

Mathematics behind System Dynamics

An Interactive Qualifying Project Report

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Abstract

The goal of this project is to introduce modeling and representation methods to solve dynamics problems. The intuitive System Dynamics representation is introduced and backed up with advanced mathematical concepts such as differential equations and Control theory techniques. This project attempts to illustrate both abstract and intuitive approaches based on examples arising in social and business systems. The topics include graphical representations, delays, numerical methods, and common behavioral pattern of typical structures.

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Preface

In this report, *Mathematics behind System Dynamics*, we present selected mathematical concepts helpful to understand System Dynamics modeling practice. Selected principles from single-variable calculus, ordinary differential equations, and control theory are covered, and their relationship to the behavior of systems is discussed. We also provide a brief introduction to System Dynamics and its application to real life problems (social, economic etc.) through various examples and case studies.

Organization

The report is divided into two major parts:

1. Calculus background and applications relevant to system dynamics.
2. Numerical solution methods used System Dynamics modeling.

Above organization aims to facilitate understanding the mathematical principles underlying the practice of system dynamics modeling that cannot be easily discerned as mathematics and system dynamics have come to use different terminologies, symbols and definitions for their respective practices.

Motivation behind the Work

As engineering students at Worcester Polytechnic Institute, we have been exposed to many subjects such as Ordinary Differential Equations, Control Engineering, and System Dynamics. Various similarities and connections were noticed among these subjects. That is in fact the motivation behind the idea of presenting the principle of system dynamics with mathematical and control theory perspectives.

Acknowledgements

We would like to take this opportunity to express our gratitude towards anyone who made this project possible.

First, we would like to show our greatest appreciation to our project supervisor Professor Khalid Saeed. We cannot thank him enough for his tremendous support and help. As a scholar of both engineering and system dynamics, Professor Saeed is the pillar of this interdisciplinary project.

We would also like to thank Worcester Polytechnic Institute for providing us with an adequate learning and research environment to complete this project.

Last but not least, a special thanks goes to our mentor, George Michael Raad whose support and advice are behind the completion of this project. Without his encouragement and guidance this project would not have materialized.

Chapter 1: An Introduction to Study

In today's society, human behavior is the source of many changes that increase the complexity of social systems. Hence, predicting a system's response and solving any of its arising problems become a difficult task. One of the beneficial tools used for solving these problems is System Dynamics.

System dynamics is a computer-aided approach to policy analysis and design, which is characterized by interdependence, mutual interaction, information feedback, and circular causality (Richardson, 1996). Graphical representations in System Dynamics such as Causal Loop Diagrams are intuitive and accessible to a wide range of people. On the other hand, those models do not exhibit all the information necessary to accurately understand the structure of the system and its various behaviors.

In order to fully define the characteristics of a system, a quantization of state variable and other factors is essential. The key concept is to transform the problem into a mathematical expression and manipulate it to retrieve any information of interest. Even though the result is in abstract form, it can be transferred to System Dynamics notation, which makes the result more intuitive. The interchangeability of system behavior between System Dynamics and Mathematics becomes a significant advantage for analysts as well as entrepreneurs.

This report consists of the study of transformations between System Dynamics and Control theory notations as well as analytical and numerical methods for solving differential equation representing dynamic models.

Chapter 2: An Introduction to System Dynamics

Motivation

This chapter defines system dynamics, its approach, and terminology. Many concepts will be introduced in order to give a general idea of system modeling. By the end of this chapter, the reader will be able to translate simple real life systems into dynamic models using causal loop diagrams.

Section 2.1: Definition of System Dynamics

Generally, social systems are sets consisting of a number of interacting components related to each other by different relationships. Most of the time, those parts are connected in such complicated ways that they form a complex system whose property and behavior is not simply defined. System dynamics is a strategic approach used in modeling such systems and determining their behavior.

The notion of **feedback** is very important in system dynamics. Furthermore, loop diagrams with feedback information are one of the major tools used to determine the structure of a complex system. The relations between each of the system elements could be mathematically described. The mathematical model of the overall system dynamics structure is a system of nonlinear, first-order differential and integral equations. Computer simulations model are used when analytical solutions are either unknown or computationally challenging (Sterman, 2000).

Section 2.2: Why do we use models?

When a problem arises in a system, actions must be taken. However, making the wrong decision could propagate the problem, and ultimately collapse the system. Therefore, understanding the behaviors and structures of systems is essential for problem solving. In general, systems contain many complex relationships which might cause them to be nonlinear, and make it difficult for the human mind to think through the problem. Therefore, many graphical and mathematical modeling methods have been developed as potential tools to understand a system.

In modeling, some insignificant relationships can be neglected to simplify the complexity of the system. Real life systems are dynamic, and therefore modeled with respect to time. Once the model is fully developed, it can be defined in its most generic form to serve as a reference for other similar systems.

Section 2.3: Information Links and Causal Loop Diagrams

This section introduces the most basic and simplistic modeling diagram used in System Dynamics. Although simplistic, its ability to layout an entire system with all of its components and corresponding connections is of major usefulness.

Cause-effect relationships among elements are usually modeled using **causal loop diagrams**. Loop diagrams are a simple tool that enables the analyst to have a general picture of the system components and their interaction with each other. The rules of diagram notation are clear and intuitive. A causal relationship is represented by an arrow pointing from the independent to the dependent variable. Near the head of the arrow is a polarity sign depending on whether the affected element is changing positively or negatively. Link after link, loops get created, and according to their effect on a given element, they are either **reinforcing** (R or +) if positive or **balancing** (B or -) negative. Those loops are called feedback loops. They “control” the value of the pointed element by either increasing or decreasing the quantity of interest. The figure below shows an example of a causal loop diagram (Meadows, 2012).

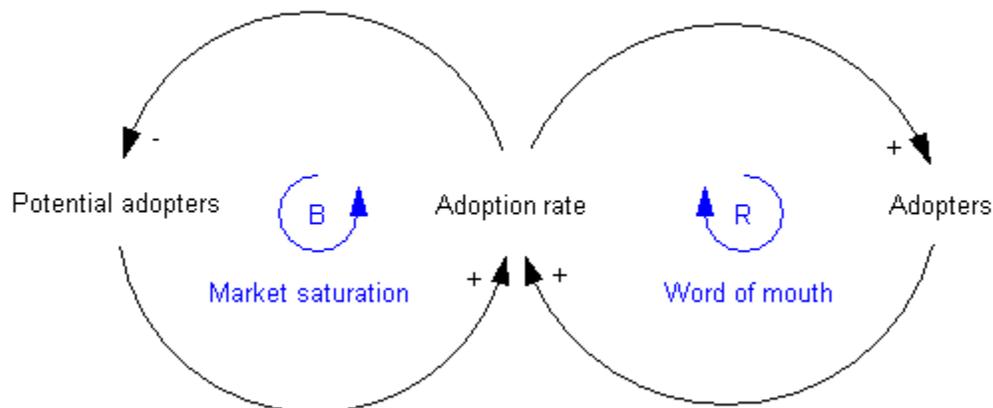


Figure 2-0-1: Causal Loop Diagram

The meaning of causal links may be intuitive, but is not fully understood most of the time. In the example in Figure 2-1, there is positive link pointing from *Adoption rate* to *Adopters*. That actually means that the dependent element *Adopters* (A) is positively changing corresponding to the independent variable *Adoption rate* (A_r).

On the other hand, there is a negative causal link from *Adoption rate* (A_r) to *Potential adopters* (P_a). This relationship means that the dependent variable (P_a) will change inversely corresponding to the independent element (A_r). Therefore, the change of the affected element over the change of the causing variable is negative. While the polarity of a causal link is rapidly determined, loop polarity requires more time and care. A loop is formed by multiple elements related to each other. Since each relation is followed by another one, the polarity of each link is multiplied by the polarity of the following one. The chain rule process continues until all the links of the loop are used. The resulting sign is the polarity of that specific loop. According to the algebraic property of multiplying plus and minus signs,

only negative polarities matters. More precisely, the number of negative links in the loop is the key. An even number of negative causal links would represent a positive loop, while an odd number of negative links would prove the opposite.

Due to the complexity of most dynamical systems, it is usually hard to keep track of all the links forming a loop, and basing the final polarity judgment on the number of negative links. A more delicate process is recommended. The initial step is to choose one element of the loop and assume its polarity to be either positive or negative. Following the direction of the links, the polarity of each two elements' change is traced as it propagates to the next element in line. Once the tracing reaches the initially chosen element, the process stops. The resulting and assumed polarities are then compared. If they are similar, the loop is therefore positive. Otherwise, the loop is negative.

Let us use the leftmost loop of the above figure as an example. Let us assume that the *Adoption rate* is positive. Because the A_r to P_a link is negative, *Potential adopters* decrease. And since the P_a to A_r link is positive, the *Adoption rate* decreases. Therefore, the change propagation opposes the initial assumption. The loop is then negative (also called balancing loop). Noting that this loop is of high simplicity, the number former method could be used. There is only one negative link in the entire loop. As mentioned before, an odd number of negative links refers to a negative loop.

Causal loop diagrams are useful because of their simple and high-level structure's description. The next section will present another way of dynamic modeling where a deeper level of information about the system is offered (Sterman, 2000).

Section 2.4: Stocks and Flows

This section gives a slight view on the notion of stock-and-flow diagrams and their advantages.

Causal loops are useful because of their ability to show relationships among the elements of a system using features such as loop polarity. On the other hand they fail at **quantifying** the elements of the system. If a component increases or decreases due its causal variable, it is important to know by how much it changed and at which rate it did. **Stocks and flows** are the concepts that account for such quantities.

With this new perception, each stock is thought of as the **accumulation** of each element size in the system. It is thus said that the system has **memory** (or history). There are two types of flows: inflows and outflows. Inflows are perceived as the **rate** at which the stock (the flow is going to) is increasing **over time**. Similarly, the outflow is the rate at which the stock (the flow is going out from) is decreasing. A simple example would be a banking system where the account balance is the stock. The balance therefore increases with deposits and decreases with withdrawals. Note that inflows and outflows are usually not constant and vary with respect to time.

As mentioned, inflows and outflows are the rates at which given quantity is being added to or subtracted from the stock. Therefore, a stock is the **integral** of the net flow added to the initial value of the stock. The net flow is eventually the outflow subtracted from the inflow.

The net flow is therefore the derivative of the total stock with respect to time.

$$\frac{d(\text{Stock})}{dt} = \text{Inflow}(t) - \text{Outflow}(t) \quad (2-2)$$

The graphical representation of stock and flow diagram is as the following:



Figure 1-2: Stock & Flow Diagram

Both inflow and outflow arrows contain a valve that dictates the rate of the flows entering or leaving a stock.

In short, stock-and-flow diagrams do not only show the structure's components and their relationships. Instead, more attention is brought to accumulation and flow processes (Sterman, 2000).

Summary

This chapter was an aperçu of System Dynamics and its basic notions. Later chapters will refer to the mentioned concepts and provide more technical explanations. In addition to that, the presented concepts and notations will serve as the common language in which more advanced topics are introduced.

Chapter 3: Graphical Representation of System Behavior

Prerequisites

Differentiation (Appendix A)

Motivation

In previous chapter, we introduced two approaches to represent a system, Causal Loop and Stock-Flow diagram. But sometimes, we might want to investigate or analyze in one specific stock or flow.

Therefore, it is important to translate a mathematic representation of stock and flow into graphical representations.

Because system changes with respect to time, it is essential to find the corresponding value of stock or flow with a specific time interval. To be more precise, these values can be defined by using mathematic function for any arbitrary time. Therefore, they can be represented as a **Cartesian graph**.

Section 3.1: Cartesian graph

A two-dimensional Cartesian graph contains two axes, which represent references for two values. The **origin** is located at the intersection of two axes. The behavior of a system is normally represented in system dynamics over time, the horizontal axis representing a timeline. The horizontal location of the origin represents the time at 0 . By the convention, any location on the right of the origin indicates a positive time. On the other hand, any location on the left indicates a negative time. The unit of time depends on the time interval of interest, which can vary from seconds to decades. This independent variable time is usually notated as t .

The vertical axis of a graph represents a reference for a value, which corresponds to time coordinate. A value is a dependent variable ,which relates to time. The vertical location of the origin represents the value of 0. In this case, any location above the origin indicates a positive value. In this case, we use stocks and flows as a value in a vertical axis.

A graph is divided by two axes, which form four separate areas. Each area is called “**quadrant**”. According to the convention, the quadrant on the top right is the first quadrant. The top left is the second quadrants. The third quadrant is on the left side below the horizontal axis. And the fourth quadrant lies on the right side below the horizontal axis. Normally, we focus on a positive time interval which corresponds to the second and the fourth quadrants.

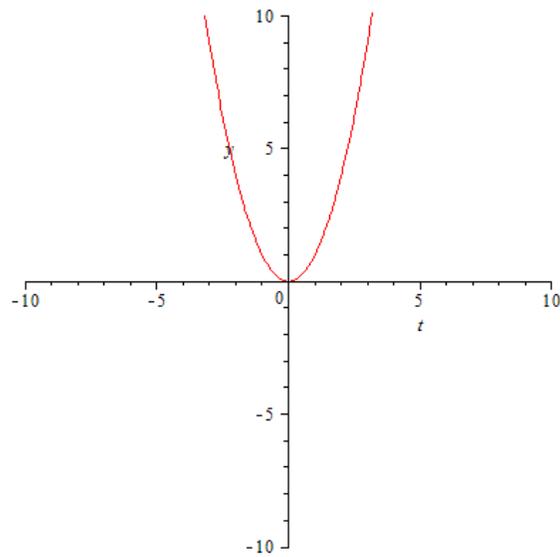


Figure 3-1: Four-Quadrant Cartesian graph

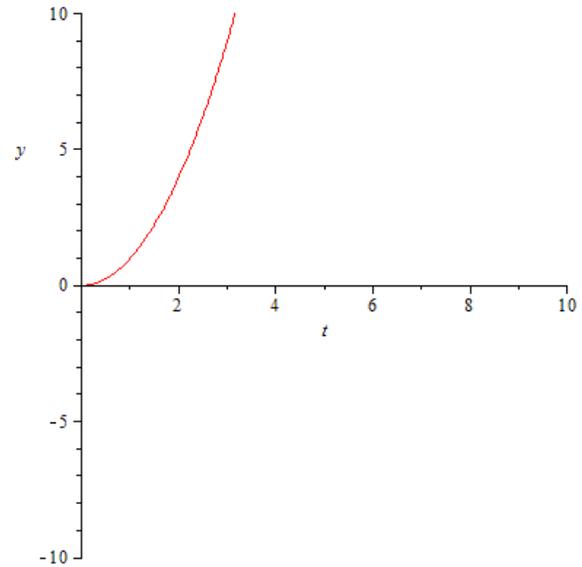


Figure 3-2: Two-Quadrant Cartesian graph

In system dynamics, every quantity is considered continuous (there is no disconnection at any point in time). Therefore, we can represent a collection of values by a curve on a graph which is basically a group of small points connected continuously along the timeline. Each point has a Cartesian coordinate that indicates a location on the graph with respect to the origin. And these coordinates can be calculated based on the relationship between a value and time. Many functions are composed of other sub-functions which can become complicated to graph. Hence, computer simulation might be required for graphing accurately.

Example:

Let us investigate a self-balancing system such as depopulation. The goal of this example is to graph the population with respect to time. One can say that the death rate depends on the population and average lifespan. Also, a causal-loop diagram is given as well as a mathematic definition of the population.

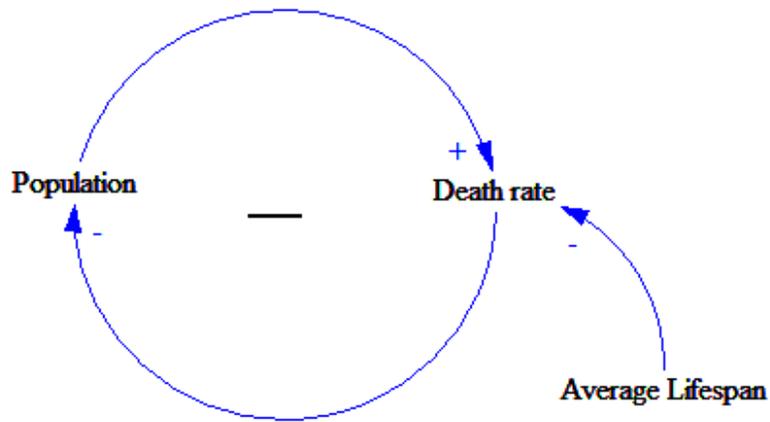


Figure 3-3: Causal Loop Diagram for Depopulation

$$P = P_0 e^{-\frac{t}{l}} \quad (3-1)$$

where P is the population, P_0 is the initial population at time $t = 0$, and l is the average lifespan.

The vertical axis of a corresponding Cartesian graph will represent a reference for population while the horizontal axis represents the timeline.

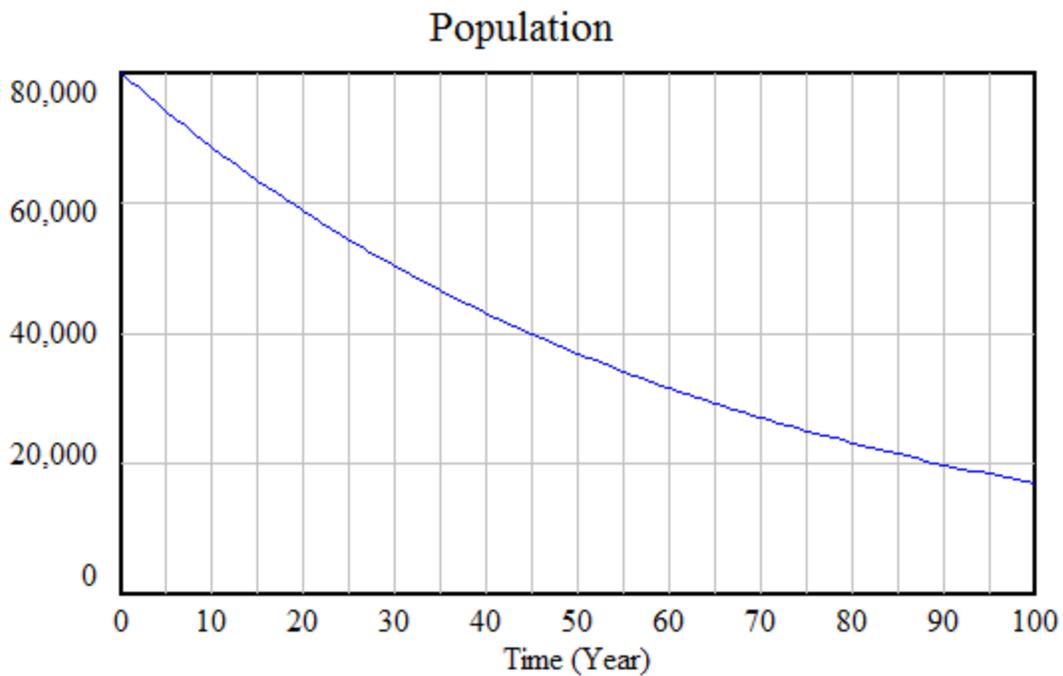


Figure 3-4: Population respect to time, $P_0 = 80,000$ units and $l = 65$ years,

With a Cartesian graph, one can easily observe the structure of a model individually. But sometimes, a system is not defined by a mathematic function in term of time. In some problems, a flow is given instead of a stock. Therefore, it is essential to understand the relationship between flows and stocks graphically.

Section 3.2: Shape of graph & first-order and second-order derivative

The curve that represents a structure can behaves differently based on the function. It can increase, decrease, or stay stationary (horizontal) which depends on the flow. With the knowledge of differential calculus, we can mathematically solve a general shape of a function. In some circumstance, we might encounter a situation when we have to estimate a structure of a stock with a given flow. In this case, the relationship between stock and flow has to be further defined.

Property I:

When a net flow is positive during the time interval (a, b) , $\frac{dS}{dt} > 0$, the stock increases during time interval (a, b) .

When a net flow is zero during the time interval (a, b) , $\frac{dS}{dt} = 0$, the stock is stationary during time interval (a, b) .

When a net flow is negative during the time interval (a, b) , $\frac{dS}{dt} < 0$, the stock decreases stationary during time interval (a, b) .

where $\frac{dS}{dt}$ is a flow for a corresponding stock.

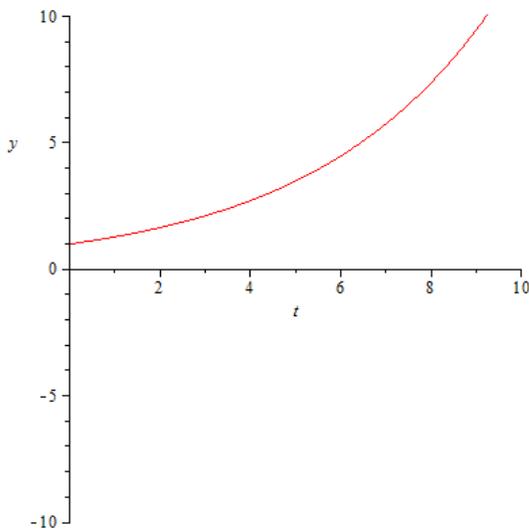


Figure 3-5: Increasing function

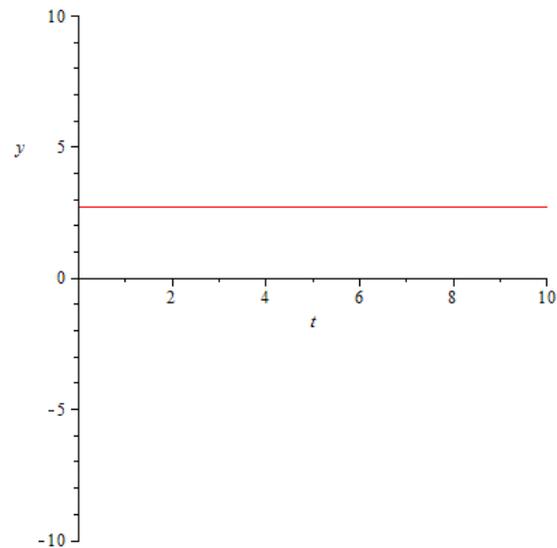


Figure 3-6: Constant Function

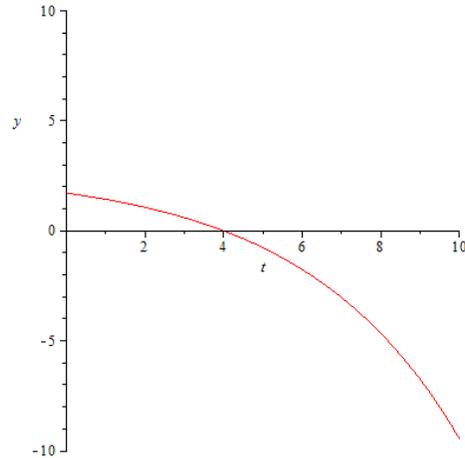


Figure 3-7: Decreasing function

A stock accumulates when there is an inflow (positive net flow). In the other hand, an outflow (negative net flow) causes a stock to deplete. A stock is in equilibrium when it has a zero net flow.

In order to fully sketch a curve, the concavity must be defined. The concavity indicates the acceleration of stock whether it concaves up, concaves down, or linear. The relationship between concavity and the curve can be mathematically defined as following.

Property II:

When a the rate of change of net flow is positive, $\frac{d^2S}{dt^2} > 0$, the curve is convex.

When a the rate of change of net flow is zero, $\frac{d^2S}{dt^2} = 0$, the curve is at point of inflection.

When a the rate of change of net flow is negative, $\frac{d^2S}{dt^2} < 0$, the curve is concave.

where $\frac{d^2S}{dt^2}$ is a second derivative of a corresponding stock.

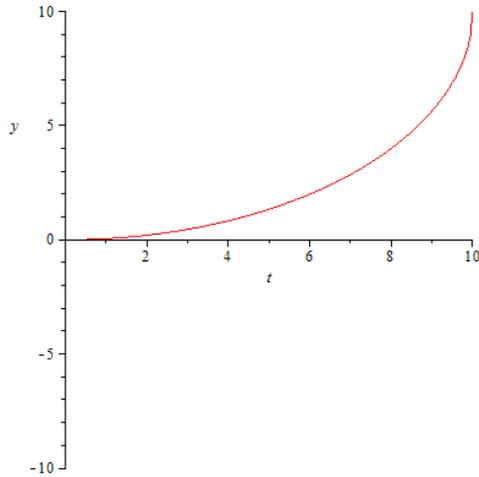


Figure 3-8: Convex function

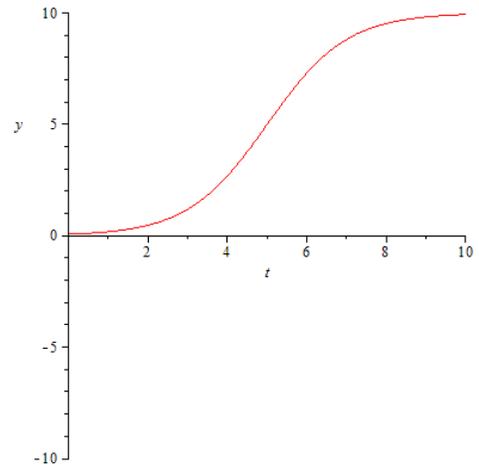


Figure 3-9: Function with point of inflection

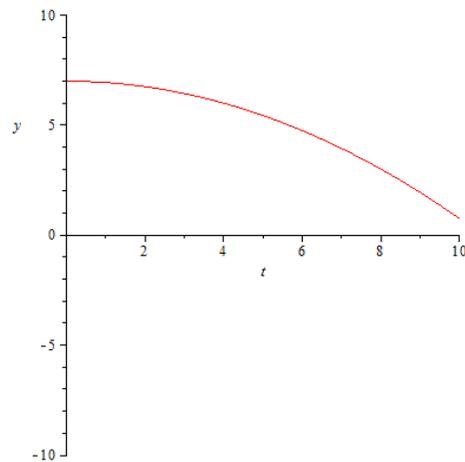


Figure 3-10: Concave function

Even though some curves have the same shape, they might not represent the same function. Because a net flow indicates only the general shape, the exact vertical position is usually estimated by the initial conditions.

The interpretation of the concavity of a function with respect to time can be interpreted as the acceleration of stock. This acceleration, in fact, is used for identifying types of critical points in a function which depict details of the structure such as maximum and minimum.

Section 3.3: Critical point, Critical value, second-derivative test

Sometimes in graphing or analyzing a structure, it might be necessary to identify the maximum or minimum value of stock in order to understand a general structure of a system.

