Examination of Gravity Waves over Topography Using Long's Equation

A Major Qualifying Project Report

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Abstract

This project models the behavior of steady gravity waves in the atmosphere over terrain using Long's Equation. We examine the derivation and assumptions behind the equation and determine how its solution depends on its parameters and the height of the terrain. We solved the equation both analytically using perturbation methods and numerically using the finite difference method with a sponge layer to prevent unrealistic wave reflection at the boundary.

Table of Contents

Executive Summary

In this project, we model the behavior of steady gravity waves in the atmosphere over terrain using Long's Equation, examining how the characteristics of the flow depends on the values of the parameters of the equation and the height of terrain. Long's Equation is derived from the Navier Stokes equations with the assumption that the fluid is steady, inviscid, incompressible, and stratified and where the flow far upstream is some fixed horizontal speed.

To get an analytic sense of how the equation behaves, the perturbation method was used, which confirms that the solution can be approximated as a wave added to a second wave whose amplitude increases as one moves away from the origin. A MATLAB program was then written to solve the equation numerically. This was done using the finite difference method, and the system of equations produced was solved by fixed point iteration using LU decomposition to solve the linear part of the equation.

To prevent the solution from "bouncing off" the edge, a sponge layer was introduced to dampen the wave. This sponge layer works by introducing a layer of points such that each point multiplied by a small number is equal to the point nearer to the boundary, with the boundary itself set to zero. This gives the equation some wiggle room, preventing reflections.

1. Introduction

Gravity waves are a phenomenon of much theoretical and practical interest. In general, gravity waves exist whenever there is a fluid where oscillations in the fluid are countered by the restoring force of gravity. An familiar example of gravity waves are the waves that form on the surface of the ocean, but in this paper we instead study the *internal* gravity waves which form within the atmosphere, and even more specifically, the steady (unchanging) waves which are formed as air flows over uneven terrain.

As air flows across the ground, its flow is disturbed by the terrain and streamlines bend accordingly. If the density of the air was constant everywhere the flow would flow around smoothly, but because the atmosphere is stratified (with the air getting less dense as you move upward) this perturbation causes waves.

These waves are of practical importance for several reasons. Besides the immediate fact that these waves of flow can cause weather balloons and things of that sort to bounce around (Nappo 203) these waves increase in amplitude until they reach the tropopause, (the place where the troposphere meets the stratosphere in the atmosphere) where the waves “break” causing turbulent effects. This turbulence causes trouble for airplanes, and additionally it causes energy in the atmosphere to be transferred up in the atmosphere which introduces factors that need to be taken into consideration by numerical weather models.

In order to study this phenomenon, Long's Equation was used, a nonlinear equation which considers the two dimensional steady-state case where the fluid is assumed to be inviscid, incompressible, and stratified. In this paper the derivation of Long's Equation from the relevant Navier-Stokes Equations is examined and then solutions to the equation are found using perturbation methods and the finite difference method.

2. Background

2.1 Navier Stokes Equations

The Navier-Stokes equations are nonlinear partial differential equations that describe the flow of fluids, It is an equation that is useful in many different fields, from meteorology to astronomy, although in certain cases a better model can be constructed by considering the molecular nature of fluids. The nonlinearity of these equations cause chaotic solutions which manifest themselves as turbulence, a not terribly well understood phenomenon for which equations need to be solved numerically.

The Navier Stokes equations concern themselves with vector functions which describe the pressure, density, and velocity of a fluid at a given point in space and time. Fairly useful to this is the idea of the substantive derivative. Often, one wishes to consider the rate of change of some quantity not at a specific point, but with respect to some bit of the fluid. This can be derived by having the path of some fluid described by the function $x_0(t)$ and the quantity of that bit of fluid being f0(t). When one applies the chain rule for multivalued functions, one gets the substantive derivative.

$$f_0(t)=f(x_0(t),t)$$

$$f'_0(x)=x_0(t)\cdot\nabla f(x_0(t),t)+\frac{\partial}{\partial t}f(x_0(t),t)=v(x,t)\cdot\nabla f+\frac{\partial f}{\partial t}$$

