)$ and we use this as a test function. We now calculate:

$$\begin{aligned}\nabla\varphi \circ \psi^{-1} &= \left[\left(\nabla(\nabla\psi)^{-1}\right) \circ \psi^{-1}\right] \cdot \left[\varphi' \circ \left(\psi \circ \psi^{-1}\right)\right] \\ &\quad + \left[(\nabla\psi)^{-1} \circ \psi^{-1}\right]\left[\nabla\varphi' \circ \left(\psi \circ \psi^{-1}\right)\right]\left[\nabla\varphi \circ \psi^{-1}\right] \\ &= \left[\left(\nabla(\nabla\psi)^{-1}\right) \circ \psi^{-1}\right] \cdot \varphi' + \left[(\nabla\psi)^{-1} \circ \psi^{-1}\right]\left[\nabla\varphi'\right]\left[\nabla\varphi \circ \psi^{-1}\right].\end{aligned}$$

Now,

$$\begin{aligned}\left[\left(\nabla(\nabla\psi)^{-1}\right) \cdot (\varphi' \circ \psi)\right]_{ij} &= \left[\left(\left(\nabla\psi^{-1}\right) \circ \psi\right) \cdot (\varphi' \circ \psi)\right]_{ij} \\ &= \sum_{k=1}^{3} \partial_j \left[\left(\nabla\psi^{-1}\right)_{ik} \circ \psi\right]\left[\varphi'_k \circ \psi\right] \\ &= \sum_{k,l=1}^{3} \left(\partial_l\partial_k\psi_i^{-1} \circ \psi\right)\left(\partial_j\psi_l\right)\left(\varphi'_k \circ \psi\right),\end{aligned}$$

and,

$$\left[(\nabla\psi)^{-1} \circ \psi^{-1}\right]\left[\nabla\varphi'\right]\left[\nabla\varphi \circ \psi^{-1}\right] = \nabla\psi^{-1}\nabla\varphi'\nabla\psi^{-1}.$$

Thus,

$$\left[\nabla\varphi \circ \psi^{-1}\right]_{ij} = \sum_{k,l=1}^{3} \left(\partial_l \partial_k \psi_i^{-1}\right)\left(\partial_j \psi_l \circ \psi^{-1}\right)\left(\varphi_k'\right) + \left[\nabla\psi^{-1}\nabla\varphi'\nabla\psi^{-1}\right]_{ij}.$$

We simplify this term by considering ψ^{-1}. Recall that $\psi^{-1}(y) = (y_1, y_2, y_3 + H(y'))$. Thus,

$$\nabla\psi^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1 H & \partial_2 H & 1 \end{bmatrix}$$

$$\left(\nabla\psi^{-1}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\partial_1 H & -\partial_2 H & 1 \end{bmatrix}$$

We then have that $\partial_l \partial_k \psi_i^{-1} = \partial_l \partial_k H$ for $k, l = 1, 2$, $i = 3$, and is zero otherwise. Thus, for $i, j = 1, 2, 3$:

$$\sum_{k,l=1}^{3} \left(\partial_l \partial_k \psi_i^{-1}\right)\left(\partial_j \psi_l \circ \psi^{-1}\right)\left(\varphi_k'\right) = \sum_{k=1}^{2} \delta_{i3}(1 - \delta_{j3})(\partial_j \partial_k H)\varphi_k',$$

where we have also used that $\partial_j \psi_l = \delta_{lj}$ for $l \neq 3$.

For the other term, observe that

$$\nabla\psi^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1 H & \partial_2 H & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 H & \partial_2 H & 1 \end{bmatrix} =: I + \mathbb{H}.$$

Similarly,

$$\left(\nabla\psi^{-1}\right)^{-1} = I - \mathbb{H}.$$

Therefore,

$$\nabla\psi^{-1}\nabla\varphi'\nabla\psi^{-1} = \nabla\varphi -' \nabla\varphi'\mathbb{H} + \mathbb{H}\nabla\varphi' - \mathbb{H}\nabla\varphi'\mathbb{H}.$$

Hence, we have shown that for $\varphi' \in H^1_{\sigma,\tau}(\Omega')$, $\varphi := \Psi^{-1}\varphi'$, we have:

$$\begin{aligned}\left[\nabla\varphi \circ \psi^{-1}\right]_{ij} &= \sum_{k=1}^{2} \delta_{i3}(1-\delta_{j3})(\partial_j\partial_k H)\varphi'_k + \nabla\varphi' - \nabla\varphi'\mathbb{H} + \mathbb{H}\nabla\varphi' - \mathbb{H}\nabla\varphi'\mathbb{H} \\ &=: \mathbb{E}[\varphi'] + \nabla\varphi' + \mathbb{H}\nabla\varphi' -' \nabla\varphi'\mathbb{H} - \mathbb{H}\nabla\varphi'\mathbb{H}.\end{aligned}$$

We now obtain:

$$\int_\Omega \mathbb{D}(u) : \mathbb{D}(\varphi) = \int_{\Omega'} \left[\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right] : \left[\mathbb{D}(\varphi') + \mathbb{F}_1(\varphi') + \mathbb{F}_0(\varphi')\right],$$

where:

$$\begin{aligned}\mathbb{F}_1(\varphi') &:= \mathbb{A}_{sym}\left(\mathbb{H}\nabla\varphi' -' \nabla\varphi'\mathbb{H} - \mathbb{H}\nabla\varphi'\mathbb{H}\right) \\ \mathbb{F}_0(\varphi') &:= \mathbb{A}_{sym}\left(\mathbb{E}[\varphi']\right).\end{aligned}$$

We tranform the boudnary term:

$$\begin{aligned}
\int_\Gamma u_\tau \cdot \varphi_\tau &= \int_\Gamma u \cdot \varphi \\
&= \int_{\psi^{-1}(\Gamma')} \Psi^{-1}\tilde{u} \cdot \Psi^{-1}\varphi' \\
&= \int_{\psi^{-1}(\Gamma')} \left(\nabla\psi^{-1} \circ \psi\right)(\tilde{u} \circ \psi) \cdot \left(\nabla\psi^{-1} \circ \psi\right)(\varphi' \circ \psi) \\
&= \int_{\Gamma'} \left(\nabla\psi^{-1}\right)\tilde{u} \cdot \left(\nabla\psi^{-1}\right)\varphi' \left|\nabla\psi^{-1}e_3\right| \left|det\,\nabla\psi^{-1}\right| \\
&= \int_{\Gamma'} \left(\nabla\psi^{-1}\right)^T \left(\nabla\psi^{-1}\right)\tilde{u} \cdot \varphi' \\
&= \int_{\Gamma'} \left(\nabla\psi^{-1}\right)^T \left(\nabla\psi^{-1}\right)\tilde{u}_\tau \cdot \varphi'_\tau.
\end{aligned}$$

For the pressure term, we let $\tilde{p} := p \circ \psi^{-1}$. Then, $\tilde{p} \in L^2_{loc}(\Omega')$. Also, since $det\,\nabla\psi^{-1} = 1$, we have that $\text{div}_x\varphi \circ \psi^{-1} = \text{div}_y\varphi'$. Thus,

$$\int_\Omega p\,\text{div}\,\varphi = \int_{\Omega'} \left(p \circ \psi^{-1}\right)\left(\text{div}\,\varphi' \circ \psi^{-1}\right) = \int_{\Omega'} \tilde{p}\,\text{div}\,\varphi'.$$

Finally, we transform the force terms:

$$\int_\Omega F \cdot \varphi = \int_{\Omega'} \left(\nabla\psi^{-1}\right)^T \left(F \circ \psi^{-1}\right) \cdot \varphi' =: \int_\Omega F' \cdot \varphi'.$$

Therefore, $(\tilde{u}, \tilde{p})$ satisfies:

$$\begin{aligned}
&\int_{\Omega'} \left[\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right] : \left[\mathbb{D}(\varphi') + \mathbb{F}_1(\varphi') + \mathbb{F}_0(\varphi')\right] \\
&+ \int_{\Gamma'} \left(\nabla\psi^{-1}\right)^T \left(\nabla\psi^{-1}\right)\tilde{u}_\tau \cdot \varphi'_\tau - \int_{\Omega'} \tilde{p}\,\text{div}\,\varphi' = \int_\Omega F' \cdot \varphi',
\end{aligned}$$

for all $\varphi' \in H^1_\tau(\Omega')$.

We now make our estimates. Let $V' := B\left(0, \frac{s}{2}\right) \cap \mathbb{R}^3 \subset \Omega'$. Let $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $\zeta \equiv 1$ in V', $\zeta \equiv 0$ in $\mathbb{R}^3 \setminus B(0,s)$, and $0 \leq \zeta \leq 1$. Thus $\zeta \equiv 0$ on Γ'.

Let $h > 0$ and define $\varphi' := -D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)$, for $k = 1, 2$. Then $\varphi' \in H_\tau^1(\Omega')$. Here, we have

$$D_k^h f(x) := \frac{f(x + he_k) - f(x)}{h}.$$

We now use φ' as a test function. We first consider the diffusion term:

$$\int_{\Omega'} \left[\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right] : \left[\mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) + \mathbb{F}_1(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) + \mathbb{F}_0(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right))\right].$$

We are going to delve into the details of the estimates for the following term. For the other terms, they follow similarly. We sketch the proof in the appendix. Now, we have

$$\begin{aligned}
\int_{\Omega'} \mathbb{D}\tilde{u} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) &= -\int_{\Omega'} \mathbb{D}\tilde{u} : D_k^{-h}\mathbb{D}\left(\zeta^2 D_k^h \tilde{u}\right) \\
&= \int_{\Omega'} D_k^h \mathbb{D}\tilde{u} : \mathbb{D}\left(\zeta^2 D_k^h \tilde{u}\right) \\
&= \int_{\Omega'} D_k^h \mathbb{D}\tilde{u} : \left[\zeta^2 D_k^h \mathbb{D}\tilde{u} + 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right)\right] \\
&\geq \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - \frac{1}{2}\int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 \\
&\quad - C\left(\Omega', \nabla\zeta\right)\int_{\Omega'} |\nabla\tilde{u}|^2 \\
&= \frac{1}{2}\int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C\left(\Omega', \nabla\zeta\right)\int_{\Omega'} |\nabla\tilde{u}|^2.
\end{aligned}$$

Next we look at the term:

$$\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)).$$

Recall that $\mathbb{F}_1(\tilde{u}) = \mathbb{A}_{sym}\left(\mathbb{H}\nabla\tilde{u} - \nabla\tilde{u}\mathbb{H} - \mathbb{H}\nabla\tilde{u}\mathbb{H}\right)$. Moreover, as $\mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right))$ is symmetric, we have that

$$\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) = \int_{\Omega'} \left[\mathbb{H}\nabla\tilde{u} - \nabla\tilde{u}\mathbb{H} - \mathbb{H}\nabla\tilde{u}\mathbb{H}\right] : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)).$$

We work out the calculations for each term. First,

$$\begin{aligned}
&\int_{\Omega'} \mathbb{H}\nabla\tilde{u} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) \\
&= \int_{\Omega'} D_k^h\left(\mathbb{H}\nabla\tilde{u}\right) : \mathbb{D}\left(\zeta^2 D_k^h \tilde{u}\right) \\
&= \int_{\Omega'} \left[\mathbb{H}^h D_k^h \nabla\tilde{u} + \left(D_k^h \mathbb{H}\right)\nabla\tilde{u}\right] : \left[\zeta^2 D_k^h \mathbb{D}\tilde{u} + 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right)\right].
\end{aligned}$$