In mathematics, a critical point is defined as a location where the derivative of the function is equal to zero. (We will omit the case where the function is not differentiable). The corresponding value at a critical point is called “critical value”. A critical point in a function indicates a possible occurrence of a local maximum or local minimum value. Therefore, we can conclude as following.

Local maximum and local minimum values of a stock may occur when a net flow of the stock is zero.

In order to find whether the critical point is local maximum, local minimum or nether, a second-derivative test is applied. The test can be summarized as following.

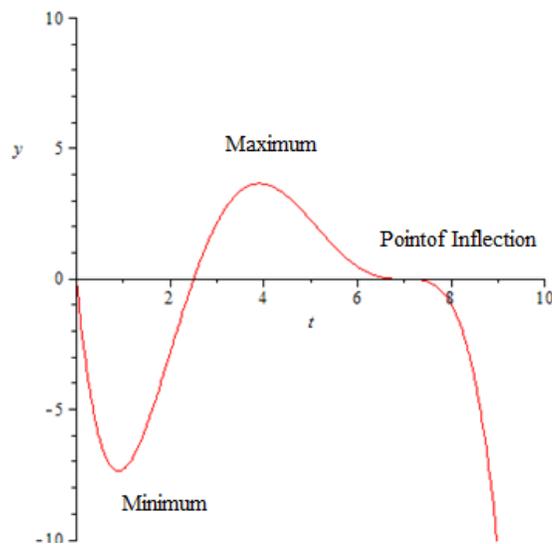


Figure 3-11: Graph containing maximum, minimum, and point of inflection

If a net flow is zero and the rate of change in net flow is positive at time t , $\frac{dS}{dt} = 0$ and $\frac{d^2S}{dt^2} > 0$, the corresponding stock has a minimum value at time t .

In the other hand, if a net flow is zero and the rate of change in net flow is negative at time t , $\frac{dS}{dt} = 0$ and $\frac{d^2S}{dt^2} < 0$, the corresponding stock has a maximum value at time t .

If the rate of change in net flow is zero at time t , $\frac{d^2S}{dt^2} = 0$, the point is neither maximum nor minimum. In fact, the location where the graph changes in concavity is called a “**point of inflection**”.

So far, we have been looking at system behavior with respect to time. We know that stocks and flows can be mathematically defined as functions of time. However, they might involve some other entities such as other stocks, flows or some types of constants. These other entities can also affect the general shape of the corresponding graph.

Section 3.4: Graphical functions for empirical analysis

To investigate the relationship between a variable and a value of stock (or flow), the graph is changed into empirical form where the variable of interest is set to be an independent variable and everything else is held as constants. In other words, the horizontal axis represents various values of the variable instead of time. In this case, some values can be negative based on interpretation. Therefore, all four quadrants might be used to graph the relationship. Also, time is chosen at a certain fixed value which make the value varies in a certain range respect to the parameter.

Example:

Let us investigate the previous case of depopulation again. This time, we observe the relationship between the death rate and the average lifespan in the causal-loop diagram. Based on the population, the death rate can be calculated by differentiating the expression respect to time.

$$P = P_0 e^{-\frac{t}{l}} \quad (3-2)$$

$$\frac{dP}{dt} = -D = -\frac{P_0}{l} e^{-\frac{t}{l}} \quad (3-3)$$

$$D = \frac{P_0}{l} e^{-\frac{t}{l}} \quad (3-4)$$

where D is death rate

What if we change the value of the average lifespan l instead of time? In this case, time would be fixed at a point t_f .

$$D = \frac{P_0}{l} e^{-\frac{t_f}{l}} \quad (3-5)$$

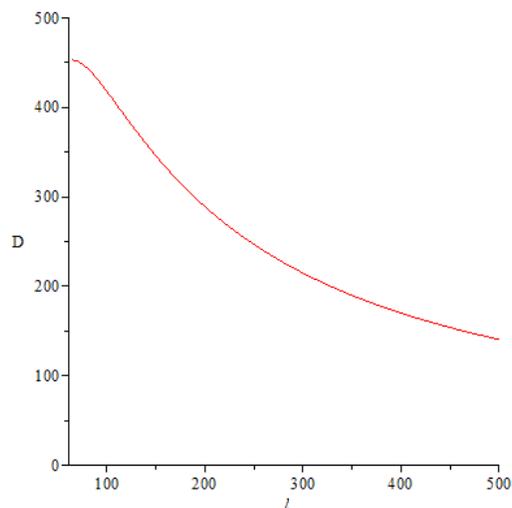


Figure 3-12: Population respect to average lifespan, $P_0 = 80,000$ units and $t = 65$ years,

Since the interested time interval is 65 year, the death rate is modeled at average lifespan of 65 years and beyond. In this case, the vertical axis is a reference axis for the death rate while the horizontal axis becomes a reference axis for average lifespan. By plotting the graph along both axes, we can observe relationship between the death rate and the average lifespan. According to Figure 3-12, when the average life span is very large, the death rate is very small in magnitude. In the other hand, when the average lifespan is small, the death rate becomes very large.

Based on observation, we can conclude that the link connected from average lifespan to the death rate has a negative polarity in the causal loop diagram. Since the change in death rate respect to average life span is negative, the relationship between them can be rewritten as the following.

$$\frac{\partial D}{\partial l} < 0 \quad (3-6)$$

Based on the example, the empirical form shows a clear distribution of the interested parameter as well as the polarity of the link between variables.

In fact, link's polarities can be determined based on partial derivatives. If the partial derivative is positive, the link from independent variable to the dependent variable is positive.

$$\frac{\partial D}{\partial P} > 0 \quad (3-7)$$

In the other hand, if the partial derivative is negative, the link from independent variable to the dependent variable is also negative.

One can clearly see that Cartesian graph cannot just only present the dynamic behavior of a system, but also conveys the relationship between different entities of the internal structure (Sterman, 2000).

Summary

Aside from Causal-loop and Stock-Flow diagrams, Cartesian graph is used as a tool for visualizing and interpreting an individual stock or flow. This graphical representation is used for analyzing a system with respect to time in order to understand its dynamic behavior. Sometimes, a flow is used for graphically identifying general behavior of a stock. Also, Cartesian graph can be used for representing a relationship between two different entities. The use of Cartesian graph is applicable to any mathematic modeling especially System Dynamics.

Chapter 4: Delays and Relationship to Integration

Prerequisites

Differentiation (Appendix A)

Integration (Appendix B)

Differential Equations (Appendix C)

Motivation

One of very common behaviors of a system is **Delays** which could be catastrophic if one did not fully understand. Delay might produce a late response which creates a difficulty to reach a desired state. Therefore, it is essential to study the theory behind delays in order to predict and improve the instability of the systems. In general, there are two types of delays.

Section 4.1: First-order and Higher-order Material delay

The first type of delays is called “Material delay”. When the outflow of a system is unequal to the inflow, the missing amount of flow must accumulate in the stock. Therefore, the stocks in the system are a source of material delay. In order to understand accumulation process of the flow, it is important to know the mathematical meaning of accumulation, as known as integration of flow.

Based on the mathematical definition of the integral, we can conclude that an amount of quantity inside of stock is the integration of total flows on the stock.

$$Stock(t) = \int_0^t [flows_{total}(s)] ds \quad (4-1)$$

where $flows_{total}(s)$ is a function of the total flow in a system.

Inflow indicates the increasing amount in the stock. In the other hand, outflow decrease the quantity in the stock. Therefore, in flow and out flow are positive and negative, respectively. Since there is always quantity in the stock at the initial time, the equation above becomes the following:

$$Stock(t) = \int_0^t [Inflow(s) - Outflow(s)] ds + Stock(0) \quad (4-2)$$

where $Stock(0)$ is the stock at the initial time. ($t = 0$)

Note that the initial stock does not have to be positive. It may be negative (in case of debts), null, or positive (capital).

Based on the definition of stock, the quantity in transit is simply the amount of quantity in the stock during a particular interval of time. Sometimes, this quantity is called “**Material in transit**”. If the quantity is accumulated in First-In-First-Out or Queue, the outflow is basically the inflow shifted by the delay time (Sterman, 2000).

$$Outflow(t) = Inflow(t - D) \quad (4-3)$$

where t and D denote time and delay time, respectively.

But in most of systems, the order of outflow does not depend on the inflow. The inflow can be well-mixed with the initial quantity in the stock and flow out which give each quantity a different delay time. In this case, the behavior of systems is calculated based on the **average delay time**. The outflow from a first-order material delay is linearly proportional to the stock by average delay time.

$$Outflow(t) = \frac{Stock(t)}{D} \quad (4-4)$$

$$\frac{d}{dt}S(t) = \frac{S(t)}{D} \quad (4-5)$$

$$\frac{d}{dt}S(t) - \frac{S(t)}{D} = 0 \quad (4-6)$$

where $S(t)$ is a stock.

Equation 4-6 is basically a first-order homogenous differential equation which can be solved by using integrating factor method. The following is the solution of Equation 4-6.

$$S(t) = S_0 e^{-\frac{t}{D}} \quad (4-7)$$

where S_0 is the initial quantity in stock

Mathematically, material delay is equivalent to the homogeneous solution of the differential equation.

In this case, the equation is in first order. Therefore, the solution is $S_0 e^{-\frac{t}{D}}$. With the average delay time, the system can also be modeled by Stock-Flow diagram.

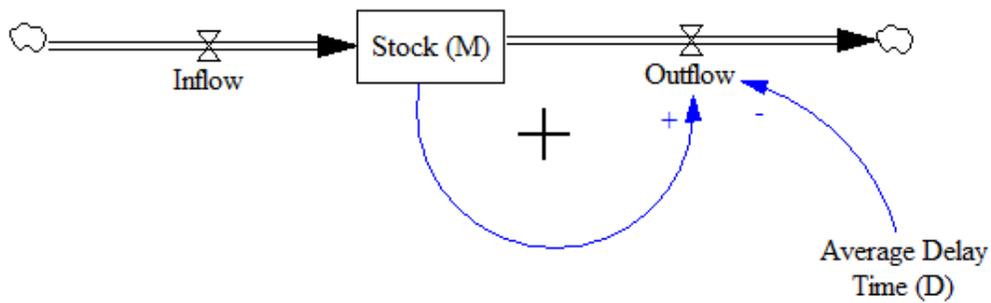


Figure 4-1: Stock- Flow Diagram of First-order Material Delay

The material delays also occur in higher order system where there are multiple stocks. The higher-order delays can be represented as n amount of first-order delays in cascade. Assuming that the average delay time in each delay is the same, the higher-order delays is equivalent to the Erlang distribution of order n which given by the following (Sterman, 2000).

$$S(t) = \frac{\left(\frac{n}{D}\right)^n}{(n-1)!} t^{n-1} e^{-\frac{n}{D}t} \quad (4-8)$$

However, most of complex system does not only contain stocks in cascade. Therefore, equation 4-8 is just a special case of higher-order delays. In order to solve for general higher-order delays, a system of differential equations is needed. The homogeneous solution of the differential equation basically corresponds to the internal structure that causes delays. Most of the time, computer simulation is required for analyzing higher-order delays with large number of stocks.

Material delays are generally caused by the structure of the system which exhibit is every dynamic system. Besides material delays, there is another type of delay that caused by human response which have a similar behavior as material delays. This type of delays is called information delays.

Section 4.2: Smoothing and information delays:

Information delays are based on human perception toward the information feedback. Unlike material delays, information delays are not created by physical inflow. Therefore, modeling a system with information delays requires a different approach.

Before we perform a mathematic analysis, let us think about psychological process of human mind. A person tends to create some delays due to the interpretation of information. Therefore, an expected value usually takes account some delays to reach the actual value of input. One of many reasons for delays is the difficulty in distinguishing between change in input and a noise in the system. Therefore, one might proportionally change the expected value to the amount of error. When the error is small, the value changes by small amount.

The error is basically the different between the actual value and the expected value which can be represented by the following mathematic expression.

$$\Delta S = \hat{S} - S \quad (4-9)$$

where ΔS is the error, \hat{S} is the actual value of the input, and S is the expected value.

The rate of change is directly proportional to the error which can be mathematically defined as following.

$$\frac{dS}{dt} = \frac{\Delta S}{\tau} = \frac{\hat{S} - S}{\tau} \quad (4-10)$$

$\frac{dS}{dt}$ denotes the rate of the expected value (inflow). The symbol τ is called “**adjustment time**” or “**time constant**”.

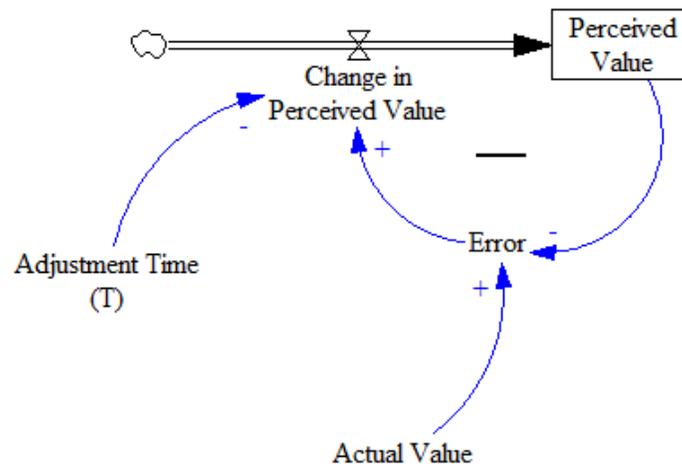


Figure 4-2: Stock & Flow Diagram of Information Delay

With a given specific function of actual value, the function of expected value can be calculated by solving the equation. However, this “differential equation” involves the derivative of a variable. Therefore, it is essential to understand how to solve a differential equation. The general behavior of this information delay is called “**exponential smoothing**” (Sterman, 2000).

Example:

Let us say that the actual and perceived values of stock are initially at steady state before time $t = 0$ months. The initial value for both cases is given as 10 units. After time $t = 0$ months, the actual value of stock changes to 20 units. How would the perceived value of stock behave? In other words, what is the mathematic expression of the perceived value?

The actual value can be mathematically written as following.

$$\hat{S} = 20 \quad (4-11)$$

The rate of change of expected value eventually becomes the following expression.

$$\frac{dS}{dt} = \frac{20 - S}{\tau} \quad (4-12)$$

where τ is a constant.

This first-order linear differential equation can be solved by using integrating factor method. The equation needs to be written in form of following.

$$\frac{dS}{dt} + p(t)S = f(t) \quad (4-13)$$

Therefore, the equation becomes the following.

$$\frac{dS}{dt} = \frac{20}{\tau} - \frac{S}{\tau} \quad (4-14)$$

$$\frac{dS}{dt} + \left(\frac{1}{\tau}\right)S = \frac{20}{\tau} \quad (4-15)$$

In this case, the integrating factor $\mu(t)$ can be calculated by the following definition.

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{1}{\tau} dt} = e^{\frac{t}{\tau}} \quad (4-16)$$

By multiplying $\mu(t)$ to the equation and integrating both sides of the equation, the expected value S can be found.

$$e^{\frac{t}{\tau}} \left[\frac{dS}{dt} + \left(\frac{1}{\tau}\right)S \right] = e^{\frac{t}{\tau}} \left[\frac{20}{\tau} \right] \quad (4-17)$$

$$\int e^{\frac{t}{\tau}} \left[\frac{dS}{dt} + \left(\frac{1}{\tau}\right)S \right] dt = \int e^{\frac{t}{\tau}} \left[\frac{20}{\tau} \right] dt \quad (4-18)$$

$$e^{\frac{t}{\tau}} S(t) = 20e^{\frac{t}{\tau}} + C \quad (4-19)$$

$$S(t) = 20 + Ce^{-\frac{t}{\tau}} \quad (4-20)$$

Notice that C is a constant of integration which needs to be defined. To solve for the constant, the initial condition must be applied. Because the expected value was initially 10 units at time = 0 ,

$$S(0) = 10 \quad (4-21)$$

Then, we can solve by substituting the expected value by 0.

$$S(0) = 20 + C e^{-\frac{(0)}{\tau}} \quad (4-22)$$

$$10 = 20 + C \cdot (1) \quad (4-23)$$

$$C = -10 \quad (4-24)$$

Now, we can defined that the expected value is

$$S(t) = 20 - 10e^{-\frac{t}{\tau}} \quad (4-25)$$

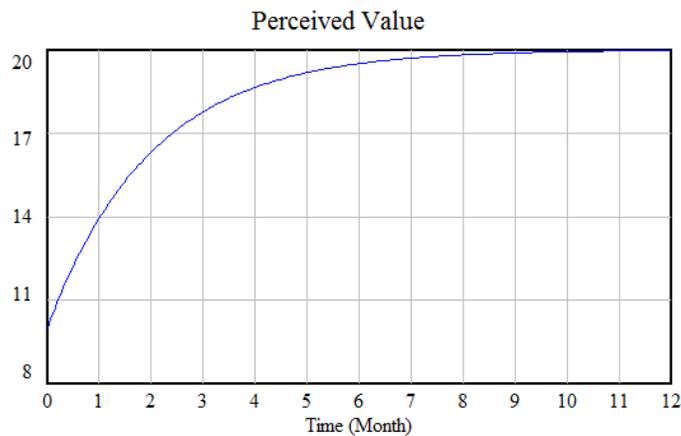


Figure 4-3: Perceived Value respect to time, $\tau = 2$,

Based on the example, one can clearly see that the perceived value is approaching to the actual value as the time increases. The characteristic of exponential smoothing that defined the general shape of the graph is the adjustment time.

The adjustment time τ define the **settling time** of a system. Settling time refers to the amount of time the function takes to reach the error of 2%. Therefore, different adjustment time gives different corresponding settling time. If a system takes very long time to adjust, the system will take long time to settle (Nise, 2010).

In more generic case, equation 4-7 can be solved and turned into the following.

$$S(t) = e^{-\frac{t}{\tau}} \int \left(\frac{\hat{S}}{\tau} \cdot e^{\frac{t}{\tau}} \right) dt + C e^{-\frac{t}{\tau}} \quad (4-26)$$

where $S(t)$ is a function of expected value, t is time, τ is adjustment time, \hat{S} is a function of an actual input, and C is a constant of integration that make the equation satisfy the initial condition.

Based on the knowledge of differential equation, one can say that the term $e^{-\frac{t}{\tau}} \int \left(\frac{\hat{S}}{\tau} \cdot e^{\frac{t}{\tau}} \right) dt$ is a particular solution. And the term $Ce^{-\frac{t}{\tau}}$ is homogeneous solution. The particular solution is also called “steady-state solution”. And the homogeneous solution is called “transient solution”.

In system dynamics, the goal of this system is to achieve the actual value. By taking the limit of time to infinity, the transient solution becomes zero which leave just only steady-state solution. However, the steady-state solution might never reach the actual value.

Example:

Let assume that the actual value oscillates over a fixed bound which can be mathematically represented as the following equation.

$$\hat{S} = A \sin(\omega t) \tag{4-27}$$

where A is an amplitude of the oscillation, and ω is the angular frequency of the oscillation.

By substituting the actual value into the equation of an expected value, we will obtain the following differential equation.

$$S(t) = e^{-\frac{t}{\tau}} \int \left(\frac{A \sin(\omega t)}{\tau} \cdot e^{\frac{t}{\tau}} \right) dt + Ce^{-\frac{t}{\tau}} \tag{4-28}$$

According to the integral in the appendix B, the integral in the equation can be simplified as following.

$$\int \left(\frac{A \sin(\omega t)}{\tau} \cdot e^{\frac{t}{\tau}} \right) dt = \frac{A}{1 + \omega^2 \tau^2} \cdot e^{\frac{t}{\tau}} \cdot (\sin(\omega t) - \omega \tau \cos(\omega t)) \tag{4-29}$$

Therefore, the expected value can be expressed with a function in term of time.

$$S(t) = e^{-\frac{t}{\tau}} \left[\frac{A}{1 + \omega^2 \tau^2} \cdot e^{\frac{t}{\tau}} \cdot (\sin(\omega t) - \omega \tau \cos(\omega t)) \right] + Ce^{-\frac{t}{\tau}} \tag{4-30}$$

$$S(t) = \frac{A}{1 + \omega^2 \tau^2} \cdot (\sin(\omega t) - \omega \tau \cos(\omega t)) + Ce^{-\frac{t}{\tau}} \tag{4-31}$$

The expression above is not trivial but we can either plot the graph or simplify the expression with some trigonometric identities.

$$S(t) = \frac{A}{1 + \omega^2 \tau^2} \cdot \left(\sqrt{1 + \omega^2 \tau^2} \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \sin(\omega t) - \sqrt{1 + \omega^2 \tau^2} \frac{\omega \tau}{\sqrt{1 + \omega^2 \tau^2}} \cos(\omega t) \right) + Ce^{-\frac{t}{\tau}} \tag{4-32}$$

$$S(t) = \frac{A}{1 + \omega^2\tau^2} \cdot \sqrt{1 + \omega^2\tau^2} \left(\frac{1}{\sqrt{1 + \omega^2\tau^2}} \sin(\omega t) - \frac{\omega\tau}{\sqrt{1 + \omega^2\tau^2}} \cos(\omega t) \right) + Ce^{-\frac{t}{\tau}} \quad (4-33)$$

$$S(t) = \frac{A}{\sqrt{1 + \omega^2\tau^2}} \cdot (\cos(\delta) \sin(\omega t) - \sin(\delta) \cos(\omega t)) + Ce^{-\frac{t}{\tau}} \quad (4-34)$$

$$S(t) = \frac{A}{\sqrt{1 + \omega^2\tau^2}} \cdot \sin(\omega t - \delta) + Ce^{-\frac{t}{\tau}} \quad (4-35)$$

where $\delta = \tan^{-1}(\omega\tau)$

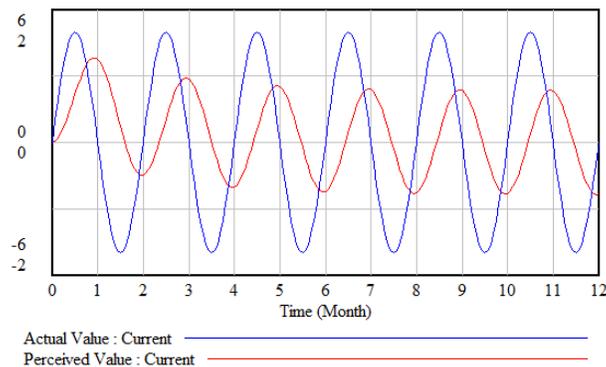


Figure 4-4: Steady-State Error

There are two points in the equation that worth notice. The first one is the amplitude of the expected value. The amplitude of actual value is A while the amplitude of the expected value is A divided by a positive number that greater than 1. This implies that the expected value never reach the actual value in amplitude. The second point is that both values have the same angular frequency. However, the expected value has a phase angle which shifts the graph by $\frac{\delta}{\omega}$ units. This also implies that there is a delay in a system and the expected value never reach the actual value at steady state. This situation is described as “**steady-state error**” where the perceived value never reach the actual value at steady state (Nise, 2010).

With general exponential smoothing, delays can cause a steady-state error. Therefore, the structure of the system needs to be changed by adjusting time constant or adding more state variables.

Summary

Delays exist in every dynamic system because of the process of integration. Late responses from the delays can create steady-state error to the system. These delays are categorized into two different types, material delays and information delays. Material delays are caused by physical structure of the system unlike information delays that caused by human’s response. Regardless of the causes, these two types of delays shared the similar behavior. Such behavior can be simulated by using System Dynamics.

Chapter 5: Numerical methods for Solution

Prerequisites

Differentiation (Appendix A)

Differential Equations (Appendix C).

Motivation

In more complex system, an analytical method for solving a system becomes more complicated. Sometimes, it is impossible to solve for a closed form solution. However, a solution to the system can be approximated by using computer simulation. Because a computer is capable of doing recursive computation in a relative short period of time, system dynamicists use it to approximate the behavior of their model. The logistics behind the computation is based on mathematics which is implemented in system dynamics software. Most of software provides a manual but understanding the logics of the software will facilitate the modeling. Therefore, we will discuss about different methods of approximation used in the software.

Section 5.1: Euler's Method

In system dynamics, a net flow of a stock (the derivative of the stock) is defined as some function of variables and constants. Since most of system has feedbacks, the net flow will depend on the stock. Therefore, a net flow of a stock can be represented by the following expression.

$$\text{Net Flow} = \frac{dS}{dt} = f(S, t) \quad (5-1)$$

where S is an amount of quantity in the stock, t is time, and $f(S, t)$ is a function that depends on S and t

For the ease of approximation, the derivative of the stock is approximated to an average rate of change in a very small time step.

$$\frac{\Delta S}{\Delta t} = \frac{S_k - S_0}{t_k - t_0} = f(S_0, t_0) \quad (5-2)$$

where S_k is the stock at initial time t_k and S_0 is the stock at initial time t_0

The Stock at time t can be written as following.

$$S_k = S_0 + f(S_0, t_0)(t_k - t_0) \quad (k) \quad (5-3)$$

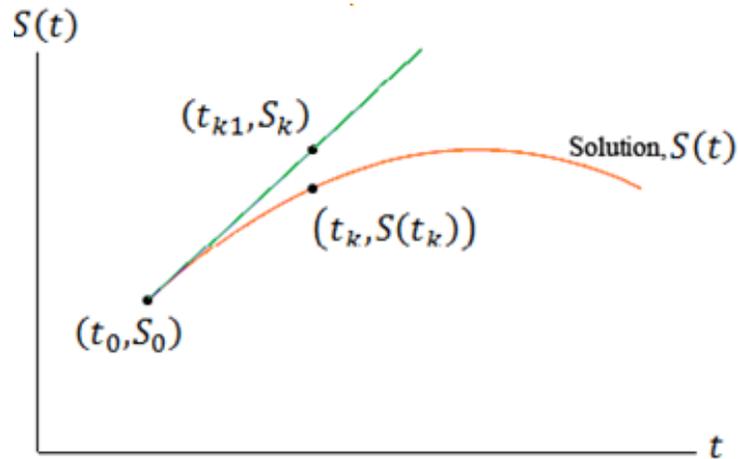


Figure 5-1: Approximation using slope

According to Figure 5-1, there is an error between the actual stock and the approximated stock. But if the time step Δt is set to a small value, the error will become less. Based on equation k, the initial stock S_0 is needed to be assigned as well as the time step.

