Reflection and Transmission of a Plane Electromagnetic Wave on a Moving Boundary Between Two Dielectrics

by

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Contents

Abstract

This work introduces formulae of Fresnel type related to reflection and transmission of a plane electromagnetic wave from a moving boundary separating two isotropic dielectrics. The dielectrics themselves remain immovable, so, the ensuing material formation represents an example of an activated dynamic material assembled from the LC-arrays serving as the discrete versions of each dielectric.

Introduction

In this work, we consider a planar electromagnetic wave that travels in 3D through the half-space occupied by immovable isotropic dielectric with material properties ϵ_2, μ_2. The wave arrives at a general angle and exhibits reflection from and transmission through a planar interface S between half-space 2 and half-space 1 occupied by another isotropic dielectric 1, with properties ϵ_1, μ_1; this dielectric is also assumed immovable.

We suppose that the planar interface S is brought into a uniform motion at velocity V, in the direction of its normal. This assumption will be implemented if some parts of material 1 are converted to material 2, and vice versa. Such property transformation may become materialized once we apply switches in a discrete version of both dielectrics manufactured each as an array of LC-cells. By properly manipulating the switches one may choose either the (L_1, C_1)-pair representing material 1, or the pair (L_2, C_2) representing material 2. It is essential that no material motion is involved in this spatio-temporal arrangement, so, by terminology of [Lur06], we have a pure activation case.

We obtain and discuss the formulae expressing the Snell's law in the presence of motion of the interface, and study the influence produced by such motion onto the total reflection and the Brewster phenomenon.

Chapter 1 exposes the classical results reproduced only for reference [Str41]. The original results related to the case of moving interface are exposed in Chapter 2.

Chapter 1

Non-moving interface

1.1 Snell's law and field vectors

Two dielectric materials are separated by an immovable planar interface S. At a basic level, the primary electromagnetic wave, also called a wave of incidence, collides with a static interface. The basic underling assumption in this case is all the waves are planar. Once the primary wave hits the interface two secondary waves are produced. One secondary wave is transmitted through the interface into the material on the opposite side. The other wave is reflected back into the original material where the primary wave originated. Without loss of generality, let the primary wave start in material 2 meaning the reflected wave travels back in material 2, and the transmitted wave travels forward into material 1.

There are some basic formulas which govern this reaction. Namely for the incident wave, the electric and magnetic field vectors, **E** and **H**, are defined as

$$\mathbf{E_i} = \mathbf{E_0} e^{ik_0 \mathbf{n_0} \bullet \mathbf{r} - i\omega t}, \quad \mathbf{H_i} = \frac{k_0}{\omega \mu_2} \mathbf{n_0} \times \mathbf{E_i}. \tag{1.1}$$

Above in the equations, $\mathbf{E_0}$ is the amplitude of the wave of incidence, and $\mathbf{n_0}$ is the unit vector describing the direction of propagation of the incident wave. The plane known as the plane of incidence is defined by $\mathbf{n_0}$ and the unit vector $\mathbf{n}$ normal to the interface. The transmission and reflection waves are defined respectively by the formulae

$$\begin{aligned} \mathbf{E_t} &= \mathbf{E_1} e^{ik_1 \mathbf{n_1} \bullet \mathbf{r} - i\omega t}, \quad \mathbf{H_t} = \frac{k_1}{\omega \mu_1} \mathbf{n_1} \times \mathbf{E_t}, \\ \mathbf{E_r} &= \mathbf{E_2} e^{ik_2 \mathbf{n_2} \bullet \mathbf{r} - i\omega t}, \quad \mathbf{H_r} = \frac{k_2}{\omega \mu_2} \mathbf{n_2} \times \mathbf{E_r}. \end{aligned} \tag{1.2}$$

In equations (1.1) and (1.2), $k_2\mathbf{n_0}$, $k_1\mathbf{n_1}$, and $k_2\mathbf{n_2}$ are, respectively, the wave vectors of the incidence, transmission, and reflection waves. The tangential components of the resulting vector fields **E** and **H** are continuous across the plane S, that is defined as $x = 0$ (Figure 1).

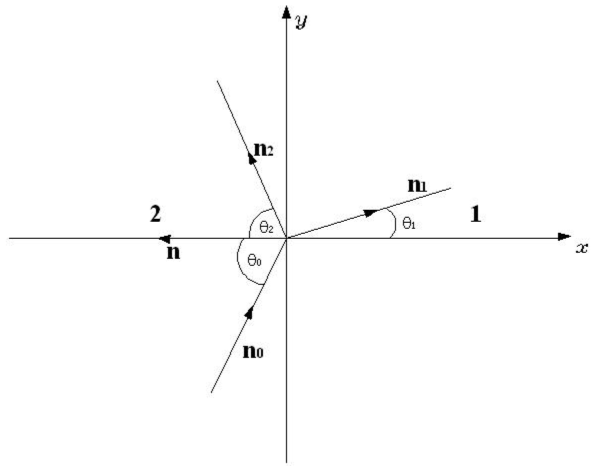

Figure 1

The frequency ω is preserved across the immovable interface, so

$$k_0 = \frac{\omega}{a_2}, \quad k_2 = \frac{\omega}{a_2} = k_0, \quad k_1 = \frac{\omega}{a_1}. \tag{1.3}$$

Here, $a_i = 1/\sqrt{\epsilon_i \mu_i}$, $i = 1, 2$ denote the phase velocity of light in materials 1 and 2, respectively.

The relationships

$$k_0 \mathbf{n_0} \bullet \mathbf{r} = k_0 \mathbf{n_2} \bullet \mathbf{r} = k_1 \mathbf{n_1} \bullet \mathbf{r} \tag{1.4}$$

should hold at all points on the interface $x = 0$; assuming that the xy-plane is the plane of incidence, equations (1.4) can be represented as

$$k_0 y \cos\alpha_0 = k_0(y \cos\alpha_2 + z \cos\beta_2) = k_1(y \cos\alpha_1 + z \cos\beta_1).$$

Here α_i and β_i denote direction angles of $\mathbf{n}_i$, $i = 0, 1, 2$, with regard to y and z-axes. These equations should hold for all values of y and z, thus

$$\cos\beta_2 = \cos\beta_1 = 0,$$
$$k_0 \cos\alpha_0 = k_0 \cos\alpha_2 = k_1 \cos\alpha_1.$$

The first line shows that all four vectors $\mathbf{n}$, $\mathbf{n_0}$, $\mathbf{n_1}$, and $\mathbf{n_2}$ belong with the xy-plane. The second line indicates that (see Figures 1 and 1a)

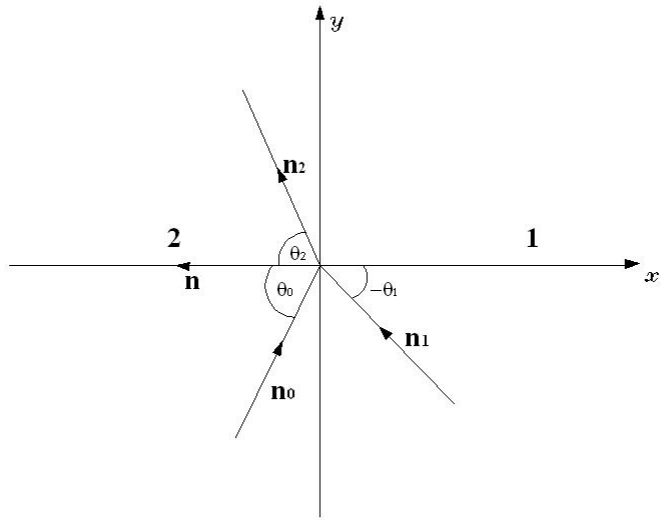

Figure 1a

$$k_0 \sin\theta_0 = k_0 \sin\theta_2 = k_1 \sin\theta_1; \tag{1.5}$$

here,

$$\theta_i = \frac{\pi}{2} - \alpha_i, \quad i = 0, 1, 2.$$

From the first of equation (1.5), it can be concluded that

$$\theta_2 = \theta_0. \tag{1.6}$$

because the second solution, $\theta_2 = \pi - \theta_0$, reproduces the incident wave. The second equation,

$$k_0 \sin\theta_0 = k_1 \sin\theta_1, \tag{1.7}$$

may have, along with the solution $\theta_1 \in [0, \pi/2]$, also the solution $\pi - \theta_1$. The first is relevant to the transmitted wave with *phase departing* from the separating plane S; the wave vector of such a wave points away from the origin and into the first quadrant of the xy-plane (Figure 1). The second solution is associated with the transmitted wave with *phase arriving* towards the separating plane S from the infinity; the wave vector of such a wave points towards the origin through the fourth quadrant of the xy-plane (Figure 1a). The right choice between the two solutions is decided by the direction of energy flow that should be *away* from the interface.

