3D Liftings and Edge-Crossing Replacements of Quad-Dominated Frameworks

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Abstract

The rigidity properties of a framework whose vertices lie in a plane is governed by the 2-dimensional generic rigidity matroid, a counting matroid defined on the underlying graph of the framework. Minimally dependent sets in this matroid do not normally correspond to planar graphs and their embeddings in the plane may have edge crossings. We use the tool of x-replacement to obtain plane frameworks whose underlying graph is a cycle in the rigidity matroid and then use their non-zero resolvable stress to lift them into 3-space while leaving their boundary fixed.

1 Introduction

William Baker in his talk at the Fields Institute [1] brought this question to mathematicians: How would one achieve the maximum number of planar lifts in a quad-dominated planar graph? His desired attributes were maximal planar liftings, well-distributed self-stresses, and reasonably sized panels which are mostly quadrilaterals. The goal of this paper is to answer whether a solution to his geometric constraint system exists given any boundary, and whether that solution would be able to properly withstand outside forces or loads.

Gridshells are structures derived from such planar liftings. In 3d, a gridshell is in equilibrium with its self-stresses, and its planar projection shares this property [1]. Thus, we can model this quad-dominated gridshell satisfying the desired properties as a planar graph in 2-space. In this graph, the edges represent rigid rods and the vertices represent flexible joints that hold these edges in place. We call any particular embedding of our graph a framework.

The distribution of forces along the bars or edges of a framework signifies the stability of the framework under outside forces. To model this, we imagine each edge in our graph as a rigid beam under tension or compression and map out the force vectors to ensure every vertex has a net force of zero, even when an external force exists. The rigidity matrix models this by having each row correspond to an edge, and each vertex incident to that edge will be under the forces of compression and tension by having equal and opposite forces on each vertex.

In a simple planar framework, it is likely that this force distribution will not incorporate all of the edges. Certain parts of the graph may have their own stable section, while other sections act independently of that stable section. However, we know that any minimally dependent framework must have a nonzero stress on all edges [7]. The self-stresses of the framework being "well-distributed" is equivalent to the minimal dependence of the rows in our rigidity matrix. In our rigidity matrix, the stresses and motions of our corresponding framework are the kernel and cokernel of our rigidity matrix respectively [7].

Since stress distribution and stability are related, we can easily see that dependence within the rigidity matrix signifies stability, or rigidity, within the corresponding edge set of our graph. However, for there to be a dependence, that means that there is one bar more than required for stability within the structure. A maximally independent set is a rigid set.

In some cases there is a potential for a vertex to move an infinitesimal amount if a force is instilled on it, regardless of whether the system is rigid or not. If in a framework, there is no possibility for any of these infinitesimal motions on any vertex, then it is infinitesimally rigid. Infinitesimal rigidity and generic rigidity are closely related; if any graph has a framework which is infinitesimally rigid, then all frameworks are also generically rigid [2]. This result is very useful for the engineer who, given a particular framework, may want to make slight alterations to the graph in order to make it more pleasing to work with or to look at.

While geometric constraints involve many lengths and distance equations, combinatorial rigidity tells us that we need not worry about those details when trying to answer the question of whether our framework is rigid. Generic rigidity tells us that as long as there is no area of the graph that is underbraced, simply the count of the number elements in our edge set will ensure rigidity. This is because we would have enough distance equations in our system to restrict the solution space to a unique solution. We need not know the particulars of our system.

Without knowing the arrangement of the edges we know that in the probability space $G_{n,d}$ of all d-regular graphs on n vertices, a given 4-regular graph on n vertices is asymptotically almost surely globally rigid [4]. This result means that given some random 4-regular graph in our space of all the 4-regular graphs on n vertices, we will with high probability have a globally rigid graph in the plane.

Another extremely useful tool in graph theory is the rigidity matroid. It is a concise way of determining where edges are and can easily determine whether any edges are dependent, find any infinitesimal motions, and calculate internal stresses of edges. The rigidity matroid is the collection of edges and vertices, which can be easily represented in the rigidity matrix with each edge getting represented as a row and each vertex represented as two columns (one for the x-coordinates and one for the y-coordinates). We can use these rigidity matroids to make many calculations regarding rigidity and stresses much quicker.

Planarity and rigidity have little to do with one another, as can perhaps be deduced by the result above. The placement of our edges, whether they cross or do not, does not influence the rigidity properties of our framework. However, in the case of planar graphs, there are interesting properties in the minimally dependent rigidity matroids. From our count of edges in a minimally dependent rigidity matroid, we have that $|E| = 2|V| - 2$, and when we apply this to Euler's formula for the number of faces in a planar graph, we find that the number of vertices in our graph is equal to the number of faces. Servatius and Christopher [6] found that any minimally dependent graph's geometric dual is itself also minimally dependent. William Baker goes over this in short detail during his talk, specifically on reciprocal graphs and their properties [1].

From Wormalds paper on random regular graphs [9], we know that a random 4-regular graph asymptotically almost surely decomposes into two Hamiltonian cycles. Using this, we can use two disjoint Hamiltonian cycles to generate a random graph, and it will be random enough and have the property of global rigidity. From the count of the number of edges, we also know that this random framework is over braced by 3 edges, thus if we were to remove two edges sharing no vertices, we would obtain a minimally dependent framework. Due to the edge crossings in such a random graph though, it does not meet the planarity property desired by our constraint system. Our paper aims to show that if we were to replace each edge crossing with a vertex of degree 4, we can maintain the minimal dependence of our framework, resulting in a planar graph satisfying all our desired properties.

2 Rigidity

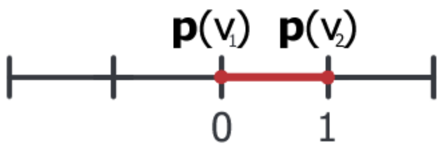
The definitions found in this section are taken from the book Combinatorial Rigidity. [3]

We will use the term *graph* to denote a finite undirected graph with no loops or repeat edges. Given a set of n vertices V and edge set E of pairs of vertices.

Definition 2.1. *A* framework *in m-space denotes a triple (V, E, $\mathbf{p}$), where (V,E) is a graph with vertex set V and edge set E and $\mathbf{p}$ is an embedding (injection) of V into real m-space.*

To answer the question of whether a framework is rigid or not, we look to the *configuration space* $\mathcal{A}$ of our injection $\mathbf{p}$. The distances we have determined between the pairs of points in our injection $\mathbf{p}$ give rise to a system of $|E|$ equations with mn variables. These are the distance equations between our points that have fixed distances due to the edges. This system determines our configuration space $\mathcal{A}$. The coordinates of the points in $P = \mathbf{p}(V)$ is one solution to the system, so we say that $P \in \mathcal{A}$ but other solutions can be found in the configuration space.

