Nonlinear Solvers For Plasticity Problems

by

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Abstract

The partial differential equation governing the problem of elastoplasticity is linear in the elastic region and nonlinear in the plastic region. In the elastic region, we encounter the problem of elasticity which is governed by the Navier Lame equations. We present a solution to the above problem through numerical schemes such as the finite element method.

In the plastic region, we encounter a nonlinear partial differential equation. This PDE is hard to solve numerically and therefore we rewrite our PDE with a penalty parameter ν. It is known that when the penalty parameter ν associated to the above PDE is zero we achieve an exact solution to the problem. This is hard to achieve from a numerical point of view however.

We will see that when we linearize the partial differential equation with Newton's method, the method fails to converge when ν is small. In this thesis, the failure of Newton's method is explained and a new method to solve the problem is proposed. The path following method will help us improve Newton's method by a better choice of the initial guess.

We obtain the convergence of this method for ν as close to zero as we want and thereby we obtain an exact solution to our original PDE.

Plots with results are presented.

ACKNOWLEDGMENTS

Plasticity, as a topic for research, was offered to me by my advisor, Professor Marcus Sarkis in April 2003. This was the beginning of my work, under his supervision, on this topic. The current thesis accounts for part of the things we discovered. I am very grateful to Professor Marcus Sarkis for helping me select an interesting and important topic, for his patience in dealing with me, for his support, encouragement, and for offering help with many useful suggestions.

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I would also like to thank my uncles, Mansour Gaval, Sharokh Gaval, Daniel Gaval, my aunt Shohreh Gaval and my grandmother Miriam Gaval for taking care of me during my studies.

Finally, I dedicate the thesis to my grandfather, Chaim Bashi Gaval.

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1 Introduction

The problems of plasticity we will encounter in this paper are in two forms.

One of the forms is plasticity with isotropic hardening/softening where the problem is defined by:
Find the displacement u, the plastic strain p (tensor) and the internal hardening variable ξ (scalar) which satisfy the equilibrium equation

$$div\ \sigma(u) + f = 0,$$

the strain-displacement relation

$$\epsilon(u) = \frac{1}{2}\,(\nabla(u) + (\nabla(u))^T),$$

the constitutive relations

$$\sigma = C\ (\epsilon(u) - p),$$

$$\alpha = -H_1\ \xi,$$

and the flow law

$$(\dot{p}, \dot{\xi}) \in K_p,$$

$$D(q,\eta) \geq D(\dot{p},\dot{\xi}) + \sigma : (q - \dot{p}) + \alpha : (\eta - \dot{\xi}) \quad \forall (q,\eta) \in K_p,$$

where $K_p = dom(D)$ and D is the dissipation function defined by

$$D(\dot{p},\dot{\xi}) = sup\{\sigma^* : \dot{p} + \alpha^* : \dot{\xi}\ : (\sigma^*, \alpha^*) \in K\}.$$

In this set of equations f is a volume force, σ is a surface force , C is the forth order elasticity tensor,H is a hardening parameter and K convex, closed, including the origin. We have that K_p is nonempty closed cone.

Later we will compute $D(\dot{p},\dot{\xi})$ explicitly and return to this version of the flow law.

The second form is that of perfect plasticity where the problem is given by:
Find the displacement u and the plastic strain p which satisfy the equilibrium equation

$$div\ \sigma(u) + f = 0,$$

the strain-displacement relation

$$\epsilon(u) = \frac{1}{2} \left(\nabla(u) + (\nabla(u))^T\right),$$

the constitutive relation

$$\sigma = C \ (\epsilon(u) - p),$$

and the flow law

$$\dot{p} \in K_p,$$

$$D(q) \geq D(\dot{p}) + \sigma : (q - \dot{p}) \quad \forall q \in K_p,$$

where $K_p = dom(D)$ and D is the dissipation function defined by

$$D(\dot{p}) = sup\{\sigma^* : \dot{p} \ : \sigma^* \in K\}$$

with the same parameters as above.

To solve these problems of plasticity numerically, we introduce the penalty parameter ν. Introducing this parameter will make it possible for us to solve the problem numerically and we will also see that the solution to the problem with the penalty parameter ν converges to the solution of the original problem.

In this thesis we will find a solution for the weak formulation of each ν problem (with specified boundary conditions) by using different linearization methods.

First we study the original problems of plasticity and arrive at the weak formulation of these problems.

2 Variational formulation of the original problem

Next, we introduce the function spaces that correspond to the variables of our problem.

2.1 Function spaces

First we define the space of displacements V

$$V = [H_0{}^1(\Omega)]^3.$$

Next, we introduce the space Q defined by

$$Q = \{q = (q_{ij})_{3\times 3} : \; q_{ji} = q_{ij}, \; q_{ij} \in L^2(\Omega)\}$$

with the usual inner product and norm of the space $[L^2(\Omega)]^{3\times 3}$.

Then the space of plastic strain Q_0 is the closed subspace of Q defined by

$$Q_0 = \{q \in Q : \; tr \; q = 0 \; \text{ a.e. in } \Omega\}.$$

The space M of internal variables is defined by

$$M = [L^2(\Omega)]^m$$

with the usual $L^2(\Omega)$-based inner product and norm (for isotropic

hardening/softening, we have $m = 1$).

Now we introduce the product space $Z = V \times Q_0 \times M$, which is a Hilbert space with the inner product

$$(w, z)_Z = (u, v)_V + (p, q)_Q + (\xi, \eta)_M$$

and the norm $\| \, z \, \|_Z = (z, z)_Z{}^{1/2}$, where $w = (u, p, \xi)$ and $z = (v, q, \eta)$.

We also introduce the product space $\bar{Z} = V \times Q_0$, which is a Hilbert space with the inner product

$$(w, z)_Z = (u, v)_V + (p, q)_Q$$

and the norm $\| \, z \, \|_{\bar{Z}} = (z, z)_{\bar{Z}}{}^{1/2}$, where $w = (u, p)$ and $z = (v, q)$.

Corresponding to the set $K_p = dom(D)$, we define

$$Z_p = \{z = (v, q, \eta) \in Z : \ (q, \eta) \in K_p \ \text{ a.e. in } \ \Omega\},$$

which is a nonempty, closed, convex cone in Z.

We also define

$$\bar{Z}_p = \{z = (v, q) \in \bar{Z} : \ q \in K_p \ \text{ a.e. in } \ \Omega\},$$

which is a nonempty, closed, convex cone in $\bar{Z}$.

2.2 Functionals and the bilinear form

For the problem of isotropic hardening/softening we introduce the bilinear form $a : Z \times Z \to R$ defined by

$$a(w, z) = \int_\Omega [C : (\epsilon(u) - p) : (\epsilon(v) - q) + \xi : H\eta] \, dx,$$

the linear functional

$$l(t) : Z \to R, \ \ < l(t), z >= \int_\Omega f(t) \cdot v \, dx$$

and the functional

$$j : Z \to R, \ \ j(z) = \int_\Omega D(q, \eta) \, dx,$$

where as before, $w = (u, p, \xi)$ and $z = (v, q, \eta)$.

For the problem of perfect plasticity we introduce the bilinear form

$$\bar{a}(w, z) = \int_\Omega [C : (\epsilon(u) - p) : (\epsilon(v) - q)] \, dx,$$

and the linear functional

$$\bar{l}(t) : \bar{Z} \to R, \ \ < \bar{l}(t), z >= \int_\Omega f(t) \cdot v \, dx$$

and the functional

$$\bar{j} : \bar{Z} \to R, \ \ \bar{j}(z) = \int_\Omega D(q) \, dx,$$

where, $w = (u, p)$ and $z = (v, q)$.

The bilinear forms $a(\cdot, \cdot)$, $\bar{a}(\cdot, \cdot)$ are symmetric as a result of the symmetry properties of C and H. From the properties of D, $j(\cdot)$ and $\bar{j}(\cdot)$ are convex, positively homogeneous, nonnegative, and lower semi continuous functionals.

2.3 The primal variational formulation

We next derive the primal variational formulation for the problem of plasticity with isotropic hardening/softening.

For simplicity, we consider the homogeneous Dirichlet boundary condition

$$u = 0 \text{ on } \Gamma.$$

We begin by integrating the flow law and using our constitutive relations to obtain

$$\int_\Omega D(q,\eta)\,dx \geq \int_\Omega D(\dot{p},\dot{\xi})\,dx + \int_\Omega [C(e(u)-p):(q-\dot{p}) - H\xi:(\eta-\dot{\xi})]\,dx \quad \forall (q,\eta)\in K_p. \tag{1}$$

Next we multiply the equilibrium equation by $v - \dot{u}$ for arbitrary $v \in V$, integrate over Ω and perform integration by parts to obtain

$$\int_\Omega C(\epsilon(u)-p):(\epsilon(v)-\epsilon(\dot{u}))\,dx = \int_\Omega f\cdot(v-\dot{u})\,dx \;\; \forall v \in V. \tag{2}$$

Now we add (1) and (2) to obtain the following variational inequality

$$a(w(t), z-\dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq < l(t), z-\dot{w}(t) >,$$

which is posed on the space Z_p.

The primal variational problem of elastoplasticity with isotropic hardening/softening is in the following form
Given $l \in H^1(0,T;Z')$, $l(0)=0$, find $w=(u,p,\xi) \; : \; [0,T] \to Z$, $w(0)=0$, such that for almost all $t \in (0,T)$, $\dot{w}(t) \in Z_p$ and

$$a(w(t), z-\dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq < l(t), z-\dot{w}(t) > \quad \forall z \in Z_p.$$

For the problem of perfect plasticity the variational problem is derived in the same way.

Once again, for simplicity we consider the homogeneous Dirichlet boundary condition

$$u = 0 \text{ on } \Gamma.$$

For this case the variational problem is in the following form

Given $l \in H^1(0,T;\bar{Z}')$, $\bar{l}(0) = 0$, find $w = (u,p) : [0,T] \to \bar{Z}$, $w(0) = 0$, such that for almost all $t \in (0,T)$, $\dot{w}(t) \in \bar{Z}_p$ and

$$\bar{a}(w(t), z - \dot{w}(t)) + \bar{j}(z) - \bar{j}(\dot{w}(t)) \geq < \bar{l}(t), z - \dot{w}(t) > \quad \forall z \in \bar{Z}_p.$$

Existence and uniqueness of these problems are proven in Han and Reddy (page 166). Here we will only state the theorem

Theorem 1. *Let H be a Hilbert space; $Z_p \subset H$ a nonempty, closed, convex cone; $a : H \times H \to R$ a bilinear form that is symmetric, bounded, and H-elliptic; $l \in H^1(0,T;H')$ with $l(0) = 0$; and $j : Z_p \to R$ nonnegative, convex, positively homogeneous, and Lipchitz continuous. Then there exists a unique solution w of our problem satisfying $w \in H^1(0,T;H)$.*

3 The yield function

In general for isotropic hardening/softening, we will let $P = (p, \xi)$ denote the generalized plastic strain (p is a second order tensor and ξ is a scalar) and $\Sigma = (\sigma, \alpha)$ denote the generalized stress (σ second order tensor and α scalar).

For perfect plasticity we have $P = p$ and $\Sigma = \sigma$.

We also let $\Pi\Sigma$ denotes the projection of Σ onto the set of admissible stresses K.

Next, we define the yield function.

We let K denote the set of admissible stresses. This set is closed, convex, contains 0 and is defined by

$$K = \{\Sigma : \Phi(\Sigma) \leq 0\},$$

where the function Φ is the yield function describing the state of the stress, i.e. elastic region characterized by $\Phi(\Sigma) < 0$ and plastic region by $\Phi(\Sigma) = 0$.

The objective is to have $\Phi(\Sigma) \leq 0$ always i.e. we want to be in the elastoplastic region.

In reality $\Phi(\Sigma) > 0$ can happen and this means that we are in the visco region.

When we are in the visco region we use the plastic flow rule to project Σ back on the elastoplastic region.

Examples of plasticity problems are those of perfect plasticity and isotropic hardening, presented in the sections below.

3.1 Perfect plasticity

Perfect plasticity is a special case of plasticity where the yield function is defined by

$$\Phi(\Sigma) = \Phi(\sigma, \alpha) = \Phi(\sigma) = \mid dev(\sigma) \mid - \sigma_y. \tag{3}$$

Together with (3) we impose the following constraint

$$\phi(\Sigma) \leq 0. \tag{4}$$

If we impose the symmetry of the stress σ and the strain q then we can write $\sigma = (\sigma_{11}, \sigma_{12}, \sigma_{12}, \sigma_{22})$ and $q = (q_{11}, q_{12}, q_{12}, q_{22})$.