In order to find a relationship between the amplitudes of $\mathbf{E_0}, \mathbf{E_1}$, and $\mathbf{E_2}$, of the incident, transmitted and reflected waves, respectively, the boundary conditions are implemented at all points on the surface S:

$$\begin{aligned} \mathbf{n}\times(\mathbf{E_0}+\mathbf{E_2}) &= \mathbf{n}\times\mathbf{E_1}, \\ \mathbf{n}\times(\mathbf{H_0}+\mathbf{H_2}) &= \mathbf{n}\times\mathbf{H_1}. \end{aligned} \tag{1.8}$$

From equations (1.1), $\mathbf{H_0} = \dfrac{k_0}{\omega\mu_2}\mathbf{n_0}\times\mathbf{E_0}$, and from equations (1.2), $\mathbf{H_2} = \dfrac{k_0}{\omega\mu_2}\mathbf{n_2}\times\mathbf{E_2}$, $\mathbf{H_1} = \dfrac{k_1}{\omega\mu_1}\mathbf{n_1}\times\mathbf{E_1}$. We use these formulae to eliminate $\mathbf{H_0}$, $\mathbf{H_1}$, and $\mathbf{H_2}$ from the second equation (1.8); it then reduces to

$$\mathbf{n}\times(\mathbf{n_0}\times\mathbf{E_0}+\mathbf{n_2}\times\mathbf{E_2})\frac{k_0}{\mu_2} = \mathbf{n}\times(\mathbf{n_1}\times\mathbf{E_1})\frac{k_1}{\mu_1}. \tag{1.9}$$

The consequences of equations (1.8) and (1.9) will be listed below for two linearly independent polarizations of the incident wave, namely, the electric and magnetic polarizations.

Case 1: $\mathbf{E_0}$ is normal to the plane of incidence (electric polarization)

In this situation, the electric vectors of the transmitted and reflected waves are parallel to $\mathbf{E_0}$, and thus perpendicular to the plane of incidence. In this polarization, the electric vectors possess components in the z-direction alone, whereas the magnetic vector show both x- and y-directions, see Figure 2.

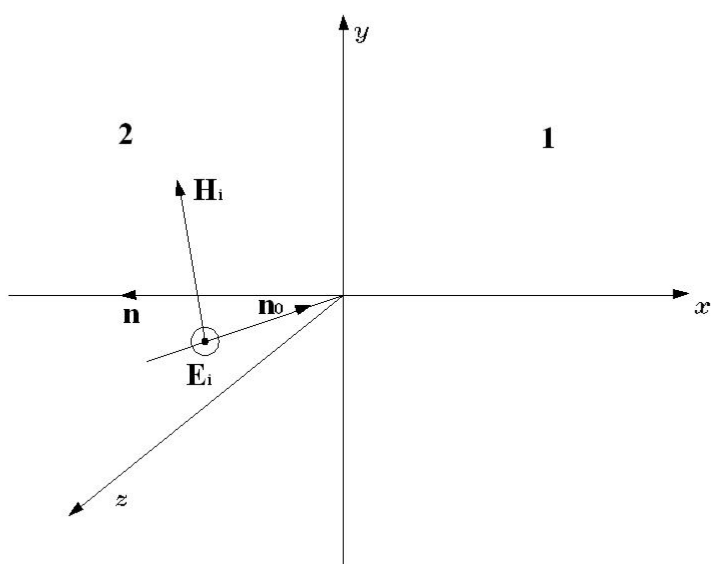

Figure 2

Therefore,

$$\mathbf{n} \bullet \mathbf{E_0} = \mathbf{n_0} \bullet \mathbf{E_0} = 0.$$

Because both dielectrics, 1 and 2, are isotropic, the transmitted wave traveling through material 1 and the reflected wave traveling through material 2 must have electric vectors parallel to $\mathbf{E_0}$ and thus also normal to the plane of incidence. Therefore, $\mathbf{n} \bullet \mathbf{E_1} = \mathbf{n} \bullet \mathbf{E_2} = 0$. Thus, from this information and as seen in Figure 1, the relations hold:

$$\begin{aligned} \mathbf{n} \bullet \mathbf{n_0} &= \cos(\pi - \theta_0) = -\cos\theta_0, \\ \mathbf{n} \bullet \mathbf{n_1} &= \cos(\pi - \theta_1) = -\cos\theta_1, \\ \mathbf{n} \bullet \mathbf{n_2} &= \cos\theta_2. \end{aligned} \tag{1.10}$$

Once the first equation (1.8) is cross multiplied by $\mathbf{n}$, the result is

$$\mathbf{E_0} + \mathbf{E_2} = \mathbf{E_1}.$$

Next, the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$ applied to equation (1.9) yields:

$$(\mathbf{n} \bullet \mathbf{E_0})\mathbf{n_0} - (\mathbf{n} \bullet \mathbf{n_0})\mathbf{E_0} + (\mathbf{n} \bullet \mathbf{E_2})\mathbf{n_2} - (\mathbf{n} \bullet \mathbf{n_2})\mathbf{E_2} = \left(\frac{k_1\mu_2}{k_0\mu_1}\right)[(\mathbf{n} \bullet \mathbf{E_1})\mathbf{n_1} - (\mathbf{n} \bullet \mathbf{n_1})\mathbf{E_1}],$$

or

$$\cos\theta_0\mathbf{E_0} - \cos\theta_2\mathbf{E_2} = \frac{k_1\mu_2}{k_0\mu_1}\cos\theta_1\mathbf{E_1}.$$

With reference to (1.6), the boundary conditions receive the form

$$\begin{aligned} \mathbf{E_0} + \mathbf{E_2} &= \mathbf{E_1}, \\ \mathbf{E_0} - \mathbf{E_2} &= \frac{k_1\mu_2}{k_0\mu_1}\frac{\cos\theta_1}{\cos\theta_0}\mathbf{E_1}. \end{aligned} \tag{1.11}$$

Equations (1.11) yield

$$\mathbf{E_1} = \frac{2k_0\mu_1\cos\theta_0}{k_1\mu_2\cos\theta_1 + k_0\mu_1\cos\theta_0}\mathbf{E_0},$$

or

$$\mathbf{E_1} = \frac{2k_0\mu_1 \cos\theta_0}{(k_0\mu_1\cos\theta_0) + \mu_2\sqrt{k_1^2 - k_0^2\sin^2\theta_0}}\mathbf{E_0}, \tag{1.12}$$

and

$$\mathbf{E_2} = \frac{k_0\mu_1\cos\theta_0 - k_1\mu_2\cos\theta_1}{k_1\mu_2\cos\theta_1 + k_0\mu_1\cos\theta_0}\mathbf{E_0},$$

or

$$\mathbf{E_2} = \frac{k_0\mu_1\cos\theta_0 - \mu_2\sqrt{k_1^2 - k_0^2\sin^2\theta_0}}{(k_0\mu_1\cos\theta_0) + \mu_2\sqrt{k_1^2 - k_0^2\sin^2\theta_0}}\mathbf{E_0}. \tag{1.13}$$

Case 2: $\mathbf{H_0}$ is normal to the plane of incidence (magnetic polarization)

This is the situation when the magnetic vectors $\mathbf{H_0}$, $\mathbf{H_1}$, $\mathbf{H_2}$ are normal to the plane of incidence and thus parallel to the surface S. In this polarization, the magnetic vectors possess components in the z-direction alone, whereas, the electric vectors show both x- and y-directions, see Figure 3.

