The Calkin-Wilf Tree: Extensions and Applications

A Major Qualifying Project (MQP) Report
Submitted to the Faculty of
WORCESTER POLYTECHNIC INSTITUTE
in partial fulfillment of the requirements
for the Degree of Bachelor of Science in

Mathematical Sciences

By:

Ben Gobler

Project Advisors:

Prof. Brigitte Servatius
Prof. Herman Servatius

Date: May 2022

Abstract

Continued fractions are of current interest in mathematics [8]. In a recent publication [7], Jack E. Graver describes a method for computing terms in the Calkin-Wilf sequence, a list of the positive rationals introduced by Neil Calkin and Herbert S. Wilf in 2000 [3]. This paper explores an original method which uses continued fractions to evaluate and locate terms in the Calkin-Wilf sequence, as well as its natural extension to include all of the rational numbers. A generalization of the Calkin-Wilf tree leads to a characterization of rational numbers by continued fractions with integer coefficients. Finally, the meaning of infinite continued fractions and irrational numbers is studied using the structure of the Calkin-Wilf tree. We characterize the irrational numbers which have periodic continued fractions by developing a matrix representation of the setup, and we explain why irrational square root numbers have periodic continued fractions with palindromic coefficients.

Executive Summary

The Calkin-Wilf tree first appeared in a publication by Neil Calkin and Herbert S. Wilf in 2000 [3]. It is a binary tree with nodes labelled by rational numbers. The left child of p/q is $p/(p+q)$, and the right child is $(p+q)/q$. The Calkin-Wilf tree has two intriguing and useful properties: first, every fraction in the tree is reduced. In addition, every possible reduced fraction appears exactly once in the tree. By reading off the elements one level at a time, we produce the Calkin-Wilf sequence $\ell(n)$, a list which includes each rational number exactly once. We would like to directly evaluate and locate terms in this sequence. Continuing the work from a recent publication by Jack E. Graver [7], we derive a method which directly computes $\ell(n)$ for a natural number n using continued fractions. The method reveals that a path in the tree from $0/1$ to p/q corresponds with a continued fraction for p/q where the coefficients are the numbers of consecutive left and right movements along the path. Moreover, the method is reversible; we can find n such that $\ell(n) = p/q$ for a rational number p/q. A strong connection is identified between the tree and the Euclidean algorithm.

We make the first extension to the Calkin-Wilf tree by considering how to reverse the generating rules. These reverse rules are used to construct more of the tree above the root. The result is a copy of the Calkin-Wilf tree, upside-down, and all negative! This double tree includes *all* rational numbers—positive, negative, and zero. With sensible indexing, the double tree produces an extended Calkin-Wilf sequence, and our method for evaluating and locating positive terms is naturally fitted to work for negative terms.

Next, we allow all forward and reverse movements from any node in the tree. This generates a four-way tree which includes the double tree and much more. Remarkably, any path in this tree from $0/1$ to p/q still corresponds with a continued fraction for p/q, however, the coefficients are allowed to take integer values. Positive coefficients represent forward movement, and negative coefficients represent reverse movement. This is a complete characterization of the rational numbers by continued fractions with integer coefficients.

Every natural-numbered level of the Calkin-Wilf tree contains rational numbers and is reached by paths of finite length. It follows that rational numbers have finite continued fractions, and therefore, infinite continued fractions cannot be rational. Some infinite continued fractions have a repeating sequence of coefficients and are called *periodic*. We use the rules of the Calkin-Wilf tree to study the irrational numbers with periodic continued fractions. The analysis leads to a delightfully simple explanation of why square root numbers have periodic continued fractions with palindromic coefficients. In addition, we are surprised that periodic paths in the four-way tree represent complex numbers! Examples are given for the third and fourth roots of unity.

We conclude with a matrix representation of the Calkin-Wilf tree which proves that an irrational number x has a periodic continued fraction exactly when $x = (ax+b)/(cx+d)$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is nontrivial in $SL_2(\mathbb{Z})$.

Acknowledgements

The success of this project would not have been possible without the support of Professor Brigitte Servatius and Professor Herman Servatius. Together they guided me through the research and writing process, and gave me the freedom to explore and discover beautiful mathematics.

I am deeply grateful to Guillermo Nuñez Ponasso for our many conversations on the topics of this project. A generous and inspiring mentor, his ideas and suggestions have shaped a large part of my current research.

I would also like to thank the Mathematical Sciences Department for endorsing my work and awarding this project with the Provost's MQP Award for Mathematical Sciences. I am grateful to the Mathematical Sciences community for offering continual support, encouragement, and everyday kindness.

Contents

1 Introduction

Can you make a list of the rationals containing each number exactly once? If you made such a list, you would want to know how to answer questions such as, "What is the 200th number in the list?" or, "Where does 22/7 appear?" At first it is not obvious how to perform these tasks, or if they are possible at all.

The rationals are countable, which means we could construct a list that includes every rational number at least once. The usual approach is to create a table where the entry in row i and column j is j/i. We may rotate this table forty-five degrees and read the entries left to right, one level at a time, to obtain the desired list.

	1	2	3	4	5	$\cdots$
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\cdots$
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\cdots$
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\cdots$
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\cdots$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$\frac{1}{1}$$
$$\frac{1}{2} \quad \frac{2}{1}$$
$$\frac{1}{3} \quad \frac{2}{2} \quad \frac{3}{1}$$
$$\frac{1}{4} \quad \frac{2}{3} \quad \frac{3}{2} \quad \frac{4}{1}$$
$$\frac{1}{5} \quad \frac{2}{4} \quad \frac{3}{3} \quad \frac{4}{2} \quad \frac{5}{1}$$
$$\iddots \quad \iddots \quad \vdots \quad \ddots \quad \ddots$$

Figure 1: Arrangements of the rational numbers

Notice that each rational number appears infinitely often in the form of different fractions. If we want to list each rational exactly once, we may choose to include only the fractions which are reduced. But it becomes difficult to identify terms in this new list due to the complexity of factoring large numbers. So to answer our questions, we will consider a different approach to the listing.

2 The Calkin-Wilf Tree

Begin with a fraction p/q where p and q are positive integers. There are two simple operations we can perform on this fraction to produce new fractions; we could add the numerator to the denominator, or add the denominator to the numerator. These operations are called the *left rule* and the *right rule*, respectively, as seen in Figure 2.

What happens when we perform these rules on the fraction 1/1? The left rule gives 1/(1+1) which is 1/2, and the right rule gives 2/1. We can perform the rules again on these fractions, and so on. The structure that results from repeating this process is called the Calkin-Wilf tree [3]. The first four levels of the tree are shown in Figure 2.