In order to approximate the value at time t_n based on the initial value time t_0 , the total time interval is divided by an integer which indicates the amount of computations needed to perform in order to obtain the value. The ratio between the total time interval and amount of computations is the time step.

$$\Delta t = \frac{t_n - t_0}{n} \quad (5-4)$$

where Δt is time step and n is the number of computations

Assuming that the initial stock at time t_0 is S_0 , the stock of the first time step is equal to the following expression.

$$S_1 = S_0 + f(S_0, t_0) \cdot \Delta t \quad (5-5)$$

The result from the first computation is used in the second computation with the same amount of time step at time t_1 .

$$S_2 = S_1 + f(S_1, t_1) \cdot \Delta t \quad (5-6)$$

The time t_1 is the initial time t_0 added with one time step $\cdot \Delta t$

$$t_1 = t_0 + \Delta t \quad (5-7)$$

The same concept is applied to the third computation which basically based on the second computation.

$$S_3 = S_2 + f(S_2, t_2) \cdot \Delta t \quad (5-8)$$

In order to reach the time interval $t_n - t_0$, n amount of computations are needed. Therefore, the stock at time t_n is based on previous n amount of computations.

$$S_n = S_{n-1} + f(S_{n-1}, t_{n-1}) \cdot \Delta t \quad (5-9)$$

This numerical method is called "Euler's method". Even though a small time step makes the estimate value more precise, it also requires more time to compute the value for each increment of time step (Farlow, 2006).

Example:

A net flow of a stock is given as the following.

$$\frac{dS}{dt} = 0.05 \cdot S, \quad S(0) = 1 \quad (5-10)$$

What is the amount of stock at t time $t = 4$ using Euler's method with time step of a) $\Delta t = 1$ and b) $\Delta t = 0.5$?

The analytical method can be performed in order to solve the problem. The analytical solution of the problem is equal to the following.

$$S(t) = e^{0.05t}, \quad S(4) \approx 1.221410 \quad (5-11)$$

But in this case, we will perform Euler's method for two different time steps.

Case 1: $\Delta t = 1$

The amount of computation is the ratio between the interested time and the time step. Therefore, the amount of computation for 1 second time step is equal to 4. The stock at time $t = 0$ is given as 1. Then, the stock at the next time step is equal to the following.

$$S_1 = S_0 + f(S_0, t_0) \cdot \Delta t \quad (5-12)$$

$$S_1 = (1) + (0.05 \cdot 1) \cdot (1) = 1 \quad (5-13)$$

The value of the first computation is used as a n input for the second computation. The time is incremented by one time step.

$$S_2 = S_1 + f(S_1, t_1) \cdot \Delta t \quad (5-14)$$

$$S_2 = (1) + (0.05 \cdot 1) \cdot (1) = 1.05 \quad (5-15)$$

The computation is repeated until the time is incremented to $t = 4$.

$$S_3 = S_2 + f(S_2, t_2) \cdot \Delta t \quad (5-16)$$

$$S_3 = (1.05) + (0.05 \cdot 1.05) \cdot (1) = 1.1025 \quad (5-17)$$

$$S_4 = S_3 + f(S_3, t_3) \cdot \Delta t \quad (5-18)$$

$$S_4 = 1.1025 + (0.05 \cdot 1.10255) \cdot (1) = 1.15763 \quad (5-19)$$

Therefore, the stock at time $t = 4$ is approximately 1.15763.

Case 2: $\Delta t = 0.5$

In this time, the time step decrease to 0.5 seconds which implies that the amount of computation will increase.. The amount of computation for 0.5 second time step is equal to 8. The stock at time $t = 0$ is given as 0. Then, the stock at the next time step is equal to the following.

$$S_1 = S_0 + f(S_0, t_0) \cdot \Delta t \quad (5-20)$$

$$S_1 = (1) + (0.05 \cdot 1) \cdot (0.5) = 1 \quad (5-21)$$

Time is incremented by one time step. In this case, time is increased by the interval of 0.5 seconds.

$$S_2 = S_1 + f(S_1, t_1) \cdot \Delta t \quad (5-22)$$

$$S_2 = (1) + (0.05 \cdot 1) \cdot (0.5) = 1.025 \quad (5-23)$$

The computation is repeated until the time is incremented to $t = 4$.

By performing Euler's method with 0.5 s time step, the solution of a stock is approximately 1.218403 units.

Based on these two cases, we can conclude that there is a trade-off in choosing a time step for numerical computation. If time step is large, the amount of computation becomes a few which cause the simulation to compute with faster rate. However, the numerical value of big time step contains large discrepancy compared to the actual solution. Therefore, a small time step is preferable although it requires more computing time.

Section5.2: Second-order Runge-Kutta Method

In higher order system, Euler's method becomes inefficient for approximating quantity in stock. The idea of using the instantaneous flow to compute the stock is needed to be modified in order to calculate more accurately. One of the methods that are widely used for approximating is **Second-order Runge-Kutta method**. The logistics behind the method is to average two instantaneous rates (Shampine, 1994).

In order to approximate the slope of the function m has to be redefined.

$$\frac{dS}{dt} = f(S, t) \quad (5-24)$$

$$S_1 = S_0 + m \cdot \Delta t \quad (5-25)$$

In Euler's method, the slope of the function is basically equal to the derivative at the starting time. In Runge-Kutta Method, the slope of the function is the average value between the derivative at the starting time and the derivative at the next time step.

The slope at the starting time is equal to the following.

$$k_1 = \frac{dS}{dt}_{t_0} = f(S_0, t_0) \quad (5-26)$$

Based on Euler's method, the stock at the next time step is simply equal to the following.

$$S_1 = S_0 + k_1 \cdot \Delta t \quad (5-27)$$

And the slope at one time step later is equal to the following.

$$k_2 = \frac{dS}{dt}_{t_0+\Delta t} = f(S_1, t_0 + \Delta t) = f(S_0 + k_1 \cdot \Delta t, t_0 + \Delta t) \quad (5-28)$$

Therefore the average of both values is expressed as following.

$$m = \frac{k_1 + k_2}{2} = \frac{f(S_0 + k_1 \cdot \Delta t, t_0 + \Delta t) - f(S_0, t_0)}{2} \quad (5-29)$$

The stock at time t_1 is approximately equal to the initial stock plus the product of slope and the time step.

$$S_1 = S_0 + m \cdot \Delta t = S_0 + \frac{f(S_0 + k_1 \cdot \Delta t, t_0 + \Delta t) - f(S_0, t_0)}{2} \cdot \Delta t \quad (5-30)$$

Let us observe the solution of the previous problem with Runge-Kutta Method.

Example:

A net flow of a stock is given as the following.

$$\frac{dS}{dt} = 0.05 \cdot S, \quad S(0) = 1 \quad (5-31)$$

What is the amount of stock at t time $t = 4$ using Runge-Kutta method with time step of $\Delta t = 1$?

Because the method computes number of solution based on more than one derivative, the time it takes to finish computation is longer than Euler's method. However, the amount of computation is calculated the same as Euler's method. Since the interested time interval is 4 and the time step is 1. The amount of computation is simply equal to 4.

The slope at the starting time can be found by calculating the derivative at the initial point.

$$k_1 = \frac{dS}{dt}_{t_0} = 0.05 \cdot S_0 = 0.05 \cdot (1) = 0.05 \quad (5-32)$$

Therefore, the corresponding stock at the next time step using Euler's method is equal to the initial stock plus the product of the slope at starting point and time step.

$$S_1 = S_0 + k_1 \cdot \Delta t = 1 + (0.05)(1) = 1.05 \quad (5-33)$$

The slope at the next time step can be found by calculating the derivative at the next time step.

$$k_2 = \frac{dS}{dt}_{t_0+\Delta t} = 0.05 \cdot S_1 = 0.05 \cdot 1.05 = 0.0525 \quad (5-34)$$

The average of both slopes is the following.

$$m = \frac{k_1 + k_2}{2} = \frac{0.05 + 0.0525}{2} = 0.05125 \quad (5-35)$$

Therefore, the stock at the next time step is now calculated based on the average of the slopes.

$$S_1 = S_0 + m \cdot \Delta t = (1) + 0.05125 \cdot (1) = 1.05125 \quad (5-36)$$

Then the process entire process starts again with new initial conditions until the total amount of computation is 4.

Number of Computation	k_1	Euler's Method	k_2	Average Slope $\left(\frac{k_1 + k_2}{2}\right)$	Runge-Kutta
1	0.05	1.05	0.0525	0.05125	1.05125
2	0.0525625	1.1038125	0.055190625	0.053876563	1.105126563
3	0.055256328	1.160382891	0.058019145	0.056637736	1.161764299
4	0.058088215	1.219852514	0.060992626	0.05954042	1.221304719

It is apparent that the numerical solution of Runge-Kutta method is very accurate compare to Euler's method with the same time step. The percentage error of the numerical solution is about 0.008027 % which is relatively small. Therefore, Second-order Runge-Kutta method is more preferable than Euler's method in accuracy. However, Runge-Kutta method takes longer time to compute than Euler's method does.

Summary

In this chapter, we discussed about two of many different computational methods which are used in system dynamics software. One of the fastest computational methods is Euler's method. The logistic is simple; however, it gives more error within larger time step. The more accurate method is Runge-Kutta method which yields very small error. In the other hand, it takes more computation time. Therefore, It is essential to understand the trade-offs from choosing one method over the others.

Chapter 6: Common Behavioral Patterns and their Underlying Structures

Prerequisites

Differential Equations (Appendix A).

Laplace Transform and Applications (Appendix D).

Motivation

This section introduces fundamental structures of dynamical systems. General behaviors are shown with an emphasis on both variant and invariant inputs to model reality at best. Similar behaviors are encountered in different fields, from strictly engineering applications to business analyses. In each discipline, the system subject to study is modeled using different methods and graphical representations. The common denominator between these different approaches is the mathematical description of the system. As a matter of fact, all scientists and engineers solve systems of similar complexity without even noticing. Modeling in different domains such as time and frequency yield to different perspectives of the exact same behavior. This section will relate critical System Dynamics approaches to other interesting engineering methodologies. Major goals are relating transient and frequency analyses and using Control Theory techniques to study some system characteristics such as stability.

Section 6.1: Exponential Growth

This section will introduce the most common first-order structure in system dynamics. For the sake of simplicity, the major focus will be on step inputs and their growing exponential response. Besides the usual transient analysis, the system will be analyzed in frequency domain. Other graphical presentations used in Control Theory (transfer function and zero-pole plot) will be introduced.

Exponential growth is an important behavior that usually describes such phenomena as population growth and compounded interest. The general formula of continuous compound interest is defined to be:

$$A(t) = A_0 e^{rt} \quad (6-1)$$

where A_0 is the initial amount, r the interest rate, and $A(t)$ the amount at any time t . It is acceptable to think of the bank account as a stock S containing the amount of money at any time. The derivative of $A(t)$ with respect to time ($rA_0 e^{rt}$) would then be a change in the stock ΔS . This approach leads to the following differential equation:

$$\dot{S} = rS \quad (6-2)$$

where \dot{S} is the derivative of S in respect to time (thus accounting for ΔS). Setting the initial amount available in the stock (Eq. 6-3) would create an initial-value problem whose solution is the defined compound interest formula.

$$S(0) = A_0 \quad (6-3)$$

Example 1 in Section 3.3 shows the solving procedure for similar differential equation.

Let us ignore the technical aspect of the problem for the moment, and focus on the representation tools learned so far. A causal loop diagram would be a good candidate to create a simple and intuitive model. A good way to start is to identify the elements of the diagram. The account balance is the first instinctive element since it is the stock that accumulates money over time. As the stock accumulates, the net increase of the balance increases as well. That would be represented by a positive link pointing from the account balance to the net increase rate. The net change also depends on the interest rate in a directly proportional fashion. And finally, the larger increase rate, the large the account balance (positive feedback). Note the difference between the net rate of the stock $\frac{dA(t)}{dt}$ and the interest rate r are not interchangeable. The following causal loop diagram shall eliminate any possible confusion.

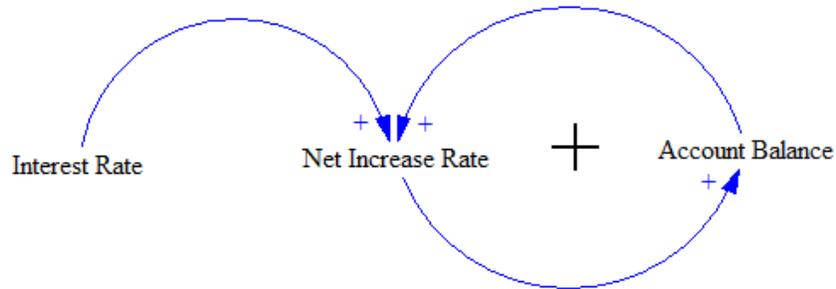


Figure 6-0-1: Causal Loop Diagram for Continuous Interest Compounding

The only loop created solely made up of positive links. The overall loop polarity is therefore positive. One key to success in system dynamics is simulation. The system should first be translated into a stock and flow diagram so that each element is given a particular role. The balance is obviously the only stock present in the diagram above. The net increase rate is therefore the inflow going into the account balance. Finally, the interest rate is an independent constant.

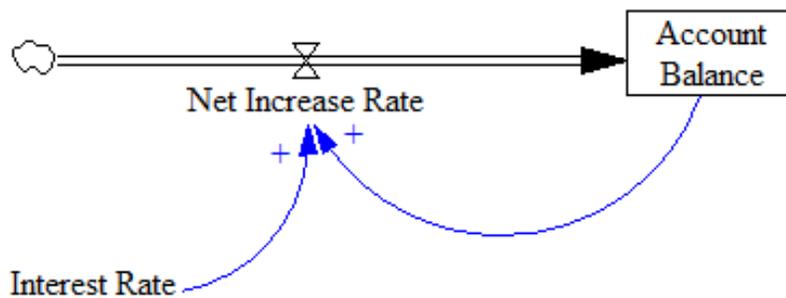


Figure 6-0-2: Stock and Flow for Continuous Interest Compounding

The above graphs are designed in Vensim¹. Each component of the configuration is mathematically defined by an equation. The account balance is defined as the integral of the net increase rate. The increase rate is described as the balance multiplied by the interest rate which is set to be 10% in this example. Notice that the account balance and the net rate have a stock/flow relationship.

As mentioned in the previous section, these two key components are related by differentiation and integration. However, differentiation is considered to be an abstract relationship. It is not obvious to figure out the state of the system using the concept of derivative. Therefore, it is preferred to mathematically define the stock (as the Integral of the rate). Integration allows the analyst to verify the accumulated variable at any point in time. In this context, the rate of change of a variable clearly does not convey the amount accumulated in that stock. On the other hand, the account balance can always be checked to reveal the money accumulated at a specific moment.

Once the Vensim simulation is run, graphs of all the elements are plotted in function of time. Here are the following generated graphs in our example.

¹ Vensim is a useful simulation package for System Dynamics available at <http://www.vensim.com/download.html>

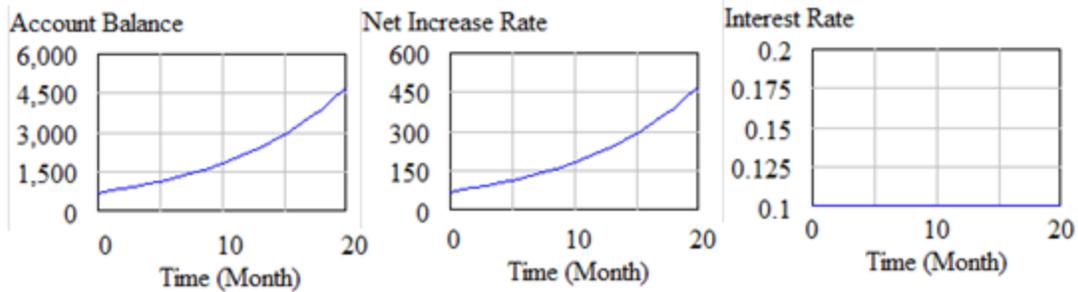


Figure 6-0-3: Vensim Simulation of Continuous Interest Compounding

The account balance exhibits an exponential behavior as predicted by Eq. 6-1. So does the increase rate (derivative of exponential function). The interest rate is set at a constant value of 10%. Vensim uses numerical integration methods such as Euler and Runge-Kutta.

Detailed Transient Analysis

In this sub-section, multiple frequency domain analysis representations are used to facilitate our transient analysis. The input and output of the present example are the initial and final stock respectively. The differential equation in Eq. 6-2 could be written as follows:

$$S_f - S_i = rS \quad (6-4)$$

where S_f and S_i are the initial and final stocks. After simple algebraic rearrangements, the system transfer function is:

$$T(t) = \frac{S_f(t)}{S_i(t)} = r + 1 \quad (6-5)$$

In control theory, systems are represented in **block diagrams** in the frequency domain (s). In the simplest cases, the diagram would contain one input signal entering one box (transfer function) where an output signal is leaving from other side. The current analysis is in the time domain. However, block diagrams are still used for understanding purposes. The following figure shows a block diagram of the system described in Eq. 6-5.

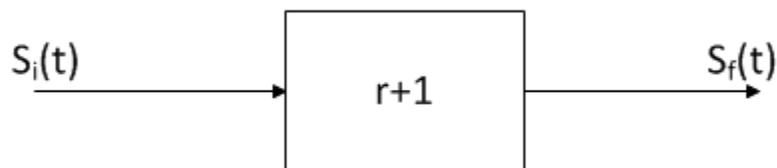


Figure 6-0-4: Block Diagram for System in Eq. 6-5

In order to study the behavior of system, it is important to analyze its function. In control theory, the system function is called a **Transfer Function**, and is in the s domain. Notice that we are still working in

the time domain. The **Bode plots** are one of the most important graphical representations of transfer functions. There are two types of bode plots: **Magnitude** and **Phase** plots. The magnitude plot shows the gain of the system $\frac{S_f(s)}{S_i(s)}$ (output/input) with respect to frequency (inverse of time). The phase plot shows how the output signal (S_f) responds to the input signal (S_i) in terms of delay or any phase difference. This concept will be explained in depth in the next section.

Software packages such as Matlab are very useful in graphing those plots. Let us make use of this frequency-domain representation technique in our transient analysis. The following figure is a bode plot of the transfer function in Eq. 6-5 and **Error! Reference source not found.**

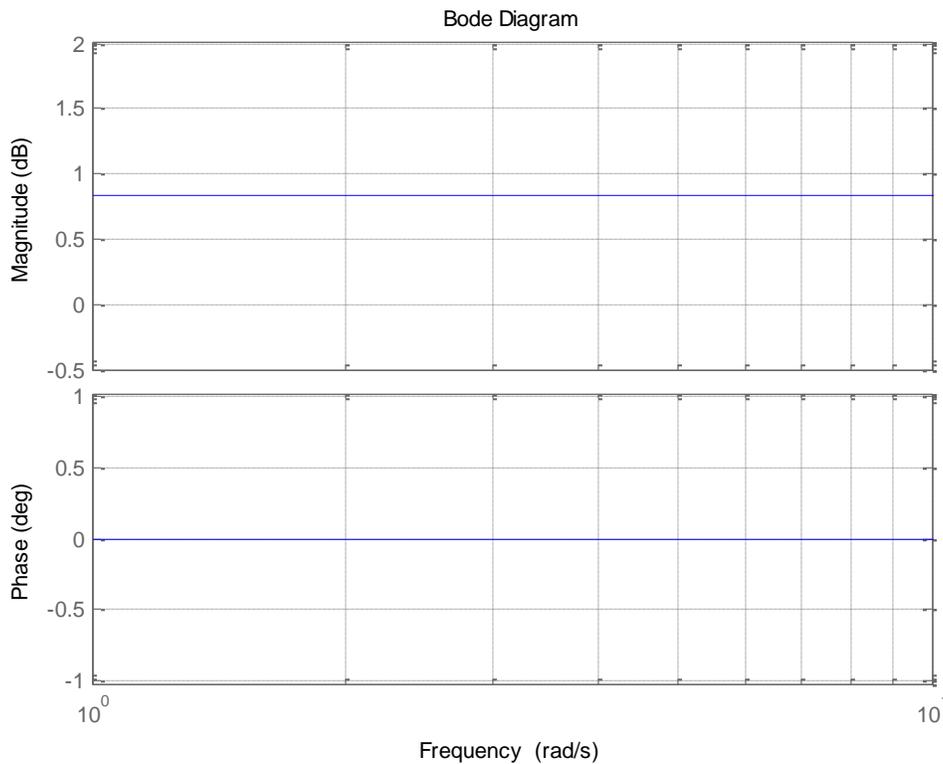


Figure 6-0-5: Bode Plots of System in Eq. 6-5

Let us first look at the magnitude plot. Note that the magnitude is constant throughout. The constant value of the magnitude is at around 0.83dB. It is said that the system provides a 0.83dB **gain**. Gain can also be expressed in terms of $\frac{un}{un}$ where un is the unit of both input and output, which is \$ in this case. Eq. 6-6 shows the relationship between gains in dB and $\frac{un}{un}$.

$$G_{dB} = 20\log\left(\frac{G_{un}}{un}\right) \quad (6-6)$$

So in our case, a gain of 0.83dB is equivalent to a gain of $1.1 \frac{\$}{\$}$. That means that the output signal (final stock) is 1.1 greater than the input signal (initial stock) all the time. That makes sense since the stock grows by 10% compared to its initial value.

Looking at the phase plot, the phase is constant at 0° all the time. Remember that our situation is ideal, where no delay is taken into account whatsoever.

The transient block diagram in **Error! Reference source not found.** is a simplified version of the original diagram shown in **Error! Reference source not found.**. The original transfer function is a unity gain block $\left(1 \frac{\$}{\$}\right)$ while the positive feedback loop box is the interest rate r . Starting from the initial value of the bank account (stock), the signal (amount of money) goes through the unity-gain box, gets multiplied by the feedback gain r , then join the summing point where its gets added to the initial stock. Going through the unity-gain block one more time and:

$$S_f = S_i(1 + r). \quad (6-7)$$

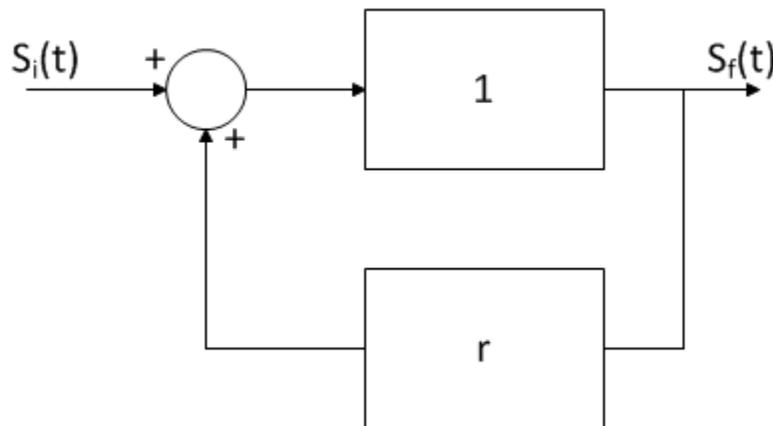


Figure 6-6: Block Diagram with Positive Feedback Loop for System in Eq. 6-5

As mentioned earlier, one loop leads to Eq.6-7. A loop can be thought of as a year; the final amount after t loops or years is therefore:

$$S_f = S_i(1 + r)^t. \quad (6-8)$$

Equation 6-7 is a very useful in determining the **time value of money** where the initial and final stocks are the present and future value of money respectively. If the interest is compounded n times a year, the equation becomes:

$$S_f = S_i \left(1 + \frac{r}{n}\right)^{nt}. \quad (6-9)$$

If the number of times the interest is compounded a year (n) is infinitesimally large, the compounding

become continuous, and Eq. 6-8 becomes Eq. 6-1. Note that in **Error! Reference source not found.**, the system has a positive feedback loop that is in charge of feeding the stock continuously (interest compounding) (Nise, 2010).

Transfer Function and Frequency Analysis

A transfer function is relation between the input and output of a system. Transfer Functions are widely used in **Control Theory** and several engineering disciplines (Nise, 2010).

$$T(s) = \frac{S_o(s)}{S_I(s)} \quad (6-10)$$

Note that Eq. 6-10 is expressed in the frequency domain (s) while Eq. 6-5 is in the time domain. The input is the initial amount A_0 and the output is the accumulated stock $A_0 e^{rt}$. Using Laplace transform, $S_I(s)$ is the step input $\frac{A_0}{s}$, and $S_o(s)$ is $\frac{A_0}{s-r}$. The transfer function of the system is therefore:

$$T(s) = \frac{s}{s-r} \quad (6-11)$$

The **zeros** of a function are the s values that force the numerator be zero. The numerator in this case is s , therefore the function has a zero at $s = 0$. The **poles** on the other hand, push the denominator to be zero. r is therefore the pole of $T(s)$. In a bode plot, the zeros increase the magnitude of the gain and phase while the poles decrease them. Since the zero (0Hz) happens before the pole (rHz), it is expected to have an increase in magnitude from 0Hz to rHz followed by a straight line resulting from the pole-zero cancellation effect as shown in **Error! Reference source not found.**

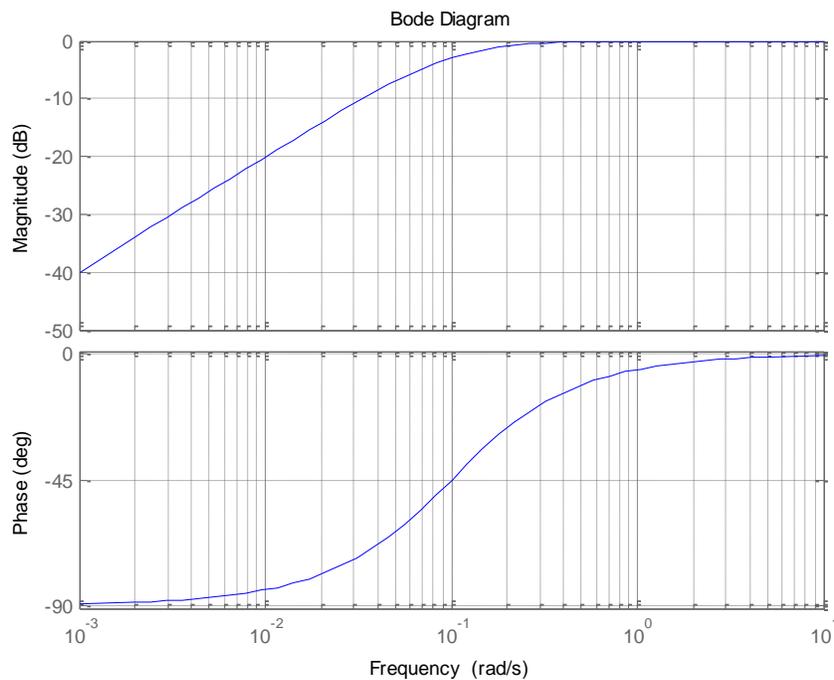


Figure 6-0-6: Bode Plots of System in Eq. 11

According to the bode plot above, the magnitude of the gain increases, and the phase delay decreases at higher frequencies. Furthermore, the gain is at its maximum when the frequency of the input function is higher than the rate. Also, the phase delay is approximately zero when the frequency of the input amount function is higher than $10r$. Notice that the initial amount A_0 is given once in a specific time without being panned over a large period. A_0 is a step function that has $\$x$ vertical change over a $0sec$ period. The frequency of A_0 is therefore high. If the initial amount were to be given over a large time period, the exponential response will have a lower magnitude, and less profit would be made. More generally, as the deposited money frequency increases, more profit is made. This makes totally sense since the more frequently we deposit money into our bank account; the larger is the revenue after any period of time.