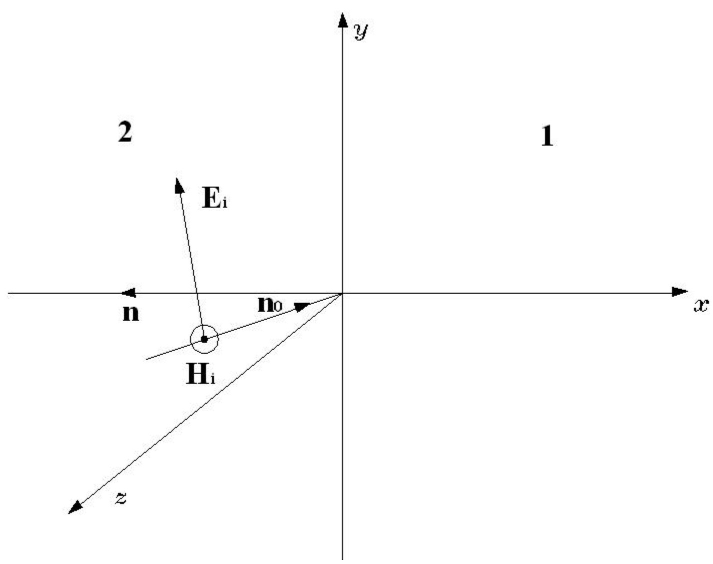

Figure 3

Therefore,

$$\mathbf{n} \bullet \mathbf{H_0} = \mathbf{n_0} \bullet \mathbf{H_0} = 0.$$

Similarly to Case 1, the magnetic field in both transmitted and reflected waves remain parallel to $\mathbf{H_0}$. With the reference to equations (1.1) and (1.2), the boundary conditions (1.8) will then be represented as

$$\mathbf{n}\times(\mathbf{H_0}+\mathbf{H_2})=\mathbf{n}\times\mathbf{H_1},$$
$$\mathbf{n}\times\left(\frac{\omega\mu_2}{k_0}\mathbf{n_0}\times\mathbf{H_0}\right)+n\times\left(\frac{\omega\mu_2}{k_0}\mathbf{n_2}\times\mathbf{H_2}\right)=\mathbf{n}\times\left(\frac{\omega\mu_1}{k_1}\mathbf{n_1}\times\mathbf{H_1}\right),$$

or, equivalently, as

$$\begin{aligned}\mathbf{H_0}+\mathbf{H_2}&=\mathbf{H_1},\\ \mathbf{H_0}-\mathbf{H_2}&=\frac{k_0\mu_1}{k_1\mu_2}\frac{\cos\theta_1}{\cos\theta_0}\mathbf{H_1}.\end{aligned}\tag{1.14}$$

These equations apply to Case 2 just as equations (1.11) apply to Case 1.
We arrive at the formulae:

$$\mathbf{H_1}=\frac{2k_1\mu_2\cos\theta_0}{(k_0\mu_1\cos\theta_1+k_1\mu_2\cos\theta_0)}\mathbf{H_0},$$

or

$$\mathbf{H_1}=\frac{2k_1^2\mu_2\cos\theta_0}{(k_1^2\mu_2\cos\theta_0)+\mu_1k_0\sqrt{k_1^2-k_0^2\sin^2\theta_0}}\mathbf{H_0},\tag{1.15}$$

and

$$\mathbf{H_2}=\frac{k_1\mu_2\cos\theta_0-k_0\mu_1\cos\theta_1}{k_0\mu_1\cos\theta_1+k_1\mu_2\cos\theta_0}\mathbf{H_0},$$

or

$$\mathbf{H_2}=\frac{k_1^2\mu_2\cos\theta_0-k_0\mu_1\sqrt{k_1^2-k_0^2\sin^2\theta_0}}{k_1^2\mu_2\cos\theta_0+k_0\mu_1\sqrt{k_1^2-k_0^2\sin^2\theta_0}}\mathbf{H_0}.\tag{1.16}$$

The above equations are similar to (1.12) and (1.13).

When the angle of incidence θ_0 is zero, both polarizations become the same, and the amplitudes of the transmitted and reflected waves reduce to:

$$\begin{aligned}\mathbf{E_1}&=\frac{2k_0\mu_1}{k_0\mu_1+k_1\mu_2}\mathbf{E_0},\\ \mathbf{E_2}&=\frac{k_0\mu_1-k_1\mu_2}{k_0\mu_1+k_1\mu_2}\mathbf{E_0},\end{aligned}\tag{1.17}$$

and

$$
\begin{aligned}
\mathbf{H_1} &= \frac{2k_1\mu_2}{k_1\mu_2 + k_2\mu_1}\mathbf{H_0}, \\
\mathbf{H_2} &= \frac{k_1\mu_2 - k_2\mu_1}{k_1\mu_2 + k_2\mu_1}\mathbf{H_0}.
\end{aligned}
\tag{1.18}
$$

1.2 The energy flow and Brewster phenomenon

If material 1 and material 2 are perfectly transparent so that the permeabilities μ_1 and μ_2 hardly differ from each other, then Snell's Law from equations (1.3), (1.6), and (1.7) can be written as

$$\frac{\sin\theta_1}{\sin\theta_0} = \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{a_1}{a_2} = n_{12}, \tag{1.19}$$

where a_1 and a_2 are the phase velocities in materials (1) and (2), respectively, and n_{12} their relative refraction index. If $\epsilon_1 > \epsilon_2$, then $n_{12} < 1$, and for every real angle of incidence θ_0 there is a corresponding transmission angle θ_1 . On the other hand, if $\epsilon_2 > \epsilon_1$, then $n_{12} > 1$ and θ_1 is real only for the range of θ_0 where $n_{12}\sin\theta_0 \leq 1$. When $n_{12}\sin\theta_0 > 1$, then $\sin\theta_1 > 1$ and no real transmission wave is produced, and this phenomenon is known as the total reflection.

With eqs. (1.19) taken into account, eqs. (1.12) and (1.13) become

$$\begin{aligned} \mathbf{E_1} &= \frac{2\cos\theta_0\sin\theta_1}{\sin(\theta_1+\theta_0)}\mathbf{E_0}, \\ \mathbf{E_2} &= \frac{\sin(\theta_1-\theta_0)}{\sin(\theta_1+\theta_0)}\mathbf{E_0}. \end{aligned} \tag{1.20}$$

When it comes to magnetic polarization, equation (1.19) becomes, with reference to (1.11),

$$\begin{aligned} \mathbf{n_1}\times\mathbf{E_1} &= \frac{2\cos\theta_0\sin\theta_1}{\sin(\theta_1+\theta_0)\cos(\theta_0-\theta_1)}\mathbf{n_0}\times\mathbf{E_0}, \\ \mathbf{n_2}\times\mathbf{E_2} &= \frac{\tan(\theta_0-\theta_1)}{\tan(\theta_0+\theta_1)}\mathbf{n_0}\times\mathbf{E_0}. \end{aligned} \tag{1.21}$$

The mean flow of energy is given by

$$\overline{\mathbf{S_i}} = \frac{1}{2}\mathbf{E_i}\times\widetilde{\mathbf{H_i}} = \frac{\sqrt{\epsilon_2}}{2}E_0^2\mathbf{n_0},$$

$$\overline{\mathbf{S_t}} = \frac{\sqrt{\epsilon_1}}{2}E_1^2\mathbf{n_1},$$

$$\overline{\mathbf{S_r}} = \frac{\sqrt{\epsilon_2}}{2}E_2^2\mathbf{n_2},$$

where $\overline{\mathbf{S}}$ is the real part of the complex vector $\frac{1}{2}\mathbf{E}\times\widetilde{\mathbf{H_i}}$, and $\widetilde{\mathbf{H_i}}$ is the complex conjugate of the magnetic vector. The symbol E denotes the magnitude of $\mathbf{E}$.