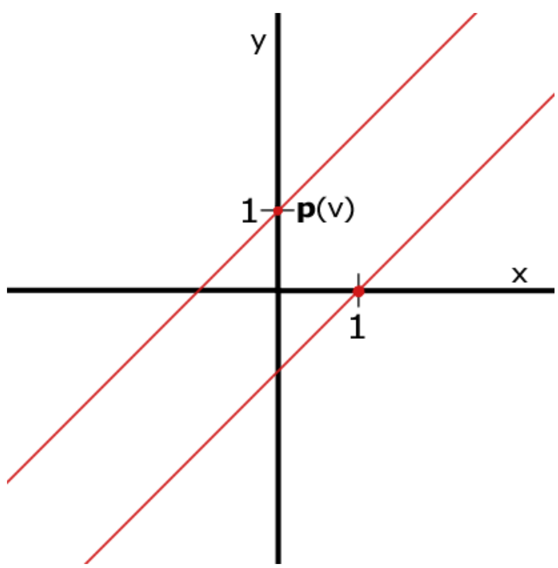
Take for example the exceedingly simple graph $V = \{v_1, v_2\}$ with edge set $E = \{(v_1, v_2)\}$. We will take an injection $\mathbf{p}$ that injects our graph into $\mathbb{R}^1$, where $\mathbf{p}(v_1) = 0$ and $\mathbf{p}(v_2) = 1$



What configuration space does $\mathbf{p}$ live within? For an arbitrary injection $\mathbf{q}$, $\mathbf{q}(v_1)$ and $\mathbf{q}(v_2)$ must satisfy the distance equation:

$$|\mathbf{q}(v_2) - \mathbf{q}(v_1)| = 1$$

We can satisfy the equations in two different ways. Either $\mathbf{q}(v_2) = \mathbf{q}(v_1) - 1$ or $\mathbf{q}(v_2) = \mathbf{q}(v_1) + 1$. Our configuration space is then the two lines in R^2 where x corresponds to $\mathbf{q}(v_1)$ and y corresponds to $\mathbf{q}(v_2)$. Any point on the lines correspond to a specific injection and framework. Our initial injection $P = \mathbf{p}(V)$ corresponds to the point $(0, 1)$.

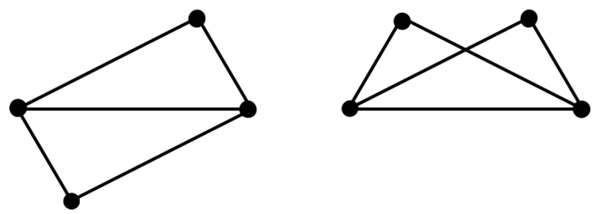


To answer the question of whether an object is rigid or not, we must ask what sort of frameworks exist in the configuration space. If we have a solution or framework $\mathbf{p}$ that satisfies the equations outlined in our configuration space, then all the frameworks congruent to **p** will also exist in the configuration space. The question of whether there exists non-congruent frameworks in the configuration space is a question of rigidity.

The subspace of $\mathcal{A}$ containing only the frameworks congruent to $(V,\, E,\, \mathbf{p})$ is determined by the set K, which contains all pairs of points in V. K is the edge set of the complete graph on V, so it fixes all the distances. The system determined by K will be called $\mathcal{C}$. The notation $\mathcal{C}$ is meant to be associated with the word "complete", as in a complete graph. The space $\mathcal{C}$ depends entirely on the given framework it contains, since it depends on the edge lengths given.

Definition 2.2. *A framework is said to be a* rigid *if the framework* $(V, E, \mathbf{p})$ *has the sets* $\mathcal{A}$ *and* $\mathcal{C}$ *equal within a neighborhood of point* P*.*

Take for example the following framework:

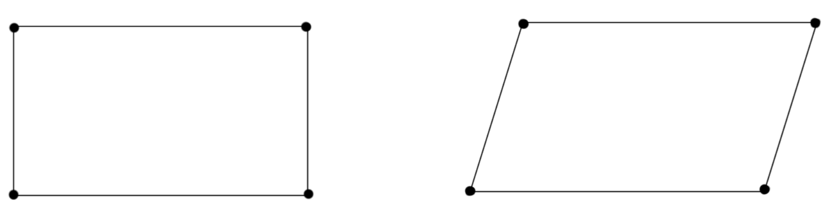


Since there are two non-congruent frameworks in the configuration space, this framework is not strongly rigid.

Take for example the strongest shape, the non-degenerate triangle. The side-side-side theorem tells us that once we have fixed our edge lengths, the only possible frameworks in the configuration space are the congruent ones. This is what we call *strongly rigid*.

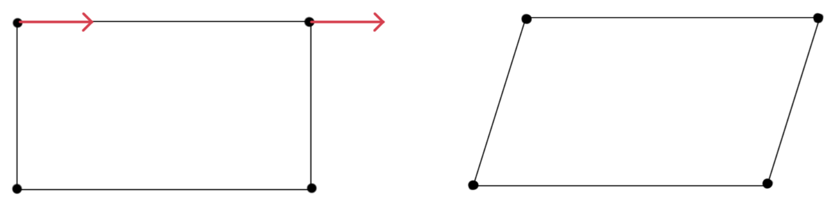
Definition 2.3. *A framework is said to be a* strongly rigid *if the framework* $(V, E, \mathbf{p})$ *has the sets* $\mathcal{A}$ *and* $\mathcal{C}$ *equal. Note that this implies rigidity.*

There are many frameworks which are neither rigid nor strongly rigid. These graphs have what are called *flexes*. Take for example the rectangle. We can alter the position of the top two points so that the edge lengths stay the same, but the distances between the diagonals change:



Definition 2.4. *We say that a framework* $(V,\, E,\, \mathbf{q})$ *is a* flex *of a framework* $(V,\, E,\, \mathbf{p})$ *if the edge lengths are kept the same, but distances which are not fixed are altered.*

Imagine the initial rectangle as an object made up of fixed-length rods with the vertices representing joints in between them. The flex of the rectangle would be the result of us sliding the top bar to the right. This motion is what we call an *infinitesimal motion*. We require that our infinitesimal motions on the system do not compress nor stretch the rods in our system.



Definition 2.5. *An assignment* $\mathbf{u}$ *of velocity vectors to the framework* $(V, E, \mathbf{p})$ *is said to be an* infinitesimal motion *of the framework* $(V, E, \mathbf{p})$ *if, for every edge* (i, j) *of* (V, E)*, the difference between the vectors* $\mathbf{u}_i$ *and* $\mathbf{u}_j$ *is perpendicular to the edge. An equivalent definition can be obtained through the system of homogeneous equations:*

$$(\mathbf{u}_i - \mathbf{u}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) = 0, \text{ for all } (i, j) \in E,$$

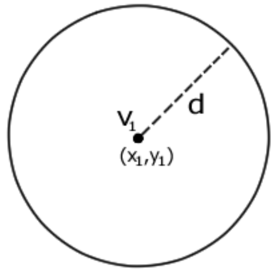
where the operation above is the dot product between two vectors.

We call an infinitesimal motion on a framework a *trivial motion* if it results in a congruent framework. These motions in R^2 would be the translations and the rotation. A framework which allows no infinitesimal motions apart from the trivial motions is called *infinitesimally rigid*.

The trivial motions on a framework in $\mathbb{R}^m$ gives a subspace of dimension $m(m+1)/2$ in the null space of each framework in $\mathbb{R}^m$. This formula is rather intuitive in $\mathbb{R}^2$ when you look at each possible trivial motion of a framework. In the plane, we have the translations parallel to the $x-$ and $y-$axes, giving us two dimensions. Our third dimension comes from the rotation about the z-axis. Since the space of congruent frameworks only has the translations and rotations as its dimension, the space $\mathcal{C}$ in $\mathbb{R}^2$ has dimension 3.