Since,

$$\begin{aligned}
&\int_{\Omega'} \zeta \mathbb{H}^h D_k^h \nabla\tilde{u} : \zeta D_k^h \mathbb{D}\tilde{u} \geq -\frac{\varepsilon}{2}\int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - \frac{C}{\varepsilon}\|\mathbb{H}\|_{C^k}^2 \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2, \\
&\int_{\Omega'} \mathbb{H}^h D_k^h \nabla\tilde{u} : 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right) \geq -\varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C\left(\varepsilon', \Omega', \mathbb{H}\right)\int_{\Omega'} |\nabla\tilde{u}|^2, \\
&\int_{\Omega'} \zeta\left(D_k^h \mathbb{H}\right)\nabla\tilde{u} : \zeta D_k^h \mathbb{D}\tilde{u} \geq -\frac{\varepsilon}{2}\int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C(\varepsilon)\|\mathbb{H}\|_{C^k}^2 \int_{\Omega'} |\nabla\tilde{u}|^2, \\
&\int_{\Omega'} 2\zeta\left(D_k^h \mathbb{H}\right)\nabla\tilde{u} : \zeta D_k^h \mathbb{D}\tilde{u} : \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right) \geq -C\left(\Omega', \nabla\zeta\right)\|\mathbb{H}\|_{C^k}^2 \int_{\Omega'} |\nabla\tilde{u}|^2,
\end{aligned}$$

we have that

$$\begin{aligned}
&\int_{\Omega'} \mathbb{H}\nabla\tilde{u} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) \\
&\geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - \|\mathbb{H}\|_{C^k}^2 \left(C(\varepsilon) - \varepsilon'\right) \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 \\
&\quad - C\left(\Omega,' \nabla\zeta, \mathbb{H}\right) \int_{\Omega'} |\nabla\tilde{u}|^2 .
\end{aligned}$$

For the next term, we have

$$\begin{aligned}
&\int_{\Omega'} \nabla\tilde{u}\,\mathbb{H} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) \\
&= \int_{\Omega'} D_k^h \left(\nabla\tilde{u}\mathbb{H}\right) : \mathbb{D}\left(\zeta^2 D_k^h \tilde{u}\right) \\
&= \int_{\Omega'} \left[(\nabla\tilde{u})^h D_k^h \mathbb{H} + \left(D_k^h \nabla\tilde{u}\right)\mathbb{H}\right] : \left[\zeta^2 D_k^h \mathbb{D}\tilde{u} + 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right)\right] .
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{\Omega'} (\nabla\tilde{u})^h D_k^h \mathbb{H} : \zeta^2 D_k^h \nabla\tilde{u} &\geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C(\varepsilon, \mathbb{H}) \int_{\Omega'} |\nabla\tilde{u}|^2 \\
\int_{\Omega'} \left(D_k^h \nabla\tilde{u}\right)\mathbb{H} : \zeta^2 D_k^h \nabla\tilde{u} &\geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C(\varepsilon)\|\mathbb{H}\|_{C^k} \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 \\
\int_{\Omega'} \left(D_k^h \nabla\tilde{u}\right)\mathbb{H} :: 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right) &\geq -\varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C(\varepsilon', \Omega', \mathbb{H}) \int_{\Omega'} |\nabla\tilde{u}|^2 \\
\int_{\Omega'} (\nabla\tilde{u})^h D_k^h \mathbb{H} : 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right) &\geq -C(\mathbb{H}, \Omega') \int_{\Omega'} |\nabla\tilde{u}|^2 .
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\Omega'} \nabla\tilde{u}\,\mathbb{H} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) \\
&\geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - (\varepsilon' + C(\varepsilon)\|\mathbb{H}\|_{C^k}) \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 \\
&\quad - C(\varepsilon, \varepsilon', \Omega, \mathbb{H}) \int_{\Omega'} |\nabla\tilde{u}|^2 .
\end{aligned}$$

As for the last term, we have

$$\begin{aligned}
\int_{\Omega'} \mathbb{H}\nabla\tilde{u}\mathbb{H} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) &= \int_{\Omega'} D_k^h \left(\mathbb{H}\nabla\tilde{u}\mathbb{H}\right) : \mathbb{D}\left(\zeta^2 D_k^h \tilde{u}\right) \\
&= \int_{\Omega'} \left[\left(\mathbb{H}\nabla\tilde{u}\right)^h D_k^h \mathbb{H} + \left(\mathbb{H}^h D_k^h \nabla\tilde{u} + \left(D_k^h \mathbb{H}\right)\nabla\tilde{u}\right)\mathbb{H}\right] \\
&\qquad : \left[\zeta^2 D_k^h \mathbb{D}\tilde{u} + 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right)\right].
\end{aligned}$$

We make estimates for each term. Indeed,

$$\begin{aligned}
&\int_{\Omega'} \left[\left(\mathbb{H}\nabla\tilde{u}\right)^h D_k^h \mathbb{H} + \left(D_k^h \mathbb{H}\right)\nabla\tilde{u}\right] : \zeta^2 D_k^h \mathbb{D}\tilde{u} \geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C(\varepsilon, \mathbb{H}) \int_{\Omega'} |\nabla\tilde{u}|^2 \\
&\int_{\Omega'} \mathbb{H}^h D_k^h \nabla\tilde{u}\mathbb{H} : \zeta^2 D_k^h \mathbb{D}\tilde{u} \geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C(\varepsilon)\|\mathbb{H}\|_{C^k}^2 \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 \\
&\int_{\Omega'} \mathbb{H}^h D_k^h \nabla\tilde{u}\mathbb{H} : 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right) \geq -\varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C(\varepsilon', \Omega', \mathbb{H}) \int_{\Omega'} |\nabla\tilde{u}|^2 \\
&\int_{\Omega'} \left[\left(\mathbb{H}\nabla\tilde{u}\right)^h D_k^h \mathbb{H} + \left(D_k^h \mathbb{H}\right)\nabla\tilde{u}\right] : 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right) \geq -C(\mathbb{H}, \Omega') \int_{\Omega'} |\nabla\tilde{u}|^2 .
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\Omega'} \mathbb{H}\nabla\tilde{u}\,\mathbb{H} : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)) \\
&\geq -\varepsilon\int_{\Omega'}\zeta^2\left|D_k^h\mathbb{D}\tilde{u}\right|^2 - (\varepsilon' + C(\varepsilon)\|\mathbb{H}\|_{C^k})\int_{\Omega'}\zeta^2\left|D_k^h\nabla\tilde{u}\right|^2 \\
&\quad - C(\varepsilon,\varepsilon',\Omega,\mathbb{H})\int_{\Omega'}|\nabla\tilde{u}|^2 .
\end{aligned}$$

Combining these estimates, we obtain

$$\begin{aligned}
&\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)) \\
&\geq -\varepsilon\int_{\Omega'}\zeta^2\left|D_k^h\mathbb{D}\tilde{u}\right|^2 - \|\mathbb{H}\|_{C^k} C\left(\Omega',\varepsilon,\varepsilon'\right)\int_{\Omega'}\zeta^2\left|D_k^h\nabla\tilde{u}\right|^2 \\
&\quad - C\left(\Omega,'\nabla\zeta,\mathbb{H}\right)\int_{\Omega'}|\nabla\tilde{u}|^2 .
\end{aligned}$$

Note that we can choose $r > 0$ to be small enough so that $\|\mathbb{H}\|_{C^k} \leq 1$. The sketch of the details for the estimates on the remaining diffusion terms are given in the appendix. Combining all these, we finally obtain the following estimate for the diffusion term

$$\begin{aligned}
&\int_{\Omega'}\left[\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right] : \left[\mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)) + \mathbb{F}_1(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)) + \mathbb{F}_0(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right))\right] \\
&\geq \left(\frac{1}{2} - \varepsilon\right)\int_{\Omega'}\zeta^2\left|D_k^h\mathbb{D}\tilde{u}\right|^2 - \left(\varepsilon' + C(\Omega',\mathbb{H},\varepsilon)\|\mathbb{H}\|_{C^k}^2\right)\int_{\Omega'}\zeta^2\left|D_k^h\nabla\tilde{u}\right|^2 - C(\varepsilon,\varepsilon',\mathbb{H},\Omega')\|\tilde{u}\|_{H^1(\Omega')}^2 .
\end{aligned}$$

We now estimate the boundary term. We let $\mathbb{T} := \left(\nabla\psi^{-1}\right)^T\left(\nabla\psi^{-1}\right)$. Then,

$$\begin{aligned}
\int_{\Gamma'} \mathbb{T}\tilde{u}_\tau \cdot \left[-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right] &= \int_{\Gamma'} D_k^h\left[\mathbb{T}\tilde{u}_\tau\right]\cdot\left[\zeta^2 D_k^h\tilde{u}\right] \\
&= \int_{\Gamma'} \left[\mathbb{T}^h D_k^h\tilde{u}_\tau + D_k^h\mathbb{T}\tilde{u}_\tau\right]\cdot\left[\zeta^2 D_k^h\tilde{u}\right] \\
&= \int_{\Gamma'} (\mathbb{T}-I)D_k^h\tilde{u}_\tau\cdot\zeta^2 D_k^h\tilde{u}_\tau + \int_{\Gamma'}\zeta^2\left|D_k^h\tilde{u}_\tau\right|^2 \\
&\quad + \int_{\Gamma'} D_k^h\mathbb{T}\tilde{u}_\tau\cdot\zeta^2 D_k^h\tilde{u}_\tau \\
&\geq \left(\frac{1}{2} - C\|\mathbb{H}\|_{C^k}\right)\int_{\Gamma'}\zeta^2\left|D_k^h\tilde{u}_\tau\right|^2 - C(\mathbb{H})\int_{\Gamma'}\left|\tilde{u}_\tau\right|^2.
\end{aligned}$$

Thus, as $\int_{\Gamma'}\left|\tilde{u}_\tau\right|^2 \leq C(\Omega')\|\tilde{u}\|_{H^1(\Omega')}$, we have:

$$\begin{aligned}
\int_{\Gamma'}\left(\nabla\psi^{-1}\right)^T\left(\nabla\psi^{-1}\right)\tilde{u}_\tau\cdot\left[-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)\right] \geq \left(\frac{1}{2} - C\|\mathbb{H}\|_{C^k}\right)\int_{\Gamma'}\zeta^2\left|D_k^h\tilde{u}_\tau\right|^2 \\
- C(\mathbb{H},\Omega')\|\tilde{u}\|_{H^1(\Omega')}.
\end{aligned}$$

