Non-Singular Anti-de Sitter Black Holes within Conformal Weyl Gravity and Black Hole CFT Duality

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WORCESTER POLYTECHNIC INSTITUTE

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To Palestine that one day she shall be free

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Abstract

Our research focuses on the universality of black hole entropy in the context of conformal Weyl gravity. To do so, we calculate the entropy of various vacuum and non-vacuum solutions using Nöther's current method and Wald's entropy formula. We examine the near horizon near the extremal Kerr metric, which is a vacuum solution to conformal Weyl gravity that has not been studied before. In addition, we analyze non-vacuum solutions by coupling the conformal Weyl gravity field equations to a near horizon (linear) $U(1)$ gauge potential. We found that the black hole entropy is not universal among our studied solutions with different symmetries. However, the respective entropies are consistent with Wald's entropy formula for the specific gravity theory. We also discuss the construction of a near-horizon CFT dual to one of our unique non-vacuum solutions, which requires the introduction of a parameter called γ that relates to the Weyl anomaly coefficient. Using AdS_2/CFT_1 correspondence, we com-

pute the full asymptotic symmetry group of the chosen non-vacuum conformal Weyl black hole and its near horizon quantum CFT dual. Finally, we provide a discussion and future research directions.

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Chapter 1

Introduction

It has been over a full century since Einstein introduced the theory of General Relativity (GR) [1]. Serving as an extension of Special Relativity (SR) [2], which exclusively deals with inertial reference frames, GR broadens the scope of SR and incorporates covariant physics in various reference frames, particularly non-inertial frames. This is accomplished by integrating the Equivalence Principle, which equates acceleration in non-inertial frames with gravitational forces. As a result, GR can be understood as a covariant theory of gravity. Furthermore, it significantly transforms our understanding of gravity by establishing a connection between the gravitational field and the metric of spacetime, which is a pseudo-Riemannian manifold. In a 4-dimensional context, when only the gravity field is considered, the

Lagrangian density simplifies to the scalar curvature, representing the most basic covariant function of the metric. The action, known as the Enistein-Hilbert action, can be expressed as

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \tag{1.1}$$

where G represents Newton's constant, various additional terms can be incorporated covariantly to account for different situations, including a cosmological constant, electromagnetic field, scalar fields, etc.

One of the most important consequences of general relativity is the prediction that gravitational fields can become so strong that they can efficiently trap particles and electromagnetic radiation such as light [3]. Such objects are called black holes. Black holes can be defined in various ways [4]. In the realm of astrophysics, they are compact objects that possess regions from which nothing can escape, and they serve as sources for immense power generation. In classical relativity, black holes are defined as the causal boundary of the past of future null infinity (event horizon), as well as an apparent horizon. In the context of semi-classical gravity, black holes are understood as thermodynamic systems with the highest possible entropy. In mathematical relativity, they are defined as an apparent horizon/isolated horizon and singularity.

One of the solutions to Einstein's equations is when the restriction to spherical symmetry is relaxed, which allows the black hole to be characterized by angular momentum [5] in addition to mass (i.e., rotating). Rotating black holes are of particular importance in astrophysics because they are thought to power quasars and other active galaxies [6–11], X-ray binaries, and gamma-ray bursts [12–17]. Stellar mass ($M_{\mathrm{BH}} < 10^2\, M_\odot$) and supermassive black holes ($M_{\mathrm{BH}} > 10^6\, M_\odot$) have been observed and well studied [18–21].

Since its inception, GR has undergone numerous experimental tests [22–25], including the recent detection of gravitational waves [26–28], and is widely recognized as the contemporary theory of gravity. However, alongside its successes, GR is plagued by fundamental issues that manifest as singularities in many of its solutions. These singularities, where a curvature invariant becomes infinite and cannot be resolved using classical methods, are particularly critical, as they are inevitable under general circumstances. For instance, such singularities occur inside the event horizon of a black hole. Their existence and inevitability demonstrate that GR is an incomplete theory of gravity, although this is consistent with the fact that it is a classical theory.

Additionally, the gravitational coupling constant, G, possesses a negative mass squared dimension [29] $(L = \hbar c^{-2} M^{-1})$ expressed as mass/energy squared (where $8\pi G = M_p^{-2}$). This characteristic renders general relativity a perturbatively nonrenormalizable theory. In an effort to address this issue, Stelle [30] proposed a solution by introducing two quadratic curvature invariants, thereby rendering the theory renormalizable. However, this modified Lagrangian introduces fourth-time derivatives of the metric, resulting in a propagator with a negative sign known as the ghost. Consequently, this ghost property renders the theory non-unitary. In fact, Ostrogradsky [31] demonstrated that any system containing time derivatives exceeding the second order is inherently unstable unless it is degenerate.

Furthermore, Hawking and Penrose's singularity theorems [32] have shown that general relativity does not provide a complete description of the behavior of spacetime at high curvatures. The prevailing belief is that the successful quantization of gravity will yield the necessary modifications to the general relativity theory essential to predicting geodesically complete spacetime manifold. However, since such a quantum theory of gravity does not exist, and even if and when it does, we will undoubtedly need to look to effective theories to understand its implications.

Then we may ask whether it is possible to construct an effective gravity theory that is intermediate between General Relativity and Quantum Gravity. Such a theory would give the correct (Einstein) behavior in the low curvature limit and would encompass the nonsingular nature that should be contained in full gravity theory.

1.1 Conformal Weyl Gravity (CWG)

In general relativity, the selection of the Einstein-Hilbert action is limited by the necessity for the equations of motion not to exceed the second order. While this produces relatively uncomplicated field equations compared to other theories, it fails to explain larger-scale observations without incorporating significant amounts of dark matter to account for galactic rotational curves. Consequently, numerous alternative actions are being explored, including conformal Weyl gravity [33–37]. This theory employs the concept of local conformal invariance of the spacetime manifold as the supplementary condition for determining the gravitational action, unlike the requirement for the equations of motion to remain below the second order.

$$g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu} \tag{1.2}$$

This results in fourth-order equations of motion for the gravitational field. Nonetheless, adopting the local conformal invariance, which is consistent with how actions are selected in field theory, results in a unique conformal Weyl gravity action

$$S_{CWG} = \alpha_g \int d^4x \sqrt{-g} W^{\alpha\mu\beta\nu} W_{\alpha\mu\beta\nu} \tag{1.3}$$

Moreover, this invariance principle may facilitate a stronger connection with the fundamental quantum nature of reality due to the additional symmetry inherent in the theory.

Recent and semi-recent studies have shown a growing interest in applying black hole holography within CWG and other fourth-order gravity theories. Various studies, including [38–42], have yielded intriguing findings that deviate from the predictions of general relativity. One such finding is the absence of universality, whereby the coefficient for black hole entropy no longer conforms to the expected value of $1/4$ and instead varies depending on the chosen spacetime. This departure from universality can be readily inferred by examining the Wald entropy for CWG [43–46]:

$$S = -\frac{\alpha_c}{8} \int W_{\mu\nu\alpha\beta} \varepsilon^{\mu\nu} \varepsilon^{\alpha\beta} d\Sigma, \tag{1.4}$$

where $d\Sigma$ is the orthogonal hypersurface area element, and its unit normal bivector is denoted by $\varepsilon^{\alpha\beta}$. It is observed that the Weyl tensor plays a crucial role in determining the Wald entropy in conformal gravity. As a consequence, black holes that conform to the condition of being conformally flat and exhibit universal area entropy laws in GR will have zero entropy in the context of conformal gravity.

1.2 AdS/CFT Duality

The principle of gauge/gravity duality is a fundamental concept in quantum gravity, asserting that a theory of gravity within a particular region of spacetime can be described by a quantum field theory residing on the boundary of that region. As previously discussed, a consistent and renormalizable quantum theory of gravity in 4 spacetime dimensions cannot be achieved. However, in cases where the cosmological constant is negative, gravity can be viewed as a quantum field theory in one lower dimension. The AdS/CFT correspondence provides the strongest evidence for this theory, suggesting that a theory of quantum gravity in d+1-dimensional anti-de Sitter space is equivalent to a conformal field theory residing on the d-dimensional boundary.

Additionally, it has been demonstrated by Brown and Henneaux [47] that the conventional canonical expression of this symmetry is established by means of the Poisson bracket, algebra of the generators, and algebra of charges, including a central extension identified as the central charge of CFT

$$c = \frac{3\ell}{2G}, \tag{1.5}$$

where ℓ is the radius of curvature of the AdS spacetime. Upon undergoing quantization, the Poisson algebra transforms into the Virasoro algebra, featuring a central extension that is proportionate to the Brown and Henneaux central charge.

$$[Q_m, Q_n] = (m-n)Q_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}, \tag{1.6}$$

Then one can use Cardy's formula [48, 49] to compute the entropy:

$$S = 2\pi\sqrt{\frac{c \cdot Q_0}{6}}. \tag{1.7}$$

The calculation of black hole entropy using the central charge c and normalized lowest eigenmode Q_0 of the dual CFT allows for a holographic determination of the number of microstates on the horizon.

Recently, the application of the Brown and Henneaux technique [50] has enabled the examination of holographic duality for more complex backgrounds. It has been demonstrated in [51] that a two-dimensional CFT dual of quantum gravity exists on the extreme Kerr background. Despite the unknown structure of the CFT, the central charge of the CFT can be determined by studying the non-trivial asymptotic symmetry of the extreme Kerr solution. Subsequently, the Cardy formula gives the microscopic entropy of the CFT, which aligns exactly with the microscopic entropy of the extreme Kerr background [51]. This duality has been expanded to other backgrounds ([52], [53], [54]).

Since the groundbreaking work of Brown and Henneaux [47] and the establishment of the $Kerr/CFT$ correspondence, it has been widely acknowledged that black holes have a holographic dual relationship with lower-dimensional conformal field theories (CFT) located at their boundaries. This bears resemblance to the broader anti-de Sitter (AdS)/CFT correspondence in string theory [55]. Consequently, a significant research effort spearheaded by Strominger [56, 57], Carlip [58, 33, 59, 60], Park [61–63] and others has been dedicated to applying CFT techniques for the computation of Bekenstein-Hawking entropy for various black holes, particularly near-extremal ones. This approach exploits the fact that near-extremal black holes exhibit a local $AdS_2 \times S^2$ topology within their proximity to the event horizon. This fact allows for the utilization of AdS/CFT techniques to construct dual lower-dimensional black hole $CFTs$, thereby offering a statistical quantum mechanical description of the microstates of black hole horizons. The outcomes of these dualities encompass the conventional thermodynamic quantities associated with black holes [64–68],

$$\begin{aligned} T_H &= \tfrac{\hbar c^3}{8\pi G k_B M} = \tfrac{\hbar \kappa}{2\pi} && \text{Hawking Temperature} \\ S_{BH} &= \tfrac{A}{4\hbar G} && \text{Bekenstein-Hawking Entropy} \end{aligned} \tag{1.8}$$

which are the standard tests for any potential theory of quantum gravity.

Goal & Outline of the Dissertation

The goal of this thesis is to explore the thermodynamic properties of black holes in both vacuum and non-vacuum solutions within the CWG paradigm and compute the respective black hole entropy of the different solutions within the Nöther current method. The dissertation is outlined as follows.

Chapter 2: In this chapter, we provide a concise overview of the principles of general relativity. Einstein's field equations will be explained, along with a detailed analysis of the Schwarzschild solution, which is a spherically symmetrical vacuum solution. We will also explore the characteristics of charged, rotating, and charged-rotating black hole solutions and provide a review of Schwarzschild Penrose diagrams..

Chapters 3, and 4 are dedicated to providing the appropriate context and motivational details necessary for understanding the methodology employed in acquiring the main results.

Chapter 5: Contains the first part of the original research. We compute entropy for the near horizon near extremal Kerr metric from counting the canonical quantum microstates via the boundary Nöther current method, within CWG and derive non-vacuum solutions of CWG. We compute the asymptotic quantum generators of the near horizon near extremal Kerr metric vacuum solution of CWG spacetime from a centrally extended Virasoro algebra with central charge generator algebra. We compute the black hole entropy for both CWG solutions and compare it to Wald's entropy.

Chapter 6: In this chapter, We construct a near horizon conformal field theory (CFT) that corresponds to one of our unique non-vacuum solutions.

Chapter 7: In conclusion, a discussion is presented, followed by an outlook of possible future research.

Chapter 2

General Relativity & Black Hole Solutions

2.1 General Relativity (GR)

In this chapter, we will adhere to the conventions as outlined in [69–71].

The equivalence principle is a fundamental component of theories of gravity. It originates from the weak equivalence principle in Newtonian gravity, which asserts that a test object's internal structure and composition do not influence its trajectory when freely falling. A test object is defined as one unaffected by electromagnetic forces or tidal gravitational fields. Einstein expanded on this principle by adding local and Lorentz invariance, that local non-gravitational experiments are independent of the velocity and position of the freely-falling reference frame, known as the Einstein equivalence principle (EEP). The principle of strong equivalence enhances the Einstein equivalence principle by affirming that objects that fall freely are not affected by their own structure and composition, even if they possess gravitational forces. To satisfy the EEP requirements, the metric representing the theory must meet certain criteria, such as symmetry ($g_{\mu\nu} = g_{\nu\mu}$), the trajectory of freely falling test objects along geodesics, and adherence to special relativity in local freely falling reference frames.

To describe a local field in both electrodynamics and gravity, it is necessary to define the potentials that convey it. The force acting on a test particle at a particular location can then be determined by calculating the gradients of these potentials. In the case of gravity, these quantities have both a dynamical and a geometrical interpretation. While the electromagnetic field is defined using a scalar and a vector potential that form the components of a four-vector $A_\mu(x)$, the gravitational field is described by 10 potentials that can be combined into a symmetric four-tensor $g_{\mu\nu}$. This tensor possesses a geometric interpretation as the metric of spacetime, which determines the invariant intervals (length of displacement dx^μ) ds in terms of local coordinates $x^{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1}$$

The metric of Minkowski space can be expressed in cartesian coordinates as (ct, x, y, z). It is important to take note that we have chosen to follow the convention where the temporal component is attributed with a negative sign, whereas the spatial components are assigned positive values. This convention will be used for all metrics presented throughout the rest of this dissertation.
The "flat" Minkowski metric, $\eta_{\mu\nu}$ and the coordinate vector dx^μ are respectively defined as

$$\eta_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad dx^{\mu} = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} \tag{2.2}$$

Then the invariant ds^2 can be expressed in the following form.

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{2.3}$$

For a test particle, the space-time intervals measured along its world line can be interpreted in a straightforward manner by using the proper time measured by a clock that is stationary relative to the particle. Since world lines are time-like, any interval along a world line satisfies $ds^2 < 0$, and the proper time interval $d\tau$ relative to the particle measured by a stationary clock is given by

$$c^2 d\tau^2 = -ds^2. \tag{2.4}$$

The local matrix inverse of the metric is denoted by $g^{\mu\nu}$:

$$g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}_{\alpha}, \tag{2.5}$$

where δ^{μ}_{α} is the Kronecker delta, and the Einstein summation convention has been used, which mandates that repeated upper and lower indices, such as ν in the equation's left-hand side, be fully summed over. From now and on we will use geometrized units ($c = 1$).

By utilizing the gradients of the potentials, i.e., of the metric tensor, we construct a quantity called the connection, also known as Christoffel symbols, with components $\Gamma^{\mu}_{\alpha\beta}$ given by the expression:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right) \tag{2.6}$$

In principle, the Christoffel symbols are the inverse metric multiplied by the partial derivatives of the metric with respect to different spacetime coordinates. The Christoffel symbols play a crucial role in defining the connection coefficients for the Levi-Civita connection. They have both geometric and physical interpretations, representing changes in basis vectors within a coordinate system and fictitious forces (or acceleration effects) in a non-inertial reference frame, respectively. As coordinate-dependent coefficients, the Christoffel symbols enable differentiation on an arbitrary Riemannian manifold.

The physical importance of Christoffel symbols can be seen easily from the geodesic equation, which is given as

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \tag{2.7}$$

The geodesic equation, Eq. 2.7, equates the rate of change of the object's 4-velocity (velocity in spacetime), i.e., acceleration, to the negative of the Christoffel symbols, which encodes how spacetime coordinates change due to spacetime curvature (which then causes acceleration) multiplied by the 4-velocity square. Note that the gravitational acceleration does not involve the mass of the object. This is the mathematical content of the equivalence principle, which, as mentioned before,

states that in a fixed gravitational field, all bodies fall with the same acceleration, independent of their mass.
Eqs.2.6 and 2.7 describe the force on an object in terms of a field (Christoffel symbols) and the field in terms of potential (the metric). This provides the basis for the general theory of relativity. However, before we give a concise formulation of Einstein's field equation, we must introduce another geometrical concept: curvature.

Curvature is defined in terms of the properties of surfaces, where it measures their divergence from a planar surface. This measure can be effectively articulated by observing the change of a vector as it is transported parallel to itself along a loop that encloses any two-dimensional surface element.

The Riemann tensor is a mathematical construct that describes the degree of curvature in a spacetime independent of any particular coordinate system. It quantifies the extent to which the spacetime deviates from being completely flat.

Christoffel symbols (Eq. 2.6) are symmetric in the lower indices, which implies that the action of two covariant derivatives on a scalar field f is not affected by the order in which the derivatives are applied

$$\nabla_\alpha \nabla_\beta f - \nabla_\beta \nabla_\alpha f = 0 \tag{2.8}$$

However, the same is not true when the covariant derivatives act on a vector field A^μ

$$\nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu = R^\mu_{\nu\alpha\beta} A^\nu \tag{2.9}$$

Additionally, the operators do not commute. (2.9) defines the *Riemann curvature tensor* $R^{\mu}_{\nu\alpha}$.

$$R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\nu\sigma} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \tag{2.10}$$

Contraction of the Riemann tensor, $R^{\alpha}_{\mu\beta\nu}$, over the 1^{st} and 3^{rd} indices, yields a symmetric 2^{nd} rank tensor known as the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} \tag{2.11}$$

A further contraction produces the scalar curvature, Ricci curvature scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.12}$$

2.1.1 Einstein field Equations

The non-vacuum dynamical field equations, Einstein field equations, follow from the Einstein-Hilbert action plus the matter action

$$S_{EH} + S_M = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathscr{L}_M \tag{2.13}$$

where $\kappa = 8\pi G$ is the gravitational coupling and is fixed by requiring GR to reproduce Newtonian gravity in the weak field limit (to guarantee covariance of the field equations). R represents the Ricci curvature scalar, while g denotes the determinant of the metric.