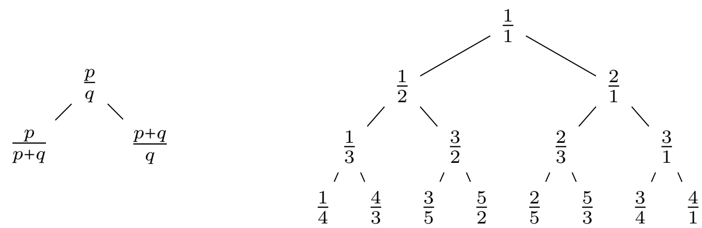


Figure 2: The Calkin-Wilf tree

The Calkin-Wilf tree has two important features. First, every fraction in the tree is reduced. In addition, every possible reduced fraction will appear exactly once in the tree. From these properties, it follows that by reading off the elements of the tree, one level at a time, we obtain a list which includes each rational number exactly once. This list is called the Calkin-Wilf sequence, denoted $\ell(n)$, and it begins 1/1, 1/2, 2/1, 1/3, 3/2, 2/3, 3/1, 1/4, ... How can we answer questions like, "What is the 200th number in the list?" From the construction, it is clear that the location of a number in the list depends on its location in the tree.

Consider a new tree where the nodes are labeled in the order of the sequence, left to right and one level at a time. This tree in Figure 3 represents the term number n of the corresponding fraction $\ell(n)$ in the Calkin-Wilf sequence. What are the generating rules for this tree? Looking at the first few levels, it becomes clear that the left and right children of n are $2n$ and $2n+1$, respectively. These are meaningful operations in binary, so for clarity, we will rewrite the nodes of the tree in base 2. Now we can interpret the generating rules as appending a '0' to go left, and a '1' to go right.

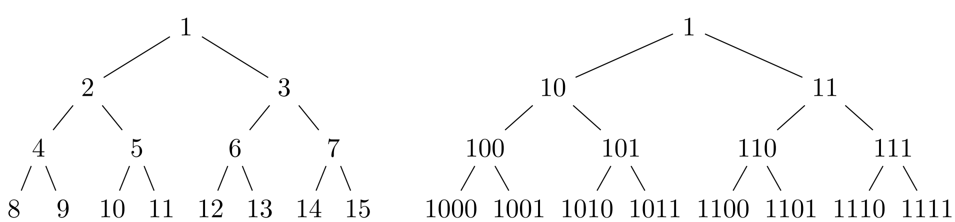


Figure 3: The n trees

To make this most natural, we should carefully consider where the appending process begins. Since 1/1 is the term $n = 1$, it should be reached by appending a '1'. So it is sensible to extend the tree by one step backward right for no n. This new node has value 0/1, since the right rule from 0/1 gives 1/1. Later we will justify this extension more thoroughly.

Now we can use n to locate $\ell(n)$ in the Calkin-Wilf tree; writing n in binary, the digits in order describe a sequence of left and right rules which, when applied to 0/1, evaluate to $\ell(n)$. For example, to find the 14th fraction in the sequence, write 14 in binary: 1110. From 0/1, perform the right rule three times, followed by one left rule. This is the path $0/1 \to 1/1 \to 2/1 \to 3/1 \to 3/4$. Then $\ell(14) = 3/4$.

3 Continued Fractions

Until now, we have followed closely to the method developed by Graver in [7]. From here on, we will continue with an original approach. It works well to use n in binary to compute $\ell(n)$, but it is inefficient to perform one rule at a time. Instead, let's consider what consecutive movements look like algebraically.

We make m right movements from p/q by adding the denominator to the numerator m times. This gives the fraction $(p+mq)/q$, which simplifies to $m+p/q$. Likewise, we make m left movements from p/q by adding the numerator to the denominator m times, which gives $p/(mp+q)$. To simplify, we invert the reciprocal of this fraction, giving $1/((mp+q)/p)$. This is $1/(m+q/p)$, which, after one more double-inversion, is $1/(m+1/(p/q))$.

$$\frac{p}{q} \xrightarrow{m \text{ right}} m+\frac{p}{q} \qquad\qquad \frac{p}{q} \xrightarrow{m \text{ left}} \frac{1}{m+\frac{1}{\frac{p}{q}}}$$

What happens when we combine these two types of movements? We may consider a generic path through the Calkin-Wilf tree to p/q and use the rules for consecutive movements to write this fraction in terms of its ancestors.

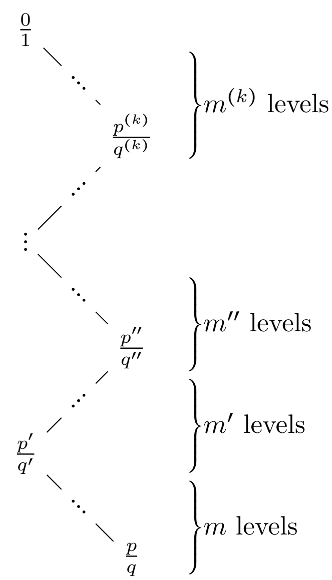


Figure 4: The path to p/q

At the bottom of Figure 4, we see that p/q is m right movements from its ancestor p'/q'. So we can write

$$\frac{p}{q} = m + \frac{p'}{q'} \tag{1}$$

Similarly, we see that p'/q' is m' left movements from its ancestor p''/q''. So we can write

$$\frac{p'}{q'} = \frac{1}{m' + \frac{1}{\frac{p''}{q''}}} \tag{2}$$

Combining (1) and (2) gives

$$\frac{p}{q} = m + \frac{1}{m' + \frac{1}{\frac{p''}{q''}}} \tag{3}$$

Then p''/q'' can be written in terms of its ancestor, and so on. We continue to make these substitutions until reaching an ancestor of the form $m^{(k)}/1$ which is $m^{(k)}$ right movements from $0/1$. So the final expression for p/q is

$$\frac{p}{q} = m + \frac{1}{m' + \frac{1}{m'' + \frac{1}{\ddots + \frac{1}{m^{(k)} + \frac{0}{1}}}}} \tag{4}$$

And $0/1$ is zero, so this simplifies to

$$\frac{p}{q} = m + \frac{1}{m' + \frac{1}{m'' + \frac{1}{\ddots + \frac{1}{m^{(k)}}}}} \tag{5}$$

The result is surprisingly simple: we can write p/q as a continued fraction whose coefficients are the numbers of consecutive left and right movements in the tree.

To demonstrate the elegance of this method, let's compute $\ell(49)$. Write 49 in binary: 110001. Splitting this number into consecutive '1's and '0's makes groups of sizes 2, 3, and 1. Then $\ell(49)$ is equal to the continued fraction with coefficients 1, 3, and 2:

$$\ell(49) \ = \ 1 + \frac{1}{3 + \frac{1}{2}} \ = \ \frac{9}{7}$$

So the 49th number in the Calkin-Wilf sequence is $9/7$. In general, we can use this method to answer questions such as,"What is the 200th number in the list?"