Note that the magnitude plot shows the magnitude of the gain, not the gain itself (which can be negative or positive).

The **Zero-Pole Plot** is used to characterize the placement of the zeros and poles of the system (Nise, 2010).

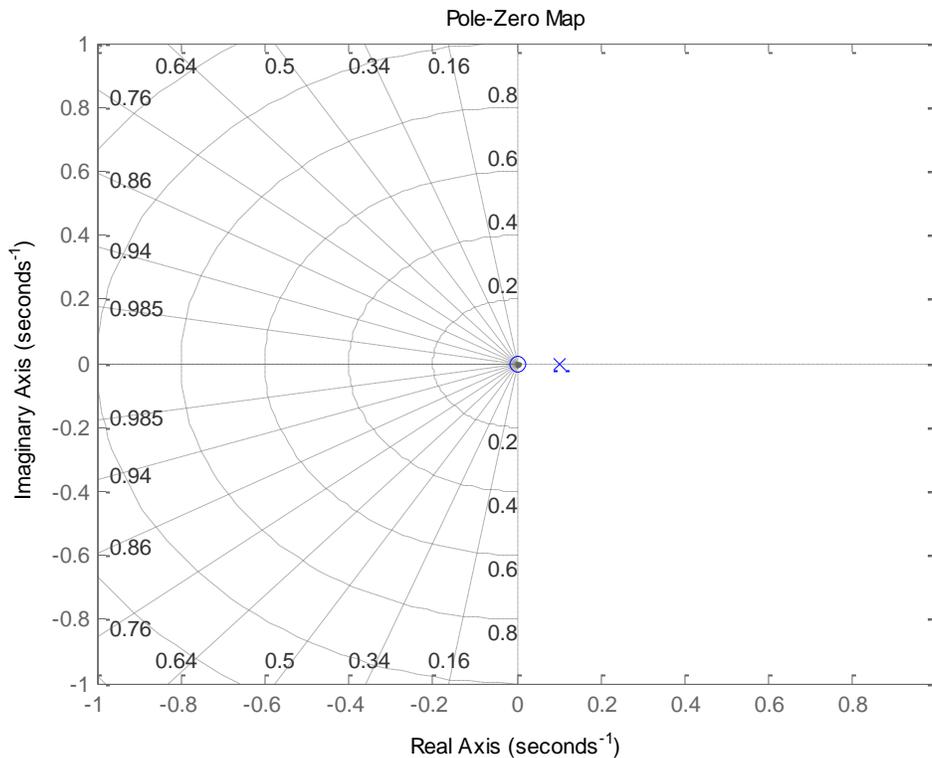


Figure 6-0-7: Zero-Pole Plot of Transfer Function in Eq. 6-11

In the figure above, the pole r is on the **positive** real axis. That explains the explosion of the exponential growth function over time. This system is therefore **unstable**. What if the poles were to be negative? Would the system be then stable?

Section 6.2: Goal Seeking

In the previous section, it was demonstrated that the **positive feedback** is the main cause of the positive growth of a function over time and the system's instability. **Negative feedback** is widely present in real life, and is in charge of decreasing the value of a stock over time. This section will analyze a common system relying on negative feedback. An interesting example of pollution control is used to picture the importance of modeling in policy making.

A typical type of structure in System Dynamics is **goal seeking** which happens when a function reaches a specific value after a certain time. This function is usually called a *bounded exponential* (Eq. 6-12).

$$A(t) = G \left(1 + \frac{A_0 - G}{G} e^{-rt} \right) \quad (6-12)$$

where A_0 is the initial amount (at $t=0$), r the decreasing rate, and G the goal-value that function reaches as it approaches infinity. In nature, the goal is the value dictated by the scientific laws ruling the universe. In industry for example, the goal is a desired value sought by a company. When the goal is higher than the initial condition, the curve exponential curve is at its increasing version (**Error! Reference source not found.a**). Otherwise, the curve is decreasing (**Error! Reference source not found.b**). *Exponential decay* is a special case of a decreasing bounded exponential when the final value (goal) approaches zero as time goes to infinity (Eq. 6-13).

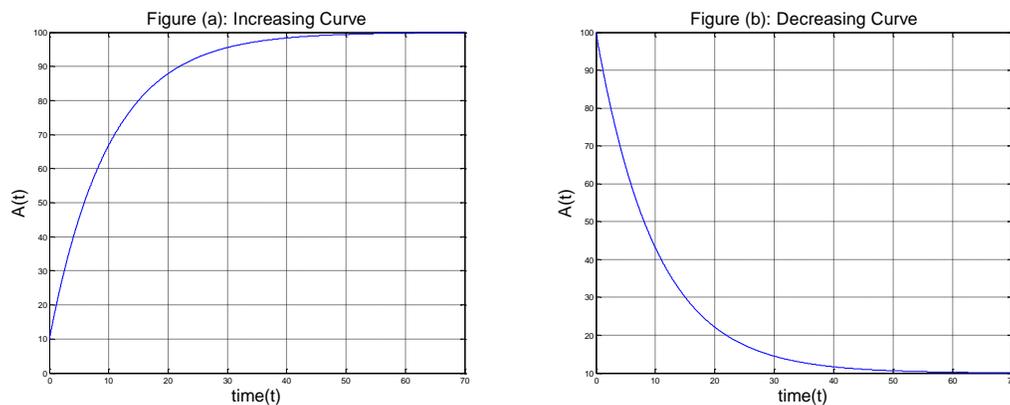


Figure 6-0-8: Goal Seeking Function: Increasing Curve (a) and Decreasing Curve (b)

$$A(t) = A_0 e^{-rt} \quad (6-13)$$

The cause behind such structure is a negative feedback loop that keeps minimizing the net flow of a certain stock. Such loops are incorporated into a system in order to control its final transient response. The intention of the negative feedback is usually to *keep an eye* on the output, and correct it each time

it is off the desired value (goal). As shown in **Error! Reference source not found.**, *corrections* are applied to the curve starting from the initial point until the desired goal is reached.

A common example of such a structure is the thermostat in air conditioner. The user sets a desired temperature value that the air conditioner tries to reach. The thermostat will keep correcting the outputted temperature until the temperature of the selected environment matches the desired value set by the user.

Negative feedback is a useful concept in policy changes as well. Let's say that a state is trying to decrease its pollution level by increasing the oil price, hoping that the emission of carbon monoxide will drop. Such step requires a deep of research and planning. On the other hand, the overall policy stratagem may be pictured in the following causal loop diagram.

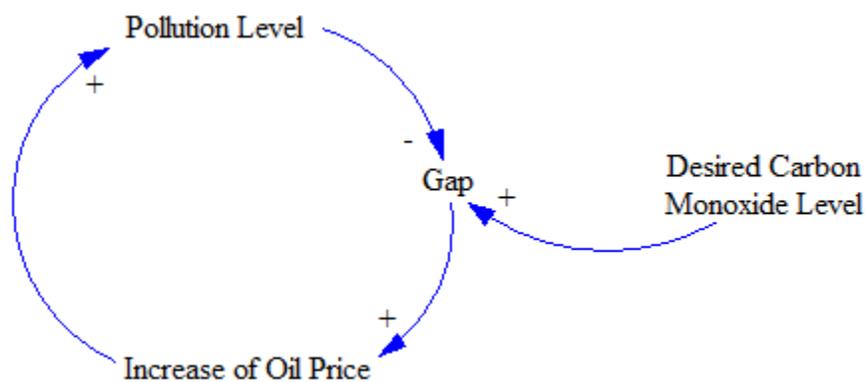


Figure 6-0-9: Causal Loop Diagram of Pollution Control

According to the signs of the individual links, the polarity of the loop is negative. In **Error! Reference source not found.**, the *Gap* is the discrepancy or difference between the desired goal and the actual value. Each time the stock (Pollution level) is fed the negative feedback, the actual level gets closer to the value sought by the stratagem.

In the stock and flow graph in **Error! Reference source not found.**, the stock *Pollution level* is defined as the integral of the net change of pollution (*Decrease Rate* – *Increase Rate*). The initial level of carbon monoxide is chosen to be 6 ppm (parts-per-million) while the desired goal is set to be 2ppm. The *Increase Rate of Pollution* is dependent on a function, that according to some variables (traffic jam, special gas discounts or coupons...). This function is set to zero in this model for the sake of simplicity. Of course, the *Decrease Rate of Pollution* is dependent on both the *initial* and *desired* pollution levels, as well as the *Percentage of Decrease* (Eq. 6-14). In real life, the percentage of pollution decrease should be dependent on specific variables such as the price effect on customers.

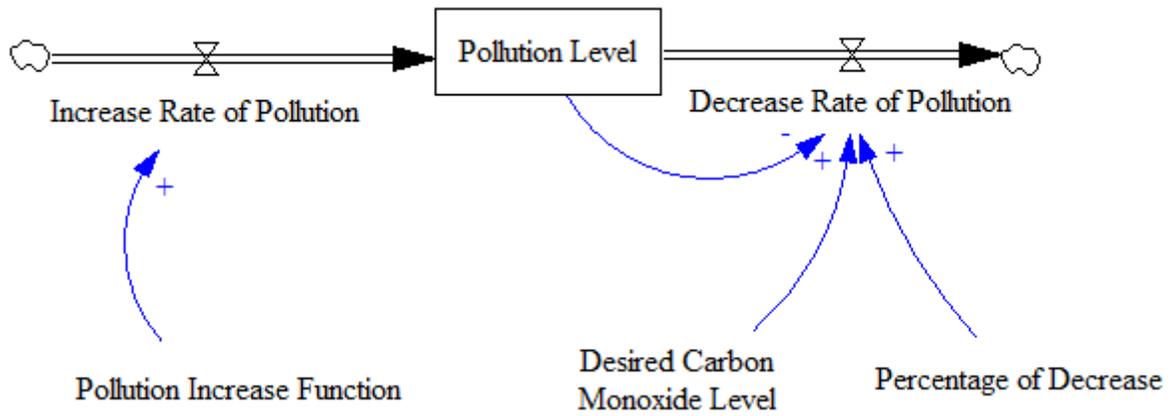


Figure 6-0-10: Stock and Flow of Pollution Control

$$\dot{A} = r(A - G) \quad (6-14)$$

where the negative sign of the decrease is taken care of by Vensim setup since that is the equation of the outflow. The percentage of decrease in this simulation was arbitrarily set to 0.025. The software package generated the following graph of *Pollution Level* where the vertical axis is the emission of carbon monoxide in ppm.

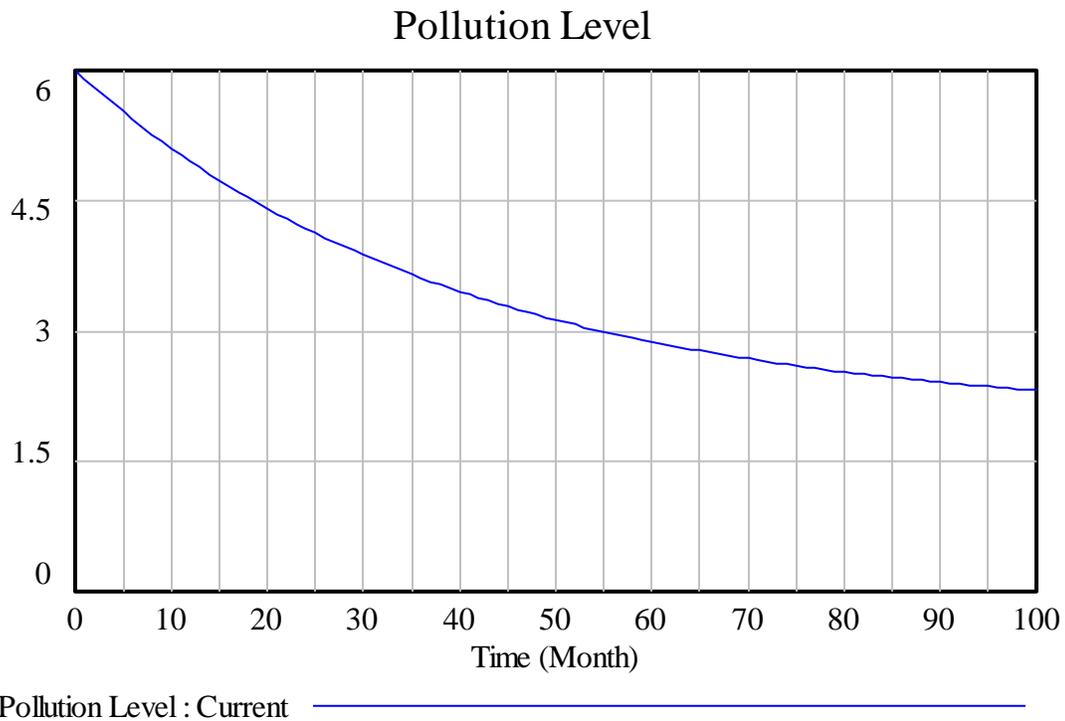


Figure 6-0-11: Carbon Monoxide Emissions (ppm) versus time

As mentioned before, the percentage of decrease is not constant in real life. Many factors may affect that percentage and disturb the smoothness of the exponential curve shown in **Error! Reference source not found.** Those percentages could be determined after careful studies and research of the society and economy is applied in. Instead, a ratio function depending on time could be found. Otherwise discrete ratio values could be determined and feed the outflow equation. Due to time constraints, a **lookup table (Error! Reference source not found.)** was created where percentages (from 0 to 0.025) were randomly assigned to specific times. That table will finally feed the outflow expression in the simulation.

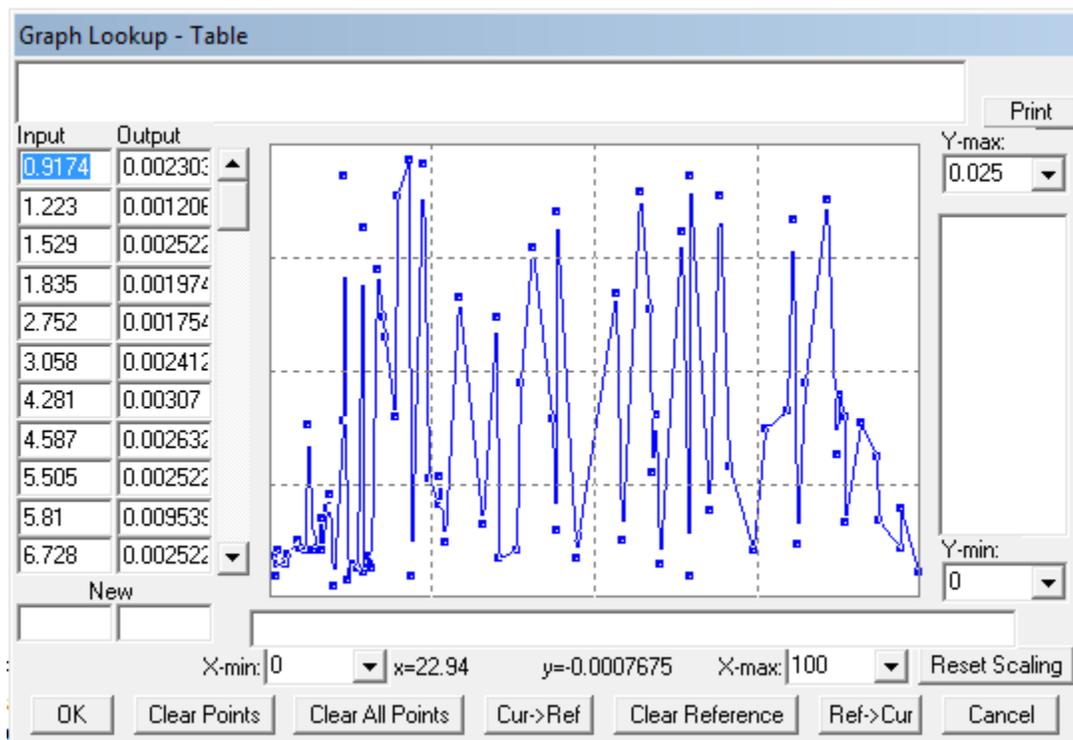
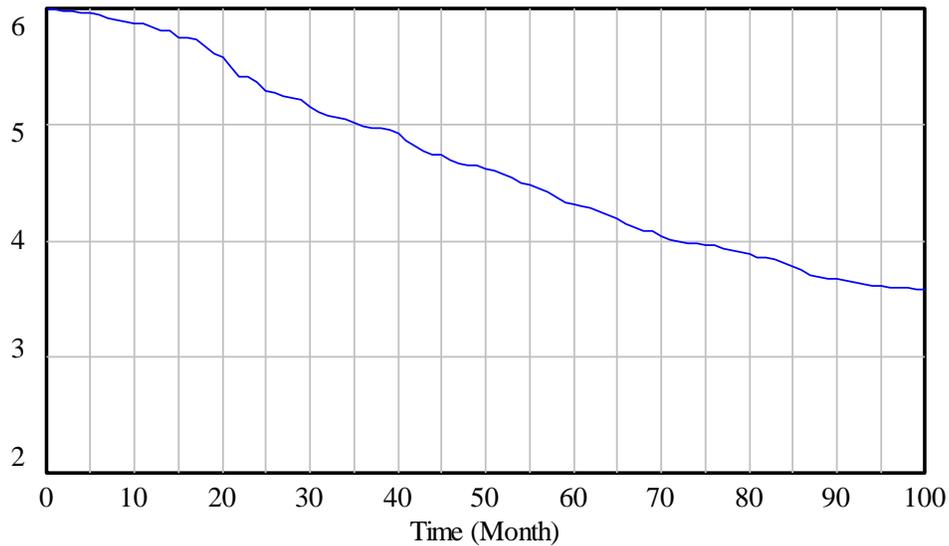


Figure 6-0-12: Graph Lookup of different Decrease Percentages over Time

As predicted, the pollution level curve loses its smoothness, and its motivation to reach its goal (2 ppm of carbon dioxide emission) is disturbed. On the other hand, the overall structure of the system is still conserved, as shown in **Error! Reference source not found.** Such simulations would very useful in three major phases of policy cycle: decision, implementation, and evaluation.

Pollution Level



Pollution Level : Current 

Figure 6-0-13: Carbon Monoxide Emissions (ppm) versus Time with Variable Decrease Ratios

The following figure shows plots of long-term carbon monoxide trends (Air & Climate, 2012) from 1985 to 2007 in the state of Massachusetts (Air & Climate, 2012). Both plots seem to have a straight decrease while there is the strong possibility that the overall structure is a bounded exponential with very slow decrease percentage. The plots show that Massachusetts CO emission was around 6 ppm (initial condition) before reaching an approximate value of 2 ppm (goal) to finally meet EPA standards (Environmental Protection Agency). As shown in the plots, the decaying trend is not smooth. As a matter of fact, multiple bumps populate the curve. In the previously simulated model, those bumps are the results of the changing decrease percentages over time. This real life data is very relevant to the previously simulated example. Furthermore, it is stated that a number of pollution control approaches were used during the period of 1985-2007.

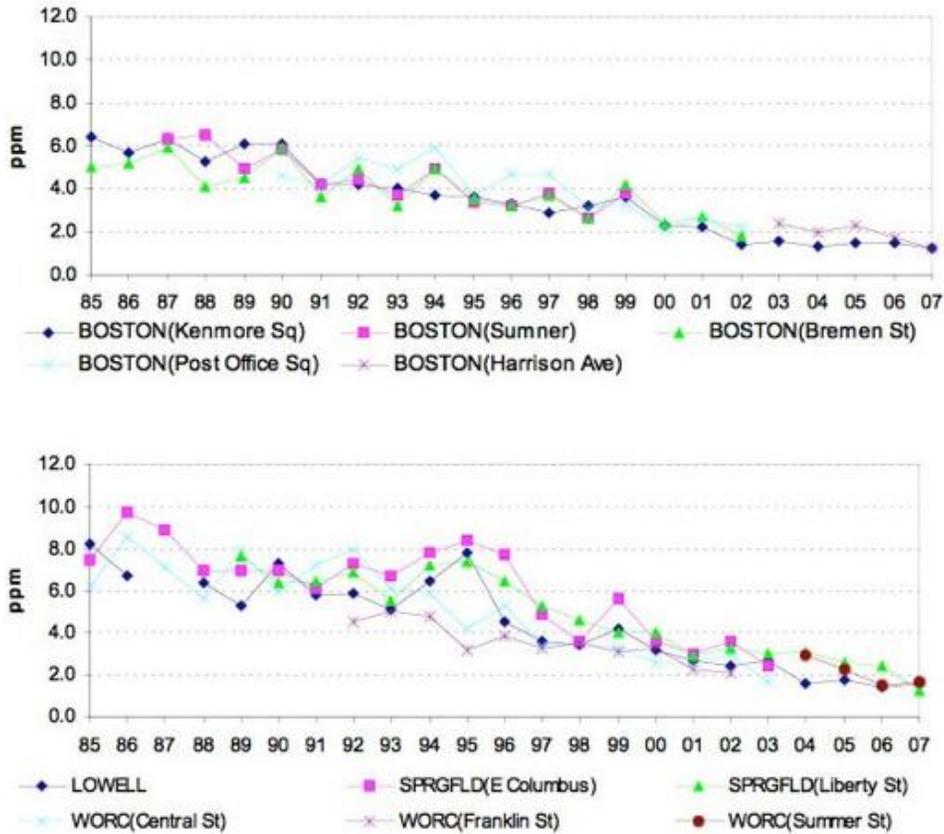


Figure 6-0-14: Long-term carbon monoxide trends in Massachusetts

Both of the plots above span over the same period and convey data for more than one city. All of the cities have the same overall structure with different behaviors from the initial to the final point of data collection. Remember that in the previous simulated model, the rate of pollution increase (inflow) was set to zero for simplicity reasons. In addition to that, the variant pollution increase ratios were randomly selected. These two important variables are behind the value changes in the plotted curves. An interesting case study would be studying the given cities in order to figure out the functions of those variables. Studies can range from sociology to psychology, with an emphasis on demographics and market research.

Using the same technique in the previous section, the transfer function of the system is:

$$T(s) = \frac{s}{s + r} \quad (6-15)$$

Equation 6-14 is relevant to both increasing and decreasing bounded exponential curves. The pole is $-r$, which is a negative value. That means that the system is stable (since the transient response $A(t)$ reaches a specific value as time goes to infinity).

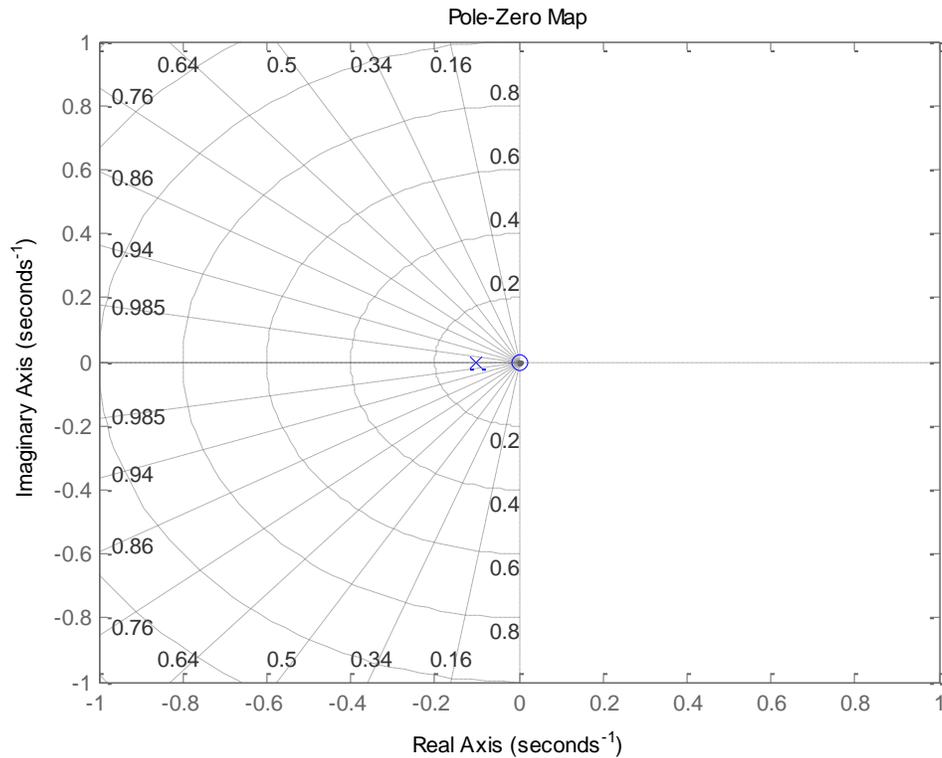


Figure 6-0-15: Zero-Pole Plot of Goal Seeking Functions

Error! Reference source not found. shows that the system pole is on the negative x-axis. Notice that the pole plot does not convey any information about whether the function is increasing or not. Instead, it describes the general behavior of the transient response as time increases (stability). This system is therefore **stable** (negative poles) (Smith, 2010).

More Analysis

A familiar concept in analyzing first order systems is the rise time t_r which is the required time for the exponential curve to go from 1.1 to 0.9 of its final value. Such period is dependent on the decrease percentage r (Eq. 6-15)

$$t_r = \frac{2.2}{r} \quad (6-16)$$

The greater the time constant r , the shorter the period the curve has to travel before reaching its destination. A large r intensifies the exponential behavior of the curve which is covering a large vertical change over a relatively small horizontal change (time). The idealistic goal response is a sharp change from the initial to the final value (step function). Immediate responses are impossible to find in real life due to different factors that contribute to the value of the time constant.