The primary energy incident per second on a unit area of the interface is the normal component of $\overline{\mathbf{S_i}}$: $\mathbf{n} \bullet \overline{\mathbf{S_i}} = -\frac{\sqrt{\epsilon_2}}{2} E_0^2 \cos\theta_0$. Similarly, the energies leaving by the reflection and transmission are

$$\mathbf{n} \bullet \overline{\mathbf{S_r}} = \frac{\sqrt{\epsilon_2}}{2} E_2^2 \cos\theta_0, \quad \mathbf{n} \bullet \overline{\mathbf{S_t}} = -\frac{\sqrt{\epsilon_1}}{2} E_1^2 \cos\theta_1.$$

In order to be in line with known principles of energies, the energy flow across the surface must be continuous,

$$\mathbf{n} \bullet \left(\overline{\mathbf{S_r}} + \overline{\mathbf{S_i}}\right) = \mathbf{n} \bullet \overline{\mathbf{S_t}}, \tag{1.22}$$

thus

$$\sqrt{\epsilon_2} E_0^2 \cos\theta_0 = \sqrt{\epsilon_2} E_2^2 \cos\theta_0 + \sqrt{\epsilon_1} E_1^2 \cos\theta_1. \tag{1.23}$$

The reflection and transmission coefficients are defined to be

$$R = \left| \frac{\mathbf{n} \bullet \overline{\mathbf{S_r}}}{\mathbf{n} \bullet \overline{\mathbf{S_i}}} \right| = \frac{E_2^2}{E_0^2}, \quad T = \left| \frac{\mathbf{n} \bullet \overline{\mathbf{S_t}}}{\mathbf{n} \bullet \overline{\mathbf{S_i}}} \right| = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \frac{\cos\theta_1}{\cos\theta_0} \frac{E_1^2}{E_0^2}.$$

Certainly, $R + T = 1$. When $\mathbf{E_0}$ is normal to the plane of incidence (electric polarization), these coefficients become

$$R_\perp = \frac{\sin^2(\theta_1 - \theta_0)}{\sin^2(\theta_1 + \theta_0)}, \quad T_\perp = \frac{\sin 2\theta_0 \sin 2\theta_1}{\sin^2(\theta_1 + \theta_0)}, \tag{1.24}$$

and when $\mathbf{H_0}$ is normal to the plane of the incidence (magnetic polarization), they are expressed as

$$R_{||} = \frac{\tan^2(\theta_0 - \theta_1)}{\tan^2(\theta_0 + \theta_1)}, \quad T_{||} = \frac{\sin 2\theta_0 \sin 2\theta_1}{\sin^2(\theta_1 + \theta_0) \cos^2(\theta_0 - \theta_1)}. \tag{1.25}$$

Under the special circumstance when incidence wave vector is normal to the plane of separation, meaning $\theta_0 = \theta_1 = 0$, the coefficients become

$$R = \left(\frac{\sqrt{\epsilon_2} - \sqrt{\epsilon_1}}{\sqrt{\epsilon_2} + \sqrt{\epsilon_1}}\right)^2 = \left(\frac{n_{12} - 1}{n_{12} + 1}\right)^2,$$

$$T = \frac{4\sqrt{\epsilon_1 \epsilon_2}}{\left(\sqrt{\epsilon_2} + \sqrt{\epsilon_1}\right)^2} = \frac{4n_{12}}{(n_{12} + 1)^2}.$$

The reflection coefficient is zero under the very special condition that if $(\theta_0 + \theta_1) \rightarrow \pi/2$ then the $\tan(\theta_0 + \theta_1) \rightarrow \infty$, and then and only then $R_{||} \rightarrow 0$. In this situation, the reflected and transmitted waves are normal to each other, so $\mathbf{n_1} \bullet \mathbf{n_2} = 0$ and $\sin\theta_1 = \sin(\pi/2 - \theta_0)$, implying that

$$\tan\theta_0 = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = n_{21}. \tag{1.26}$$

The angle which satisfies equation (1.26) is the *Brewster angle*. As illustrated above, an incident wave can be separated into two parts, one polarized in the direction of the normal and the other parallel to the plane of incidence. A consequence of this is that the reflection coefficients become different based on the angle of the incidence. For example, if the incidence angle is equal to the *Brewster angle*, then the reflection wave is polarized entirely in the direction normal to the plane of incidence.

1.3 Total reflection

The phenomenon of total reflection occurs when the there is no transmission wave traveling into material 1, instead, there arises the surface wave traveling along S on the side of it that is occupied by material 1. This phenomenon occurs only when θ_0 is such that

$$\sin\theta_1 = n_{12}\sin\theta_0 = \sqrt{\frac{\epsilon_2}{\epsilon_1}}\sin\theta_0 > 1,$$

and the only values of θ_1 to satisfy this equation are complex. A complex angle of transmission implies a shift in the phase as well as an attenuation factor.

Suppose $\sin\theta_1 > 1$, then $\cos\theta_1$ is purely imaginary, such that

$$\cos\theta_1 = \frac{i}{\sqrt{\epsilon_1}}\sqrt{\epsilon_2\sin^2\theta_0 - \epsilon_1} = in_{12}\sqrt{\sin^2\theta_0 - n_{21}^2}. \tag{1.27}$$

The radical produces two roots. Bearing in mind that the field must always be finite, the root which allows for this condition will be chosen. By letting the surface S be at $x = 0$ and assuming $\mu_1 = \mu_2$, the phase of the transmitted wave becomes

$$\begin{aligned} k_1\mathbf{n_1} \bullet \mathbf{r} &= \omega\sqrt{\epsilon_1\mu_1}(x\cos\theta_1 + y\sin\theta_1) \\ &= \omega\sqrt{\epsilon_2\mu_2}\left(x\sqrt{\sin^2\theta_0 - n_{21}^2} + y\sin\theta_0\right), \end{aligned}$$

and for $x > 0$ the field intensity of the same wave is then

$$\mathbf{E_t} = \mathbf{E_1}e^{-\beta x + i\alpha y - i\omega t}, \tag{1.28}$$

with α, β, defined as follows

$$\alpha = \omega\sqrt{\epsilon_2\mu_2}\sin\theta_0, \quad \beta = \omega\sqrt{\epsilon_2\mu_2}\sqrt{\sin\theta_0 - n_{21}^2}.$$

To illustrate that the positive root from (1.27) is the correct choice in this situation, observe that as $x \to +\infty$, the field defined in (1.28) goes to zero.