A keen and useful observation about this method is that it is reversible; we can write a reduced fraction p/q as a continued fraction, then use the coefficients to form groups of consecutive '1's and '0's, and then read the groups as one binary number, and convert back to decimal to obtain the n for which $\ell(n) = p/q$. This process is illustrated in Figure 5. Now we can answer questions such as, "Where does 22/7 appear in the list?"

$$n = 49 \;\to\; 110001 \;\to\; \underbrace{11}_{2}\,\underbrace{000}_{3}\,\underbrace{1}_{1} \;\to\; 1 + \cfrac{1}{3 + \frac{1}{2}} \;\to\; \ell(n) = \frac{9}{7}$$

$$\ell(n) = \frac{9}{7} \;\to\; 1 + \cfrac{1}{3 + \frac{1}{2}} \;\to\; \underbrace{11}_{2}\,\underbrace{000}_{3}\,\underbrace{1}_{1} \;\to\; 110001 \;\to\; n = 49$$

Figure 5: Reversing the method

The subtlety in the reversed process is to decide how to write p/q as a continued fraction. This can be done in many ways, so it is important that we specify which ones correspond with the path to p/q in the tree. From the construction of the continued fraction by consecutive left and right rules, we know that the leading coefficient represents right movements. And since the path begins with right movement from $0/1$, the final coefficient also represents right movements. In between, the coefficients alternate representing left and right movements. So we conclude that a continued fraction describes a path in the tree when it has an odd number of coefficients.

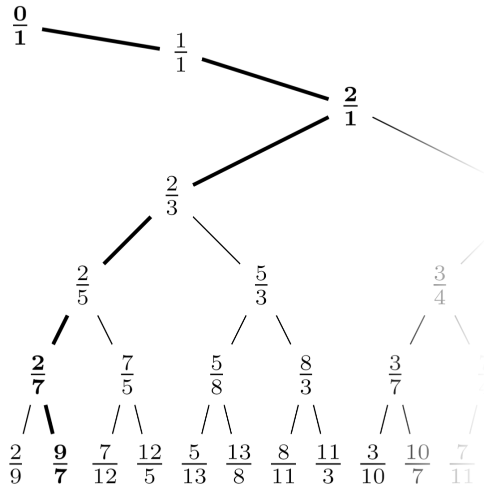
It is always possible to write a given continued fraction with an odd number of coefficients by modifying the final coefficient; if $m^{(k)} > 1$, we write this coefficient as $(m^{(k)} - 1) + 1/1$, producing a new final coefficient. And if $m^{(k)} = 1$, we write the previous coefficient as $(m^{(k-1)} + 1)$ and remove $m^{(k)}$. In either case, we change the parity of the number of coefficients.

4 Euclidean Algorithm

The Calkin-Wilf tree is closely related to the Euclidean algorithm. Recall that this algorithm finds the greatest common divisor of two numbers a and b by writing $a = q \cdot b + r$ for positive integers q and $r < b$. The process is repeated with the two numbers b and r until $r = 0$. Once this happens, the value of b is the greatest common divisor of the original two numbers.

Consider the right half of the Calkin-Wilf tree in Figure 6. The fraction $9/7$ appears in the lowest level. We perform the Euclidean algorithm on 9 and 7, which takes three iterations before $r = 0$. Notice that the values of q, 1, 3, and 2, are the numbers of left and right movements along the path in the tree, and the values of b and r are the numerators and denominators of the fractions at the turning points. Also, we see at the end of the algorithm that the greatest common divisor of 9 and 7 is 1, and we should expect that since every fraction in the tree is reduced.

The Euclidean algorithm is deterministic, which reminds us that in the Calkin-Wilf tree, there is a unique sequence of left and right movements to a particular reduced rational p/q. Now we have an algorithm which, given a reduced rational p/q, can recover the sequence of left and right movements from $0/1$ to p/q in the tree.



a	$=$	$q \cdot b$	$+$	r
9	=	$1 \cdot 7$	+	2
7	=	$3 \cdot 2$	+	1
2	=	$2 \cdot 1$	+	0

Figure 6: The Euclidean algorithm

5 The Double Tree

We know how to navigate down the Calkin-Wilf tree using left and right rules. Reasonably, we may wonder how to travel back up the tree, undoing the left and right rules. This is simple; to undo a left movement, subtract the numerator from the denominator, and to undo a right movement, subtract the denominator from the numerator.

We can use these backward movements to discover more of the tree above 0/1. A backward right movement from 0/1 gives −1/1, and from there, more backward movements reveal a copy of the Calkin-Wilf tree, upside-down, and all negative!

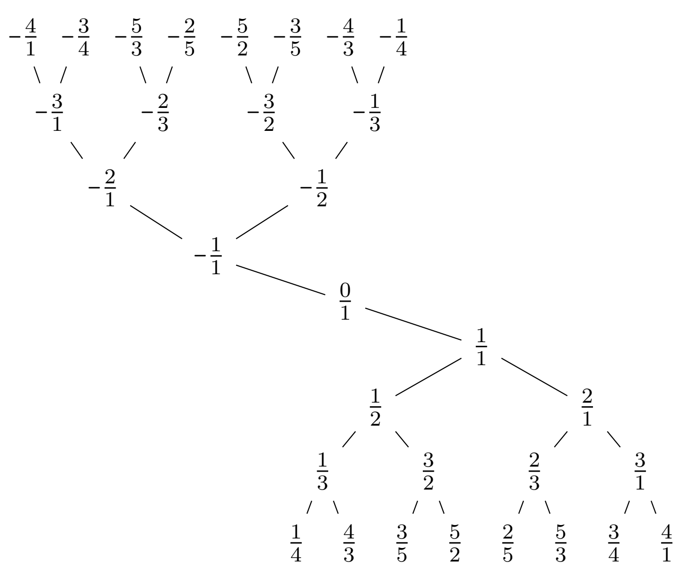


Figure 7: The double tree

Now we have a complete answer to the question, "Can you make a list of the rationals containing each number exactly once?" because this double tree includes *all* of the rationals—positive, negative, and zero. Reading the levels of this tree produces a doubly infinite sequence which answers the question.

$$\begin{array}{rcccccccccccccc} n & = & \dots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ \ell(n) & = & \dots & -\frac{3}{2} & -\frac{1}{3} & -\frac{2}{1} & -\frac{1}{2} & -\frac{1}{1} & \frac{0}{1} & \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & \frac{1}{3} & \frac{3}{2} & \dots \end{array}$$

Figure 8: The extended Calkin-Wilf sequence

By indexing the sequence as in Figure 8, the positive fractions maintain their term numbers from the Calkin-Wilf sequence. The fraction 0/1 has term number 0, representing zero movements, which justifies its role as the starting place for paths in the tree. The negative fractions have negative term numbers and are reached only by backward movements. Is there a natural way to extend the continued fraction method to represent backward paths?