Our entire space $\mathcal{A}$ has dimension mn, which for the case of $\mathbb{R}^2$ is $2n$. This is because when we have no fixed edge lengths to restrict our framework, we can choose any coordinates for each point, thus 2 degrees of freedom for each coordinate. With n vertices, that leads to $2n$ degrees of freedom.

Each edge in a framework reduces the degrees of freedom by 1. For example, say we are given two vertices v_1, v_2 with an edge between them of distance d. When we place our first vertex, $v_1 = (x_1, y_1)$ has 2 degrees of freedom. Once we've chosen those coordinates (call them $x_1 = a$, $y_1 = b$) we then have our choice for placement of v_2 restricted by a circle of radius d with v_1 as its center.



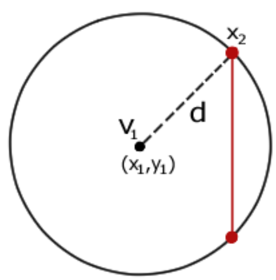
For the generic distance equation, we have that

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$$

With $x_1 = a$ and $y_1 = b$, we have that

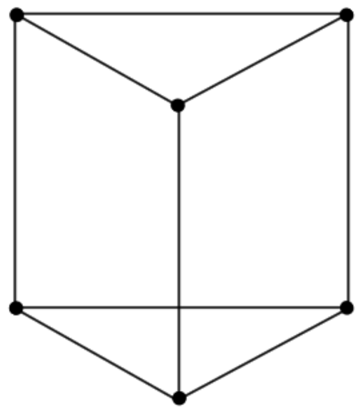
$$\sqrt{(x_2 - a)^2 + (y_2 - b)^2} = d$$

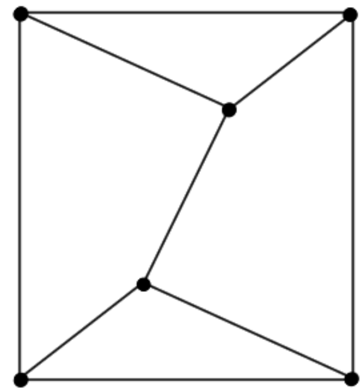
Once we've made a choice for x_2, of which there are infinitely many choices despite the restrictions (thus we have 1 degree of freedom), our choice of y_2 is restricted to one of two solutions, the positive or negative value, leading us to a unique solution.



To make the dimension of the configuration space $\mathcal{A}$ and the dimension of $\mathcal{C}$ equal, we restrict the dimension of $\mathcal{A}$ by having $2n-3$ edges or distance equations. This would reduce the dimension of A to 3. We can see this just by looking at the edges as a system of $2n-3$ equations with $2n$ unknowns. Once we determine our placement of the first vertex, and choose one coordinate for our second vertex, we have determined a unique solution. Thus we only have 3 degrees of freedom.

It should be noted that everything done above assumes a generic placement of vertices and generic choices of edge lengths. This is because depending on your specific embedding (choice of edge length and vertex placement) of a generically rigid framework, you will have a non-rigid framework despite it satisfying the edge count. The two frameworks below show an example of this. The framework on the left is nonrigid despite satisfying the edge count, while the framework on the right is rigid. The embedding on the left allows for a horizontal shear due to having three parallel edges of equal length.





So to obtain a rigid system with the minimal number of edges, we want $|E| = 2n - 3$ generically-placed edges in our framework. This is a result of Laman's theorem:

Theorem 2.1. *[Laman's Theorem] A graph $G = (V, E)$ is generically rigid in $\mathbb{R}^2$ if and only if there is a subgraph $G' = (V, E')$ such that $|E'| = 2 \cdot |V| - 3$ and for every subgraph $G'' = (V'', E'')$ of G', $|E''| \leq 2 \cdot |V''| - 3$*

Since the three trivial motions cannot be restricted any further, we have reached the maximum number of linearly independent edges once we have $2n-3$ edges. Any more edges must be dependent on the system. This result is derived from Laman's theorem.

Lemma 2.2. *[Counting Lemma] Any non-empty set of edges E with $|E| > 2|V(E)| - 3$ is dependent for every plane configuration $\mathbf{p}$. A set E is independent only if for all non-empty subsets E'', $E'' \leq 2|V(E'')| - 3$*

The relationship between generic rigidity and infinitesimally rigid is an extremely useful result. The following theorem which is a result from Connelly[2] allows us to take a non-generic embedding, and say that because a particular embedding is infinitesimally rigid, the generic embedding of the graph must also be rigid.

Theorem 2.3. *If a framework $(V, E, \mathbf{p})$ is infinitesimally rigid for some embedding $\mathbf{p}$ of V into $\mathbb{R}^m$ then $(V, E, \mathbf{q})$ is infinitesimally rigid for all generic embeddings $\mathbf{q}$ of V into $\mathbb{R}^m$.*

For the rest of this paper, assume that we are working within $\mathbb{R}^2$.

3 Rigidity Matroids

The talk of the maximum number of linearly independent distance equations should bring to mind linear algebra's definitions of independence and dependence between vectors in a system. This is where we introduce the concept of *matroids*. By studying the dependence and independence within our linear systems determined by our frameworks, we can gain an insight into what frameworks are rigid or non-rigid by studying their associated *rigidity matroids*. The concept of a matroid was introduced first by Whitney in his paper published in 1935, and so the following definitions are sourced from his paper [8].

3.1 Independent Sets

For a finite set M of elements, a subset N of M is either "dependent" or "independent". For the formulation of a matroid using these sets, we have the following axioms:

(I_1) The empty set is independent.

(I_2) Any subset of an independent set is an independent set.

(I_3) If $N = e_1, e_2, \ldots, e_n$ and $N' = e'_1, e'_2, \ldots, e'_{n+1}$ are both independent sets, then there exists some element $e'_k \in N'$, $e'_k \notin N$ such that $N + e'_k$ is an independent set.

Earlier we discussed how each edge, or distance equation, reduces the degrees of freedom for our framework, and the dimensions of our configuration space $\mathcal{A}$ and our congruent space $\mathcal{C}$. Generically, once we reach the count of $2|V| - 3$ edges, we can no longer restrict the degrees of freedom any further. If we were to look at our edges as rows in a matrix, that would be equivalent to saying the maximum number of linearly independent rows is $2|V| - 3$, and that any rows added would have to be dependent on the other rows in the system. This is exactly what the rigidity matrix does.

A rigidity matrix $R_{V,E}(\mathbf{p})$ for a graph (V, E) along with an embedding $\mathbf{p}$ is defined as

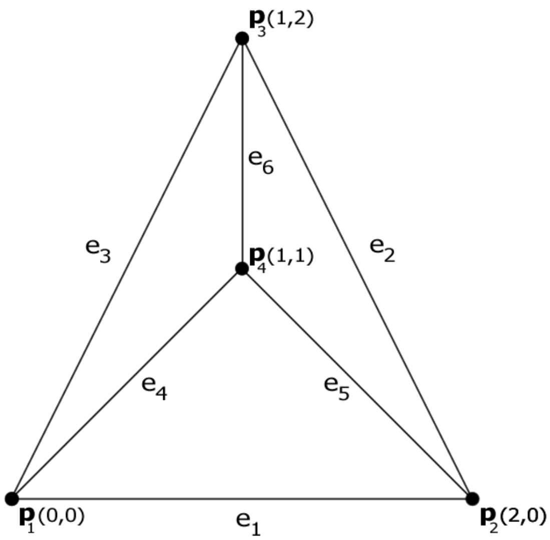
$$\begin{aligned} R_{V,E}(\mathbf{p})_{e,i} &= \mathbf{p}_i - \mathbf{p}_j && \text{if } e = \{i, j\} \in E \\ &= 0 && \text{otherwise} \end{aligned}$$

where $\mathbf{p}_i = \mathbf{p}(v_i)$.