We move on to the pressure term:

$$\begin{aligned}
\int_{\Omega'}\tilde{p}\,\mathrm{div}\left(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)\right) &= -\int_{\Omega'}\tilde{p}D_k^{-h}\left(2\zeta\nabla\zeta\cdot D_k^h\tilde{u} + \zeta^2\mathrm{div}\left(D_k^h\tilde{u}\right)\right) \\
&= -\int_{\Omega'}\tilde{p}D_k^{-h}\left(2\zeta\nabla\zeta\cdot D_k^h\tilde{u}\right),
\end{aligned}$$

since $\mathrm{div}\left(D_k^h\tilde{u}\right) = D_k^h\,\mathrm{div}\;\tilde{u} = 0$ in Ω'. Now, since

$$D_k^h\left((2\zeta\nabla\zeta)^- h\cdot\tilde{u}\right) = 2\zeta\nabla\zeta\cdot D_k^h\tilde{u} + D_k^h\left((2\zeta\nabla\zeta)^{-h}\right)\cdot\tilde{u} = 2\zeta\nabla\zeta\cdot D_k^h\tilde{u} + D_k^{-h}\left(2\zeta\nabla\zeta\right)\cdot\tilde{u},$$

we then have

$$-\int_{\Omega'} \tilde{p} D_k^{-h} \left(2\zeta\nabla\zeta \cdot D_k^h \tilde{u}\right) = -\int_{\Omega'} \tilde{p} D_k^{-h} \left[D_k^h \left((2\zeta\nabla\zeta)^{-}h \cdot \tilde{u}\right) - D_k^{-h} \left(2\zeta\nabla\zeta\right) \cdot \tilde{u}\right].$$

We now make some estimates. First,

$$-\int_{\Omega'} \tilde{p} D_k^{-h} D_k^h \left((2\zeta\nabla\zeta)^{-}h \cdot \tilde{u}\right) \geq -\tilde{\varepsilon}' \int_{\Omega'} \left|D_k^{-h} D_k^h \left((2\zeta\nabla\zeta)^{-}h \cdot \tilde{u}\right)\right|^2 - C(\tilde{\varepsilon}') \int_{\Omega'} \tilde{p}^2.$$

Also,

$$\begin{aligned}
\int_{\Omega'} \left|D_k^{-h} D_k^h \left((2\zeta\nabla\zeta)^{-}h \cdot \tilde{u}\right)\right|^2 &\leq C(\Omega') \int_{\Omega'} \left|\nabla D_k^h \left((2\zeta\nabla\zeta)^{-h} \cdot \tilde{u}\right)\right|^2 \\
&= C(\Omega') \int_{\Omega'} \left|D_k^h \left(\nabla \left((2\zeta\nabla\zeta)^{-h}\right)^T \tilde{u} + (\tilde{u})^T (2\zeta\nabla\zeta)^{-h}\right)\right|^2 \\
&\leq C(\Omega') \left(\int_{\Omega'} |\nabla \tilde{u}|^2 + \int_{\Omega'} \zeta^2 \left|D_k^h \nabla \tilde{u}\right|^2\right).
\end{aligned}$$

Combining these, we obtain

$$\left|\int_{\Omega'} \tilde{p} \operatorname{div}\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right)\right| \leq \varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla \tilde{u}\right|^2 + C(\Omega') \int_{\Omega'} \tilde{p}^2 + C(\Omega') \|\tilde{u}\|_{H^1(\Omega')}^2.$$

Now, for the force term, we have

$$\int_{\Omega'} F' \cdot D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right) \leq C(\tilde{\varepsilon}') \int_{\Omega'} |F'|^2 + \tilde{\varepsilon}' \int_{\Omega'} \left|D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right|^2.$$

We estimate the second term as

$$\int_{\Omega'} \left|D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right|^2 \leq C(\Omega') \int_{\Omega'} \left|\nabla\left(\zeta^2 D_k^h \tilde{u}\right)\right|^2$$
$$\leq C(\Omega') \int_{\Omega'} \left(\zeta^4 \left|\nabla D_k^h \tilde{u}\right|^2 + (2\zeta)^2 D_k^h \tilde{u} \otimes \nabla\zeta\right).$$

Since, $0 \leq \zeta \leq 1$, we have that $\zeta^4 \leq \zeta^2$, and so

$$\int_{\Omega'} F' \cdot D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right) \leq \varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 + C(\varepsilon') \int_{\Omega'} |F'|^2 + C(\Omega') \int_{\Omega'} |\nabla\tilde{u}|^2 .$$

We now combine these estimates:

$$\left(\frac{1}{2} - \varepsilon\right) \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - \left(\varepsilon' + C(\Omega', \varepsilon)\|\mathbb{H}\|_{C^k}^2\right) \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C(\Omega', \mathbb{H}, \varepsilon, \varepsilon')\|\tilde{u}\|_{H^1(\Omega')}^2$$
$$+ \left(\frac{1}{2} - C\|\mathbb{H}\|_{C^k}\right) \int_{\Gamma'} \zeta^2 \left|D_k^h \tilde{u}_\tau\right|^2 - C(\Omega', \mathbb{H})\|\tilde{u}\|_{H^1(\Omega')}^2 - \varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C \int_{\Omega'} \tilde{p}^2$$
$$\leq \int_{\Omega'} \left[\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right] : \left[\mathbb{D}(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) + \mathbb{F}_1(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)) + \mathbb{F}_0(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right))\right]$$
$$+ \int_{\Gamma'} \left(\nabla\psi^{-1}\right)^T \left(\nabla\psi^{-1}\right) \tilde{u}_\tau \cdot \left[-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right] + \int_{\Omega'} \tilde{p}\, \mathrm{div}\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right)$$
$$= \int_{\Omega'} F' \cdot D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)$$
$$\leq \varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 + C(\varepsilon') \int_{\Omega'} |F'|^2 + C(\Omega') \int_{\Omega'} |\nabla\tilde{u}|^2 .$$

Thus, we have, up to multiples of ε and ε':

$$\left(\frac{1}{2} - \varepsilon\right) \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - \left(\varepsilon' + C(\Omega', \varepsilon)\|\mathbb{H}\|_{C^k}^2\right) \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C(\Omega', \mathbb{H}, \varepsilon, \varepsilon')\|\tilde{u}\|_{H^1(\Omega')}^2$$
$$\leq C\left(\int_{\Omega'} |F'|^2 + \int_{\Omega'} \tilde{p}^2 + \|\tilde{u}\|_{H^1(\Omega')}^2\right).$$

Now, note that

$$\int_{\Omega'} \zeta^2 \left|\mathbb{D}D_k^h\tilde{u}\right|^2 = \int_{B(0,s)^+} \zeta^2 \left|\mathbb{D}D_k^h\tilde{u}\right|^2,$$

and

$$\int_{\Omega'} \zeta^2 \left|\nabla D_k^h\tilde{u}\right|^2 = \int_{B(0,s)^+} \zeta^2 \left|\nabla D_k^h\tilde{u}\right|^2.$$

Also,

$$\begin{aligned}\|\zeta\nabla D_k^h\tilde{u}\|_{L^2(B(0,s)^+)} &= \|\nabla(\zeta D_k^h\tilde{u}) - D_k^h\tilde{u}\otimes\nabla\zeta\|_{L^2(B(0,s)^+)}\\ &\leq \|\nabla(\zeta D_k^h\tilde{u})\|_{L^2(B(0,s)^+)} + \|D_k^h\tilde{u}\otimes\nabla\zeta\|_{L^2(B(0,s)^+)},\end{aligned}$$

and

$$\begin{aligned}\|\zeta\mathbb{D}D_k^h\tilde{u}\|_{L^2(B(0,s)^+)} &= \|\mathbb{D}(\zeta D_k^h\tilde{u}) - \mathbb{A}_{sym}\left(D_k^h\tilde{u}\otimes\nabla\zeta\right)\|_{L^2(B(0,s)^+)}\\ &\leq \|\mathbb{D}(\zeta D_k^h\tilde{u})\|_{L^2(B(0,s)^+)} - \|\mathbb{A}_{sym}\left(D_k^h\tilde{u}\otimes\nabla\zeta\right)\|_{L^2(B(0,s)^+)}.\end{aligned}$$

Combining these with our estimate, we obtain

$$\begin{aligned}&\left(\frac{1}{2}-\varepsilon\right)\|\mathbb{D}(\zeta D_k^h\tilde{u})\|_{L^2(B(0,s)^+)}^2 - (\varepsilon' + C(\Omega',\varepsilon)\|\mathbb{H}\|_{C^k}))\|\nabla(\zeta D_k^h\tilde{u})\|_{L^2(B(0,s)^+)}^2\\ &\left(\frac{1}{2} - C(\Omega')\|\mathbb{H}\|_{C^k}\right)\|\zeta D_k^h\tilde{u}_\tau\|_{L^2(\Gamma')}^2\\ &\leq C\left(\|F'\|_{L^2(\Omega')}^2 + \|\tilde{p}\|_{L^2(\Omega')}^2 + \|D_k^h\tilde{u}\|_{L^2(\Omega')} + \|\tilde{u}\|_{H^1(\Omega')}^2\right)\\ &\leq C\left(\|F'\|_{L^2(\Omega')}^2 + \|\tilde{p}\|_{L^2(\Omega')}^2 + \|\tilde{u}\|_{H^1(\Omega')}^2\right).\end{aligned}$$

In order to compare the gradient terms with the strain terms, we need Korn's inequality in the following form:

$$
\begin{aligned}
\int_{B(0,s)^+} \left|\nabla\left(\zeta D_k^h \tilde{u}\right)(x)\right|^2 dx &= s^{-3}\int_{B(0,1)^+} \left|\left(\nabla\left(\zeta D_k^h \tilde{u}\right)\right)(sy)\right|^2 dy \\
&= s^{-3-2}\int_{B(0,1)^+} \left|\nabla\left(\left(\zeta D_k^h \tilde{u}\right)(s\cdot)\right)(y)\right|^2 dy \\
&\leq C_{K(B_1^+)} s^{-3-2}\Bigg[\int_{B(0,1)^+} \left|\mathbb{D}\left(\left(\zeta D_k^h \tilde{u}\right)(s\cdot)\right)(y)\right|^2 dy \\
&\qquad + \int_{B(0,1)^+} \left|\left(\zeta D_k^h \tilde{u}\right)(sy)\right|^2 dy\Bigg] \\
&\leq C_{K(B_1^+)}\Bigg[\int_{B(0,s)^+} \left|\mathbb{D}\left(\zeta D_k^h \tilde{u}\right)(x)\right|^2 dx \\
&\qquad + \frac{1}{s^2}\int_{B(0,s)^+} \left|\left(\zeta D_k^h \tilde{u}\right)(x)\right|^2 dx\Bigg],
\end{aligned}
$$

where $C_{K(B_1^+)}$ is the Korn's constant in the upper unit half ball. Thus,

$$
\begin{aligned}
&\left[\left(\frac{1}{2}-\varepsilon\right)C_{K(B_1^+)}^{-1} - (\varepsilon' + C(\varepsilon,\Omega')\|\mathbb{H}\|_{C^k})\right]\|\nabla(\zeta D_k^h \tilde{u})\|_{L^2(\Omega')}^2 \\
&+\left(\frac{1}{2} - C(\Omega')\|\mathbb{H}\|_{C^k}\right)\|\zeta D_k^h \tilde{u}_\tau\|_{L^2(\Gamma')}^2 \\
&\leq C\left(\|F'\|_{L^2(\Omega')}^2 + \|\tilde{p}\|_{L^2(\Omega')}^2 + \|\tilde{u}\|_{H^1(\Omega')}^2\right).
\end{aligned}
$$

Hence, by choosing $\varepsilon > 0$ small enough and then choosing $r > 0$ small enough, we obtain

$$
\|D_k^h \nabla \tilde{u}\|_{L^2(V')} \leq \|\nabla\left(\zeta D_k^h \tilde{u}\right)\|_{L^2(\Omega')} \leq C\left(\|F'\|_{L^2(\Omega')}^2 + \|\tilde{p}\|_{L^2(\Omega')}^2\right),
$$

i.e., $\dfrac{\partial^2 \tilde{u}_i}{\partial_j \partial_k} \in L^2(V')$ for $i,j = 1,2,3$ and $k = 1,2$.