Since $\ell(-n) = -\ell(n)$, we should expect that the continued fraction for $\ell(-n)$ is the negative of the continued fraction for $\ell(n)$. But multiplying a continued fraction by negative one is equivalent to making all of the coefficients negative. This gives us a way to interpret the path to a negative rational in the tree, where backward left and right movements are represented by negative coefficients in the continued fraction. For the reversed method, we use the negative coefficients to write groupings of '1's and '0's representing backward movements. As a binary number, this is the value of n for the negative term $\ell(-n)$. Now we can perform the original and reverse methods for terms in the extended sequence.

$$\ell(-49) = -\ell(49) = -\frac{9}{7} = -\left(1 + \frac{1}{3 + \frac{1}{2}}\right) = -1 + \frac{1}{-3 + \frac{1}{-2}}$$

$$\ell(-49) = -1 + \frac{1}{-3 + \frac{1}{-2}} \leftrightarrow \underbrace{-1 \mid -1}_{-2} \mid \underbrace{-0 \mid -0 \mid -0}_{-3} \mid \underbrace{-1}_{-1} \leftrightarrow -110001 \leftrightarrow n = -49$$

Figure 9: The methods for negative terms

6 The Four-way Tree

Investigating backward movements leads us to wonder what would happen if we allowed these types of movements from anywhere in the tree. That is, from every fraction p/q, we produce four other fractions by adding or subtracting the numerator from the denominator, and vice versa. Beginning with 0/1, the result of repeating this process is shown in Figure 10.

This structure contains the double tree and much more. Since every rational number appears in the double tree, it is clear that the four-way tree necessarily has duplicates. Along the antidiagonal, we see that

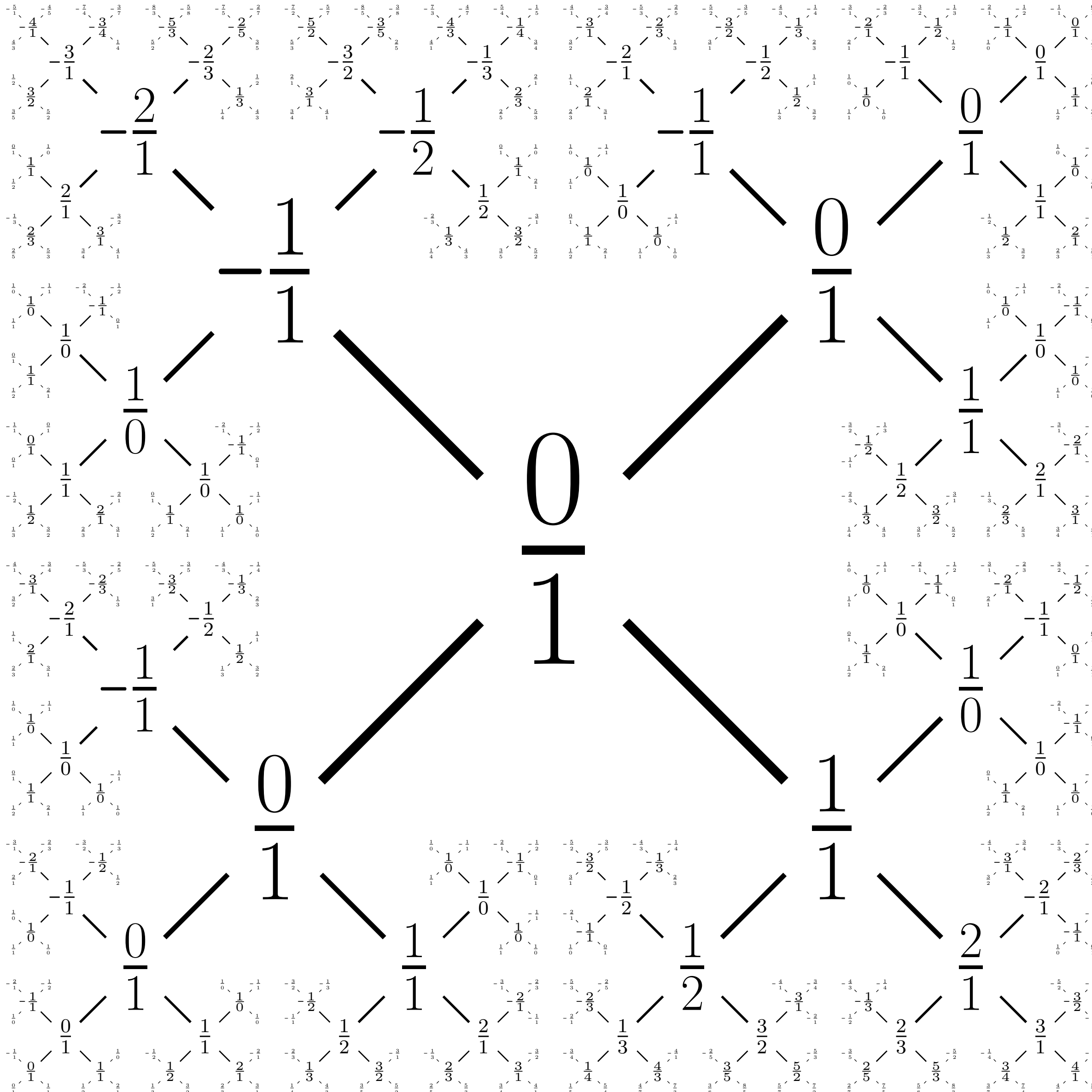
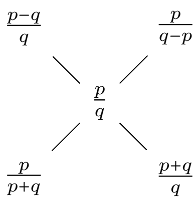


Figure 10: The four-way tree

every element is 0/1, since the left and backward left rules add and subtract zero from the denominator. Each occurrence of 0/1 produces an identical copy of the tree, so it is clear that each rational number appears infinitely often.

What is the significance of a path in this tree? We would like to relate any path from 0/1 to p/q with a continued fraction for p/q. To do this, we should follow the procedure from the original method. Just as we identified algebraic rules for making m consecutive left and right movements, we can determine rules for making m consecutive backward movements. As expected, these rules have the same structure as the rules for forward movements, and instances of m are replaced with $-m$.