Starting with the Einstein-Hilbert action

$$\begin{aligned} S_{EH} &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}R \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu} \end{aligned} \tag{2.14}$$

The 2^{nd} equation of (2.14) is the Einstein-Hilbert action written in canonical form. Varying the action with respect to the inverse metric $g^{\mu\nu}$

$$\begin{aligned} \delta S &= \frac{1}{2\kappa}\int d^4x\delta\left(\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}\right) \\ &= \frac{1}{2\kappa}\int d^4x\Big\{\left(\delta\sqrt{-g}\right)g^{\mu\nu}R_{\mu\nu}+\sqrt{-g}R_{\mu\nu}\,\delta g^{\mu\nu}+\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}\Big\} \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{-\frac{1}{2}g_{\mu\nu}\,\delta g^{\mu\nu}g^{\mu\nu}R_{\mu\nu}+R_{\mu\nu}\,\delta g^{\mu\nu}+g^{\mu\nu}\,\delta R_{\mu\nu}\Big\} \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{-\frac{1}{2}g_{\mu\nu}\,\delta g^{\mu\nu}R+R_{\mu\nu}\,\delta g^{\mu\nu}+g^{\mu\nu}\,\delta R_{\mu\nu}\Big\} \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{-\frac{1}{2}g_{\mu\nu}R+R_{\mu\nu}\Big\}\delta g^{\mu\nu}+\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu} \end{aligned} \tag{2.15}$$

Rewriting the above equation

$$\begin{aligned} \delta S &= \frac{1}{2\kappa}\int d^4x\left(\sqrt{-g}\Big\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big\}\delta g^{\mu\nu}+g^{\mu\nu}\,\delta R_{\mu\nu}\right) \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big\}\delta g^{\mu\nu}+\frac{1}{2\kappa}\int d^4x\sqrt{-g}g^{\mu\nu}\,\delta R_{\mu\nu} \\ &= \delta S_1+\delta S_2 \end{aligned} \tag{2.16}$$

Working on δS_2 of (2.16)

$$
\begin{aligned}
\delta S_2 &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}g^{\mu\nu}\,\delta R_{\mu\nu} \\
&= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,g^{\mu\nu}\left(\nabla_\alpha\,\delta\Gamma^\alpha_{\mu\nu}-\nabla_\nu\,\delta\Gamma^\alpha_{\mu\alpha}\right) \\
&= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\left(\nabla_\alpha g^{\mu\nu}\,\delta\Gamma^\alpha_{\mu\nu}-\nabla_\nu g^{\mu\nu}\,\delta\Gamma^\alpha_{\mu\alpha}\right) \qquad (2.17)
\end{aligned}
$$

The above equation (2.17) can be simplified by invoking the metric compatibility, which is the ability to move the metric to either side of the derivative operation. A covariantr derivative is compatible with the metric if $\nabla_\alpha\,g_{\mu\nu}=0$. Then Eq. 2.17 becomes.

$$\delta S_2 \;=\; 0. \qquad (2.18)$$

Then the final result of the variation of the action becomes

$$\delta S \;=\; \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big\}\delta g^{\mu\nu}=0, \qquad (2.19)$$

and the vacuum Einstein equation is:

$$G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0, \qquad (2.20)$$

where $G_{\mu\nu}$ is Einstein tensor.

Looking at the matter action

$$S_M \;=\; \int d^4x\sqrt{-g}\,\mathscr{L}_M \qquad (2.21)$$

Varying the matter action

$$\begin{aligned}\delta S_M &= \int d^4x\,\delta\left(\sqrt{-g}\,\mathscr{L}_M\right) \\ &= \int d^4x \frac{\delta(\sqrt{-g}\mathscr{L}_M)}{\delta g^{\mu\nu}}\,\delta g^{\mu\nu}. \end{aligned} \tag{2.22}$$

However, the energy momentum tensor is defined as

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathscr{L}_M)}{\delta g^{\mu\nu}}. \tag{2.23}$$

Then

$$\begin{aligned}\delta(S_{EH}+S_M) &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{2\kappa}{-2}T_{\mu\nu}\Big\}\delta g^{\mu\nu} \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-\kappa T_{\mu\nu}\Big\}\delta g^{\mu\nu} = 0. \end{aligned} \tag{2.24}$$

Recall that $\kappa = 8\pi G$. Finally, yielding the non-vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{2.25}$$

The Einstein field equations can be written as

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{2.26}$$

where $T = T_\mu{}^\mu = g^{\mu\nu}T_{\mu\nu}$. Of course, in empty spacetime $T_{\mu\nu} = 0$; the second version of Einstein's equations (2.26) then implies that the Ricci tensor vanishes:

$$R_{\mu\nu} = 0. \tag{2.27}$$

Spacetime geometries that meet these conditions are referred to as *Ricci-flat*. An intriguing feature of these equations is that a void spacetime is still capable of sustaining significant gravitational fields: the Riemann curvature tensor can have non-zero components even if its contracted form, the Ricci tensor, exhibits a value of zero everywhere.

The significance of this observation is paramount as it forms the foundational basis for the theoretical postulations regarding the existence of gravitational waves. Non-trivial solutions of the Einstein equations can be represented as lumps in the gravitational field that propagate at the velocity of light. These lumps transport a finite amount of energy and momentum per unit of volume between two distinct flat regions of empty space.

2.2 Black Hole Solutions

Two distinct features define black holes. The first attribute, from which they derive their name, is the presence of a horizon that cannot be mapped to conformal infinity. The second attribute is a singularity, a point inside the horizon with an infinite curvature. Furthermore, it has been conjectured that these two features are closely linked and that any singularity within any spacetime should be concealed by a horizon. This conjecture is known as the Cosmic Censorship hypothesis [72].

This section presents several black hole solutions, including the spherically symmetric Schwarzschild, Reissner-Nordström, and axially symmetric Kerr. The Schwarzschild solution is distinct and represents the gravitational field outside a spherically symmetric mass. The Kerr metric is the unique solution describing all rotating black holes in vacuum. This solution describes the exterior gravitational field surrounding a rotating mass distribution. The Reissner-Nordström metric is a solution that characterizes the geometry of spacetime surrounding a non-rotating, spherically symmetric object possessing both mass (M) and electric charge (Q). In addition to the requirement of spherical symmetry, this solution assumes the absence of any matter within the space, with only an electromagnetic field present. All three mentioned above solutions have an event horizon that cannot be mapped to conformal infinity.

2.2.1 The Schwarzschild Solution

The Schwarzschild solution is a solution to the vacuum Einstein equations ($G_{\mu\nu} = R_{\mu\nu} = 0$, i.e., Ricci curvature tensor is flat), are valid for any spherically symmetric vacuum solution to Einstein's equations.
The Schwarzschild line elements [73] is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{2.28}$$

where $f(r) = 1 - \frac{2GM}{r}$, $d\Omega^2 = d\theta^2 + sin^2\theta\, d\phi$, which is the unit sphere line element.

The corresponding metric tensor $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}. \tag{2.29}$$

It can be observed from

$$g_{00} = -\left(1 - \frac{2GM}{r}\right), \qquad g_{11} = \left(1 - \frac{2GM}{r}\right)^{-1}, \tag{2.30}$$

that the parameter M, which is the sole independent variable in the solution, can be associated with the overall mass that serves as the origin of the gravitational curvature. Additionally, we have two singularities. A *curvature singularity* at $r = 0$ from g_{00} and a *coordinate singularity* at $r = r_s = 2GM$ from g_{11}. Coordinate singularities are *not curvature* (not physical) and can be removed by a different choice of coordinate system. Note that when $M \to 0$ we recover Minkowski space. Additionally, the metric becomes progressively Minkowskian as $r \to \infty$ (asymptotic flatness).

2.2.2 The Reissner–Nordström Solution

In contrast to the Schwarzschild (and Kerr, as we will see in §2.2.3) solutions, which are a vacuum solution of Einstein's equations, the Reissner–Nordström solution ([74],[75]) is not a vacuum solution.
The presence of an electric field results in the manifestation of an energy-momentum tensor ($T_{\mu\nu}$) that is not equal to zero, which is present throughout space.

The interaction between gravity and the electromagnetic field is described by the Einstein-Maxwell action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.31}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and A_μ is is the electromagnetic four potential given by

$$A_\mu = \left(\frac{Q}{r}, 0, 0, 0 \right), \tag{2.32}$$

where Q is the total charge measured by a distant observer.

The equations of motion derived from the variation of the Einstein-Maxwell action (2.31) are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 2 \left(F_{\mu\alpha} F_\nu^\beta - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \\ \nabla_\mu F^{\mu\nu} &= 0. \end{aligned} \tag{2.33}$$

They provide the spherically symmetric solution

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.34}$$

when $g_{00} = 0$ we have two horizons (inner, r_- and outer, r_+)

$$r_\pm = GM \pm \sqrt{G^2M^2 - G^2Q^2}. \tag{2.35}$$

We can see from Eq. 2.34 that the time coordinate has different behaviors depending on the value of r. When $r_+ < r < r_-$, t is a timelike coordinate (i.e., outside the outer horizon and inside the inner horizon). However, when $r_- < r < r_+$, t is a spacelike coordinate between the inner and outer horizons.

This is different behavior than the Schwarzschild solution. Outside the outer horizon, spacetime falls at the speed of light, as it does inside the inner horizon.

This is caused by electromagnetic repulsion (i.e., the negative pressure of the electric field)

In between the horizons, space behaves as in the Schwarzschild black hole; spacetime falls at velocities higher than the speed of light. If $M^2 = Q^2$, we would see that the inner and outer horizons are the same, providing just a 2-dimensional spherical surface where spacetime is falling at precisely the speed of light. Even though there is only a "shell" of spacetime with velocity c, light could not get out of the inside of the black hole. If $Q = 0$, we get, as expected from the last section, only one horizon corresponding to Schwarzschild's solution, $r = 2GM$. If $M^2 < Q^2$ we get a naked singularity.

2.2.3 The Kerr Metric

The Kerr metric [76] in the Boyer-Linquist coordinates [77] corresponds to the line element is:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GMr}{\Sigma^2}\right) dt^2 - \frac{4GMra\sin^2\theta}{\Sigma^2} d\phi dt + \frac{\Sigma^2}{\Delta} dr^2 + \Sigma^2 d\phi^2 \\ &+ \left(r^2 + a^2 + \frac{4GMra\sin^2\theta}{\Sigma^2}\right) \sin^2\theta d\phi^2, \end{aligned} \tag{2.36}$$

with

$$\begin{aligned} x &= (t, r, \theta, \phi) & a &\equiv J/M \\ \Delta &\equiv r^2 - 2GMr + a^2 & \Sigma &\equiv r^2 + a^2\cos^2\theta. \end{aligned} \tag{2.37}$$

The parameter denoted by a, commonly referred to as the Kerr parameter, possesses the dimensions of length when expressed in terms of geometrized units. The parameter J represents the angular momentum of the black hole, while the parameter M indicates its mass.

Several characteristics of the Kerr metric can be readily observed through an examination of the line element presented in (2.36).

- The Kerr metric represents a vacuum solution of the Einstein field equations and is valid in the absence of matter.
- Reduces to Schwarzschild metric: When $a = J/M = 0$ (with $M \neq 0$), the Kerr line element reduces to Schwarzschild line element.
- *Asymptotically flat*: The Kerr metric is asymptotically flat for $r \gg M$ and $r \gg a$.
- *Axisymmetric*: The Kerr metric is independent of t and ϕ, implying the existence of Killing vectors (i.e., the metric is preserved).
  - ❏ $\zeta_t = (1,0,0,0)$ (stationary metric) $\rightarrow$ time translation generated by ∂_t
  - ❏ $\zeta_\phi = (0,0,0,1)$ (axially symmetric metric) $\rightarrow$ rotational translation around ϕ generated by ∂_ϕ.
  - ❏ m In contrast to the Schwarzschild metric, the Kerr metric exhibits only axial symmetry.
- Has off-diagonal terms $g_{03} = g_{30} = -\left(\frac{2Mra\sin^2\theta}{\Sigma^2}\right)$ (inertial frame dragging).
- *Singularity and horizon structure*:
  - ❏ Coordinate singularity, $\Delta \rightarrow 0$ and $ds \rightarrow \infty$, assuming $a \ll M$

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

r_+ is the event horizon and it has non-zero Hawking temperature.

r_- is the Cauchy horizon.

- Physical singularity is a ring singularity

Additionally, it should be noted that the Kerr solution described in Eq. 2.36 yields three potential possibilities: when $GM > a$, when $GM < a$, and when $GM = a$. The condition where $GM = a$ is referred to as the extremal limit, which is unstable, and $GM < a$ features a naked singularity. These two cases are devoid of any physical interest.

2.3 Penrose-Carter Diagrams

Penrose-Carter conformal diagrams, commonly known as *Penrose diagrams* [72], provide a practical method for representing global and causal structure of spacetimes with sufficient symmetries. A coordinate transformation is required to depict the global structure accurately while preserving the appropriate causal structure in a 2-dimensional spacetime diagram. The transformation must bring infinities to finite coordinate distances and maintain the (radial) null curves on diagonal lines. This is accomplished through *conformal compactification*.

A conformal transformation, see Chapter 3, is a local re-scaling of the form

$$\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}, \tag{2.38}$$

or, equivalently,

$$\tilde{ds}^2 = \Omega^2(x)\, ds^2. \tag{2.39}$$

The transformation leaves the light-cone structure invariant because the tangent vector $T^{\mu\nu}$ of a curve $x^\mu(\lambda)$ that is null with respect to $g_{\mu\nu}$, i.e.

$$T_\mu T^\mu = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{2.40}$$

is also null with respect to $\tilde{g}_{\mu\nu}$.

We can draw the Penrose diagram through a series of coordinate transformations. See Figure 2.2.

First, we use tortoise coordinates

$$\left.\begin{matrix} u \\ v \end{matrix}\right\} = t \mp r^*, \tag{2.41}$$

we obtain

$$dr^* = \left(1 - \frac{2GM}{r}\right)^{-1} dr\,, \tag{2.42}$$

by integration, r^* is

$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|, \tag{2.43}$$

then the Schwarzschild metric (2.29) becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)(dt - dr^*)(dt + dr^*) + r^2 d\Omega^2. \tag{2.44}$$

Second, we use the Eddington-Finkelstein coordinates

$$\begin{aligned} v &\equiv t + r^* = t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right|, \\ u &\equiv t - r^* = t - r - 2GM \ln\left|\frac{r}{2GM} - 1\right|, \end{aligned} \tag{2.45}$$

then the (t,r) part of the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv, \tag{2.46}$$

where $r = r(u,v)$.

Third, we use the Kruskal-Szekeres coordinates When $r > 2GM$, these coordinates are defined by

$$\begin{aligned} V &= e^{\frac{v}{4GM}}, \\ U &= -e^{-\frac{u}{4GM}}, \end{aligned} \tag{2.47}$$

and the metric (2.46) is written as

$$ds^2 = -\frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} dV\, dU, \qquad \text{when} \quad r > 2GM. \tag{2.48}$$

When $r < 2GM$, these coordinates are defined by

$$\begin{aligned} V &= e^{\frac{v}{4GM}}, \\ U &= e^{-\frac{u}{4GM}}, \end{aligned} \tag{2.49}$$

and the metric (2.46) is then written as

$$ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} dV dU, \qquad \text{when} \quad r < 2GM. \tag{2.50}$$

Fourth, we use the following defined coordinates

$$\begin{aligned} \tilde{V} &= \tan^{-1}\left(\frac{V}{4M\sqrt{2GM}}\right), \\ \tilde{U} &= \tan^{-1}\left(\frac{U}{4M\sqrt{2M}}\right). \end{aligned} \tag{2.51}$$

where the infinities appeared in V or U are converted to finite values $\pm\pi/2$, Figure 2.1.

Finally, we use the following defined coordinates

$$\begin{aligned} \tilde{T} &= \tfrac{1}{2}\left(\tilde{V} + \tilde{U}\right), \\ \tilde{R} &= \tfrac{1}{2}\left(\tilde{V} - \tilde{U}\right). \end{aligned} \tag{2.52}$$

where

$$R^{\pm} = \begin{cases} t \to \pm\infty \\ r = 0 \end{cases} \tag{2.53}$$

represent the singularity region.

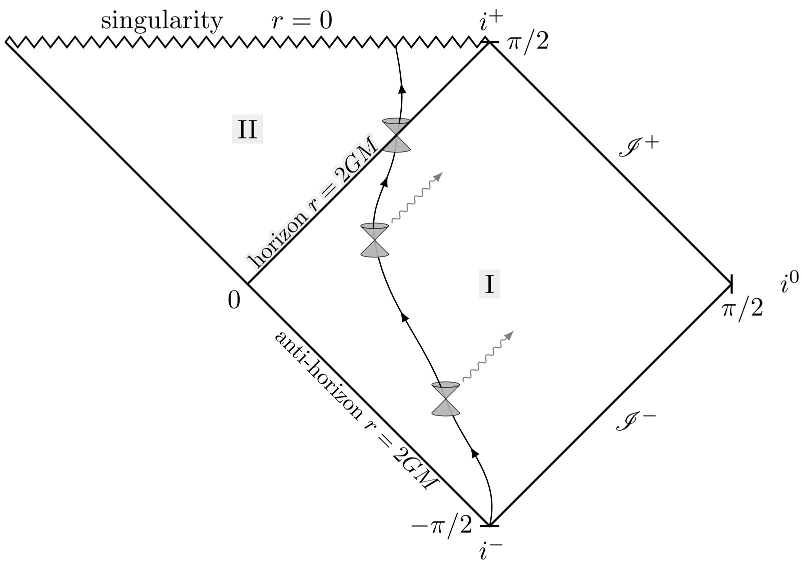

Fig. 2.1: Penrose diagram of the Schwarzschild patch

The Penrose diagram is a graphical representation that can depict infinite time or radial coordinates as points or lines. Moreover, null geodesics are portrayed as lines with an inclination of $\pm 45^\circ$ to the vertical axis. Each point on the diagram corresponds to a 2-dimensional sphere $4\pi r^2$, with angular coordinates θ and φ attached to each coordinate point. As a result, the Penrose diagram is also referred to as the conformal diagram. The Penrose diagram is partitioned into four distinct regions, delineated by the two diagonal lines H^+ and H^-. These regions are as follows: I represents our universe; II corresponds to a black hole; III denotes another universe that experiences time in reverse compared to our own; finally, IV represents a white hole that emits matter from its horizon and is the time reversal of a black hole. It is noteworthy that while null geodesics in region I can reach $\mathscr{I}^+$ or the black hole through the horizon H^+, null geodesics in region II, located

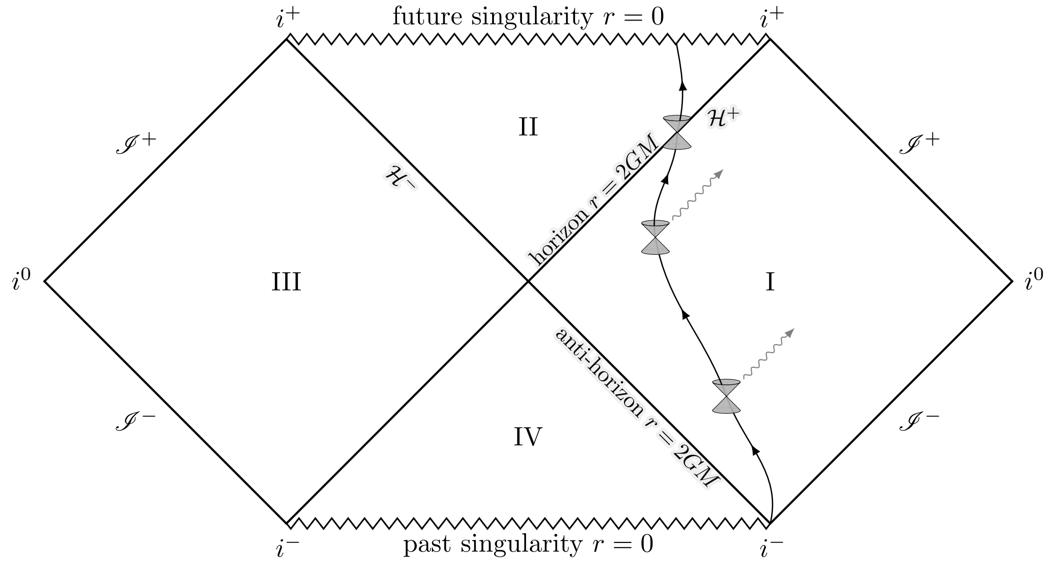

Fig. 2.2: Penrose diagram for the maximally extended Schwarzschild spacetime

inside the black hole, are unable to reach our universe through the same horizon, i.e., H^+.