For our frameworks, we consider the rows of a rigidity matrix (which correspond to the edges of the graph of our framework) of a particular framework to be our finite set of elements. Rows are considered independent based on the definitions of independence within basic linear algebra.

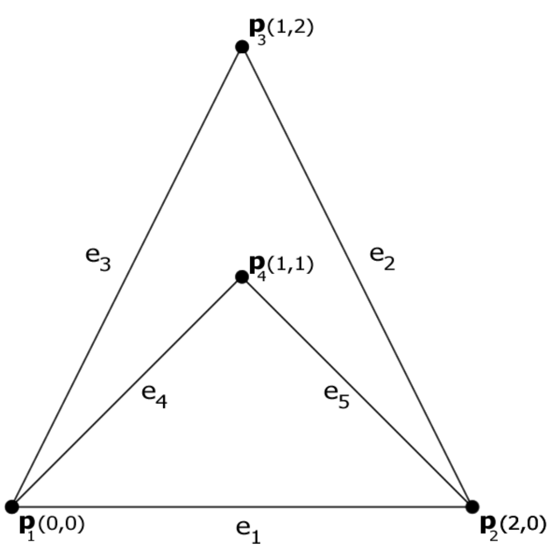
Independence within the rows of a rigidity matroid does not yield a lot of information about the matroid. With all of its rows being independent from one another, we only extrapolate that the framework must have less than or equal to $2|V| - 3$ edges, and thus is not overbraced.

The set of edges of K_4 form a minimally dependent set, also known as a circuit. We will use the following embedding and its corresponding rigidity matroid:



	v_1		v_2		v_3		v_4	
e_1	-2	0	2	0	0	0	0	0
e_2	0	0	1	-2	-1	2	0	0
e_3	-1	-2	0	0	1	2	0	0
e_4	-1	-1	0	0	0	0	1	1
e_5	0	0	1	-1	0	0	-1	1
e_6	0	0	0	0	0	1	0	-1

Removing one edge from K_4 (the choice of which edge is trivial) gives us a rigidity matroid with entirely independent rows.



$$\begin{array}{c|cccccccc|}
 & v_1 & & v_2 & & v_3 & & v_4 & \\
e_1 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
e_2 & 0 & 0 & 1 & -2 & -1 & 2 & 0 & 0 \\
e_3 & -1 & -2 & 0 & 0 & 1 & 2 & 0 & 0 \\
e_4 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\
e_5 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1
\end{array}$$

The row reduced echelon form of the matrix above supports the assertion:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}$$

It is easy to see that any subset of the rows in the rigidity matroid of K_4 forms an independent set, but not every framework corresponding to those rigidity matroid are rigid objects, as in the case of a triangle with an edge connecting it to a vertex of degree 1. Independent rigidity matroids can be either rigid or nonrigid objects.

3.2 Circuits

An equivalent formulation for a matroid is that using circuits, also known as minimally dependent sets. For a set of elements M, any subset is either a circuit or not a circuit. With circuits, the axioms for a matroid are as follows:

(C_1) No proper subset of a circuit is a circuit.

(C_2) If P_1 and P_2 are circuits, with e_1 in both P_1 and P_2, with $e_2 \in P_1$ but $e_2 \notin P_2$, then there is a circuit P_3 in $P_1 + P_2$ containing e_2 but not e_1.

Since the set of our edges are minimally dependent, this means that there exists nonzero scalars such that the rows vectors of our rigidity matrix can each be assigned a nonzero scalar and sum up to the zero vector. Since it's minimally dependent, every proper subset must be independent. We call a graph a *circuit graph* if the rows of its rigidity matrix form a minimally dependent set.

Since the maximum number of edges that can be linearly independent is $2|V| - 3$ by Lemma 2.2, we must overbrace a rigid framework with one additional edge in order to obtain a circuit. The rank of our rigidity matroid would then be $2|V| - 3$, the minimum number of edges necessary to obtain rigidity within our framework. So any circuit has $2|V| - 2$ edges.

4 Stresses

In a real life example, any edge in a graph can be seen as under tension or compression. Assuming a static design, any edge will have equal and opposite forces on each of its vertices to prevent the edge from moving.

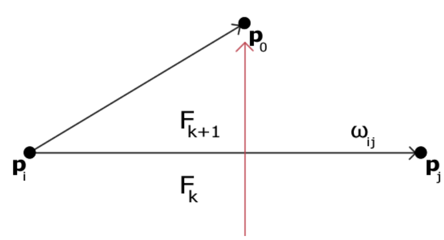
Using our rigidity matrix of a given framework, we can define the self-stresses of a system.

Definition 4.1. *Given a graph $G = (V, E)$. The assignment of a scalar to each of the edges of a given framework $G(\mathbf{p})$, $\omega : E \to \mathbb{R}$, is considered a* self-stress *if for each vertex $i \in V$:*

$$\sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = 0.$$

From the definition we can see that our assignment of these scalars is the solution to the null-space of our rigidity matrix, the solution to $\omega R_G(\mathbf{p}) = 0$. When we have minimal dependence, we can assign all nonzero stress scalars to each edge corresponding to a row vector in our rigidity matrix to find a solution to the system.

We can use these stress scalars in order to compute what the z-coordinates of a given position vector $\mathbf{p}_0 = (x, y)$ should be depending on what face they are on. To calculate our z-coordinate, we take the cross product between our position vector $\mathbf{p}_0$ and our position vector of the edge we cross over to reach the face our position vector is now on. For our cross product, we make each position vector embedded in the plane $z = 0$, giving us 0 for our z-coordinate in the cross product.



$$F_{k+1}(\mathbf{p}_0) = F_k + \omega_{ij}\left[[\mathbf{p}_j - \mathbf{p}_i] \times [\mathbf{p}_0 - \mathbf{p}_i]\right]_z$$

By the rules and definitions of vector and tensor calculus, this will give us a vector lifted in the positive z direction, so long as our stress scalar ω_{ij} is positive. Because we are taking the cross product, our order matters. This uniquely determines our z-coordinate of any position vector for the lifting of our framework.

The uniqueness is a result of the properties of minimal dependence in our framework. Imagine we take some path circling around some vertex v_i in our embedding, crossing over every edge incident to the vertex, before finally arriving at our initial position again. We must be at the same z-position that we were initially at once we arrive at our initial position again.

The change in our z coordinate can be calculated using the formula for the z-coordinate:

$$\left[\sum_{j|e_{ij}} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i)\right] \times (\mathbf{p}_0 - \mathbf{p}_i) = 0$$

It can be observed that the above formula must be 0. This is a result of the properties of our stress scalars and minimal dependence. Any closed path around the vertices will sum to zero.