Given a path from 0/1 to p/q in the four-way tree, we can write p/q in terms of its ancestors using the four rules in Figure 11. After making the substitutions, the result is a continued fraction with integer coefficients. Positive coefficients represent forward movements, and negative coefficients represent backward movements.



$$\frac{p}{q} \xleftarrow{m \text{ left}} \frac{1}{-m + \frac{1}{\frac{p}{q}}} \qquad \frac{p}{q} \xleftarrow{m \text{ right}} -m + \frac{p}{q}$$

$$\frac{p}{q} \xrightarrow{m \text{ left}} \frac{1}{m + \frac{1}{\frac{p}{q}}} \qquad \frac{p}{q} \xrightarrow{m \text{ right}} m + \frac{p}{q}$$

Figure 11: Rules for consecutive movements

So the paths in this tree represent all ways to write rational numbers as continued fractions with integer coefficients. And every such continued fraction with rational value p/q and an odd number of coefficients determines a path in the tree from 0/1 to p/q.

The only caveat to evaluating continued fractions with integer coefficients is that it is possible to arrive at division by zero. An example is given in Figure 12. For these continued fractions to be meaningful representations of rationals in the tree, we must specify a convention for addition and division with 1/0. Naturally, we define $m + 1/0$ to be 1/0 for any integer m, and the inverse $1/(1/0)$ to be 0/1, which is zero. Following these conventions, we can evaluate any continued fraction with integer coefficients, and its value will match the rational number at the end of the corresponding path in the four-way tree. For example, the continued fraction in Figure 12 represents the path which makes 1 right, 2 backward left, 1 right, 3 left, and 2 backward right movements, which is a path from 0/1 to −2/1 in the four-way tree.

$$-2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{-2 + \frac{1}{1}}}} = -2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{-1}}} = -2 + \frac{1}{3 + \frac{1}{0}} = -2 + \frac{1}{\frac{1}{0}} = -2 + 0 = -2$$

Figure 12: Arithmetic with 1/0

7 Infinite Paths

Having characterized the relationship between finite continued fractions and rational numbers, it is natural to wonder whether infinite continued fractions also have an interpretation in these trees. Since continued fractions for rational numbers are finite, it follows that infinite continued fractions cannot be rational. And worse, infinite continued fractions cannot be evaluated in the traditional sense. We must decide how to assign meaningful values to these continued fractions other than by direct computation. A reasonable approach is to consider the sequence of *convergents*. This sequence is obtained by truncating the continued fraction after the first n coefficients. This is a sequence of finite continued fractions whose values are directly computable. For example, consider the infinite continued fraction in Figure 13. The first five convergents are shown with their rational values. This sequence converges to the golden ratio φ, which is the irrational number $(1+\sqrt{5})/2$.

$$1 \quad 1+\frac{1}{1} \quad 1+\frac{1}{1+\frac{1}{1}} \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}} \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} \quad \cdots \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}$$

$$1 \quad 2 \quad 3/2 \quad 5/3 \quad 8/5 \quad \cdots \quad \varphi$$

Figure 13: Convergents of an infinite continued fraction

If we decide that infinite continued fractions are meaningful, then we can locate their values in the Calkin-Wilf tree. The coefficients determine a path which leads beyond all of the natural numbered levels of the tree, to where the elements are irrational. This part of the tree does not behave like the rational section above it. For instance, it is not true that each irrational number appears exactly once. In fact, each irrational number appears infinitely often!

We can use paths in the tree to determine the structure of these infinite continued fractions. Some paths may appear random, such as that for the circle constant π, whose continued fraction coefficients begin 3, 7, 15, 1, 292, ... Other continued fractions may have a sequence of coefficients which repeats indefinitely. These are called *periodic* continued fractions. The continued fraction for the golden ratio is periodic with a repeating sequence of '1's. Naturally, periodic coefficients create periodic paths which we can study in the infinite extension of Calkin-Wilf tree.

Begin with a finite sequence of left and right movements. Label the start of the path $x/1$, and then use the left and right rules to evaluate the other elements. Finally, set the end of the path equal to x. This forces x to have a periodic path, and we can solve for x. An example is given in Figure 14. See that x satisfies $x^2 - x - 1 = 0$, and from the quadratic formula we find that x is the golden ratio.

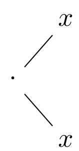
$$\frac{x}{1} \;\swarrow\; \frac{x}{x+1} \;\searrow\; \frac{2x+1}{x+1} = x$$

$$x^2 - x - 1 = 0$$

$$x = \varphi$$

Figure 14: Creating a periodic path

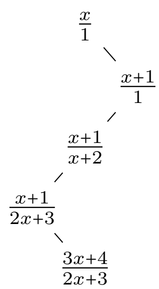
We can use this path to write x in terms of its ancestors, the second of which is x again. This gives a recursive definition of x which can be fed into itself to produce a periodic continued fraction for x. Again, we find that the continued fraction for the golden ratio is periodic of all '1's.



$$x = 1 + \cfrac{1}{1 + \frac{1}{x}} \qquad x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Figure 15: Producing a periodic continued fraction

From a periodic path, we can determine an irrational number, its minimal polynomial, and its continued fraction. In another example, we discover $x = \sqrt{2}$ with a period of '2's.

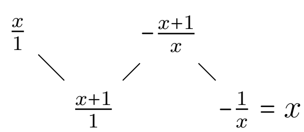


$$x^2 - 2 = 0 \qquad x = \sqrt{2}$$

$$x = 1 + \cfrac{1}{2 + \frac{1}{1+x}} \qquad x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

Figure 16: Periodic path for $\sqrt{2}$

What about periodic paths in the four-way tree? Consider the example in Figure 17 with forward and backward movements.



$$x^2 + 1 = 0 \qquad x = i$$

$$x = 1 + \cfrac{1}{-1 + \frac{1}{1+x}} \qquad x = 1 + \cfrac{1}{-1 + \cfrac{1}{2 + \cfrac{1}{-1 + \ddots}}}$$

Figure 17: Periodic path for i

To our surprise, this is a periodic continued fraction which represents a complex number! We must decide if this is meaningful. As before, we will study the convergents of this continued fraction. The first six are shown in Figure 18.

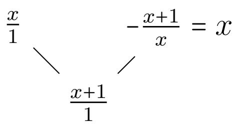
1	$1 + \frac{1}{-1}$	$1 + \cfrac{1}{-1+\frac{1}{2}}$	$1 + \cfrac{1}{-1+\cfrac{1}{2+\frac{1}{-1}}}$	$1 + \cfrac{1}{-1+\cfrac{1}{2+\cfrac{1}{-1+\frac{1}{2}}}}$	$1 + \cfrac{1}{-1+\cfrac{1}{2+\cfrac{1}{-1+\cfrac{1}{2+\frac{1}{-1}}}}}$	$\cdots$
1	0	-1	$\frac{1}{0}$	1	0	$\cdots$

Figure 18: Convergents of the continued fraction for i

The sequence of convergents oscillates between four values. This reflects that i is the fourth root of unity.

