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A Construction of Cospectral Graphs

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By

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Abstract

In this report, we investigated two questions in the field of spectral graph theory. The first question was whether it is possible to extend recent results to find a large class of graphs uniquely determined by the spectrum of its adjacency matrix. Our investigation led to the discovery of a pair of cospectral graphs which contradicted the existence of such a class of graphs. The second question was whether there exists a construction of cospectral graphs that consists of adding a single edge and vertex to a given pair of cospectral graphs. We discovered that such a construction exists, and generated several pairs of cospectral graphs using this method. Further investigation showed that this construction of cospectral graphs is strongly related to two previously studied constructions.

Acknowledgement

I would like to thank Professor Peter Christopher for his guidance and assistance throughout this project, and for the support and encouragement that he generously offered as it progressed.

1. Introduction

The (0,1)-adjacency matrix is one of many possible representations of a graph. The adjacency matrix is a useful representation of a graph, as it condenses the entire structure of a graph, its vertices and edges, into a single mathematical object. By determining the eigenvalues of the adjacency matrix, we obtain the spectrum of a graph with respect to the adjacency matrix. In this paper, we will refer to this simply as the spectrum of a graph. The development of a graph spectrum has introduced another parameter of a graph to be studied, and has prompted further research in an attempt to discover what information is encapsulated within the eigenvalues of a graph's adjacency matrix.

The notion of graph spectra has been used to investigate open questions within other fields of research. The properties of a graph's spectrum have been applied to the field of chemistry, by using graph theoretical techniques to examine the structure of molecules [3]. It has also been applied to networking problems, with prior work measuring the relationship between the spectrum and the connectivity of the network [1]. Thus, there are real-world applications to the application of linear algebra ideas and techniques to graph theory.

To this end, the field of spectral graph theory has developed to investigate the properties of graph spectra. Two questions which have been a topic of research in this field will be looked at in this paper. Firstly, which graphs, or classes of graphs, are uniquely determined by their spectra? That is, which graphs have a spectrum that no other graph, up to isomorphism, could possibly have? Secondly, what methods exist to construct pairs of non-isomorphic graphs that are cospectral, or have the same spectrum? There has been much research into these questions, yielding results for many different classes of graphs.

Expanding upon recent research which showed that certain classes of unicyclic and bicyclic graphs are uniquely determined by their spectra, we investigated a more general class of graphs consisting of a core cycle and disjoint paths which are adjacent to that cycle. Our initial aim was to determine whether there was some generalized class of graphs, similar to those previously studied, whose spectra were unique. The process instead revealed a method for constructing pairs of cospectral graphs by adding a single edge and vertex. This construction allows for the creation of an infinite class of graphs that are not uniquely identified by their spectra. After further research, it was found that this construction is very closely related to two previously discovered constructions of cospectral graphs.

2. Background

2.1 Spectral Graph Theory

There are several different representations to describe a given graph. Some of these use matrices to describe the characteristics of the different vertices and edges of a graph. Such matrices include the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix, and the Seidel matrix. Each matrix has a different definition, leading to diverse matrices which each somehow represent the structural properties of a given graph.

In this report, we will look at the (0,1)-adjacency matrix. In an adjacency matrix, an entry (i, j) is assigned the value 1 if there exists an edge between vertices i and j, and the value 0 otherwise. Using such a representation for a graph, certain properties of the graph are easy to identify. These include simple properties such as the number of vertices and edges, as well as the degree of each vertex and whether the graph is bipartite. Other properties can also be determined using the adjacency matrix, such as the number of walks of any given length between two vertices.

The field of spectral graph theory probes further into the usefulness of the adjacency matrix by applying linear algebra to find the eigenvalues and eigenvectors of a graph. Taken together, the set of eigenvalues, with repetition, of a matrix are referred to as the spectrum of a matrix. Thus, a graph can also be described by the spectra of the several matrices that may be used to represent it. For this paper, we will use the term spectrum to refer to the spectrum of a graph with respect to its adjacency matrix. The largest eigenvalue of a graph's spectrum is referred to as the spectral radius of the graph.

Since the adjacency matrix is a graph invariant, each graph is associated with a single spectrum and is thus also a graph invariant. If there is no other graph, up to isomorphism, that has the same spectrum, then the graph is uniquely identified by its spectrum. If two non-isomorphic graphs share the same spectrum, then these graphs are referred to as cospectral graphs.

2.2 Known Theorems

In this report, we will solely look at simple graphs. Let $G = (V(G),\ E(G))$ be a simple graph with order $|V(G)| = n$ and size $|E(G)| = m$. Assign a labeling to the vertices of the graph, so that $V(G) = \{1,2, \dots, n\}$, and let *A* be the (0,1)-adjacency matrix of graph *G*, based on the following construction.

$$A_{(i,j)} = \begin{cases} 1 & if\ (i,j) \in E(G) \\ 0 & otherwise \end{cases}$$

Let $Q_G(X)$ be the characteristic polynomial of the matrix *A*, and $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ its associated eigenvalues. Thus, the values $\lambda_1(G), \dots, \lambda_n(G)$ form the spectrum of the graph *G*. The spectral radius of *G* is defined to be $\lambda_1(G)$, the largest eigenvalue of the adjacency matrix.

One useful facet of the adjacency matrix is that it can used to determine the number of walks between any two vertices of the graph by taking powers of the matrix. Thus, ${A^n}_{(i,j)}$ gives the number of walks of length *n* between vertices *i* and *j*. By considering the diagonal entries of the powers of A, one can also find the number of closed walks of any given length on a given vertex.

The trace $tr(A)$ of a matrix A is equal to the sum of the entries on the main diagonal of the matrix, and is also equal to the sum of the eigenvalues of the matrix. By the definition of the adjacency matrix, $tr(A) = \sum \lambda_i = 0$, and so the values of the spectrum of a graph must add to zero. By taking the trace of powers of the adjacency matrix, $tr(A^i)$, one can obtain the number of closed walks of a given graph.

From the spectrum, much information can be determined concerning the parameters and structure of a graph:

Theorem 2.1 [11]:

Using the (0,1)-adjacency matrix, the following information can be deduced from the spectrum of a graph:

- *The number of vertices* $|V(G)| = n$.
- *The number of edges* $|E(G)| = m$.
- *Whether G is regular.*
- *The number of closed walks of length i, for any i.*
- *Whether G is bipartite.*

Thus two graphs having the same spectrum must have some very similar structural properties. Investigating this topic further has required the development of theorems and formulas for calculating the characteristic polynomial and the spectrum of a graph. Some of these theorems which will prove useful in the remainder of our paper are included here. The following theorem gives an inequality that relates the values of the spectrum of a graph to the spectrum of a subgraph.