Assuming that $\sin\theta_1 > 1$, we apply equations (1.12) and (1.13) showing the amplitudes of reflected and transmitted electric fields

$$
\begin{aligned}
\mathbf{E_{1\perp}} &= \frac{2\cos\theta_0}{i\sqrt{\sin^2\theta_0 - n_{21}^2} + \cos\theta_0}\mathbf{E_{0\perp}}, \\
\mathbf{E_{2\perp}} &= \frac{\cos\theta_0 - i\sqrt{\sin^2\theta_0 - n_{21}^2}}{i\sqrt{\sin^2\theta_0 - n_{21}^2} + \cos\theta_0}\mathbf{E_{0\perp}}.
\end{aligned}
\tag{1.29}
$$

Using eqs. (1.3), eqs. (1.15) and (1.16) show the amplitudes of the reflected and transmitted magnetic field

$$
\begin{aligned}
\mathbf{n_1} \times \mathbf{E_{1||}} &= \frac{2n_{21}\cos\theta_0}{n_{21}\cos\theta_0 + i\sqrt{\sin^2\theta_0 - n_{21}^2}}\mathbf{n_0} \times \mathbf{E_{0||}}, \\
\mathbf{n_2} \times \mathbf{E_{2||}} &= \frac{n_{21}^2\cos\theta_0 - i\sqrt{\sin^2\theta_0 - n_{21}^2}}{n_{21}^2\cos\theta_0 + i\sqrt{\sin^2\theta_0 - n_{21}^2}}\mathbf{n_0} \times \mathbf{E_{0||}}.
\end{aligned}
\tag{1.30}
$$

It can be seen that the transmitted and reflected waves are out of phase with the incident wave since the coefficients of $\mathbf{E_0}$ are complex. From eqs. (1.29) and (1.30) and given the definition of the reflection coefficient to be $R = \mathbf{E_2} \bullet \tilde{\mathbf{E_2}}/E_0^2$, we find that

$$
R_\perp = R_{||} = 1, \quad T_\perp = T_{||} = 0. \tag{1.31}
$$

The energy in the incident wave appears to be exactly equal to the energy of reflected wave meaning there is no energy flow into the material of lesser refractive index. However, this does not mean that the field intensity in material 1 is zero. In the case of total reflection, the transmitted wave produced does not travel through material 1, but instead is bound to travel along the surface of the interface.

Chapter 2

Moving interface

2.1 Snell's law

Again, without loss of generality, we will continue to refer to material 1 on the left and material 2 on the right of the interface and throughout the whole work, it will be assumed $a_2 < a_1$. For simplicity reasons, we shall allow $\mathbf{E}$ to be $\mathbf{k}E$ and $\varphi = \frac{\omega}{a}\mathbf{n} \bullet \mathbf{r} - \omega t$, with $a = 1/\sqrt{\epsilon\mu}$; then the general formulae for the electric and magnetic field vectors described in equations (1.1) and (1.2) become

$$\begin{aligned}
\mathbf{E_i} &= \mathbf{k}E_0 e^{i\varphi_0}, \quad \mathbf{H_i} = \frac{1}{\mu_2 a_2}\mathbf{n_0} \times \mathbf{E_i}, \\
\mathbf{E_r} &= \mathbf{k}E_2 e^{i\varphi_2}, \quad \mathbf{H_r} = \frac{1}{\mu_2 a_2}\mathbf{n_2} \times \mathbf{E_r}, \\
\mathbf{E_t} &= \mathbf{k}E_1 e^{i\varphi_1}, \quad \mathbf{H_t} = \frac{1}{\mu_1 a_1}\mathbf{n_1} \times \mathbf{E_t}.
\end{aligned} \tag{2.1}$$

As in Chapter 1, we have two dielectric materials separated by a planar interface, S. The difference here is that the planar interface is not static; instead, the interface is assumed to be moving at velocity $\mathbf{i}V$. With this interface, we associate a new ("primed") frame as a frame moving at velocity $\mathbf{V} = \mathbf{i}V$ relative to the laboratory (non-primed) frame.

On the interface $x = Vt$. This along with the observation that

$$\begin{aligned}
\mathbf{n_0} &= \mathbf{i}\cos\theta_0 + \mathbf{j}\sin\theta_0, \\
\mathbf{n_2} &= -\mathbf{i}\cos\theta_2 + \mathbf{j}\sin\theta_2, \\
\mathbf{n_1} &= \mathbf{i}\cos\theta_1 + \mathbf{j}\sin\theta_1,
\end{aligned}$$

and $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, allow the φ's to become

$$\begin{aligned}
\varphi_0 &= \frac{\omega}{a_2}[y \sin\theta_0 + (V\cos\theta_0 - a_2)t],\\
\varphi_2 &= \frac{\omega_2}{a_2}[y \sin\theta_2 - (V\cos\theta_2 + a_2)t],\\
\varphi_1 &= \frac{\omega_1}{a_1}[y \sin\theta_1 + (V\cos\theta_1 - a_1)t].
\end{aligned} \tag{2.2}$$

Now we have the equalities

$$\begin{gathered}
\frac{\omega}{a_2}\sin\theta_0 = \frac{\omega_2}{a_2}\sin\theta_2 = \frac{\omega_1}{a_1}\sin\theta_1,\\
\frac{\omega}{a_2}(V\cos\theta_0 - a_2) = -\frac{\omega_2}{a_2}(V\cos\theta_2 - a_2) = \frac{\omega_1}{a_1}(V\cos\theta_1 - a_1).
\end{gathered} \tag{2.3}$$

Since the quantities $V, a_1, a_2, \omega, \theta_0$ are known, the above equalities should be solved for $\omega_1, \omega_2, \theta_1, \theta_2$. We have

$$\begin{gathered}
\frac{\sin\theta_0}{V\cos\theta_0 - a_2} = -\frac{\sin\theta_2}{V\cos\theta_2 + a_2},\\
\frac{\sin\theta_0}{V\cos\theta_0 - a_2} = \frac{\sin\theta_1}{V\cos\theta_1 - a_1},\\
\omega_2 = \omega\frac{\sin\theta_0}{\sin\theta_2},\\
\omega_1 = \omega\frac{\sin\theta_0}{\sin\theta_1}\frac{a_1}{a_2}.
\end{gathered} \tag{2.4}$$

To further solve for θ_1 and θ_2, let λ be defined as

$$\frac{\sin\theta_0}{V\cos\theta_0 - a_2} \tag{2.5}$$

then

$$\begin{aligned}
\tan\frac{\theta_1}{2} &= \frac{-1 \pm \sqrt{1 - \lambda^2(a_1^2 - V^2)}}{\lambda(a_1 + V)},\\
\tan\frac{\theta_2}{2} &= \frac{-1 \pm \sqrt{1 - \lambda^2(a_2^2 - V^2)}}{\lambda(a_2 - V)}.
\end{aligned} \tag{2.6}$$

Taking $\tan\dfrac{\theta_1}{2}$ and $\tan\dfrac{\theta_2}{2}$ separately, the roots for each are described explicitly below.

$$\tan\frac{\theta_1}{2} = \begin{cases} \dfrac{-1+\sqrt{1-\lambda^2(a_1^2-V^2)}}{\lambda(a_1+V)} = \dfrac{-V\cos\theta_0 + a_2 + \sqrt{\dfrac{V^2+a_2^2-2Va_2\cos\theta_0 - a_1^2\sin^2\theta_0}{V^2\cos^2\theta_0 - 2Va_2\cos\theta_0 + a_2^2}}}{(a_1+V)\sin\theta_0}, \\[2ex] \dfrac{-1-\sqrt{1-\lambda^2(a_1^2-V^2)}}{\lambda(a_1+V)} = \dfrac{-V\cos\theta_0 + a_2 - \sqrt{\dfrac{V^2+a_2^2-2Va_2\cos\theta_0 - a_1^2\sin^2\theta_0}{V^2\cos^2\theta_0 - 2Va_2\cos\theta_0 + a_2^2}}}{(a_1+V)\sin\theta_0}, \end{cases} \tag{2.7}$$

$$\tan\frac{\theta_2}{2} = \begin{cases} \dfrac{-1+\sqrt{1-\lambda^2(a_2^2-V^2)}}{\lambda(a_2-V)} = \dfrac{V+a_2}{a_2-V}\tan\dfrac{\theta_0}{2}, \\[2ex] \dfrac{-1-\sqrt{1-\lambda^2(a_2^2-V^2)}}{\lambda(a_2-V)} = -\dfrac{V-a_2}{(a_2-V)\tan\dfrac{\theta_0}{2}} = \cot\dfrac{\theta_0}{2}. \end{cases} \tag{2.8}$$

When $V=0$, the above formulae reduce to the situation of a non-moving interface as described in Section 1.3.