The first Navier Stokes equation is derived by applying this concept to Newton's Second Law of motion F = ma. The forces acting on a fluid consist of forces due to pressure and viscosity, which functions of the various properties of the fluid, and external forces. From this, one gets:

$$\rho(\frac{\partial v}{\partial t}+v\cdot v)=-\nabla p+\nabla\cdot T+F$$

With ρ being density, p being pressure, v being velocity, T being the tensions in the fluid (sans pressure) and F being external forces. Tension is a complicated tensor containing several unknowns, so for many applications it is assumed that the fluid is Newtonian, and thus the equation is (with μ being a constant)

$$\rho(\frac{\partial v}{\partial t}+v\cdot v)=-\nabla p+\mu\nabla^2 V+F$$

This, however, is only three equations (when you expand into the three dimensions of velocity) with five unknowns. Another equation can be derived from conservation of mass. The amount of mass in some volume Ω changes at the rate equal to how much fluid flows in minus how much flows out, which can be written

$$\frac{d}{dt}\int_{\Omega}\rho\,dx=-\int_{\partial\Omega}\rho v\cdot n\,dS$$

Moving the derivative into the integral on the left side and applying the divergence theorem to the right, you get:

$$\int_{\Omega}\frac{\partial\rho}{\partial t}dx=-\int_{\Omega}\nabla\cdot(\rho v)dx$$

$$\int_{\Omega}\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)dx=0$$

Since the function is zero for any arbitrary shape, we can conclude:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho v)=\frac{\partial \rho}{\partial t}+v\cdot\nabla\rho+\rho\nabla v=0$$

This equation can be further simplified (and give us a fifth equation) by assuming that the fluid is incompressible. That is, the density of any particular bit of fluid is constant, so the substantive derivative is zero. Thus

$$\frac{\partial \rho}{\partial t}+v\cdot\nabla\rho=0$$

Putting this into the equation we got from conservation of momentum (and then dividing by density cannot be zero) we get the conveniently simple equation:

$$\nabla\cdot v=0$$

We now have enough equations to in principle be able to solve for the values of pressure, density, and velocity in the fluid, provided that we also have the values for initial and boundary conditions.

2.2 Derivation of Long's Equation

(Based largely on class notes from Professor Humi.)

Long's equation describes stratified steady flow of inviscid incompressible fluid in the presence of gravity.

As we have already shown from the Navier Stokes equation, (and dropping out the viscosity term) we have the equations

$$\nabla \cdot \bar{v} = 0 \quad (1)$$

$$\rho\left(\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v}\right) = -\nabla p - \rho g$$

Because the flow is steady (not changing over time) we can drop the $\frac{\partial \bar{u}}{\partial t}$ term, and we will also divide both sides by ρ to get:

$$(\bar{v} \cdot \nabla)\bar{v} = \frac{-1}{\rho}\nabla p - g \quad (2)$$

Additionally, we have the constraint that the density of the fluid does not change with respect to time either so dp/dt is zero. But because of conservation of mass we have that Dp/Dt the substantial derivative with respect to time is also zero, which lets us arrive at:

$$(\bar{v} \cdot \nabla)\rho = 0 \quad (3)$$

We now nondimensionalize these equations and break (2) into two equations for each of the components of $\bar{v}$ (which are u and w respectively) and get:

Using the new units $x = L\bar{x}$ $\quad z = \frac{U_0}{N_0}\bar{z}$ $\quad u = U_0 \bar{u}$ $\quad w = \frac{U_0^2}{L N_0}\bar{w}$ $\quad p = \frac{g U_0 \rho_0}{N_0}\bar{p}$ $\quad \rho = \frac{\bar{\rho}}{\rho_0}$

Equations 1 and 3 remain the same and equation 2 becomes

$$\beta\rho(uu_x + wu_z) = -p_x \quad \text{(2a)}$$

$$\beta\rho(uw_x + ww_z) = -\mu^2(p_z + \rho) \quad \text{(2b)}$$

Where $\beta = \frac{N_0 U_0}{g}$ $\quad \mu = \frac{U_0}{N_0 L}$ with β being the Boussinesq parameter which controls stratification and μ being the long wave parameter which controls dispersive effects.

In light of equation (1) we can introduce a function ψ such that $u = \frac{\partial\psi}{\partial z}$ $\quad w = \frac{-\partial\psi}{\partial x}$

Using this, equation (3) becomes $J\{\psi, \rho\} = 0$ where $J\{f, g\} = f_x g_z - f_z g_x$

This is useful because it turns out that whenever J{f,g} = 0, f can be expressed as a function of g (or vica versa) for the following reason:

$$if\; g = G(f),\; then\; J\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial G}{\partial f}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial G}{\partial f}\frac{\partial f}{\partial x} = 0$$ and similarly vica versa.