<u>*Theorem 2.2: (Interlacing Theorem)*</u> [6]

Let G be a graph with n vertices and spectrum $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$*, and let H be an induced subgraph of G with m vertices and spectrum* $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$*. Then for* $i = 1..m,\ \lambda_{n-m+i} \leq \mu_i \leq \lambda_i$.

<u>*Corollary 2.3*</u>*:*

Let G be a graph with n vertices and spectrum $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$*, and let H be an induced subgraph of G with n-1 vertices, created by deleting a vertex from G, and with spectrum* $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$*. Then:*

$$\lambda_1(G) \geq \mu_1(G) \geq \lambda_2(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \lambda_n(G).$$

The following two theorems provide methods for calculating the characteristic polynomial of a graph by performing some manipulations of the structure of the graph.

<u>*Theorem 2.4*</u> [2]*:*

Let G be a graph obtained by joining by an edge a vertex x of graph G_1 *and a vertex y of graph* G_2*. Then:*

$$Q_G(X) = Q_{G_1}(X) * Q_{G_2}(X) - Q_{G_1/x}(X) * Q_{G_2/y}(X)$$

<u>*Theorem 2.5*</u> [2]*:*

Let G be a graph and x a pendant vertex of G. Let y be the neighbor of x in G. Then:

$$Q_G(X) = X * Q_{G/x}(X) - Q_{G/x,y}(X)$$

2.3 Graphs Uniquely Identified by Their Spectrum

An open question in the field of spectral graph theory is to determine which graphs are uniquely identified by their spectra, and which graphs are not; that is, whether two graphs which are cospectral (have identical spectra) are also isomorphic. It is known that not every graph is uniquely identified by its spectrum, and Figure 1 below shows the smallest pair of non-isomorphic graphs which share the same spectrum.

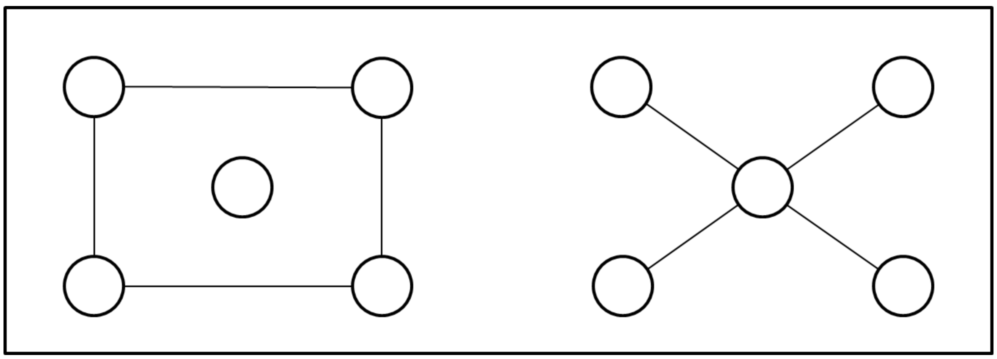


Figure 1 - Smallest Cospectral Graphs

There are, however, many classes of graphs whose spectra do uniquely identify a graph. The survey “Which Graphs Are Determined by Their Spectrum?” by van Dam and Haemers [11] covers such results. For example, it is known that all paths, cycles, complete graphs, and complete bipartite graphs are each uniquely determined by their spectra.

Continuing this investigation, recent papers have expanded the analysis of spectra to further classes of graphs. Boulet and Jouve [4] proved that all lollipop graphs are identified by their spectra. The definition of a lollipop graph relies on a graph construction called coalescence. The coalescence of two graphs, G_1 and G_2, is created by first selecting a distinct vertex v_1 in G_1 and a distinct vertex v_2 in G_2 on which to perform the manipulation. Then, the coalescence

$G_1 * G_2$ consists of the vertices and edges of the individual graphs, except that the two vertices v_1 and v_2 are combined into a single vertex v which is adjacent to every vertex in G_1 adjacent to v_1 and every vertex in G_2 adjacent to v_2.

A lollipop graph is thus defined as a coalescence of a cycle and a path, with one of the path's end vertices as a distinguished vertex. Figure 2 shows an example of a lollipop graph consisting of a coalescence of a C_5 and a P_3. Boulet and Jouve completed their proof by counting the closed walks of a lollipop graph, and showing that no other graph could share the same number of closed walks.

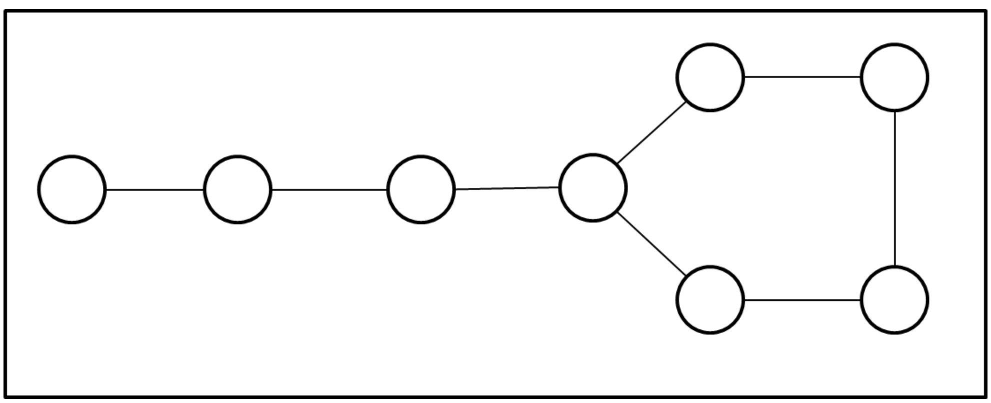


Figure 2 – Example of a Lollipop Graph

Wang, Belando, Huang, and Li Marzi [12] proved that dumbbell graphs, with some exceptions, are identified uniquely by their spectra. A dumbbell graph is defined as a coalescence of two cycles and a path, such that each cycle is adjacent to one of the end-vertices of the path. An example of a dumbbell graph consisting of a C_4, a C_5, and a P_3 is shown in Figure 3. Their proof followed a similar method as Boulet and Jouve.