Once the curve starts grazing the desired final value, it is hard to determine at what percentage the actual value is from the goal. The settling time is the required time for the response curve to stay within 2% of the desired value (Eq. 6-16).

$$t_s = \frac{4}{r} \quad (6-17)$$

A good way to picture the behavior of the system is to think of the policy change as a strict input step function waiting for a response. As mentioned above, an ideal response would be an outputted twin step function. The sharpness of the policy change input is causing a delay in the system response. In a more mathematical sense, the higher the input frequency of the signal, the slower the response will be. The following figure shows the bode plots of the system described in Eq. 6-14. The only difference is that the plot takes into consideration the initial value as an input (not the goal).

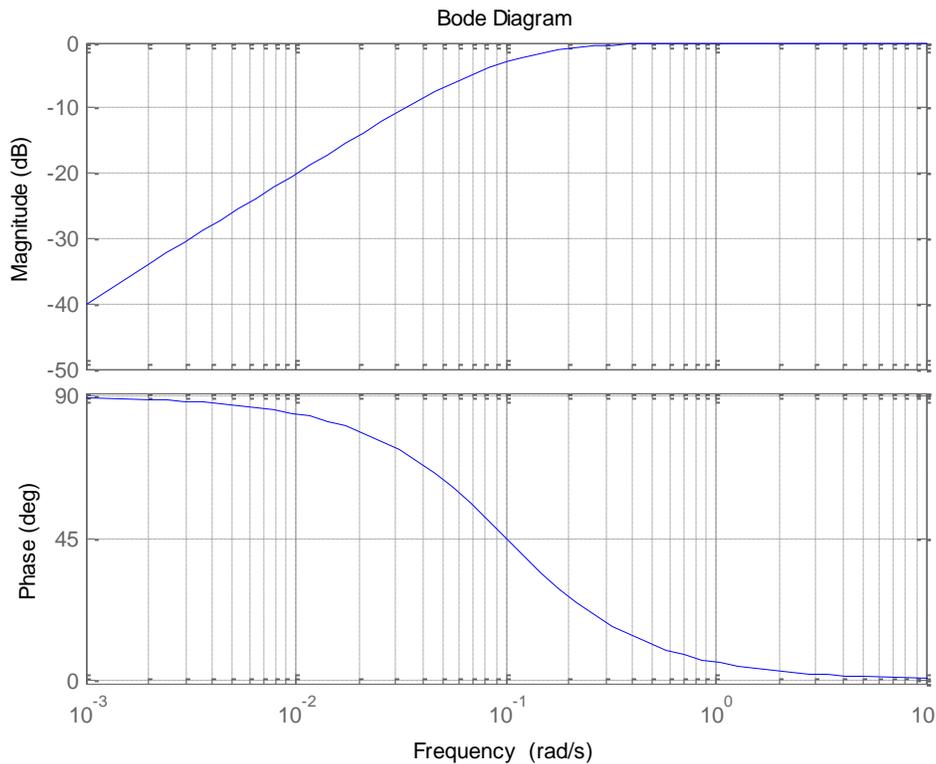


Figure 6-0-16: Bode Plot of Goal Seeking System

Note that the magnitude of the gain is high at high frequencies and low at low frequencies. In general, the goal is opposing the initial value. Therefore, the larger the frequency of the goal, the slower the system response will be. More explanations are shown in *Section 3.1 Smoothing and Information Delay*. Any policy maker must be aware of this rule. In order for the transient response to be imitating the input set by the policy maker, the input signal of the policy should have a low frequency, meaning that the expected vertical change over time should be minimized. It is probable then that if policies are imposed step by step, the output response would be more predictable since it would be imitating its ideal twin at its best. The following figure shows a policy input signal (dotted) and its response (solid). The input signal reaches a value of 100 in 0 seconds (high frequency). At that point, the response is attenuated in magnitude and delayed in phase.

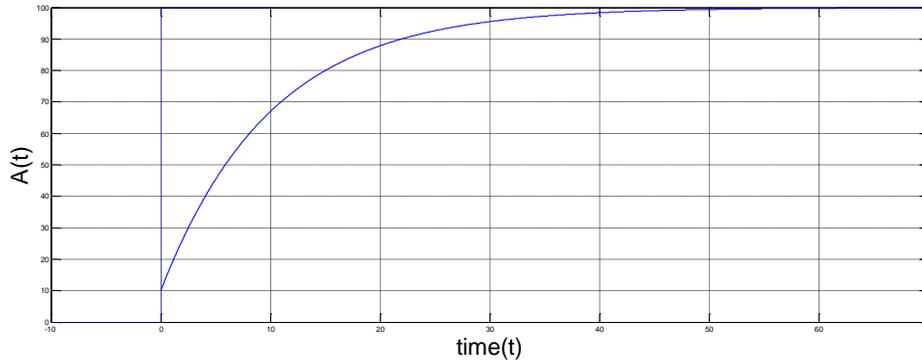


Figure 6-0-17: Input Policy Signal and Transient Response

This section provided a good exposure to a familiar stable first-order system. Note that the polarity of the feedback loop in System Dynamics is the sign of the system poles in Control theory.

Section 6.3: S-Shaped

The last two sections introduced negative and positive feedback separately. This section will present a famous system where both negative and positive feedbacks are present. In these situations, the system behavior is controlled by the effect of both feedback polarities (poles). A useful optimization example is used to show how one pole (+) could initialize the rise of the system response while the other pole (-) dictate the behavior of the system in the long term where a specific value is reached (stable). Irregularities are also discussed to briefly cover the concept of overshoot and collapse. Besides simulations, real life data are shown to support these concepts.

The S-shaped graph is characterized by an exponential growth followed by a decay to finally settle at one final value. It is literally a combination of exponential growth and goal seeking behaviors. This structure is caused by two feedback loops, one positive, and another negative. It is therefore appropriate to assume that the system has two poles: one positive, and one negative. The distance of each pole from the center determines the strength of its effect on the system, and the placement of the turning point where the transient response starts decaying to reach a final destination called carrying capacity. The S-shaped transient curve usually describes population growth over time. The general differential equation describing this behavior is:

$$\dot{P} = r_{in}P - r_{out} \frac{P^2}{C} \quad (6-18)$$

where P is population, and r_{in} - r_{out} are the rates at which people are born or die.

An example is the number of shoppers in the mall. People enter and leave the mall at different rates. The number of shoppers keeps increasing until the mall capacity is met. No more growth should be allowed once the saturation level is met. The following figure shows the stock and flow diagram of such situation. The more customers in the mall, the lower gets the Capacity/Shoppers ratio. And the higher the capacity, the larger is that ratio. As shown in the figure, the positive feedback loop is from the stock to the inflow. The negative feedback starts from the stock and goes to the Capacity/Shoppers ratio (-), to finally target the outflow.

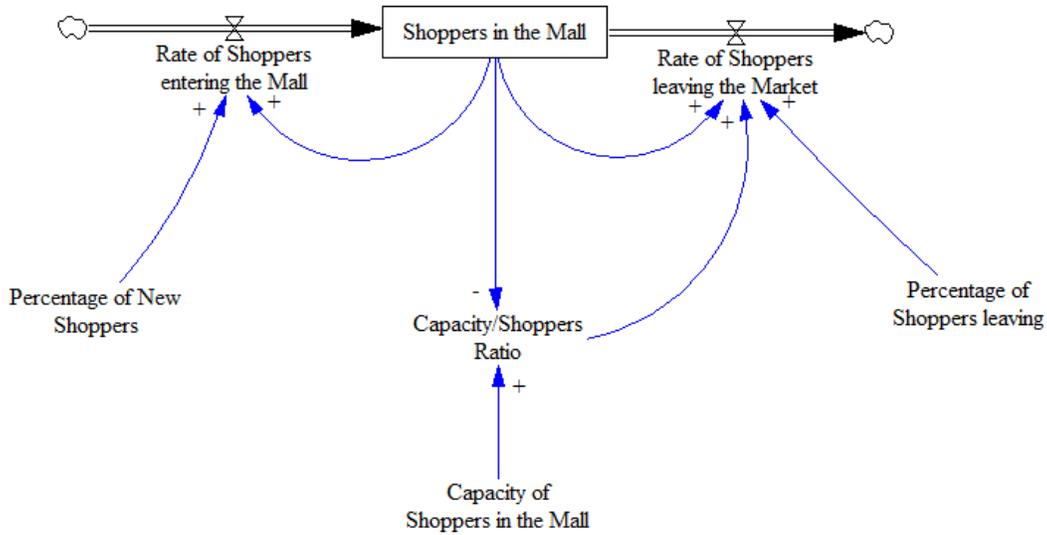


Figure 6-0-18: Stock and Flow Diagram of Shoppers in a Mall

The *Shoppers in the Mall* is defined as the integral of the net flow. The *Rate of Shoppers entering the Mall* (RSEM) is expressed as follows:

$$RSEM = PNS * SM \quad (6-19)$$

where PNS is the Percentage of New Shoppers (0.2), and SM is the actual stock Shoppers in the Mall.

The Rate of Shoppers leaving the Market is defined:

$$RSLM = PSL * SM / CSR \quad (6-20)$$

where PSL is the Percentage of People Leaving (0.1), and CSR is the Capacity to Shoppers Ratio.



Figure 6-0-19: Shoppers in the Mall over Time

Notice that the transient response is exponentially increasing, hits its turning point, and then decays toward its goal which is the Mall maximum capacity in this Case. The next figure shows the Capacity to Shoppers Ratio. Note that the ratio decays as time increases, and as more people enter the mall.

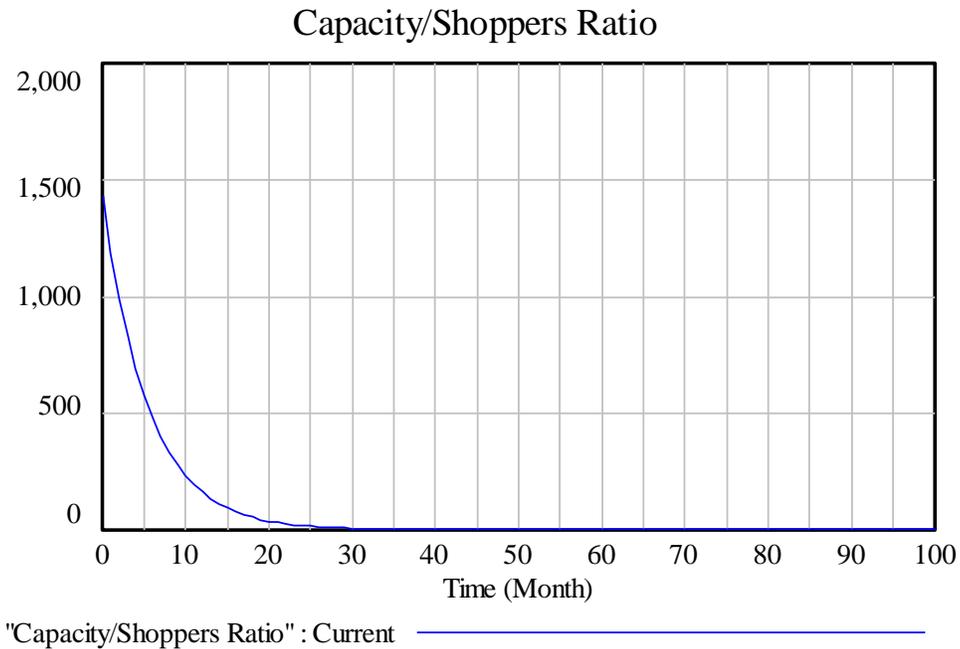


Figure 6-0-20: Capacity to Shoppers Ratio

In the previous simulation the percentage of people entering and leaving the mall is constant. In real life, the percentage actually changes during the day. The percentage of people entering the mall in the morning might be very low, while it is at its highest around noon. A lookup function is used to set different percentages at different time of the day to imitate the real world in a more appropriate fashion (**Error! Reference source not found.**).

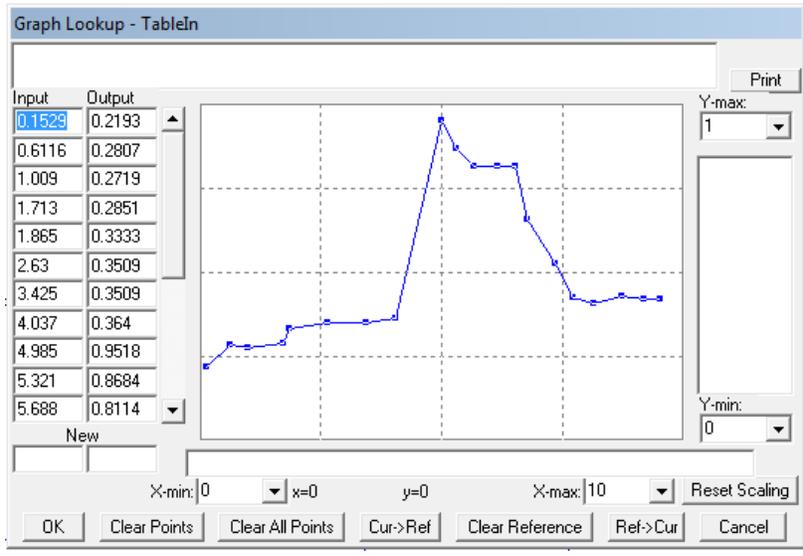


Figure 6-0-21: Percentages of Shoppers Entering the Mall over Time

The maximum capacity of the mall is set to 2000 shoppers this time. The resulting transient response preserves its original shape, but exhibits minor distortions.



Figure 6-0-22: Shoppers in the Mall with Variant Entering Rate

The most important observation is that the number of shoppers exceeded the capacity of the mall (**overshoot**). A blind manager might be too preoccupied in finding a number of ways to attract the customer without considering the saturation level of the mall. Such simulations are very useful in determining the best percentage distribution over time. Otherwise, the supply and demand balance will

be violated, and safety standards would not be respected. Another extreme case is the **collapse** of the system after overshooting its goal. The overshoot is due to the increase of the percentage of shoppers entering (larger than percentage of people leaving), and its sudden drop. The challenge of managing this space is increasing the number of shoppers in the mall (to increase profit) and respecting constraints such as space limit. A good knowledge of optimization and queuing theory is recommended in this area of analysis.

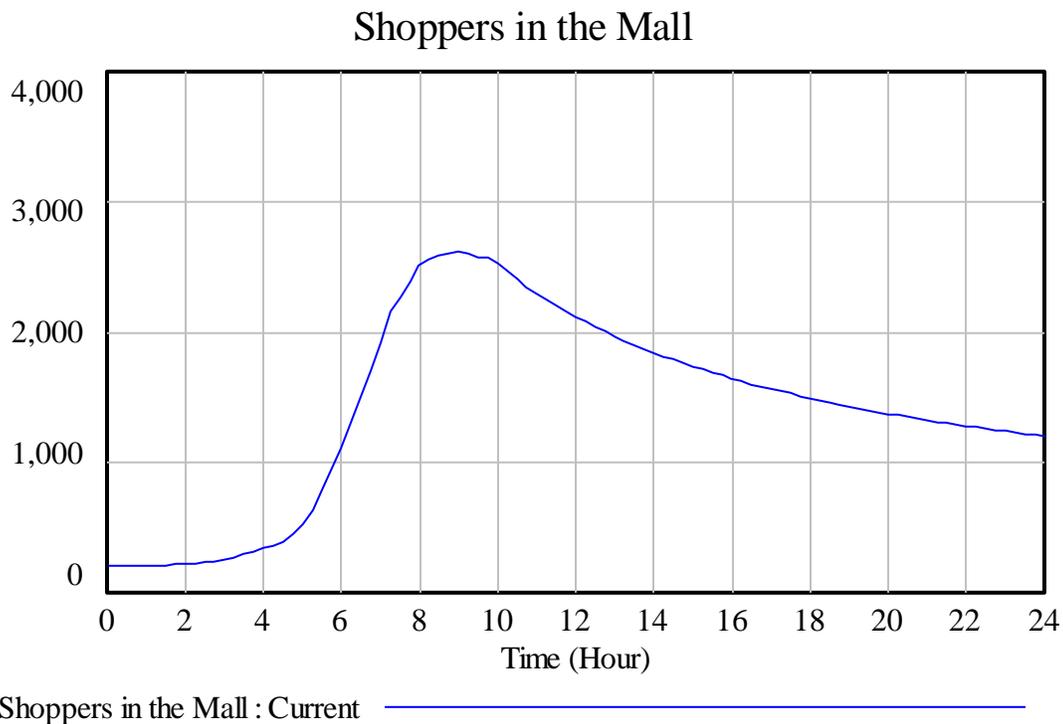


Figure 6-0-23: Overshoot then Collapse of System

Similar situation is noticed in the price of *Gold Money Gold Bullion* from the beginning of March to around March 14th of 2012 (**Error! Reference source not found.**) . (Gold Money Gold Bullion, 2012)

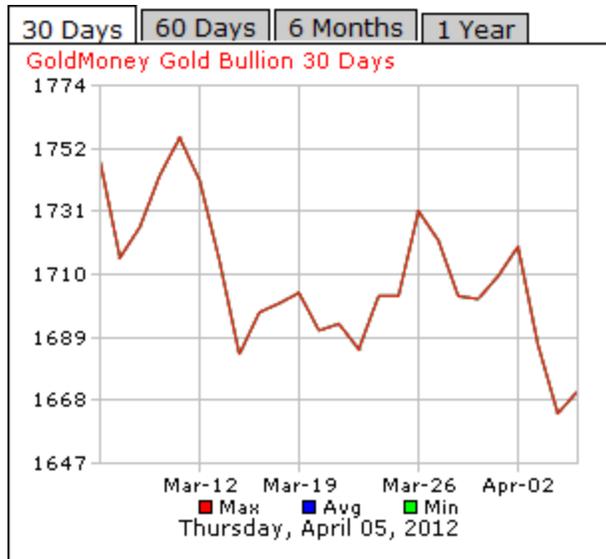


Figure 6-0-24: Price of Gold over a 30-day Period

An important rule to keep in mind is that extrapolation is dangerous since many factors that shape the trend are easy to be ignored.

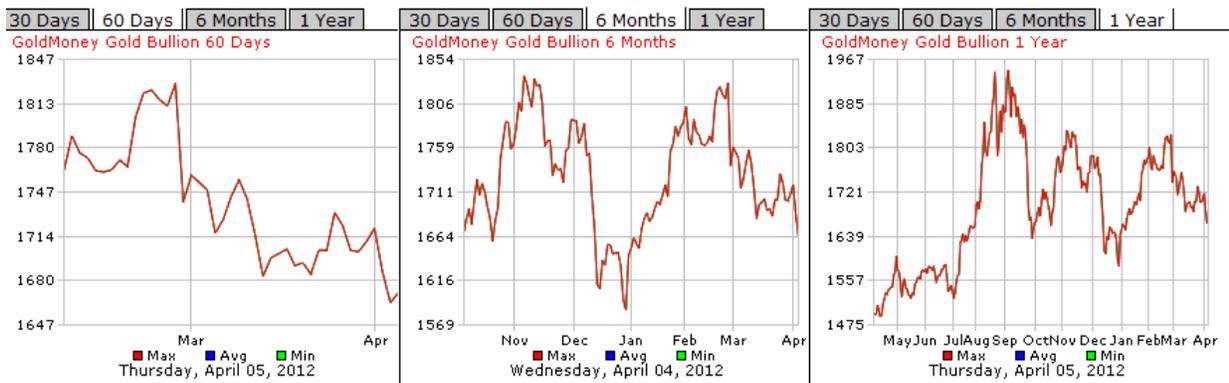


Figure 6-0-25: Price of Gold over Different Periods

Looking at the evolution of any variable of interest over a bounded time interval is misleading. In the 60-days graph, the behavior seems to be decaying (goal seeking). The 6-months graph shows oscillation that may push the analyst to assume a sinusoidal trend over time. Finally, the 1-year graph seems to behave as an S-shaped transient response.

The model of the mall example mentioned earlier is clearly nonlinear. That nonlinearity makes it difficult to transform the system into the frequency domain using Laplace transform. On the other hand, since the causal loop diagram of the system had two feedback loops (positive and negative), the system is known to have to poles (+ and -).

Section 6.4: Oscillation

So far, behaviors have been shown to be either diverging or converging to a certain value along with time. This section will introduce a system whose behavior oscillates over time without completely diverging from a target or settling at a value. Later sections will introduce various types of damping systems and their Control analysis.

Oscillation in the transient response of a system is caused by the delays present in that system. Furthermore, delays in the negative feedback loops of the system are the direct cause of fluctuation. The reason behind the Goal seeking behavior is the presence of the negative feedback loop that corrects the discrepancy between the desired goal and the actual state of the system. In general goal seeking behaviors, the corrective action is considered to be immediate and independent external factors. That explains the smoothness of the transient exponential response of such systems.

In the real world, the corrective action speed is hindered by several factors. Major delay sources range from measurement to final decision processes. In industry, decision-making is delicate, and is not done instantaneously. Data has first to be collected, reported and analyzed. Such critical steps are the delays imposed on the correction action (negative loop). Delays are not only the time required to perceive the change of the system and so forth. Additional seemingly innocent stocks in the system may cause delay. As a matter of fact, the number of stock in a given system is the degree number of the differential equation representing the system.

Celest V. Chung has interesting examples in *Generic Structures in Oscillating Systems* for the MIT System Dynamics Education Project under the supervision of Dr. Jay W. Forrester. One of the mentioned examples is *The Employment Instability Model* shown in **Error! Reference source not found..**

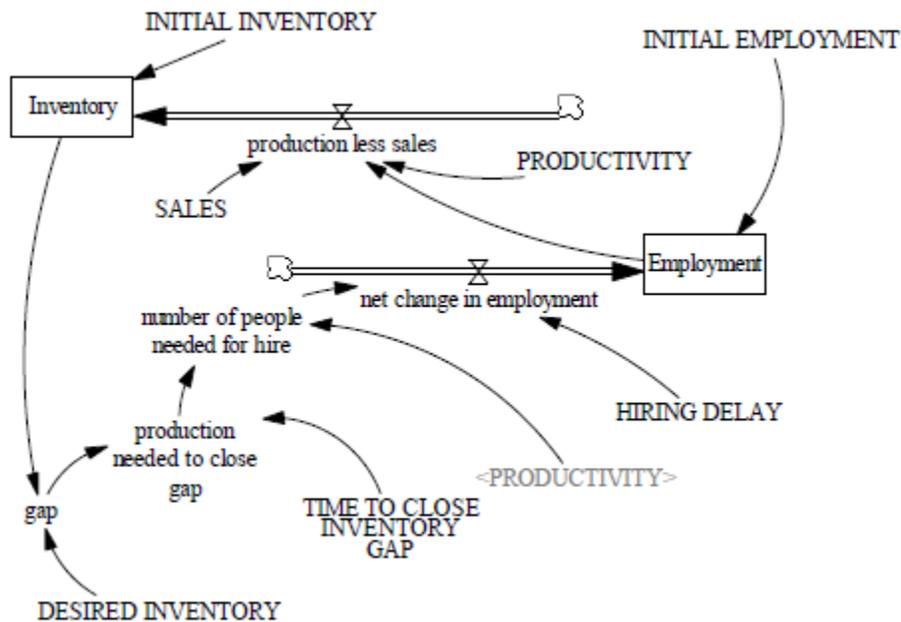


Figure 6-0-26: Employment Instability Model

Both Inventory and Employment are interconnected and dependent of each other. In previous sections, as stock is usually dependent on its inflow and outflow, which are dependent on other variables. In this model, the flows of Stock1 are dependent on Stock2. This relationship causes delays which results in fluctuations in the transient response of the system. The more stock present in the system, the larger the degree of the system (Differential Equation Approach) is, and the longer the delays are. **Error! Reference source not found.** shows the transient responses of Chung's model (Chung, 1994).

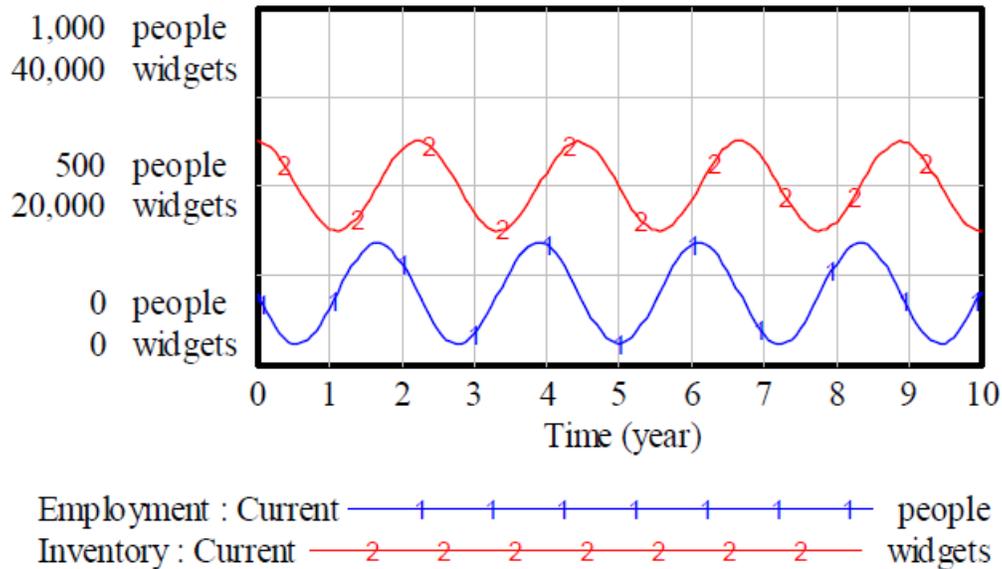


Figure 6-0-27: Graph of Inventory and Employment

It is clear that both stocks oscillate over time due to their dependency on each other. All oscillations are not the same though. There are few types of oscillations whose responses over time are different.

Section 6.5: Overdamped Oscillation

Overdamped oscillation is a special case of oscillation. The system does not fluctuate as expected. The response is similar to an S-Shaped function whose turning point is the only fluctuation characteristics of the curve. The transfer function (Eq. 6-16) of such system contains two different negative poles (stable). The distance of each pole from the center provides the strength of each pole, and its effect on the transient response (**Error! Reference source not found.**). The zero-pole plot of such system is shown in **Error! Reference source not found.**.

$$T(s) = \frac{a}{(s+\sigma_1)(s+\sigma_2)} \quad (6-21)$$

where a is a scalar, and $-\sigma_1$ and $-\sigma_2$ are the poles of the function.

The general formula of the transient response of an overdamped system is:

$$f(t) = Ae^{-\sigma_1 t} + Be^{-\sigma_2 t} \quad (6-22)$$

where A and B are scalars.