The remaining two equations can be similarly transformed into:

$$\rho(\psi_z\psi_{zx} - \psi_x\psi_{zz}) = \frac{-p_x}{\beta}$$

$$\mu^2\rho(-\psi_z\psi_{xx} - \psi_x\psi_{xz}) = -p_z/\beta - \rho/\beta$$

Which can be combined to get

$$[\rho(\psi_z\psi_{zx} - \psi_x\psi_{zz})]_z - [\mu^2\rho(-\psi_z\psi_{xx} + \psi_x\psi_{xz})]_x = \frac{-p_{xz}}{\beta} - \left(\frac{-p_{zx}}{\beta} - \frac{\rho_x}{\beta}\right) = \frac{\rho_x}{\beta}$$

Flipping signs this becomes

$$[\rho(\psi_z\psi_{zx}-\psi_x\psi_{zz})]_z+[\mu^2\rho(\psi_x\psi_{xz}-\psi_z\psi_{xx})]_x=\frac{-p_{xz}}{\beta}-(\frac{-p_{zx}}{\beta}-\frac{\rho_x}{\beta})=\frac{\rho_x}{\beta}$$

And deriving out:

$$\rho_z(\psi_z\psi_{zx}-\psi_x\psi_{zz})+\mu^2\rho_x(\psi_x\psi_{xz}-\psi_z\psi_{xx})+\rho((\psi_z\psi_{zx}-\psi_x\psi_{zz})_z+\mu^2(\psi_x\psi_{xz}-\psi_z\psi_{xx})_x)=\frac{\rho_x}{\beta}$$

This can be broken down into a series of expressions using J by:

$$\rho_z(\psi_z\psi_{zx}-\psi_x\psi_{zz})+\mu^2\rho_x(\psi_x\psi_{xz}-\psi_z\psi_{xx}) \quad =$$
$$\rho_z(\psi_z\psi_{zx}-\mu^2\psi_x\psi_{xx})+\rho_x(\psi_x\psi_{xz}-\mu^2\psi_z\psi_{zz}) \quad = \textit{By} \quad \psi_z\rho_x=\psi_x\rho_z \quad \textit{and swapping terms}$$

$$\rho_z\frac{\partial}{\partial x}\{\frac{1}{2}(\psi_z^2-\mu^2\psi_x^2)\}+\rho_x\frac{\partial}{\partial z}\{\frac{1}{2}(\psi_z^2-\mu^2\psi_x^2)\}=J\{\frac{1}{2}(\psi_z^2-\mu^2\psi_x^2),\rho\}$$

and

$$\rho((\psi_z\psi_{zx}-\psi_x\psi_{zz})_z+\mu^2(\psi_x\psi_{xz}-\psi_z\psi_{xx})_x) \quad =$$
$$\rho((\psi_z\psi_{zxz}+\psi_{zz}\psi_{zx}-\psi_{xz}\psi_{zz}-\psi_x\psi_{zzz})+\mu^2(\psi_{xx}\psi_{xz}+\psi_x\psi_{xzx}-\psi_{zx}\psi_{xx}-\psi_z\psi_{xxx})) \quad =$$
$$\rho((\psi_z\psi_{zxz}-\psi_x\psi_{zzz})+\mu^2(\psi_x\psi_{xzx}-\psi_z\psi_{xxx})) \quad = \textit{By cancellation}$$
$$\rho(\psi_z(\psi_{zz}+\mu^2\psi_{xx})_x+\psi_x(\psi_{zz}-\mu^2\psi_{xx})_z) \quad = \quad \rho J\{\psi_{zz}+\mu^2\psi_{xx},\psi\}$$

and lastly that $-J\{\frac{z}{\beta},\rho\}=-(0-\frac{\rho_x}{\beta})=\frac{\rho_x}{\beta}$ together to get:

$$J\{\frac{1}{2}(\psi_z^2+\mu^2\psi_x^{2,}p),\rho\}+\rho J\{\psi_{zz}+\mu^2\psi_{xx},\psi\}=-J\{z/\beta,\rho\}$$

Because J is linear, you can move the right hand side over and combine to get:

$$\rho J\{\psi_{zz}+\mu^2\psi_{xx},\psi\}+J\{\frac{1}{2}(\psi_z^2+\mu^2\psi_x^2)+z/\beta,\rho\}=0$$

But now we take into account that ρ = ρ(ψ) (and thus $\rho_x=\rho_\psi\psi_x$ and $\rho_z=\rho_\psi\psi_z$ which allows us to get:

$$\rho J\{\psi_{zz}+\mu^2\psi_{xx},\psi\}+\rho_\psi J\{\frac{1}{2}(\psi_z^2+\mu^2\psi_x^2)+z/\beta,\psi\}=0$$

Pulling 1/β out of the left J and dividing both sides of the equation by ρ, we get:

$$J\{\psi_{zz}+\mu^2\psi_{xx},\psi\}+\frac{\rho_\psi}{\beta\rho}J\{\frac{\beta}{2}(\psi_z^2+\mu^2\psi_x^2)+z,\psi\}=0$$