The set of admissible stresses K is defined by

$$K = \{\sigma : \mid dev\sigma \mid \leq \sigma_y\} = \{\sigma : \ (\sigma_{11} - \sigma_{22})^2 + 4{\sigma_{12}}^2 \leq 2{\sigma_y}^2\}.$$

To define the set of admissible strains K_p we first write the definition for the dissipation function

$$D(q) = sup\{\sigma : q \ : \sigma \in K\}. \tag{5}$$

Then the set K_p is defined by

$$K_p = \{q : \ D(q) \text{ is finite}\},$$

but $q : \sigma = q_{11}\sigma_{11} + q_{22}\sigma_{22} + 2q_{12}\sigma_{12} = \frac{1}{2}(q_{11} - q_{22})(\sigma_{11} - \sigma_{22}) + \frac{1}{2}(q_{11} + q_{22})(\sigma_{11} + \sigma_{22}) + 2q_{12}\sigma_{12}$.

Note that $\mid \sigma_{11} - \sigma_{22} \mid$ is bounded and $\mid \sigma_{12} \mid$ is bounded (because $\sigma \in K$) and so $D(q)$ is finite if $q_{11} + q_{22} = 0$. Therefore in order for q to be in K_p, we need $q_{11} = -q_{22}$. Note that $tr(q) = 0$ if $q \in K_p$.

Consider the map M which takes $X = (x_{11}, x_{12}, x_{12}, x_{22})$ to $\hat{X} = (x_{11}, x_{22}, \sqrt{2}x_{12})$. Then, the map M preserves inner product and therefore it is an isometry. Indeed, it preserves inner product since $(x_{11}, x_{12}, x_{12}, x_{22})(y_{11}, y_{12}, y_{12}, y_{22}) = x_{11}y_{11} + x_{22}y_{22} + 2x_{12}y_{12} = (x_{11}, x_{22}, \sqrt{2}x_{12})(y_{11}, y_{22}, \sqrt{2}y_{12})$. Therefore instead of performing our analysis on X, we can perform the analysis on $\bar{X}$.

Next we analyze $\bar{K} = M(K)$, $\bar{K}_p = M(K_p)$ and $\bar{X} = M(X)$.

$$\bar{K} = \{\bar{\sigma} : \ (\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + 2\bar{\sigma}_{12}^2 \le 2{\sigma_y}^2\},$$

where $(\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12}) = (\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12})$. This is a cylinder with axis $\frac{1}{\sqrt{2}}(1, 1, 0)$ and radius σ_y. Therefore $\bar{K}_p$ is the orthogonal cross section of $\bar{K}$ passing through the origin.

Next we compute $\sigma - \Pi\sigma$ for σ outside K.

The projection Π is uniquely determined by $\tau = \Pi\sigma$, where we have

$$dist(\sigma, K) = |\sigma - \tau| = |\sigma - \Pi\sigma| = inf\{|\sigma - \tau'|\}$$

for $\tau' \in K$.

We decompose $\sigma = dev(\sigma) + \sigma - dev(\sigma)$. Then

$$M(\sigma - dev(\sigma)) = M(\frac{\sigma_{11} + \sigma_{22}}{2}, 0, 0, \frac{\sigma_{11} + \sigma_{22}}{2}) = \frac{1}{2}(\sigma_{11} + \sigma_{22}, \sigma_{11} + \sigma_{22}, 0),$$

$$M(dev(\sigma)) = M(\frac{\sigma_{11} - \sigma_{22}}{2}, \sigma_{12}, \sigma_{12}, \frac{\sigma_{22} - \sigma_{11}}{2}) = (\frac{\sigma_{11} - \sigma_{22}}{2}, \frac{\sigma_{22} - \sigma_{11}}{2}, \sqrt{2}\sigma_{12}).$$

This shows that $M(\sigma - dev(\sigma)) = \sqrt{2}\alpha(1, 1, 0)$ (axis direction) and $M(dev(\sigma)) \in \bar{K}_p$. Therefore

$$\Pi\sigma = \sigma - dev(\sigma) + \frac{\sigma_y}{\mid dev(\sigma) \mid} dev(\sigma),$$

$$\sigma - \Pi\sigma = \sigma - (\sigma - dev(\sigma) + \frac{\sigma_y}{\mid dev(\sigma) \mid} dev(\sigma)) = dev(\sigma)(1 - \frac{\sigma_y}{\mid dev(\sigma) \mid}).$$

3.2 Plasticity with isotropic hardening/softening

For plasticity with isotropic hardening/softening, the yield function has the form

$$\Phi(\sigma, \alpha) = \mid dev(\sigma) \mid - \sigma_y \ (1 + H\alpha).$$

For isotropic hardening $H > 0, \alpha \geq 0$ and σ_y is as above.

As in the case of perfect plasticity we impose (4).

If we impose the symmetry of the stress σ and the strain q then we can write $\sigma = (\sigma_{11}, \sigma_{12}, \sigma_{12}, \sigma_{22}, \alpha)$ and $q = (q_{11}, q_{12}, q_{12}, q_{22}, \eta)$.

The set of admissible stresses K is defined by

$$K = \{\Sigma : \mid dev\sigma \mid \leq \sigma_y(1 + H\alpha)\} = \{\sigma : \ (\sigma_{11} - \sigma_{22})^2 + 4{\sigma_{12}}^2 \leq 2{\sigma_y}^2(1 + H\alpha)^2\}.$$

To define the set of admissible strains K_p we first write the definition for the dissipation function

$$D(q) = sup\{\Sigma : q \ : \Sigma \in K\}. \tag{6}$$

Then the set K_p is defined by

$$K_p = \{q : \ D(q) \text{ is finite}\},$$

but this then imposes the additional constraint $q_{11} + q_{22} \leq 0$, which is equivilant to $q_{11} \leq -q_{22}$.

Consider the map M which takes $X = (x_{11}, x_{12}, x_{12}, x_{22}, \alpha)$ to $\hat{X} = (x_{11}, x_{22}, \sqrt{2}x_{12}, \alpha)$. Then, the map M preserves inner product and therefore it is an isometry. Indeed, it preserves inner product since $(x_{11}, x_{12}, x_{12}, x_{22}, \alpha)(y_{11}, y_{12}, y_{12}, y_{22}, \eta) = x_{11}y_{11} + x_{22}y_{22} + 2x_{12}y_{12} + \alpha\eta = (x_{11}, x_{22}, \sqrt{2}x_{12}, \alpha)(y_{11}, y_{22}, \sqrt{2}y_{12}, \eta)$. Therefore instead of performing our analysis on X, we can perform the analysis on $\bar{X}$.

Next we analyze $\bar{K} = M(K)$, $\bar{K}_p = M(K_p)$ and $\bar{X} = M(X)$.

$$\bar{K} = \{\bar{\sigma} : \ (\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + 2\bar{\sigma}_{12}^2 \leq 2{\sigma_y}^2(1 + H\alpha)^2\},$$

where $(\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12}, \alpha) = (\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12}, \alpha)$. This is a cone with axis $\frac{1}{\sqrt{2}}(1, 1, 0)$ and radius $\sigma_y(1 + H\alpha)$.

Next we compute $\Sigma - \Pi\Sigma$ for Σ outside K.

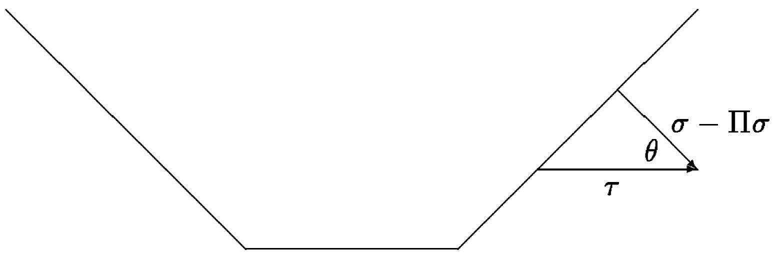
The projection Π is uniquely determined by $(\tau, \beta) = \Pi(\sigma, \alpha)$, where we have

$$dist((\sigma, \alpha), K) = |(\sigma - \tau, \alpha - \beta)| = |(\sigma, \alpha) - \Pi(\sigma, \alpha)| = inf\{|(\sigma - \tau', \alpha - \beta')|\}$$

for $(\tau', \beta') \in K$.

In order to compute $\sigma - \Pi\sigma$, we first notice that

$$\sigma - \Pi\sigma = \tau \; cos(\theta)$$



The vector τ can be computed in the same way as $\sigma - \Pi\sigma$ for the case of perfect plasticity (with the radius σ_y replaced with the radius $\sigma_y(1 + H\alpha)$). Therefore, we have:

$$\tau = dev(\sigma)(1 - \frac{\sigma_y(1 + H\alpha)}{\mid dev(\sigma) \mid}).$$

The angle θ can also be computed since we know $tan(\theta) = H\sigma_y$. Therefore

$$\sigma - \Pi\sigma = (\tau - \Pi\tau) \; cos(\theta) = \frac{1}{\sqrt{1 + H^2\sigma_y^2}} \; (1 - \frac{(1 + \alpha H)\sigma_y}{\mid dev(\sigma) \mid})dev(\sigma).$$

Now that we have $\sigma - \Pi\sigma$, we can compute $\alpha - \Pi\alpha$. Indeed $\alpha - \Pi\alpha$ is the scalar -$\mid \sigma - \Pi\sigma \mid \; tan(\theta)$ (with the minus sign indicating $\Pi\alpha > \alpha$). Therefore we obtain:

$$\alpha - \Pi\alpha = \frac{1}{\sqrt{1 + H^2\sigma_y^2}} \; (1 - \frac{(1 + \alpha H)\sigma_y}{\mid dev(\sigma) \mid})(-H\sigma_y)dev(\sigma).$$

4 Some physical assumptions

For small strain theory the basic assumption is that the energy function ψ

$$\psi := \bar{e} - \eta\theta$$

($\bar{e}$ is the internal energy of the body, η is the total entropy in the body and $\theta > 0$ is the absolute temperature) can be decomposed into an elastic part ψ_1 and a plastic part ψ_2 so that

$$\psi = \psi_1(e) + \psi_2(p).$$

We also assume that the forces acting on the body are of potential type so that we have:

$$\Sigma = -\frac{d\psi}{dP} \tag{7}$$

with the minus sign indicating that the forces due to plasticity occur in the direction opposite to the motion.

We impose a quadratic approximation to the function ψ in both the elastic region and the plastic region.

In the elastic region ψ_1 can be approximated by assuming that it is a quadratic function of the strain

$$\psi_1(e) = \frac{1}{2}\ e : Ce$$

In the plastic region ψ_2 is defined differently for every problem.

For the case of isotropic hardening, we assume that ψ_2 is a quadratic function of the internal strain

$$\psi_2(\xi) = \frac{1}{2}\ H_1\xi^2,$$

where H_1 is a positive hardening parameter.

For the case of perfect plasticity ψ_2 is not defined in the plastic region.

Equation (7) implies that in the elastic region

$$\sigma = -\frac{d\psi_1}{dp} = -\frac{d(\frac{1}{2}\ e : Ce)}{dp} = -\frac{d(\frac{1}{2}(\epsilon - p) : C(\epsilon - p))}{dp} = C(\epsilon - p) = Ce.$$

In the plastic region for the case of isotropic hardening (or softening) we have:

$$\alpha = -\frac{d\psi_2}{d\xi} = -H_1\xi$$

Next, we introduce the problem in the dual space.

5 Variational formulation of the dual problem

Next, we introduce the function spaces that correspond to the variables of our problem.

5.1 Function spaces

We define the space of displacements V as before,

$$V = [H_0{}^1(\Omega)]^3,$$

the space of stresses is defined by

$$S = \{\tau = ((\tau_{ij})_{3\times 3} : \ \tau_{ij} = \tau_{ji}, \ \ \tau_{ij} \in L^2(\Omega)\},$$

and the space of internal forces M

$$M = \{\mu = (\mu_j) : \ \mu_j \in L^2(\Omega), \ \ j = 1, ..., m\}.$$

with $m = 1$ for isotropic hardening/softening.

Next we introduce the space

$$\mathcal{T} = S \times M.$$

This space is given the inner products induces by the natural inner products on S and M.