2.2 Field vectors and Brewster phenomenon with moving interface

The Lorentz transformation:

$$\begin{array}{rcl} \mathbf{E}'_{||} & = & (\mathbf{E} + \mu\mathbf{V} \times \mathbf{H})_{||}\,, \\ \mathbf{E}'_{\perp} & = & \Gamma(\mathbf{E} + \mu\mathbf{V} \times \mathbf{H})_{\perp}, \\ \mathbf{H}'_{||} & = & (\mathbf{H} - \epsilon\mathbf{V} \times \mathbf{E})_{||}\,, \\ \mathbf{H}'_{\perp} & = & \Gamma(\mathbf{H} - \epsilon\mathbf{V} \times \mathbf{E})_{\perp}, \end{array} \tag{2.9}$$

where, $\Gamma = 1/\sqrt{1 - \dfrac{V^2}{c^2}}$, express the field components in a primed frame through their values in a non-primed frame. When the field is electrically polarized, the primed electric components are expressed as

$$\begin{aligned} E'_{ix} &= E_{ix} + \frac{V}{a_2}\mathbf{i} \times (\mathbf{n_0} \times \mathbf{k})E_0 e^{i\varphi_0} \bullet \mathbf{i} = 0, \\ \frac{1}{\Gamma}E'_{iy} &= E_{iy} + \frac{V}{a_2}\mathbf{i} \times (\mathbf{n_0} \times \mathbf{k})E_0 e^{i\varphi_0} \bullet \mathbf{j} = 0, \\ \frac{1}{\Gamma}E'_{iz} &= E_{iz} + \frac{V}{a_2}\mathbf{i} \times (\mathbf{n_0} \times \mathbf{k})E_0 e^{i\varphi_0} \bullet \mathbf{k} = \left(1 - \frac{V\cos\theta_0}{a_2}\right) E_0 e^{i\varphi_0}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} E'_{rx} &= 0, \\ E'_{ry} &= 0, \\ \frac{1}{\Gamma}E'_{rz} &= \left(1 - \frac{V\cos\theta_2}{a_2}\right) E_2 e^{i\varphi_2}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} E'_{tx} &= 0, \\ E'_{ty} &= 0, \\ \frac{1}{\Gamma}E'_{tz} &= \left(1 - \frac{V\cos\theta_1}{a_1}\right) E_1 e^{i\varphi_1}. \end{aligned} \tag{2.12}$$

The magnetic vectors of the electric polarization are expressed in the prime frame as

$$\begin{aligned} H'_{ix} &= H_{ix} = \frac{1}{\mu_2 a_2}E_0 e^{i\varphi_0}\sin\theta_0, \\ \frac{1}{\Gamma}H'_{iy} &= H_{iy} - V\,(\mathbf{i} \times \mathbf{k}) \bullet \mathbf{j}\epsilon_2 E_0 e^{i\varphi_0} = \epsilon_2(V - a_2\cos\theta_0)E_0 e^{i\varphi_0}, \\ \frac{1}{\Gamma}H'_{iz} &= H_{iz} - V\,(\mathbf{i} \times \mathbf{k}) \bullet \mathbf{k}\epsilon_2 E_0 e^{i\varphi_0} = 0, \end{aligned} \tag{2.13}$$

$$H'_{rx} = H_{rx} = \frac{1}{\mu_2 a_2} E_2 e^{i\varphi_2} \sin\theta_2,$$
$$\frac{1}{\Gamma} H'_{ry} = H_{ry} - V\left(\mathbf{i} \times \mathbf{k}\right) \bullet \mathbf{j} \epsilon_2 E_2 e^{i\varphi_2} = \epsilon_2 (V + a_2 \cos\theta_2) E_2 e^{i\varphi_2}, \tag{2.14}$$
$$\frac{1}{\Gamma} H'_{iz} = 0,$$

$$H'_{tx} = H_{tx} = \frac{1}{\mu_1 a_1} E_1 e^{i\varphi_1} \sin\theta_1,$$
$$\frac{1}{\Gamma} H'_{ty} = \epsilon_1 (V - a_1 \cos\theta_1) E_1 e^{i\varphi_1}, \tag{2.15}$$
$$\frac{1}{\Gamma} H'_{tz} = 0.$$

In electric polarization, the boundary conditions to ensure continuity across the interface are

$$E'_{iz} + E'_{rz} = E'_{tz}, \qquad H'_{iy} + H'_{ry} = H'_{ty}. \tag{2.16}$$

The second case to be dealt with is about magnetic polarization of the field.

The electric vectors in this situation are expressed in the prime frame, with reference to the Lorentz transformation, as:

$$E'_{ix} = E_{ix},$$
$$\frac{1}{\Gamma} E'_{iy} = E_{iy} + \mu_2 \left(V\mathbf{i} \times H_{iz}\mathbf{k}\right) \bullet \mathbf{j} = \left(1 - \frac{V}{a_2 \cos\theta_0}\right) E_{iy}, \tag{2.17}$$
$$E'_{iz} = 0,$$

$$E'_{rx} = E_{rx},$$
$$\frac{1}{\Gamma} E'_{ry} = E_{ry} + \mu_2 \left(V\mathbf{i} \times H_{rz}\mathbf{k}\right) \bullet \mathbf{j} = \left(1 + \frac{V}{a_2 \cos\theta_2}\right) E_{ry}, \tag{2.18}$$
$$E'_{iz} = 0,$$

$$E'_{tx} = E_{tx},$$
$$\frac{1}{\Gamma} E'_{ty} = E_{ty} + \mu_1 \left(V\mathbf{i} \times H_{tz}\mathbf{k}\right) \bullet \mathbf{j} = \left(1 - \frac{V}{a_1 \cos\theta_1}\right) E_{ty}, \tag{2.19}$$
$$E'_{tz} = 0.$$

The magnetic vectors of the same polarization are expressed in prime frame as

$$
\begin{aligned}
H'_{ix} &= 0, \\
H'_{iy} &= 0, \\
\frac{1}{\Gamma} H'_{iz} &= H_{iz} - \epsilon_2 \left(V\mathbf{i} \times E_{iy}\mathbf{j}\right) \bullet \mathbf{k} = \left(\frac{1}{\mu_2 a_2 \cos\theta_0} - \epsilon_2 V\right) E_{iy},
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
H'_{rx} &= 0, \\
H'_{ry} &= 0, \\
\frac{1}{\Gamma} H'_{rz} &= H_{rz} - \epsilon_2 \left(V\mathbf{i} \times E_{ry}\mathbf{j}\right) \bullet \mathbf{k} = -\left(\frac{1}{\mu_2 a_2 \cos\theta_2} + \epsilon_2 V\right) E_{ry},
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
H'_{tx} &= 0, \\
H'_{ty} &= 0, \\
\frac{1}{\Gamma} H'_{tz} &= H_{tz} - \epsilon_1 \left(V\mathbf{i} \times E_{ty}\mathbf{j}\right) \bullet \mathbf{k} = \left(\frac{1}{\mu_1 a_1 \cos\theta_1} - \epsilon_1 V\right) E_{ty},
\end{aligned} \tag{2.22}
$$

In magnetic polarization, the boundary conditions to ensure continuity across the interface are

$$
\begin{aligned}
E'_{iy} + E'_{ry} &= E'_{ty}, \\
H'_{iz} + H'_{rz} &= H'_{tz}.
\end{aligned} \tag{2.23}
$$

We are specifically interested in how the Brewster phenomenon is affected by the motion of the interface. To this end, we examine the magnetic polarization.