5.3.1.2 H^1 regularity of the pressure in the tangential directions

Let $\zeta \in C_0^\infty(V')$. Thus, $\partial_k \zeta \in H^1_\tau(\Omega')$ for $k = 1, 2$. Note that $\partial_k\zeta \equiv 0$ on Γ'. We then use $\partial_k\zeta$ as a test function to get

$$
\begin{aligned}
&-\int_{V'} \tilde{p}\, \mathrm{div}\,(\partial_k \zeta) \\
&= \int_{V'} F' \cdot \partial_k \zeta - \int_{\Omega'} [\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})] : [\mathbb{D}\partial_k\zeta + \mathbb{F}_1(\partial_k\zeta) + \mathbb{F}_0(\partial_k\zeta)] \\
&= \langle -\partial_k F', \zeta \rangle_{H^{-1}(\Omega'), H^1_0(\Omega)} \\
&\quad - \int_{V'} \mathbb{A}_{sym} \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0\left(\tilde{u}\right) \right] : \mathbb{A}_{sym} \left[\left(\nabla\psi^{-1}\right) \nabla\partial_k\zeta \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0\left(\partial_k\zeta\right) \right].
\end{aligned}
$$

Observe that:

$$
\begin{aligned}
\left(\nabla\psi^{-1}\right) \nabla\partial_k\zeta \left(\nabla\psi^{-1}\right)^{-1} &= \partial_k \left[\left(\nabla\psi^{-1}\right) \nabla\zeta \left(\nabla\psi^{-1}\right)^{-1} \right] - \left(\partial_k \left(\nabla\psi^{-1}\right)\right) \nabla\zeta \left(\nabla\psi^{-1}\right) \\
&\quad - \left(\nabla\psi^{-1}\right) \nabla\zeta \left(\partial_k \left(\nabla\psi^{-1}\right)\right) \\
&=: \partial_k \left[\left(\nabla\psi^{-1}\right) \nabla\zeta \left(\nabla\psi^{-1}\right)^{-1} \right] - \mathbb{G}_1(\zeta),
\end{aligned}
$$

where $\mathbb{G}_1(\zeta)$ is first-order in terms of derivatives of ζ. Thus,

$$
\begin{aligned}
&-\int_{V'} \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0\left(\tilde{u}\right) \right] : \left[\left(\nabla\psi^{-1}\right) \nabla\partial_k\zeta \left(\nabla\psi^{-1}\right)^{-1} \right] \\
&= -\int_{V'} \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0\left(\tilde{u}\right) \right] : \left[\partial_k \left(\left(\nabla\psi^{-1}\right) \nabla\zeta \left(\nabla\psi^{-1}\right)^{-1} \right) - \mathbb{G}_1(\zeta) \right] \\
&= \int_{V'} \partial_k \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0\left(\tilde{u}\right) \right] : \left(\nabla\psi^{-1}\right) \nabla\zeta \left(\nabla\psi^{-1}\right)^{-1} \\
&\quad + \int_{V'} \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0\left(\tilde{u}\right) \right] : \mathbb{G}_1(\zeta).
\end{aligned}
$$

Now $\partial_k \left[(\nabla\psi^{-1}) \nabla\tilde{u} (\nabla\psi^{-1})^{-1} + \mathbb{F}_0(\tilde{u}) \right] \in L^2(V')$ by the tangential regularity of $\tilde{u}$. Also, $(\nabla\psi^{-1}) \nabla\tilde{u} (\nabla\psi^{-1})^{-1} + \mathbb{F}_0(\tilde{u}) \in L^2(V')$ as well since $\tilde{u} \in H^1(\Omega')$. Thus the functional, $\mathcal{F}_1(\tilde{u})$, defined by

$$\begin{aligned}&\langle \mathcal{F}_1(\tilde{u}), \varphi \rangle_{H^{-1}(\Omega'), H_0^1(\Omega')} \\ &:= \int_{V'} \mathbb{A}_{sym} \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0(\tilde{u}) \right] : \mathbb{A}_{sym} \left[\left(\nabla\psi^{-1}\right) \nabla\partial_k\varphi \left(\nabla\psi^{-1}\right)^{-1} \right],\end{aligned}$$

for $\varphi \in H_0^1(\Omega')$, is in $H^{-1}(\Omega')$. Similarly,

$$\langle \mathcal{F}_2(\tilde{u}), \varphi \rangle_{H^{-1}(\Omega'), H_0^1(\Omega')} := \int_{V'} \mathbb{A}_{sym} \left[\left(\nabla\psi^{-1}\right) \nabla\tilde{u} \left(\nabla\psi^{-1}\right)^{-1} + \mathbb{F}_0(\tilde{u}) \right] : \mathbb{A}_{sym} \left[\mathbb{F}_0(\partial_k\varphi) \right],$$

is well-defined for $\varphi \in H_0^1(\Omega')$. Lastly,

$$-\int_{V'} \tilde{p} \operatorname{div}(\partial_k\zeta) = \langle \nabla\tilde{p}, \partial_k\zeta \rangle_{H^{-1}(\Omega'), H_0^1(\Omega)} = -\langle \partial_k(\nabla\tilde{p}), \zeta \rangle_{H^{-1}(\Omega'), H^1(\Omega)}.$$

Thus, we have that

$$\langle \partial_k(\nabla\tilde{p}), \zeta \rangle_{H^{-1}(\Omega'), H_0^1(\Omega')} = \left\langle \partial_k F' - \sum_{i=1}^{2} \mathcal{F}_i(\tilde{u}), \zeta \right\rangle_{H^{-1}(\Omega'), H^1(\Omega')},$$

and so $\partial_k(\nabla\tilde{p}) \in H^{-1}(\Omega')$ for $k = 1, 2$. As $\nabla\tilde{p} \in H^{-1}(\Omega')$, by Necas' lemma, we have that $\partial_k\tilde{p} \in L^2(V')$ for $k = 1, 2$.

5.3.1.3 Normal regularity of the velocity

Since $\tilde{u}$ is solenoidal, we have

$$\sum_{i=1}^{3} \partial_i\tilde{u}_i = 0,$$

so that

$$\partial_3 \tilde{u}_3 = -\sum_{i=1}^{2} \partial_i \tilde{u}_i.$$

Differentiating, we get

$$\frac{\partial^2 \tilde{u}_3}{\partial\, x_3^2} = \sum_{i=1}^{2} \partial_3 \partial_i \tilde{u}_i \;\in L^2(V'),$$

by the results of the previous section. To obtain the full H^2-regularity of $\tilde{u}$, we need to find out what equation does it solve. First, for $\varphi \in C_0^\infty(\Omega')$ we have that

$$\int_{\Omega'} [\mathbb{D}\tilde{u} + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})] : [\mathbb{D}\varphi + \mathbb{F}_1(\varphi) + \mathbb{F}_0(\varphi)] - \int_{\Omega'} \tilde{p}\,\mathrm{div}\;\varphi = \int_{\Omega'} F' \cdot \varphi.$$

We look into each of the terms. The pressure term can be written as

$$-\int_{\Omega'} \tilde{p}\,\mathrm{div}\,\varphi = \langle \nabla \tilde{p}, \varphi \rangle_{H^{-1}.(\Omega'),H_0^1(\Omega')}.$$

As for the diffusion term, we begin with

$$\begin{aligned}\int_{\Omega'} [\mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})] : \mathbb{D}(\varphi) &= \int_{\Omega'} [\mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})] : \nabla\varphi \\ &= \langle -\mathrm{div}\,(\mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u}))\,, \varphi \rangle_{H^{-1}(\Omega'),H_0^1(\Omega')}\,,\end{aligned}$$

where the last line is because φ has compact support in Ω' so the boundary terms vanish. The last expression is then equal to:

$$\begin{aligned}
&\left\langle -\operatorname{div}\left(\mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right), \varphi\right\rangle_{H^{-1}(\Omega'),H_0^1(\Omega')} \\
&= \left\langle -\frac{1}{2}\Delta\tilde{u} - \operatorname{div}\left(\mathbb{A}_{sym}\left(\mathbb{H}\nabla\tilde{u} - \nabla\tilde{u}\mathbb{H} - \mathbb{H}\nabla\tilde{u}\mathbb{H}\right)\right) - \operatorname{div}\left(\mathbb{A}_{sym}(\mathbb{E}(\tilde{u}))\right), \varphi\right\rangle_{H^{-1}(\Omega'),H_0^1(\Omega')} \\
&= \left\langle -\frac{1}{2}\Delta\tilde{u} - \operatorname{div}\left(\mathbb{A}_{sym}\left(\mathbb{H}\nabla\tilde{u} - \nabla\tilde{u}\mathbb{H} - \mathbb{H}\nabla\tilde{u}\mathbb{H}\right)\right) - \operatorname{div}\left(\mathbb{E}(\tilde{u})\right), \varphi\right\rangle_{H^{-1}(\Omega'),H_0^1(\Omega')},
\end{aligned}$$

where the last line is because for a matrix $\mathbb{T}$, we have

$$\operatorname{div}\mathbb{A}_{sym}(\mathbb{T}) = \frac{1}{2}\sum_{i,j}\partial_j\left(T_{ij} + T_{ji}\right)e_i + \sum_{i,j}\partial_j\left(T_{ij}\right)e_i.$$

Next, we let $\mathbb{D}_{\mathbb{H}}(\tilde{u}) := \mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})$. Then,

$$\begin{aligned}
&\int_{\Omega'}\left[\mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})\right] : \mathbb{F}_1(\varphi) \\
&= \int_{\Omega'}\mathbb{D}_{\mathbb{H}}(\tilde{u}) : \left[\mathbb{H}\nabla\tilde{u} - \nabla\tilde{u}\mathbb{H} - \mathbb{H}\nabla\tilde{u}\mathbb{H}\right] \\
&= \int_{\Omega'}\left[\mathbb{H}^T\mathbb{D}(\tilde{u}) - \mathbb{D}(\tilde{u})\mathbb{H}^T - \mathbb{H}^T\mathbb{D}(\tilde{u})\mathbb{H}^T\right] : \nabla\varphi \\
&= -\left\langle \operatorname{div}\left(\mathbb{H}^T\mathbb{D}(\tilde{u}) - \mathbb{D}(\tilde{u})\mathbb{H}^T - \mathbb{H}^T\mathbb{D}(\tilde{u})\mathbb{H}^T\right), \varphi\right\rangle_{H^{-1}(\Omega'),H_0^1(\Omega')}.
\end{aligned}$$