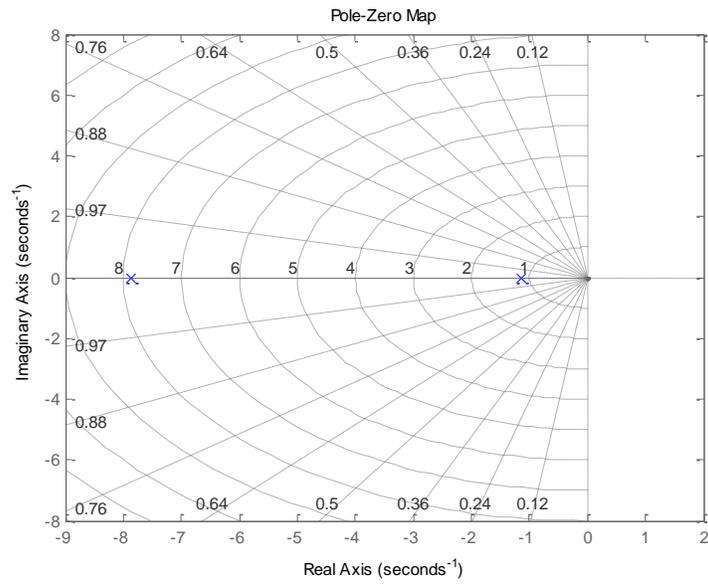


Figure 6-0-28: Zero-Pole Plot of Overdamped Oscillation System

The following figure shows that bode plot of the example transfer function:

$$T(s) = \frac{9}{s^2 + 9s + 9} \quad (6-23)$$

where the poles are:

$$\begin{aligned} -\sigma_1 &= -7.854 \\ -\sigma_2 &= -1.146 \end{aligned}$$

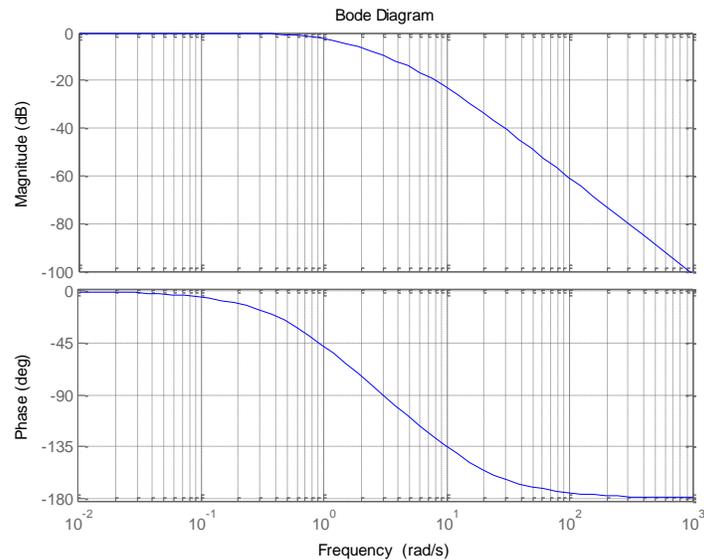


Figure 6-0-29: Bode Plot of Overdamped Oscillation System

An overdamped system has two negative real poles. This system does not oscillate due to the absence of any imaginary parts to its poles. It is in fact one of the extreme types of damping.

Section 6.6: Underdamped Oscillation

Underdamped oscillation results in a fluctuating system that reaches a final value over time (**Error! Reference source not found.**). The transient response overshoots start high, decrease over time, and then settles at a final value.

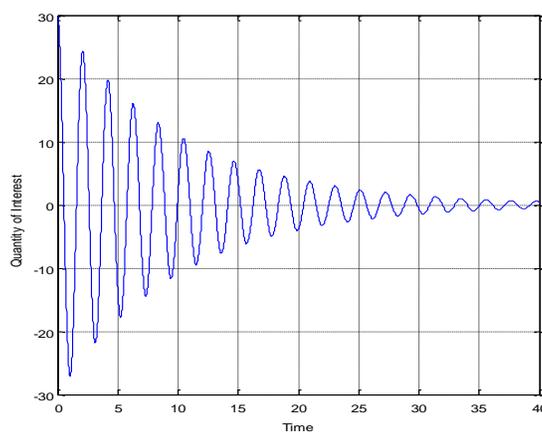


Figure 6-0-30: Transient Response of Underdamped System

An underdamped oscillatory system has two **complex** poles (not purely imaginary):

$$-\sigma_d \pm j\omega_d$$

The transient response is the following general form:

$$f(t) = Ae^{-\sigma_d t} \cos(\omega_d t - \varphi) \tag{6-24}$$

where ω_d is the radian frequency, and φ is the phase of the sinusoid of the natural response.

Eq. 6-21 is an example of an underdamped oscillatory system.

$$T(s) = \frac{9}{s^2 + 2s + 9} \tag{6-25}$$

Where the poles are: $-1 \pm j\sqrt{8}$

The following figure shows the zero-pole plot of the example transfer function above.

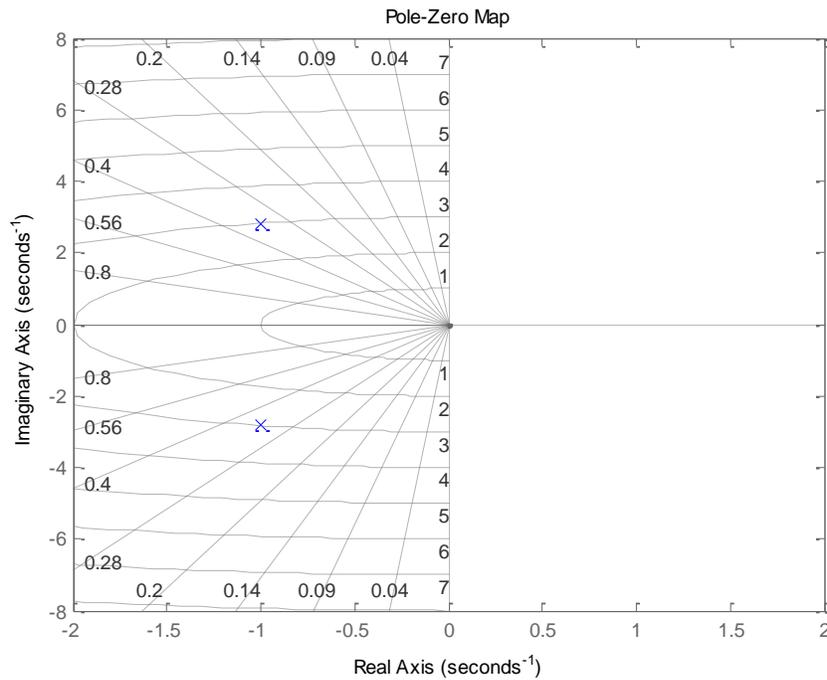


Figure 6-0-31: Zero-Pole Plot of Underdamped Oscillation System

The bode plot of underdamped systems is not very different from overdamped oscillatory systems.

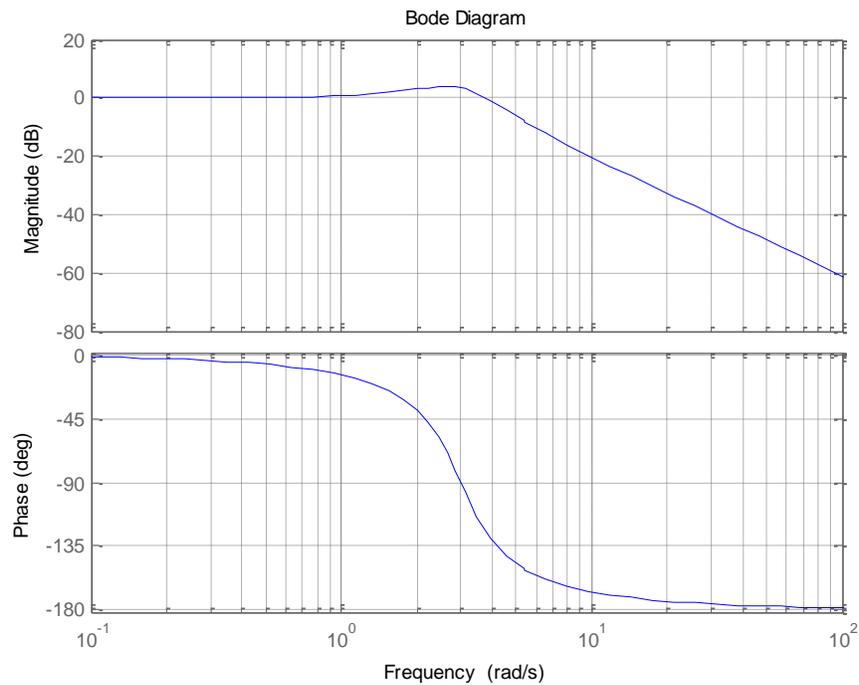


Figure 6-0-32: Bode Plot of Underdamped Oscillation System\

The Underdamped system has two conjugate poles with negative real components. The farther the poles are on the left s-plane (toward negative real axis), the more stable the system is. The imaginary components are directly related to the frequency of the system response. The farthest the imaginary components are (vertical pole distance), the highest is the frequency of the response.

Section 6.7: Undamped Oscillation

Undamped oscillation is a good example of instability where the transient response keeps overshooting and undershooting the goal (**Error! Reference source not found.**).

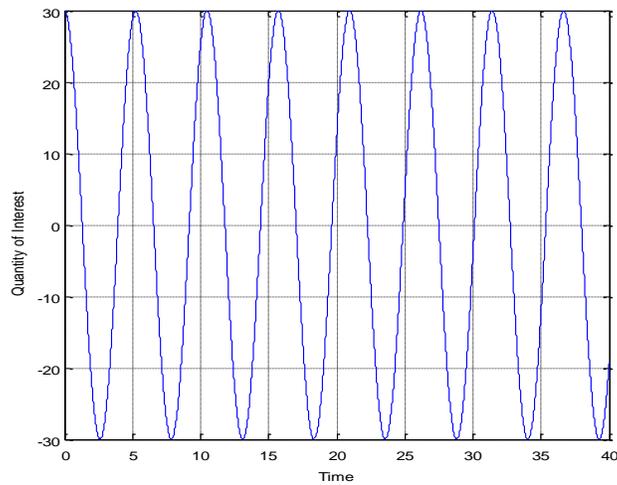


Figure 6-0-33: Transient Response of Undamped System

An undamped oscillatory system has two **purely imaginary** poles with no real parts: $\pm j\omega$.

The general transient response is any undamped system is as follows:

$$f(t) = A\cos(\omega t - \varphi) \quad (6-26)$$

Eq. 6-23 is an example of an undamped oscillatory system.

$$T(s) = \frac{9}{s^2+9} \quad (6-27)$$

where the poles are: $\pm j3$

The following figure shows the zero-pole plot of the example transfer function above.

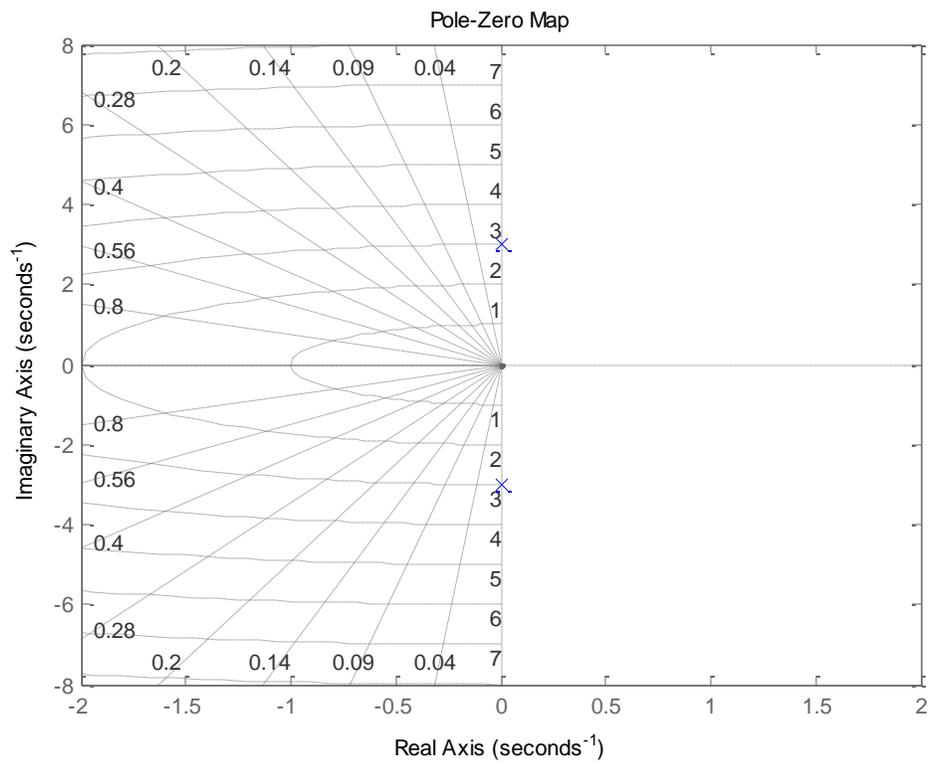


Figure 6-0-34: Zero-Pole Plot of Undamped Oscillation System

The bode plot of this unstable system is very different that all others due to its sharp asymptote representing the humongous magnitude of gain.

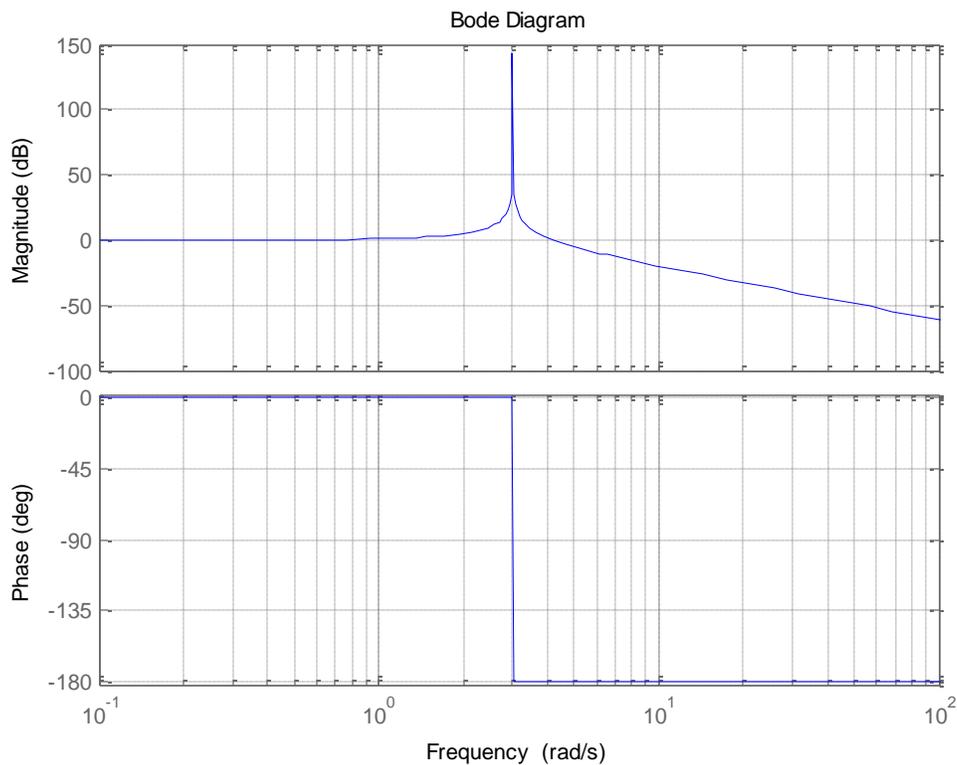


Figure 6-0-35: Bode Plot of Undamped Oscillation System

There is a subtle transition between damped systems (conjugate poles with real negative components) and undamped (with pure complex poles) systems. That happens when the poles do not have a high enough negative real component to shift them from the oscillatory effect of the imaginary axis of the s-plane. Those systems are known as a marginally stable system.

Summary

This chapter presented major system structures and their corresponding behaviors. Step inputs were first assumed to introduce the most general behaviors. However, indices such as growing rates were changed over time to show the transient responses corresponding to time varying input functions. Potential resemblances were highlighted between System Dynamics and Control Theory such as feedback loops and system poles. While System Dynamics relies on Integral equations and transient analysis, Control Engineering focuses on transfer functions and frequency analysis. Both methods may be used to predict the system's behavior. As matter of fact, they are easily related at the exception of nonlinearities where domain translation is not trivial. Transfer functions contain all the components in a compact "coded" format where the general system behavior is clear unlike the direct implication of each component. On the other hand, stock-and-flow diagrams are more intuitive in their ability to show how the structure's components are internally connected.

Chapter 7: Conclusion and Future Work

System Dynamics modeling is usually presented in such an intuitive way to be understood by the general public. This paper introduces the mathematical strategies used to analyze real-life systems and their dynamics. This study is not only covering the complex technicalities behind seemingly simple dynamic models. This study is linking both worlds: System Dynamics tools to create intuitive models and thorough mathematical techniques used behind the scene.

The Graphical representation of system behaviors is the first solid bridge between the two worlds. Once enough information is extracted from system dynamic representations such as Stock-and-Flow diagrams, Cartesian graphs are used to show the behavior of individual stocks and flows with respect to time. Notions such as integration and accumulation are discussed to serve as a critical base for this paper.

Delays are mainly introduced into a system because of the inability to distinguish between actual changes and misleading noises in the system. Being the major source of inaccuracy, delays are defined and separately into two distinct types: Material and Information delays. Mathematical expressions are provided to describe first and high-order material delays and smoothing processes related to information delays.

Complex dynamic systems are modeled using software packages that use approximation techniques to evaluate stocks. Numerical methods such as Euler's and Runge-Kutta are therefore introduced and compared.

Finally, typical structures are presented along with their eventual behaviors. Using control Theory methodologies, structures are represented as systems containing single inputs and outputs for the sake of simplicity. System blocks are described by compact transfer functions enclosing all the information of that particular structure. Step and varying inputs are applied to common structures to produce both ideal and real-life transient responses. Control theory analysis techniques such as pole-zero and bode plots are introduced to judge the stability of particular systems and their corresponding transient response behavior.

Multiple paths could be taken to extend this work. One of the interesting routes would be developing and linking dynamic systems delays and control theory techniques. A practical application of this study would be the creation of a control panel that could represent real-life systems, detect the noise sources in the system and suggest modifications to reduce the delay and amount of error in the system. This panel would be beneficial to investors, sociologists, and any other specialists interested in automatically fixing their system of interest and reaching a desired behavior.

Appendices

Appendix A: Differentiation

Average versus Instantaneous Rate of Change

To begin, let us explore how to compute an average rate of change over some time interval.

$$f_{avg} = \frac{f(t_1) - f(t_0)}{t_1 - t_0} \quad (\text{A-1})$$

where t_0 and t_1 are time points bounding the time interval of interest, and $f(t)$ is the value at the corresponding point of time.

Example:

George starts driving from his house at 7:30 am to arrive at work at 8:00 am.

What is the rate of change of the distance if the distance between George's house and office is 25 miles?

Solution:

The quantity of interest is the distance between the house and the office (25 miles). The time that George takes to arrive to his office is half an hour.

$$8:00 - 7:30 = 30 \text{ min} * \frac{1 \text{ h}}{60 \text{ min}} = \frac{1}{2} \text{ h} \quad (\text{A-2})$$

The rate of change of the distance is:

$$\frac{25 \text{ miles}}{\frac{1}{2} \text{ h}} = 50 \text{ mph} \quad (\text{A-3})$$

Note:

Distance is a measure of length without considering direction. On the other hand, **displacement** takes into account both distance and directions in respect to a referenced coordinate system.

Example:

George leaves work at the end of the day and drives back to his house. The overall distance travelled that day was 50 miles ($2 * 25 \text{ miles}$). However the total displacement is 0 miles since George travels from his house to work (+25 miles), then from work to his house (-25 miles).

The rate of change of distance is **speed** (no direction), while the rate of change of displacement is **velocity**.

For this example, the unit of speed and velocity is miles per hour (mph). In general, the unit of speed and velocity is expressed as the unit of length over the unit of time. Based on the example, does this mean that George has been driving at 50 mph for the entire time?

No, apparently George will need to decelerate/accelerate for various reasons (red light, pedestrians crossing the street etc.). However, on average, George's speed was 50 mph.

Let us now look at the graphical representation of George's trip.

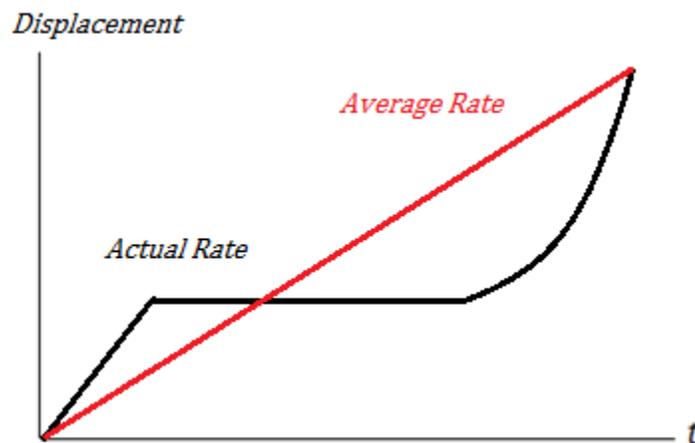


Figure A-1: George's trip respect to time

You may have noticed that the calculated rate does not take into account the multiple changes occurring during the trip. That is because the calculations are based on two endpoints only (initial and final time). This method calculates the **average rate of change**.

Graphically, the average rate of change can be represented as the slope of the secant line passing through both endpoints of the graph (Figure A-1).

To calculate the rate of change at a specific point in time, one must refer to the **instantaneous rate of change**. To find the instantaneous rate algebraically, it is necessary to know the function representing

the action although it is usually difficult to obtain. However, plotting the data points allows us to obtain a picture of the action. In this case, the instantaneous rate is approximated graphically by calculating the slope of the **tangent line** at a specific point on the graph.

A tangent line at a particular point is a straight line that touches the curve at that point.

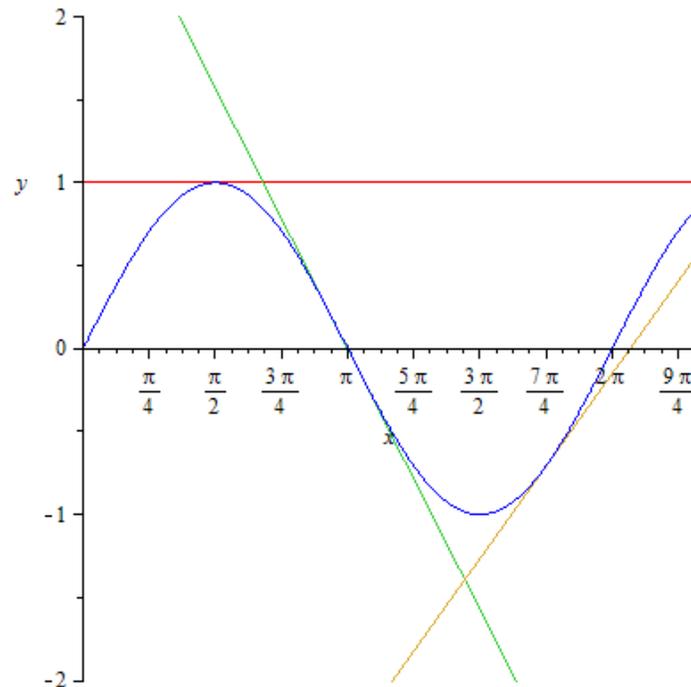


Figure A-2: Tangent lines

The figure above shows the graph of $\sin(x)$ and its tangent lines at the following points: $x = \frac{\pi}{2}, \pi, \frac{7\pi}{4}$.

You can clearly see that the tangent lines drawn above have different slopes. In other words, there are three different instantaneous rates of change at the three selected points. Without evaluating their values, the slopes are zero, negative, and positive respectively.

By drawing the tangent line at any selected point, one can approximate the behavior of the rate of change.

In mathematical terminology, the instantaneous rate of change is referred to as “Derivative” of a function with respect to time.

Mathematical Definition of Derivative

Some “limit” knowledge is required to understand the main focus of this section. Let us briefly discuss the concept of limit. Figure(A-3) contains the graph of the function $f(x)$

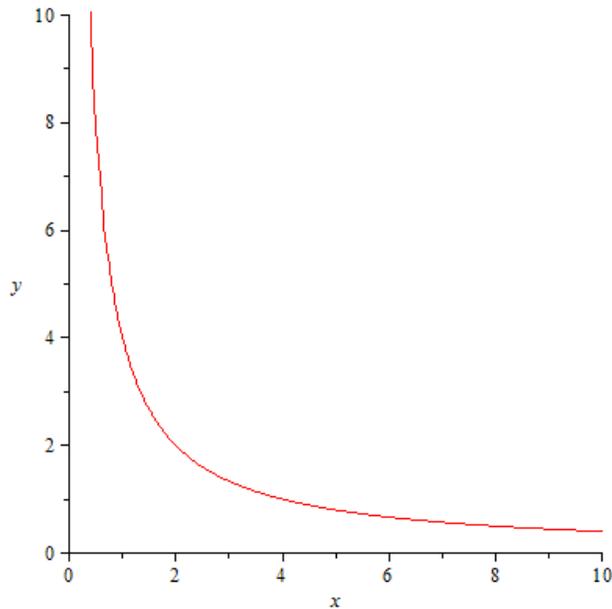


Figure A-3: Graph of $f(x)$

Notice that the graph above does not completely touch the x-axis. Instead, it approaches the x-axis while the value of x is getting larger. In other words, the function $f(x)$ approaches 0 as x goes to infinity.

The following mathematic expression represents the above statement.

$$\lim_{x \rightarrow \infty} f(x) = 0 \quad (\text{A-4})$$

This is one of the many cases involving limit.

The general form is as the following.

$$\lim_{x \rightarrow a} f(x) = L \quad (\text{A-5})$$

Where $f(x)$ is approaching a certain value L as x is going to a value a . This is the concept of Limit.

Note that $f(x)$ and x never actually reach their corresponding values a and L respectively. But instead, the ends of limit get extremely close to them.

Recall from the previous section that the slope of the secant line is the average rate. The slope of a secant line can be calculated as follows:

$$m_{sec} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} \quad (\text{A-6})$$

$$m_{sec} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (\text{A-7})$$

As mentioned before, the instantaneous rate of change is the slope of the tangent line. The secant line is created based on two points. Mathematically speaking, the tangent line is the limit of the secant line slope as the difference between the points is approaching zero. Using conventional notation:

$$m_{tan} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (\text{A-8})$$

Since the derivative of a function is actually the slope of the tangent line at a given point, the definition of the derivative is as the following:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (\text{A-9})$$

where $f'(x)$ is the derivative of $f(x)$. Other derivative notations are: $\frac{d}{dx}f(x) = \frac{dy}{dx}$.