We also define the convex subsets

$$\mathcal{P} = \{T = (\tau, \mu) \in \mathcal{T} : \ (\tau, \mu) \in K \text{ a.e. in } \Omega\},$$

$$\bar{\mathcal{P}} = \{\tau \in S : \ \tau \in K \text{ a.e. in } \Omega\}.$$

5.2 Functionals and the bilinear form

We now introduce the bilinear forms associated with the dual problem

$$a : S \times S \to R, \ \ a(\sigma, \tau) = \int_\Omega \sigma : C^{-1}\tau \; dx,$$

$$b : V \times S \rightarrow R, \;\; b(v, \tau) = -\int_\Omega \epsilon(v) : \tau \; dx,$$

$$c : M \times M \rightarrow R, \;\; c(\alpha, \mu) = \int_\Omega \alpha : H^{-1}\mu \; dx,$$

and the bilinear form

$$A : \mathcal{T} \times \mathcal{T} \rightarrow R, \;\; A(\Sigma, T) = a(\sigma, \tau) + c(\alpha, \mu)$$

for $\Sigma = (\sigma, \alpha)$ and $T = (\tau, \mu)$.

We also introduce the linear functional

$$l(t) : V \rightarrow R, \;\; < l(t), v >= -\int_\Omega f(t) \cdot v \; dx.$$

5.3 The dual variational formulation

To arrive at the dual variational problem we first state our problem in the dual space: Find the displacement u and the forces Σ that satisfy the equlibrium equation

$$div \; \sigma + f = 0,$$

and this version of the flow law

$$\Sigma : \dot{P} \geq T : \dot{P} \quad \forall T \in K.$$

The above version of the flow law is derived from the original version of the flow law by setting $(q, \eta) = 0$, using the constitutive relations and the definition of $D(\dot{P})$.

This version of the flow law can also be stated as the following constraint mininimization problem:

$$\min_{T \in K} \{-\mathcal{D}[T; \dot{P}]\},$$

where $\mathcal{D}[T; \dot{P}] = T : \dot{P}$.

For simplicity, we consider the homogeneous Dirichlet boundary condition

$$u = 0 \text{ on } \Gamma.$$

We take the problem in the dual space, integrate by parts and obtain (after some algebra) the variational problem for the case of isotropic hardening/softening:
given $l \in H^1(0,T;V')$ with $l(0) = 0$, find $(u, \Sigma) = (u, \sigma, \alpha) : [0,T] \to V \times \mathcal{P}$ with $(u(0), \Sigma(0)) = (0,0)$ such that for almost all $t \in (0,T)$,

$$b(v, \sigma(t)) = < l(t), v > \quad \forall v \in V,$$

$$A(\dot{\Sigma}(t), T - \Sigma(t)) + b(\dot{u}(t), \tau - \sigma(t)) \geq 0 \ \ \forall T = (\tau, \mu) \in \mathcal{P}.$$

For the case of perfect plasticity the variational problem is derived in the same way. For this case we have the following problem:
given $l \in H^1(0,T;V')$ with $l(0) = 0$, find $(u, \Sigma) = (u, \sigma) : [0,T] \to V \times \bar{\mathcal{P}}$ with $(u(0), \sigma(0)) = (0,0)$ such that for almost all $t \in [0,T]$,

$$b(v, \sigma(t)) = < l(t), v > \quad \forall v \in V,$$

$$a(\dot{\sigma}(t), T - \Sigma(t)) + b(\dot{u}(t), \tau - \sigma(t)) \geq 0 \ \ \forall T = (\tau) \in \bar{\mathcal{P}}.$$

The connection between the dual problem and the original problem is given in Han and Reddy (on page 181). Here we will state the result

Theorem 2. *Assume $f \in H^1$. Then $(u, P) \in H^1$ is a solution of the original problem if and only if $(u, \Sigma) \in H^1$ is a solution of the dual problem, where (u, P) and (u, Σ) are related by the constitutive equations, i.e.*

$$\sigma = C(\epsilon(u) - p),$$

$$\alpha = -H\ \xi.$$

Therefore we have the existence and uniqueness of the solution of the dual problem for both perfect plasticity and isotropic hardening.

Next we introduce the method of regularization for the dual problem (although this method can also be introduced in the original problem).

5.4 Regularization of the dual problem

The regularization of the dual problem is used from a theoretical point of view and from a numerical point of view.

From a theoretical point of view, it is used in order to show the existence of a solution to the dual problem directly. This is done in Han and Reddy (on page 196).

From a numerical point of view the regularization of the dual problem is easier to solve than the dual problem. The reason for this is that we replace the constraint inequality (the flow rule) with an unconstraint equality (the regularized version of the flow rule).

Remark 1. *For what is to follow, we will concentrate on the case of isotropic hardening/softening, although similar results can be derived for the case of perfect plasticity.*

The regularization of the dual problem is introduced in Han and Reddy on page 196 (we will use the same notation for the spaces, the functionals and the bilinear forms as for the dual problem). This version of the problem reads:
Find $(u, \Sigma) \in V \times \mathcal{T}$ such that

$$b(v, \sigma(t)) = < l(t), v > \quad \forall v \in V,$$

$$A(\dot{\Sigma}(t), T) + b(\dot{u}(t), \tau) + (J'_\nu(\Sigma), T) = 0 \;\; \forall T = (\tau, \mu) \in \mathcal{T}.$$

where $J'_\nu(\Sigma) = \dfrac{1}{\nu}(\Sigma - \Pi\Sigma)$.

We see that with the regularization method the flow rule was replaced with an equality on the whole space.

This version of the flow rule is equivalent to:

$$T : (-\dot{P}) + T : (\frac{1}{\nu}(\Sigma - \Pi\Sigma)) = 0 \;\; \forall T = (\tau, \mu) \in \mathcal{T},$$

which is equivalent to

$$\dot{P} = \frac{1}{\nu}(\Sigma - \Pi\Sigma)$$

In Han and Reddy (on page 198), it is shown that the solution to the regularized problem converges to the solution of the dual problem as the penalty parameter ν goes to 0.

Therefore in our plasticity problems we can solve the regularization of the problem (with ν very small) in order to obtain the solution to our original problem.

We return to the general problem of plasticity where the constraint inequality (flow rule) is replaced with an unconstraint equality (We will call the problem with the equality the ν problem).

The ν problem reads:
Find the displacement u and the stress Σ which satisfy the equilibrium equation

$$div\ \sigma(u) + f = 0,$$

the strain-displacement relation

$$\epsilon(u) = \frac{1}{2}\left(\nabla(u) + (\nabla(u))^T\right),$$

the constitutive relations

$$\sigma = C\ (\epsilon(u) - p),$$

$$\alpha = -H\ \xi,$$

and the flow law of the form

$$\dot{P} = \frac{1}{\nu}\,(\Sigma - \Pi\Sigma).$$

6 The flow rule

We proceed by describing the flow rule for both perfect plasticity and isotropic hardening/softening.

The general flow rule is of the form

$$\dot{P} = \frac{1}{\nu}\,(\Sigma - \Pi\Sigma)$$

The above law states that in the plastic region, the flow is in the direction perpendicular to the set of admissible stresses at a point $\Sigma \in K$.

The parameter ν in the flow rule is a penalty parameter and we would like to have $\nu = 0$ in order to obtain the exact solution to the original problem. This is hard to obtain from a numerical point of view because the nonlinearity of σ (in u) in the viscoplastic region causes our linearization methods to diverge.

Therefore we try to make ν as close to 0 as possible without causing our linearization methods to diverge.

6.1 Perfect plasticity

We next proceed by calculating the flow rule for perfect plasticity.

We restate the flow rule

$$\dot{P} = \frac{1}{\nu}(\Sigma - \Pi\Sigma). \tag{8}$$

In the absence of the internal variables and internal forces (8) becomes:

$$\dot{p} = \frac{1}{\nu}(\sigma - \Pi\sigma). \tag{9}$$

Let $K = \{\sigma \mid \Phi(\sigma) = \mid dev(\sigma) \mid -\sigma_y, \;\; \Phi(\sigma) \leq 0\}$ be the closed convex set of admissible stresses in the elastic range. Then, if we are in the viscoplastic region, we project back on the convex set K and

$$\sigma - \Pi\sigma = dev(\sigma)(1 - \frac{\sigma_y}{\mid dev(\sigma) \mid})_+. \tag{10}$$

Using (3), (8) and (10) we obtain

$$\dot{\epsilon} - C^{-1}\dot{\sigma} = \frac{1}{\nu}(1 - \frac{\sigma_y}{\mid dev(\sigma) \mid})_+ dev(\sigma).$$

We also have that in the plastic region

$$\mid dev(\sigma) \mid > \sigma_y \Leftrightarrow 1 - \frac{\sigma_y}{\mid dev(\sigma) \mid} > 0$$

.

Note that in the elastic region

$$\sigma = \Pi\sigma \Rightarrow \dot{p} = 0$$

From the analysis on the two cases above, we have

$$\dot{\epsilon} - C^{-1}\dot{\sigma} = \frac{1}{\nu}(1 - \frac{\sigma_y}{\mid dev(\sigma) \mid})_+ dev(\sigma),$$

where $(x)_+ = max\{0, x\}$.

We will later discretize the equation above and with $C^{-1}\sigma$ known from the Navier Lame equations we will obtain an expression for σ in the viscoplastic region.

6.2 Isotropic hardening

We next proceed by calculating the flow rule for isotropic hardening. We restate the flow rule

$$\dot{P} = \frac{1}{\nu}\left(\Sigma - \Pi\Sigma\right) \tag{11}$$

In this case both α and ξ are present and we use the flow rule to compute $\sigma - \Pi\sigma$ and $\alpha - \Pi\alpha$ where $\Pi\sigma$ and $\Pi\alpha$ are the first and second component (respectively) of the projection of Σ onto the cone.

For isotropic hardening the yield function Φ is given by

$$\Phi(\sigma, \alpha) = \mid dev(\sigma) \mid - \sigma_y(1 + H\alpha)$$

where H denotes the modulus of hardening ($H > 0$).

We compute the projection on the cone and use (11) to arrive at the following equations

$$\dot{\epsilon} - C^{-1}\dot{\sigma} = \frac{1}{\nu\sqrt{1 + H^2\sigma_y^2}}\left(1 - \frac{(1 + \alpha H)\ \sigma_y}{\mid dev(\sigma) \mid}\right)_+ dev(\sigma) \tag{12}$$

$$\dot{\xi} = \frac{1}{\nu\sqrt{1 + H^2\sigma_y^2}}\left(1 - \frac{(1 + \alpha H)\ \sigma_y}{\mid dev(\sigma) \mid}\right)_+ (-H\sigma_y) \mid dev(\sigma) \mid \tag{13}$$

where we have used

$$\mid dev(\sigma) \mid > \sigma_y(1 + H\alpha) \Leftrightarrow 1 > \sigma_y \frac{(1 + H\alpha)}{\mid dev(\sigma) \mid}$$

Using the fact that the forces σ, α are of potential type, i.e.

$$\alpha = -\frac{d\psi_2}{d\xi}$$

and the quadratic approximation to ψ_2 in the plastic region

$$\psi_2(\xi) = \frac{1}{2} H_1 \xi^2$$

we obtain

$$\dot{\xi} = -H_1^{-1} \dot{\alpha}$$

Therefore equations (12), (13) become:

$$\dot{\epsilon} - C^{-1}\dot{\sigma} = \frac{1}{\nu\sqrt{1+H^2\sigma_y^2}} \left(1 - \frac{(1+\alpha H)\ \sigma_y}{\mid dev(\sigma) \mid}\right)_+ dev(\sigma)$$

$$-H_1^{-1}\dot{\alpha} = \frac{1}{\nu\sqrt{1+H^2\sigma_y^2}} \left(1 - \frac{(1+\alpha H)\sigma_y}{\mid dev(\sigma) \mid}\right)_+ (-H\sigma_y) \mid dev(\sigma) \mid$$

Remark 2. *The projection that we have computed above is only correct if $\alpha \geq 0$. However, in our case we take $\alpha_0 \geq 0$. This implies $\alpha_t \geq 0$ and therefore the projection above is correct.*

We proceed by looking at the weak formulation of the equilibrium equation.

7 Weak formulation of the problem

In this section we study the weak formulation of the two models presented above.