Chapter 3

Conformal Weyl Gravity (CWG)

3.1 Motivations for CWG

Effective field theories have played a crucial role in the progress of physics thus far, as they have allowed us to extract a vast amount of information about the physics of various phenomena despite our incomplete understanding of the fundamental unified theory of quantum gravity. By considering the appropriate low-energy effective theories, we have been able to calculate observable quantities with high precision and make predictions about the behavior of a given quantum field theory at the energy scale of interest. The renormalization group approach helps us deal with the limitations of every quantum field theory, which has a cutoff scale beyond which its validity breaks down, by providing a way to organize our lack of knowledge about what happens beyond the cutoff in terms of what we do know about the couplings among low-energy degrees of freedom. This enables us to make predictions about the QFT based on the determination of these IR couplings.

Conformal Weyl Gravity (CWG) presents itself as a viable alternative to General Relativity as a renormalizable quantum gravitation theory. It offers solutions to some of the issues that arise within GR. CWG [37] has the ability to generate linear terms that account for the rotation curve of galaxies ("dark matter" problem), thereby negating the need for dark matter while also confirming the accuracy of Newtonian gravity in the solar system. All vacuum solutions of Einstein's gravity are also exact vacuum solutions in CWG (like Schwarzschild, Kerr spacetimes, etc.). Moreover, CWG provides an explanation for the accelerated expansion of the universe without requiring the presence of dark energy. Despite its potential, the theory has not been taken seriously due to its tendency to produce ghosts, as its equations of motion are fourth-order.

3.2 Weyl Gravity

CWG [35–37] adds conformal symmetry in addition to diffeomorphism symmetry from general relativity.

The conformal transformation, is given as

$$g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu}, \tag{3.1}$$

under these transformations the Weyl tensor transforms as

$$W_{\alpha\mu\beta\nu} \to \Omega^2(x)\, W_{\alpha\mu\beta\nu}. \tag{3.2}$$

Then the CWG action, which is invariant under conformal transformation is given by:

$$S_{CWG} = \alpha_c \int d^4x\sqrt{-g}W^{\alpha\mu\beta\nu}W_{\alpha\mu\beta\nu}, \tag{3.3}$$

where α_c is dimensionless coupling coefficient, and it is locally conformally invariant (has no scale), and $W_{\alpha\mu\beta\nu}$ is the Weyl tensor, conformal curvature tensor [78, 3] defined as

$$\begin{aligned} W_{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu} - \frac{1}{n-2}\left(g_{\mu\nu}R_{\alpha\beta} - g_{\mu\beta}R_{\alpha\nu} + g_{\alpha\beta}R_{\mu\nu} - g_{\alpha\nu}R_{\mu\beta}\right) \\ + \frac{R}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right). \end{aligned} \tag{3.4}$$

where n is the spacetime dimension, in this case 4. The dimensionless coupling coefficient may initially make CWG seem like a promising approach to exploring quantum gravity since it incorporates cosmological dynamics within its vacuum. However, the fourth-order nature of CWG and its potential violations of unitarity give rise to significant challenges [79]. It is nonetheless an attractive theory and includes all the standard solar-system tests of general relativity and warrants investigation in its own right.

CWG is diffeomorhically, conformally, and scale invariant. The Weyl tensor is constructed so that contraction between any two indices gives 0.

$$g_{\alpha\beta}W^{\alpha\mu\beta\nu} = 0. \tag{3.5}$$

The Weyl tensor (3.4) is the trace-free part of the Riemann tensor (2.9). In n-dimensional spacetime, the Rienmann tensor can be decomposed into its trace-free and trace part as

$$
\begin{aligned}
R_{\alpha\mu\beta\nu} &= W_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(g_{\mu\nu}R_{\alpha\beta} - g_{\mu\beta}R_{\alpha\nu} + g_{\alpha\beta}R_{\mu\nu} - g_{\alpha\nu}R_{\mu\beta}\right) \\
&- \frac{R}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right). \qquad (3.6)
\end{aligned}
$$

Contracting (3.6) with itself in 4D ($n = 4$) gives

$$
\begin{aligned}
R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu} &= W_{\alpha\mu\beta\nu}W^{\alpha\mu\beta\nu} + \frac{4}{(n-2)}R^{\alpha\mu}R_{\alpha\mu} - \frac{2}{(n-2)(n-1)}R^2 \\
&= W_{\alpha\mu\beta\nu}W^{\alpha\mu\beta\nu} + 2R^{\alpha\mu}R_{\alpha\mu} - \frac{1}{3}R^2, \qquad (3.7)
\end{aligned}
$$

and the Weyl tensor squared is now

$$
W_{\alpha\mu\beta\nu}W^{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu} - 2R^{\alpha\mu}R_{\alpha\mu} + \frac{1}{3}R^2. \qquad (3.8)
$$

Substituting (3.8) into (3.3) and simplifying, the action (3.3) becomes

$$
\begin{aligned}
S_{CWG} &= \alpha_c \int d^4x\sqrt{-g}W^{\alpha\mu\beta\nu}W_{\alpha\mu\beta\nu} \\
&= \alpha_c \int d^4x\sqrt{-g}\left\{R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu} - R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right\}. \qquad (3.9)
\end{aligned}
$$

However, looking at part of the integrand of (3.11)

$$
\sqrt{-g}\left\{R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu} - R_{\mu\nu}R^{\mu\nu} + R^2\right\}, \qquad (3.10)
$$

which is a total divergence [80, 81], then (3.11) can be simplified further

$$
S_{CWG} = \alpha_c \int d^4x\sqrt{-g}\left\{R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right\}. \qquad (3.11)
$$

The EOM for CWG can be expressed using the variation of (3.11) with respect to the inverse metric $g^{\mu\nu}$

$$\begin{aligned}\delta S_{CWG} &= \alpha_c \int d^4x \delta \left\{ \sqrt{-g} \Big(R^{\alpha\mu} R_{\alpha\mu} - \frac{1}{3} R^2 \Big) \right\} \\ &= \alpha_c \int d^4x \left\{ \delta \sqrt{-g} \Big(R^{\alpha\mu} R_{\alpha\mu} - \frac{1}{3} R^2 \Big) + \sqrt{-g} \delta \Big(R^{\alpha\mu} R_{\alpha\mu} - \frac{1}{3} R^2 \Big) \right\} \end{aligned} \quad (3.12)$$

and can be rewritten as

$$\frac{1}{\alpha} \frac{1}{\sqrt{-g}} \frac{\delta S_{CWG}}{\delta g_{\beta\nu}} = -2\alpha_c W_{\mu\nu}, \quad (3.13)$$

where $W^{\mu\nu}$ is the specific gravitational rank-two tensor of the CWG and it is analog to Einstein tensor $G^{\mu\nu}$, and defined as

$$W_{\mu\nu} = 2\nabla_\alpha \nabla_\beta W_\mu{}^{\alpha\beta}{}_\nu + R_{\alpha_\beta} W_\mu{}^{\alpha\beta}{}_\nu \, . \quad (3.14)$$

Adding the variation of the matter action, the full action is now

$$-2\alpha_c W^{\mu\nu} + -\frac{1}{2} T^{\mu\nu} = 0, \quad (3.15)$$

Then (3.15) is the equivalent of (2.25) and is the field equation in Weyl gravity. A spherically symmetric vacuum solution of the fourth-order [35] of the field equation given above is given as

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - k^2 r, \quad (3.16)$$

where β, γ, and k are integration constants. Note that when $-3\beta\gamma + \gamma r$ is too small the solution reduces to Schwarzschild in de-Sitter spacetime. therefore, k might be identified with the de-Sitter scalar curvature. Where in Einstein's gravity, an explicit cosmological term is required to obtain a similar solution.

There has been growing attention in the field of CWG and other fourth-order gravity theories towards the application of black hole holography [38, 82, 46, 83, 84]. This has led to a number of intriguing findings that challenge the predictions of general relativity. One notable discrepancy is the absence of universality in black hole entropy, whereby the coefficient deviates from the expected value of $\frac{1}{4}$ and instead varies depending on the specific spacetime chosen. This lack of universality becomes evident when examining the Wald entropy for CWG [43, 85, 45, 46]:

$$S = -\frac{\alpha_c}{8}\int W_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu}\varepsilon^{\alpha\beta}d\Sigma, \tag{3.17}$$

where $d\Sigma$ is the orthogonal hypersurface area element and $\varepsilon^{\mu\nu}$ is its unit normal bivector. (3.17) shows clearly that the Wald entropy in CWG depends on the Weyl tensor. Consequently, when considering conformally flat black holes in the framework of general relativity, it can be observed that they possess universal area entropy laws. However, within the context of CWG, these black holes will exhibit an entropy value of zero. One noteworthy example is the RN/CFT correspondence [86–89] with spacetime element:

$$ds^2 = Q^2\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\Omega^2\right), \tag{3.18}$$

where Q is the black hole charge parameter and $d\Omega^2$ is the unit two sphere. The above line element is conformally flat for the spacetime diffeomorphism and conformal factor ($e^{2\omega}$) given by:

$$\begin{aligned} &T = t, && x = -\frac{\sin\theta\cos\phi}{r}, \\ &y = -\frac{\sin\theta\sin\phi}{r}, && z = -\frac{\cos\theta}{r} \\ &\text{and } e^{2\omega} = Q^2 r^2, \end{aligned} \tag{3.19}$$

and therefore, has a vanishing Weyl tensor. Another example is the Weyl rescaled near horizon Schwarzschild spacetime within a finite mass/temperature gauge [39, 90]:

$$ds^2 = -\frac{r^2 - 2GMr}{r^2}dt^2 + \frac{r^2}{r^2 - 2GMr}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{3.20}$$

which exhibits conformal flatness through the diffeomorphism and conformal factor:

$$T = \frac{\exp\{\{\frac{t}{4GM}\}\}}{GM}\cosh u; \qquad x = \frac{\exp\{\{\frac{t}{4GM}\}\}}{GM}\sinh u \cos\phi \sin\theta;$$

$$y = \frac{\exp\{\{\frac{t}{4GM}\}\}}{GM}\sinh u \sin\phi \sin\theta; \qquad z = \frac{\exp\{\{\frac{t}{4GM}\}\}}{GM}\sinh u \cos\theta;$$

$$e^{2\omega} = \exp\left\{\left\{\frac{-t}{2GM}\right\}\right\}\ell^2 r(r - 2GM); \quad \text{and } u = \ln\sqrt{1 - \frac{2GM}{r}}. \tag{3.21}$$

Chapter 4

Quantum Field Theory in Curved Spacetime

4.1 Quantum Field Theory (QFT)

In this chapter, we will adhere to the conventions and approaches as outlined in [91–96].

Quantum field theory (QFT) is a relativistically invariant extension of quantum mechanics that applies to continuous field systems. The primary difference between QFT and traditional quantum mechanics is identifying the underlying degrees of freedom (N), which are the fields.

$$\lim_{N\to\infty} QM = QFT \tag{4.1}$$

While the utilization of continuous fields to depict discrete objects may seem peculiar, when these fields are quantized, discrete quantum excitations can emerge, which can be interpreted as particles. Experimental results support this interpretation. In contrast, traditional quantum mechanics does not adequately account for particle creation and annihilation at high energies, which makes it incompatible with special relativity. QFT resolves this issue and reduces to traditional quantum mechanics in appropriate limits.

Quantum mechanics, which emerged in the early 20^{th} century, introduced the concept of representing physical states using elements in a Hilbert space and physical observables using Hermitian operators. The Schrödinger equation governs the unitary evolution of physical states. The process of transitioning from classical mechanics to quantum mechanics, known as quantization, involves applying the same form of quantum particle Hamiltonian as the classical Hamiltonian and replacing classical Poisson brackets with quantum commutators in the first quantization. Second quantization applies a similar quantization process to fields, treating classical field Hamiltonian density as quantum field Hamiltonian density and replacing classical Poisson brackets with equal-time commutation relations, transforming fields into operators. The coefficients of the quantum fields are interpreted as creation and annihilation operators, facilitating the building of the Hilbert space for multi-particle states, also referred to as the Fock space. This method is known as "canonical quantization." There are other approaches for quantization, such as The Gupta-Bleuler method of quantization, unlike canonical quantization, which preserves full Lorentz symmetry, constitutes a significant benefit. However, the approach is not without drawbacks, as it permits the propagation of ghosts or non-physical states that possess negative norms. These states can only be eliminated by imposing constraints on the state vectors. Additionally, there is stochastic quantization, which preserves gauge invariance, the Bechi-Rouet-Stora-Tyupin

(BRST) approach, the Batalin-Vilkovisky (BV) approach, and the path integral method.

4.1.1 Canonical Quantization of Free Fields

We start with a classical free scalar field theory in Hamiltonian formalism. This theory is subsequently transformed into a quantum theory by substituting Poisson brackets with commutators. Due to its reliance on canonical formalism, it is crucial to verify that the resulting quantum theory is Lorentz invariant. Here, we provide a succinct overview of the canonical quantization process for the Klein-Gordon scalar field.

4.1.1.1 Quantization of Scalar Field in Minkowski Spacetime

The scalar field can be written as $\varphi(x)$. In the context of classical field theory, the Lorentz-invariant Lagrangian density of a scalar field can be expressed as follows

$$\mathscr{L} = \frac{1}{2}\partial_0\varphi\partial_0\varphi - \frac{1}{2}\partial_i\varphi\partial_i\varphi - \frac{1}{2}m^2\varphi^2, \tag{4.2}$$

where m can be considered as the mass of the field, $\partial_0 \equiv \frac{\partial}{\partial x^0}$, $\partial_i \equiv \frac{\partial}{\partial x^i}$, $x^0 \equiv t$, and x^i denotes Cartesian coordinates x, y, and z. Using Minkowski metric (2.2) $\eta^{\mu\nu}$

$$\mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2, \tag{4.3}$$

varying the action

$$\delta S = \frac{1}{2}\int dx^4\delta\left(\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right) = 0. \tag{4.4}$$

We use the Euler-Lagrange equation to find the equation of motion

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathscr{L}}{\partial \varphi} = 0. \tag{4.5}$$

We define the conjugate momentum field π to φ as

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_t \varphi)} = \partial_t \varphi, \tag{4.6}$$

then the equation of motion for the field is

$$\eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi = 0, \tag{4.7}$$

where $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is defined as the d'Alembertian. The above equation becomes

$$\Box \varphi - m^2 \varphi = 0, \tag{4.8}$$

(4.8) is known as the Klein-Gordon equation. In order to solve the Klein-Gordon equation, we must rewrite it as

$$-\frac{\partial^2 \varphi}{\partial t} + \nabla^2 \varphi - m^2 \varphi = 0 \tag{4.9}$$

where one classical solution is a plane-wave

$$\begin{aligned} \varphi(t,x) &= \varphi_0 e^{i\vec{k}.\vec{x} - i\omega t} \\ &= \varphi_0 e^{ik_\mu x^\mu}, \end{aligned} \tag{4.10}$$

where the wave vector is given by $k^\mu = (\omega, \vec{k})$ and ω is

$$\omega = \sqrt{\vec{k}^2 + m^2}. \tag{4.11}$$

We define the scalar product as

$$\begin{aligned}(f_1, f_2) &= -i \int d^{4-1}x \Big(f_1(x) \partial_t f_2^*(x) - (\partial_t f_1(x)) f_2^*(x) \Big) \\ &= -i \int d^3x \Big(f_1(x)\, \partial_t f_2^*(x) - f_2^*(x)\, \partial_t f_1(x) \Big).\end{aligned} \tag{4.12}$$

Then the f_k modes of (4.10)) are orthogonal if

$$(f_k, f_{k'}) = 0, \tag{4.13}$$

where $k \neq k^{'}$. If we choose

$$\begin{aligned}f_k &= \sqrt{\frac{1}{2\omega(2\pi)^3}}\, e^{i\vec{k}.\vec{x} - i\omega t} \\ &= \sqrt{\frac{1}{16\pi^3\omega}}\, e^{i\vec{k}.\vec{x} - i\omega t}.\end{aligned} \tag{4.14}$$

But we need the complex conjugate, f_k^*, with negative frequency modes. The orthonormal relationships between these modes are

$$\begin{aligned}(f_{\vec{k}_1}, f_{\vec{k}_2}) &= \delta(\vec{k}_1 - \vec{k}_2) \\ (f_{\vec{k}_1}, f_{\vec{k}_2}^*) &= 0 \\ (f_{\vec{k}_1}^*, f_{\vec{k}_2}^*) &= -\delta(\vec{k}_1 - \vec{k}_2).\end{aligned} \tag{4.15}$$

Now that we have a complete set of orthonormal modes, the Klein-Gordon equation becomes

$$\varphi(t,\vec{x}) = \int d^3\vec{k}\left(a_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + a^\dagger_{\vec{k}} f^*_{\vec{k}}(t,\vec{x})\right). \tag{4.16}$$

The fields, $f_{\vec{k}}$ and $f^*_{\vec{k}}$, are quantized in the canonical quantization scheme by treating the field φ as an operator and imposing the following equal time commutation relations:

$$\begin{aligned}
[\varphi(t,\vec{x}), \varphi(t,\vec{x}')] &= 0 \\
[\pi(t,\vec{x}), \pi(t,\vec{x}')] &= 0 \\
[\varphi(t,\vec{x}), \pi(t,\vec{x}')] &= i\delta(\vec{x}-\vec{x}'),
\end{aligned} \tag{4.17}$$

remember π is the canonically conjugate variable to φ given by (4.6).