The boundary conditions for such polarization can be explicitly written as

$$
\begin{aligned}
\left(1 - \frac{V}{a_2 \cos\theta_0}\right) E_{iy} + \left(1 + \frac{V}{a_2 \cos\theta_2}\right) E_{ry} &= \left(1 - \frac{V}{a_1 \cos\theta_1}\right) E_{ty}, \\
\left(\frac{1}{\mu_2 a_2 \cos\theta_0} - \epsilon_2 V\right) E_{iy} - \left(\frac{1}{\mu_2 a_2 \cos\theta_2} + \epsilon_2 V\right) E_{ry} &= \left(\frac{1}{\mu_1 a_1 \cos\theta_1} - \epsilon_1 V\right) E_{ty}.
\end{aligned}
$$

Eliminating E_{ty} from the above equations yields

$$
\begin{aligned}
E_{iy} \left[\left(1 - \frac{V}{a_2 \cos\theta_0}\right)\left(\frac{1}{\mu_1 a_1 \cos\theta_1} - \epsilon_1 V\right) - \left(\frac{1}{\mu_2 a_2 \cos\theta_2} - \epsilon_2 V\right)\left(1 - \frac{V}{a_1 \cos\theta_1}\right)\right] = \\
E_{ry} \left[\left(\frac{1}{\mu_2 a_2 \cos\theta_2} - \epsilon_2 V\right)\left(1 - \frac{V}{a_1 \cos\theta_1}\right) - \left(1 + \frac{V}{a_2 \cos\theta_2}\right)\left(\frac{1}{\mu_1 a_1 \cos\theta_1} - \epsilon_1 V\right)\right]
\end{aligned} \tag{2.24}
$$

If we introduce symbols

$$N \equiv \left[\left(1 - \frac{V}{a_2 \cos\theta_0}\right)\left(\frac{1}{\mu_1 a_1 \cos\theta_1} - \epsilon_1 V\right) - \left(\frac{1}{\mu_2 a_2 \cos\theta_2} - \epsilon_2 V\right)\left(1 - \frac{V}{a_1 \cos\theta_1}\right)\right],$$

and

$$M \equiv \left[\left(\frac{1}{\mu_2 a_2 \cos\theta_2} - \epsilon_2 V\right)\left(1 - \frac{V}{a_1 \cos\theta_1}\right) - \left(1 + \frac{V}{a_2 \cos\theta_2}\right)\left(\frac{1}{\mu_1 a_1 \cos\theta_1} - \epsilon_1 V\right)\right],$$

then the reflection coefficient, equation (1.25), in the primed frame now becomes

$$R = \left(\frac{E'_r}{E'_i}\right)^2 = \frac{\left(1 + \dfrac{V}{a_2 \cos\theta_2}\right)^2}{\left(1 - \dfrac{V}{a_2 \cos\theta_2}\right)^2} \frac{N^2}{M^2}. \tag{2.25}$$

The reflected wave is absent when $R = 0$, thus we must examine when N is zero, i.e.

$$\frac{\mu_2(a_2 \cos\theta_0 - V)(1 - \epsilon_1 V \mu_1 a_1 \cos\theta_1) - \mu_1(1 - \epsilon_2 V \mu_2 a_2 \cos\theta_0)(a_1 \cos\theta_1 - V)}{\mu_1 \mu_2 a_1 a_2 \cos\theta_0 \cos\theta_1} = 0 \tag{2.26}$$

Since $a = \dfrac{1}{\sqrt{\epsilon\mu}}$,

$$(1 - V\epsilon_1\mu_1 a_1 \cos\theta_1) = -\frac{1}{a_1}(V \cos\theta_1 - a_1), \tag{2.27}$$

$$(1 - V\epsilon_2\mu_2 a_2 \cos\theta_0) = -\frac{1}{a_2}(V \cos\theta_0 - a_2), \tag{2.28}$$

and the equation (2.26) can be written as

$$-\frac{\mu_2}{a_1}(a_2\cos\theta_0 - V)(V\cos\theta_1 - a_1) + \frac{\mu_1}{a_2}(a_1\cos\theta_1 - V)(V\cos\theta_0 - a_2) = 0,$$

or

$$(V\cos\theta_0 - a_2)[\frac{\mu_2}{a_1}(V - a_2\cos\theta_0)\frac{(V\cos\theta_1 - a_1)}{(V\cos\theta_0 - a_2)} - \frac{\mu_1}{a_2}(V - a_1\cos\theta_1)] = 0. \quad (2.29)$$

The conditions defined in equation (2.4) allow for

$$\frac{(V\cos\theta_1 - a_1)}{(V\cos\theta_0 - a_2)} = \frac{\sin\theta_1}{\sin\theta_0}, \quad (2.30)$$

thus, equation (2.29) becomes

$$\frac{V\cos\theta_0 - a_2}{\sin\theta_0}[\frac{\mu_2}{a_1}(V - a_2\cos\theta_0)\sin\theta_1 - \frac{\mu_1}{a_2}(V - a_1\cos\theta_1)\sin\theta_0] = 0. \quad (2.31)$$

By using (2.30) to eliminate V, we obtain

$$\begin{aligned} V - a_2\cos\theta_0 &= \frac{1}{\cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0}[a_1\sin\theta_0 \\ &- a_2\sin\theta_1 - a_2\cos\theta_0\,(\cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0)] \\ &= \frac{\sin\theta_0}{cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0}[a_1 - a_2\cos\theta_0\cos\theta_1 - a_2\sin\theta_0\sin\theta_1] \\ &= \frac{\sin\theta_0}{cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0}[a_1 - a_2\cos(\theta_1 - \theta_0)], \end{aligned} \quad (2.32)$$

and, similarly

$$\begin{aligned} V - a_1\cos\theta_1 &= \frac{1}{\cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0}[a_1\sin\theta_0 \\ &- a_2\sin\theta_1 - a_1\cos\theta_1\,(\cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0)] \\ &= -\frac{\sin\theta_1}{cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0}\left[a_2 - a_1\cos\theta_0\cos\theta_1 - a_1\sin\theta_1\sin\theta_0\right] \\ &= -\frac{\sin\theta_0}{cos\theta_1\sin\theta_0 - \sin\theta_1\cos\theta_0}\left[a_2 - a_1\cos(\theta_1 - \theta_0)\right]. \end{aligned} \quad (2.33)$$

We now use (2.32) and (2.33) to transform (2.31) as follows:

$$\frac{V\cos\theta_0 - a_2}{\sin\theta_0\sin(\theta_0-\theta_1)}\left\{\frac{\mu_2}{a_1}\sin\theta_1\sin\theta_0\left[a_1 - a_2\cos(\theta_0-\theta_1)\right] + \frac{\mu_1}{a_2}\sin\theta_1\sin\theta_0\left[a_2 - a_1\cos(\theta_0-\theta_1)\right]\right\}$$
$$= \frac{(V\cos\theta_0 - a_2)\sin\theta_1}{\sin(\theta_0-\theta_1)}\left\{\mu_1+\mu_2-\left(\mu_2\frac{a_2}{a_1}+\mu_1\frac{a_1}{a_2}\right)\cos(\theta_0-\theta_1)\right\} = 0 \tag{2.34}$$

This equation shows that

$$\cos(\theta_1-\theta_0) = \frac{(\mu_1+\mu_2)\,a_1a_2}{\mu_2a_2^2+\mu_1a_1^2} \equiv \kappa, \tag{2.35}$$

which is $V-$*independent*. Together with (2.30), it constitutes a system to determine the angles of θ_0 and θ_1 related to Brewster phenomenon; particularly the value of θ_0, traditionally termed the Brewster angle, will be V-dependent. The system is rewritten as

$$\begin{aligned} V\sin(\theta_0-\theta_1) &= a_1\sin\theta_0 - a_2\sin\theta_1, \\ \cos(\theta_1-\theta_0) &= \kappa. \end{aligned} \tag{2.36}$$

If we define $(\theta_0-\theta_1)$ as τ, then

$$\begin{aligned} V\sin\tau &= (a_1 - a_2\cos\tau)\sin\theta_0 + a_2\sin\tau\cos\theta_0, \\ \cos\tau &= \kappa. \end{aligned} \tag{2.37}$$

Thus the Brewster phenomenon (no reflection wave), when the interface is moving at a velocity V, occurs at the special instant when

$$V = \frac{a_1 - a_2\kappa}{\sqrt{1-\kappa^2}}\sin\theta_0 + a_2\cos\theta_0. \tag{2.38}$$

In conclusion, although the difference in the $\cos(\theta_0-\theta_1)$ is V-independent, the analogue of *Brewster angle* (equation (1.26)) in the primed frame is velocity dependent and is given by (2.38).