Since the convergents do not converge, our rationale supports that this continued fraction should not have a real number value. Instead, we have shown that it satisfies $x^2 + 1 = 0$ and thus represents a complex number.

Consider another example in Figure 19. We find that x satisfies $x^2 + x + 1 = 0$, which is the third cyclotomic polynomial, so x is the third root of unity. The sequence of convergents for the continued fraction oscillates between three values: 0, −1, and 1/0.



$$x^2 + x + 1 = 0$$

$$x = \omega_3$$

$$x = 0 + \cfrac{1}{-1 + \cfrac{1}{1+x}}$$

$$x = 0 + \cfrac{1}{-1 + \cfrac{1}{1 + \cfrac{1}{-1+\ddots}}}$$

Figure 19: Periodic path for ω_3

There are periodic paths which do not produce results like we have seen so far. A path of m left movements produces the equation $mx^2 = 0$, so $x = 0/1$, which is rational. A path of m right movements produces the equation $x + m = x$, which appears to have no solution when m is nonzero. However, from our definition of arithmetic with 1/0 in the previous section, we see that $x = 1/0$ is a solution. After these two results, we might think that a path is useful as a period if it is a combination of left and right movements, but that is not always true. It is possible to construct a path which, after labelling one end x and performing the sequence of movements, results the other end labelled x as well. Setting this equal to x does not produce a quadratic equation; rather, it is an identity which holds for any value of x. We call such a path an *identity path*. Two examples are given in Figure 20. The trivial identity path is made by no movements at all. It is easy to construct an identity path using the four-way tree; simply find two nodes with the same label, and choose the unique path between them. It may be that all identity paths are generated by a certain collection of sequences or patterns of movements.

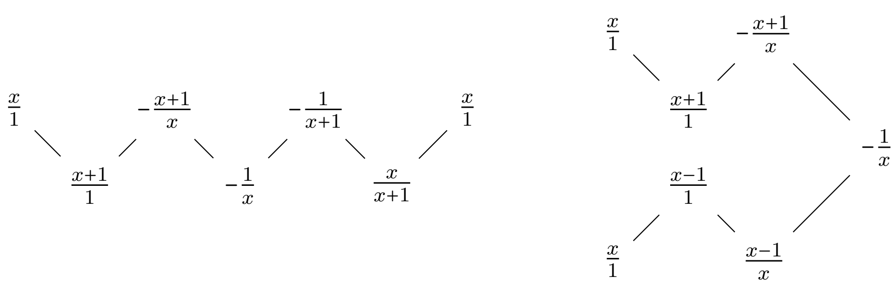


Figure 20: Identity paths

8 Square Roots and Palindromes

We have seen that certain periodic paths from x to x produce a quadratic equation. But recall that quadratic equations have two solutions. How is the second solution represented by the path? When we follow the path from one end to the other, we may choose at which end to start. Both directions produce the same quadratic equation, and each direction converges to one of the two solutions.

For example, the periodic path for φ is a left movement followed by a right movement. The reverse of this path is a backward right movement followed by a backward left movement. This second periodic path is in the upper half of the double tree, so we expect convergence to a negative number, which is $(1-\sqrt{5})/2$, the second solution to $x^2 - x - 1 = 0$.

The consequences are more exciting for square root numbers. These numbers are the solutions to quadratic equations of the form $ax^2 + c = 0$. Notably, the two solutions are negatives of each other. For example, the solutions to $x^2 - 2 = 0$ are $\sqrt{2}$ and $-\sqrt{2}$. We have seen the periodic path for $\sqrt{2}$ in Figure 16, and we found a continued fraction for $\sqrt{2}$ by writing the lower node x in terms of its ancestors above. If instead we write the upper x in terms of its ancestors, we will write a continued fraction for $-\sqrt{2}$. But recall that the coefficients of the continued fraction for $-x$ are the negatives of the coefficients for x. This is profound—following the path forward and backward produces the *same* numbers of movements. Thus, the sequence of coefficients is a palindrome! We conclude this result for all solutions to $ax^2 + c = 0$ which are irrational.

9 Matrix Representation

Periodic paths are surprising and intriguing. It is natural to wonder which irrational numbers can be found in this way. To answer the question, observe that after performing a sequence of movements from $x/1$, the result is of the form $(ax+b)/(cx+d)$, where a, b, c, and d are integers. We establish a periodic path by setting this equal to x. Solving for x produces a quadratic equation. Therefore, if x has a periodic continued fraction, then x must be a quadratic irrational number.

Is the converse true? We would like to know which quadratic integers have periodic continued fractions. There is an elegant answer when we represent the Calkin-Wilf process with matrices. Identify the fraction $(ax+b)/(cx+d)$ with the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $x = (1x+0)/(0x+1) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the identity matrix. To perform the left rule, we multiply on the left by $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and to perform the right rule, we multiply on the left by $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Multiple movements are made using powers of these matrices:

$$L^m = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)^m = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right) \qquad R^m = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^m = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$$

For example, we make 2 right, 3 left, and 1 right movements from x by the following matrix multiplication:

$$R^1L^3R^2x = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}4&9\\3&7\end{pmatrix} \equiv \frac{4x+9}{3x+7}$$

Setting $x = 0/1$, this matrix computes a term of the Calkin-Wilf sequence in the second column; we are familiar with this path from 0/1 to 9/7. But the matrix $\begin{pmatrix}4&9\\3&7\end{pmatrix}$ can produce the result of following this path from *any* x, rational or irrational.

The trees can be made relative to x with fractions or their matrix representations as shown in Figure 21.