To evaluate the derivative at a specific value of x ,

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \quad (\text{A-10})$$

where $f'(a)$ is the derivative of $f(x)$ at $x = a$. Other derivative notations are: $\frac{d}{dx}f(a)$.

Second and Higher Order Derivative

Based on the previous sections, it can be concluded that the time derivative of the displacement is velocity. Moreover, the derivative of velocity is **acceleration**. Acceleration is then the **second derivative** of displacement. The second derivative of a function can be found by differentiating the function twice.

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{d}{dx}(f(x))\right) \quad (\text{A-11})$$

Higher order derivative could be found if necessary (i.e. jerk is third derivative of displacement). The n^{th} derivative of a function can be found by differentiating the function n times.

$$f^{(n)}(x) = \frac{d^n}{dx^n}(f(x)) \quad (\text{A-12})$$

Appendix B: Integration

Mathematical definition of integration

In system dynamics, a stock is the quantity of flow that accumulates over time. Inflows add quantity to the stock, hence increasing it. On the other hand, outflows decrease the stock. A flow of a system can be given in term of mathematic function. In this case, we denote the flow as $f(t)$.

Notice that the independent variable is time t . The graph of the function can be plotted respect to time. The accumulation or the integration of the flow is equal to the corresponding area under the function. The area can be approximated by summing a group of very small rectangles next to each other. The height of each rectangle will correspond to the value of the flow. Assuming that each rectangle has equal base; we can define the length of the base as the length of the interested time interval divided by the number of rectangles.

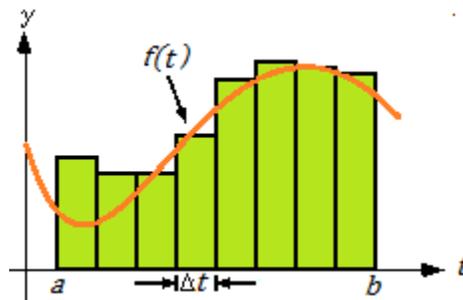


Figure B-1: Summation of rectangles

The width of each rectangle can be represented as following,

$$\Delta t = \frac{b - a}{n} \quad (\text{B-1})$$

Where Δt the width, b is the upper time limit, a is the lower time limit, and n is the number of rectangles.

The left edge of each rectangle is located in the position that can be represented as the value of t in term of the width and the limits of integration.

$$t_i = a + ((i - 1) * \Delta t) \quad (\text{B-2})$$

Where t_i is the position of the left edge and i is the index of the rectangle from 1 to n

Then the height of each rectangle is

$$f(t_i) = f(a + ((i - 1) * \Delta t)) \quad (\text{B-3})$$

The area of a rectangle at the index i is the width multiplied by the height at corresponding position.

$$A_i = f(t_i) * \Delta t \quad (\text{B-4})$$

Then we sum all small rectangles together in order to get approximate area under the curve. We notate the sum of all small rectangles from index 1 to n by using Sigma notation

$$[f(t_1) * \Delta t] + [f(t_2) * \Delta t] + [f(t_3) * \Delta t] + \dots + [f(t_n) * \Delta t] = \sum_{i=1}^n f(t_i) * \Delta t \quad (\text{B-5})$$

The sum is called **Riemann's sum**. Notice that the Riemann's sum yields just only the approximate value of the total area under the curve. By taking the limit of n approaching to infinity, the widths of the rectangles become infinitesimal. And the value of the limit becomes the area under the curve. This process of summation is known as integration. And the sum is called "Integral". The integral of a function during time interval (a, b) is denoted by following.

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_i) * \Delta t, \quad \text{where } \Delta t = \frac{b - a}{n} \quad (\text{B-6})$$

Applying the direct definition on integral takes long time. The integral is usually calculated by using Anti-derivative and Fundamental theorem of Calculus.

Anti-derivative

In chapter ("differentiation"), the focus was differentiation and its applications. The derivative of a function would give the instantaneous rate of change at any point of time. In other real life problems, going opposite direction is necessary to find a solution. For example, we could be asked to find the displacement depending on velocity and time. In other words, we are looking for the function whose derivative would yield the same expression as the given velocity. We referred the displacement as the **anti-derivative** of the velocity.

Example:

a) Find the derivative of the following functions

- $f(t) = t^2$

- $g(t) = t^2 + 4$

- $h(t) = t^2 + 100$

b) What is the anti-derivative of $2t$?

Solution:

a) based on the table 1,

- $f'(t) = 2t$
 - $g'(t) = 2t + 0 = 2t$
 - $h'(t) = 2t + 0 = 2t$
- b) Since $f'(t), g'(t), h'(t)$ is $2t$.

$f(t)$ is an anti-derivative of $2t$ as well as $g(t)$ and $h(t)$

As you can see, there are multiple answers to the question since all $f(t)$, $g(t)$, and $h(t)$ yield the same derivative. The reason why they have the same derivative is because each function can be written in form of other functions plus a constant. The derivative of any constant function is equal to zero. Therefore, the sets of all anti-derivatives can be expressed as

$$F(t) + C \tag{B-7}$$

where C is any arbitrary constant.

In fact, we also call this sets “Indefinite integral” which notated as

$$\int f(t) dt = F(t) + C \tag{B-8}$$

Therefore, the indefinite integral of $2t$ is $t^2 + C$

You can find the indefinite integral of a function by thinking it as a backward procedure of derivative.

Evaluation of an integral

Unbounded integrals will always have a function in term of the independent variable as a result. A **definite integral** contains finite bounds giving a solution as a number.

For a definite integral, the function $f(t)$ is usually referred as a curve. And the definite integral of $f(t)$ between given bounds is referred as the area under the curve $f(t)$ between those given bounds

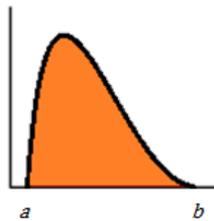


Figure B-2: Evaluation of Integral

The bounds of the integral are called **limits of integration**. The limits of integration contain lower and upper limits. The function inside the integral is called an **integrand**. And the last term of expression dx is called **differential of t**.

$$\int_a^b f(x) dt \quad (\text{B-9})$$

Because of its finite bounds, the value of definite integral yields a value that represents the area under the curve. To evaluate the integral, the fundamental theorem of calculus says that

$$\begin{aligned} \text{If } f(t) &= F'(t), \\ \int_a^b f(t) dt &= F(b) - F(a) \end{aligned} \quad (\text{B-10})$$

Where $f(t)$ is an integrand, and $F(t)$ is the anti-derivative of $f(t)$.

Note: the value of the integral can be negative which implies that the area is actually above the curve, but below the x-axis.

Appendix C: Ordinary Differential Equation

Linearity, and Homogeneity

A differential equation is an equation that relates an unknown function to its derivative(s). If the unknown function depends on one variable only, the corresponding differential equation is known as **Ordinary Differential Equation**. There are other types of differential equation involving multivariable functions. But since the only independent variable in system dynamics is time, only ordinary differential equation will be discussed in this book. These are some example of ordinary differential equation:

$$\frac{dy}{dt} = 2y + 1 \quad (\text{C-1})$$

$$5 \cdot \frac{d^3x}{dt^3} = -3x \frac{dx}{dt} - 4x \quad (\text{C-2})$$

$$y' + y = y^2 \quad (\text{C-3})$$

An ordinary differential equation can be divided into two categories: linear and nonlinear. Any **linear** differential equations are in the following form:

$$a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1(t) \frac{dy}{dt} + a_0(t)y = f(t) \quad (\text{C-4})$$

None of the coefficients $a_k(t)$ in the equation above should be in terms of the unknown function y . Otherwise, the differential equation would be **nonlinear**.

Example:

$$\frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + y = \sin(x) \quad (\text{Linear}) \quad (\text{C-5})$$

$$\frac{d^4y}{dt^4} + 2y \frac{dy}{dt} + y = x^2 \quad (\text{Nonlinear}) \quad (\text{C-6})$$

$$\frac{dy}{dt} + ty = e^t \quad (\text{linear}) \quad (\text{C-7})$$

$$\frac{dy}{dt} + \ln(y) = 3 \quad (\text{Nonlinear}) \quad (\text{C-8})$$

In equation (x), $f(t)$ is called the **nonhomogeneous term**. In the engineering vocabulary, it is referred to as the **forcing term**. If $f(t)$ is zero, the differential equation is said to be **homogeneous**. Otherwise, the equation is **nonhomogeneous**.

Example:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(x) \quad (\text{Nonhomogenous}) \quad (\text{C-9})$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0 \quad (\text{Homogeneous}) \quad (\text{C-10})$$

Also, the coefficients in any linear differential equation are either constant or variable. If all of the coefficients are constant, the equation is said to be **constant coefficient** differential equation. Otherwise, the equation is **variable coefficient** differential equation.

Example:

$$2\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = \sin(x) \quad (\text{Constant coefficient}) \quad (\text{C-11})$$

$$3\frac{d^4y}{dt^4} + x^2\frac{dy}{dt} + \sin(x)y = x^2 \quad (\text{Variable coefficient}) \quad (\text{C-12})$$

In equation (x), the value n represents the order of the equation. Besides linearity and homogeneity, it is important to recognize the order of the differential equation.

Example:

$$\frac{d^4y}{dt^4} + y = x^2 \quad (\text{4th order}) \quad (\text{C-13})$$

$$\frac{dy}{dt} + y\frac{d^2y}{dt^2} = \cos(x) \quad (\text{2nd order}) \quad (\text{C-14})$$

These characteristics are useful in determining the appropriate approach to find the **solution** $y(x)$.for the corresponding differential equations.

Solutions of Linear Differential Equations

In order to find solutions for linear differential equations, it is important to understand different types and the practical meaning of solutions. $y(x)$ is a **solution** of a differential equation if and only if $y(x)$ satisfy the equation. However, differential equation can have multiple solutions.

Example:

$$\frac{dy}{dx} - 2y = x \quad (\text{C-15})$$

The followings are two solutions of the given differential equation.

$$y_1 = e^{2x} - \frac{x}{2} - \frac{1}{4}, \quad y_2 = 2e^{2x} - \frac{x}{2} - \frac{1}{4} \quad (\text{C-16})$$

Both solutions y_1 and y_2 satisfy the equation.

$$\begin{aligned} \frac{dy_1}{dx} - 2y_1 &= \left(2e^{2x} - \frac{1}{2}\right) - 2\left(e^{2x} - \frac{x}{2} - \frac{1}{4}\right) \\ &= 2e^{2x} - \frac{1}{2} - 2e^{2x} + x + \frac{1}{2} = x \end{aligned} \quad (\text{C-17})$$

$$\begin{aligned} \frac{dy_2}{dx} - 2y_2 &= \left(4e^{2x} - \frac{1}{2}\right) - 2\left(2e^{2x} - \frac{x}{2} - \frac{1}{4}\right) \\ &= 4e^{2x} - \frac{1}{2} - 4e^{2x} + x + \frac{1}{2} = x \end{aligned} \quad (\text{C-18})$$

However, there are infinite solutions to this particular differential equation. Therefore the set of all solutions is put into a single expression known as a **general solution**.

In this case, the general solution of the differential equation is the following:

$$y_g(x) = Ce^{2x} - \frac{x}{2} - \frac{1}{4} \quad (\text{C-19})$$

where C is an undetermined constant. This constant can be solved by applying initial condition to the solution. Once the constant is defined, a general solution simply becomes a solution.

With an undetermined constant, a general solution consists of a **particular solution** and a **homogeneous solution**. In other word, the general solution of this differential equation can be written in form of the following:

$$y_g(x) = y_h(x) + y_p(x) \quad (\text{C-20})$$

where $y_p(x)$ is a particular solution and $y_h(x)$ is a homogenous solution.

By substituting a homogeneous solution to the corresponding differential equation, the equation become homogeneous (forcing function is 0). In this case, the homogeneous solution of equation 1 is

$$y_h(x) = Ce^{2x} \quad (\text{C-21})$$

because

$$\frac{d}{dx}y_h(x) - 2y_h(x) = 2Ce^{2x} - 2(Ce^{2x}) = 0 \quad (\text{C-22})$$

In the expression of a general solution, homogenous solution is an expression involving the undetermined constant C . Because the homogeneous solution give the value of 0, multiplying by any arbitrary constant give the same expression as the one without it.

In the other hand, a particular solution is a solution that satisfies the differential equation but does not involve undetermined constant C . In this case, the particular solution of the differential equation is

$$y_p(x) = -\frac{x}{2} - \frac{1}{4} \quad (\text{C-23})$$

because

$$\frac{d}{dx}y_p(x) - 2y_p(x) = \left(-\frac{1}{2}\right) - 2\left(-\frac{x}{2} - \frac{1}{4}\right) = -\frac{1}{2} + x + \frac{1}{2} = x \quad (\text{C-24})$$

Using some solving techniques may yield both homogeneous and particular solutions at the same time. But other technique might requires one to find the homogenous solution before proceed to find the particular solution. Therefore, understanding the different types of solution is essential for solving and analysis differential equations.

First-order Differential Equation solving technique

Separation of variables

Any first-order differential equation that can be expressed as the following

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad (\text{C-25})$$

is known as a separable equation. Separation of variables is a technique used in solving first-order separable equations. This technique is very useful for solving first-order nonlinear differential equation.

Let us show the procedure to solve the general equation:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad (\text{C-26})$$

Each side of the equation should have one distinct variable. That way, the left-hand side of the equation contains y while the right-hand side contains x only.

$$g(y)dy = f(x)dx \quad (\text{C-27})$$

Both sides are integrated. By convention, the constant of integration is added to the side involving the independent variable (x in this case).

$$\int g(y)dy = \int f(x)dx + C \quad (\text{C-28})$$

Example:

Let us solve the following equation:

$$\frac{dy}{dx} = x^2y \quad (\text{C-29})$$

Equation (C-29) is a separable equation since it is in the form of equation (C-25). It is therefore valid to use separation of variables.

$$\frac{dy}{y} = x^2dx \quad (\text{C-30})$$

$$\int \frac{dy}{y} = \int x^2dx \quad (\text{C-31})$$

$$\ln|y| = \frac{x^3}{3} + c \quad (\text{C-32})$$

$$e^{\ln|y|} = e^{\left(\frac{x^3}{3}+c\right)} \quad (\text{C-33})$$

$$|y| = e^{\left(\frac{x^3}{3}\right)} \cdot e^c \quad (\text{C-34})$$

Since e^c is a constant, it can be changed into a constant C for the sake of simplicity. The absolute value can be then dropped since C could be either positive or negative.

$$y = Ce^{\left(\frac{x^3}{3}\right)} \quad (\text{C-35})$$

Integrating factor Method

Any first-order differential equation that can be expressed as the following

$$a(x) \frac{dy}{dx} + b(x)y = g(x) \quad (\text{C-36})$$

is known as first-order linear equation. Integrating factor method is a technique used in solving first-order linear equations.

The general procedure used for solving first-order linear differential equation is as follow.

First, equation (C-37) is rearranged in the format

$$\frac{dy}{dx} + p(x)y = f(x) \quad (\text{C-37})$$

by dividing the entire equation by the coefficient of the first-order derivative $a(x)$. $p(x)$ and $f(x)$ are $\frac{b(x)}{a(x)}$ and $\frac{g(x)}{a(x)}$ respectively.

The next step is to find the integrating factor which is defined as

$$\mu(x) = e^{\int p(x)dx} \quad (\text{C-38})$$

and multiply both sides of the equation by it in order to get the following form

$$\mu(x) \left[\frac{dy}{dx} + p(x)y \right] = \mu(x)f(x) \quad (\text{C-39})$$

$$e^{\int p(x)dx} \left[\frac{dy}{dx} + p(x)y \right] = e^{\int p(x)dx} f(x) \quad (\text{C-40})$$

$$\left[\frac{dy}{dx} e^{\int p(x)dx} + ye^{\int p(x)dx} p(x) \right] = e^{\int p(x)dx} f(x) \quad (\text{C-41})$$

Note that the left hand-side of the equation is the derivative of $ye^{\int p(x)dx}$ according to the product rule.

$$\frac{d}{dx} [ye^{\int p(x)dx}] = e^{\int p(x)dx} f(x) \quad (\text{C-42})$$

$$\int \frac{d}{dx} [ye^{\int p(x)dx}] dx = \int e^{\int p(x)dx} f(x) dx \quad (\text{C-43})$$

By integrating both sides of the equation, we obtain

$$ye^{\int p(x)dx} = \int e^{\int p(x)dx} f(x) dx + C \quad (\text{C-44})$$

By arranging expression, the function y can be expressed as

$$1y = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} f(x) dx + C \right] \quad (\text{C-45})$$

or

$$y = \frac{\int \mu(x)f(x)dx + C}{\mu(x)} \text{ where } \mu(x) = e^{\int p(x)dx} \quad (\text{C-46})$$

Example:

$$\frac{dy}{dx} + \frac{2}{x}y = x \quad (\text{C-47})$$

$$p(x) = \frac{2}{x} ; f(x) = x \quad (\text{C-48})$$

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x} dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2 \quad (\text{C-49})$$

$$y = \frac{\int x^2(x)dx + C}{x^2} \quad (\text{C-50})$$

$$y = \frac{\int x^3 dx + C}{x^2} = \frac{\int x^3 dx}{x^2} + \frac{C}{x^2} = \frac{x^4}{4} + \frac{C}{x^2} \quad (\text{C-51})$$

$$y = \frac{x^4}{4} + \frac{C}{x^2} \quad (\text{C-52})$$

Second-order Linear Differential Equation solving technique

In order to solve for a general solution of a problem in second-order, it is essential to be acquainted with structure and characteristic of the solutions. Because of linear behavior, the **superposition principle** can be applied in solving linear differential equations. The superposition principle states that the output of a linear system caused by multiple sub-systems is the sum of the outputs of each sub-system individually. Since y_1 and y_2 satisfy the homogenous differential equation, the solution in form of their linear combination also satisfy the equation.

A general solution of a second-order linear homogeneous differential equation can be written in form of a linear combination between two linearly independent homogeneous solutions.

$$y_h = C_1y_1 + C_2y_2 \quad (\text{C-53})$$

where y_1 and y_2 are linearly independent homogenous solutions and C_1 and C_2 are undetermined constants.

The word "**linearly independent**" refers that one of the solutions cannot be written in form of scalar multiply of the other function.

$$y_1 \neq Cy_2 \quad (\text{C-54})$$

Therefore, both of solutions need to be found in order to fully define a general solution of homogenous differential equation. One of the methods that mathematicians and engineers use is **Characteristic Equation**.

Characteristic Equation of homogeneous equation

A second-order linear homogeneous differential equation with constant coefficients can be expressed as followed:

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad (\text{C-55})$$

where a, b, c are non-zero constants.

In this particular technique, the solution of the differential equation is assumed to be in the form of:

$$y = e^{rx} \quad (\text{C-56})$$

where A and m are constants.

Therefore, the derivatives of the solution are

$$\frac{dy}{dx} = re^{rx} \quad (\text{C-57})$$

$$\frac{d^2y}{dx^2} = r^2 e^{rx} \quad (\text{C-58})$$

By substituting the corresponding derivatives to Equation (C-55), the equation becomes

$$a(r^2 e^{rx}) + b(re^{rx}) + c(e^{rx}) = 0 \quad (\text{C-59})$$

Since e^{mx} has a positive value, the equation can be divided Ae^{mx} which gives the following equation.

$$a(r^2) + b(r) + c = 0 \quad (\text{C-60})$$

This equation is called **characteristic equation**. Then, the quadratic formula is applied in order to obtain roots of the equation r .

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\text{C-61})$$

or

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Because the quadratic formula yields two values, there are two corresponding solutions.

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x} \quad (\text{C-62})$$

The general solution equation (C-55) is the linear combination of y_1 and y_2 .

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad (\text{C-63})$$

Based on the quadratic characteristic, the type of solution can be categorized in three different types.

Case 1: Real & Distinct Root ($b^2 - 4ac > 0$)

This particular term $b^2 - 4ac$ is called **discriminant**. It describes the characteristic of the solutions of quadratic formula. If the discriminant is positive, there will be two linearly independent solutions.

Example:

$$y'' - y' + 6y = 0 \quad (\text{C-64})$$

Let $y = e^{rx}$, therefore $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Then, the the expression can be substitute with the corresponding derivatives.

$$(r^2 e^{rx}) - (re^{rx}) + 6(e^{rx}) = 0 \quad (\text{C-65})$$

$$(r^2) - (r) + 6 = 0 \quad (\text{C-66})$$

By using quadratic formula, the roots of the characteristic equation can be obtained.

$$r_1 = \frac{-(-1) + \sqrt{(-1)^2 - 4(-1)(6)}}{2(1)} = \frac{1 + 5}{2} = 3 \quad (\text{C-67})$$

$$r_2 = \frac{-(-1) - \sqrt{(-1)^2 - 4(-1)(6)}}{2(1)} = \frac{1 - 5}{2} = -2 \quad (\text{C-68})$$

Therefore, the general solution of this differential equation is the following.

$$y = C_1y_1 + C_2y_2 = C_1e^{r_1x} + C_2e^{r_2x} = C_1e^{3x} + C_2e^{-2x} \quad (\text{C-69})$$

Case 2: Repeated Root ($b^2 - 4ac = 0$)

When the discriminant equals to zero, the quadratic formula yields only one solution. But in order to fully express the general solution of second-order differential equations, the second linear independent homogeneous is required. A root of the characteristic equation can be derived.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a} \quad (\text{C-70})$$

By using other technique of solving differential equation, the solutions of the repeated root second-order differential equation can be expressed as the following.

$$y_1 = e^{rx}, \quad y_2 = xe^{rx} \quad (\text{C-71})$$

Example:

$$4y'' + 4y' + y = 0 \quad (\text{C-72})$$

Let $y = e^{rx}$, therefore $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Then, the the expression can be substitute with the corresponding derivatives.

$$4(r^2e^{rx}) + 4(re^{rx}) + (e^{rx}) = 0 \quad (\text{C-73})$$

$$4(r^2) + 4(r) + 1 = 0 \quad (\text{C-74})$$

Since the discriminant is $(4)^2 - 4(4)(1) = 0$, two homogeneous solution can be expressed as the following.

$$r = -\frac{b}{2a} = -\frac{(4)}{2(4)} = -\frac{1}{2} \quad (\text{C-75})$$

Therefore, the general solution of this differential equation is the following.

$$y = C_1y_1 + C_2y_2 = C_1e^{rx} + C_2xe^{rx} = C_1e^{-\frac{1}{2}x} + C_2xe^{-\frac{1}{2}x} \quad (\text{C-76})$$

Case 3: Complex Root ($b^2 - 4ac < 0$)

A negative discriminant causes the expression to involve complex number. Since the discriminant is under the square root, the square root of negative numbers yields complex value in the exponents. It is more preferable for mathematicians and engineers to put expression in non-complex form.

The complex roots of the characteristic equation can be found

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{-(4ac - b^2)}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{(4ac - b^2)}}{2a} \sqrt{-1} \quad (\text{C-77})$$

$$r = -\frac{b}{2a} \pm \frac{\sqrt{(4ac - b^2)}}{2a} i \quad (\text{C-78})$$

The expression that involves of the complex number i is called the **imaginary part**. The one that does not involve i is called the **real part**. Let we denote the imaginary part and the real part to be in term of constants.

$$\alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{(4ac - b^2)}}{2a} \quad (\text{C-79})$$

The roots can be rewritten as:

$$r = \alpha \pm \beta i \quad (\text{C-80})$$

Therefore, by using Euler's formula and algebraic manipulation, the homogeneous solutions of complex root second-order differential equation can be rearranged to a form that has no complex number.

$$y_1 = e^{\alpha x} \cos(\beta x), \quad y_2 = e^{\alpha x} \sin(\beta x) \quad (\text{C-81})$$

Example:

$$y'' + 4y' + 29y = 0 \quad (\text{C-82})$$

Let $y = e^{rx}$, therefore $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Then, the the expression can be substitute with the corresponding derivatives.

$$(r^2 e^{rx}) + 4(re^{rx}) + 29(e^{rx}) = 0 \quad (\text{C-83})$$

$$(r^2) + 4(r) + 29 = 0 \quad (\text{C-84})$$

Since the discriminant is $(4)^2 - 4(1)(29) = -100$, two complex roots can be expressed as the following.

$$\alpha = -\frac{b}{2a} = -\frac{(4)}{2(1)} = -2 \quad (\text{C-85})$$

$$\beta = \frac{\sqrt{(4ac - b^2)}}{2a} = \frac{\sqrt{(4(1)(29) - (4)^2)}}{2(1)} = 5 \quad (\text{C-86})$$

$$r = \alpha \pm \beta i = -2 \pm 5i \quad (\text{C-87})$$

Therefore, the general solution of this differential equation is the following.

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x) = C_1 e^{-2x} \cos(5x) + C_2 e^{-2x} \sin(5x) \quad (\text{C-88})$$

If the initial conditions of a problem are given, the differential equation can be relatively easy to solve after finding the general solution. A second-order differential equation requires two initial conditions. The initial condition is usually given in term of initial quantity and initial rate.

Example:

$$y'' - y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = -1 \quad (\text{C-89})$$

From the previous example, the general solution of the equation above is

$$y(x) = C_1 e^{3x} + C_2 e^{-2x} \quad (\text{C-90})$$

The derivative of the general solution is

$$y'(x) = 3C_1 e^{3x} - 2C_2 e^{-2x} \quad (\text{C-91})$$

The constant C_1 and C_2 can be solved by substituting corresponding value of $y(x)$, $y'(x)$, and x .

$$y(0) = C_1 e^{3(0)} + C_2 e^{-2(0)} \quad (\text{C-92})$$

$$2 = C_1 + C_2 \quad (\text{C-93})$$

$$y'(0) = 3C_1 e^{3(0)} - 2C_2 e^{-2(0)} \quad (\text{C-94})$$

$$-1 = 3C_1 - 2C_2 \quad (\text{C-95})$$

By solving the system of linear equations, equation (C-93) is multiplied by 2 added to equation (C-95).

$$2 * 2 = 2 * (C_1 + C_2) \quad (\text{C-96})$$

$$(2 * 2) - 1 = 2 * (C_1 + C_2) + 3C_1 - 2C_2 \quad (\text{C-97})$$

$$3 = 5C_1 \quad (\text{C-98})$$

$$C_1 = \frac{3}{5} \text{ and } C_2 = \frac{7}{5} \quad (\text{C-99})$$

Therefore, the solution of the initial-value problem can be written as the following.

$$y(x) = \frac{3}{5}e^{3x} + \frac{7}{5}e^{-2x} \quad (\text{C-100})$$

Method of Undetermined Coefficients

Some second-order differential equations might be nonhomogeneous. Therefore, it is necessary to solve for both homogeneous solution and particular solution in order to obtain the general solution of nonhomogeneous differential equation. In this case, the method of finding homogeneous solution of a nonhomogeneous differential equation is the same with homogeneous differential equation. The general solution of second-order nonhomogeneous equation can be expressed in form of the following.

$$y(x) = C_1y_1 + C_2y_2 + y_p \quad (\text{C-101})$$

where y_1 and y_2 are linearly independent homogenous solutions, y_p is a particular solution and C_1 and C_2 are undetermined constants.