Lastly,

$$\begin{aligned}\int_{\Omega'} [\mathbb{D}(\tilde{u}) + \mathbb{F}_1(\tilde{u}) + \mathbb{F}_0(\tilde{u})] : \mathbb{F}_0(\varphi) &= \int_{\Omega'} \mathbb{D}_{\mathbb{H}}(\tilde{u}) : \mathbb{E}(\tilde{\varphi}) \\ &= \sum_{i,j} \sum_{l=1}^{2} \int_{\Omega'} [\mathbb{D}_{\mathbb{H}}(\tilde{u})]_{ij}\, \delta_{i3}(1-\delta_{j3})(\partial_j \partial_l H)\varphi_l \\ &= \sum_{l=1}^{3} \int_{\Omega'} \left(\sum_{i,j} [\mathbb{D}_{\mathbb{H}}(\tilde{u})]_{ij}\, \delta_{i3}(1-\delta_{j3})(\partial_j \partial_l H) \right) \varphi_l \\ &=: \int_{\Omega'} \mathbb{J}(\tilde{u}) \cdot \varphi.\end{aligned}$$

Thus,

$$\begin{aligned}&\Big\langle -\frac{1}{2}\Delta\tilde{u} - \operatorname{div}(\mathbb{H}\nabla\tilde{u} - \nabla\tilde{u}\mathbb{H} - \mathbb{H}\nabla\tilde{u}\mathbb{H}) \\ &\qquad - \operatorname{div}\left(\mathbb{H}^T\mathbb{D}(\tilde{u}) - \mathbb{D}(\tilde{u})\mathbb{H}^T - \mathbb{H}^T\mathbb{D}(\tilde{u})\mathbb{H}^T\right) + \nabla\tilde{p}, \varphi \Big\rangle_{H^{-1}(\Omega'), H_0^1(\Omega')} \\ &= \int_{\Omega'} (F' + \operatorname{div}\mathbb{E}(\tilde{u}) - \mathbb{J}(\tilde{u})) \cdot \varphi\end{aligned}$$

Hence, we can write out the system:

$$\begin{bmatrix} 1 - C(r) & C(r) \\ C(r) & 1 - C(r) \end{bmatrix} \begin{bmatrix} \partial_3\partial_3\tilde{u}_1 \\ \partial_3\partial_3\tilde{u} \end{bmatrix} = \begin{bmatrix} F_1(\partial_k\partial_j\tilde{u}_i, \partial_k\tilde{p}, F') \\ F_2(\partial_k\partial_j\tilde{u}_i, \partial_k\tilde{p}, F') \end{bmatrix},$$

For some F_1 and F_2, $k = 1, 2$, $i, j = 1, 2, 3$, and constant $C(r) > 0$ that is small for small $r > 0$. Note that the right-hand side of the equation is in $L^2(V')$. Thus, choosing small enough $r > 0$, we finally obtain that $\dfrac{\partial^2\tilde{u}_1}{\partial x_3^2}, \dfrac{\partial^2\tilde{u}_1}{\partial x_3^2} \in L^2(V')$ and hence, $\tilde{u} \in H^2(V')$.

Finally, we go back to the original coordinates. We have

$$u = \left(\nabla \psi^{-1} \circ \psi\right) \left(\tilde{u} \circ \psi\right),$$

and

$$\nabla p = \left(\nabla \psi^{-1}\right)^T \left(\nabla \tilde{p} \circ \psi^{-1}\right).$$

As ψ and ψ^{-1} are smooth, we have that $u \in H^2(V)$ and $p \in H^1(V)$, with $V = \psi^{-1}(V') \subset \Omega$. □

The constants appearing in our estimates depend on $r > 0$. To get global constants, we cover the compact domain with finitely many balls of radius $s := \min\{r, \frac{d}{2}\} > 0$. We see that, upon relabeling of axes, we get the same subdomains even after translations or rotations of the solid particles. Hence, these global constants remain the same up to rigid motion of the solids.

An important step in the proof is the use of Korn's inequality to compare the strain of some vector fields with their gradients. At first glance, one would think that by choosing $r > 0$, this would change the Korn's constant and hence would affect how $r > 0$ should be chosen and so on, *ad infinitum.*

In the proof, we see that the particular step where we do this is when we derived the inequality:

$$\int_{B(0,s)^+} \left|\nabla \left(\zeta D_k^h \tilde{u}\right)(x)\right|^2 dx \leq C_{K(B_1^+)} \Bigg[\int_{B(0,s)^+} \left|\mathbb{D} \left(\zeta D_k^h \tilde{u}\right)(x)\right|^2 dx + \frac{1}{s^2} \int_{B(0,s)^+} \left|\left(\zeta D_k^h \tilde{u}\right)(x)\right|^2 dx \Bigg].$$

Notice that the terms we needed to compare, namely, the gradient and strain terms, do not have $\frac{1}{s^2}$ multiplied to them. Hence, the result is that, the choice of $r > 0$ does not affect how the strain scales with the gradient; it does increase the contribution of the L^2-norm of the difference quotients by a factor of $\frac{1}{s^2}$, but that is something that we can control and does not affect the choice of $r > 0$.

5.4 Nonstationary Stokes problem

5.4.1 Existence of weak solution

We first prove the existence of a weak solution to the nonstationary Stokes problem.

Theorem 15. *Let* $f \in L^2((0,T) \times \Omega)$ *and* $u_0 \in L^2(\Omega)$*. Then, there exists* $u \in \mathcal{V} := \{v \in L^2(0,T; H^1(\Omega)) \,|\, \partial_t v \in L^2(0,T; (H^1_{\sigma,\tau}(\Omega))^*)\}$ *and* $p \in L^2_{loc}((0,T) \times \Omega)$ *that solves the following problem in a weak sense:*

$$\begin{aligned} \partial_t u - \Delta u + \nabla p = f, &\quad \textit{in } (0,T) \times \Omega \\ \textit{div}\, u = 0, &\quad \textit{in } (0,T) \times \Omega \\ u \cdot n = 0, &\quad \textit{on } (0,T) \times \Gamma \\ [\mathbb{D}(u)n]_\tau + \alpha u_\tau = 0, &\quad \textit{on } (0,T) \times \Gamma \\ u(0) = u_0, &\quad \textit{in } \Omega. \end{aligned}$$

Proof. We solve this by Rothe's method. Indeed, let $N \in \mathbb{N}$ and $k := \frac{T}{N}$. The plan is to solve the problem iteratively in subintervals of $[0,T]$ of length k. First, we let $u^0 := u_0$ and

$$f^m := \frac{1}{k}\int_{(m-1)k}^{mk} f(t)\,dt, \quad m = 1, ..., N.$$

Note that f^m is the time average of f in $[(m-1)k, mk]$. We then solve the following stationary problem for (u^m, p^m):

$$\begin{aligned} \frac{u^m - u^{m-1}}{k} - \Delta u^m + \nabla p^m = f^m, &\quad \text{in } \Omega \\ \operatorname{div} u^m = 0, &\quad \text{in } \Omega \\ u^m \cdot n = 0, &\quad \text{on } \Gamma \\ [\mathbb{D}(u^m)n]_\tau + \alpha u^m = 0, &\quad \text{on } \Gamma. \end{aligned}$$

By our results in the previous section, (u^m, p^m) exist and that $u^m \in H^2(\Omega)$ and $p^m \in H^1(\Omega)$. We now define our approximate solutions:

$$\begin{aligned} u_N(t) &:= \sum_{m=1}^{N} u^m 1_{[(m-1)k,mk)}(t) \\ w_N(t) &:= \sum_{m=1}^{N} \left[u^m + \left(\frac{u^m - u^{m-1}}{k}\right)(t - mk)\right] 1_{[(m-1)k,mk)}(t). \end{aligned}$$

Note that $u_N : [0, T] \to H^2(\Omega)$ and $w_N : [0, T] \to L^2(\Omega)$, is continuous, and linear.

We now make our estimates. First, the weak form of the problem in $[(m-1)k, mk)$ is:

$$\int_\Omega \frac{u^m - u^{m-1}}{k} \cdot \varphi + 2\int_\Omega \mathbb{D}u^m : \mathbb{D}\varphi + 2\int_\Gamma \alpha u^m_\tau \cdot \varphi_\tau = \int_\Omega f^m \cdot \varphi,$$

for all $\varphi \in H^1_{\sigma,\tau}(\Omega)$. We set $\varphi := u^m$. Then we have

$$\frac{1}{k}\int_\Omega (u^m - u^{m-1})\cdot u^m + 2\int_\Omega |\mathbb{D}(u^m)|^2 + 2\alpha\int_\Gamma |u^m_\tau|^2 = \int_\Omega f^m \cdot u^m.$$

Observe first that:

$$\begin{aligned}(u^m - u^{m-1})\cdot u^m &= (u^m - u^{m-1})\cdot(u^m - u^{m-1} + u^{m-1})\\ &= \left|u^m - u^{m-1}\right|^2 - \left|u^{m-1}\right|^2 + u^m\cdot(u^{m-1} - u^m + u^m)\\ &= \left|u^m - u^{m-1}\right|^2 - \left|u^{m-1}\right|^2 + |u^m|^2 - u^m\cdot(u^m - u^{m-1}),\end{aligned}$$

so that

$$(u^m - u^{m-1})\cdot u^m = \frac{1}{2}\left(\left|u^m - u^{m-1}\right|^2 - \left|u^{m-1}\right|^2 + |u^m|^2\right).$$

Thus, we can write

$$\frac{1}{k}\int_\Omega (u^m - u^{m-1})\cdot u^m = \frac{1}{2k}\left(\|u^m - u^{m-1}\|^2_{L^2(\Omega)} + \|u^m\|^2_{L^2(\Omega)} - \|u^{m-1}\|^2_{L^2(\Omega)}\right).$$

By Korn's inequality, we have

$$2\int_\Omega |\mathbb{D}(u^m)|^2 + 2\alpha\int_\Gamma |u^m_\tau|^2 \geq C_{K,\alpha}\|u^m\|^2_{H^1(\Omega)}.$$

Lastly, we have

$$\int_\Omega f^m\cdot u^m \leq \varepsilon\|u^m\|^2_{H^1(\Omega)} + C(\varepsilon)\|f^m\|^2_{L^2(\Omega)},$$

for $\varepsilon > 0$. Thus, choosing ε to be small enough, we get for $m = 1, ..., N$:

$$\|u^m - u^{m-1}\|^2_{L^2(\Omega)} + \|u^m\|^2_{L^2(\Omega)} - \|u^{m-1}\|^2_{L^2(\Omega)} + C(\Omega,\alpha)k\|u^m\|^2_{H^1(\Omega)} \leq Ck\|f^m\|^2_{L^2(\Omega)}$$

$$\|u^{m-1} - u^{m-2}\|^2_{L^2(\Omega)} + \|u^{m-1}\|^2_{L^2(\Omega)} - \|u^{m-2}\|^2_{L^2(\Omega)} + C(\Omega,\alpha)k\|u^{m-1}\|^2_{H^1(\Omega)} \leq Ck\|f^{m-1}\|^2_{L^2(\Omega)}$$

$$\vdots$$

$$\|u^1 - u^0\|^2_{L^2(\Omega)} + \|u^1\|^2_{L^2(\Omega)} - \|u^0\|^2_{L^2(\Omega)} + C(\Omega,\alpha)k\|u^1\|^2_{H^1(\Omega)} \leq Ck\|f^1\|^2_{L^2(\Omega)}.$$

Summing, we get for $m = 1, ..., N$:

$$\sum_{j=1}^{m} \|u^j - u^{j-1}\|^2_{L^2(\Omega)} + \|u^m\|^2_{L^2(\Omega)} + C(\Omega,\alpha)k\sum_{j=1}^{m}\|u^j\|^2_{H^1(\Omega)} \leq Ck\sum_{j=1}^{m}\|f^j\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}.$$