To get the commutation relation between the coefficients, we used (4.6) and plugging (4.16) into the last equation of (4.17).

$$\begin{aligned}
[a_{\vec{k}}, a_{\vec{k}'}] &= 0 \\
[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] &= 0 \\
[a_{\vec{k}}, a^\dagger_{\vec{k}'}] &= \delta(\vec{k}-\vec{k}'),
\end{aligned} \tag{4.18}$$

$a^\dagger_{\vec{k}}$ is called the creation operator and $a_{\vec{k}}$ the annihilation operator, i.e., creation operator $a^\dagger_{\vec{k}}$ creates a particle of momentum $\vec{k}$ and $a_{\vec{k}}$ destroys a particle of the same momentum. The vacuum or no-particle state $|0\rangle$ has the property that it is annihilated by all the $a_{\vec{k}}$ operators

$$a_{\vec{k}}\,|0\rangle = 0. \tag{4.19}$$

The one-particle state, represented by $|1_{\vec{k}}\rangle$ is produced when $a^{\dagger}_{\vec{k}}$ is operated on $|0\rangle$

$$|1_{\vec{k}}\rangle = a^{\dagger}_{\vec{k}}\,|0\rangle\,. \tag{4.20}$$

A multi-particle state can be formed by applying multiple creation operators to the vacuum state:

$$|n_{\vec{k}_1}, n_{\vec{k}_2}, \ldots, n_{\vec{k}_m}\rangle = \prod_{i=1}^{m} \frac{(a^{\dagger}_{\vec{k}_i})^{n_{\vec{k}_i}}}{\sqrt{n_{\vec{k}_i}!}}\,|0\rangle\,. \tag{4.21}$$

To determine the number of particles $n_{\vec{k}_i}$ with a particular momentum $\vec{k}_i$, one can apply the number operator, $N_{\vec{k}_i} \equiv a^{\dagger}_{\vec{k}_i} a_{\vec{k}}$, to the state of multi-particle.

$$N_{\vec{k}_i}\,|n_{\vec{k}_1}, n_{\vec{k}_2}, \ldots, n_{\vec{k}_m}\rangle = \Big(\sum_{i=1}^{m} n_{\vec{k}_i}\Big)\,|n_{\vec{k}_1}, n_{\vec{k}_2}, \ldots, n_{\vec{k}_m}\rangle\,. \tag{4.22}$$

4.2 Quantum Field Theory in Curved Spacetime

4.2.1 Effective Action in Curved spacetime

Examine a theory that is represented by a Lagrangian, which involves various fields. From this, we can develop the classical action [97],

$$S(\varphi) = \int d^4x \mathscr{L}[\varphi(x)], \tag{4.23}$$

similar as before φ is a scalar field and

$$\mathscr{L}[\varphi(x)] = \varphi(x)\Big(-\Box + V(\varphi)\Big)\varphi(x), \tag{4.24}$$

and the Euclidean action as

$$S_E[\varphi, g_{\mu\nu}] = \frac{1}{2}\int d^2x\sqrt{g}\Big\{\varphi(x)\big[-\Box + V(\varphi)\big]\varphi(x)\Big\}, \tag{4.25}$$

where

$$\Box\varphi \equiv \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\varphi\right),$$

is the usual d'Alembertian on a scalar.
It is necessary to select the Euclidean field φ and the metric $g_{\mu\nu}$ as real-valued functions, even though transforming a metric with the Lorentzian signature through a coordinate transformation typically results in a complex-valued metric after applying the Wick rotation.

Then the Euclidean generating functional is defined as

$$Z \equiv \int D\varphi\, e^{-S(\varphi)}. \tag{4.26}$$

We define a new generating functional as

$$\Gamma_{eff} = -\ln(Z). \tag{4.27}$$

Let's consider an eigenvalue equation for some differential operator $\Box$ with $V(\varphi) = 0$ in (4.24)

$$-\Box\varphi(x) = \lambda_n \varphi_n, \tag{4.28}$$

an arbitrary function $\varphi(x)$ can be expanded as

$$\varphi(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x), \tag{4.29}$$

where φ_n are orthogonal basis of Hilbert space.

To diagnolize the action (4.25) in the basis of the eigenfunctions φ_n, we substituting 4.29 into it

$$S_E[\varphi, g_{\mu\nu}] = \frac{1}{2}\sum_{n=0}^{\infty} c_n^2 \lambda_n. \tag{4.30}$$

Now we can express $D\varphi$ in term of c_n and since c_n and λ_n are coordinate independent we define $D\varphi$ as

$$D\varphi = \prod_{n=0}^{\infty} \frac{dc_n}{\sqrt{2\pi}}, \tag{4.31}$$

then the path integral (4.26) becomes

$$\int \prod_{n=0}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n c_n^2} = \left[\prod_{n=0}^{\infty} \lambda_n\right]^{-\frac{1}{2}}. \tag{4.32}$$

It is a commonly known fact that the result of multiplying all eigenvalues of a finite-dimensional operator is equivalent to its determinant. In the event that a proper extension of the determinant for infinite-dimensional operators is present, we can formally express the Euclidean effective action (4.27) in the following form:

$$\Gamma_{eff} = \frac{1}{2} \ln \prod_{n=0}^{\infty} \lambda_n \equiv \frac{1}{2} \ln \det \Box. \tag{4.33}$$

Therefore, the calculation of the effective action is now simplified to the task of evaluating the determinant of a differential operator, referred to as a functional determinant. Nevertheless, it is evident that a functional determinant lacks a precise definition. For instance, the eigenvalues λ_n of the differential operator $\Box$ increase with n, leading to a divergence in their product. To obtain a finite outcome, it is necessary to employ the ζ function regularization to compute the determinant.

However, prior to solving the functional determinant of the operator $\Box$, let's construct a complete basis of "generalized vectors" $|x\rangle$ normalized as

$$\langle x|x'\rangle = \delta(x - x'), \tag{4.34}$$

where the unit operator is

$$\mathbb{1} = \int d^2x \, |x\rangle \langle x| \, . \tag{4.35}$$

Now, the eigenvalue problem (4.28) becomes

$$\Box_g \, |\psi_n\rangle = \lambda_n \, |\psi_n\rangle \, , \tag{4.36}$$

and its eigenvectors $|\psi_n\rangle$ normalized as

$$\langle \psi_n | \psi_m \rangle = \delta_{nm}. \tag{4.37}$$

For our operator $\Box_g$, we define the *zeta* function $\zeta_{\Box_g}(s)$ as

$$\zeta_{\Box_g}(s) \equiv \mathrm{Tr}\{\Box_g^{-s}\} \equiv \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n}\right)^s. \tag{4.38}$$

It should be noted that the above ζ function bears resemblance to Riemann's ζ function, except for the inclusion of summation over the eigenvalues λ_n instead of the integers. Equation (4.38) converges for real $s > 0$. However, for the cases of s where it does not diverge, we use analytical continuation, which extends it to all complex values of s except at the poles where it is infinite. Taking the derivative of (4.38) with respect to s we get

$$\frac{\zeta_{\Box_g}(s)}{ds} = \frac{d}{ds}\sum_{n=0}^{\infty} e^{-s\ln\lambda_n} = -\sum_{n=0}^{\infty} e^{-s\ln\lambda_n}\ln\lambda_n, \tag{4.39}$$

then

$$\Gamma_{eff} = \ln\det\Box_g = \ln\prod_{n=0}^{\infty}\lambda_n = \sum_{n=0}^{\infty}\ln\lambda_n = -\left.\frac{\zeta_{\Box_g}(s)}{ds}\right|_{s=0}. \tag{4.40}$$

Equation (4.40) can be considered as the *definition* of the regularized determinant of $\Box_g$.

The computation of $\zeta_{\Box_g}(s)$ can be simplified by solving the partial differential equation for the heat kernel. This involves using a Hermitian operator $\Box_g$, which has positive eigenvalues λ_n and a complete set of eigenvectors $|\psi_n\rangle$. The heat kernel operator is then defined as

$$\hat{K}(\tau) \equiv e^{-\Box_g \tau} = \sum_{n=0}^{\infty} e^{-\lambda_n \tau} |\psi_n\rangle \langle\psi_n| . \tag{4.41}$$

Taking the *trace* of the heat kernel (4.41), which is independent of the choice of the orthonormal basis

$$\mathrm{Tr}\{\hat{K}(\tau)\} \equiv \sum_{n=0}^{\infty} \langle\psi_n| \hat{K}(\tau) |\psi_n\rangle = \sum_{n=0}^{\infty} e^{-\lambda_n \tau}. \tag{4.42}$$

However, from the definition of the Γ function

$$\Gamma(s) \equiv \int_0^{\infty} \tau^{s-1} e^{-\tau} d\tau = \lambda^s \int_0^{\infty} \tau^{s-1} e^{-\lambda \tau} d\tau, \qquad \mathbb{R}(s) > 0 \tag{4.43}$$

we get

$$\begin{aligned} \zeta_{\Box_g}(s) &= \sum_{n=0}^{\infty} (\lambda_n)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \left\{ \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \tau^{s-1} d\tau \right\} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \left[\mathrm{Tr}\{\hat{K}(\tau)\} \right] \tau^{s-1} d\tau. \end{aligned} \tag{4.44}$$

But from (4.40)

$$\Gamma_{eff} = -\left.\frac{d\zeta_{\Box_g}(s)}{ds}\right|_{s=0}, \tag{4.45}$$

and it follows from the heat kernel definition that

$$\frac{d\hat{K}(\tau)}{d\tau} = -\Box_g \hat{K}(\tau), \tag{4.46}$$

plugging the unit operator (4.35) into (4.46)

$$\frac{d}{d\tau}\langle x|\hat{K}(\tau)|x'\rangle = -\int d^2x'' \langle x|\Box_g|x''\rangle \hat{K}(x'',x',\tau). \tag{4.47}$$

But when $\tau = 0$ then $\hat{K} = \mathbb{1}$ and since the *trace* of the heat kernel is independent of the choice of the orthonormal basis, then

$$\mathrm{Tr}\{\hat{K}(\tau)\} = \int d^2x \hat{K}(x,x,\tau), \tag{4.48}$$

then the heat kernel equation (4.47) becomes

$$\frac{d\hat{K}(x,x',\tau)}{d\tau} = \frac{\partial^2 \hat{K}(x,x',\tau)}{\partial x^2}. \tag{4.49}$$

To find a solution for effective action, we have to:

- solve the above PDE (4.49) for the heat kernel
- expand the heat kernel in powers of τ using the Seeley-DeWitt expansion
- find $\mathrm{Tr}\{\hat{K}(\tau)\}$

$$\mathrm{Tr}\{\hat{K}(\tau)\} = \frac{1}{4\pi}\int d^2x\sqrt{-g}\left\{\frac{1}{\tau} + \frac{R}{6} - \frac{1}{120}\tau^2 R\Box_g R - \frac{1}{60}\tau^2 R_{\mu\nu}\Box_g R^{\mu\nu}\right\}. \tag{4.50}$$

- substitute (4.50) into (4.44)

$$\Gamma_{g_{\mu\nu}} = -\frac{1}{8\pi}\int d^2x\sqrt{-g}R\int_0^\infty d\tau\left\{\frac{-\tau}{120}\Box_g + \frac{-\tau}{120}\Box_g\right\}R. \tag{4.51}$$

- change the integration variable from τ to $\xi = -\tau\Box_g$ to obtain the the Polyakov 2D gravitational action [96]

$$\Gamma_{g_{\mu\nu}} = \frac{1}{96\pi}\int d^2x\sqrt{-g}R\Box_g^{-1}R. \tag{4.52}$$

4.2.2 Quantization of Scalar field in Curved Spacetime

The process of quantizing fields in curved spacetime is similar in approach to that of Minkowski space. Therefore, we extend the approach we highlighted in §4.1.1.1 to curved spacetime.

$$\mathscr{L} = -\frac{1}{2}\Big(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + \zeta R\Big)\varphi^2, \tag{4.53}$$

$\nabla_\mu\varphi = \partial_\mu\varphi$ for the scalar field. The term $\zeta R\varphi^2$ represents the coupling between the scalar field and the gravitational field, where ζ is a dimensionless coupling constant. The case $\zeta = 0$ is referred to as minimal coupling, and when $\zeta = \frac{d-2}{4(d-1)}$ makes the equation conformally invariant [94]. The corresponding field equation is

$$(\Box + m^2 + \zeta R)\varphi = 0, \tag{4.54}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and the scalar product is now

$$(\varphi_1, \varphi_2) = -i \int_\Sigma d\Sigma^\mu \sqrt{\gamma}\Big(\varphi_1 \nabla_\mu \varphi_2^* - \varphi_2^* \nabla_\mu \varphi_1\Big), \tag{4.55}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, Σ is a three-dimensional spacelike hypersurface with the future-directed unit normal vector n^μ, and γ is the determinant of the induced metric ($\det\gamma_{\mu\nu}$) on Σ.

We quantize the field canonically with the commutation relations similar to (4.17)

$$\begin{aligned}
[\varphi(t,\vec{x}), \pi(t,\vec{x}')] &= \frac{i}{\sqrt{-g}}\delta^3(\vec{x}-\vec{x}') \\
[\varphi(t,\vec{x}), \varphi(t,\vec{x}')] &= 0 \\
[\pi(t,\vec{x}), \pi(t,\vec{x}')] &= 0.
\end{aligned} \tag{4.56}$$

The subsequent step is to write down the general solution, plug it into the equal-time commutation relation (1^{st} equation of (4.56)), and promote the coefficients and the fields to operators. The coefficients of the positive-frequency modes are annihilation operators, and therefore the coefficients of the negative-frequency modes are going to be creation operators. We are able to obtain the vacuum state and multi-particle states, which build up the full Fock space. However, the most significant difference between quantum theory in curved spacetime and flat spacetime without acceleration is that we do not have a standardized definition of positive- or negative-frequency modes. Consequently, the annihilation operator in one set of modes is often a mix of annihilation and creation operators in another. The choice of modes to expand the final solution depends on the observer. Hence, the vacuum state that is perceived by one observer may not be a vacuum state by another observer. This can be the essence of Hawking and Unruh radiation [25]. Why do we generally lack a uniform definition of positive- or negative-frequency modes in curved spacetime? In flat spacetime, the Lorentz transformation changes the frequency of the modes, and therefore the new frequency is simply the frequency within the boosted frame. So, the sign of the frequency is invariant. The creation operator within the original frame will still create a particle within the new frame but with boosted frequency and momentum (or four-momentum) in this frame. Therefore, The vacuum is a constant for all observers and comprises positive and negative frequency modes. However, in curved spacetime, the lack of a time-like vector that preserves the metric prevents us from discerning whether a mode is a positive or negative frequency. We will find an entire orthonormal set of solutions, which may be a basis of the Fock space, but there's a group of such sets. Different observers have their basis modes and don't always have an identical vacuum state.

Consider two bases for the Fock space f_i, f_i^* and g_i, g_i^*

$$\varphi = \sum_i \left\{ a_i f_i + a_i^\dagger f_i^* \right\}, \tag{4.57}$$

$$\varphi = \sum_i \left\{ b_i g_i + b_i^\dagger g_i^* \right\}. \tag{4.58}$$

It follows that

$$
\begin{aligned}
(f_i, f_i) &= \delta_{ij} \\
(f_i^*, f_i^*) &= -\delta_{ij} \\
(f_i, f_i^*) &= 0,
\end{aligned}
\tag{4.59}
$$

and

$$
\begin{aligned}
(g_i, g_i) &= \delta_{ij} \\
(g_i^*, g_i^*) &= -\delta_{ij} \\
(g_i, g_i^*) &= 0.
\end{aligned}
\tag{4.60}
$$

Imposing canonical commutation relations, we get

$$
\begin{aligned}
[a_i, a_j^\dagger] &= \delta_{ij} \\
[a_i, a_j] &= 0 \\
[a_i^\dagger, a_j^\dagger] &= 0,
\end{aligned}
\tag{4.61}
$$

and

$$
\begin{aligned}
[b_i, b_j^\dagger] &= \delta_{ij} \\
[b_i, b_j] &= 0 \\
[b_i^\dagger, b_j^\dagger] &= 0.
\end{aligned}
\tag{4.62}
$$

The Fock state and the number operator for mode f_i are given as

$$|n_i\rangle = \frac{1}{\sqrt{n_i!}}\left(a_i^\dagger\right)^{n_i}|0_f\rangle, \tag{4.63}$$

$$N_i^f = a_i^\dagger a_i. \tag{4.64}$$

Similar relations can be found for the g-modes.

We use the Bogoliubov transformation to redefine the creation and annihilation operators linearly

$$\begin{aligned} f_i &= \sum_j \left(\alpha_{ij}^* g_j - \beta_{ij} g_j^*\right) \\ g_i &= \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right), \end{aligned} \tag{4.65}$$

Applying the orthonormality relations on (4.65), we get

$$\begin{aligned} \alpha_{ij} &= (g_i, f_i) \\ \beta_{ij} &= -(g_i, f_i^*). \end{aligned} \tag{4.66}$$

We can use Bogoliubov coefficients (4.66) to transfer between the operators

$$\begin{aligned} a_i &= \sum_j \left(\alpha_{ij} b_j + \beta_{ij}^* b_j^\dagger\right) \\ b_i &= \sum_j \left(\alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger\right). \end{aligned} \tag{4.67}$$

It is possible to determine the quantity of f_j particles that an observer utilizing f-modes in the g-vacuum state $|0^g\rangle$ can detect. This can be achieved by calculating the expectation value of the f_j number operator in the $|0^g\rangle$ state.

$$\begin{aligned}
\langle 0^g | N_j^f |0^g\rangle &= \langle 0^g | a_j^\dagger a_j |0^g\rangle \\
&= \left\langle 0^g \left| \left[\sum_k (\alpha_{jk} b_k^\dagger - \beta_{jk} b_k)\right] \left[\sum_i (\alpha_{ji}^* b_i - \beta_{ji} b_i^\dagger)\right] \right| 0^g \right\rangle \\
&= \left\langle 0^g \left| \sum_{k,i} \beta_{jk} \beta_{ji}^* b_k b_i^\dagger \right| 0^g \right\rangle \\
&= \sum_{k,i} \beta_{jk} \beta_{ji}^* \langle 0^g | b_k b_i^\dagger + \delta_{ki} |0^g\rangle \\
&= \sum_{k,i} \beta_{jk} \beta_{ji}^* \delta_{ki} \\
&= \sum_k |\beta_{jk}|^2, \qquad (4.68)
\end{aligned}$$

which is to say that the vacuum of the g-modes contains $\sum_k |\beta_{jk}|^2$ particles in the f-mode. It follows that an observer utilizing the f-modes will observe the presence of particles in a state where an observer utilizing the g-modes observes no particles.

4.2.3 The Unruh Effect

The Unruh effect postulates that a uniformly accelerating observer will perceive a thermal bath, similar to blackbody radiation, while an observer in inertial frame will not perceive any such radiation. This implies that there exists a gas of particles in empty space, the temperature of which is directly proportional to the acceleration.