We now let introduce the non-dimensionalized Brunt-Vaisala frequency $N^2(\psi)=\frac{-\rho_\psi}{\beta\rho}$ and combine the two J's together:

$$J\{\psi_{zz}+\mu^2\psi_{xx}-N^2(\psi)[z+\frac{\beta}{2}(\psi_z^2+\mu^2\psi_x^2)],\psi\}=0$$

Now, because of the above proved theorem regarding J{f, g} = 0 we know that there exists some function G such that

$$\psi_{zz}+\mu^2\psi_{xx}-N^2(\psi)[z+\frac{\beta}{2}(\psi_z^2+\mu^2\psi_x^2)]=G(\psi)$$

To determine G, we make the assumption that $\lim_{x\to-\infty}\psi(x,z)=z$ (which means physically that "infinitely far away" the stream is horizontal and at speed 1) which means that "at infinity"

$$-N^2(\psi)(\psi+\frac{\beta}{2})=G(\psi)$$

And combining those two equations, we get:

$$\psi_{zz}+\mu^2\psi_{xx}-N^2(\psi)[z-\psi+\frac{\beta}{2}(\psi_z^2+\mu^2\psi_x^2-1)]=0$$ which is the Long's Equation

Frequently however, (and in this MQP in particular) it is useful to consider the slightly different equation gotten by considering the perturbation from the steady stream $\eta = \psi - z$

Using that definition the new equation

$$\eta_{zz}-\alpha^2\eta_z^2+\mu^2(\eta_{xx}-\alpha^2\eta_x^2)-N^2(\psi)(\beta\eta_z-\eta)=0$$

where $\alpha^2=\frac{N^2(\psi)\beta}{2}$

2.3 Boundary conditions

We have already used to derive Long's Equation the boundary value $\psi(-\infty, z)=z$. On the right hand side this is generally balanced out by $\psi(\infty, z)=z$ even though physically this cannot be guaranteed to be the case because Long's Equation contains no terms that dissipate energy. Because infinite domains are not computationally doable, this is generally modeled using a finite domain, which we compensate for using sponge layers, which shall be described in the section on numerical methods.

The standard boundary condition for Long's Equation is that over some terrain defined by the function $\varepsilon f(x)$

$$\psi(x, \varepsilon f(x))=0$$

Which physically means that there is no wind along the ground. In principle, 0 could be replaced by any other constant to get something with the same physical meaning, but zero is a nice number so it is generally used.

For the perturbation model, this becomes

$$\eta(x, \varepsilon f(x)) = \psi(x, \varepsilon f(x)) - \varepsilon f(x) = -\varepsilon f(x)$$

However, for very small values of ε (ε << 1), we can see that it is approximated such that

$$\eta(x, 0) \approx -\varepsilon f(x)$$

This approximation is highly useful because it means that a rectangular domain can be used allowing for, allowing for the various solution methods that follow.

3. Solutions to Long's Equation

3.1 Analytic Solution to Linear Long's Equation using Separation of Variables

For the parameters β = 0, α = 1, the perturbation formulation of Long's Equation becomes the following linear partial differential equation:

$$\nabla \eta + N^2 \eta = 0$$

Rather than consider the complicated case of non-rectangular boundaries, we will consider merely the simpler case where we know the f

$$\eta(0, z) = 0$$
$$\eta(x, 0) = -\varepsilon f(x)$$
$$\frac{\partial \eta}{\partial x}(a, z) = 0$$
$$\frac{\partial \eta}{\partial z}(x, b) = 0$$

By letting $\eta(x, z) = \varphi(x) \psi(z)$ you get the following equations and boundary conditions:

$$\varphi'' - \lambda \varphi = 0$$
$$\psi'' + (\lambda + N^2) \psi = 0$$
$$\varphi(0) = 0$$
$$\varphi'(a) = 0$$

By considering the various cases that λ can take on, we ultimately arrive at:

$$\lambda = -w_n^2$$

$$w_n = \frac{2n-1}{2a}\pi$$

$$\varphi_n(x) = B_n \sin(w_n x)$$

The differential equation for ψ can be written in one of two ways:

$$\psi'' + (N^2 - w_n^2)\psi = 0$$

$$\psi'' - (w_n^2 - N^2)\psi = 0$$

Which solution you get depends on whether N is bigger or smaller than w_n

Let $v_n = \sqrt{|N^2 - w_n^2|}$

$\psi_n = A_n \cos(v_n z) + B_n \sin(v_n z)$ for $w_n < N$

$\psi_n = A_n \cosh(v_n z) + B_n \sin(v_n z)$ for $w_n > N$

Thus, if we let k be the largest n such that $w_n < N$ we get the general solution

$$\eta(x,z) = \sum_{n=1}^{k} (\sin(w_n x)(A_n \cos(v_n z) + B_n \sin(v_n z)))$$

$$+ \sum_{n=k+1}^{\infty} (\sin(w_n x)(A_n \cosh(v_n z) + B_n \sinh(v_n z)))$$

An and Bn can be solved for using Fourier series, but I will not do that here because it's not so interesting.