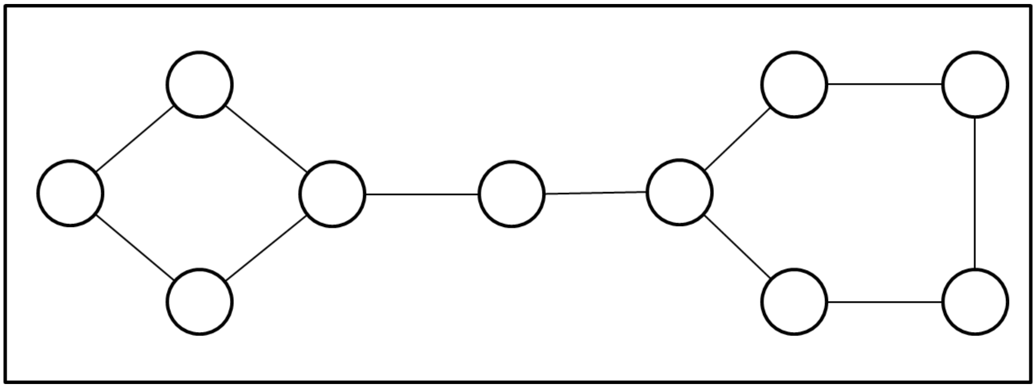


Figure 3 – Example of a Dumbbell Graph

These discoveries indicate the possibility that there may be a more generalized class of graphs consisting of cycles and paths that may be uniquely identified by its spectrum with regard to their adjacency matrices.

3. Findings

3.1 Expanding a Lollipop/Dumbbell Family of Graph

To investigate the possibility of finding a more generalized class of graphs that have unique spectra, our first goal was to try to extend the uniqueness of the lollipop graph to possibly allow for a second path branching out of the core cycle. Thus, we chose the graph G shown below in Figure 4, which consists of a $L(5,3)$ graph (a cycle of length 5 connected to a path of length 3) with an additional edge connected to the cycle.

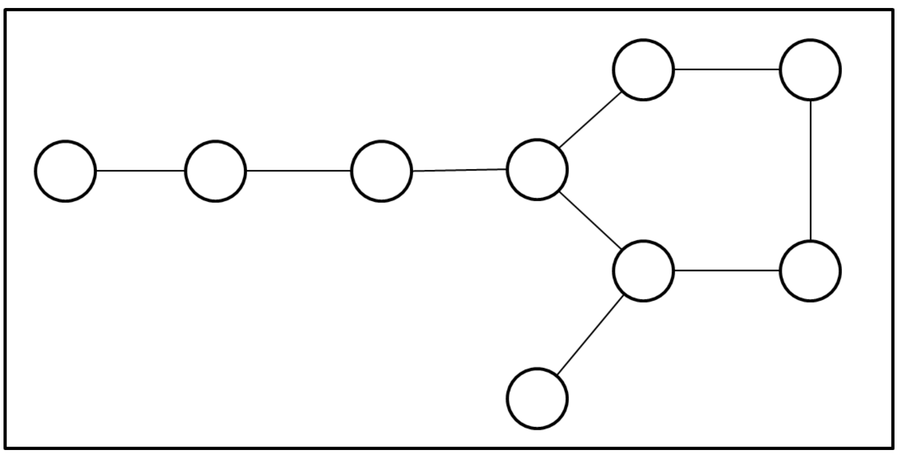


Figure 4 - Extension of Lollipop Graph

To demonstrate that this graph has a unique spectrum, we used several properties of graphs and graph spectra to determine what structural characteristics a graph cospectral with this graph must have. We enumerated all the graphs with these properties, and then used Maple to calculate the spectra of these possibilities in an effort to find a cospectral graph.

Since the spectrum of a graph demonstrates the number of vertices and edges of a graph, any graph cospectral with G must also have 9 vertices and 9 edges. Furthermore, the main structure of the graph can be clarified by considering the number of closed walks of G. From Theorem 2.1, the number of closed walks of any length can be deduced from the spectrum of a graph. Thus, two cospectral graphs must share the same number of closed walks of any length. Since the number of closed walks of length 3 is directly proportional to the number of triangles in the graph, two cospectral graphs must share the same number of triangles. Similarly, if there are no triangles, then the number of closed walks of length 5 is directly proportional to the number of 5-cycles, and thus cospectral graphs share the same number of 5-cycles. Using these properties, it can be seen that any graph cospectral with G must be triangle-free, and must have exactly one 5-cycle. This information narrows the list of possible graphs cospectral to G greatly.

To further narrow down possibilities, we next considered the class of graphs that contain $L(5,3)$ as a subgraph, a class which closely resemble the graph G. Again, we used the fact that cospectral graphs must share the same number of closed walks of all lengths to examine these options. Since these graphs share almost all the same edges (namely those in the $L(5,3)$ subgraph), they share the same number of closed walks that do not traverse the one unique edge in each graph. Thus, we only need to compare the number of closed walks that traverse the edge not part of the $L(5,3)$ subgraph, and the graphs are not cospectral if they contain a different number of closed walks for some length. Using this method, we were able to rule out all graphs containing $L(5,3)$ as a subgraph. Furthermore, this method leads us to believe that it is impossible for two graphs created by taking a graph and adding a single vertex and an edge connecting that vertex to the original graph to be cospectral unless they are isomorphic.

Thus, a graph cospectral with G must contain exactly one 5-cycle, no triangles, and must not contain $L(5,3)$ as a subgraph. There are a total of 25 such graphs, most of which consisting of a 5-cycle that acts as a central node, with the remaining edges forming a tree around this node.

Without any further means of removing non-cospectral graphs, we next developed a worksheet in Maple to calculate the exact spectrum of a graph, given its adjacency matrix. This worksheet consisted of the assignment of the matrix to the variable A, followed by the two evaluations.

$$S := LinearAlgebra\,[Determinant\,](A)$$

$$fsolve\,(S)$$

These functions give the eigenvalues of the adjacency matrix as given by the variable assignment which precedes it. This worksheet was used to determine the spectra of the 25 graphs which satisfy the conditions of cospectrality with G, as given above.

By enumerating the spectra for all graphs which may potentially be cospectral to G, we have determined that there is no graph that is cospectral to G without being isomorphic to G, and thus that G is uniquely identified by its spectrum. However, through this process, we discovered a new pair of graphs which are cospectral, but not isomorphic. These graphs are shown below in Figure 5.