5 Liftings

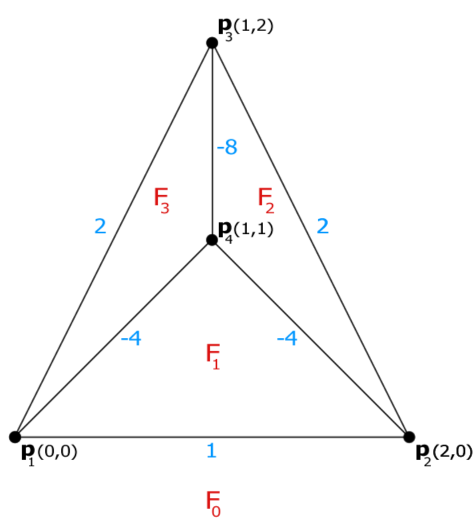
Utilizing the internal stresses, we can calculate a lifting of the structure into the third dimension. For example, one could picture lifting the center vertex of the K4 graph out of the plane to form a pyramid. This can be done for more complex shapes as well.

We can determine these liftings by determining the height of any point on a given face in our graph. Each face would be in a plane determined by our lift equation $L : \mathbb{R}^2 \to \mathbb{R}$ which determines the z-coordinate for our points based on the position we are at within the face.

$$F_{new} = F_{start} + [w_{ij}(p_i - p_j)x(p - p_j)]_z \tag{1}$$

Assuming that the external face is embedded in the x-y plane, we can calculate the lift equation for each face incident to the external face, and proceed until every face is calculated this way.

5.1 K_4



$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & & v_2 & & v_3 & & v_4 & \end{array} \\ \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

The null vector containing our stress scalars is $[1, 2, 2, -4, -4, -8]$. Since our outside face F_0 is embedded in the x-y plane, we have that

$$F_0 = 0$$

$$\begin{aligned} F_1 &= F_0 + (1)\big[[(2,0,0) - (0,0,0)] \times [(x,y,0) - (0,0,0)]\big]_z \\ &= 0 + \begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ x & y & 0 \end{vmatrix} \\ &= 2y \end{aligned}$$

$$\begin{aligned} F_2 &= F_0 + (2)\big[[(1,2,0) - (2,0,0)] \times [(x,y,0) - (2,0,0)]\big]_z \\ &= 0 + 2\begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ x-2 & y & 0 \end{vmatrix} \\ &= -2(2x + y - 4) \end{aligned}$$

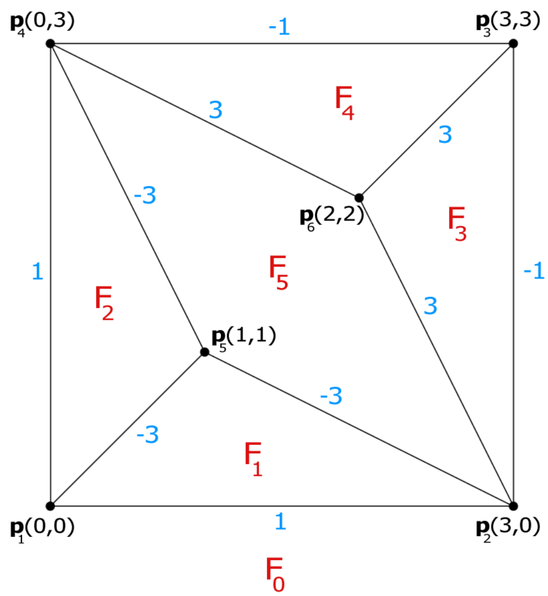
$$\begin{aligned} F_3 &= F_0 + (2)\big[[(0,0,0) - (1,2,0)] \times [(x,y,0) - (1,2,0)]\big]_z \\ &= 0 + 2\begin{vmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ x-1 & y-2 & 0 \end{vmatrix} \\ &= 2(2x - y) \end{aligned}$$

We can observe with this example that if we are to travel in a circle around our center vertex $\mathbf{p}(v_4)$ crossing over each edge adjacent to the vertex, we will net 0 for the change in our z-coordinate:

$$\sum_{j|e_{4j}} \omega_{4j}(\mathbf{p}_j - \mathbf{p}_4) = 4\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 8\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The height for our center vertex can be calculated using any of our inner face equations: $(1, 1, 2)$. All other vertices have a height of 0 as they are on the border.

5.2 Square Double Banana



$$
\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} v_1 & & v_2 & & v_3 & & v_4 & & v_5 & & v_6 & \end{array} \\
\begin{bmatrix}
-3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 2 & -1
\end{bmatrix}
\end{array}
$$

The null vector corresponding to this rigidity matroid is

$$\big[1, -1, -1, 1, -3, -3, -3, 3, 3, 3\big]$$

$$F_0 = 0$$

$$
\begin{aligned}
F_1 &= F_0 + 1\big[[(3,0,0) - (0,0,0)] \times [(x,y,0) - (0,0,0)]\big]_z \\
&= 0 + 1 \begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ x & y & 0 \end{vmatrix} \\
&= 3y
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= F_0 + 1\big[[(0,0,0)-(0,3,0)] \times [(x,y,0)-(0,3,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ x & y-3 & 0 \end{vmatrix} \\
&= 3x
\end{aligned}
$$

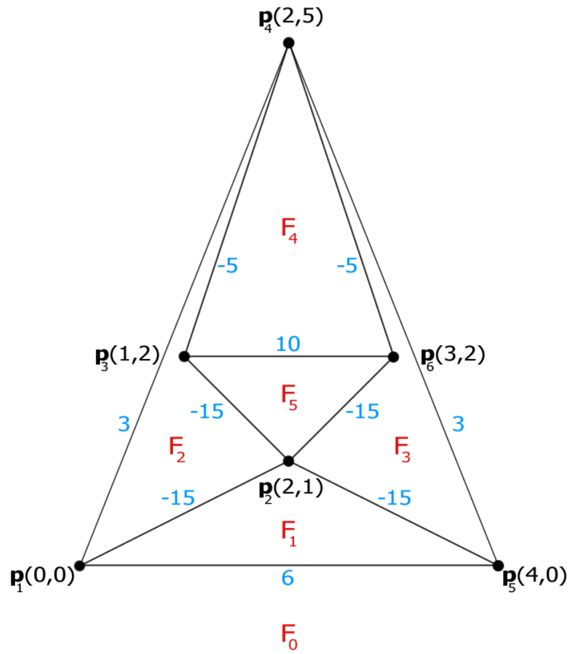
$$
\begin{aligned}
F_3 &= F_0 + (-1)\big[[(3,3,0)-(3,0,0)] \times [(x,y,0)-(3,0,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ x-3 & y & 0 \end{vmatrix} \\
&= 3(x-3)
\end{aligned}
$$

$$
\begin{aligned}
F_4 &= F_0 + (-1)\big[[(0,3,0)-(3,3,0)] \times [(x,y,0)-(3,3,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ x-3 & y-3 & 0 \end{vmatrix} \\
&= 3(y-3)
\end{aligned}
$$

$$
\begin{aligned}
F_5 &= F_1 + (-3)\big[[(3,0,0)-(1,1,0)] \times [(x,y,0)-(1,1,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ 2 & -1 & 0 \\ x-1 & y-1 & 0 \end{vmatrix} \\
&= 3(3-x-y)
\end{aligned}
$$

From these lift equations, we get the heights for the vertices inside of our border. $\mathbf{p}_5$ has a height of 3 and $\mathbf{p}_6$ has a height of -3.