We can interpret this solution as having a wavelike solution for the first “couple” frequencies and then a Laplace-like solution for the rest of the series. Thus, the degree to which waves appear in the equation depends on how big k is. For reasonable cases, the non-wave terms of the series can be ignored.

The simple case of asking whether waves appear at all can be solved for simply by determining whether $w_1 < N$. Through a fairly trivial bit of algebra, this condition turns out to be the case if and only if $a > \frac{\pi}{2N}$. So for instance, for N = 10^(-2), the width of the space being solved for needs to be larger than approximately 15.7 in order for waves to appear, although making the graph even larger is necessary to get “interesting” results.

3.2 Perturbation Method

The perturbed Long's equation, after some algebra, becomes

$$\eta_{zz}+\mu^2\eta_{xx}+N^2\eta-\alpha^2(\eta_z^2+\mu^2\eta_x^2+2\eta_z)=0$$

We can represent η as a perturbation approximation in the following way:

$$\eta=\eta^0+\alpha^2\eta^1+\ldots$$

Plugging these two equations together, we get

$$\eta_{zz}^0+\alpha^2\eta_{zz}^1+\ldots+\mu(\eta_{xx}^0+\alpha^2\eta_{xx}^1+\ldots)+N^2(\eta^0+\alpha^2\eta^1)-\alpha^2((\eta^0{}_z)^2+\ldots+\mu(\eta^0{}_x)^2+\ldots+2\eta^0{}_z+\ldots)$$

$$\eta_{zz}^0+\mu^2\eta_{xx}^0+N^2\eta^0+\alpha^2(\eta_{zz}^1+\mu^2\eta_{xx}^1+N^2\eta^1-((\eta_z^0)^2+\mu^2(\eta^0{}_x)^2+2\eta^0{}_z))+\ldots=0$$

From which we get two equations

$$\eta_{zz}^0+\mu^2\eta_{xx}^0+N^2\eta^0=0$$

$$\eta_{zz}^1+\mu^2\eta_{xx}^1+N^2\eta^1=(\eta_z^0)^2+\mu^2(\eta_x^0)^2+2\eta_z^0$$

For the particular case of μ = 1, we have already solved in the previous section. Using that solution, the second equation becomes a linear partial differential equation. The equation

$$\partial_{zz}n_1+\partial_{xx}n_1+N^2n_1=(\partial_z n_0)^2+(\partial_x n_0)^2+2n_0$$

Can be made easier to solve by considering the simpler case for n_0 having only a wave of one particular frequency such that

$$n_0(x,z)=\cos(wx+vz)$$

Using this, the equation for n_1 becomes

$$\partial_{zz}n_1+\partial_{xx}n_1+N^2n_1=(w^2+v^2)\sin^2(wx+vz)+2\cos(wx+vz)$$

To get a particular solution for this, we do something similar to a traveling wave solution:

Try $n(x,z)=f(wx+vz)$ where for brevity we let $wx+vz=u$

Thus, $(w^2+v^2)f''+N^2f=(w^2+v^2)\sin^2(u)+2\cos(u)$

However we note that, as we showed in the first equation, $w^2+v^2=N^2$

$$N^2f''+N^2f=N^2\sin^2(u)+2\cos(u)$$

Using maple to solve this equation, we get the particular solution:

$$f(u)=\frac{3u\sin(u)+N^2\cos^2(u)+3\cos(u)+N^2}{3N^2}$$

Or in other words

$$n_1(x,z)=\frac{(kx+mz)\sin(kx+mz)+\cos(kx+mz)}{N^2}+\frac{\cos^2(kx+mz)+1}{3}$$

To get the general solution for n_1 we add to that the general solution to the homogeneous solution solved for elsewhere, and then solve for the boundary conditions.

3.3 Numerical Solution

Although analytical analysis of the equations is of use, ultimately for more complicated forms of the equation numerical methods are ideal. Our preliminary examinations were using the proprietary COMSOL Multiphysics, but ultimately our own code was written up.

The code uses the finite difference method; that is, by considering a finite domain and then considering discrete points along this domain and using finite differences between these points to create a series of equations that approximate the perturbation formulation of Long's Equation and a bottom boundary condition given by letting the terrain be equal to $\frac{\varepsilon}{(1+x^2)^{3/2}}$ (for ε = 0.1 or 0.25). From this, by the approximation given earlier, we set that $\eta(x,0)=\frac{-\varepsilon}{(1+x^2)^{3/2}}$.