As we define δ to be $\frac{V}{a_2}$ and γ to be $\frac{a_2}{a_1}$, equation (2.38) becomes

$$\frac{1-\gamma\kappa}{\sqrt{1-\kappa^2}}\sin\theta_0 + \cos\theta_0 - \delta = 0. \tag{2.39}$$

As previously stated, the permiabilities of material 1 and material 2 are nearly identical, thus we assume $\mu_1 = \mu_2$ and equation (2.39) becomes

$$\frac{1 - \dfrac{2\gamma^2}{(\gamma^2+1)}}{\sqrt{1 - \dfrac{4\gamma^2}{(\gamma+1)^2}}} \sin\theta_0 + \cos\theta_0 - \delta = 0. \tag{2.40}$$

Figure 4 is a graph of the left hand side of the above equality as a function of θ_0, $0 \le \theta_0 \le \pi$, for γ defined to be $1/2$ and a set of values of parameters δ.

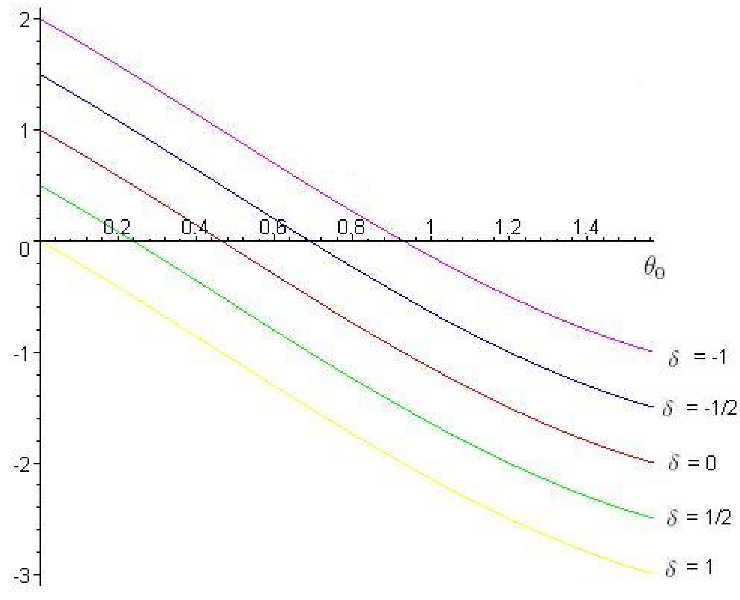

Figure 4

The Brewster phenomenon occurs when the graph of the function in equation (2.40), crosses the abscissa. Compared to the case of an immovable boundary, the motion of it in the same direction as the incident wave decreases the *Brewster angle*, whereas, the motion of the opposite direction increases it.

2.3 Total reflection in the case moving interface

Total reflection occurs when there is no forward traveling wave into material 1 from material 2. This is to say, θ_1 is imaginary and the discriminant of $\tan\frac{\theta_1}{2}$ is negative, i.e. $1 - \lambda^2(a_1^2 - V^2) < 0$, or $1 < \lambda^2(a_1^2 - V^2)$. The reference to (2.4) reveals that when $V < a_1$, this discriminant is only negative when either

$$\frac{\sin\theta_0}{V\cos\theta_0 - a_2}\sqrt{a_1^2 - V^2} > 1, \quad V\cos\theta_0 - a_2 > 0,$$

or

$$\frac{\sin\theta_0}{a_2 - V\cos\theta_0}\sqrt{a_1^2 - V^2} > 1, \quad V\cos\theta_0 - a_2 < 0.$$

Defining δ to be $\frac{V}{a_2}$, the latter case can be written as

$$\psi \equiv \frac{\sqrt{1 - \frac{a_2^2}{a_1^2}\delta^2}}{\frac{a_2}{a_1}}\frac{\sin\theta_0}{1 - \delta\cos\theta_0} > 1. \tag{2.41}$$

Figure 5 is a graph of ψ as a function of θ_0, $0 \le \theta_0 \le \frac{\pi}{2}$, for $\gamma = 1/2$ and a set of values of parameter δ.

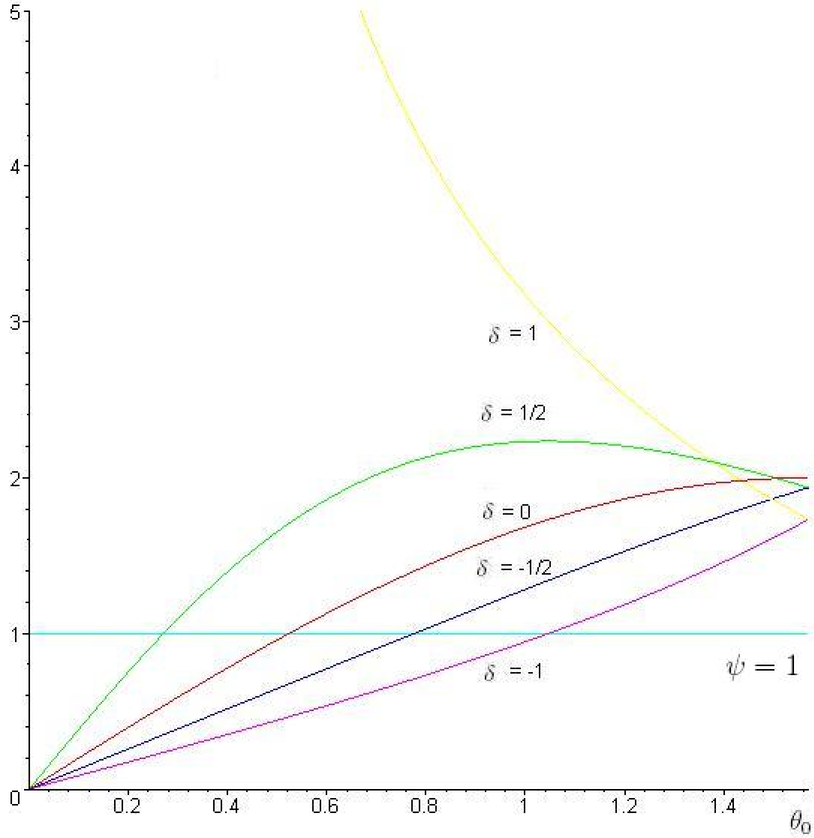

Figure 5

As Figure 5 illustrates, as $\delta \to 1$ the intersection point between the graphs of the functions of ψ and the line $y = 1$ approach the y-axis. The interpretation of this observation is that when $V = a_2$, total reflection becomes the only possibility and thus will always occur. However, for values of $V < a_2$, there exists a range of θ_0 values such that total reflection will not occur. In Figure 5, the ranges of θ_0 where total reflection is absent can be seen as the segments of the θ_0-axis related to the part of the curves under the line $\psi = 1$. The length of such segments approaches zero as δ approaches unity. We see that motion of the interface in the direction of the incident wave facilitates the total reflection.

In conclusion, the case when $V > a_1$ should be examined. As previously stated, with $V > a_1$, total reflection occurs when θ_1 is imaginary, thus $1 - \lambda^2(a_1^2 - V^2) < 0$. We observe that in this region $1 - \lambda^2(a_1^2 - V^2)$ is never less than zero and total reflection does not occur at all.

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