Let us look at the original equilibrium equation with the specified boundary conditions

$$\begin{cases} \text{div } \sigma(u) + f = 0 \quad \text{on } \Omega \\ \sigma \cdot n = g \quad \text{on } \Gamma_N \\ Mu = w \quad \text{on } \Gamma_D \end{cases}$$

Where $\sigma \in L^2(\Omega; R^{d\times d}_{sym})$, $f \in L^2(\Omega; R^d)$, $g \in L^2(\Gamma_N)$, $w \in H^1(\Omega)^d$, $M \in L^\infty(\Gamma_D)^{d\times d}$

The weak formulation of the problem for isotropic hardening/softening is:
Find $u \in H^1(\Omega)^d, Mu = w$ on Γ_D such that

$$\begin{cases} \int_\Omega \sigma(u) : \epsilon(v)\ dx = \int_\Omega f\ v\ dx + \int_{\Gamma_N} g\ v\ ds \\ \epsilon(\dot{u}) - C^{-1}\dot{\sigma} = \frac{1}{\nu}\ (\sigma - \Pi\sigma) \\ \xi(\dot{\alpha}) = \frac{1}{\nu}\ (\alpha - \Pi\alpha) \end{cases} \tag{14}$$

for all $v \in H^1_D(\Omega)^d := \{v \in H^1(\Omega)^d : Mv = 0 \text{ on } \Gamma_D\}$.

The weak formulation of the problem for perfect plasticity is:
Find $u \in H^1(\Omega)^d, Mu = w$ on Γ_D such that

$$\begin{cases} \int_\Omega \sigma(u) : \epsilon(v)\ dx = \int_\Omega f\ v\ dx + \int_{\Gamma_N} g\ v\ ds \\ \epsilon(\dot{u}) - C^{-1}\dot{\sigma} = \frac{1}{\nu}\ (\sigma - \Pi\sigma) \end{cases}$$

for all $v \in H^1_D(\Omega)^d := \{v \in H^1(\Omega)^d : Mv = 0 \text{ on } \Gamma_D\}$.

8 Time discretization

We will proceed with time discretization in order to solve the above equation.

We assume that u is known at a discrete time u_0 and we want to compute u at the next discrete time u_1. To do this, u at an intermediate time u_θ is computed and then extrapolation is used to compute u_1. Thus

$$u_1 = \frac{1}{\theta}(u_\theta + (\theta - 1)u_0).$$

Therefore what we look for is u_θ that satisfies the weak formulation of the problem.

We discretize the equilibrium equations and the flow rule (in time) so that the discretized problem of isotropic hardening/softening is:
Find $u \in H^1(\Omega)^d, Mu = w$ on Γ_D such that

$$\begin{cases} \int_\Omega \sigma(u_\theta) : \epsilon(v)\ dx = \int_\Omega f_\theta\ v\ dx + \int_{\Gamma_N} g_\theta\ v\ ds \\ \frac{1}{\theta k}[\epsilon(u_\theta - u_0) - C^{-1}(\sigma_\theta - \sigma_0)] = \frac{1}{\nu}\ [\sigma_\theta - \Pi\sigma_\theta] \\ \frac{1}{\theta k}[\xi(\alpha_\theta, t_\theta) - \xi(\alpha_0, t_0)] = \frac{1}{\nu}\ [\alpha_\theta - \Pi\alpha_\theta] \end{cases} \tag{15}$$

for all $v \in H^1_D(\Omega)^d := \{v \in H^1(\Omega)^d : Mv = 0 \text{ on } \Gamma_D\}$ where $\sigma_\theta = (1-\theta)\sigma_0 + \theta\sigma_1$, $\frac{1}{2} \leq \theta \leq 1$ and $k = dt$ is the timestep in our scheme.

For the problem of perfect plasticity the problem is:

Find $u \in H^1(\Omega)^d, Mu = w$ on Γ_D such that

$$\begin{cases} \int_\Omega \sigma(u_\theta) : \epsilon(v)\, dx = \int_\Omega f_\theta\, v\, dx + \int_{\Gamma_N} g_\theta\, v\, ds \\ \frac{1}{\theta k}[\epsilon(u_\theta - u_0) - C^{-1}(\sigma_\theta - \sigma_0)] = \frac{1}{\nu}\,[\sigma_\theta - \Pi\sigma_\theta] \end{cases}$$

for all $v \in H^1_D(\Omega)^d := \{v \in H^1(\Omega)^d : Mv = 0 \text{ on } \Gamma_D\}$.

In this thesis we take $\theta = 1$ so that we use Backward Euler's method for the time discretization.

9 Analytic expression for the stress tensor

We define

$$A := \epsilon(\frac{u_\theta - u_0}{\theta k}) + C^{-1}\frac{\sigma_0}{\theta k}$$

then the discretized version of the flow rule is

$$A - C^{-1}\,\frac{\sigma_\theta}{\theta k} = \frac{1}{\nu}\,(id - \Pi)\sigma_\theta$$

Now we would like to obtain an expression for $C^{-1}\sigma_\theta$ in terms of $tr\sigma_\theta$ and $dev\sigma_\theta$ by using the Navier Lame equations.

The Navier Lame equations state that for an isotropic material we have

$$\sigma_\theta = \lambda\; tre_\theta I + 2\mu\; e_\theta$$

Therefore using the properties of trace and deviator for tensors, we obtain

$$\sigma_\theta = \gamma\; tre_\theta I + \delta\; deve_\theta, \tag{16}$$

$$e_\theta = \alpha\; tr\sigma_\theta I + \beta\; dev\sigma_\theta, \tag{17}$$

where $\delta = 2\mu$, $\gamma = \lambda + 2\mu/d$, $\beta = 1/\delta$, $\alpha = 1/(d^2\gamma)$ and d is the dimension of the problem.

Equation (17) then implies

$$C^{-1}\sigma_\theta = \frac{1}{d^2\lambda + 2d\mu}\; tr\sigma_\theta I + \frac{1}{2\mu}\; dev\sigma_\theta \tag{18}$$

Equation (18) can then be used to solve for σ_θ in both cases perfect plasticity and isotropic hardening.

Theorem 3. *For perfect viscoplasticity and perfect plasticity there exist constants C_1, C_2, C_3 such that the stress tensor is*

$$\sigma_\theta(\theta k A) = C_1 tr(\theta k A) I + (C_2 + C_3 / \mid dev \theta k A \mid) dev \theta k A.$$

For perfect viscoplasticity these constants are
$C_1 := \lambda + 2\mu/d, \quad C_2 := \nu/(\beta\nu + \theta k), \quad C_3 = \theta k \sigma_y/(\beta\nu + \theta k).$
For perfect plasticity we take the limit as $\nu \to 0$ and the constants are
$C_1 := \lambda + 2\mu/d, \qquad C_2 := 0, \qquad C_3 := \sigma_y.$
The plastic phase occurs for

$$\mid dev(\theta k A) \mid > \frac{\sigma_y}{2\mu}.$$

In the elastic phase the stress tensor is

$$\sigma_\theta(\theta k A) = C_1 \, tr(\theta k A) I + 2\mu \; dev(\theta k A).$$

Proof. For perfect plasticity the discretized version of the flow rule is

$$A - C^{-1} \frac{\sigma_\theta}{\theta k} = \frac{1}{\nu} \, (1 - \frac{\sigma_y}{\mid dev \sigma_\theta \mid})_+ dev \sigma_\theta \tag{19}$$

so that the plastic phase occurs when $(1 - \sigma_y / \mid dev \sigma_\theta \mid) > 0$.

Using $C^{-1}\sigma_\theta = \alpha \; tr\sigma_\theta I + \beta \; dev\sigma_\theta$ we obtain

$$\theta k \; dev A = (\beta + \frac{\theta k}{\nu} \, (1 - \frac{\sigma_y}{\mid dev \sigma_\theta \mid})) dev \sigma_\theta. \tag{20}$$

We solve (20) for $\mid dev\sigma_\theta \mid$ and obtain

$$\mid dev \sigma_\theta \mid = \theta k \; \frac{\nu \mid dev A \mid + \sigma_y}{\theta k + \beta \nu}. \tag{21}$$

Using (21) in (20) we have

$$dev \sigma_\theta = (C_2 + \frac{C_3}{\mid dev \theta k A \mid}) dev \theta k A.$$

Taking the trace of both sides of equation (19) and using equation (20) we obtain

$$tr \sigma_\theta = C_1 d \; tr(\theta k A).$$

This proves the first part.

The plastic phase occurs for

$$(1 - \frac{\sigma_y}{\mid dev\sigma_\theta \mid}) > 0.$$

Using (21) this is equivalent to

$$\mid dev(\theta kA) \mid > \frac{\sigma_y}{2\mu}.$$

The last statement of the theorem follows directly from the Navier Lame equations. □

Theorem 4. *For viscoplasticity and plasticity with isotropic hardening there exist constants* C_1, C_2, C_3, C_4 *such that the stress tensor is*

$$\sigma_\theta(\theta kA) = C_1 tr(\theta kA)I + (C_3/(C_2 \mid dev\theta kA \mid) + C_4/C_2)dev\theta kA.$$

For viscoplasticity with isotropic hardening these constants are
$C_1 := \lambda + 2\mu/d, \qquad C_2 := \beta\nu\sqrt{1 + H^2\sigma_y^2} + \theta k(1 + \beta H_1 H^2 \sigma_y^2),$
$C_3 := \theta k \sigma_y (1 + \alpha_0 H), \qquad C_4 := H_1 H^2 \theta k \sigma_y^2 + \nu\sqrt{1 + H^2\sigma_y^2}.$
For perfect plasticity we take the limit as $\nu \to 0$ *and the constants are*
$C_1 := \lambda + 2\mu/d, \qquad C_2 := \theta k(1 + \beta H_1 H^2 \sigma_y^2),$
$C_3 := \theta k \sigma_y (1 + \alpha_0 H), \qquad C_4 := H_1 H^2 \theta k \sigma_y^2.$
The plastic phase occurs for

$$\mid dev(\theta kA) \mid > \beta(1 + \alpha_0 H)\sigma_y.$$

In the elastic phase the stress tensor is

$$\sigma_\theta(\theta kA) = C_1\, tr(\theta kA)I + 2\mu\, dev(\theta kA).$$

Proof. The discretized version of the flow rule is

$$A - C^{-1}\frac{\sigma_\theta}{\theta k} = \frac{1}{\nu}\,\frac{1}{\sqrt{1 + H^2\sigma_y^2}}\,(1 - \frac{(1 + \alpha_\theta H)\sigma_y}{\mid dev\sigma_\theta \mid})_+ dev\sigma_\theta, \tag{22}$$

$$-H_1^{-1}\frac{\alpha_\theta - \alpha_0}{\theta k} = -\frac{1}{\nu}\,\frac{1}{\sqrt{1 + H^2\sigma_y^2}}(1 - \frac{(1 + \alpha_\theta H)\sigma_y}{\mid dev\sigma_\theta \mid})_+ H\sigma_y \mid dev\sigma_\theta \mid. \tag{23}$$

The plastic phase occurs for $1-(1+\alpha_\theta H)\sigma_y/\mid dev\sigma_\theta \mid > 0$.

Using $C^{-1}\sigma_\theta = \alpha\, tr\sigma_\theta I + \beta\, dev\sigma_\theta$ and (22) we obtain

$$\theta k\, devA = (\beta + \frac{\theta k}{\nu} \frac{1}{\sqrt{1+H^2\sigma_y^2}}(1-\frac{(1+\alpha_\theta H)\sigma_y}{\mid dev\sigma_\theta \mid}))dev\sigma_\theta. \tag{24}$$

Solving this for $\mid dev\theta kA \mid$ we have

$$\mid dev\theta kA \mid = (\beta + \frac{\theta k}{\nu} \frac{1}{\sqrt{1+H^2\sigma_y^2}}(1-\frac{(1+\alpha_\theta H)\sigma_y}{\mid dev\sigma_\theta \mid})) \mid dev\sigma_\theta \mid. \tag{25}$$

The second part of the flow rule (equation (23)) can be solved for α_θ. Inserting the result in (25), we can obtain an expression for $\mid dev\sigma_\theta \mid$.

Using the expression for $\mid dev\sigma_\theta \mid$ in (24) we have

$$dev\sigma_\theta = (\frac{C_3}{C_2 \mid dev\theta kA \mid} + \frac{C_4}{C_2})dev\theta kA. \tag{26}$$

For the spherical part of σ_θ we take the trace on both sides of equation (22) and use (18) to obtain

$$tr\sigma_\theta = C_1 d\, tr(\theta kA).$$

This proves the first part of the theorem.

The plastic phase occurs for $\mid dev\sigma_\theta \mid > (1+\alpha_\theta H)\sigma_y$.

Using (26) this is equivalent to

$$\mid dev\theta kA \mid > \beta(1+\alpha_0 H)\sigma_y.$$

The last part of the theorem follows directly from the Navier Lame equations. □

We now have an expression for σ_θ (for perfect plasticity and isotropic hardening/softening).