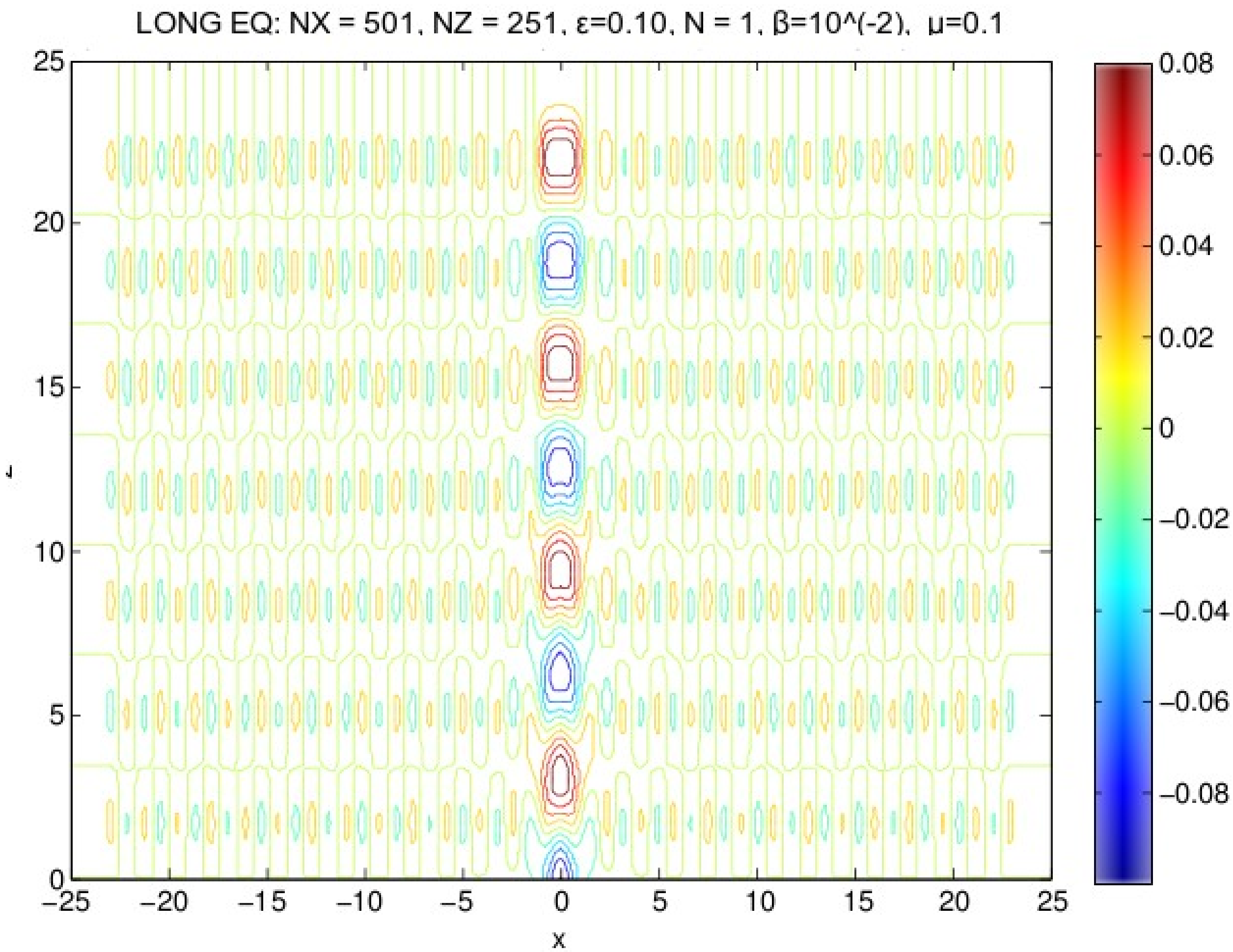
As was noted earlier, the natural boundary value conditions involve infinite domains which is not particularly amicable to numerical methods. To convert the boundary value conditions into the finite domain, they are turned into $\eta(-L,z)=0$ $\eta(L,z)=0$ and the third boundary value condition $\eta(x,L)=0$ for the top side. (L = 25.) However, to naively use these boundary value conditions can produce unrealistic results, first because in physical reality the equation does not go horizontal so quickly, but also because such a boundary value condition can allow waves to reflect off the edges greatly changing the result of the equation. To account for this, a sponge layer was used.