5.3 Triangular Double Banana



	v_1		v_2		v_3		v_4		v_5		v_6	
e_1	-2	-1	2	1	0	0	0	0	0	0	0	0
e_2	0	0	1	-1	-1	1	0	0	0	0	0	0
e_3	0	0	0	0	-1	-3	1	3	0	0	0	0
e_4	-2	-5	0	0	0	0	2	5	0	0	0	0
e_5	-4	0	0	0	0	0	0	0	4	0	0	0
e_6	0	0	-2	1	0	0	0	0	2	-1	0	0
e_7	0	0	0	0	0	0	-2	5	2	-5	0	0
e_8	0	0	0	0	-2	0	0	0	0	0	2	0
e_9	0	0	-1	-1	0	0	0	0	0	0	1	1
e_{10}	0	0	0	0	0	0	-1	3	0	0	1	-3

The null vector corresponding to this rigidity matroid is

$$\left[-15, -15, -5, 3, 6, -15, 3, 10, -15, -5\right]$$

$$F_0 = 0$$

$$\begin{aligned}
F_1 &= F_0 + 6\left[[(4,0,0) - (0,0,0)] \times [(x,y,0) - (0,0,0)]\right]_z \\
&= 0 + 6 \begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \\ x & y & 0 \end{vmatrix} \\
&= 24y
\end{aligned}$$

$$F_2 = F_1 - 15\big[[(2,1,0)-(0,0,0)] \times [(x,y,0)-(0,0,0)]\big]_z$$
$$= 15x - 6y$$

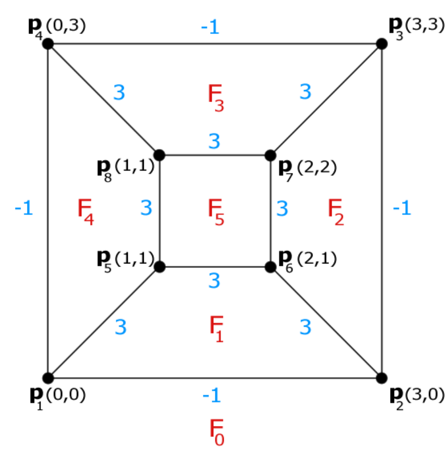
$$F_3 = F_0 + 3\big[[(2,5,0)-(4,0,0)] \times [(x,y,0)-(4,0,0)]\big]_z$$
$$= 60 - 15x - 6y$$

$$F_4 = F_2 - 5\big[[(1,2,0)-(2,5,0)] \times [(x,y,0)-(2,5,0)]\big]_z$$
$$= 5 - y$$

$$F_5 = F_2 - 15\big[[(2,1,0)-(1,2,0)] \times [(x,y,0)-(1,2,0)]\big]_z$$
$$= 45 - 21y$$

From these lift equations, we get the heights for the vertices inside of our border. $\mathbf{p}_2$ has a height of 24, $\mathbf{p}_3$ has a height of 3, and $\mathbf{p}_6$ has a height of 3. So we can see that there's a steep incline in the planes F_2 and F_3.

5.4 Cube Graph



$$
\begin{array}{c|cccccccccccccccc|}
 & v_1 & & v_2 & & v_3 & & v_4 & & v_5 & & v_6 & & v_7 & & v_8 & \\
e_1 & -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_2 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_3 & 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_4 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_5 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_6 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
e_7 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
e_8 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
e_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
e_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
e_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

The null vector is $[-1,-1,-1,-1,3,3,3,3,3,3,3,3]$

$$F_0 = 0$$

$$
\begin{aligned}
F_1 &= F_0 + (-1)\big[[(3,0,0)-(0,0,0)] \times [(x,y,0)-(0,0,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ x & y & 0 \end{vmatrix} \\
&= -3y
\end{aligned}
$$

$$
\begin{aligned}
F_4 &= F_0 + (-1)\big[[(0,0,0)-(0,3,0)] \times [(x,y,0)-(0,3,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ x & y-3 & 0 \end{vmatrix} \\
&= -3x
\end{aligned}
$$

$$
\begin{aligned}
F_3 &= F_0 + (-1)\big[[(0,3,0)-(3,3,0)] \times [(x,y,0)-(3,3,0)]\big]_z \\
&= 0 - 1 \begin{vmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ x-3 & y-3 & 0 \end{vmatrix} \\
&= -3(3-y)
\end{aligned}
$$

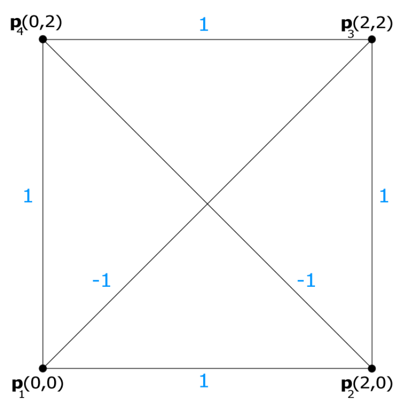
$$
\begin{aligned}
F_2 &= F_0 + (-1)\big[[(3,3,0)-(3,0,0)] \times [(x,y,0)-(3,0,0)]\big]_z \\
&= 0 - 1\begin{vmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ x-3 & y & 0 \end{vmatrix} \\
&= -3(3-x)
\end{aligned}
$$

$$
\begin{aligned}
F_5 &= F_1 + 3\big[[(2,1,0)-(1,1,0)] \times [(x,y,0)-(1,1,0)]\big]_z \\
&= F_1 + 3\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ x-1 & y-1 & 0 \end{vmatrix} \\
&= F_1 + 3y - 3 \\
&= -3y + 3y - 3 = -3
\end{aligned}
$$

Our equation for F_5 tells us that the inner square is a flat plane parallel to the $z = 0$ plane, and thus the heights for each of our inner vertices will be -3. While this isn't a lifting in the sense of going in the positive z direction, we can simply multiply our stress vector by -1 to get face equations that are multiplied by -1, effectively flipping everything over.

6 X-Replacement

Liftings are an effective tool. However, issues arise when there are overlapping edges. Take for example the case of the K4 graph.



In this case, there can be no physical lifting of the graph. If all boundaries edges are fixed, the interior edges could not physically exist—they would intersect at a new point. One way we can resolve this issue is to look at X-replacement. We replace the crossing edges with 4 edges connected to a vertex placed at the intersection. However, a lifting can only exist in the case of a graph that is a minimally dependent circuit. Before assuming a maximal lift always exists, we must prove that the new graph formed by adding in this vertex is also a circuit.

Theorem 6.1 (X-Replacement)**.** *Let $G = (V, E)$ be a circuit graph with an embedding $\mathbf{p}$ into $\mathbb{R}^2$. Let $e_\alpha = (v_1, v_2)$ and $e_\beta = (v_3, v_4)$ be distinct edges in E. Let the graph H be a modification of G with the removal of e_α and e_β and the addition of a vertex v_{n+1} with four edges $e_\alpha^{(1)} = (v_1, v_{n+1})$, $e_\alpha^{(2)} = (v_2, v_{n+1})$, $e_\beta^{(1)} = (v_3, v_{n+1})$, $e_\beta^{(2)} = (v_4, v_{n+1})$. This new graph H must also be a circuit.*

In order to prove this, we will look to a result of our counting theorem.