10 Discretization in space

We proceed by discretizing the problem in space for a given time by the finite element method.

10.1 Ritz-Galerkin finite element method

Suppose the domain Ω has a polygonal boundary Γ, we can cover $\bar{\Omega}$ by an admissible triangulation S_h.
The elements of S_h are triangles for d=2 and tetrahedrons for d=3.

Remark 3. *The notation S_h is used since h stands for a discretization parameter, and this suggests that the approximate solution will converge to the true solution of the given continuous problem as $h \to 0$.*

Let S_h be the finite element space.
We partition the given domain Ω into many subdomains (in this thesis we use triangles), and consider functions which reduce to a polynomial on each subdomain.

A partition $\mathcal{T} = \{T_1, T_2, \cdots, T_M\}$ of Ω into triangular or quadrilateral elements (closed) is called admissible provided the following properties hold

i. $\bar{\Omega} = \cup_{i=1}^{M} T_i$

ii. If $T_i \cap T_j$ consists of exactly one point, then it is a common vertex of T_i and T_j.

iii. If for $i \neq j$, $T_i \cap T_j$ consists of more than one point, then $T_i \cap T_j$ is a common edge of T_i and T_j.

For simplicity we will use the notation S for the space S_h.

10.2 The equilibrium equation

In the discrete version of the problem we replace $H^1(\Omega)$ and $H^1_D(\Omega)$ by finite dimensional subspaces S and $S_D = \{V \in S : V = 0 \text{ on } \Gamma_D\}$, respectively.

The discrete problem is: find $U_\theta \in S$, such that

$$\int_\Omega \sigma_\theta(\theta k\ A) : \epsilon(V) dx = \int_\Omega f_\theta V dx + \int_{\Gamma_N} g_\theta V ds \tag{27}$$

for all $V \in S_D$,

where A is defined by

$$A := \epsilon(\frac{u_\theta - u_0}{\theta k}) + C^{-1}\frac{\sigma_0}{\theta k}$$

Therefore, we can use Theorem (3) and Theorem (4) in order to compute $\sigma(\theta k\ A)$ for perfect plasticity and isotropic hardening.

S is a triangulation of Ω and $\bar{N}$ the set of all nodes in S, and let

$$(\phi_1, ..., \phi_{dN}) := (\phi_1 e_1, ...\phi_1 e_d, ..., \phi_N e_1, ..., \phi_N e_d)$$

be the nodal basis of the finite dimensional space S, where N is the number of nodes in the mesh.

ϕ_z above is the scalar hat function of node z in the triangulation S and we have $\phi_z(z) = 1$ and $\phi_z(y) = 0$ for all $y \in N$ with $y \neq z$. Everywhere else ϕ_z is determined by a linear interpolation.

Therefore equation (27) is equivalent to

$$F_p = \int_\Omega \sigma_\theta(\epsilon(U_\theta - U_0) + C^{-1}\sigma_0) : \epsilon(\phi_p)dx - \int_\Omega f_\theta\ \phi_p\ dx - \int_{\Gamma_N} g_\theta\ \phi_p\ ds = 0 \qquad (28)$$

for $p = 1, ..., dN$.

We can decompose F_p into a part Q_p which depends on u_θ and a part P_p which is independent of u_θ. Therefore $F_p := Q_p - P_p$ with

$$Q_p := \int_\Omega \sigma_\theta(\epsilon(U_\theta - U_0) + C^{-1}\sigma_0) : \epsilon(\phi_p)\ dx,$$

$$P_p := \int_\Omega f_\theta\ \phi_p\ dx + \int_{\Gamma_N} g_\theta\ \phi_p\ ds$$

for $T \in \mathcal{T}$.

We approximate $\int_T f_\theta\ \phi_p\ dx$ by evaluating the value of f_θ and ϕ in the center (x_S, y_S) of T. Thus

$$\int_T f\ \phi_p\ dx \approx \frac{1}{3} \mid T \mid f_k(x_S, y_S) \quad \text{with} \ \ k := mod(p-1, 2) + 1$$

Where $\mid T \mid$ denotes the area of the triangle T.

Similarly we approximate $\int_E g_\theta \ \phi_p \ ds$ by evaluating the value of g and ϕ in the center (x_M, y_M) of the edge E with length $\mid E \mid$ and obtain

$$\int_E g_\theta \ \phi_p \ ds \approx \frac{1}{2} \mid E \mid g_k(x_M, y_M) \qquad \text{with } \ k := mod(p-1, 2) + 1$$

This shows that we have an approximation to P_p.

Therefore using the decomposition of F_p and the fact that P_p does not depend on U_θ, we only need to compute P_p once at every discrete time while Q_p must be computed at every Newton and Homotopic iteration in a given discrete time.

We would like to find $\Delta U_\theta = U_\theta - U_0$ satisfying equation (28). We use different iterative methods in order to approximate ΔU_θ.

We begin with Newton-Raphson method.

10.3 Newton-Raphson method

In general Newton's method can be used to solve the following problem:

$$F(u^*) = 0,$$

where $F : R^n \to R^n$ is a continuously differentiable function.

If we define

$$DF = (\partial F_i(u)/\partial u_j) \in R^{n \times n}.$$

Then Newton's method can be implemented in the following way:

1. Given an initial u^0.
2. Until termination, let:

$$u_{k+1} = u_k - DF(u_k)^{-1} F(u_k).$$

In reality we do not use the form of Newton's method given above. The reason for this is that computing $DF(u_k)^{-1}$ is very expensive. Therefore we find a different way to implement Newton's method.

One way to do this in three steps is

1. Given an initial u^0.
2. We iterate:

Decide whether to stop or continue. Solve

$$DF(u_k)s_k = -F(u_k).$$

3. Update

$$u_{k+1} = u_k + s_k.$$

ϵ and itmax are given parameters, where itmax denotes the maximum number of iterations.

We terminate the method if $\| F(u_k) \|_2 < \epsilon \| F(u_0) \|_2$ or if the number of iterations exceed itmax.

When using Newton's method we have the following convergence result

Theorem 5. *Suppose F is lipschitz continuously differentiable at u^* and that $F(u^*) = 0$ and $DF(u^*)$ is nonsingular. Then for u_0 sufficiently near u^*, $\{u^k\}$ produced by Newton's method is well-defined and converges to u^* with*

$$\| u^{k+1} - u^* \| \leq C \| u^k - u^* \|^2$$

for a constant C independent of k.

We can see that although Newton's method is quadratically convergent, the convergence is only local.
The iterates may diverge if u^0 is not near a solution. This causes a problem when we try to implement Newton's method for the problem of plasticity. The reason for this is that we do not know how to choose u^0 sufficiently near a solution.

10.3.1 Isotropic hardening

We next, look at the performance of Newton's method for the case of isotropic hardening/softening and perfect plasticity (for $d = 2$ throughout).

We note that since P_p is independent of U_θ, the only things that we need to compute at every newton iteration are Q and DF (defined above).

We have an analytic expression for σ from Theorem 4 and we can compute DF and Q.

$$\begin{cases} (DF)_{ps} = \mid T \mid (C_1\, tr(\epsilon(\phi_p))tr(\epsilon(\phi_s)) + C_7\, dev(\epsilon(\phi_p)) : \epsilon(\phi_s) - (C_8)_p\, dev(\bar{A}) : \epsilon(\phi_s)) \\ Q_p = \mid T \mid (C_1\, tr(\bar{A})tr(\epsilon(\phi_p)) + C_7\, dev(\bar{A}) : \epsilon(\phi_p)) \end{cases}$$

where C_1, C_2, C_3, C_4 are the same as in Theorem 4 (with $d = 2$) and we have defined

$$\bar{A} := \epsilon(u_\theta - u_0) + C^{-1}\sigma_0, \quad C_5 := 2\mu, \quad C_6 := \frac{1}{2\mu}(1 + \alpha_0 H)\sigma_y,$$

$$C_7 := \begin{cases} C_3/(C_2 \mid dev(\bar{A}) \mid) + C_4/C_2 & \text{if} \quad dev(\bar{A}) - C_6 > 0, \\ C_5 & \text{else}, \end{cases}$$

and

$$(C_8)_p := \begin{cases} C_3/(C2 | \, dev(\bar{A}) \mid^3) dev(\epsilon(\phi_p)) : dev(\bar{A}) & \text{if} \quad \mid dev(\bar{A}) \mid - C_6 > 0, \\ [0\ 0\ 0\ 0\ 0\ 0]^T & \text{else}. \end{cases}$$

We try to implement Newton's method with the following parameters

$$t = (0, .4), \quad dt = .05, \quad \sigma_y = .1 \text{ and } \ H = 150.$$

We use the initial guess $U^0 = \Delta U^0 = 0$ at every discrete time t_n.

We remember that the objective is to have $\nu \approx 0$ in order to obtain an exact solution to the problem.
When we use $\nu = .006$ the method converges at every time step (see Table 1).

From Table 1 we can observe that although Newton's method converges, the convergence is very slow. It takes the method more than 90 Newton iterations to converge for a time scale $t \geq .15$

Next we try to obtain better accuracy by using $\nu = .003$ in our problem.

With this value of ν the method fails to converge within 100 iterations at $t = .15$ (see Table 1). Therefore we need a better method in order to be able to reduce ν further and obtain a more accurate solution.

Newton's method also fails to converge for the case of perfect plasticity.

10.3.2 Perfect plasticity

For the case of perfect plasticity we use Theorem 3 to obtain an analytic expression for σ which can then be used to obtain an expression for DF and Q

$$\begin{cases} (DF)_{ps} = \mid T \mid (C_1\ tr(\epsilon(\phi_p))tr(\epsilon(\phi_s)) + C_5\ dev(\epsilon(\phi_p)) : \epsilon(\phi_s) - (C_6)_p\ dev(\bar{A}) : \epsilon(\phi_s)) \\ Q_p = \mid T \mid (C_1\ tr(\bar{A})tr(\epsilon(\phi_p)) + C_5\ dev(\bar{A}) : \epsilon(\phi_p)) \end{cases}$$

Where C_1, C_2, C_3 are the same as in Theorem 3 (with $d = 2$) and we have defined

$$\bar{A} := \epsilon(u_\theta - u_0) + C^{-1}\sigma_0, \quad C_4 := 2\mu,$$

$$C_5 := \begin{cases} C_2 + C_3/ \mid dev(\bar{A}) \mid & \text{if} \quad dev(\bar{A}) - \sigma_y/(2\mu) > 0, \\ 2\mu & \text{else,} \end{cases}$$

and

$$(C_6) := \begin{cases} C_3/\mid dev(\bar{A}) \mid^3 [dev(\epsilon(\phi_p)) : dev(\bar{A})]_{j=1}^6 & \text{if} \quad \mid dev(\bar{A}) \mid -\sigma_y/(2\mu) > 0, \\ [0\ 0\ 0\ 0\ 0\ 0]^T & \text{else.} \end{cases}$$

We try to implement Newton's method to our problem with the parameters $\sigma_y = 100$ $dt = .025$ and $t = (0, .15)$ and the initial guess $U^0 = U(t_n)$ at every discrete time t_n.

When we use $\nu = 256$ the method converges at every discrete time (see Table 2).

For $\nu = 128$ the method fails to converge (at t=.15) within 100 iterations (see Table 2).

Therefore in this case we also need a better method in order to be able to reduce ν and obtain a more accurate solution.

We have seen that for perfect plasticity and isotropic hardening/softening Newton's method does not give a solution to the problem with the desired accuracy. The reason is that when we reduce ν to obtain better accuracy our method diverges.

We would like to modify Newton's method so that we can decrease the value of ν to obtain a more accurate solution to the problem without having the method diverge. We would also like to decrease the number of Newton Iterations needed in order to obtain the solution.

We next propose a globally convergent Modification of Newton's method with backtracking [H.F. Walker].

10.4 Modification of Newton's method with backtracking

We use Newton's method with an additional constraint. The constraint is that the reduction of F is not less than the reduction predicted by the local linear model. If this constraint is not satisfied, we decrease our step length until the constraint is satisfied.

The method works in the following way:
Given $u_0,\ t \in (0,1),\ 0 < \theta_{min} < \theta_{max} < 1$, iterate:
Solve (for s) in $DF(u_k)s_k = -F(u_k)$
Let $u_{k+1} = u_k + s_k$
while $\mid ared \mid < t \mid pred \mid$
choose $\theta \in [\theta_{min}, \theta_{max}]$
set $s_k = \theta_k s_k$ and $u_{k+1} = u_k + s_k$, where $ared$ and $pred$ are defined as follows:
actual reduction:=ared:=$\parallel (F(u_k)) \parallel - \parallel (F(u_k + s_k)) \parallel$,
predicted reduction:=pred:=$\parallel (F(u_k)) \parallel - \parallel (F(u_k) + DF(u_k)s_k) \parallel$.
We see that the predicted reduction is the reduction predicted by the local linear model of F at u.