One of the methods used in solving for a particular is Method of Undetermined Coefficients. The method is based on guessing a particular solution by characteristic of the nonhomogeneous term.

The second-order linear differential equation with constant coefficients can be written as below.

$$ay'' + by' + cy = f(x) \quad (\text{C-102})$$

By inspecting the nonhomogeneous term, its derivatives can be found as $f'(x)$ and $f''(x)$. The guess might be obtained by linear combination of the derivatives and the function itself.

$$y_p(x) = C_1f(x) + C_2f'(x) + C_3f''(x) \quad (\text{C-103})$$

Example:

$$y'' + 5y' + 4y = 3 \quad (\text{C-104})$$

It might be trivial that the solution is in term of constant since the nonhomogeneous term is a constant. If the guess is assumed to be $y_p = A$, the according derivatives will be zero. Substituting the corresponding value into the differential equation, we will obtain

$$(0) + 5(0) + 4(A) = 3 \quad (\text{C-105})$$

$$A = \frac{3}{4} \quad (\text{C-106})$$

Now we know that the particular solution is $\frac{3}{4}$, the characteristic equation is applied to obtain the homogeneous solution.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4(1)(4)}}{2(1)} = \frac{-5 \pm 3}{2} = -4, -1 \quad (\text{C-107})$$

$$y_h = C_1 e^{-4x} + C_2 e^{-x} \quad (\text{C-108})$$

The general solution is written as a sum of particular solution and homogeneous solution. Therefore, the general solution of this differential equation is

$$y = \frac{3}{4} + C_1 e^{-4x} + C_2 e^{-x} \quad (\text{C-109})$$

Example:

$$y'' - 2y' + 1 = e^{2x} \quad (\text{C-110})$$

Since the nonhomogeneous term is in term of e^{2x} which has derivatives of $2e^{2x}$ and $4e^{2x}$, the guess is assumed to be $y_p = Ae^{2x}$, the according first-order derivatives will be $2Ae^{2x}$. And the second-order derivative is $4Ae^{2x}$. Substituting the corresponding value into the differential equation, we will obtain

$$(4Ae^{2x}) - 2(2Ae^{2x}) + (Ae^{2x}) = e^{2x} \quad (\text{C-111})$$

$$(4 - 4 + 1)(Ae^{2x}) = e^{2x} \quad (\text{C-112})$$

$$Ae^{2x} = e^{2x} \quad (\text{C-113})$$

$$A = 1 \quad (\text{C-114})$$

Now we know that the particular solution is e^{2x} , the characteristic equation is applied to obtain the homogeneous solution.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \frac{2}{2} = 1 \quad (\text{C-115})$$

Since the root is repeated, the homogenous solution can be written as

$$y_h = C_1 e^x + C_2 x e^x \quad (\text{C-116})$$

The general solution is written as a sum of particular solution and homogeneous solution. Therefore, the general solution of this differential equation is

$$y = e^{2x} + C_1 e^x + C_2 x e^x \quad (\text{C-117})$$

Example:

$$y'' - y' - 6y = 3\cos(2x) \quad (\text{C-118})$$

Since the derivatives of nonhomogeneous term are $-6\sin(2x)$ and $-12\cos(2x)$, the guess is assumed to be

$$y_p = A \sin(2x) + B \cos(2x) \quad (\text{C-119})$$

The first-order derivative is $2A \cos(2x) - 2B \sin(2x)$. And the second-order derivative is $-4A \sin(2x) - 4B \cos(2x)$. Substituting the corresponding value into the differential equation, we will obtain

$$(-4A \sin(2x) - 4B \cos(2x)) - (2A \cos(2x) - 2B \sin(2x)) - 6(A \sin(2x) + B \cos(2x)) = 3 \cos(2x) \quad (\text{C-120})$$

$$(-10A + 2B) \sin(2x) + (-2A - 10B) \cos(2x) = 3 \cos(2x) \quad (\text{C-121})$$

On the left side of equation (C-121), the coefficient of the term involving sine is $(-10A + 2B)$ while the right side has the coefficient of 0. (because there is no term involving sine on the right). The following linear equation can be obtained.

$$(-10A + 2B) = 0 \quad (\text{equation 1}) \quad (\text{C-122})$$

By inspecting the cosine term, we notice that the coefficient on the left-hand side of equation (C-121) has to equal to the one on the right. Therefore, another linear equation can be found.

$$(-2A - 10B) = 3 \quad (\text{equation 2}) \quad (\text{C-123})$$

The system of equations can be solved by multiplying equation (C-122) by 5.

$$5 * (-10A + 2B) = 5 * 0 \quad (\text{equation 3}) \quad (\text{C-124})$$

Then, equation (C-123) and (C-124) are added in order to solve for coefficient A .

$$5 * (-10A + 2B) + (-2A - 10B) = 5 * 0 + 3 \quad (\text{C-125})$$

$$-52A = 3 \quad (\text{C-126})$$

$$A = -\frac{3}{52} \quad (\text{C-127})$$

The coefficient B can be solved by substituting the value of the coefficient A into the linear equation.

$$B = -\frac{15}{52} \quad (\text{C-128})$$

The corresponding particular solution of the differential equation becomes

$$y_p = -\frac{3}{52} \sin(2x) - \frac{15}{52} \cos(2x) \quad (\text{C-129})$$

Now we know that the particular solution is $-\frac{3}{52} \sin(2x) - \frac{15}{52} \cos(2x)$, the characteristic equation is applied to obtain the homogeneous solution.

$$y = e^{rx} \quad (C-130)$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)} = \frac{-1 \pm 5}{2} = -3 \text{ and } 2 \quad (C-131)$$

Since the roots are distinct, the homogenous solution can be written as

$$y_h = C_1 e^{-3x} + C_2 e^{2x} \quad (C-132)$$

The general solution is written as a sum of particular solution and homogeneous solution. Therefore, the general solution of this differential equation is

$$y = -\frac{3}{52} \sin(2x) - \frac{15}{52} \cos(2x) + C_1 e^{-3x} + C_2 e^{2x} \quad (C-133)$$

Example:

$$2y'' + y' - 3y = 14xe^{2x} \quad (C-134)$$

In this case, the homogeneous term is written in form a product of polynomial and exponential function. The particular solution may be assumed to be in form of the product as well. Therefore, the particular solution is in form of $y_p = (Ax + B)e^{2x}$ or $Axe^{2x} + Be^{2x}$.

By using product rule of differentiation, the first-order derivative is

$$y' = (Ax + B)(2e^{2x}) + (A)(e^{2x}) = 2Axe^{2x} + (A + 2B)e^{2x} \quad (C-135)$$

And the second-order derivative is

$$y'' = (Ax + B)(4e^{2x}) + (A)(2e^{2x}) + (A)(2e^{2x}) = 4Axe^{2x} + 4(A + B)e^{2x} \quad (C-136)$$

Substituting the corresponding value into equation (C-134), we will obtain

$$2[4Axe^{2x} + 4(A + B)e^{2x}] + [2Axe^{2x} + (A + 2B)e^{2x}] - 3[Axe^{2x} + Be^{2x}] = 14xe^{2x} \quad (C-137)$$

$$[8Axe^{2x} + 2Axe^{2x} - 3Axe^{2x}] + [8(A + B)e^{2x} + (A + 2B)e^{2x} - 3Be^{2x}] = 14xe^{2x} \quad (C-138)$$

$$[7Axe^{2x}] + [(9A + 7B)e^{2x}] = 14xe^{2x} \quad (C-139)$$

The coefficients in front of xe^{2x} have to be the same for both sides of the equation. Therefore, a linear equation can be obtained by setting the terms equal to one another.

$$7Axe^{2x} = 14xe^{2x} \quad (C-140)$$

$$7A = 14 \quad (C-141)$$

$$A = 2 \quad (C-142)$$

Since the term involving e^{2x} does not exist on the right-hand side of equation (C-139), the coefficient in front has to be zero.

$$(9A + 7B)e^{2x} = 0 \quad (\text{C-143})$$

$$9A + 7B = 0 \quad (\text{C-144})$$

By substituting A into the equation, the value of B can be found.

$$9(2) + 7B = 0 \quad (\text{C-145})$$

$$B = -\frac{18}{7} \quad (\text{C-146})$$

The corresponding particular solution of the differential equation becomes

$$y_p = (Ax + B)e^{2x} = y_p = \left(2x - \frac{18}{7}\right)e^{2x} \quad (\text{C-147})$$

The homogeneous solution is required in order to fully express the general solution of the differential equation. Therefore, the characteristic equation is applied to obtain the homogeneous solution.

$$2y'' + y' - 3y = 13xe^{2x} \quad (\text{C-148})$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(2)(-3)}}{2(2)} = \frac{-1 \pm 5}{2} = -3, 2 \quad (\text{C-149})$$

$$y_h = C_1e^{-3x} + C_2e^{2x} \quad (\text{C-150})$$

The general solution is written as a sum of particular solution and homogeneous solution. Therefore, the general solution of this differential equation is

$$y = \left(2x - \frac{18}{7}\right)e^{2x} + C_1e^{-3x} + C_2e^{2x} \quad (\text{C-151})$$

Sometimes the nonhomogeneous term might be in the form of homogeneous solution. In this case, a particular solution cannot be in the term of nonhomogeneous term. Therefore, the guess for a particular solution needs to be adjusted.

Example:

$$y'' + 9y = 2\cos(3x) \quad (\text{C-152})$$

If the guess was said to be in form of $y = A \sin(3x) + B \cos(3x)$, the equation would be incorrect.

$$y' = 3A \cos(3x) - 3B \sin(3x) \quad (\text{C-153})$$

$$y'' = -9A \sin(3x) - 9B \cos(3x) \quad (\text{C-154})$$

$$[-9A \sin(3x) - 9B \cos(3x)] + 9[A \sin(3x) + B \cos(3x)] = 2\cos(3x) \quad (\text{C-155})$$

$$0 \neq 2\cos(3x) \quad (\text{C-156})$$

This time, the assumed particular solution is multiplied by polynomial x . The particular solution will be in the form of

$$y = Ax \sin(3x) + Bx \cos(3x) \quad (\text{C-157})$$

Since the derivative of this function involves $\cos(3x)$, the solution might verify the equation.

The corresponding derivatives are

$$y' = A[3x \cos(3x) + \sin(3x)] + B[-3x \sin(3x) + \cos(3x)] \quad (\text{C-158})$$

$$= [3Ax \cos(3x) + A \sin(3x)] + [-3Bx \sin(3x) + B \cos(3x)]$$

$$y'' = A[3[-3x \sin(3x) + \cos(3x)] + 3A \cos(3x)] + B[-3[3x \cos(3x) + \sin(3x)] - 3B \sin(3x)] \quad (\text{C-159})$$

$$= [-9Ax \sin(3x) + 3A \cos(3x) + 3A \cos(3x)] + [-9Bx \cos(3x) - 3B \sin(3x) - 3B \sin(3x)]$$

$$= [-9Ax \sin(3x) + 6A \cos(3x)] + [-9Bx \cos(3x) - 6B \sin(3x)]$$

Substituting the corresponding value equation (C-152), we will obtain

$$[-9Ax \sin(3x) + 6A \cos(3x)] + [-9Bx \cos(3x) - 6B \sin(3x)] + 9[Ax \sin(3x) + Bx \cos(3x)] = 2\cos(3x) \quad (\text{C-160})$$

$$[6A \cos(3x)] + [-6B \sin(3x)] = 2 \cos(3x) \quad (\text{C-161})$$

The coefficients in front of $\cos(3x)$ have to be equal to one another. Therefore, we obtain an equation.

$$6A \cos(3x) = 2\cos(3x) \quad (\text{C-162})$$

$$6A = 2 \quad (\text{C-163})$$

$$A = \frac{1}{3} \quad (\text{C-164})$$

Since there is no term involving $\sin(3x)$ on the right-hand side of equation (C-161), the coefficient B becomes zero. Then, the particular becomes the following.

$$y_p = \frac{1}{3} x \sin(3x) \quad (\text{C-165})$$

The characteristic equation is applied to obtain the homogeneous solution. In this case, the roots of the characteristic equation are complex.

$$y'' + 9y = 2\cos(3x) \quad (\text{C-166})$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0 - 4(1)(9)}}{2(1)} = \frac{\pm 6i}{2} = 0 \pm 3i \quad (\text{C-167})$$

Therefore, the homogeneous solution of this differential equation is the following.

$$\begin{aligned} y_g &= C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x) \quad (\text{C-168}) \\ &= C_1 e^{(0)x} \cos(3x) + C_2 e^{(0)x} \sin(3x) \\ &= C_1 \cos(3x) + C_2 \sin(3x) \end{aligned}$$

The general solution is written as a sum of particular solution and homogeneous solution. Therefore, the general solution of this differential equation is

$$y = \frac{1}{3} x \sin(3x) + C_1 \cos(3x) + C_2 \sin(3x) \quad (\text{C-169})$$

The following is the table of corresponding function used for guess the particular solution of second-order linear differential equation with constant coefficient. The corresponding guess of particular solutions cannot be the same as homogeneous solutions.

Table: Guess of Particular Solution

Type	$f(x)$	y_p
Constant	a	$x^k \cdot A$
Polynomial	$P_n(x) = a_0 x^n + \dots + a_{n-1} x + a_n$	$x^k \cdot (A_0 x^n + \dots + A_{n-1} x + A_n)$
Exponential	$a e^{\alpha x}$	$x^k \cdot A e^{\alpha x}$
Sine or Cosine	$a \cos(\beta x) + b \sin(\beta x)$	$x^k \cdot (A \cos(\beta x) + B \sin(\beta x))$
Product of Polynomial and Exponential	$P_n(x) e^{\alpha x}$	$x^k \cdot (A_0 x^n + \dots + A_{n-1} x + A_n) e^{\alpha x}$
Product of Polynomial and Sine or Cosine	$P_n(x) \cos(\beta x)$ $P_n(x) \sin(\beta x)$	$x^k \cdot [(A_0 x^n + \dots + A_{n-1} x + A_n) \cos(\beta x) + (B_0 x^n + \dots + B_{n-1} x + B_n) \sin(\beta x)]$
Product of Polynomial, exponential and Sine or Cosine	$P_n(x) e^{\alpha x} \cos(\beta x)$ $P_n(x) e^{\alpha x} \sin(\beta x)$	$x^k \cdot [(A_0 x^n + \dots + A_{n-1} x + A_n) e^{\alpha x} \cos(\beta x) + (B_0 x^n + \dots + B_{n-1} x + B_n) e^{\alpha x} \sin(\beta x)]$
The exponent k in x^k is the smallest of the integers 0,1, or 2 that ensures that no term in the particular solution y_p is also a solution of the corresponding homogenous equation.		

Appendix D: The Laplace Transform

Definition

There is no doubt that the methods described in previous sections are a good way to solve ordinary differential equations. But due to its unique nature, **Laplace Transform** is one of the most used concepts in solving initial-value problems. Moreover, it is a necessary mathematical tool in many engineering disciplines where the use of differential equations is a must.

The power of Laplace Transform resides in its ability to *transform* differential equations into algebraic equations. The **integral transformation** in equation (D-1) is the key that transforms the function of interest $f(t)$ into its image $F(s)$ where a and b are the boundaries of the interval within which $f(t)$ is defined. In fact, this integral is the transformation from the **time-domain** to **frequency-domain**. It is not important at this stage to know the technical meaning of these domains. It is simply a change of domain that allows us to depict some of the function characteristics that would not be clearly seen in the time-domain.

$$F(s) = \int_a^b e^{-st} f(t) dt \quad (\text{D-1})$$

Laplace transform of $f(t)$ is presented as either $F(s)$ or $\mathcal{L}\{f\}$.

As mentioned above, a function $f(t)$ should be integrated as shown in Eq. (D-1) in order to obtain $F(s)$. Since this book is focused on the application of mathematics rather than its theory and technicality, it is recommended to use [Table 1](#).

	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	$t^n (n = 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
3	$t^p (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$e^{at}t^n (n = 1, 2, \dots)$	$\frac{n!}{(s-a)^{n+1}}$
6	$\sin bt$	$\frac{b}{s^2 + b^2}$
7	$\cos bt$	$\frac{s}{s^2 + b^2}$
8	$\sinh bt$	$\frac{b}{s^2 - b^2}$
9	$\cosh bt$	$\frac{s}{s^2 - b^2}$
10	$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$
11	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$
12	$u(t-c)$	$\frac{e^{-cs}}{s}$
13	$u(t-c)f(t-c)$	$e^{-cs}F(s)$
14	$\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$
15	$\delta(t-c)$	e^{-cs}
16	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$
17	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
18	$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} dt}{1 - e^{-sT}}$

Table 1: Laplace Transform Table

Example 1

Find the Laplace transform of $\sin 3t$.

From the 6th entry of Table 1, $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

Find the Laplace transform of $e^{3t} \cos 2t$.

From the 11th entry of $\{e^{3t} \cos 2t\} = \frac{s-3}{(s-3)^2+4}$

Properties

The most basic property of Laplace transform is **linearity**. Therefore, the image of two or more functions added together is the sum of all independent images. Likewise, the image of the product of a constant and a function is the product of the constant and the image of that function.

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \quad (\text{D-2})$$

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\} \quad (\text{D-3})$$

Now, here comes the first relationship between Laplace transform and differential equation. This property is the **Laplace Transform of higher derivatives** and is stated as follow:

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0) \quad (\text{D-4})$$

when $f, f', f'', \dots, f^{(n-1)}$ are continuous in the positive time domain and $f^{(n)}$ is **piecewise continuous**¹. Let us not stress about the continuity characteristics because all functions used in System Dynamics are continuous. This derivative formula will be used in the following sections as a potential tool to solve initial value problems.

The third property is **translation** and is shown in Eq. (D-5). It can be seen from the integral transformation equation that multiplying a function f by e^{at} would result in a translation in s .

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a) \quad (\text{D-5})$$

when a is a constant.

Another property useful in solving differential equations is as follows:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s) \quad (\text{D-6})$$

when n is a positive integer.

	Property
1	$\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2	$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
3	$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0)$
4	$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$
5	$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s)$

Table 2: Laplace Properties

Inverse Laplace Transform

Definition

Both Laplace and inverse Laplace transforms are used to solve differential equations. Clear steps are shown in the section of “Initial-Value Problems”. Expressions are now given in the s domain, and solutions are sought in the time domain.

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad (\text{D-7})$$

The integral relation stated in Eq. (D-1) could be rearranged in order to find the inverse Laplace transform. The resulting formula would involve some complex mathematical principles that are beyond the scope of this book. Therefore, it is recommended to make use of [Table 1](#) in the previous section.

Example 1

Find the inverse Laplace transform of $\frac{s}{s^2+25}$.

According to the 7th entry of [Table 1](#), $f(t) = \cos(5t)$.

Find the inverse Laplace transform of $\frac{6s}{(s^2+9)^2}$.

Looking at the 14th entry of the Laplace table, it is easy to notice that $a = 3$.

Therefore, $f(t) = t\sin(3t)$.

Properties

Same as Laplace transform, inverse is linear. Therefore, the image of two or more functions added together is the sum of all independent images. Likewise, the image of the product of a constant and a function is the product of the constant and the image of that function.

$$\mathcal{L}^{-1}\{F + G\} = \mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\} \quad (\text{D-8})$$

$$\mathcal{L}^{-1}\{aF\} = a\mathcal{L}^{-1}\{F\} \quad (\text{D-9})$$

Partial Fraction Decomposition

In most cases, expressions are not as simple as the ones proposed in examples 1 and 2. Expressions are usually found in the form of $\frac{N(s)}{D(s)}$ where both $N(s)$ and $D(s)$ are polynomials. If the degree of the numerator is lower than the degree of the denominator, the inverse Laplace transform is found after simplifying $\frac{N(s)}{D(s)}$ using **partial fraction decomposition**.

The decomposition depends on the type of factor in the denominator. There are four major types of factors.

1. **Linear Factor:** The general form of a linear factor is $as + b$. For each factor in $D(s)$, there is an integer A in $N(s)$, thus resulting into the term $\frac{A}{as+b}$.
2. **Power and Linear Factor:** The linear factor may have a power of n : $(as + b)^n$. There is a different integer A corresponding to each order of the factor. The resulting forms are the following: $\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \dots + \frac{A_n}{(as+b)^n}$.
3. **Quadratic Factor:** there is a numerator $As + B$ for every denominator $as^2 + bs + c$. The corresponding term is then: $\frac{As+B}{as^2+bs+c}$.
4. **Power and Quadratic Factor:** using the same principle in type 2, the corresponding decomposition term is $\frac{A_1s+B_1}{as^2+bs+c} + \frac{A_2s+B_2}{(as^2+bs+c)^2} + \dots + \frac{A_ns+B_n}{(as^2+bs+c)^n}$.

Example 2

Find the inverse Laplace transform of $\frac{1}{s(2s^2+3)}$.

Since the provided Transform table does not contain such complicated expression, partial fraction decomposition should be used to simplify it.

$$\frac{1}{s(2s^2 + 3)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + 3} \quad (\text{D-10})$$

Common denominator procedure follows.

$$1 = A(2s^2 + 3) + s(Bs + C)$$

The above expression is factored in terms of power of s .

$$1 = s^2(2A + B) + Cs + 3A$$

The corresponding coefficients of each s^n are set equal.

$$s^2: A + B = 0$$

$$s^1: C = 0$$

$$s^0: 3A = 1$$

From the set of equations,

$$A = \frac{1}{3}$$

$$B = -\frac{1}{3}$$

$$C = 0$$

The original equality becomes:

$$\frac{1}{s(2s^2 + 3)} = \frac{1/3}{s} - \frac{1/3}{2s^2 + 3} \quad (\text{D-11})$$

Now that the expression is simplified, the inverse Laplace transform can be easily found.

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s(2s^2+3)}\right) &= \mathcal{L}^{-1}\left(\frac{1/3}{s} - \frac{1/3}{2s^2+3}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1/3}{s}\right) - \mathcal{L}^{-1}\left(\frac{1/3}{2(s^2+\frac{3}{2})}\right) \\ &= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s^2+\frac{3}{2}}\right) \end{aligned} \quad (\text{D-12})$$

Looking at the Transform table, and using the linearity principle,

$$\mathcal{L}^{-1}\left(\frac{1}{s(2s^2 + 3)}\right) = \frac{1}{3} - \frac{1}{6}\cos\left(\sqrt{\frac{3}{2}}\right) \quad (\text{D-13})$$

Application of Laplace transformation to Initial-Value Problems

Motivation and Methodology

In this context, an initial-value problem is an unsolved differential equation accompanied by one or more initial values (i.e. $y(0) = 200$). In real life, that initial value might be representing the money originally deposited in a bank account.

The procedure is to first take Laplace transform of both sides of the equation. The next step is to have both sides of the equation in function of $\mathcal{L}\{y\}$ (any s^n is still allowed to be in the equation). At this point, the differential equation has been transformed into an algebraic equation. It is therefore easy to solve for $\mathcal{L}\{y\}$. The solution in time domain $y(t)$ is then found using the inverse Laplace Transform.

Example 1

Solve the following initial-value problem.

$$y'' + 5y' + y = e^{-3t}, \quad y(0) = 0, \quad y'(0) = 1 \quad (\text{D-14})$$

In this example, the steps would not be explained. The reader could refer to the previous example as it contains a very detailed step-by-step procedure. The use of the Transform table and properties is highly recommended.

First, let us take the Laplace transform of both sides.

$$\mathcal{L}\{y'' + 6y' + 9y\} = \mathcal{L}\{e^{-3t}\} \quad (\text{D-15})$$

Taking advantage of linearity is always beneficial

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\} \quad (\text{D-16})$$

Let's now make use of the derivative property stated in Eq. (D-4). The equation then becomes:

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 6(s\mathcal{L}\{y\} - y(0)) + 9\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\} \quad (\text{D-17})$$

Since $y(0) = 0$, $y'(0) = 1$ and $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$ (4th entry of [Table 1](#)),

$$s^2\mathcal{L}\{y\} - 1 + 6s\mathcal{L}\{y\} + 9\mathcal{L}\{y\} = \frac{1}{s+3} \quad (\text{D-18})$$

$$\mathcal{L}\{y\}(s^2 + 6s + 9) = \frac{s+4}{s+3} \quad (\text{D-19})$$

$$\mathcal{L}\{y\} = \frac{s + 4}{(s + 3)(s^2 + 6s + 9)} \quad (\text{D-20})$$

$$\mathcal{L}\{y\} = \frac{s + 4}{(s + 3)^3} \quad (\text{D-21})$$

Since we have two factors in the denominators, the use of partial fraction decomposition is necessary.

$$\frac{s + 4}{(s + 3)^3} = \frac{A}{s + 3} + \frac{B}{(s + 3)^2} + \frac{C}{(s + 3)^3} \quad (\text{D-22})$$

After solving the equation the expression is equal to the following.

$$\frac{s + 4}{(s + 3)^3} = \frac{1}{(s + 3)^2} + \frac{1}{(s + 3)^3} \quad (\text{D-23})$$

Therefore,

$$\mathcal{L}\{y\} = \frac{1}{(s + 3)^2} + \frac{1}{(s + 3)^3} \quad (\text{D-24})$$

Now that $\mathcal{L}\{y\}$ is solved, the inverse Laplace transform can be found to solve for $y(t)$

$$\begin{aligned} \mathcal{L}^{-1}\{y\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s + 3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s + 3)^3}\right\} \\ &= te^{-3t} + \frac{1}{6}t^2e^{-3t} \end{aligned} \quad (\text{D-25})$$

(5th entry of Transform table along simple algebraic arrangements)

The solution to the initial-value problem is therefore:

$$y(t) = te^{-3t} + \frac{1}{6}t^2e^{-3t} \quad (\text{D-26})$$

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