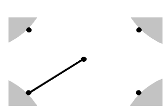
Lemma 6.2. *A circuit framework embedded into $\mathbb{R}^2$ has a minimal vertex degree of* 3*.*

Proof. Given that we have a circuit framework, we know by Lemma 2.2 that our edge count must be $|E| = 2|V| - 2$. Let $|V| = n$, and suppose we have a vertex v_n of degree 2. Take a subgraph of the circuit framework where that vertex is removed. The subgraph would have $|E| - 2 = 2(|V| - 1) - 2$ edges, which means that there must be a dependence within the subgraph. This cannot be true; since our original graph was a circuit, every subgraph must be independent. Therefore, our assumption that there is a vertex of degree 2 is false, and since no dependent graph can contain a vertex of degree 1, our minimal vertex degree in a circuit graph must be 3. ☐

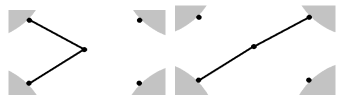
It is to be noted that by nature, a 2-extension, or a vertex of degree 2 added onto a graph, must be independent as it results in an independent vertex.

In our theorem statement H has $2n + 2$ edges and $n + 1$ vertices, thus the graph H must have a dependency. We can show by edge counts that there is no possible way for any subgraph of H to be dependent, and thus the whole graph H must be dependent, and thus must be a circuit. Our proof will go by a case by case basis of inclusion of a certain number of edges of our edge set $\{e_\alpha^{(1)}, e_\alpha^{(2)}, e_\beta^{(1)}, e_\beta^{(2)}\}$.

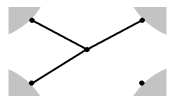
Proof. Let $G^- = G \backslash \{e_\alpha, e_\beta\}$ be the subgraph of G with the edges e_α and e_β removed. We know that G^- must be an independent graph by the first circuit axiom on page 10. Thus any subgraph of H that contains a circuit must contain some of the edges in our set $\{e_\alpha^{(1)}, e_\alpha^{(2)}, e_\beta^{(1)}, e_\beta^{(2)}\}$.



Case 1: One Edge Without loss of generality, given an edge from our set and the subgraph G^-, by Lemma 6.2 there's no way for the subgraph to be dependent unless G^- is dependent.

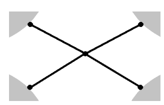


Case 2: Two Edges Without loss of generality, we can pick any two edges from our set $\{e_\alpha^{(1)}, e_\alpha^{(2)}, e_\beta^{(1)}, e_\beta^{(2)}\}$. A dependent subgraph cannot be formed from G^- and these two edges due to Lemma 6.2.



Case 3: Three Edges Without loss of generality, we can pick any three edges from our edge set $\{e_\alpha^{(1)}, e_\alpha^{(2)}, e_\beta^{(1)}, e_\beta^{(2)}\}$ to form a subgraph with G^-. Our edge count in this subgraph is $2n-3$, which by the counting Lemma 2.2, would be the maximum number of independent edges possible.

Suppose that adding our three edges results in a dependency in our subgraph of H. This dependency cannot include all of the edges in H, and so there must be some subgraph C containing part of the graph G^- and the three edges from our edge set. The part of G^- included in C is thus overbraced, containing enough edges to create a circuit with the three new edges. The graph of $G^-\backslash C$ must be underbraced as a result of this. But this cannot be possible, because there can be no underbraced section in our original graph G, since G is a circuit.



Case 4: Four Edges We now look at the entire graph of H. By our counting Lemma 2.2, we must have a dependency somewhere in our graph H because we have gone past the maximum number of possible independent edges. We have proven that it is impossible for the dependency to exist within a subgraph of H, and so the dependency must involve all of the edges of H. Thus, H satisfies the definition of minimal dependence. □

An equivalent proof can be conducted to show that an X-replacement in a rigid framework preserves rigidity. The same counting logic above can be used, but counting the number of edges needed to obtain rigidity.

We have a second proof using the rank of the rigidity matrix rather than the counting lemma.

Proof. Given a circuit graph $G = (V, E)$. We know that because G is a circuit, there is no overbracing in any subgraph of G, because no submatrix of R_G can be rank-deficient when R_G is minimally rank-deficient. Thus we have that for any $E' \subset E$, $|E'| \leq 2|V'| - 3$.

Let us now remove two disjoint edges $e_1 = (v_1, v_2), e_2 = (v_3, v_4)$. We will add into our graph a vertex v and add 4 edges $(v, v_1), (v, v_2), (v, v_3), (v, v_4)$, which will connect to the vertices which had the two edges removed. We now have a new graph

$$G_X = \left(V + v, E - \{e_1, e_2\} + \{(v, v_i)\}_{i=1}^4\right)$$

Notice that in our new graph, all the vertices in V have maintained the same degree as they had in G. Thus, there's no way that any submatrix of R_{G_X} not including v is rank deficient.

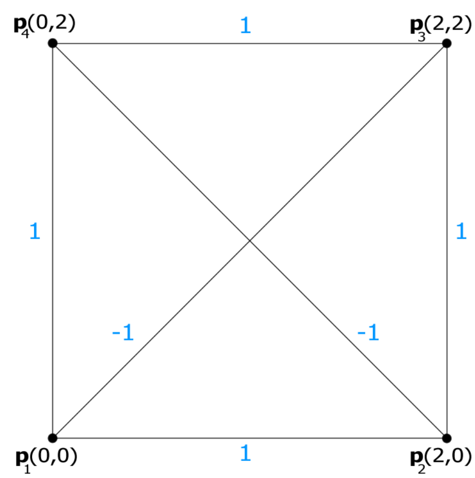
R_{G_X} has more rows than the maximum number of independent row vectors possible, thus there must be a rank-deficiency, and the submatrix that has this rank-deficiency must include the edges of v. Let this subgraph corresponding to the rank-deficient submatrix of R_{G_X} be denoted

$$C = \left(V_1 + v, E_1 + \{(v, v_i)\}_{i=1}^4 - \{e_1, e_2\}\right).$$

We know that $|E_1 + \{e_1, e_2\}| \le 2|V_1| - 3$ since the degrees are maintained as they were in the original graph G. Including in the addition of v and its associated edges, we have that $|E_1 + \{(v, v_i)\}_{i=1}^4 - \{e_1, e_2\}| = |E_1| + 2$ and $|V_1 + v| = |V_1| + 1$. Thus, our subgraph C cannot be rank-deficient, as it has less than or equal to the number of edges to maintain full rank. □

This theorem has the application that if we have some randomly generated Hamilton cycle with many crossings, we can X-replace each crossing with a vertex, adding in many vertices of degree 4, and maintaining the dependence of the graph for future liftings.