In practice we usually choose $\theta_{min} = .1,\ \theta_{max} = .8$ and $\theta = .5$.
The parameter t is chosen to be very small so that the actual reduction needs to be less than a small fraction of the predicted reduction in order for us to accept the Newton step.

We terminate the Newton iterations if $\parallel F(u_k) \parallel < 10^{-6} \quad \parallel F(u_0 \parallel$ or if the number of iterations exceed itmax.

We next look at the performance of the Modified Newton's method for the case of isotropic hardening/softening and perfect plasticity (with $d = 2$).

10.4.1 Isotropic hardening

For isotropic hardening/softening DF and Q are the same as in Newton's method.

$$\begin{cases} (DF)_{ps} = \mid T \mid (C_1\ tr(\epsilon(\phi_p))tr(\epsilon(\phi_s)) + C_7\ dev(\epsilon(\phi_p)) : \epsilon(\phi_s) - (C_8)_p\ dev(\bar{A}) : \epsilon(\phi_s)) \\ Q_p = \mid T \mid (C_1\ tr(\bar{A})tr(\epsilon(\phi_p)) + C_7\ dev(\bar{A}) : \epsilon(\phi_p)). \end{cases}$$

with the same parameters as in Newton's method.

We try to implement the modified method with the parameters $t = (0, .4)$, $dt = .05$, $\sigma_y = .1$, $H = 150$ and the initial guess $U^0 = U(t_n)$ at every discrete time t_n.

We remember that the objective is to have $\nu \approx 0$ in order to obtain an exact solution to the problem. We observe that when we use $\nu = .003$ Modified Newton's method converges at every time step (see Table 1). This shows us that for this problem our modified method is better than Newton's method because it converges with $\nu = .003$ where Newton's method diverges.

We implement our method with $\nu = .003/2$ and we see that the method fails to converge within 100 iterations at $t = .15$ (see Table 1). The modified iterations (modifying s) do converge, but the Newton iterations do not converge.

We note that although this method reduced ν by half, it still does not provide us with a sufficiently accurate solution to the original problem.

Therefore we still need to modify our method in order to obtain a solution with the desired accuracy.

Modified Newton's method also performs better than the original Newton's method for the case of perfect plasticity ($d = 2$).

10.4.2 Perfect plasticity

In the case of perfect plasticity DF and Q are the same as in Newton's method

$$\begin{cases} (DF)_{ps} = \mid T \mid (C_1\ tr(\epsilon(\phi_p))tr(\epsilon(\phi_s)) + C_5\ dev(\epsilon(\phi_p)) : \epsilon(\phi_s) - (C_6)_p dev(\bar{A}) : \epsilon(\phi_s)) \\ Q_p = \mid T \mid (C_1\ tr(\bar{A})tr(\epsilon(\phi_p)) + C_5\ dev(\bar{A}) : \epsilon(\phi_p)) \end{cases}$$

With the same parameters as in Newton's method.

We try to implement Modified Newton's method to our problem with the parameters $\sigma_y = 100$, $dt = .025$ and $t = (0, .15)$ and the initial guess $U^0 = U(t_n)$ at every discrete time t_n.

We see that our method converges with $\nu = 256$ (see Table 2).

We implement our method with $\nu = 128$ and observe that our method fails to converge within 100 iterations at $t = .15$ (see Table 2). The modified iterations (modifying s) do converge, but the Newton iterations do not converge.

In this case the modification of Newton's method did not improve Newton's method at all.

We have seen that for perfect plasticity and isotropic hardening Newton's method does not give solution to the problem with the desired accuracy and therefore we tried to obtain a more accurate solution by using modified Newton's method.

Although the Modified Newton's method performs better than Newton's method for plasticity with isotropic hardening, it does not give a solution with the desired accuracy to the problem and so we need to find a better way to improve this method.

Another problem with the Modified Newton's method is that it did not decrease the number of Newton iterations needed to obtain the solution.

We would like our next method to decrease the number of Newton iterations needed to obtain the solution.

10.5 Path following method

We proceed with the next modification.

We would like to decrease ν and when doing this prevent our method from diverging. To do this we refer back to Theorem 6 and see that Newton's method is quadraticly convergent as long as we choose the initial guess close to the solution. We would like to maintain the local quadratic convergence.

Therefore we first solve the problem with Modified Newton's method with a large value of ν, say ν_1 and the method does not diverge. This then gives us an approximation to the accurate solution associated to the problem with $\nu_2 = k\dot{\nu}_1,\ k < 1$.

Then, the solution of the problem with ν_1 will provide a good initial guess to the problem with ν_2, where $\nu_2 < \nu_1$ so that the Modified Newton's method will not diverge.

When we find the solution to the problem with ν_2, we can then decrease ν further in a similar way.

We do this by using the solution to the problem with ν_2 as the initial guess for the problem with ν_3, where $\nu_3 < \nu_2$.

Continuing in this way we generate a sequence $\{\nu_i\}$. This sequence then generates $\{U_{\nu_i}\}$, a sequence of solutions to the problem. Each solution depends on the value of ν and since we chooose $\nu_{i+1} < \nu_i$ we will obtain a more accurate solution to our problem at every ν iteration.

We terminate the Modified Newton iterations for each ν_i problem

if $\| F({u^k}_{\nu_i}) \| < 10^{-6} \ \| F({u^0}_{\nu_0}) \|$ or if the number of iterations exceed itmax.

When implementing the path following method we would like to decrease the value of ν as much as possible at every ν iteration so that we will obtain the desired accuracy to the problem in as few ν iterations as possible.

The problem is that if we try to decrease ν too much in a single ν iteration the path following method might diverge. The reason for this is that the initial guess being the solution to the problem with ν_i might not be a close enough approximation to the solution of the problem ν_{i+1} and therefore Modified dNewton's method will diverge.

In practice we usually choose $\nu_{i+1} = k \cdot \nu_i$, where $k < 1$ is a parameter that depends on the problem.

We next examine the performance of the path following method for the case of isotropic hardening/softening and for the case of perfect plasticity.

10.5.1 Isotropic hardening/softening

We try to implement our path following method with the parameters
$t = (0, .4)$, $dt = .05$, $\sigma_y = .1, H = 150$ and the initial guess $U^0 = U(t_n)$ at every discrete time t_n.

When we use $\nu = .003$ Modified Newton's method converges at every time step (see Table 1). This then gives us $\nu_0 = .003$ which we can use to generate our sequence ν_i.

We choose $\nu_{i+1} = k \cdot \nu_i$, $k < 1$.

First we implement the method with $k = 1/2$ and observe that our method then converges at every ν iteration and at every time step.

We then try different values of k and observe that the method converges even with a very small value of k.

We can choose $k = 1/500$ and the method will still converge.

We can also decrease the number of Newton Iterations by simply choosing ν_0 to be a larger value. In fact if we choose $\nu_0 = 15$ instead of $\nu_0 = .003$ our method will converge with very few Newton iterations (see Table 1).

Table 1 shows that the method converges after only five ν iterations at every timestep.

Next we note that the path following method works very well for isotropic hardening even when the mesh is large. See Table 3 for the mesh shown in Figure 5, where we have 2947 coordinates and 5680 elements.

Our path following method also gives us the desired accuracy for the case of perfect plasticity.

10.5.2 Perfect plasticity

We try to implement the path following method to our problem with the parameters
$\sigma_y = 100$, $dt = .025$, $t = (0, .15)$ and the initial guess $U^0 = U(t_n)$ for every discrete time t_n.

We see that the modified Newton's converges with $\nu = 128$ (Table 2). This then gives us

$\nu_0 = 128$ which we can use to generate ν_{i+1}.

We choose $\nu_{i+1} = k \cdot \nu_i$, $k < 1$.

We begin by implementing the method with $k = 1/2$ and observe that our method converges at every ν iteration and at every time step.

We then try different values of k and observe that the method converges even with $k = 1/5000$.

To decrease the number of Newton iterations we choose $\nu_0 = 256$ and the method converges with very few Newton iterations (see Table 2).

Table 2 shows that the method converges after only three ν iterations at every timestep.

Table 4 shows that the method converges even when the problem involves a mesh is large (Figure 6 for the mesh, where we have 5680 elements and 2947 coordinates).

The last thing we want to do is to optimize the program.

Since certain things depend neither on ν nor the initial guess u^0, they do not have to be computed at every ν iteration. Indeed if we write $F = Q - P$ where

$$Q = \int_\Omega \sigma(u) : \epsilon(\phi)$$

$$P = \int_\Omega f\ \phi - \int_{\Gamma_N} g\ \phi$$

then we do not need to compute P at every ν iteration. The only things that we compute in a specified timestep after every ν iteration are Q and DF.

11 Concluding remarks

We have seen that the problem of plasticity involves a nonlinear partial differential equation.

We have tried to linearize this equation with Newton's method and observed that the method fails to converge when ν is small and therefore we did not obtain the desired accuracy to the problem.

We then tried to impose a globally convergent modification to Newton's method with backtracking.

We have seen that this method converges with ν smaller than that given by Newton's method. However this method still did not provide us with the desired accuracy and when we tried to lower ν further the method failed to converge.

Next we tried to use a path following method.

We observed that this method works very well for the problem of plasticity. It works well for isotropic hardening and for perfect plasticity. The method converges even when ν is very small.

Therefore we obtain a solution to our problem with the desired accuracy.

The path following method is computationally expenssive however. The reason for this is that we have to go through an extra ν iteration in order to obtain the solution.

Although this causes the program to be slower, we have lessened this effect by optimizing our program. We have also used the extra ν iteration to lower the number of Newton iterations that we need in order to obtain the solution.

Therefore, for our problems it has caused the program to be overall faster than the original Newton's method.

12 References

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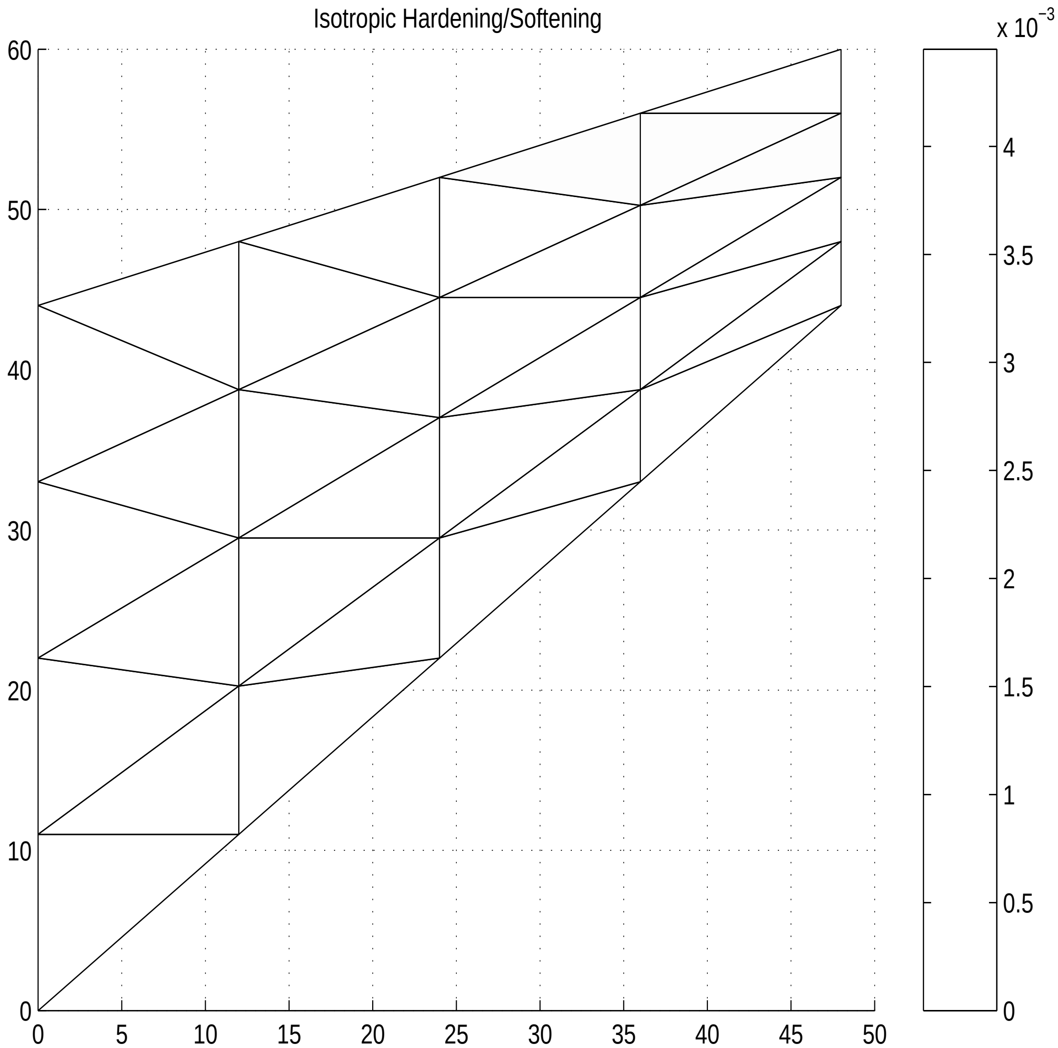