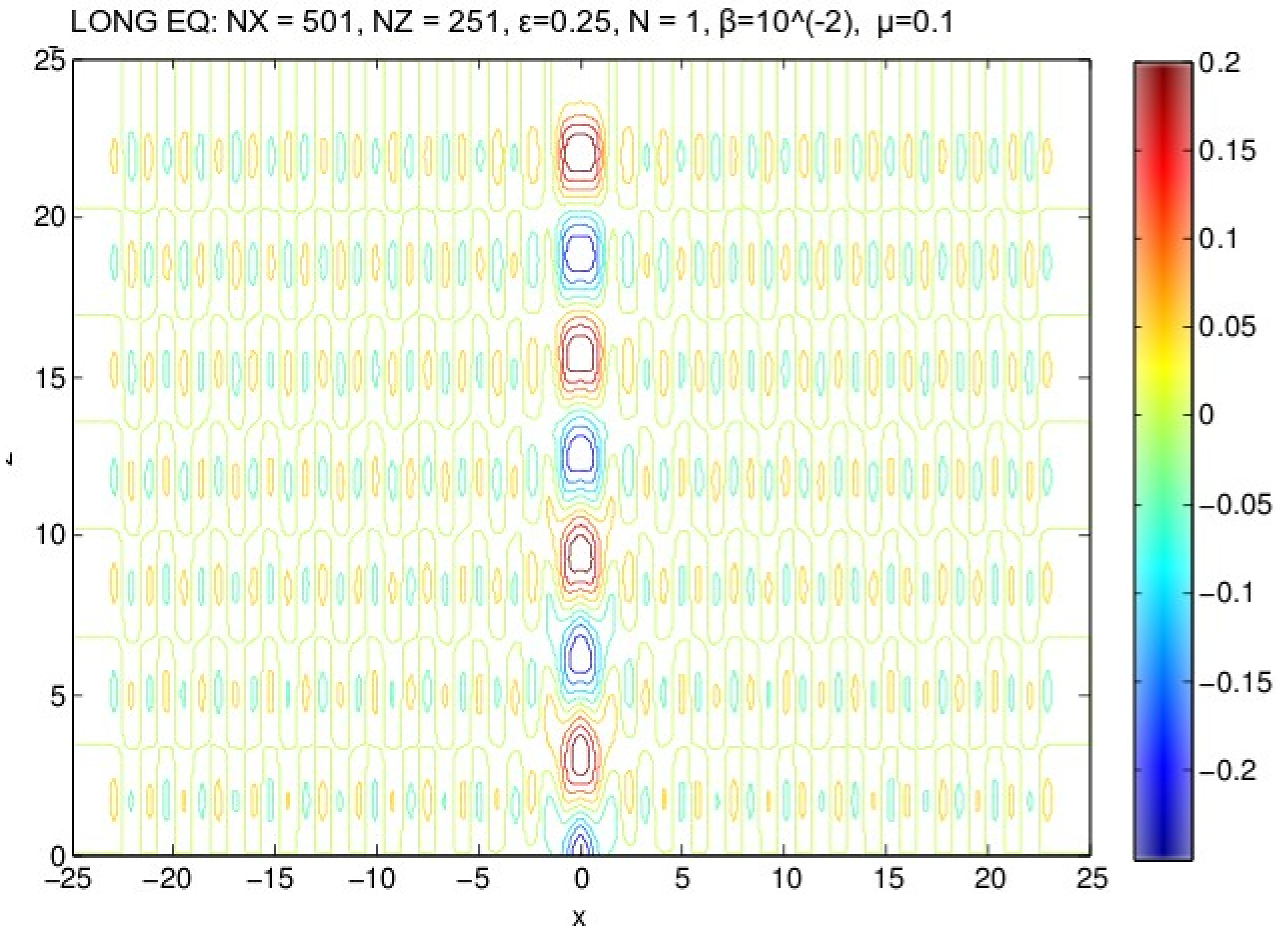
In a sponge layer, a gap is made between the part of the equations which satisfies Long's Equation and the boundary such that as you move from the interior to the boundary each point is the previous point multiplied by some number between 0 and 1 (for our program, 0.85). For the program, this layer was 20 points thick. The boundary itself is still set to the value of 0, but because the value of the equation at the beginning of the sponge layer is multiplied by .85 twenty times (to get roughly 4%) which allows for the waves to be slightly off from exactly zero at the edge. This has the effect of "dampening" any reflections that may occur in the solution.

To solve the system of equations produced by the finite difference method, each of these equations are formulated of the form f(x) = 0 where x is the vector of all points on the grid. The function f is broken into a linear part J (represented in the program as a matrix) and a nonlinear part F such that J(x) + F(x) = 0. Through simple algebra, this equation becomes $x = J^{-1} F(x)$. Thus, we can see that the problem of solving the equation turns out to be equivalent to finding a fixed point of the function $J^{-1} F(x)$. Fortunately, fixed point iteration converges in few steps (with initial guess of x = 0).