In order to derive the Unruh temperature, we start by considering the 2-dimensional Minkowski spacetime

$$ds^2 = -dt^2 + dx^2, \tag{4.69}$$

the inertial light-cone coordinates are defined as

$$\left.\begin{array}{c} u \\ v \end{array}\right\} \equiv t \mp x, \tag{4.70}$$

then the metric (4.69) becomes

$$ds^2 = dudv, \tag{4.71}$$

within this spacetime, a uniformly accelerating observer with acceleration a follows the trajectory

$$t(\tau) = \frac{1}{a} \sinh a\tau, \tag{4.72}$$

$$x(\tau) = \frac{1}{a} \cosh a\tau, \tag{4.73}$$

where τ is the proper time. In the spacetime diagram depicted in Figure 3.1, an observer follows a hyperbolic trajectory exclusively within region I in Figure 4.1. Signals transmitted from region II are inaccessible to this observer. Consequently, the ray $t = x$, where x is greater than 0, resembles the event horizon and is referred to as the observer's future horizon. By definition, the future horizon delineates the boundary beyond which signal reception is impossible.

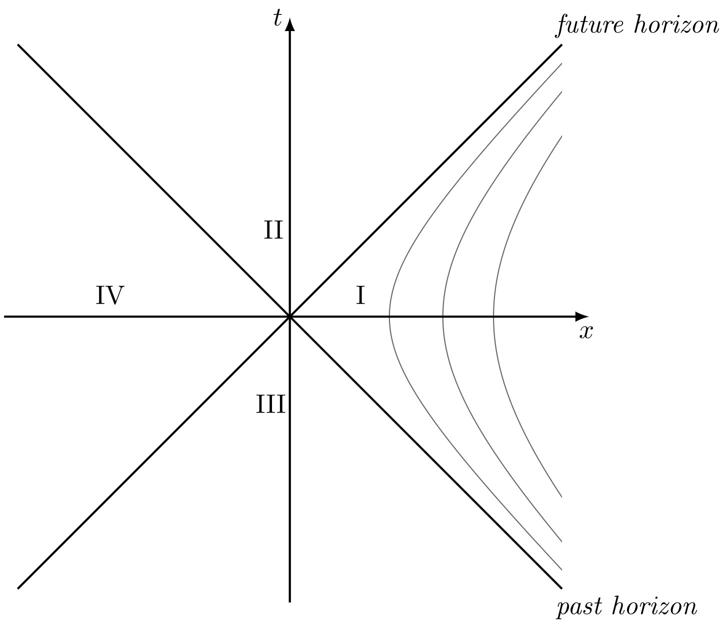

Fig. 4.1: The spacetime diagram illustrating a uniformly accelerating observer in Minkowski spacetime. The presence of prominent lines, namely $t = x$ and $t = -x$, serves to partition the spacetime into four distinct regions. The uniformly accelerating observer, referred to as a Rindler observer, consistently occupies region I. The line $t = x$ corresponds to the future horizon, while the hyperbolas the worldline of a Rindler observer.

The motion of an accelerating observer can be described using the Rindler coordinates (ξ, σ), which provide a more natural coordinate system.

$$t(\xi, \sigma) = \frac{1}{a} e^{a\xi} \sinh a\xi, \tag{4.74}$$

$$x(\xi, \sigma) = \frac{1}{a} e^{a\xi} \cosh a\xi. \tag{4.75}$$

The metric in the accelerated frame is now

$$ds^2 = e^{2a\xi}(-d\xi^2 + d\sigma^2). \tag{4.76}$$

We consider the massless scalar field, where the equation of motion corresponds to the Klein-Gordon equation (4.8) with a mass parameter of $m = 0$.

In order to determine the number of particles observed by a Rindler observer in the Minkowski vacuum $|0^M\rangle$, we need to quantize the free scalar field in these coordinates. This process involves reformulating the Klein-Gordon equation in terms of Rindler coordinates, solving the equation to identify the plane-wave modes, and subsequently quantizing the fields.

For a massless scalar field, the Klein-Gordon equation (4.8) is now given by

$$(\partial_\xi^2 - \partial_\sigma^2)\varphi = 0. \tag{4.77}$$

The same methodologies described in Section 4.2.2 can be employed here.

We express the quantum field in Minkwoski and Kruskal [98, 99] coordinates as:

$$\begin{aligned}\hat{\varphi} &= \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left\{ e^{-i\omega u} \hat{a}_\omega^- + e^{-i\omega u} \hat{a}_\omega^+ \right\} \\ &= \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left\{ e^{-i\Omega\tilde{u}} \hat{b}_\Omega^- + e^{-i\Omega\tilde{u}} \hat{b}_\Omega^+ \right\},\end{aligned} \tag{4.78}$$

where

$$\begin{aligned} V &= 4Me^{v/4M} \\ U &= -4Me^{-u/4M}. \end{aligned} \tag{4.79}$$

Within Minkowski space, the coordinates u represent light-cone coordinates, while the frequencies ω are associated with the creation and annihilation operators, $\hat{a}^{+}_{\omega}$ and $\hat{a}^{-}_{\omega}$ for the Minkowski vacuum $|0_M\rangle$. Correspondingly, Rindler space can be characterized by the coordinates U and frequencies Ω, with the vacuum state $|0_R\rangle$. We use *Bogoliubov transformation* to compute the relation between $|0_M\rangle$ and $|0_R\rangle$, i.e., between $\hat{a}^{\pm}_{\omega}$ and $\hat{b}^{\pm}_{\Omega}$.

$$\hat{b}^{-}_{\Omega} = \int_0^{\infty} d\omega(\alpha_{\Omega\omega}\,\hat{a}^{-}_{\omega} - \beta_{\Omega\omega}\,\hat{a}^{+}_{\omega}). \tag{4.80}$$

It should be noted that an inverse Bogoliubov transformation is not possible due to the fact that Rindler space only encompasses half of Minkowski space. The commutation relations for b_{Ω} reveal that

$$\int_0^{\infty} d\omega(\alpha_{\Omega\omega}\,\alpha^{*}_{\Omega'\omega} - \beta_{\Omega\omega}\,\beta^{*}_{\Omega'\omega}) = \delta\,(\Omega - \Omega'). \tag{4.81}$$

Substituting (4.80) into (4.78) gives us

$$\frac{1}{\sqrt{\omega}}e^{-i\omega u} = \int_0^{\infty} \frac{d\Omega'}{\sqrt{\Omega}'}\left(\alpha_{\Omega'\omega}\,e^{-i\Omega' U} - \beta^{*}_{\Omega'\omega}\,e^{i\Omega U}\right), \tag{4.82}$$

multiplying the above (4.82) with $e^{\pm i\Omega U}$ we obtain

$$\begin{aligned}\alpha_{\Omega\omega} &= \int_{-\infty}^{\infty} e^{\pm i\omega u + i\Omega U} dU \\ &= \pm\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty}(-au)^{-i\frac{\Omega}{a}-1}\,e^{\mp i\omega u}du \\ &= \pm\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\exp\left(\frac{i\Omega}{a}\ln\frac{\omega}{a}\right)\,\Gamma\left(-\frac{i\Omega}{a}\right),\end{aligned} \tag{4.83}$$

and

$$|\alpha_{\Omega\omega}|^2 = e^{2\pi\Omega/a}|\beta_{\Omega\omega}|^2, \tag{4.84}$$

where a is the acceleration parameter.

We now transition to the Rindler frame, which refers to the accelerated observer, and we calculate the expectation value of the occupation number of Rindler states within the Minkowski vacuum $\hat{N}_\Omega = \hat{b}^+_\Omega \hat{b}^-_\Omega$.

$$\begin{aligned}
\langle \hat{N}_\Omega \rangle &= \langle 0_M | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_M \rangle \\
&= \left\langle 0_M \middle| \int d\omega \left(\alpha^*_{\omega\Omega} \hat{a}^+_\omega - \beta^*_{\omega\Omega} \hat{a}^-_\omega \right) \int d\omega' \left(\alpha_{\omega'\Omega} \hat{a}^-_{\omega'} - \beta_{\omega'\Omega} \hat{a}^-_{\omega'} \right) \middle| \right\rangle \\
&= \int d\omega |\beta_{\Omega\omega}|^2,
\end{aligned} \tag{4.85}$$

applying the normalization condition $\Omega = \Omega' \Rightarrow \int d\omega \left(|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2 \right) = \delta(0)$ thus

$$\langle N_\Omega \rangle = \left[\exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \delta(0). \tag{4.86}$$

Due to $\delta(0)$, there is a divergent factor that signals that we are looking at an infinite volume of space. Therefore, we will calculate the density of particles instead

$$n_\Omega = \frac{\langle N_\Omega \rangle}{V} = \left[\exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1}. \tag{4.87}$$

This outcome aligns with the spectrum of blackbody radiation (Planck distribution) with the *Unruh* temperature

$$T = \frac{a}{2\pi}. \tag{4.88}$$

The *Unruh effect* can be explained in the following way. As mentioned earlier, Unruh radiation refers to the observation of a collection of particles resembling a thermal bath by an observer in an accelerated state within the Minkowski vacuum. Consequently, the detector undergoing acceleration becomes linked to the quantum fluctuations present in the Minkowski vacuum. These linkage/couplings are facilitated by an external entity, which is responsible for the observer's acceleration. The temperature, or energy, perceived by the observer in the accelerated state is sourced from its own acceleration.

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4.2.4 Hawking Radiation

Hawking radiation [66, 100] refers to the phenomenon in which black holes emit a thermal spectrum of particles. Consider the time-radial part of the Schwarzschild metric (2.28) and introduce tortoise coordinates

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{r}\right) - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \\ &= \left(1 - \frac{2GM}{r(r^*)}\right)(dt^2 - dr^{*2}), \end{aligned} \tag{4.89}$$

where r^* is the tortoise coordinate and given as

$$r^{*2} = r - r_{Sch} + r_{Sch} \ln\left(\frac{r - r_{Sch}}{r_{Sch}}\right), \tag{4.90}$$

where $r_{Sch} = 2GM$ and r^* is defined only for $r > r_{Sch}$ and varies from $\pm\infty$. Introducing the tortoise light-cone coordinates

$$\left.\begin{array}{c} u \\ v \end{array}\right\} = t \mp r^*, \tag{4.91}$$

then

$$ds^2 = \left(1 - \frac{2GM}{r(r^*)}\right) du dv. \tag{4.92}$$

The Schwarzschild coordinates exhibit singularity at $r = r_{Sch}$ on the horizon, while the tortoise light-cone coordinates u and v are likewise singular and limited to the outer region of the black hole, $r > r_{Sch}$. To fully describe the spacetime, an alternative coordinate system is necessary. We switch to Kruskal coordinates, where they are defined by

$$\begin{aligned} V &= 4Me^{v/4M} \\ U &= -4Me^{-u/4M}. \end{aligned} \tag{4.93}$$

In Kruskal coordinates, the metric (4.92) becomes

$$ds^2 = \frac{2GM}{r} e^{1 - \frac{2GM}{r}} dU dV, \tag{4.94}$$

where the Kruskal light-cone is defined as

$$\left.\begin{matrix} U \\ V \end{matrix}\right\} = T \mp R, \tag{4.95}$$

where the surfaces of constant U or V represent radial light-cones.

The quantized mode expansion in Kruskal coordinates, which includes both temporal and spatial coordinates, is as follows.

$$\varphi(U,V) = \int \frac{d\omega}{\sqrt{4\pi\omega}} \left(e^{-i\omega U} \hat{a}_\omega + e^{i\omega U} \hat{a}^\dagger_\omega + e^{-i\omega V} \hat{a}_{-\omega} + e^{i\omega V} \hat{a}^\dagger_{-\omega} \right). \tag{4.96}$$

The vacuum state in the Kruskal reference frame is the zero eigenvector of all annihilation operators $\hat{a}_{-\omega}$,

$$\hat{a}^-_\omega |0_k\rangle = 0 \quad \text{for all } k. \tag{4.97}$$

The quantized mode expansion in the tortoise coordinates (t, r^*) is similar to Kruskal expansion (4.98) and given as

$$\varphi(u,v) = \int \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(e^{-i\Omega u} \hat{b}_\Omega + e^{i\Omega u} \hat{b}^\dagger_\omega + e^{-i\Omega v} \hat{b}^-_\Omega + e^{i\Omega v} \hat{b}^{-\dagger}_\Omega \right). \tag{4.98}$$

The creation and annihilation operators $\hat{b}^\pm_\Omega$ and $\hat{a}^\pm_\omega$ satisfy the usual commutation relations. The operators $\hat{b}^\pm_\Omega$ describe particles moving either in the positive $(\Omega > 0)$ or negative $(\Omega < 0)$ direction.

The corresponding eigenstate $|0_B\rangle$ defined as

$$\hat{b}^-_\Omega |0_B\rangle = 0, \tag{4.99}$$

and it is called the *Boulware vacuum*.

The states $|0_B\rangle$ and $|0_k\rangle$, which are both void of particles and are respectively annihilated by operators $\hat{b}^-_\Omega$ and $\hat{a}^-_\omega$, are distinct. As a result, the action of operator $\hat{b}^-_\Omega$ on state $|0_k\rangle$ does not produce zero, indicating that state $|0_k\rangle$ contains particles with frequencies $\pm\Omega$. This suggests that the gravitational field creates particles.

Using Bogolyubov transformation to express the operators $\hat{b}^\pm_\Omega$ in terms of the operators $\hat{a}^\pm_\omega$

$$\hat{b}^-_\Omega = \int d\omega (\alpha_{\Omega\omega}\,\hat{a}^-_\omega - \beta_{\Omega\omega}\,\hat{a}^+_\omega), \tag{4.100}$$

where the Bogolyubov coefficients $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$ are determined as

$$\left.\begin{array}{l}\alpha_{\omega\Omega}\\ \\ \beta_{\Omega\omega}\end{array}\right\} = \pm\frac{2\pi}{\kappa}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{+\infty} e^{\pm\frac{\pi\Omega}{2\kappa}}\, e^{\frac{i\Omega}{\kappa}\ln\frac{\omega}{\kappa}}\,\Gamma\left(-\frac{i\Omega}{\kappa}\right). \tag{4.101}$$

The average number of particle seen by observer at infinity is

$$\langle 0_k|\,\hat{b}_{+\Omega}\hat{b}^-_\Omega\,|0_k\rangle = \int_0^\infty d\omega |\beta_{\Omega\omega}|^2, \tag{4.102}$$

then the total number of emitted particles in the mode Ω is

$$\langle 0_k | \hat{b}_{+\Omega} \hat{b}_{\Omega}^{-} | 0_k \rangle = \frac{1}{e^{2\pi\Omega/\kappa} - 1} \delta(0), \tag{4.103}$$

which represents the total number of particles in the entire space, and it is expected to diverge ($\delta(0)$).

The particles that are emitted have a thermal spectrum, which aligns with the spectrum of black body radiation with the temperature

$$T_H = \frac{\kappa}{2\pi}. \tag{4.104}$$

The surface gravity κ of a Schwarzschild black hole is $\kappa = \frac{1}{4GM}$, then the Hawking temperature a Schwarzschild black hole is

$$T_H = \frac{1}{8\pi GM}. \tag{4.105}$$

Equation (4.105) indicates that the thermal radiation emitted by a black hole, which is observed by an observer at infinity, is directly related to the inverse of the black hole's mass.

4.2.5 Black Hole Thermodynamics

A significant correlation exists between specific laws pertaining to the dynamics of black holes [68] and the principles of thermodynamics. Within the domain of black holes, the key parameters of interest include κ, representing the surface gravity at the horizon, M, denoting the mass of the black hole, A, signifying the area of the black hole's horizon, as well as Q and J, which respectively represent the charge, coordinate angular velocity and momentum. In the context of black hole physics, it is observed that mass corresponds to energy, which is not surprising considering that they are different manifestations of the same quantity. The area of a black hole plays the role of entropy S in thermodynamics. The concept of angular momentum in black hole physics is analogous to the work terms in thermodynamics. This relationship is closely connected to the fact that energy can be extracted from a black hole by taking advantage of the frame-dragging effect present around rotating black holes, as demonstrated by Penrose [72].

The laws of thermodynamics can be summarized as follows:
The **Zeroth Law of Thermodynamics** states that the temperature remains constant within a body in thermal equilibrium.
Similarly, the **Zeroth Law of Black Hole Thermodynamics** states that the value of a particular parameter n remains constant throughout the horizon of a stationary black hole.
The 0^{st} law can be shown to be true [101, 70] by first letting ξ be the Killing vector normal to the Killing horizon H^+ and defining the surface gravity on H^+

$$\kappa^2 = -\frac{1}{2}(\nabla^\beta \xi^\alpha)(\nabla_\beta \xi_\alpha), \tag{4.106}$$

and using the identity

$$\nabla_\mu \nabla_\nu \xi_\alpha = R_{\alpha\mu\nu\beta}\xi^\beta. \tag{4.107}$$

Differentiating (4.106) on Σ and using the above identity

$$\kappa \nabla_\alpha \kappa = -\frac{1}{2} \nabla^\nu \xi^\mu R_{\mu\nu\alpha\beta} \xi^\beta. \tag{4.108}$$

If $\kappa \neq 0$ on any generator, then it cannot vary from one generator to another, and it must be constant on H^+.

The **First Law of Thermodynamics** can be expressed as

$$dE = TdS + dW, \tag{4.109}$$

where dE represents a change in energy, T represents temperature, dS represents a change in entropy, and dW represents work done.
Similarly, the **First Law of Black Hole Thermodynamics** for a rotating charged black hole can be expressed as

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J + \Phi_H \delta Q, \tag{4.110}$$

where δM represents a change in mass, δA represents a change in the area of the black hole's horizon, Ω_H represents the angular velocity, Φ_H represents the electric potential at the horizon, and δJ represents a change in angular momentum. As (4.110) shows, the surface gravity κ plays the role of temperature. Despite the fact that the values of κ, Ω_H, and Φ_H are exclusively defined within the vicinity of the event horizon, they remain invariant across the stationary black hole's horizon.