Figure 1: The initial mesh at time t=0, for isotropic hardening

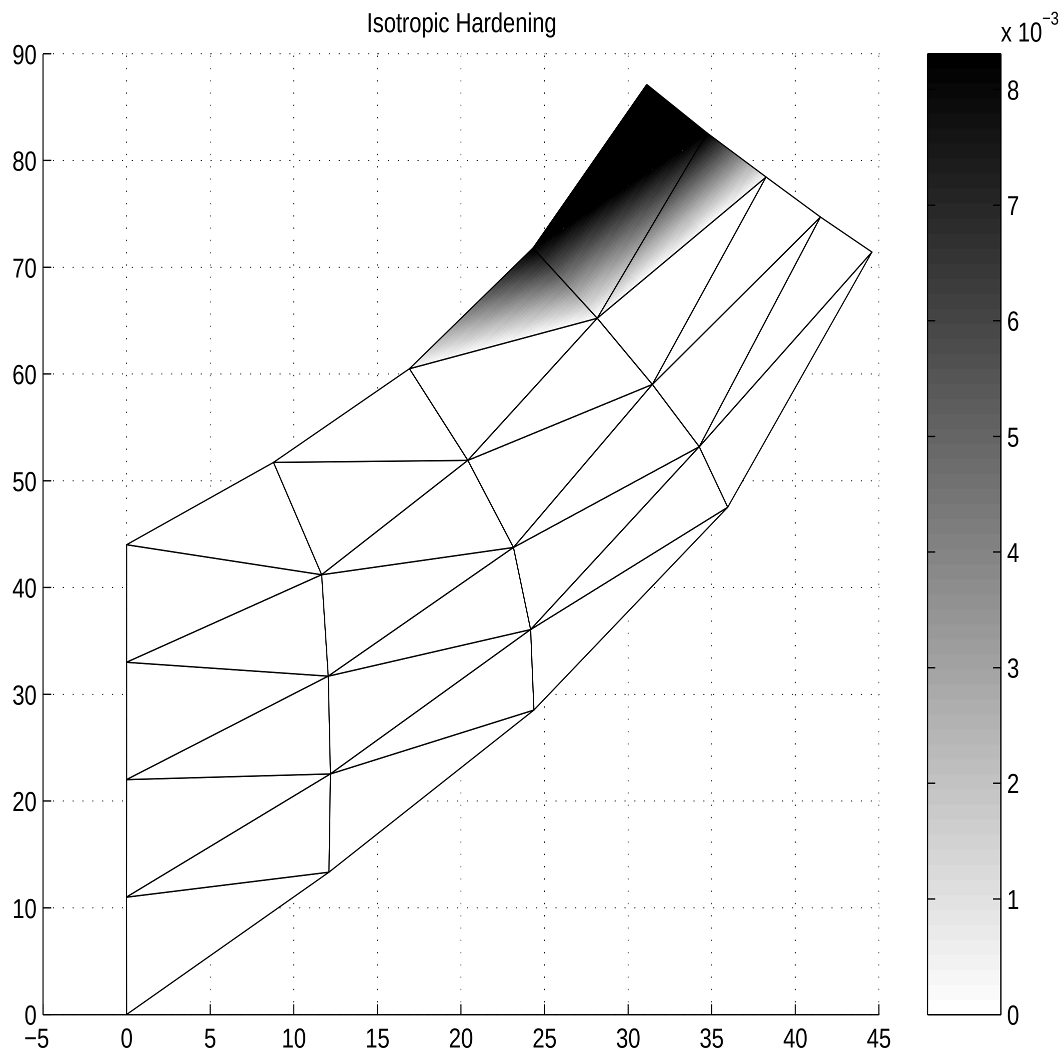

Figure 2: Isotropic hardening at final time t=.4, using the path following method

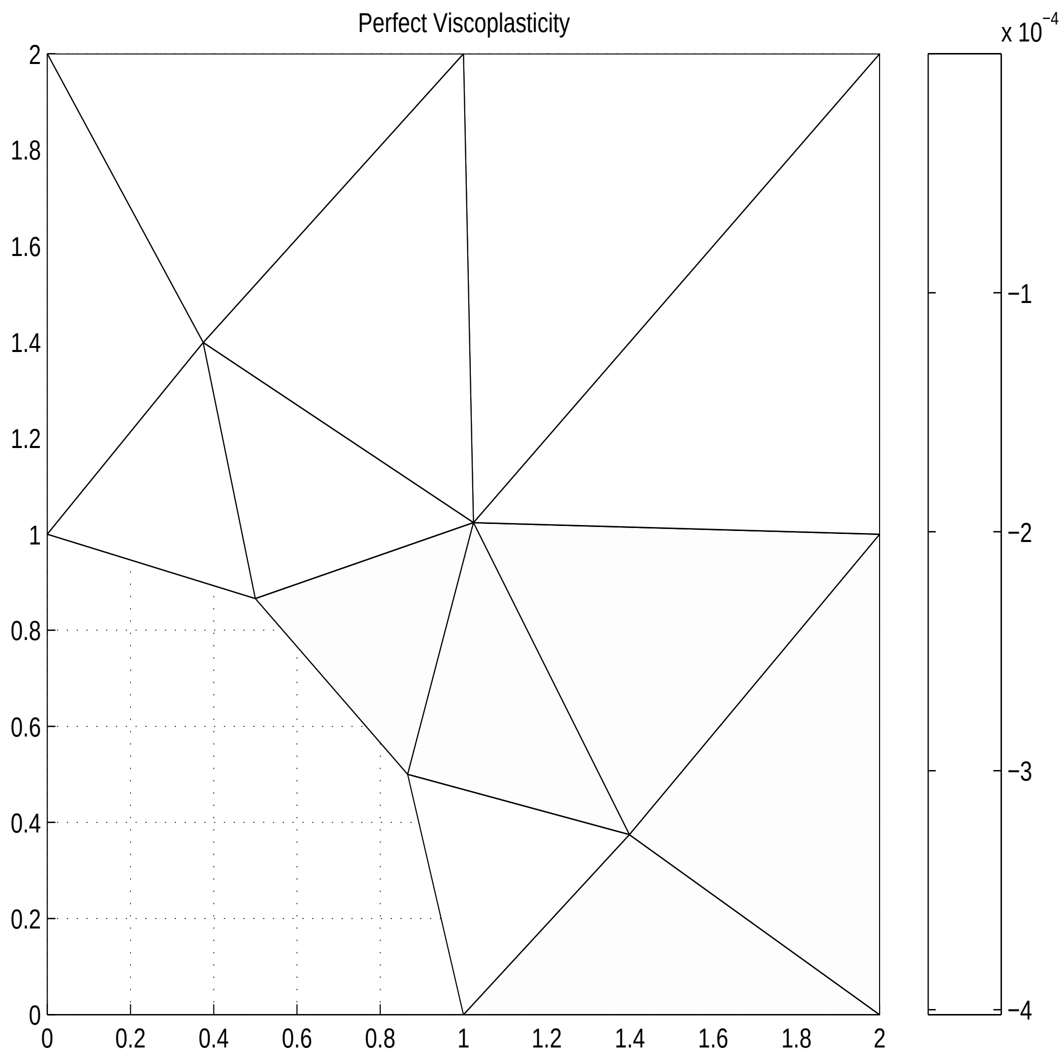

Figure 3: The initial mesh at time t=0, for perfect plasticity

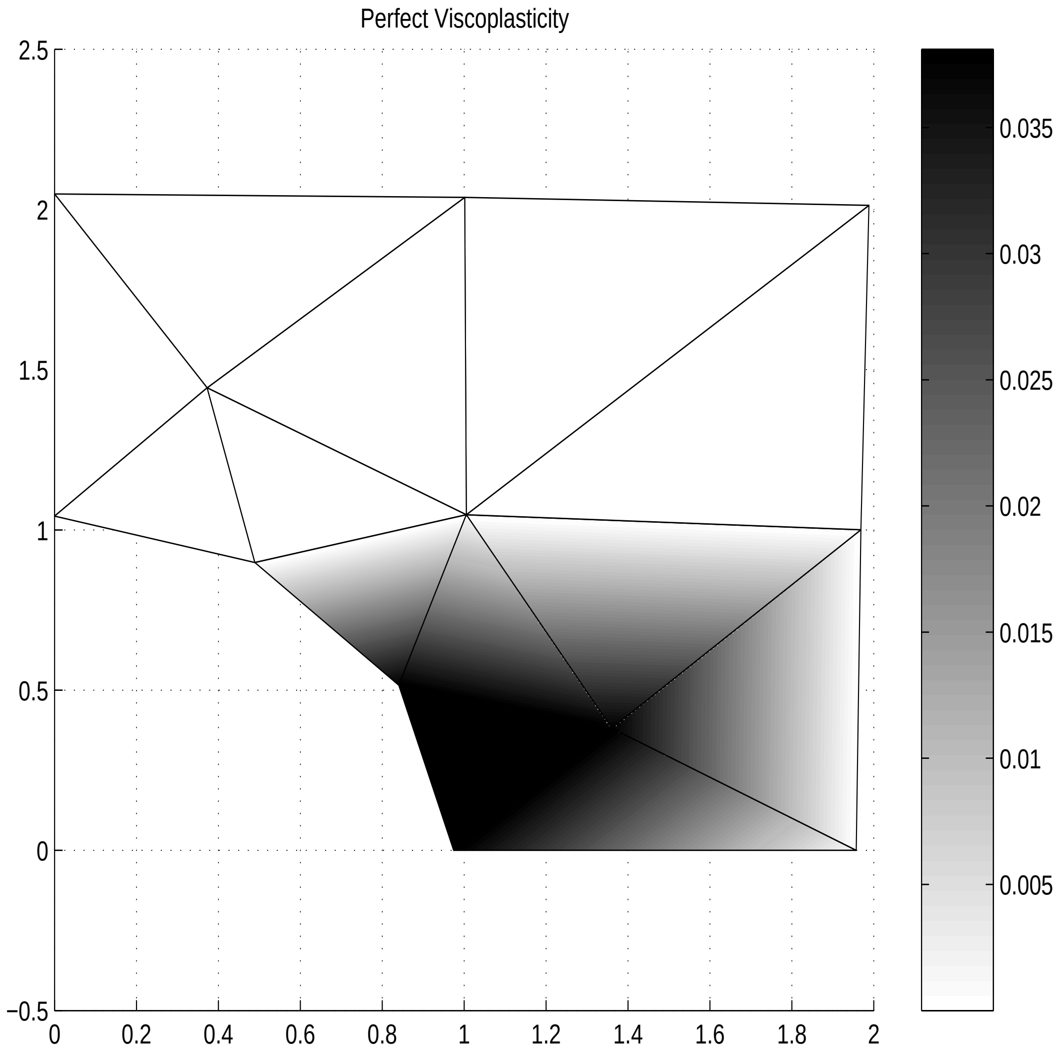

Figure 4: Perfect Plasticity at final time t=.15, using the path following method