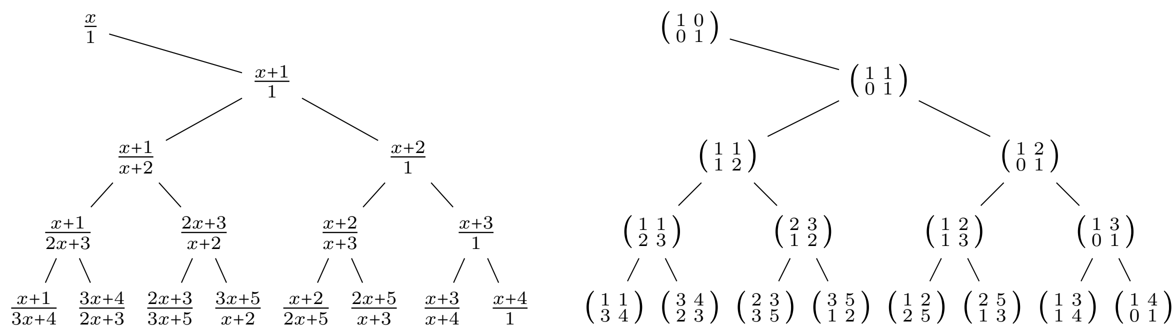


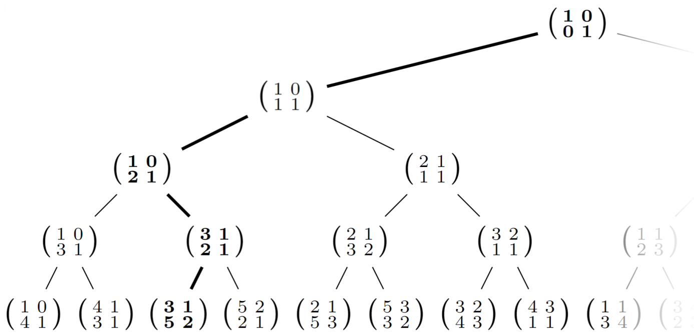
Figure 21: The relative Calkin-Wilf tree

The second column of each matrix holds the numerator and denominator of a fraction in the Calkin-Wilf tree. However, these matrices are not unique by their second column; there will be many different matrices with the same second column. Thus, the relative four-way tree has more information than the one in Figure 10 which is lost when considering only the second column of each matrix.

Previously, we used the Euclidean algorithm to reconstruct the sequence of left and right movements along the path from 0/1 to p/q in the Calkin-Wilf tree. How can we do the same for matrices in the relative tree? Since the second column $\begin{pmatrix}b\\d\end{pmatrix}$ of each matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ represents the fraction b/d in the Calkin-Wilf tree, we simply perform the Euclidean algorithm on b and d. The result is a sequence of left and right movements from 0/1 to b/d. In the matrix representation, we write $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ as a sequence of multiplications by L and R to a matrix of the form $\begin{pmatrix}x&0\\y&1\end{pmatrix}$. If we impose the restriction that this matrix has determinant 1, which will be justified below, we have that $x = 1$ and y is free. Then this matrix is $\begin{pmatrix}1&0\\y&1\end{pmatrix}$ for some positive integer y. This is L^y, so we finish the deconstruction with y left movements from $\begin{pmatrix}1&0\\0&1\end{pmatrix}$. An example is shown in Figure 22.

These matrices in the relative tree are related to the extended Euclidean algorithm and Bézout's identity, which states that there exists an integer combination of two integers which is equal to their greatest common divisor. As the entries b and d of these matrices represent numerator-denominator pairs of Calkin-Wilf tree elements, we know that $gcd(b, d) = 1$. But since these matrices have determinant 1, we also know that

$ad - bc = 1$. So each matrix represents one such integer combination of b and d satisfying Bézout's identity.



$$\frac{1}{2} = 0 + \cfrac{1}{1 + \frac{1}{1}}$$

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = R^0 L^1 R^1 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = L^1 R^1 L^2$$

Figure 22: The Euclidean algorithm for matrices

How do we represent backward movements? Naturally, the inverses of these matrices are $L^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $R^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ such that the above powers of L and R hold for all integers m. This allows us to construct a matrix representation of the four-way tree which we can use to compute a rational number at the end of a path in the original four-way tree.

Which matrices appear in this relative four-way tree? They are of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and are a product of $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and their inverses. It is well known that these two matrices generate the special linear group $SL_2(\mathbb{Z})$, the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries satisfying $ad - bc = 1$. We may quotient this space by $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since multiplying each term of the fraction $(ax + b)/(cx + d)$ by negative one does not change its value. The result of this quotient is the projective special linear group $PSL_2(\mathbb{Z})$.

What does the matrix representation reveal about periodic paths? Starting from x and fixing a sequence of movements as the period, we find that the elements along the path are of the form $(ax + b)/(cx + d)$. We identify such an element with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then to ask which fractions $(ax + b)/(cx + d)$ are reachable by left and right movements is to ask which matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are generated by L and R. We know that these are the matrices in $SL_2(\mathbb{Z})$.

Just as the double tree includes each rational number exactly once, there is a relative tree which contains each matrix in $PSL_2(\mathbb{Z})$ exactly once. Beginning with the anti-diagonal of matrices generated by L^m for integer m, we create the double tree structure from each of these matrices. Then every matrix in $PSL_2(\mathbb{Z})$ appears exactly once by the result of the Euclidean algorithm for matrices in relative trees.

We have seen how to produce an irrational number with a periodic continued fraction by setting the element at the end of a periodic path equal to x. That is, we solve $(ax + b)/(cx + d) = x$. So we conclude that x has a periodic continued fraction exactly when $x = (ax + b)/(cx + d)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nontrivial in $SL_2(\mathbb{Z})$. That is, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ cannot be $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, since this matrix is produced by an identity path. From a number x which is written in this form, we obtain the value of x using the quadratic formula.

10 Further Studies

The Calkin-Wilf tree is of current interest. Neil Calkin and Herbert S. Wilf's expository paper [3] has collected nearly 70 citations since its publication in 2000, including many in recent years. Below are brief descriptions of three areas of further study.

The Stern-Brocot Tree

Related to the Calkin-Wilf tree is the Stern-Brocot tree, another arrangement of the rational numbers [6]. Both trees are shown in Figure 23. In each tree, the root 1/1 is the left child of 0/1 and the right child of 0/1. A child in the Stern-Brocot tree is obtained by taking the mediant of its most recent left and right ancestors. That is, if these two ancestors are a/b and c/d, then the child is $(a+b)/(c+d)$. A depth-first search of the tree produces a list of rationals ordered by magnitude.

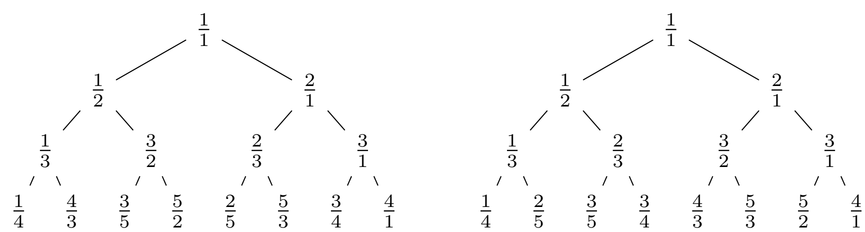


Figure 23: The Calkin-Wilf and Stern-Brocot Trees

We see that the same rational numbers appear in each level of both trees. The levels are linked [2] by a permutation which is constructed as follows: for level n of one tree, label the entries 0 to $(n-1)$ with n digit binary numbers. Then perform the bit reversal on these numbers. This establishes a permutation on the entries which transforms one tree into the other. Note that bit reversal is an order 2 action, so each permutation consists only of 2-cycles. Thus, the same permutation is applied to a level of either tree to produce the other. An example of the permutation for the third levels is shown in Figure 24.