Let Σ be a spacelike hypersurface in a stationary exterior black hole spacetime with an inner boundary, H, The Komar equations [70] for the total mass and the angular momentum are defined as

$$\begin{aligned} M &= -\frac{1}{8\pi}\oint_{\Sigma} \nabla^{\mu} t^{\nu} dS_{\mu\nu} \\ J &= \frac{1}{16\pi}\oint_{\Sigma} \nabla^{\mu} \phi^{\nu} dS_{\mu\nu}, \end{aligned} \tag{4.111}$$

and on the horizon H as

$$\begin{aligned} M_H &= -\frac{1}{8\pi}\oint_{H} \nabla^{\mu} t^{\nu} dS_{\mu\nu} \\ J_H &= \frac{1}{16\pi}\oint_{H} \nabla^{\mu} \phi^{\nu} dS_{\mu\nu}, \end{aligned} \tag{4.112}$$

take

$$\begin{aligned} M_H - 2\Omega_H J_H &= -\frac{1}{8\pi}\oint_{H} \nabla^{\mu} t^{\nu} dS_{\mu\nu} - \frac{2\Omega_h}{16\pi}\oint_{H} \nabla^{\mu} \phi^{\nu} dS_{\mu\nu} \\ &= -\frac{1}{8\pi}\oint_{H} \nabla^{\mu} \left(t^{\nu} + \Omega_H \phi^{\nu}\right) dS_{\mu\nu} \\ &= -\frac{1}{8\pi}\oint_{H} \nabla^{\mu} \xi^{\nu} dS_{\mu\nu}. \end{aligned} \tag{4.113}$$

But on H, $dS_{\mu\nu}$ is

$$dS_{\mu\nu} = \left(\xi_{\mu} n_{\nu} - \xi_{\nu} n_{\mu}\right) dA, \tag{4.114}$$

where $n_\mu \xi^\mu = -1$. Then (4.115) becomes

$$
\begin{aligned}
M_H - 2\Omega_H J_H &= -\frac{1}{8\pi}\oint_H \nabla^\mu \xi^\nu dS_{\mu\nu} = -\frac{1}{4\pi}\oint_H dA \underbrace{(\xi\cdot\nabla\xi)^\nu}_{\kappa\xi^\nu} n_\nu \\
&= -\frac{\kappa}{4\pi}\oint_H dA \underbrace{(\xi\cdot n)}_{-1} \\
&= \frac{\kappa}{4\pi}A, \qquad (4.115)
\end{aligned}
$$

therefore,

$$M_H = \frac{\kappa}{4\pi}A + 2\Omega_H J_H. \qquad (4.116)$$

(4.116) is the generalized Smarr formula [102].

Now, consider a stationary black hole with a mass denoted by M, a charge denoted by Q, and angular momentum denoted by J. The black hole has a future event horizon with a surface gravity denoted by κ and angular velocity denoted by Ω_H. If the black hole is perturbed and subsequently settles into a new black hole with mass $M+\delta M$, charge $Q+\delta Q$, and angular momentum $J+\delta J$, then

$$\delta M = \frac{\kappa}{4\pi}\delta A + 2\Omega_H \delta J_H. \qquad (4.117)$$

The new relations for the mass and angular momentum are

$$
\begin{aligned}
\delta M &= -\int_H T^\mu_\nu t^\nu d\Sigma_\mu \\
\delta J &= \int_H T^\mu_\nu \phi^\nu d\Sigma_\mu, \qquad (4.118)
\end{aligned}
$$

where $d\Sigma_\mu = -\xi_\mu \, dS \, dv$ and $dv = \sqrt{\sigma}\, d^2\theta$.

Then

$$\begin{aligned} \delta M - 2\Omega_H \delta J &= \int_H T_{\mu\nu}(t^\nu + \Omega_H \phi^\nu)\xi^\mu \, dS \, dv \\ &= \int dv \oint_H T_{\mu\nu}\, \xi^\mu \, \xi^\nu dS. \end{aligned} \tag{4.119}$$

Since θ and $\sigma_{\mu\nu}$ are first order in $T_{\mu\nu}$ then we can neglect the quadratic terms and use the following expression for Raychaudhuri equation

$$\frac{d\theta}{dv} = \kappa\,\theta - 8\pi T_{\mu\nu}\, \xi^\mu \, \xi^\nu, \tag{4.120}$$

then (4.119) becomes

$$\begin{aligned} \delta M - 2\Omega_H \delta J &= -\frac{1}{8\pi}\int dv \oint_H \left\{\frac{d\theta}{dv} - \kappa\,\theta\right\} dS \\ &= -\frac{1}{8\pi}\oint_H \theta \, dS\Big|_{-\infty}^{\infty} + \frac{\kappa}{8\pi}\int dv \oint_H \theta \, dS. \end{aligned} \tag{4.121}$$

Since the black hole is stationary before and after perturbation, then $\theta(v = \pm\infty) = 0$

$$\begin{aligned} \delta M - 2\Omega_H \delta J &= \frac{\kappa}{8\pi}\int dv \oint_H \theta \, dS \\ &= \frac{\kappa}{8\pi}\int dv \oint_H \left\{\frac{1}{dS}\frac{d}{dv}dS\right\} dS \\ &= \frac{\kappa}{8\pi}\oint_H dS\Big|_{-\infty}^{\infty} \\ &= \frac{\kappa}{8\pi}\,\delta A. \end{aligned} \tag{4.122}$$

The above equation is the 1^{st} law as presented in (4.110).

The **Second Law of Thermodynamics** states that the entropy of a system always increases in any process.

Likewise, the **Second Law of Black Hole Thermodynamics** (Hawking's Area Theorem) states that the area always increases, that is, In other words, if $T_{\mu\nu}$ conforms to the weak energy condition, and under the assumption of the validity of the cosmic censorship hypothesis, it can be concluded that the area of the future event horizon in an asymptotically flat spacetime will consistently increase over time.

$$\delta A \geq 0. \tag{4.123}$$

Hawking demonstrated that, based on fundamental premises, the surface area of a black hole's event horizon could never decrease [32]. To elaborate, consider a null geodesic congruence originating from one side of a spacelike 2-surface, with the convergence point ρ of the congruence defined as the rate of change of an infinitesimal cross-sectional area δA. ρ can be defined as $\frac{d}{d\lambda}\ln\delta A$, where λ is an affine parameter for the null geodesics . Then we get an equation that relates the null geodesic congruence to Ricci tensor

$$\frac{d}{d\lambda}\rho = \frac{1}{2}\rho^2 + \sigma^2 + R_{\mu\nu}k^\mu k^\nu, \tag{4.124}$$

where σ^2 is the square of the shear tensor of the congruence, and k^μ is the tangent vector to the geodesics. (4.124) is known as the *focusing equation* or *Raychaudhuri equation*. What this equation says is that the evolution of the expansion scalar ρ is determined by the square of the geodesic deviation and by the Ricci curvature. If there is no curvature, then the expansion is always slowing down. The focusing equation (4.124) reveals that an initially converging congruence must eventually reach a "crossing point" where ρ begins to diverge within a finite λ provided by $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, *null energy condition* (this is equivalent via Einstein's equation to the condition $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$) [103].

The boundary of the past of the future null infinity, also known as the future event horizon in an asymptotically flat black hole spacetime, is defined as the limit of the points that can communicate with remote regions of spacetime in the future. According to the focusing equation (4.124, and $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$), and the absence of naked singularities, *the cross-sectional area of the future event horizon cannot decrease anywhere*. This is due to the fact that if the horizon generators converge, they will eventually reach a crossing point within a finite area. However, the horizon must remain tangent to the light cones, and the generators cannot leave the horizon or be extended far enough to reach the crossing point without encountering a singularity. Consequently, the area theorem sets an upper limit on the total amount of energy that can be extracted from a black hole.

The **Third Law of Thermodynamics** states that reaching a temperature of absolute zero through any physical process is impossible.
Similarly, the **Third Law of Black Hole Thermodynamics** states that reaching a value of zero for a specific parameter κ through any physical process is impossible.

A summary of black hole mechanics and their analogy to thermodynamics is given in Table 4.1.

Law	Thermodynamics	Black Hole
0^{st}	T is constant throughoutbody in thermal equilibrium	κ is constant overa stationary horizon
1^{st}	$dE = TdS + dW$	$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q$
2^{nd}	$\delta S \geq 0$	$\delta A \geq 0$
3^{rd}	$T = 0$ unachievable	$\kappa = 0$ unachievable

Table 4.1: Laws of Thermodynamics vs. Laws of Black Hole Mechanics.

Chapter 5

Black hole Entropy within CWG Paradigm via Nöther Current Method

5.1 Review of Boundary Nöther Current and Its Charge

We start by providing a brief review the boundary Nöther current method [104–107].

Taking the general surface term to be

$$I_{sur} = \int \mathrm{d}^3x\sqrt{g}\mathscr{L}. \tag{5.1}$$

The boundary term does not contribute to the equations of motion; however, it can contribute to thermodynamics. Considering a general Lagrangian, a total derivative of a vector field, where the resulting action has only a surface contribution. Such a Lagrangian density is expressed as

$$\sqrt{g}\mathscr{L} = \sqrt{g}\nabla_\mu B^\mu, \tag{5.2}$$

where $\mathscr{L}$ is a scalar. Next, we consider infinitesimal diffeomorphism:
$x^\mu \to x^\mu + \xi^\mu$.
The Noether current J^μ can be found by considering the changes of both sides of equation (5.2).
The left-hand side of (5.2) varies by:

$$\begin{aligned}
\delta_\xi(\sqrt{g}\mathscr{L}) &= \delta_\xi\sqrt{g}\mathscr{L} + \sqrt{g}\delta_\xi\mathscr{L} \\
&= \frac{1}{2}\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta}\mathscr{L} + \sqrt{g}\xi^\mu\nabla_\mu\mathscr{L} \\
&= \frac{1}{2}\sqrt{g}g^{\alpha\beta}(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\mathscr{L} + \sqrt{g}\xi^\mu\nabla_\mu\mathscr{L} \\
&= \sqrt{g}(g^{\alpha\beta}\nabla_\alpha\xi_\beta\mathscr{L} + \xi^\mu\nabla_\mu\mathscr{L}) \\
&= \sqrt{g}(\nabla_\mu\xi_\mu\mathscr{L} + \xi^\mu\nabla_\mu\mathscr{L}) \\
&= \sqrt{g}\nabla_\mu(\mathscr{L}\xi^\mu).
\end{aligned} \tag{5.3}$$

The variation of the right-hand side of (5.2) is given by:

$$\begin{aligned}
\delta(\sqrt{g}\nabla_\mu B^\mu) &= \delta(\sqrt{g}\frac{1}{\sqrt{g}}\partial_\mu[\sqrt{g}B^\mu]) \\
&= \delta[\partial_\mu(\sqrt{g}B^\mu)] \\
&= \partial_\mu[\delta(\sqrt{g}B^\mu)] \\
&= \partial_\mu[\delta(\sqrt{g}B^\mu) + \sqrt{g}\delta B^\mu] \\
&= \partial_\mu[\frac{1}{2}\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta}B^\mu + \sqrt{g}\mathscr{L}\xi B^\mu] \\
&= \partial_\mu[\frac{1}{2}\sqrt{g}g^{\alpha\beta}(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)B^\mu + \sqrt{g}(\xi^\alpha\nabla_\alpha B^\mu - B^\alpha\nabla_\alpha\xi^\mu)] \\
&= \partial_\mu[\sqrt{g}g^{\alpha\beta}\nabla_\alpha\xi_\beta B^\mu + \sqrt{g}(\xi_\alpha\nabla_\alpha B^\mu - B^\alpha\nabla_\alpha\xi^\mu)] \\
&= \partial_\mu(\sqrt{g}[\nabla_\alpha\xi^\alpha B^\mu + \xi^\alpha\nabla_\alpha B^\mu - B^\alpha\nabla_\alpha\xi^\mu)] \\
&= \partial_\mu(\sqrt{g}[\nabla_\alpha(B^\mu\xi^\alpha) - B^\alpha\nabla_\alpha\xi^\mu]) \\
&= \sqrt{g}\nabla_\mu[\nabla_\alpha(B^\mu\xi^\alpha) - B^\alpha\nabla_\alpha\xi^\mu], \qquad (5.4)
\end{aligned}$$

$\Rightarrow 0 = \nabla_\mu[\mathscr{L}\xi^\mu - \nabla_\alpha(B^\mu\xi^\alpha) + B^\alpha\nabla_\alpha\xi^\mu] = \nabla_\mu J^\mu.$

Thus the Nöther current reads:

$$J^\mu[\xi] = \mathscr{L}\xi^\mu - \nabla_\alpha(B^\mu\xi^\alpha) + B^\alpha\nabla_\alpha\xi^\mu. \qquad (5.5)$$

Recall that $\sqrt{g}\mathscr{L} = \sqrt{g}\nabla_\mu B^\mu$,

then

$$\begin{aligned}
J^\mu[\xi] &= \nabla_\alpha B^\alpha\xi^\mu - \nabla_\alpha(B^\mu\xi^\alpha) + B^\alpha\nabla_\alpha\xi^\mu \\
&= \nabla_\alpha(B^\alpha\xi^\mu) - \nabla_\alpha(B^\mu\xi^\alpha) \\
&= \nabla_\alpha J^{\mu\alpha}, \qquad (5.6)
\end{aligned}$$

therefore

$$J^{\mu\alpha} = B^{\alpha}\xi^{\mu} - B^{\mu}\xi^{\alpha}. \tag{5.7}$$

Nöther charge is given by:

$$Q[\xi] = \int_{\Sigma} \mathrm{d}\Sigma_{\mu} J^{\mu}, \tag{5.8}$$

Σ is the timelike hypersurface with unit normal to M^{α}. Using Stokes Theorem:

$$Q[\xi] = \int_{\partial\Sigma} \sqrt{g}\,\mathrm{d}\Sigma_{\mu} J^{\mu} = \frac{1}{2}\int_{\partial\Sigma} \sqrt{h}\,\mathrm{d}\Sigma_{\alpha\beta} J^{\alpha\beta}, \tag{5.9}$$

where $\sqrt{h}d\Sigma_{\alpha\beta} = -\sqrt{h}d^2x\left(n_{\alpha}m_{\beta} - n_{\beta}m_{\alpha}\right)$ is the area element on the 2-dimensional hypersurface of $\partial\Sigma$ and n^{α} is its spacelike unit vector. h is the determinant of the induced metric of $(d-2)-$dimensional boundary in a $d-$ dimensional spacetime.

5.2 Nöther Current Method within Einstein Gravity

At this juncture, we can show how the aforementioned formalism related to black hole entropy by applying it to an unexplored example of general relativity's spacetime. In particular, we focus on the near horizon near extremal Kerr metric [108, 109] in the finite mass/temperature gauge [65, 110], which is presented by the line element.

$$ds^2 = \frac{1+\cos^2\theta}{2}\left(-\frac{r^2-2GMr-a^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2-2GMr-a^2}dr^2 + \ell^2 d\theta^2\right) + \frac{2\ell^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi + \frac{r-2GM}{\ell^2}dt\right)^2, \tag{5.10}$$

where a is the angular momentum per unit mass parameter and $\ell^2 = r_+^2 + a^2$. The gauge we are examining is important for our analysis because it has a Hawking temperature that is not equal to zero and is directly correlated to its surface gravity.

$$\kappa = \frac{1}{2}f'(r_+) = \frac{1}{2}\left(\frac{r^2-2GMr-a^2}{\ell^2}\right). \tag{5.11}$$

Moreover, the utilization of this gauge allows for the introduction of a non-zero finite time regulator for Q. Specifically, the Hawking temperature $T_H = \kappa/2\pi$ is precisely equivalent to the temperature of the non-extremal general Kerr black hole in standard Boyer-Lindquist coordinates [65, 110].

For Einstein's gravity, we can write the general surface term as [111, 104, 107]

$$B^\alpha = \frac{2Q^{\alpha\sigma}_{\lambda\gamma}\, g^{\beta\gamma}\Gamma^\lambda_{\beta\sigma}}{16\pi G}, \tag{5.12}$$

where

$$Q^{\alpha}_{\lambda\gamma}{}^{\sigma} = \frac{1}{2}\left(\delta^\alpha{}_\lambda\, \delta^\sigma{}_\gamma - \delta^\alpha{}_\gamma \delta^\sigma{}_\lambda\right), \tag{5.13}$$

and $\Gamma^\lambda_{\beta_\sigma}$ is the usual metric compatible Levi-Civita connection. The spacetime expressed in equation (5.10) possesses a standard temporal Killing isometry denoted by $\xi_{(t)} = \partial_t$. This particular isometry choice gives rise to a quantity known as the

Nöther energy or charge $Q[\xi_{(t)}]$, which has a specific physical interpretation as the product of the entropy and temperature of the $\partial\Sigma$ [107, 112, 113]. $\partial\Sigma$ in the context of the metric (5.10) corresponds to the black hole horizon. It follows that $Q[\xi_{(t)}]|_{r_+}$ should be equivalent to the product of T_H and S_{BH} of (1.8).

to compute the charge we use (5.10), (5.12) and (5.1):

$$B^r = -\frac{2f'(r_+)}{16\pi G(1+\cos^2\theta)}, \tag{5.14}$$

$$B^\theta = -\frac{8\cos^2\theta\cot\theta}{16\pi G\ell^2(1+\cos^2\theta)^2}, \quad \text{and} \tag{5.15}$$

$$d\Sigma_{tr} = -\frac{1+\cos^2\theta}{2}, \tag{5.16}$$

then

$$\begin{aligned} Q[\xi]|_{r_+} &= \frac{1}{2}\int_{\partial\Sigma}\sqrt{h}d\Sigma_{\alpha\beta}J^{\alpha\beta} \\ &= \frac{1}{32\pi G}\int\sqrt{h}d^2x4\kappa \\ &= \frac{A\kappa}{8\pi G}, \end{aligned} \tag{5.17}$$

where A is the black hole horizon area, as mentioned before. Multiplying the above by the finite time regulator $\beta = 1/T_H = 2\pi/\kappa$ gives:

$$Q[\xi]\beta = \frac{A\kappa}{8\pi G}\frac{2\pi}{\kappa} = \frac{A}{4G}. \tag{5.18}$$

Therefore, the Bekenstein-Hawking entropy formula for the near horizon near extremal Kerr metric is derived from the boundary Nöther current charge in the context of Einstein gravity. The aforementioned analysis of thermodynamics suggests that the entropy of a black hole is represented by the boundary term in the total action. Now, we will delve into the fundamental process of establishing this inference from a statistical counting of canonical quantum microstates through the general definition of the generator algebra

$$[Q_1, Q_2] = \frac{1}{2}\left[\delta_{\xi_1} Q[\xi_2] - \delta_{\xi_2} Q[\xi_1]\right]. \tag{5.19}$$

Next starting with (5.9), let us look at $\delta_{\xi_1} Q[\xi_2]$:

$$\begin{aligned}\delta_{\xi_1} Q[\xi_2] &= \int d\Sigma_\mu \delta_{\xi_1}\left(\sqrt{g} J^\mu[\xi_2]\right)\\ &= \int \sqrt{g} d\Sigma_\mu \left\{\nabla_\alpha\left(\xi_1^\alpha J^\mu[\xi_2] - \xi_1^\mu J^\alpha[\xi_2]\right)\right\}\\ &= \int_{\partial\Sigma} \sqrt{h} d\Sigma_{\mu\nu} \xi_1^\nu J^\mu[\xi_2],\end{aligned} \tag{5.20}$$

which implies;

$$\begin{aligned}[Q_1, Q_2] =& \frac{1}{2}\left(\int_{\partial\Sigma} \sqrt{h} d\Sigma_{\mu\nu} \xi_1^\nu J^\mu[\xi_2] - \int_{\partial\Sigma} \sqrt{h} d\Sigma_{\mu\nu} \xi_2^\nu J^\mu[\xi_1]\right)\\ =& \frac{1}{2}\int_{\partial\Sigma} \sqrt{h} d\Sigma_{\mu\nu}\left(\xi_2^\mu J^\nu[\xi_1] - \xi_1^\mu J^\nu[\xi_2]\right).\end{aligned} \tag{5.21}$$