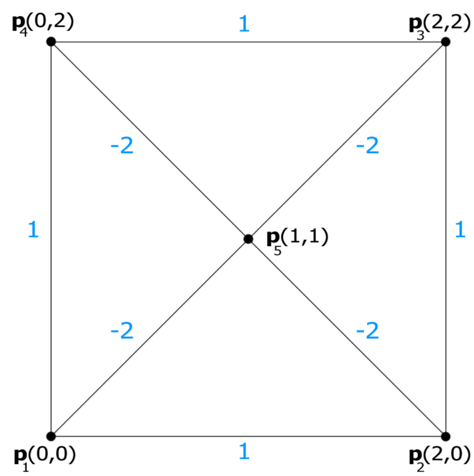
We can look to an example to bolster our confidence in the truth of this theorem and to provide some insight. The following embedding of K_4 has an edge crossing between its diagonals.



The corresponding rigidity matrix for this embedding is:

$$
\begin{array}{c}
 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} v_1 & & v_2 & & v_3 & & v_4 & \end{array} \\
\left[\begin{array}{cccccccc}
-2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 \\
-2 & -2 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & -2 & 2
\end{array}\right]
\end{array}
$$

The diagonals have stress values -1 for each edge.

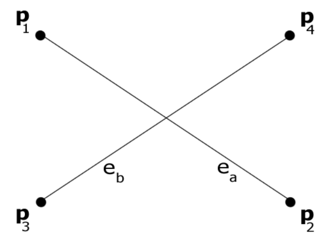


When we perform our X-replacement, we have the new graph and corresponding rigidity matrix:

$$
\begin{array}{c}
 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_7 \\ e_8 \\ e_9 \\ e_{10}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} v_1 & & v_2 & & v_3 & & v_4 & & v_5 & \end{array} \\
\left[\begin{array}{cccccccccc}
-2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1
\end{array}\right]
\end{array}
$$

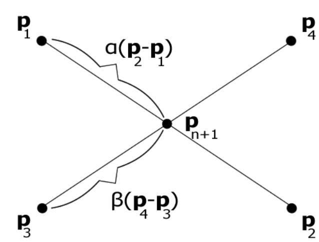
Our first four rows are exactly the same, but our new rows are different. One thing to notice is that if we have equal scalars multiplying the rows corresponding to edges e_7, e_8, e_9, e_{10}, adding up each row will result in the last two entries being 0. The values in our vertex columns v_1, v_2, v_3, and v_4 are similar to the values that were in the columns of rows e_5 and e_6, but multiplied by $1/2$ (which is what the lengths of each vector was shortened to respective to the original edge lengths). Thus, we can easily see that all we need to multiply each row by is 2, the inverse of what each edge length was shortened to.

We can look at the rigidity matroid of an X-replacement of an edge crossing in a general case to get more of an intuition for this. Given some generic graph containing a crossing with edges $e_a = (\mathbf{p}_1, \mathbf{p}_2)$ and $e_b = (\mathbf{p}_3, \mathbf{p}_4)$.



The mostly nonzero rows and columns of the rigidity matrix containing the edges e_α and e_β are as follows:

$$\begin{array}{c} \\ e_a \\ e_b \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} \mathbf{p}_1 - \mathbf{p}_2 & \mathbf{p}_2 - \mathbf{p}_1 & 0 & 0 \\ 0 & 0 & \mathbf{p}_3 - \mathbf{p}_4 & \mathbf{p}_4 - \mathbf{p}_3 \end{array}\right] \end{array}$$



Our edges are crossing at some point along the two edges. The position vector would be

$$\mathbf{p}_{n+1} = \mathbf{p}_1 + \alpha(\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{p}_3 + \beta(\mathbf{p}_4 - \mathbf{p}_3)$$

Where $0 < \alpha, \beta < 1$. We can now calculate what the values of the nonzero rows and columns of our edges in the rigidity matrix:

$$\begin{array}{c} \\ e_a^{(1)} \\ e_a^{(2)} \\ e_b^{(1)} \\ e_b^{(2)} \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_{n+1} \end{array} \\ \left[\begin{array}{ccccc} -\alpha(\mathbf{p}_2 - \mathbf{p}_1) & 0 & 0 & 0 & \alpha(\mathbf{p}_2 - \mathbf{p}_1) \\ 0 & (1-\alpha)(\mathbf{p}_2 - \mathbf{p}_1) & 0 & 0 & (1-\alpha)(\mathbf{p}_1 - \mathbf{p}_2) \\ 0 & 0 & -\beta(\mathbf{p}_4 - \mathbf{p}_3) & 0 & \beta(\mathbf{p}_4 - \mathbf{p}_3) \\ 0 & 0 & 0 & (1-\beta)(\mathbf{p}_4 - \mathbf{p}_3) & (1-\beta)(\mathbf{p}_3 - \mathbf{p}_4) \end{array}\right] \end{array}$$

It is quite easy to see that whatever the stress multipliers were on the original edges e_α and e_β, for each segmentation we can multiply by $1/\alpha$ or $1/(1-\alpha)$ for the segments of e_α, and for the segments of e_β we can multiply the stress by $1/\beta$ or $1/(1-\beta)$.

7 Conclusion

Now that we have the tool to replace an edge crossing with a vertex, we can answer William Baker's initial question in his talk using the properties of combinatorial rigidity and randomness on a 4-regular quad-dominated graph. Since we know that in the probability space of 4-regular graphs on n vertices, a given graph is both asymptotically almost surely globally rigid [4] and a.a.s. decomposes into two edge-disjoint Hamiltonian cycles [9], we can use this to create a particular and partially randomized graph.

The boundary of our graph will be given by a nicely embedded Hamiltonian cycle, with each vertex labeled from 1 to n so each edge joins consecutive numbers. To get our second Hamiltonian cycle, we will take a random permutation of the numbers 1 to n. As n gets larger, the probability that any two consecutive numbers will appear next to one another in the permutation gets smaller and smaller. This permutation will give us our second Hamiltonian cycle. This graph will have many crossings, and many of its "faces" will be quadrilaterals. We can reason that most of our faces will be quadrilaterals simply from the relationship between our face count and edge count; the average degree of any face will be 4, thus most of our faces must be quadrilaterals. The event of any three edges crossing at a single point is miraculous and thus a case we need not worry about. If it were to occur, this issue could be easily resolved by adjusting one of the initial vertices, removing the triple crossing.

The edge crossings can be replaced with a vertex of degree 4 using our X-replacement, and our graph will become planar while staying 4-regular and overbraced. Since we desire minimal dependence in our graph, we need only to remove any 2 edges sharing no vertex to obtain a planar graph that can be lifted.

The lifting formula is rather simple, and only depends on the face that was previously calculated. From this, future works can use this formula and the randomization procedure above to generate planar graphs and their face equations to determine the heights of each vertex. We would need a procedure that takes a path visiting each face of the graph and calculates their lift equation accordingly. This can be done by taking the dual graph, which is a minimally dependent rigidity graph [6], and finding a spanning tree within it. This spanning tree is how we would visit each face and calculate their lift.

There is already existing code for calculating the lifts that can be found in a previous project [5], so building an algorithm using his code and the method written above could be a CS project for future students to build upon this work.

Our project has answered the two questions that William Baker presented in his talk [1]. For his first question, we maximized the planar liftings and have shown through combinatorics that the position of the vertices in our generated graph do not particularly matter. For his second question, the polygonal boundary can be given by one of our Hamiltonian cycles, and through randomness of the second Hamiltonian cycle, as well as X-replacement, we ensure that the resultant graph is quad-dominated and that it will have maximal lifts due to minimal dependence within its rigidity matroid.

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