Note that:

- $\|u^m\|^2_{L^2(\Omega)}$ will give $L^\infty_t L^2_x$ control
- $k\sum_{j=1}^m \|u^j\|^2_{H^1(\Omega)}$ will give $L^2_t H^1_x$ control

The control on the sizes of the approximate solutions would come from the data. In this end, we have

$$\begin{aligned} k\sum_{j=1}^{m}\|f^j\|^2_{L^2(\Omega)} &= k\sum_{j=1}^{m}\int_\Omega \left(\frac{1}{k}\int_{(j-1)k}^{jk} f\,dt\right)^2 dx \\ \textit{(by Jensen's inequality)} &\leq k\sum_{j=1}^{m}\int_\Omega \left(\frac{1}{k}\int_{(j-1)k}^{jk} f^2\,dt\right) dx \\ &= \|f\|^2_{L^2((0,T)\times\Omega)}, \end{aligned}$$

for $m = 1, ..., N$. Thus we have the following estimates on the approximate solutions:

$$\begin{aligned}
\|u_N\|^2_{L^2(0,T;H^1(\Omega))} &= \sum_{m=1}^{N} \int_{(m-1)k}^{mk} \|u^m\|^2_{H^1(\Omega)}\, dt \\
&= k \sum_{m=1}^{N} \|u^m\|^2_{H^1(\Omega)} \\
&\leq C \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|u_0\|^2_{L^2(\Omega)} \right),
\end{aligned}$$

and

$$\|u_N\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \sup_{1\leq m\leq N} \|u^m\|^2_{L^2(\Omega)} \leq C \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|u_0\|^2_{L^2(\Omega)} \right).$$

We now look at the time derivatives of the approximate solutions. Given a test function $\varphi \in H^1_{\sigma,\tau}(\Omega)$, we have

$$\begin{aligned}
\left| \int_\Omega \frac{u^m - u^{m-1}}{k} \cdot \varphi \right| &= \left| \int_\Omega f^m \cdot \varphi - 2 \int_\Omega \mathbb{D}u^m : \mathbb{D}\varphi - 2 \int_\Gamma \alpha u^m_\tau \cdot \varphi_\tau \right| \\
&\leq C(\Omega, \alpha) \left(\|u^m\|_{H^1(\Omega)} + \|f^m\|_{L^2(\Omega)} \right) \|\varphi\|_{H^1(\Omega)}.
\end{aligned}$$

Thus, we have

$$\left\| \frac{u^m - u^{m-1}}{k} \right\|_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*} \leq C(\Omega, \alpha) \left(\|u^m\|_{H^1(\Omega)} + \|f^m\|_{L^2(\Omega)} \right).$$

Multiplying this by k and summing from $j = 1, ..., m$, we get

$$\begin{aligned}
k \sum_{j=1}^{m} \left\| \frac{u^j - u^{j-1}}{k} \right\|^2_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*} &\leq C(\Omega, \alpha) \sum_{j=1}^{m} \left(\|u^j\|_{H^1(\Omega)} + \|f^j\|_{L^2(\Omega)} \right) \|\varphi\|_{H^1(\Omega)} \\
&\leq C(\Omega, \alpha) \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|u_0\|^2_{L^2(\Omega)} \right).
\end{aligned}$$

In particular,

$$
\begin{aligned}
\|\partial_t w_N\|^2_{L^2\left(0,T;\left(H^1_{\sigma,\tau}(\Omega)\right)^*\right)} &= \sum_{m=1}^{N} \int_{(m-1)k}^{mk} \left\| \frac{u^j - u^{j-1}}{k} \right\|^2_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*} dt \\
&= k \sum_{m=1}^{N} \left\| \frac{u^j - u^{j-1}}{k} \right\|^2_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*} \\
&\leq C(\Omega, \alpha) \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|u_0\|^2_{L^2(\Omega)} \right).
\end{aligned}
$$

Next, we calculate:

$$
\begin{aligned}
\|u_N - w_N\|^2_{L^2((0,T)\times\Omega)} &= \sum_{m=1}^{N} \int_{(m-1)k}^{mk} \int_{\Omega} \left| u^m - \left(u^m + \frac{u^m - u^{m-1}}{k}(t - mk) \right) \right|^2 dx\, dt \\
&= \sum_{m=1}^{N} \left(\int_{\Omega} \left| \frac{u^m - u^{m-1}}{k} \right|^2 dx \right) \left(\int_{(m-1)k}^{mk} (t - mk)^2\, dt \right) \\
&= \frac{k}{3} \sum_{m=1}^{N} \|u^m - u^{m-1}\|^2_{L^2(\Omega)} \\
&\leq C \frac{T}{N} \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|u_0\|^2_{L^2(\Omega)} \right) \\
&\to 0, \quad \text{as } N \to \infty.
\end{aligned}
$$

Thus, $u_N - w_N \to 0$ in $L^2\left((0,T)\times\Omega\right)$ as $N \to \infty$.

Note that as $\{u_N\}$ is bounded in $L^2\left((0,T)\times\Omega\right)$, the previous result implies that $\{w_N\}$ is bounded in $L^2\left((0,T)\times\Omega\right)$ as well. Therefore, combining this with the previous estimates, we have, up to subsequences, the following convergences:

$$
\begin{aligned}
u_N &\rightharpoonup u, \quad wk - L^2\left(0,T; H^1(\Omega)\right), \; wk^* - L^\infty\left(0,T; L^2(\Omega)\right) \\
w_N &\rightharpoonup w, \quad wk - L^2\left((0,T)\times\Omega\right) \\
\partial_t w_N &\rightharpoonup v, \quad wk - L^2\left(0,T; \left(H^1_{\sigma,\tau}(\Omega)^*\right)\right).
\end{aligned}
$$

Since $u_N - w_N \to 0$ in $L^2((0,T)\times\Omega)$, we have that $u \equiv w$. Our goal now is to show that $\partial_t u \equiv v$. Indeed, since $H^1_{\sigma,\tau}(\Omega) \subset L^2_\sigma(\Omega) \subset \left(H^1_{\sigma,\tau}(\Omega)\right)^*$ and these inclusions are dense, we have that for $\varphi \in H^1_{\sigma,\tau}(\Omega)$ and $\zeta \in C_0^\infty(0,T)$:

$$\begin{aligned}
\int_0^T \zeta(t)\langle \partial_t w, \varphi\rangle_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*, H^1_{\sigma,\tau}(\Omega)}\, dt &:= \int_0^T \zeta'(t)\langle w, \varphi\rangle_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*, H^1_{\sigma,\tau}(\Omega)}\, dt \\
&= -\int_0^T \zeta'(t)\left(\int_\Omega w\varphi\, dx\right) dt \\
&= -\lim_{N\to\infty}\int_0^T\int_\Omega \zeta'(t) w_N(t,x)\varphi(x)\, dx\, dt \\
&= \lim_{N\to\infty}\int_0^T \zeta(t)\langle \partial_t w_N, \varphi\rangle_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*, H^1_{\sigma,\tau}(\Omega)}\, dt \\
&= \int_0^T \zeta(t)\langle v, \varphi\rangle_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*, H^1_{\sigma,\tau}(\Omega)}\, dt.
\end{aligned}$$

Thus, $\partial_t u \equiv \partial_t w \equiv v$.

For the force term, one can show *(see [57])* that

$$f_N := \sum_{m=1}^{N} f^m 1_{[(m-1)k, mk)} \to f, \quad \text{in } L^2((0,T)\times\Omega).$$

We also have that

$$\|u_N\|_{L^2((0,T)\times\Gamma)} \leq C(\Omega)\|u_N\|_{L^2(0,T;H^1(\Omega))} \leq C\left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right).$$

Thus, up to a subsequence, we have

$$u_n \rightharpoonup u, \quad wk - L^2((0,T)\times\Gamma).$$

Now, for $\varphi \in H^1_{\sigma,\tau}(\Omega)$, we have that the approximate solutions satisfy

$$\int_0^T \int_\Omega \partial_t w_N \cdot \varphi + 2\int_0^T \int_\Omega \mathbb{D}u_N : \mathbb{D}\varphi + 2\int_0^T \int_\Gamma \alpha (u_N)_\tau \cdot \varphi_\tau = \int_0^T \int_\Omega f_N \cdot \varphi.$$

Using the convergences we have obtained, we get that as $N \to \infty$:

$$\int_0^T \langle \partial_t u, \varphi \rangle_{\left(H^1_{\sigma,\tau}(\Omega)\right)^*, H^1_{\sigma,\tau}(\Omega)} + 2\int_0^T \int_\Omega \mathbb{D}u : \mathbb{D}\varphi + 2\int_0^T \int_\Gamma \alpha u_\tau \cdot \varphi_\tau = \int_0^T \int_\Omega f \cdot \varphi.,$$

i.e., u is a weak solution to the nonstationary Stokes problem. □

The dependence of the labeled constant $C(\Omega, \alpha)$ in the domain is due to the trace constant of Ω. Similar to Remark 1, one can argue that this constant remains unchanged when the solids undergo rigid motion.

There are previous work on the solvability of the nonstationary Stokes problem with slip boundary conditions. For example, [49] talks about Stokes with friction type slip conditions and [2] talks about maximal regularity of the Stokes operator with Navier slip conditions. In a lot of these treatments, to prove the solvability of the non stationary problem, they appeal to semigroup methods. In our work, we want to keep track of the constants which might otherwise be opaque to semigroup techniques; hence we opted for a simpler Rothe's method approach.

5.4.2 Higher regularity of weak solution

In this section, we show that if the initial data has better regularity, then so does u.

Theorem 16. *Suppose further that $u_0 \in H^1(\Omega)$. Then $u \in W^{1,2}\left(0, T; L^2(\Omega)\right) \cap$*

$L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Proof. Indeed, let $\varphi := \frac{u^m - u^{m-1}}{k}$ be a test function for $m = 1, ..., N$. Then,

$$\int_\Omega \left|\frac{u^m - u^{m-1}}{k}\right|^2 + 2\int_\Omega \mathbb{D}u^m : \mathbb{D}\left(\frac{u^m - u^{m-1}}{k}\right) + 2\int_\Gamma \alpha\,(u^m)_\tau \cdot \left(\frac{u^m - u^{m-1}}{k}\right)_\tau$$
$$= \int_\Omega f^m \cdot \left(\frac{u^m - u^{m-1}}{k}\right).$$

We make our estimates. First,

$$\begin{aligned}
&2\int_\Omega \mathbb{D}u^m : \mathbb{D}\left(\frac{u^m - u^{m-1}}{k}\right) + 2\int_\Gamma \alpha u_\tau^m \cdot \left(\frac{u^m - u^{m-1}}{k}\right)_\tau \\
&\geq \frac{2}{k}\left(\int_\Omega |\mathbb{D}u^m|^2 - \int_\Omega \mathbb{D}u^m : \mathbb{D}u^{m-1} + \int_\Gamma \alpha\,|u_\tau^m|^2 - \int_\Gamma \alpha u_\tau^m \cdot u_\tau^{m-1}\right) \\
&\geq \frac{1}{k}\left(\int_\Omega |\mathbb{D}u^m|^2 - \int_\Omega \left|\mathbb{D}u^{m-1}\right|^2 + \int_\Gamma \alpha\,|u_\tau^m|^2 - \int_\Gamma \alpha\left|u_\tau^{m-1}\right|^2\right).
\end{aligned}$$