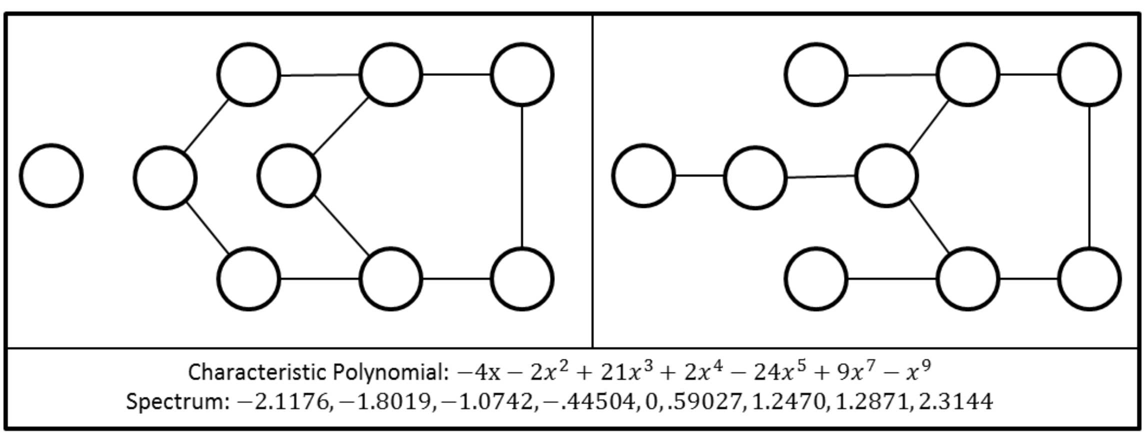


Figure 5 - Cospectral Graphs with 9 Vertices

The discovery that the graph shown on the right side of Figure 5 is not uniquely identified by its spectrum seems to indicate that there is not a generalized class of graphs, consisting of a cycle and disjoint paths branching off that cycle, which is determined by its spectra. Furthermore, the methods which have been used to determine that the graph G has no non-isomorphic graphs cospectral to it would not scale well to graphs of a larger order, or to graphs which contain an even cycle instead of an odd. Even cycles would be more difficult because, while we could conclusively say in this case that any graph cospectral with our original graph must have exactly one odd cycle of a given length, we would not also be able to say that there must be a cycle of the same even length if the original graph contained an even cycle.

3.2 A Construction of Cospectral Graphs

Another aspect of spectral graph theory that has prompted a large research effort is the construction of non-isomorphic cospectral graphs. This research has developed several different methods that can be used to generate classes of graphs having the same spectra. This research has yielded positive results for many different constructions of graphs. Here, we investigated whether it is possible to construct a pair of cospectral graphs by adding a single vertex and edge to a smaller pair of cospectral graphs. This construction is formally defined below, and an example of is shown in Figure 6.

Definition of construction: Let G_1 *and* G_2 *be graphs with the same spectrum, and let* v_1 *and* v_2*, respectively, be vertices of these graphs. Construct new graphs* G'_1 *by adding a vertex* v'_1 *and edge* $\{v_1, v'_1\}$ *to* G_1*, and* G'_2 *by adding a vertex* v'_2 *and edge* $\{v_2, v'_2\}$*to* G_2*.*

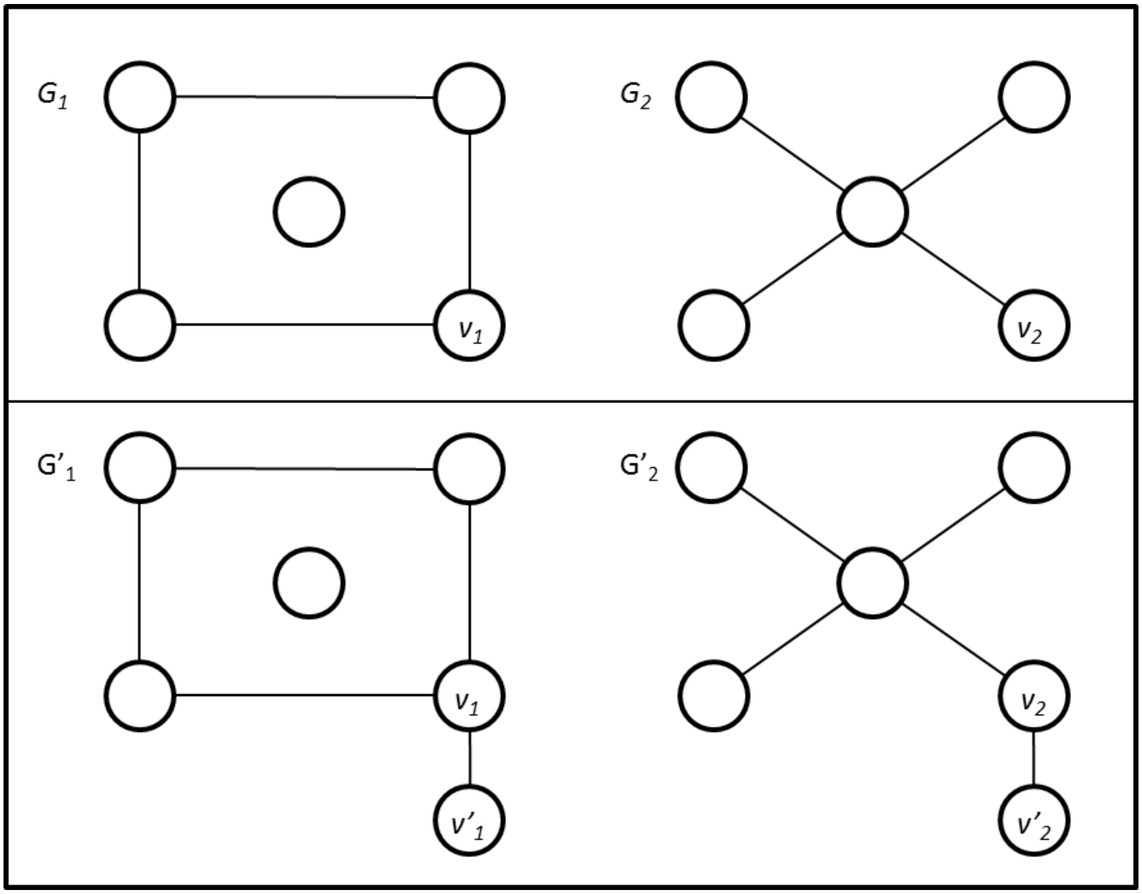


Figure 6 - Example of Constructed Graphs

If we assume that G_1 and G_2 are cospectral, then we can determine under what conditions this construction will yield graphs G'_1 and G'_2 that are also cospectral. This can be done by considering the number of closed walks of each length in the respective graphs, as cospectral graphs must have the same number of closed walks. Since G_1 and G_2 are cospectral, it is established they have this property. Thus, since the graphs G'_1 and G'_2 have these graphs as subgraphs, any closed walk in G_1 is also a closed walk in G'_1, and a similar argument is made for G_2 and G'_2. These closed walks in fact consist of the set of all closed walks in G'_1 and G'_2 which do not travel through vertices v'_1 and v'_2, respectively. Since it has been shown that there are the same number of such closed walks in G_1 and $G_{2,}$ to show G'_1 and G'_2 are cospectral it will be sufficient to show that the number of closed walks that contain v'_1 in G'_1 and v'_2 in G'_2 are identical.