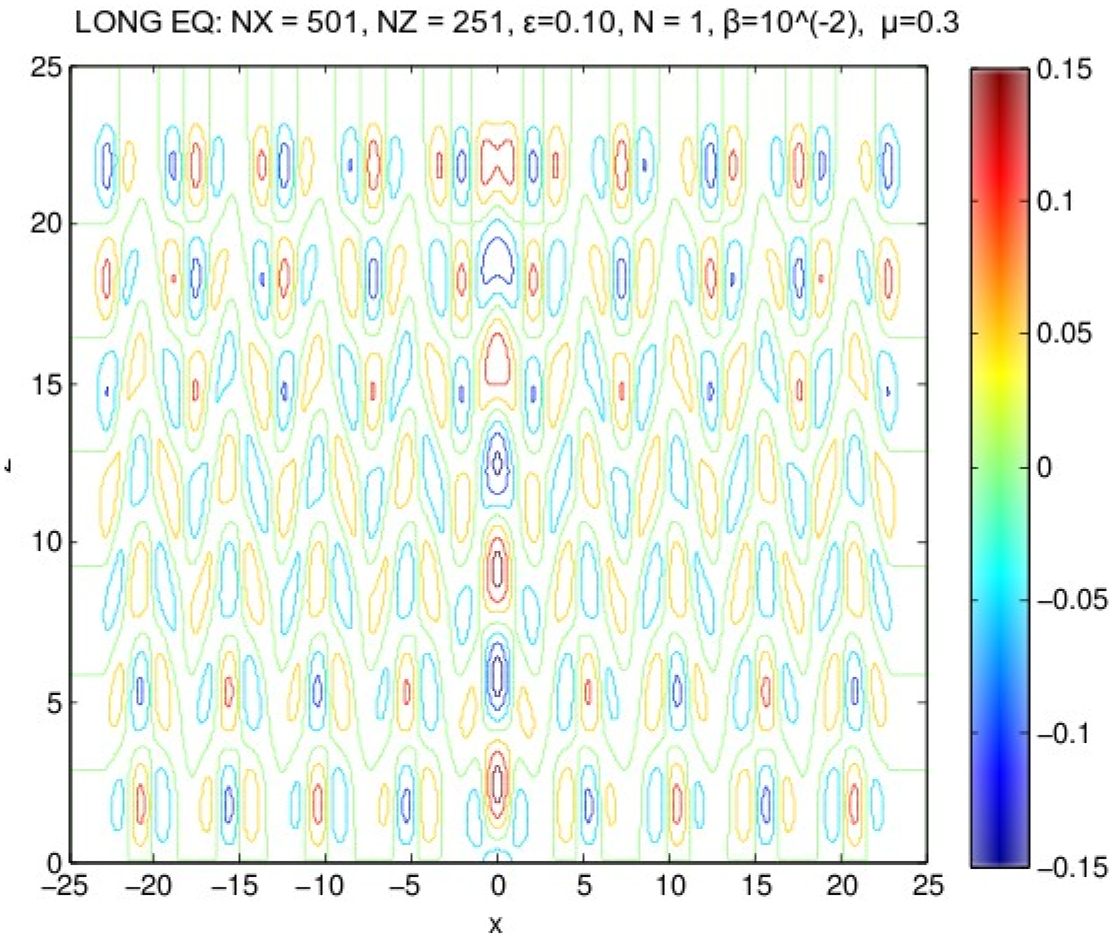
The computation is not completely speedy however because the matrix J is fairly large, with the 501x251 grid used in the results below giving a 125751x125751 matrix. (125751 = 501 * 251.) Naively attempting to invert J would be rather inefficient, so instead an LU decomposition is calculated before the iteration and then is used to solve for J in each step. Because J is sparse, (specifically, each equation in the system considers at most five points on the grid) the conjugate gradient method may have been faster, but attempting to implement the method ran into difficulties so the idea was scrapped due to time constraints.

3.4 Results of Numerical Solution





Note that even though the graph appears similar, increasing the height of the terrain from 0.1 to 0.25 has increased the amplitude of the wave significantly from 0.08 to 0.25 as can be seen by looking at the scales of the respective graphs.



3.5 Conclusions

With everything analyzed, various conclusions can be made. Examining the results of the numerical analysis, it can be seen that a change in the height of the terrain (at least in so far as the terrain is small enough that the approximation used remains accurate) has the primary effect of increasing the amplitude of the waves while not changing much else. Additionally, it can be see that an increase in μ causes the waves to spread out over the terrain. This makes sense since for the extreme case of $\mu = 0$, all terms involving x drop out entirely. An increase in β, in line with the results of the perturbation analysis, causes the waves to increase in amplitude as one moves away from the mountain (in particular upwards.)

4. Bibliography

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