As for the force term, we have

$$\int_\Omega f^m \cdot \left(\frac{u^m - u^{m-1}}{k}\right) \leq \frac{1}{2}\left(\int_\Omega |f^m|^2 + \int_\Omega \left|\frac{u^m - u^{m-1}}{k}\right|^2\right).$$

Thus, multiplying by k, writing things out as time integrals and summing from $m = 1, ..., N$, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{m=1}^N \int_{(m-1)k}^{mk} \int_\Omega \left|\frac{u^m - u^{m-1}}{k}\right|^2 \\
&+ \frac{1}{k}\sum_{m=1}^N \int_{(m-1)k}^{mk} \left(\int_\Omega |\mathbb{D}u^m|^2 - \int_\Omega \left|\mathbb{D}u^{m-1}\right|^2 + \int_\Gamma \alpha\,|u_\tau^m|^2 - \int_\Gamma \alpha\left|u_\tau^{m-1}\right|^2\right) \\
&= \frac{1}{2}\sum_{m=1}^N \int_{(m-1)k}^{mk} \int_\Omega |f^m|^2.
\end{aligned}$$

Since,

$$\sum_{m=1}^{N} \int_{(m-1)k}^{mk} \int_{\Omega} \left| \frac{u^m - u^{m-1}}{k} \right|^2 = \int_0^T \int_{\Omega} |\partial_t w_N|^2$$
$$\sum_{m=1}^{N} \int_{(m-1)k}^{mk} \int_{\Omega} |f^m|^2 = \int_0^T \int_{\Omega} |f_N|^2 ,$$

and

$$\frac{1}{k} \sum_{m=1}^{N} \int_{(m-1)k}^{mk} \left(\int_{\Omega} |\mathbb{D}u^m|^2 - \int_{\Omega} \left|\mathbb{D}u^{m-1}\right|^2 + \int_{\Gamma} \alpha \left|u_\tau^m\right|^2 - \int_{\Gamma} \alpha \left|u_\tau^{m-1}\right|^2 \right)$$
$$= \|\mathbb{D}u^N\|^2_{L(\Omega)} + \|\alpha u_\tau^N\|^2_{L^2(\Gamma)} - \|\mathbb{D}u_0\|^2_{L(\Omega)} - \|\alpha(u_0)_\tau\|^2_{L^2(\Gamma)},$$

we obtain

$$\begin{aligned} \frac{1}{2}\|\partial_t w_N\|^2_{L^2((0,T)\times\Omega)} + \|\mathbb{D}u^N\|^2_{L(\Omega)} + \|\alpha u_\tau^N\|^2_{L^2(\Gamma)} & \\ \leq \frac{1}{2}\|f_N\|^2_{L^2((0,T)\times\Omega)} + \|\mathbb{D}u_0\|^2_{L(\Omega)} + \|\alpha(u_0)_\tau\|^2_{L^2(\Gamma)} & \\ \leq \frac{1}{2}\|f\|^2_{L^2((0,T)\times\Omega)} + \|\mathbb{D}u_0\|^2_{L(\Omega)} + \|\alpha(u_0)_\tau\|^2_{L^2(\Gamma)}. & \end{aligned}$$

Thus, for each N, we have

$$\|\partial_t w_N\|^2_{L^2((0,T)\times\Omega)} \leq C \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|\mathbb{D}u_0\|^2_{L(\Omega)} + \|\alpha(u_0)_\tau\|^2_{L^2(\Gamma)} \right).$$

Finally,

$$\begin{aligned} \|\partial_t u\|^2_{L^2((0,T)\times\Omega)} &= \|\partial_t w\|^2_{L^2((0,T)\times\Omega)} \\ &\leq C \left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|\mathbb{D}u_0\|^2_{L(\Omega)} + \|\alpha(u_0)_\tau\|^2_{L^2(\Gamma)} \right), \end{aligned}$$

i.e., $\partial_t u \in L^2((0,T)\times\Omega)$. Therefore,

$$-\Delta u + \nabla p = f - \partial_t u \in L^2(\Omega), \quad \text{a.e.}\, t \in (0,T),$$

so that by the H^2-regularity result we have obtained in an earlier section, we have $u(t) \in H^2(\Omega)$ for a.e. $t \in (0,T)$. And then we have,

$$\begin{aligned}\int_0^T \|u(t)\|^2_{H^2(\Omega)}\,dt &\le C(\Omega,\alpha)\left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|\partial_t u\|^2_{L^2((0,T)\times\Omega)}\right)\\ &\le C(\Omega,\alpha)\left(\|f\|^2_{L^2((0,T)\times\Omega)} + \|\mathbb{D}u_0\|^2_{L(\Omega)} + \|\alpha(u_0)_\tau\|^2_{L^2(\Gamma)}\right).\end{aligned}$$

Thus, $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. □

5.5 Solvability of the fixed domain problem

We finally prove:

Theorem 17. *Let $F \in L^2((0,T)\times\Omega)$ and $u_0 \in H^1(\Omega)$. Then, there exist $(u,p) \in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\times L^2(0,T;H^1(\Omega))$ that solves*

$$\begin{aligned}\partial_t u - \Delta u + \nabla p = F + (\mathcal{L}-\Delta)u - \mathcal{M}u + (\nabla - \mathcal{G})p, &\quad \text{in } (0,T)\times\Omega\\ \text{div}\; u = 0, &\quad \text{in } (0,T)\times\Omega\\ u\cdot\mu = 0, &\quad \text{on } (0,T)\times\Gamma\\ [\mathbb{D}(u)\mu]_\tau + \alpha u = 0, &\quad \text{on } (0,T)\times\Gamma\\ u(0) = u_0, &\end{aligned}$$

Proof. We solve this using the Banach contraction principle. Indeed, given some

$v \in L^2(0,T;H^2(\Omega))$ and $\pi \in L^2(0,T;H^1(\Omega))$, by the results of the previous section, there exits a unique $u_v \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $p_\pi \in L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned}
\partial_t u_v - \Delta u_v + \nabla p_\pi = F + (\mathcal{L} - \Delta)v - \mathcal{M}v + (\nabla - \mathcal{G})\pi, &\quad \text{in } (0,T) \times \Omega \\
\text{div } u_v = 0, &\quad \text{in } (0,T) \times \Omega \\
u_v \cdot \mu = 0, &\quad \text{on } (0,T) \times \Gamma \\
[\mathbb{D}(u_v)\mu]_\tau + \alpha u = 0, &\quad \text{on } (0,T) \times \Gamma \\
u_v(0) = u_0. &
\end{aligned}$$

This now defines a map $(v,\pi) \mapsto (u_v, p_\pi) =: T(v,\pi)$. We show that $T : L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^1(\Omega)) \to L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^1(\Omega))$ is a contraction.

Let $v_1, v_2 \in L^2(0,T;H^2(\Omega))$ and $\pi_1, \pi_2 \in L^2(0,T;H^1(\Omega))$. By linearity, $(w,p) := T(v_1,\pi_1) - T(v_2,\pi_2)$ solves

$$\begin{aligned}
\partial_t w - \Delta w + \nabla p = F &+ (\mathcal{L} - \Delta)(v_1 - v_2) \\
&- \mathcal{M}(v_1 - v_2) + (\nabla - \mathcal{G})(\pi_1 - \pi_2), \quad \text{in } (0,T) \times \Omega \\
\text{div } w = 0, &\quad \text{in } (0,T) \times \Omega \\
w \cdot \mu = 0, &\quad \text{on } (0,T) \times \Gamma \\
[\mathbb{D}(w)\mu]_\tau + \alpha u = 0, &\quad \text{on } (0,T) \times \Gamma \\
w(0) = 0. &
\end{aligned}$$

By the smoothness of the motions of the solid particles, we have that the terms $g^{ij}, g_{ij}, \Gamma^k_{ij}, \partial_j b^*_i, b^*_j, \mathring{Y}_j, \partial_k Y_i$ are Lipschitz continuous on $[0,T]$. So that on $Q_T :=$

$(0,T)\times\Omega$, we have

$$
\begin{aligned}
&\|w\|_{L^2(0,T;H^2(\Omega))}+\|p\|_{L^2(0,T;H^1(\Omega))}\\
&\le C(\Omega,\alpha)(\|(\mathcal{L}-\Delta)(v_1-v_2)\|_{L^2(Q_T)}+\|\mathcal{B}(v_1-v_2)\|_{L^2(Q_T)}+\|\mathcal{M}(v_1-v_2)\|_{L^2(Q_T)}\\
&\qquad\qquad+\|(\mathcal{G}-\nabla)(\pi_1-\pi_2)\|_{L^2(Q_T)})\\
&\le T\cdot C(\Omega,\alpha)(\|v_1-v_2\|_{L^2(0,T;H^2(\Omega))}+\|\pi_1-\pi_2\|_{L^2(0,T;H^1(\Omega))}).
\end{aligned}
$$

Thus, by choosing $T>0$ to be small enough, we get that T is a contraction, and hence a fixed point exists. Moreover, the velocity from the fixed point, u, is in $W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)\cap L^\infty\left(0,T;L^2(\Omega)\right)$ □

5.6 Appendix

We provide a sketch of the details of the estimates for the remaining diffusion terms. We first consider:

$$
\int_{\Omega'}\mathbb{D}\tilde{u}:\mathbb{F}_1\left(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)\right).
$$

For this term, we only calculate

$$
\begin{aligned}
&\int_{\Omega'}\mathbb{D}\tilde{u}:\mathbb{H}\nabla\left(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)\right)\\
&=\int_{\Omega'}D_k^h\left(\mathbb{H}^T\mathbb{D}\tilde{u}\right):\nabla\left(\zeta^2 D_k^h\tilde{u}\right)\\
&=\int_{\Omega'}\left[{\mathbb{H}^T}^h D_k^h\mathbb{D}\tilde{u}+\left(D_k^h\mathbb{H}^T\right)\mathbb{D}\tilde{u}\right]:\left[\zeta^2 D_k^h\nabla\tilde{u}+2\zeta\left(D_k^h\tilde{u}\otimes\nabla\zeta\right)\right].
\end{aligned}
$$

Now,

$$\begin{aligned}
&\int_{\Omega'} \zeta D_k^h \mathbb{D}\tilde{u} : \zeta \mathbb{H}^h D_k^h \nabla \tilde{u} \geq -\frac{\varepsilon}{2} \int_{\Omega'} \zeta^2 \left| D_k^h \mathbb{D}\tilde{u} \right|^2 - C(\varepsilon) \|\mathbb{H}\|_{C^k}^2 \int_{\Omega'} \zeta^2 \left| D_k^h \nabla \tilde{u} \right|^2, \\
&\int_{\Omega'} \zeta \left(D_k^h \mathbb{H} \right)^T \mathbb{D}\tilde{u} : \zeta D_k^h \nabla \tilde{u} \geq -\varepsilon' \int_{\Omega'} \zeta^2 \left| D_k^h \nabla \tilde{u} \right|^2 - C\left(\varepsilon', \Omega', \mathbb{H}\right) \int_{\Omega'} |\nabla \tilde{u}|^2, \\
&\int_{\Omega'} D_k^h \mathbb{D}\tilde{u} : \left[2\zeta \mathbb{H}^h \left(D_k^h \tilde{u} \otimes \nabla \zeta \right) \right] \geq -\frac{\varepsilon}{2} \int_{\Omega'} \zeta^2 \left| D_k^h \mathbb{D}\tilde{u} \right|^2 - C(\varepsilon, \mathbb{H}, \Omega') \int_{\Omega'} |\nabla \tilde{u}|^2, \\
&\int_{\Omega'} \left(D_k^h \mathbb{H} \right)^T \mathbb{D}\tilde{u} : \left[2\zeta D_k^h \tilde{u} \otimes \nabla \zeta \right] \geq -C(\mathbb{H}, \Omega') \int_{\Omega'} |\nabla \tilde{u}|^2.
\end{aligned}$$