Thus, adding a new vertex and edge to a pair of cospectral graphs will result in a new pair of cospectral graphs only if the number of new closed walks added of each length is identical. We thus considered what factors determine how many new closed walks of small lengths are added to identify under what circumstances the graphs maintain cospectrality. Closed walks of length 2 must necessarily travel along only one edge, from one vertex to another. Since G'_1 and G'_2 each have only one additional edge, the number of closed walks of length 2 in the graphs must be identical. Similarly, closed walks of length 3 must necessarily travel on a triangle, and a new triangle is not created in either G'_1 or G'_2. Thus, the number of closed walks of length 3 in each graph remains unchanged.

Thus, the first obstacle that may prevent cospectrality arises when closed walks of length 4 are considered. There are 3 possible forms that such a walk could take: a walk could travel over the same edge twice, it could travel over two adjacent edges, or it could travel over a 4-cycle. Since walks of the first type correspond between G'_1 and G'_2, and a 4-cycle is not added with this construction, the only possible difference would be over walks of the second type. Since one of the two edges of this closed walk must be the new edge added in the construction, the number of closed walks of the second type is determined by the number of edges adjacent to the new edge; or, alternatively, it is determined by the degree of the vertex v_1. Thus, one of the conditions necessary for G'_1 and G'_2 to be cospectral is that the degrees of vertices v_1 and v_2 must be identical.

Looking at closed walks of larger length, it is possible to develop further conditions necessary for cospectrality. Walks of length 5 must contain either a triangle or a 5-cycle. Since

v'_1 and v'_2 both have degree 1 and thus cannot be a part of a triangle or a 5-cycle, the only possible form a new closed walk of length 5 can take is along a triangle on v_1, and also visits v'_1. Thus, the number of new closed walks of length 5 is determined by the numbers of triangles that contain the vertices v_1 and v_2, and so for the graphs to be cospectral these numbers must be equal. Arguments such as this become more complicated as the lengths of the closed walks become longer. For closed walks of length 6, for example: in order for G'_1 and G'_2 to be cospectral, the number of vertices distance 2 from v_1 and v_2 plus twice the number of 4-cycles containing v_1 and v_2 must be equal.

Thus, by looking at the number of closed walks added to the graphs by forming this construction, we have found three minimum conditions that must be true for the construction to yield a new pair of cospectral graphs. This information is reflecting in the following theorem.

Theorem 3.1:

Let G_1 and G_2 be graphs with the same spectrum, and let G'_1 and G'_2 be graphs constructed as defined above. Then the following conditions must hold for the graphs G'_1 and G'_2 to possibly be cospectral:

1) v_1 and v_2 must have the same degree.

2) The number of triangles in G_1 containing v_1 and the number of triangles in G_2 containing v_2 must be equal.

3) The number of vertices distance 2 from v_1 and v_2, plus twice the number of 4-cycles containing v_1 and v_2 must be equal.

We can find a further condition for this construction to yield cospectral graphs by considering the characteristic polynomial of the graphs. From Theorem 2.5, we have an equation for finding the characteristic polynomial of a graph with a pendant vertex. This equation can be used to find the characteristic polynomial of G'_1 and G'_2. From this, we can see that these characteristic polynomials is depending on the characteristic polynomials of G_1 and G_2, as well as the characteristic polynomials of the graph forming by deleting v_1 from G_1 and deleting v_2 from G_2. Using this, we can formulate the following corollary to the theorem.

Corollary 3.2:

Let G_1 and G_2 be graphs with the same spectrum, and let G'_1 and G'_2 be graphs constructed as defined above. Then G'_1 and G'_2 are cospectral if and only if G_1/v_1 and G_2/v_2 are cospectral.

Thus, there are several conditions that we can test for in our search for a pair of graphs for which this construction may be successful. We began by testing small graphs and measuring the following parameters for each unique vertex v:

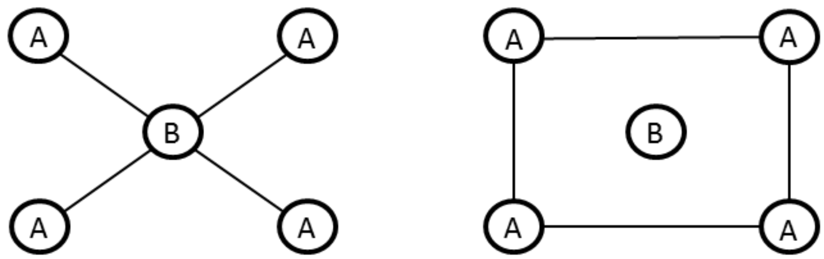
D_1 – degree of vertex

C_3 – number of triangles containing vertex

D_2 – number of vertices distance 2 from vertex

C_4 – number of 4-cycles containing vertex

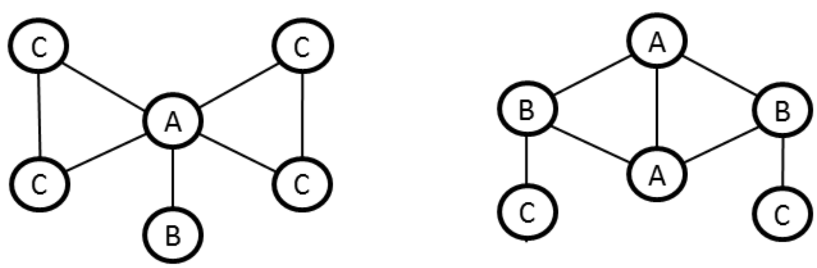
This was done with the goal of finding two vertices in a pair of cospectral graphs such that the values D_1, C_3, and $D_2 + 2 * C_4$ are all equal. Such a pair of vertices would be a candidate for adding a pendant vertex. For the smallest cospectral graphs, this did not yield any viable candidates, as shown in Figure 7 and Figure 8 below.