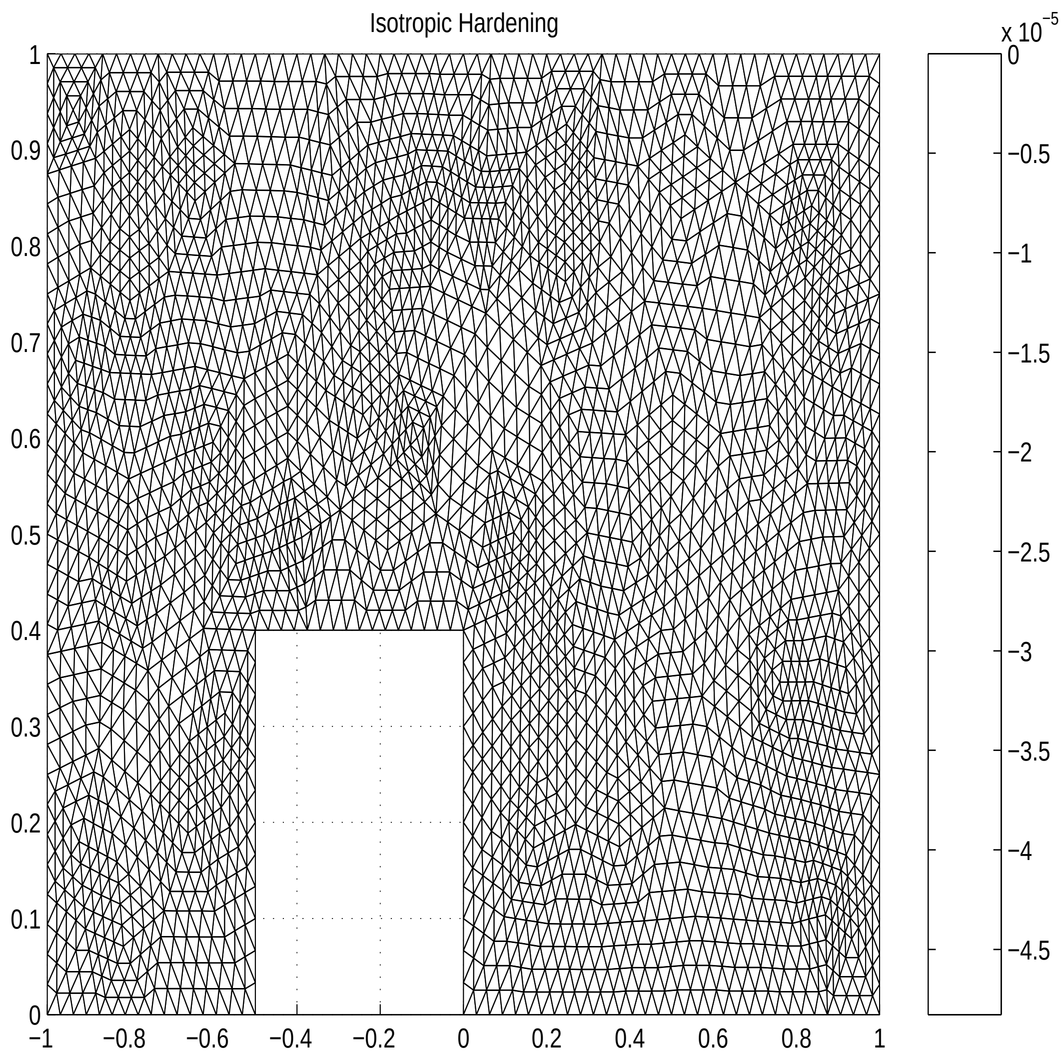

Figure 5: The larger mesh at time t=0, for isotropic hardening and perfect plasticity

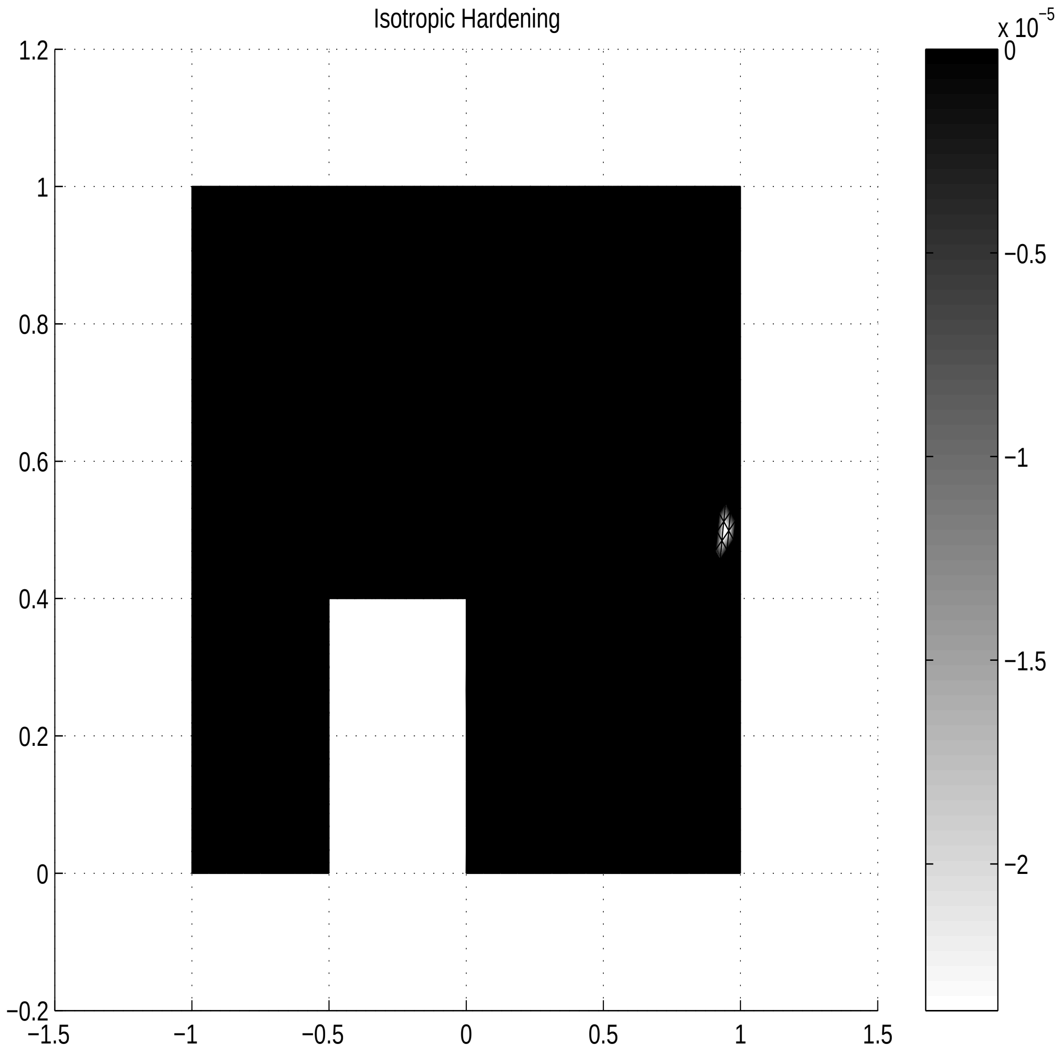

Figure 6: The larger mesh at time t=.4, for isotropic hardening

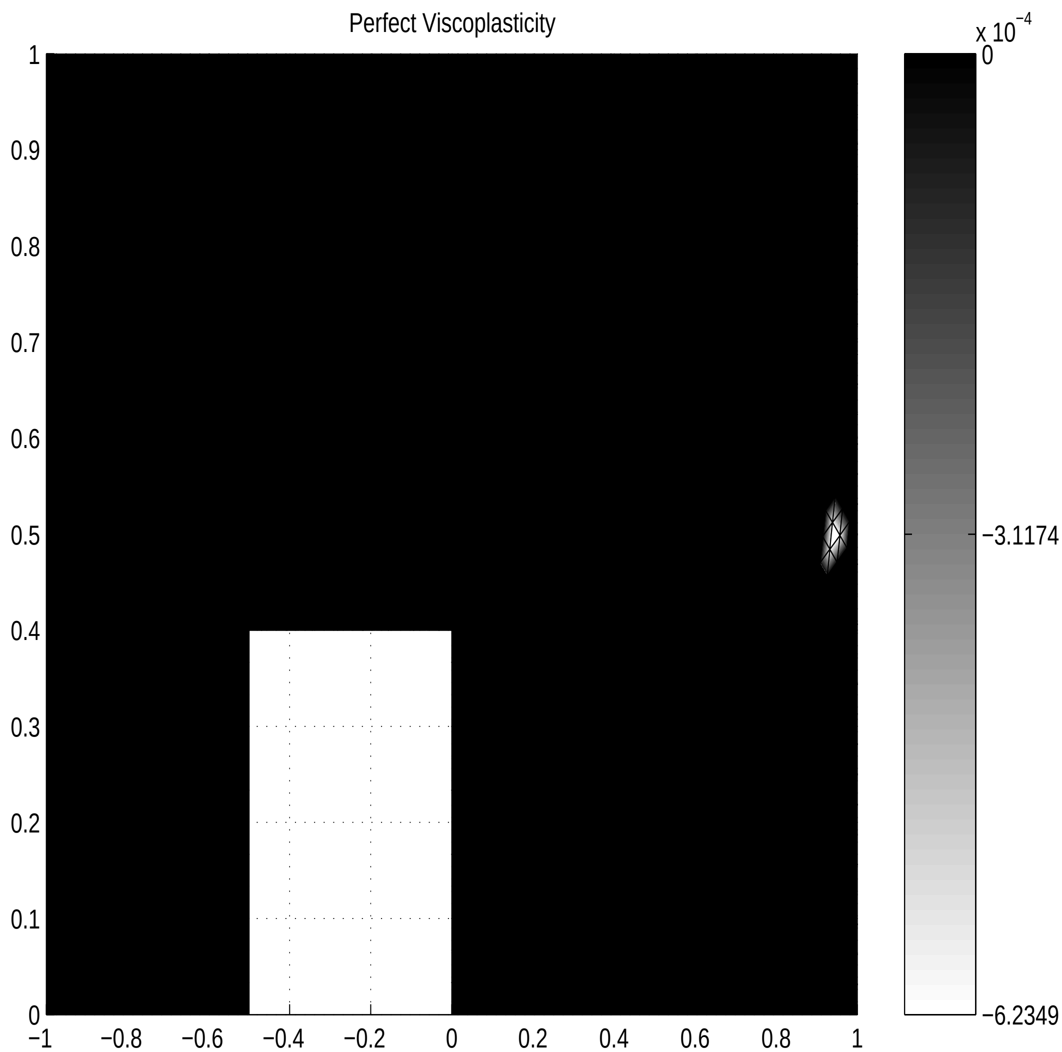

Figure 7: The larger mesh at time t=.15, for perfect plasticity

Table 1: Isotropic hardening

Number of Newton Iterations									
Time	Newton's Method ($\nu = .006$)	Newton's Method ($\nu = .003$)	Modified Newton ($\nu = .003$)	Modified Newton ($\nu = .0015$)	Path Following Method $\nu_i = 1.5 \times (500)^{-i}$				
					$\imath = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
.05	11	12	11	11	6	9	6	3	0
.1	68	87	76	90	6	9	6	3	0
.15	91	100	95	100	6	10	7	3	0
.2	94	100	97	100	6	10	7	3	0
.25	95	100	97	100	6	11	7	3	0
.3	95	100	97	100	6	11	7	3	0
.35	95	100	97	100	6	11	8	3	0
.4	95	100	97	100	6	12	8	3	0

Table 2: Perfect plasticity

Number of Newton Iterations									
Time	Newton's Method ($\nu = 256$)	Newton's Method ($\nu = 128$)	Modified Newtons ($\nu = 256$)	Modified Newton ($\nu = 128$)	Path Following Method $\nu_i = 256 \times (5000)^{-i}$				
					$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
.025	1	1	2	2	2	0	0	0	0
.05	1	1	3	3	3	0	0	0	0
.075	3	3	4	4	4	2	0	0	0
.1	3	4	5	5	5	3	2	0	0
.125	5	5	6	6	6	3	2	0	0
.15	5	100	5	100	5	9	3	0	0

Table 3: Isotropic hardening for larger mesh

Number of Newton Iterations					
Time	Path Following Method $\nu_i = 1.5 \times (500)^{-i}$				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
.05	8	3	0	0	0
.1	8	4	0	0	0
.15	8	6	0	0	0
.2	8	21	7	0	0
.25	8	23	8	1	0
.3	8	20	8	1	0
.35	8	19	8	1	0
.4	8	18	8	1	0

Table 4: Perfect plasticity for larger mesh

Number of Newton Iterations					
Time	Path Following Method $\nu_i = 256 \times (5000)^{-i}$				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
.025	2	2	0	0	0
.05	2	2	0	0	0
.075	2	2	0	0	0
.1	2	2	0	0	0
.125	2	2	0	0	0
.15	2	2	0	0	0