Combining these, we obtain

$$\begin{aligned}
\int_{\Omega'} \mathbb{D}\tilde{u} : \mathbb{H} \nabla \left(-D_k^{-h} \left(\zeta^2 D_k^h \tilde{u} \right) \right) \geq -\varepsilon \int_{\Omega'} \zeta^2 \left| D_k^h \mathbb{D}\tilde{u} \right|^2 - C\left(\|\mathbb{H}\|_{C^k}^2 C(\varepsilon) - \varepsilon' \right) \int_{\Omega'} \zeta^2 \left| D_k^h \nabla \tilde{u} \right|^2 \\
- C\left(\Omega,' \nabla \zeta, \mathbb{H}\right) \int_{\Omega'} |\nabla \tilde{u}|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\Omega'} \mathbb{D}\tilde{u} : \mathbb{F}_1 \left(-D_k^{-h} \left(\zeta^2 D_k^h \tilde{u} \right) \right) \geq -\varepsilon \int_{\Omega'} \zeta^2 \left| D_k^h \mathbb{D}\tilde{u} \right|^2 - C\left(\|\mathbb{H}\|_{C^k}^2 C(\varepsilon) - \varepsilon' \right) \int_{\Omega'} \zeta^2 \left| D_k^h \nabla \tilde{u} \right|^2 \\
- C\left(\Omega,' \nabla \zeta, \mathbb{H}\right) \int_{\Omega'} |\nabla \tilde{u}|^2.
\end{aligned}$$

Next, we consider the term:

$$\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{F}_1 \left(-D_k^{-h} \left(\zeta^2 D_k^h \tilde{u} \right) \right)$$

For this, we calculate

$$
\begin{aligned}
&\int_{\Omega'} \mathbb{H}\nabla\tilde{u} : \mathbb{H}\nabla\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right)\\
&= \int_{\Omega'} D_k^h\left(\mathbb{H}^T\mathbb{H}\nabla\tilde{u}\right) : \nabla\left(\zeta^2 D_k^h\tilde{u}\right)\\
&= \int_{\Omega'} \left[\left(\mathbb{H}^T\mathbb{H}\right)^h D_k^h\nabla\tilde{u} + \left(\left(\mathbb{H}^T\mathbb{H}\right)\right)\nabla\tilde{u}\right] : \left[\zeta^2 D_k^h\nabla\tilde{u} + 2\zeta\left(D_k^h\tilde{u}\otimes\nabla\zeta\right)\right]\\
&\geq -\|\mathbb{H}\|_{C^k}^2 \int_{\Omega'} \zeta^2\left|D_k^h\nabla\tilde{u}\right|^2 - \varepsilon'\int_{\Omega'}\zeta^2\left|D_k^h\nabla\tilde{u}\right|^2\\
&\quad - C\left(\Omega',\nabla\zeta,\mathbb{H}\right)\int_{\Omega'}\left|\nabla\tilde{u}\right|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{F}_1\left(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)\right) \geq -\|\mathbb{H}\|_{C^k}^2\int_{\Omega'}\zeta^2\left|D_k^h\nabla\tilde{u}\right|^2 - \varepsilon'\int_{\Omega'}\zeta^2\left|D_k^h\nabla\tilde{u}\right|^2\\
- C\left(\Omega',\nabla\zeta,\mathbb{H}\right)\int_{\Omega'}\left|\nabla\tilde{u}\right|^2.
\end{aligned}
$$

As for the term,

$$
\int_{\Omega'} \mathbb{D}\tilde{u} : \mathbb{F}_0\left(-D_k^{-h}\left(\zeta^2 D_k^h\tilde{u}\right)\right),
$$

we have

$$
\begin{aligned}
&\int_{\Omega'} \mathbb{D}\tilde{u} : \mathbb{F}_0 \left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \\
&= \int_{\Omega'} \mathbb{D}\tilde{u} : \mathbb{E}\left[-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right] \\
&= -\sum_{i,j} \int_{\Omega'} (\mathbb{D}\tilde{u})_{ij} \left[\sum_{l=1}^{2} \delta_{i3}(1-\delta_{j3})\partial_j \partial_l H D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}_l\right)\right] \\
&= -\sum_{i,j} \sum_{l=1}^{2} \int_{\Omega'} \delta_{i3}(1-\delta_{j3}) D_k^h \left((\mathbb{D}\tilde{u})_{ij}\partial_j \partial_l H\right)\left(\zeta^2 D_k^h \tilde{u}_l\right) \\
&= -\sum_{i,j} \sum_{l=1}^{2} \int_{\Omega'} \delta_{i3}(1-\delta_{j3}) [(\partial_k \partial_l H)^h D_k^h (\mathbb{D}\tilde{u})_{ij} \\
&\qquad\qquad + \left(D_k^h\left(\partial_j \partial_l h\right)\right)(\mathbb{D}\tilde{u})_{ij}]\left(\zeta^2 D_k^h \tilde{u}_l\right) \\
&\geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C\left(\Omega', \varepsilon, \mathbb{H}\right) \int_{\Omega'} |\nabla \tilde{u}|^2 .
\end{aligned}
$$

We now consider the term:

$$
\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{F}_0\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right).
$$

Indeed, we have

$$
\begin{aligned}
&\int_{\Omega'} \mathbb{H}\nabla\tilde{u} : \mathbb{E}\left[-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right] \\
&= -\sum_{i,j}\sum_{l=1}^{2}\int_{\Omega'} (\mathbb{H}\nabla\tilde{u})_{ij}\delta_{i3}(1-\delta_{j3})\partial_j\partial_l H D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}_l\right) \\
&= \sum_{i,j}\sum_{l=1}^{2}\int_{\Omega'} \delta_{i3}(1-\delta_{j3}) D_k^h\left((\mathbb{H}\nabla\tilde{u})_{ij}\partial_j\partial_l H\right)\left(\zeta^2 D_k^h \tilde{u}_l\right) \\
&= \sum_{i,j}\sum_{l=1}^{2}\int_{\Omega'} \delta_{i3}(1-\delta_{j3})[(\mathbb{H}_{im}\partial_j\partial_l H)^h\, D_k^h(\nabla\tilde{u})_{mj} \\
&\qquad\qquad + (\nabla\tilde{u})_{mj} D_k^h\left(\mathbb{H}_{im}\partial_j\partial_l H\right)]\left(\zeta^2 D_k^h \tilde{u}_l\right) \\
&\geq -\varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C\left(\Omega', \varepsilon', \mathbb{H}\right)\int_{\Omega'} |\nabla\tilde{u}|^2 .
\end{aligned}
$$

Making similar calculations for the pther terms, we obtain

$$
\int_{\Omega'} \mathbb{F}_1(\tilde{u}) : \mathbb{F}_0\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \geq \varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C\left(\Omega', \varepsilon', \mathbb{H}\right)\int_{\Omega'} |\nabla\tilde{u}|^2 .
$$

Next, we consider

$$
\begin{aligned}
\int_{\Omega'} \mathbb{F}_0(\tilde{u}) : \mathbb{D}\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) &= \int_{\Omega'} \mathbb{E}(\tilde{u}) : \mathbb{D}\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \\
&= \int_{\Omega'} D_k^h\left(\mathbb{E}(\tilde{u})\right) : \left[\zeta^2 D_k^h \mathbb{D}\tilde{u} + 2\zeta \mathbb{A}_{sym}\left(D_k^h \tilde{u} \otimes \nabla\zeta\right)\right].
\end{aligned}
$$

As,

$$
\begin{aligned}
\left(D_k^h\left(\mathbb{E}(\tilde{u})\right)\right)_{ij} &= D_k^h\left(\sum_{l=1}^{2} \delta - i3(1-\delta_{j3})\partial_j\partial_l H \tilde{u}_l\right) \\
&= \sum_{l=1}^{2} \delta_{i3}(1-\delta_{j3})\left[(\partial_j\partial_l H)^h D_k^h \tilde{u}_l + \tilde{u}_l D_k^h(\partial_j\partial_l H)\right],
\end{aligned}
$$

it follows that,

$$\int_{\Omega'} \mathbb{F}_0(\tilde{u}) : \mathbb{D}\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \geq -\varepsilon \int_{\Omega'} \zeta^2 \left|D_k^h \mathbb{D}\tilde{u}\right|^2 - C(\varepsilon, \mathbb{H}, \Omega')\|\tilde{u}\|_{H^1(\Omega')}^2.$$

Next, we consider

$$\int_{\Omega'} \mathbb{F}_0(\tilde{u}) : \mathbb{F}_1\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right).$$

Indeed, we only look at

$$\begin{aligned}\int_{\Omega'} \mathbb{E}(\tilde{u}) : \mathbb{H}\nabla\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) &= \int_{\Omega'} \mathbb{H}^T \mathbb{E}(\tilde{u}) : \nabla\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \\ &= \int_{\Omega'} D_k^h\left(\mathbb{H}^T \mathbb{E}(\tilde{u})\right) : \left[\zeta^2 D_k^h \nabla\tilde{u} + 2\zeta\left(D_k^h \tilde{u} \otimes \nabla\zeta\right)\right] \\ &\geq -\varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C(\varepsilon, \mathbb{H}, \Omega')\|\tilde{u}\|_{H^1(\Omega')}^2.\end{aligned}$$

Thus,

$$\int_{\Omega'} \mathbb{F}_0(\tilde{u}) : \mathbb{F}_1\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \geq -\varepsilon' \int_{\Omega'} \zeta^2 \left|D_k^h \nabla\tilde{u}\right|^2 - C(\varepsilon, \mathbb{H}, \Omega')\|\tilde{u}\|_{H^1(\Omega')}^2.$$

The last term we look into is:

$$\begin{aligned}\int_{\Omega'} \mathbb{F}_0(\tilde{u}) : \mathbb{F}_0\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) &= \int_{\Omega'} \mathbb{F}_0(\tilde{u}) : \mathbb{E}\left(-D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}\right)\right) \\ &= -\sum_{i,j}\sum_{l=1}^{2} \int_{\Omega'} (\mathbb{F}_0(\tilde{u}))_{ij}\delta_{i3}(1-\delta_{j3})\partial_j\partial_l H D_k^{-h}\left(\zeta^2 D_k^h \tilde{u}_l\right) \\ &= \sum_{i,j}\sum_{l=1}^{2} \int_{\Omega'} \delta_{i3}(1-\delta_{j3}) D_k^h\left((\mathbb{F}_0(\tilde{u}))_{ij}\partial_j\partial_l H\right)\left(\zeta^2 D_k^h \tilde{u}_l\right) \\ &\geq -C(\mathbb{H}, \Omega')\|\tilde{u}\|_{H^1(\Omega')}^2.\end{aligned}$$

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