	D_1	C_3	D_2	C_4
A	1	0	3	0
B	4	0	0	0

	D_1	C_3	D_2	C_4
A	2	0	1	1
B	0	0	0	0

Figure 7 - Attempt 1 at Finding Pair of Vertices to Build on

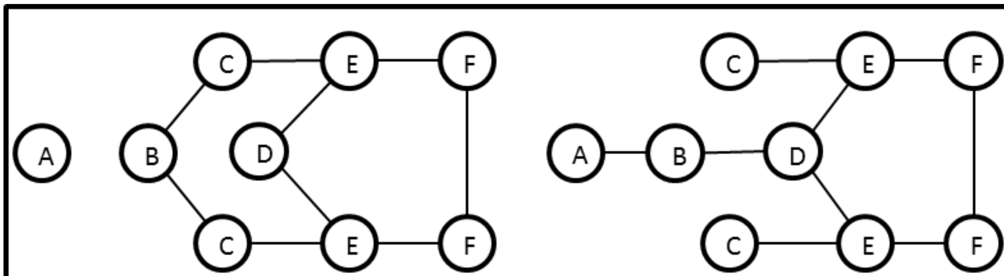


	D_1	C_3	D_2	C_4
A	5	2	0	0
B	1	0	4	0
C	2	1	3	0

	D_1	C_3	D_2	C_4
A	3	2	2	1
B	3	1	1	1
C	1	0	2	0

Figure 8 - Attempt 2 at Finding Pair of Vertices to Build on

From here, we sought to examine a pair of graphs larger and more complex in structure. This led us to choose the pair of cospectral graphs we found in Section 3.1 to perform this analysis, as shown below in Figure 9. These graphs offer more promise than the two previous pairs, as there are a few pairs of vertices that meet the conditions that must be met for the construction of cospectral graphs to be possible. These correspondences are illustrated by the lines connecting the two tables.

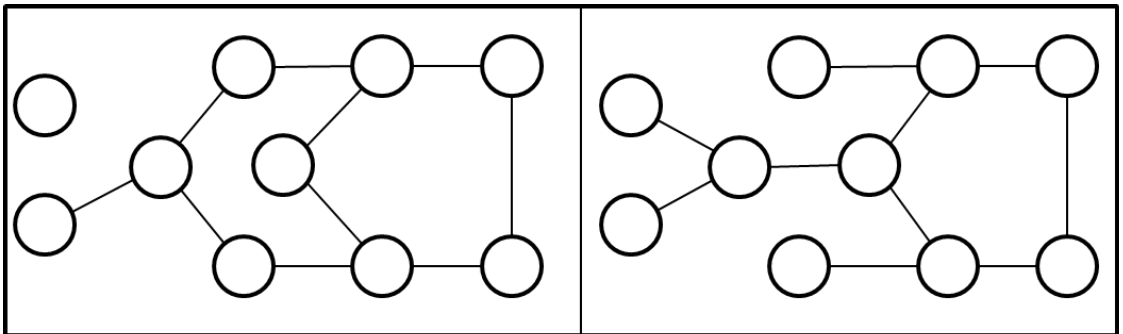


	D_1	C_3	D_2	C_4
A	0	0	0	0
B	2	0	2	0
C	2	0	3	0
D	2	0	4	0
E	3	0	3	0
F	2	0	3	0

	D_1	C_3	D_2	C_4
A	1	0	1	0
B	2	0	2	0
C	1	0	2	0
D	3	0	5	0
E	3	0	3	0
F	2	0	3	0

Figure 9 - Attempt 3 at Finding Vertices to Build on

To test whether any of these possibilities result in the construction in a pair of cospectral graphs, we calculated the spectrum of each of the graphs using Maple. This process verified that such a construction is in fact possible, and that there are three unique pairs of cospectral graphs that can be constructed by added a single vertex and edge to the graph above. One of these pairs is shown below in Figure 10 along with its spectrum, and all three pairs can be found in Appendix A.



Characteristic Polynomial: $7x^2 + 4x^3 - 33x^4 - 2x^5 + 31x^6 - 10x^8 + x^{10}$
Spectrum: $-2.1906,\ -1.8019,\ -1.3582,\ -.44504,\ 0,\ 0,\ .64428,\ 1.2470,\ 1.5528, 2.3517$

Figure 10 - Cospectral Graphs with 10 Vertices

Thus, it is clear that it is possible to construct a pair of graphs with the same spectrum from a smaller pair of cospectral graphs. One may ask whether this process can be repeated, so that an even larger pair of cospectral graphs may be constructed. This is in fact true in all cases, and can be easily shown using Theorem 2.5. Recall that if G is a graph with a pendant vertex x, then the characteristic polynomial of G is given by $Q_G(X) = X * Q_{G/x}(X) - Q_{G/x,y}(X)$, where y is the neighbor of x. Consider a scenario where two cospectral graphs have already been constructed, and we attempt to repeat the process by building off the newly added vertex. We can then define G and G' to be these newly constructed graphs. Thus, from Theorem 2.5, the characteristic polynomial of these graphs can be calculated from the characteristic polynomials of the smaller graphs on which they were built. Since these graphs were cospectral, their characteristic polynomials are the same, and it follows that the graphs G and G' must also be cospectral.

With the knowledge that the construction can be repeated to produce larger cospectral graphs, it is clear that graphs of infinite size can be built while preserving cospectrality. Referring back to the graphs in Figure 9 of order 9, it is possible to construct 12 unique pairs of cospectral graphs of order 11 by twice adding a pendant vertex. These pairs of graphs are shown in Appendix A, along with their spectra and characteristic polynomials.

Returning to the question of whether there is a generalized class of unicyclic graphs uniquely determined by its spectrum. It has been shown that there is not such a class, and using the results of this section, it is possible to construct an infinite class of unicyclic graphs that have a non-unique spectrum. The graphs constructed in this report illustrate an infinite class of unicyclic graphs, with a cycle of length 5, which are cospectral to another infinite class of graphs. As described above, these graphs can be seen as being built off of Godsil's remarkable pair of graphs, shown in Figure 12, by attaching a path of length 3 to two of the vertices.

However, there is no reason to limit this only to paths of length 3, as any length path would also result in a pair of graphs which are cospectral, including one that is unicyclic. Just as in the original case, we could build off of these graphs infinitely while maintaining cospectrality. Thus, it is possible to construct an infinite class of unicyclic graphs, for any cycle length, which are cospectral to another infinite class of graphs. An illustration of these infinite classes of graphs is shown below in Figure 11, with dotted lines representing unspecified parts of the graph that could be constructed in any way without destroying the cospectrality of the graphs.