5.3 Horizon Microstates and CWG: Vacuum

Let us begin by considering the near horizon near extremal Kerr metric in (5.10), which is a vacuum solution of general relativity and, therefore, also a vacuum solution of CWG. In order to analyze this metric, we will first rewrite it using the typical general functions:

$$ds^2 = \frac{1+\cos^2\theta}{2}\left(-f(r_+ + \rho)\,dt^2 + \frac{1}{f(r_+ + \rho)}d\rho^2 + \ell^2 d\theta^2\right) + \frac{2\ell^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi - A(r_+ + \rho)\,dt\right)^2, \tag{5.22}$$

where we have introduced a radial shift (Bondi transformation) by replacing r with $r_+ + \rho$, effectively placing the boundary of the black hole horizon limit at $\rho = 0$. Subsequently, we imposed the following near horizon metric fall-off conditions [57].

$$\delta g_{\mu\nu} = \begin{pmatrix} \mathscr{O}(\rho) & \mathscr{O}(1) & \mathscr{O}(1) & \mathscr{O}(1) \\ & \mathscr{O}(1) & \mathscr{O}(1) & \mathscr{O}\left(\frac{1}{\rho}\right) \\ & & \mathscr{O}(1) & \mathscr{O}(1) \\ & & & \mathscr{O}(1) \end{pmatrix}, \tag{5.23}$$

where we have indexed μ and ν over $\{t,\rho,\theta,\phi\}$. The above boundary conditions are preserved by a copy of the conformal group generated by the diffeomorphism:

$$\xi_\varepsilon = \left(\varepsilon(t) - \frac{\rho}{f(r_+ + \rho)}\varepsilon'(t)\right)\partial_t - \rho\varepsilon'(t)\partial_\rho, \tag{5.24}$$

which is seen when choosing $\varepsilon(t)$ to be normalized circle diffeomorphisms; $\frac{e^{in\kappa t}}{\kappa}$ and computing the classical Lie bracket in the horizon limit $\rho = 0$, which yields the centrally extended Virasoro algebra:

$$i\{\xi_m, \xi_n\} = (m-n)\xi_{m+n} - \frac{m^3}{2\kappa}\delta_{m+n,0}\partial_t. \tag{5.25}$$

The boundary action for CWG is given by [46, 83, 45]:

$$S_B = \frac{\alpha_c}{4\pi}\int d^3x\sqrt{\sigma}W^{\mu\nu\alpha\beta}n_\mu n_\nu \nabla_\alpha n_\beta, \tag{5.26}$$

where $n_\mu = \left\{0, \sqrt{\frac{1+\cos^2\theta}{2f(r)}}, 0, 0\right\}$ is the previously defined spacelike unit normal. From the action (5.26) we have the boundary Lagrangian density:

$$\mathscr{L} = \frac{\alpha}{4\pi}W^{\mu\nu\alpha\beta}n_\mu n_\nu \nabla_\alpha n_\beta, \tag{5.27}$$

from which we obtain $B^\alpha = n^\alpha \mathscr{L}$. Now substituting these results and (5.24) into (5.7) through (5.8) and (5.21), where $m_\mu = \left\{\sqrt{\frac{(1+\cos^2\theta)f(r)}{2}}, 0, 0, 0\right\}$, we obtain in the horizon limit:

$$Q = \frac{\alpha_c}{3\pi\ell^2}\int_{\partial\Sigma}\sqrt{h}d^2x\Big\{\Big[\left(1+\cos^2\theta\right)\Big(-32\ell^4\sin^2\theta A'(r_+)^2 + \ell^2(\cos(2\theta)+3)^2 f''(r_+) + 44\cos(2\theta) - \cos(4\theta) + 21\Big)\Big] / \left((\cos(2\theta)+3)^4\right)\Big\}\left(\kappa\varepsilon(t) - \frac{\varepsilon'(t)}{2}\right), \tag{5.28}$$

and

$$[Q_1, Q_2] = \frac{\alpha_c}{3\pi\ell^2}\int_{\partial\Sigma}\sqrt{h}d^2x\left[\left(1+\cos^2\theta\right)\left(-32\ell^4\sin^2\theta A'(r_+)^2 + \ell^2(\cos(2\theta)+3)^2 f''(r_+) + 44\cos(2\theta) - \cos(4\theta) + 21\right)\right] / \left[(\cos(2\theta)+3)^4\right] \left[\kappa\left(\varepsilon_1\varepsilon_2' - \varepsilon_2\varepsilon_1'\right) - \frac{\varepsilon_1\varepsilon_2'' - \varepsilon_2\varepsilon_1''}{2} + \frac{\varepsilon_1'\varepsilon_2'' - \varepsilon_2'\varepsilon_1''}{4\kappa}\right]. \tag{5.29}$$

Next, upon substituting $\varepsilon(t) = \frac{e^{in\kappa t}}{\kappa}$ and performing the integration over $\sqrt{h}d^2x$ we have:

$$Q_m = \frac{\alpha_c}{4\pi}\left[A'(r_+) + f''(r_+)\right] A\delta_{m,0}, \tag{5.30}$$

and

$$[Q_m, Q_n] = \frac{\alpha_c}{4\pi}\left[A'(r_+) + f''(r_+)\right]\left[iA(m-n)\delta_{m+n,0} - im^3\frac{A}{2}\delta_{m+n,0}\right], \tag{5.31}$$

which we recognize as the quantum Virasoro algebra of the boundary degrees of freedom. Furthermore we can read off the central charge and lowest Virasoro mode:

$$\begin{aligned} Q_0 =& \frac{\alpha_c}{4\pi}\left[A'(r_+) + f''(r_+)\right] A \\ \frac{c}{12} =& \frac{\alpha_c}{4\pi}\left[A'(r_+) + f''(r_+)\right]\frac{A}{2}, \end{aligned} \tag{5.32}$$

which after employing Cardy's formula yields the black hole entropy of the near horizon near extremal Kerr black hole within CWG:

$$S_{BH} = 2\pi\sqrt{\frac{Q_0 c}{6}} = \frac{\alpha_c}{2}\left[A'(r_+) + f''(r_+)\right] A. \tag{5.33}$$

Comparing the above to the Wald entropy for CWG (3.17) of the same spacetime (5.22) we have:

$$S = -\frac{\alpha_c}{8}\int W_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu}\varepsilon^{\alpha\beta}d\Sigma = \frac{\alpha_c}{2}\left[A'(r_+) + f''(r_+)\right] A, \tag{5.34}$$

which precisely agrees with the previous result above for the near horizon near extremal Kerr black hole from counting the canonical quantum microstates from the boundary Nöther current method.

5.4 Horizon Microstates and CWG: non-Vacuum

Now we turn our attention to non-vacuum solutions of CWG, which provides us with an additional venue to explore more unique spacetimes. Specifically, we focus on the near horizon near extremal CWG-Maxwell theory. The CWG-Maxwell system has been well explored for the Coulomb potential in [36] with the solution given by the line element (2.28), where:

$$f(r) = w + \frac{u}{r} + \gamma r - kr^2,\; w = 1 - 3\beta\gamma,\; u = -\beta(2 - 3\beta\gamma) - \frac{Q^2}{8\alpha\gamma}, \tag{5.35}$$

and Q is the black hole charge parameter for the $U(1)$ vector potential $A_\mu = \{Q/r, 0, 0, 0\}$. To maintain simplicity and ensure general applicability, we can explore the extremal limit of (5.35) in the mathematically simplified case where $\beta = \frac{3k+\gamma^2}{9k\gamma}$ and with extremal limit given by $Q = \sqrt{-\frac{8\alpha}{3}}$. In this scenario the metric (5.35) takes the form:

$$ds^2 = \frac{(3rk - \gamma)^3}{27rk^2} dt^2 - \frac{27rk^2}{(3rk - \gamma)^3} dr^2 + r^2 d\Omega^2, \tag{5.36}$$

and looking at the near horizon of (5.36) in the standard way, where $r \to r\lambda + \frac{\gamma}{3k}$, $t \sim t/\lambda$ and further in the limit as $\lambda \to 0$, we get $g_{tt} \to \infty$. This makes it seem naively unclear how to investigate the near horizon near extremal limit of the

CWG-Maxwell theory, without killing off degrees of freedom or running into divergences in the metric.

However, it is well known [86, 89, 88, 87] that in the near horizon near extremal limit of charged black holes the $U(1)$ gauge potential becomes linear:

$$A_\mu = \{Qr, 0, 0, 0\}. \tag{5.37}$$

Additionally, the CWG-Maxwell system is stationary for the Euler-Lagrange equation:

$$-2\alpha_c W_{\mu\nu} + \frac{1}{2} T_{\mu\nu} = 0, \tag{5.38}$$

where

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F^2, \tag{5.39}$$

is the Maxwell stress tensor and

$$W_{\mu\nu} = 2\nabla_\alpha \nabla_\beta W_\mu{}^{\alpha\beta}{}_\nu + R_{\alpha_\beta} W_\mu{}^{\alpha\beta}{}_\nu\,, \tag{5.40}$$

is the Bach tensor, both of which stem from the inverse metric variation of the CWG-Maxwell theory. Consequently and employing the ansatz [82]

$$ds^2 = K_1(\theta)\left[-f(r)dt^2 + f(r)^{-1}dr^2 +^2 d\theta^2\right] + \frac{{}^2 sin^2\theta}{K_2(\theta)}\left(d\phi - A_\mu(r)dx^\mu\right)^2. \tag{5.41}$$

We are free to forge for near horizon near extremal CWG-Maxwell solutions directly from the above field equation with linear $U(1)$ potential to aid in our analysis. Our current study includes the two following general solutions:

$$\begin{aligned} ds^2 &= -B(r)dt^2 + \frac{1}{B(r)}dr^2 + \ell^2 d\Omega^2 \\ &= -\left(\frac{r^3\left(2\alpha\left(c_1^2\ell^4-4\right)-3Q^2\ell^4\right)}{24\alpha c_2\ell^4} + \frac{r^2c_1}{2} + rc_2\right)dt^2 \\ &\quad + \left(\frac{r^3\left(2\alpha\left(c_1^2\ell^4-4\right)-3Q^2\ell^4\right)}{24\alpha c_2\ell^4} + \frac{r^2c_1}{2} + rc_2\right)^{-1} dr^2 + \ell^2 d\Omega^2, \end{aligned} \tag{5.42}$$

and

$$\begin{aligned} ds_{c_3}^2 &= \sin\theta\left(-f(r)dt^2 + \frac{1}{f(r)}dr^2 + \ell^2 d\theta^2\right) + \ell^2\sin\theta d\phi^2 \\ &= \sin\theta\left(-\frac{r^2+c_3r}{\ell^2}dt^2 + \frac{\ell^2}{r^2+c_3r}dr^2 + \ell^2 d\theta^2\right) + \ell^2\sin\theta d\phi^2, \end{aligned} \tag{5.43}$$

where the c_i's are constants. The above solutions classify (general relativisticly) globally both as *Petrov type D* and additionally as *Segre type* $[(11)(1,1)]$ $(IID-M2(11))$ and $[(11),Z\bar{Z}]$ $(II-M3(111))$ respectively. Without loss of generality and to ensure local $AdS_2\times S^2$ topology, we can impose that $c_1=\sqrt{\frac{3Q^2\ell^4+8\alpha_c}{2\ell^4\alpha_c}}$, which implies for (5.42) that:

$$\begin{aligned} ds_{c_2}^2 = &-\left(r^2\sqrt{\frac{3Q^2\ell^4+8\alpha_c}{8\ell^4\alpha_c}} + rc_2\right)dt^2 + \left(r^2\sqrt{\frac{3Q^2\ell^4+8\alpha_c}{8\ell^4\alpha_c}} + rc_2\right)^{-1} dr^2 \\ &+ \ell^2 d\Omega^2. \end{aligned} \tag{5.44}$$

We have included the subscripts c_2 and c_3 on the above line elements in (5.43) and (5.44) in order to distinguish them in subsequent discussion/analysis. Next, we will compute the black hole entropy of the above two solutions via the previously outlined Nöther current method and again compare our results to the Wald entropy formula.

It turns out that the above solutions exhibit the same (or even more relaxed) near horizon fall off conditions (5.23) and thus near horizon preserving isometry given by (5.24). This can be more easily proven since the above solutions are spherically symmetric, via the Bondi-like transformation:

$$dt = du + \frac{1}{f\left(r_+ + \rho\right)} d\rho, \tag{5.45}$$

and solving the Killing equation $\mathscr{L}_\xi g_{\mu\nu} = 0$, which yields for both $ds^2_{c_2}$ and $ds^2_{c_3}$ cases:

$$\begin{aligned} \xi^u =& F(u,\theta,\phi) \text{ and} && (5.46) \\ \xi^\rho =& -\rho\partial_u F(u,\theta,\phi). && (5.47) \end{aligned}$$

Transforming back to the temporal coordinate t, we have:

$$\begin{aligned} \xi^t =& \varepsilon(t,\theta,\phi) - \frac{\rho}{f\left(r_+ + \rho\right)}\partial_t\varepsilon(t,\theta,\phi) \text{ and} && (5.48) \\ \xi^\rho =& -\rho\partial_t\varepsilon(t,\theta,\phi), && (5.49) \end{aligned}$$

which matches (5.24) for an ε that exhibits angular dependence. Now, for both $ds^2_{c_2}$ and $ds^2_{c_3}$ cases we may chose $n_\mu = \left\{0, \frac{1}{\sqrt{f(r)}}, 0, 0\right\}$, $m_\mu = \left\{-\sqrt{f(r)}, 0, 0, 0\right\}$,

$\varepsilon(t,\theta,\phi)=\varepsilon(t)$ and following the the same procedures from Sec. 5.3, we find:

$$Q=\begin{cases}\frac{\alpha_c}{4\pi}\frac{2-\ell^2 f''(r_+)}{6\ell^2}\int_{\partial\Sigma}\sqrt{h}d^2x\left(\kappa\varepsilon(t)-\frac{\varepsilon'(t)}{2}\right) & \text{case } c_2\\ \frac{\alpha_c}{4\pi}\frac{-f''(r_+)}{6}\int_{\partial\Sigma}\sqrt{h}d^2x\left(\kappa\varepsilon(t)-\frac{\varepsilon'(t)}{2}\right) & \text{case } c_3\end{cases} \tag{5.50}$$

and

$$[Q_1,Q_2]=$$
$$\begin{cases}\frac{\alpha_c}{4\pi}\frac{2-\ell^2 f''(r_+)}{6\ell^2}\int_{\partial\Sigma}\sqrt{h}d^2x\left[\kappa\left(\varepsilon_1\varepsilon_2'-\varepsilon_2\varepsilon_1'\right)-\frac{\varepsilon_1\varepsilon_2''-\varepsilon_2\varepsilon_1''}{2}+\frac{\varepsilon_1'\varepsilon_2''-\varepsilon_2'\varepsilon_1''}{4\kappa}\right] & \text{case } c_2\\ \frac{\alpha_c}{4\pi}\frac{-f''(r_+)}{6}\int_{\partial\Sigma}\sqrt{h}d^2x\left[\kappa\left(\varepsilon_1\varepsilon_2'-\varepsilon_2\varepsilon_1'\right)-\frac{\varepsilon_1\varepsilon_2''-\varepsilon_2\varepsilon_1''}{2}+\frac{\varepsilon_1'\varepsilon_2''-\varepsilon_2'\varepsilon_1''}{4\kappa}\right] & \text{case } c_3\end{cases}. \tag{5.51}$$

Now, substituting $\varepsilon(t)=\frac{e^{in\kappa t}}{\kappa}$ and performing the integration over $\sqrt{h}d^2x$ we obtain:

$$Q_m=\begin{cases}\frac{\alpha_c}{4\pi}\frac{2-\ell^2 f''(r_+)}{6\ell^2}A\delta_{m,0} & \text{case } c_2\\ \frac{\alpha_c}{4\pi}\frac{-f''(r_+)}{6}A\delta_{m,0} & \text{case } c_3\end{cases} \tag{5.52}$$

and

$$[Q_m,Q_n]=\begin{cases}\frac{\alpha_c}{4\pi}\frac{2-\ell^2 f''(r_+)}{6\ell^2}\left[iA(m-n)\delta_{m+n,0}-im^3\frac{A}{2}\delta_{m+n,0}\right] & \text{case } c_2\\ \frac{\alpha_c}{4\pi}\frac{-f''(r_+)}{6}\left[iA(m-n)\delta_{m+n,0}-im^3\frac{A}{2}\delta_{m+n,0}\right] & \text{case } c_3\end{cases}, \tag{5.53}$$

which we again recognize as the quantum Virasoro algebra of the boundary degrees of freedom. Again, reading off the central charge and lowest Virasoro mode, we have:

$$\begin{aligned} Q_0 &= \begin{cases} \frac{\alpha_c}{4\pi}\frac{2-\ell^2 f''(r_+)}{6\ell^2}A & \text{case } c_2 \\ \frac{\alpha_c}{4\pi}\frac{-f''(r_+)}{6}A & \text{case } c_3 \end{cases} \\ \frac{c}{12} &= \begin{cases} \frac{\alpha_c}{4\pi}\frac{2-\ell^2 f''(r_+)}{6\ell^2}\frac{A}{2} & \text{case } c_2 \\ \frac{\alpha_c}{4\pi}\frac{-f''(r_+)}{6}\frac{A}{2} & \text{case } c_3 \end{cases} \end{aligned} \tag{5.54}$$

and after employing Cardy's formula, we obtain the black hole entropy for $ds^2_{c_2}$ and $ds^2_{c_3}$ within CWG:

$$S_{BH} = 2\pi\sqrt{\frac{Q_0 c}{6}} = \begin{cases} \frac{\alpha_c\left(2-\ell^2 f''(r_+)\right)}{12\ell^2}A & \text{case } c_2 \\ \frac{\alpha_c f''(r_+)}{12}A & \text{case } c_3 \end{cases}. \tag{5.55}$$

Finally, when comparing the above results with the CWG Wald entropy formula. (3.17) we get:

$$S = -\frac{\alpha_c}{8}\int W_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu}\varepsilon^{\alpha\beta}d\Sigma = \begin{cases} \frac{\alpha_c\left(2-\ell^2 f''(r_+)\right)}{12\ell^2}A & \text{case } c_2 \\ -\frac{\alpha_c f''(r_+)}{12}A & \text{case } c_3 \end{cases}, \tag{5.56}$$

which again precisely agrees with the results for $ds^2_{c_2}$ and $ds^2_{c_3}$ from counting the canonical quantum microstates via the boundary Nöther current method, modulo the minus sign in the $ds^2_{c_3}$ case. We should note that, in the $ds^2_{c_3}$ case, we could have chosen the minus sign after taking the square root in the above Cardy formula.