$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\frac{3}{4}$	$\frac{4}{1}$
0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111
000	100	010	110	001	101	011	111
0	4	2	6	1	5	3	7
$\frac{1}{4}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{4}{1}$

Figure 24: Permuting the third levels

By appending a '1' to the front of each binary labelling, we obtain the term number n of each fraction in the Calkin-Wilf or Stern-Brocot sequence. From this relationship between the two trees, we can apply to the Stern-Brocot sequence what we know about evaluating and locating terms in the Calkin-Wilf sequence using continued fractions. The Stern-Brocot tree may reveal other insightful properties of the rational numbers and continued fractions by its alternative structure.

The Hyperbinary Sequence

In their original paper, Calkin and Wilf presented their sequence of rational numbers as the product of a related sequence. In particular, the numerators and denominators of rationals in the Calkin-Wilf sequence $\{1/1, 1/2, 2/1, 1/3, 3/2, 2/3, 3/1, 1/4, \dots\}$ appear to be identical with a shift. The integer sequence of numerators is $\{1, 1, 2, 1, 3, 2, 3, 1, 4, \dots\}$ and is called the *hyperbinary sequence*, denoted $b(n)$ for $n \geq 0$. Each term n counts the number of ways to write n in binary where each place value is used at most twice. For example, $b(4)$ counts the hyperbinary representations of four: 12, 20, and 100, which are $(2+1+1)$, $(2+2)$, and (4), respectively. The Calkin-Wilf sequence is produced by forming ratios of consecutive terms in this sequence. That is, $\ell(n) = b(n-1)/b(n)$ for $n \geq 1$.

The properties of the Calkin-Wilf sequence reveal some structure of the hyperbinary sequence. Since each rational in $\ell(n)$ is reduced, we have that consecutive terms in $b(n)$ are coprime. And since every possible reduced rational appears exactly once in $\ell(n)$, we have that every possible ordered pair of integers appears exactly once as a pair of consecutive terms in $b(n)$. For example, there is exactly one location in $b(n)$ where a 9 is followed by a 7.

In addition, we learn about the hyperbinary sequence from the left and right rules. If a term in the Calkin-Wilf tree has numerator $b(n)$, then its left and right children have numerators $b(2n+1)$ and $b(2n+2)$. But the left and right children of $b(n)/b(n+1)$ have numerators $b(n)$ and $(b(n)+b(n+1))$ using the left and right rules. Comparing expressions, we obtain the following recursive rules:

$$b(2n+1) = b(n)$$
$$b(2n+2) = b(n) + b(n+1)$$

Figure 25: Recursive rules for $b(n)$

Together with the initial condition $b(0) = 1$, these rules generate the entire sequence $b(n)$.

We would like to learn more about the hyperbinary sequence using the Calkin-Wilf tree, and vice versa. Perhaps there is a relationship between $b(n)$ and continued fractions, or $b(n)$ and the Stern-Brocot tree, which can be examined using the properties that have been discussed in this report.

The Action of $SL_2(\mathbb{Z})$ on the Upper-Half Plane

The special linear group $SL_2(\mathbb{Z})$, also called the *modular group*, is foundational to the study of modular forms. The *upper-half plane*, denoted $\mathcal{H}$, is the set of complex numbers with positive imaginary component. We find that the matrices in $SL_2(\mathbb{Z})$ act transitively on $\mathcal{H}$ via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$$

for complex numbers z in $\mathcal{H}$. We have already seen that $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are identity actions, so we may instead consider the quotient group $SL_2(\mathbb{Z})/\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is the projective special linear group $PSL_2(\mathbb{Z})$.

The irrational numbers with periodic continued fractions are the solutions to $x = (ax+b)/(cx+d)$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is nontrivial in $SL_2(\mathbb{Z})$. If x is in $\mathcal{H}$, then we may think of solutions to this equation as the fixed points of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acting on $\mathcal{H}$.

Notice that different matrices in $SL_2(\mathbb{Z})$ may represent have the same fixed points. For example, some of the matrices which fix $\sqrt{2}$ are $\left(\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 17 & 24 \\ 12 & 17 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 99 & 140 \\ 70 & 99 \end{smallmatrix}\right)$. These are produced by the convergents of the continued fraction for $\sqrt{2}$. These convergents are also found by tiling copies of the periodic path for $\sqrt{2}$ end-to-end, then writing a continued fraction using for x from its ancestry. The first column of each matrix represents the value of that convergent. The second columns also converge to $\sqrt{2}$. It would be convenient to find some mapping from $SL_2(\mathbb{Z})$ to itself which collapses the matrices fixing the same x onto one representative. This would partition the group into equivalence classes representing the quadratic irrationals with periodic continued fractions. Alternatively, there may be an action for which these matrices lie on the same orbit. Such a mapping or action may have other interesting properties and consequences for the structure of $SL_2(\mathbb{Z})$ or the relative Calkin-Wilf trees.

We would like to apply what we know about periodic paths and their continued fractions to the actions of $SL_2(\mathbb{Z})$ on the upper-half plane, and vice versa. These mappings have been well-documented, and the results in this area of study may reveal more about the nature of the relative Calkin-Wilf tree or periodic continued fractions. It is possible that this perspective could explain the three and four oscillating convergents of the third and fourth roots of unity. Perhaps the Calkin-Wilf tree can be used to simplify certain proofs of results in the study of modular forms.

References

[1] Harry Appelgate and Hironori Onishi. Continued fractions and the conjugacy problem in $SL_2(Z)$. *Communications in Algebra*, 9(11):1121–1130, 1981.

[2] Bruce Bates, Martin Bunder, and Keith Tognetti. Linking the Calkin–Wilf and Stern–Brocot trees. *European Journal of Combinatorics*, 31(7):1637–1661, 2010.

[3] Neil Calkin and Herbert S. Wilf. Recounting the rationals. *The American Mathematical Monthly*, 107(4):360–363, 2000.

[4] Ben Gobler. Listing the rationals using continued fractions. YouTube. https://www.youtube.com/channel/UCRIJRq80X-dOmbROAhwD1qg, 2021.

[5] Ben Gobler. Listing the rationals using continued fractions. *The Pi Mu Epsilon Journal*, 15(6):347–354, 2022.

[6] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. A foundation for computer science. Addison-Wesley, 1989.

[7] Jack E. Graver. Listing the positive rationals. *Mathematics Magazine*, 94(1):24–33, 2021.

[8] Oleg Karpenkov. *Geometry of Continued Fractions*. Algorithms and Computation in Mathematics. Springer Berlin Heidelberg, 2013.