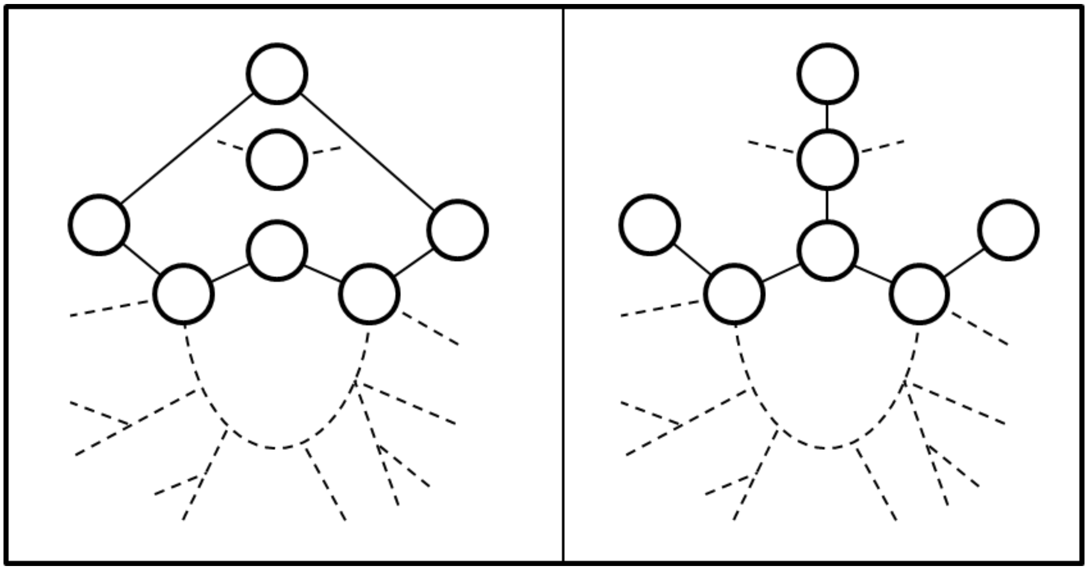


Figure 11 - Infinite Class of Graphs That is Unicyclic, Cospectral

With the open-ended nature of graphs that this construction allows, it is possible to construct many classes of graphs which will remain cospectral even as vertices and edges continue to be appended to them.

3.3 Comparison to Known Constructions

With this development of an infinite construction of cospectral graphs, we turned to prior methods of constructing graphs with the same spectrum to see if there was any prior work that also led to this conclusion. This turned out to be true, as the construction we developed bears a strong relation to two known constructions of cospectral graphs which have been previously discovered in prior work on spectral graph theory.

One such established method, described in [6], uses the coalescence of graphs to formulate a pair of graphs that share the same spectrum. By the following theorem, it is possible to construct cospectral graphs by taking the coalescence of graphs.

<u>*Theorem 3.3*</u> [6]*:*

Let G_1 and G_2 be graphs with the same spectrum, and let G be another graph. If $G_1 - v_1$ and $G_2 - v_2$ also have the same spectrum, then $G * G_1$ and $G * G_2$ are cospectral.

This theorem allows for a construction of cospectral graphs which is a stronger version of the construction described in this report. It can easily be seen that Corollary 3.2 mirrors this theorem, and in fact Corollary 3.2 can be proved by taking G in this theorem to be a graph consisting of a single vertex. A construction based on this theorem is stronger than a construction based on Corollary 3.2, as it allows for graphs of any structure to be appended onto the cospectral graphs, whereas the construction defined in this report does not allow for any connected loops to be created with the addition of vertices.

This theorem has also been used to develop extensive families of cospectral trees, and in fact Schwenk [9] found that this construction shows the existence of arbitrarily large families of cospectral trees. Schwenk also showed that, as larger trees are considered, the probability that the tree will be uniquely identified by its spectrum approaches zero, leading him to make the claim that almost all trees are cospectral.

The construction described in this paper also has similarities to a method of construction of cospectral graphs developed by Godsil which relies on edge switching. In [10] there is a brief mention at the end of the paper of a pair of cospectral graphs with what is termed a ‘remarkable property’, discovered by Godsil and described to Schwenk through personal communication. These graphs are shown below in Figure 12.

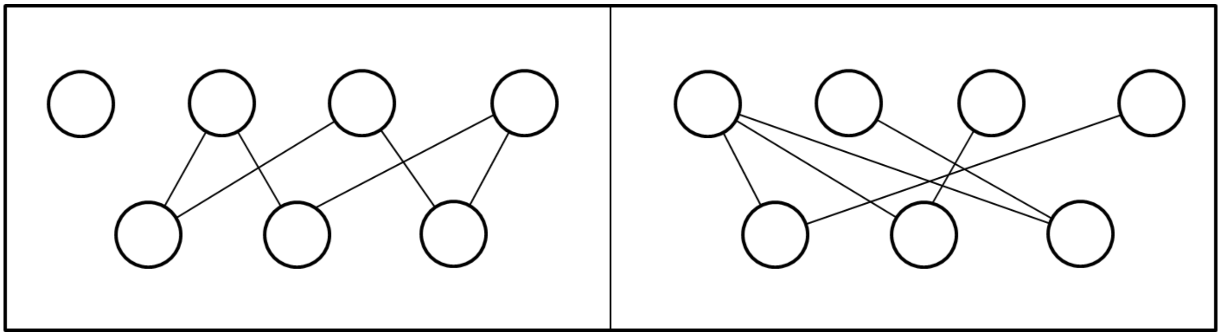


Figure 12 – Graphs from [10] with 'Remarkable' Property

The remarkable property of these graphs is that, by taking any graph G and connecting it to the bottom row of vertices in each graph, one can obtain a pair of cospectral graphs. This procedure is illustrated in Figure 13 below.