Chapter 6

Quantum Fields and AdS_2/CFT_1 within CWG Paradigm

In this chapter, we will utilize semiclassical approaches to investigate the development of a near horizon conformal field theory (CFT) action that contains holographic information about the thermodynamics of the chosen black hole. Our focus will be on the $ds^2_{c_2}$ case, and we will follow the methodologies outlined in [64, 65, 114, 115]. To identify the relevant two-dimensional field components for our construction, we will examine the spacetime using a scalar field with minimal coupling, then take the near horizon limit and eliminate angular degrees of freedom through a Robinson-Wilczek two-dimensional decomposition [116]. The resulting two-dimensional theory will exhibit a well-known quantum effective action similar to the Polyakov action.

$$S_{eff} \sim \bar{\beta}^{\psi} \int d^2x \sqrt{-g^{(2)}} R^{(2)} \frac{1}{\Box_{g^{(2)}}} R^{(2)} + \cdots , \tag{6.1}$$

where $\bar{\beta}^{\psi} = \frac{const}{G}$ is the Weyl anomaly coefficient [117, 118].

6.1 Effective Action & Asymptotic Symmetries

Our objective is to determine the complete effective theory and the value of $\bar{\beta}^{\psi}$ to s-wave approximation within the CWG paradigm. The s-wave approximation is reasonable given that ψ originates from gravity and should therefore possess characteristics such as being real and dimensionless. Additionally, this region of approximation appears to encapsulate the majority of gravitational dynamics [119], i.e., $ds^2_{c_2}$ exhibits a Kaluza-Klein decomposition:

$$\begin{aligned} ds^2 = & -f(r)dt^2 + f(r)^{-1}dr^2 + \ell^2 e^{-2\psi(r)} \left[d\theta^2 + \sin^2\theta \left(d\phi - Adt\right)^2 \right] \\ = & \; ds^2_{2D} + \ell^2 e^{-2\psi(r)} \left[d\theta^2 + \sin^2\theta \left(d\phi - A_\mu dx^\mu\right)^2 \right], \end{aligned} \tag{6.2}$$

where we have introduced a two-dimensional black hole coupled to a two-dimensional real scalar and two-dimensional $U(1)$ gauge field. In the case of $ds^2_{c_2}$, we see that the only allowable gauge field couplings are linear phase shifts given by $\phi \to \phi - At$, which are trivial. However, for spacetimes not exhibiting global spherical symmetry, the s-wave is still an appropriate approximation.

In order to identify the appropriate two-dimensional near horizon theory, we initiate our analysis by examining the behavior of a four-dimensional massless free scalar field within the background of (6.2):

$$S^{(4)}[\varphi,g] = -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\mu\varphi\nabla^\mu\varphi$$
$$= \frac{1}{2}\int d^4x\,\varphi\left[\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\right]\varphi \tag{6.3}$$

To achieve a two-dimensional theory, it is necessary to eliminate angular degrees of freedom. This can be accomplished by expressing φ as a series of spherical harmonics and examining the regime where $r \sim r_+$ in tortoise coordinates. Then, integrating out the angular degrees of freedom

$$S^{(4)}[\varphi,g] \overset{r\sim r_+}{\longrightarrow} S^{(2)}[\varphi_{lm},g^{(2)}]$$
$$= \frac{\ell^2}{2}\int dt dr \varphi^*_{lm}\left[\frac{1}{f(r)}\left(\partial t - iA_t\right)^2 - \partial_r f(r)\partial_r\right]\varphi_{lm}. \tag{6.4}$$

where the transform to tortoise coordinates defined by $\frac{dr^*}{dr} = \frac{\ell^2}{r^2-\varepsilon^2}$.

The resulting two-dimensional theory is much simplified by considering the region $U \sim \varepsilon$, since interacting/mixing and potential terms ($\sim l(l+1)\ldots$) are weighted by the near horizon exponentially decaying term $f\left(r\left(r^*\right)\right) \sim e^{2\kappa r^*}$. Therefore, within the specified regime, (6.3) can be reduced to an action that involves an infinite number of massless complex scalar fields given by:

$$S = -\frac{\ell^2}{2}\int d^2x\varphi^*_{lm}D_\mu\left[\sqrt{-g^{(2)}}g^{\mu\nu}_{(2)}D_\nu\right]\varphi_{lm}, \tag{6.5}$$

where $D_\mu = \partial_\mu - imA_\mu$ is the gauge covariant derivative. We have now reached the Robinson and Wilczek two-dimensional analog (RW2DA) fields for the (6.2) spacetime expressed by:

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} -f(r) & 0 \\ 0 & \frac{1}{f(r)} \end{pmatrix} \qquad f(r) = r^2\sqrt{\frac{3Q^2\ell^4 + 8\alpha_c}{8\ell^4\alpha_c}} + rc_2 \tag{6.6}$$

and trivial $U(1)$ gauge field given by:

$$A = A_t dt = constant = 0. \tag{6.7}$$

Taking the s-wave approximation and incorporating the redefinition proposed by CWG, namely $\varphi_{00} = \sqrt{\gamma}\psi$ (where ψ is dimensionless and γ is proportional to α_c), the effective action for the aforementioned two-dimensional theory (6.5) can be expressed in two components [120, 121]:

$$\Gamma = \Gamma_{grav} + \Gamma_{U(1)}, \tag{6.8}$$

where

$$\begin{aligned} \Gamma_{grav} =& \frac{\gamma\ell^2}{96\pi}\int d^2x\sqrt{-g^{(2)}}R^{(2)}\frac{1}{\Box_{g^{(2)}}}R^{(2)} \text{ and} \\ \Gamma_{U(1)} =& \frac{\gamma e^2\ell^2}{2\pi}\int F\frac{1}{\Box_{g^{(2)}}}F. \end{aligned} \tag{6.9}$$

Comparison with equation (6.1), it can be observed that in the context of our CWG-inspired scenario, the value of $\bar{\beta}^{\psi}$ is equal to $\frac{\gamma\ell^2}{96\pi}$. It is appropriate to mention that for the case of $ds^2_{c_2}$, $\Gamma_{U(1)}$ is equal to zero as described in equation (6.8). The subsequent plan of action involves reinstating locality within the quantum effective action (7.8). This can be achieved through the incorporation of auxiliary (dilation and axion) scalar fields Φ and B, where:

$$\square_{g^{(2)}}\Phi = R^{(2)} \quad \text{and} \quad \square_{g^{(2)}}B = \varepsilon^{\mu\nu}\partial_\mu A_\nu. \tag{6.10}$$

which in terms of general $f(r)$ and $_t(r)$, the previously mentioned expression can be simplified into the subsequent set of differential equations:

$$\begin{aligned} -\frac{1}{f(r)}\partial_t^2\Phi + \partial_r f(r)\partial_r\Phi =& R^{(2)} \\ -\frac{1}{f(r)}\partial_t^2 B + \partial_r f(r)\partial_r B =& -\partial_r A_t, \end{aligned} \tag{6.11}$$

and exhibit the general solutions:

$$\begin{aligned} \Phi(t,r) =& \alpha_1 t + \int dr \frac{\alpha_2 - f'(r)}{f(r)} \\ B(t,r) =& \beta_1 t + \int dr \frac{\beta_2 - A_t(r)}{f(r)}, \end{aligned} \tag{6.12}$$

where α_i and β_i are integration constants. Now, to obtain a local action we substitute (6.10) into (6.9) which yields our final near horizon CFT in Liouville form and is given by:

$$\begin{aligned} S_{NHCFT} =& \frac{\gamma\ell^2}{96\pi}\int d^2x\sqrt{-g^{(2)}}\left\{-\Phi\square_{g^{(2)}}\Phi + 2\Phi R^{(2)}\right\} \\ &+ \frac{\gamma e^2\ell^2}{2\pi}\int d^2x\sqrt{-g^{(2)}}\left\{-B\square_{g^{(2)}}B + 2B\left(\frac{\varepsilon^{\mu\nu}}{\sqrt{-g^{(2)}}}\right)\partial_\mu A_\nu\right\} \end{aligned} \tag{6.13}$$

6.2 Energy Momentum & Virasoro Algebra

Next, we will examine and compute the non-trivial asymptotic symmetries pertaining to the two-dimensional gravitational component of (6.6), with a focus on large r behavior given by:

$$g^{(0)}_{\mu\ \nu} = \begin{pmatrix} -\frac{\sqrt{\frac{3Q^2\ell^4+8\alpha_c}{\ell^4\alpha_c}}r^2}{2\sqrt{2}} - c_2 r + \mathscr{O}\left(\left(\frac{1}{r}\right)^3\right) & 0 \\ 0 & \frac{2\sqrt{2}}{\sqrt{\frac{3Q^2\ell^4+8\alpha_c}{\ell^4\alpha_c}}r^2} + \mathscr{O}\left(\left(\frac{1}{r}\right)^3\right) \end{pmatrix}, \tag{6.14}$$

which is asymptotically AdS_2 with Ricci Scalar,

$$R = -\frac{2}{l^2} + O\left(\left(\frac{1}{r}\right)^1\right) \tag{6.15}$$

where

$$l^2 = \frac{2\sqrt{2}}{\sqrt{\frac{3Q^2\ell^4+8\alpha_c}{\ell^4\alpha_c}}} \tag{6.16}$$

Coupling (6.14) with the following metric fall-off conditions:

$$\delta g_{\mu\nu} = \begin{pmatrix} \mathscr{O}(r) & \mathscr{O}\left(\left(\frac{1}{r}\right)^0\right) \\ \mathscr{O}\left(\left(\frac{1}{r}\right)^0\right) & \mathscr{O}\left(\left(\frac{1}{r}\right)^3\right) \end{pmatrix}, \tag{6.17}$$

we get the following set of asymptotic symmetries:

$$\chi = -C_1 \frac{r\sqrt{\frac{16}{\ell^4}+\frac{6Q^2}{\alpha_c}}\,\xi(t)}{4c_2 + r\sqrt{\frac{16}{\ell^4}+\frac{6Q^2}{\alpha_c}}}\partial_t + C_2 r \xi'(t)\partial_r, \tag{6.18}$$

where $\xi(t)$ is an arbitrary function and C_i are arbitrary normalization constants. Switching to conformal light-cone coordinates, which are:

$$x^{\pm} = t \pm r^*, \tag{6.19}$$

where r^* is the radial tortoise coordinate; we get:

$$\chi^{\pm} = \frac{\left(-C_1 r(r^*)\sqrt{\frac{16}{\ell^4}+\frac{6Q^2}{\alpha_c}}\,\xi(x^+,x^-) \pm 4C_2\xi'(x^+,x^-)\right)}{4c_2 + r(r^*)\sqrt{\frac{16}{\ell^4}+\frac{6Q^2}{\alpha_c}}}. \tag{6.20}$$

The above diffeomorphisms exhibit smoothness at their radial boundaries, and it is possible to normalize the arbitrary constants to enable $\chi^{\pm}$ to conform to a Witt or $Diff(S^1)$ subalgebra $i\{\chi^{\pm}_m, \chi^{\pm}_n\} = (m-n)\chi^{\pm}_{(m+n)}$ when choosing a circle diffeomorphism representation for $\xi(x^+,x^-)$.

The energy-momentum tensor (EMT) for (6.13) is given by the standard definition:

$$\begin{aligned}\langle T_{\mu\nu}\rangle =& \frac{2}{\sqrt{-g^{(2)}}}\frac{\delta S_{NHCFT}}{\delta g^{(2)\mu\nu}} \\ =& \frac{\gamma\ell^2}{48\pi G}\left\{\partial_\mu\Phi\partial_\nu\Phi - 2\nabla_\mu\partial_\nu\Phi + {g^{(2)}}_{\mu\nu}\left[2R^{(2)} - \frac{1}{2}\nabla_\alpha\Phi\nabla^\alpha\Phi\right]\right\}.\end{aligned} \tag{6.21}$$

Now, it will be advantageous to pick a convenient choice for the c_2 constant in (6.6), such that:

$$\begin{aligned} f(r) =& \frac{r^2 - \ell r}{l^2}, \text{ where} \\ l^2 =& \sqrt{\frac{8\ell^4 \alpha_c}{3Q^2\ell^4 + 8\alpha_c}}, \quad \text{as previously defined and} \\ c_2 =& -\frac{\ell}{l^2}. \end{aligned} \tag{6.22}$$

Now, using the general solution (6.12) and substituting into (6.21), and additionally implementing modified Unruh Vacuum boundary conditions (MUBC) [122]:

$$\begin{cases} \langle T_{++} \rangle = 0 & r \to \infty, \; l \to \infty \\ \langle T_{--} \rangle = 0 & r \to r_+ \end{cases} \tag{6.23}$$

we can now determine the integration constants α_i:

$$\alpha_1 = -\alpha_2 = \frac{1}{2} f'(r_+) \tag{6.24}$$

as well as the energy-momentum tensor. Given the two-dimensional reduction, the above EMT exhibits a Weyl (trace) anomaly, which is given by:

$$\left\langle T_\mu{}^\mu \right\rangle = -\bar{\beta}^\Psi 4R^{(2)}, \tag{6.25}$$

which uniquely determines the value of central charge via [116]:

$$\frac{c}{24\pi} = 4\bar{\beta}^\Psi \Rightarrow c = \ell^2\gamma = \frac{\gamma}{4\pi} A. \tag{6.26}$$

Furthermore, as a result of our utilization of the previously defined MUBC (6.23), the defined asymptotic boundary condition of $\mathscr{O}(\frac{1}{\ell})^2$ (referred to as the limit $x^+ \to \infty$), signifies that the energy-momentum tensor (EMT) is predominantly dominated by a single holomorphic component, denoted as $\langle T_{--} \rangle$. If we extend this component with respect to boundary fields (6.14) and compute its response to the asymptotic symmetries, we get:

$$\delta_{\chi^-} \langle T_{--} \rangle = \chi^- \langle T_{--} \rangle' + 2 \langle T_{--} \rangle \left(\chi^-\right)' + \frac{c}{24\pi} \left(\chi^-\right)''' + \mathscr{O}\left(\left(\frac{1}{r}\right)^3\right), \tag{6.27}$$

where the primes indicate time derivatives. Thus, the above result shows that $\langle T_{--} \rangle$ transforms asymptotically as the EMT of a one dimensional CFT, with center given by (6.26).

Finally, we are now able to compute the generator charge algebra of the full asymptotic symmetry group (ASG) by compactifying (which regulates the asymptotic charges) the x^- coordinate to a circle parametrized from $0 \to 2\pi/\kappa$, and defining the respective asymptotic conserved charge as:

$$Q_n = \lim_{x^+ \to \infty} \int dx^\mu \left\langle T_{\mu\nu} \right\rangle \chi_n^\nu. \tag{6.28}$$

where $\xi(x^+, x^-)$ has been replaced with circle diffieomorphisms $\frac{e^{-in\kappa x^\pm}}{\kappa}$ in (6.20) and compute the canonical response of Q_n with respect to the asymptotic symmetry, yielding:

$$\delta_{\chi_m^-} n = [m, n] = (m-n)_n + \frac{c}{12} m \left(m^2 - 1\right) \delta_{m+n,0}. \tag{6.29}$$

This indicates that the asymptotic quantum generators form a centrally extended Virasoro algebra.

6.3 Entropy & Temperature

The center charge (6.26) with lowest eigenmode is given by:

$$Q_0 = \frac{\gamma}{96\pi}A. \tag{6.30}$$

The previous analysis demonstrates that the RW2DA fields of $ds^2_{c_2}$ as expressed in equation (6.6) can be effectively described by a one-dimensional string theory, represented by equation (6.13). Additionally, the microstates of the horizon in this theory have a holographic duality to a one-dimensional conformal field theory (CFT_1) with a center given by:

$$c = \frac{\gamma}{4\pi}A \tag{6.31}$$

and lowest Virasoro eigenmode:

$$_0 = \frac{\gamma}{96\pi}A. \tag{6.32}$$

Using these results in the Cardy Formula (1.7) [RAID: FIX the equation link], we obtain:

$$S = 2\pi\sqrt{\frac{c_0}{6}} = \frac{\gamma A}{24}. \tag{6.33}$$

By comparing the above with Equation (5.55), we are able to ascertain ix the γ constant for the case of $ds^2_{c_2}$ to be:

$$\gamma = \frac{8\pi\alpha_c\left(2-\ell^2 f''(r_+)\right)}{\ell^2}. \tag{6.34}$$

Subsequently, after fixing the value of γ and calculating the center as indicated in equation (6.26), we can demonstrate that the incorporation of black hole temperature within our particular AdS/CFT framework by focusing on the EMT, which is determined by a single holomorphic component on the horizon:

$$\langle T_{++}\rangle = -\frac{c}{48\pi}\left(\frac{f(r_+)}{2}\right)^2. \tag{6.35}$$

The above is the Hawking flux (HF) specifically in relation to the central charge:

$$\langle T_{++}\rangle = cHF = -\frac{c\pi}{12}\left(T_H\right)^2, \tag{6.36}$$

where $T_H = \frac{f(r_+)}{4\pi}$.

The results presented here are highly intriguing as they provide evidence that our developed AdS_2/CFT_1 correspondence encompasses both the entropy and temperature of black holes.

Chapter 7

Conclusion

In this thesis, we extensively investigated the thermodynamic properties of black hole solutions within the CWG paradigm. Our analysis encompassed both vacuum and non-vacuum solutions, allowing us to compute the black hole entropy for various solutions using the Nöther current method. To validate and reinforce our findings, we also performed a comparative analysis between the Nöther current approach and the Wald entropy formula for CWG. Surprisingly, our results aligned with Wald's entropy, but we have gone beyond that to uncover the intriguing fact that black hole entropy within CWG is not universally consistent. Instead, it exhibits variations depending on the specific black hole being examined and its associated symmetries. This fascinating divergence from the uniformity observed in Einstein-Hilbert gravity, where the black hole entropy is governed by the standard Bekenstein-Hawking formula regardless of spacetime, adds a new dimension to our understanding of black hole thermodynamics within the CWG framework.

Our Nöther current analysis also indicated that the degrees of freedom at the horizon of a quantum black hole is connected to a conformal field theory of lower dimension. To delve further into this duality, we undertook a detailed examination by utilizing semiclassical methods to construct a two-dimensional quantum effective action near the horizon. This allowed us to establish that the resulting Energy-Momentum Tensor (EMT), at the appropriate boundary, forms a variant of the Virasoro algebra with central extension. The central extension and lowest eigenmode of this algebra encode the entropy of the black hole using Cardy's formula. Additionally, we illustrate how our construction of the conformal field theory also incorporates the temperature of the black hole through the quantum holomorphic flux on the horizon.

One intriguing aspect of our AdS/CFT framework is the inclusion of a parameter γ, which had to be determined by comparing the resulting entropy with that obtained through the Nöther current approach. It would be highly interesting for future studies to investigate the behavior of this parameter under renormalization group flow analysis, as it is intricately linked to the coupling of the two-dimensional CFT. Such an analysis could potentially shed new light on the existence of an analogous c-theorem [123] for (CWG).

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