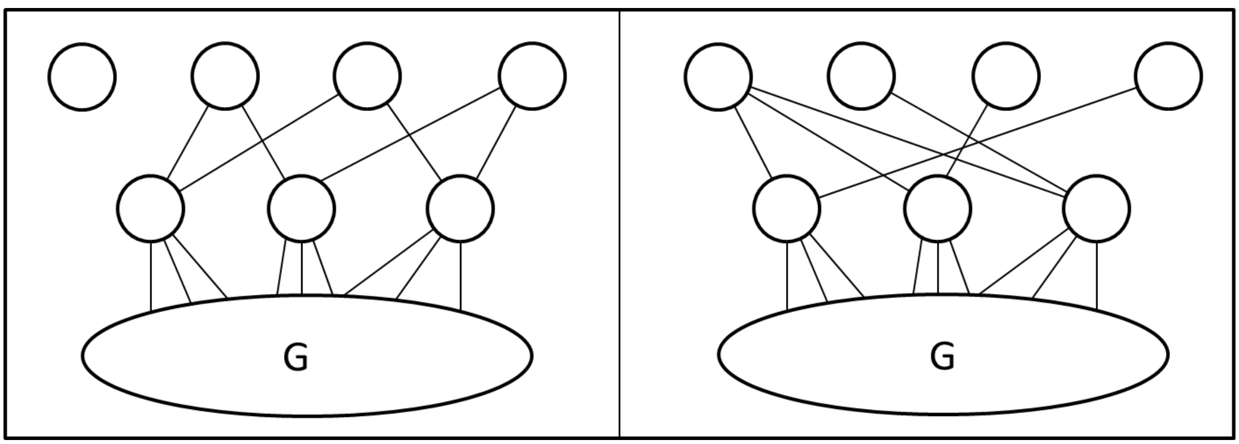


Figure 13 – Constructing Cospectral Graphs from the Remarkable Graphs

In [7], released a few years later, Godsil again noted the pair of graphs shown above. In this case, he illustrated these graphs as an example of his construction of cospectral graphs using local switching. This construction relies on matrix-theoretical techniques to identify partitions of vertices that allow for edges to be switched while preserving the eigenvalues of the adjacency matrix. For example, the graphs shown in Figure 12 meet the conditions such that the edge switching performed to change the left graph to the right graph does not change the eigenvalues of the adjacency matrix. Thus, for any graph that satisfies the structure shown in Figure 13, it is possible to partition the vertices so that the subgraph found in Figure 12 is in its own partition, and thus local switching allows for the construction of a cospectral graph.

The construction of cospectral graphs which has been described in this paper is thus closely related to Godsil's construction of cospectral graphs. The analysis of vertices from earlier in this section shows that the chosen vertices of the graphs shown in Figure 11 are candidates to be built upon using the construction of this paper. When a pendant vertex is added, the resulting graphs are cospectral, as expected. Similarly, the cospectral graphs that we have constructed all have one of these 'remarkable' graphs as a subgraph, meaning that these graphs could also have been constructed by building off of the 'remarkable' graphs. Thus, the construction of cospectral graphs described in this report produces the same construction as at least one example of the construction used in [7], but arrives at it through different methods.

Thus the construction found in this report is not a novel, undiscovered method for constructing cospectral graphs. It springs naturally from a theorem found in [6], and is used extensively in [9] to identify cospectral trees. In addition, the graphs which we have constructed in this report are also examples of the local switching method described in [7]. However, as far as we can tell, the method of identifying potential candidate vertices to build off by examining the parameters of the vertex (such as the degree and the number of triangles on which it is located) is a new way of identifying new constructed graphs. This gives another tool besides checking if the graphs created by deleting a vertex are cospectral, a method which becomes more difficult as graphs become larger. Thus, this method may allow for larger cospectral graphs to be identified more easily.

3.4 Further Areas to Explore

Other Graphs To Build Upon

With this successful discovery of a method to construct pairs of cospectral graphs by building off of smaller graphs, the next step is to determine which graphs allow for this construction. The vertex conditions covered in this report provide a useful method of searching for viable possibilities. However, from there it is necessary to calculate the spectrum in order to determine whether the construction is in fact successful. Further research can be done to search for other graphs for which this construction works, or possibly for other methods for identifying which graphs allow for this construction.

One special interest that this research can focus on is the use of regular cospectral graphs to aid this construction. Preliminary investigation in this area shows the possibility of promising results. For example, the cospectral pairs of graphs shown below in Figure 14, each consisting of 12 vertices, allow for vertices to be added adjacent to any other vertex, each resulting in cospectral graphs.

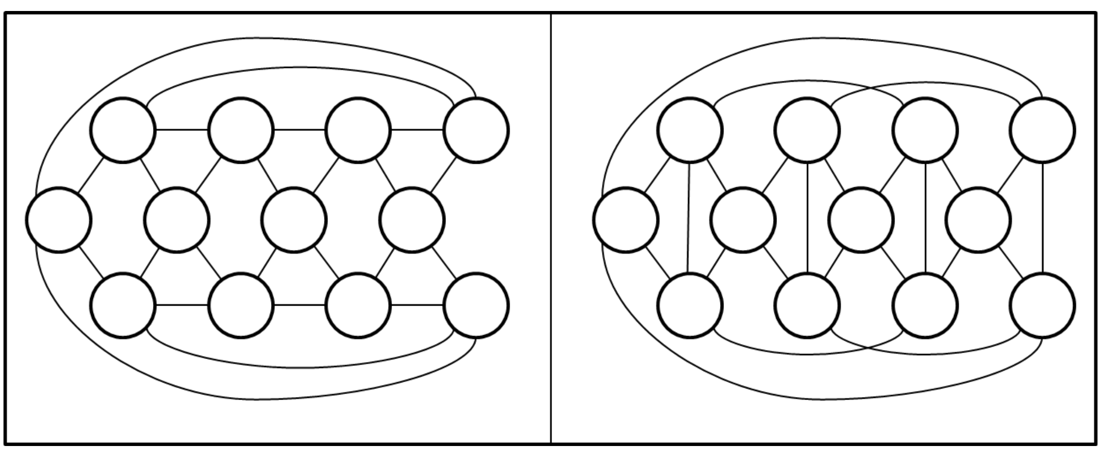


Figure 14 – Smallest Pair of Regular Cospectral Graphs [6]

Performing the same action on a pair of regular cospectral graphs of order 16 also proves to successfully produce a pair of cospectral graphs. One can wonder if this construction would be successful on any pair of regular cospectral graphs. Further exploration would perhaps find an answer to this question.

Spectral Radius

With the full spectra enumerated for each of the cospectral graphs we constructed in this report, we can examine how the spectral radius (the largest eigenvalue of the adjacency matrix) changes with the addition of a new vertex and edge. From the Interlacing Theorem (Theorem 2.2), it follows that the spectral radius of a graph must increase with the addition of a new vertex and edge. This increase, however, is different for each placement of the new edge. The constructed graphs listed in Appendix A are sorted in ascending order of their spectral radius. Thus, it can be seen how the placement of the new edge determines the change in spectral radius. The graphs which result in the largest spectral radius seem to be those where the new vertex is placed adjacent to a vertex in a more centralized location in the graph. Further research can be done to test whether this is true in all cases, and possibly to determine whether there is any way to predict the increase in the spectral radius of graph caused by adding a vertex and edge.

Other Extensions to the Lollipop and Dumbbell Graphs

This report investigated whether there was some more general class of graph related to the lollipop and dumbbell graphs which is determined by its spectrum. While we found such a class does not exist, it is possible there may be other specific classes which are determined by its spectrum. One example of a class that could be tested is the graph shown below in Figure 15, which extends the one-cycle structure of the lollipop and two-cycle structure of the dumbbell.

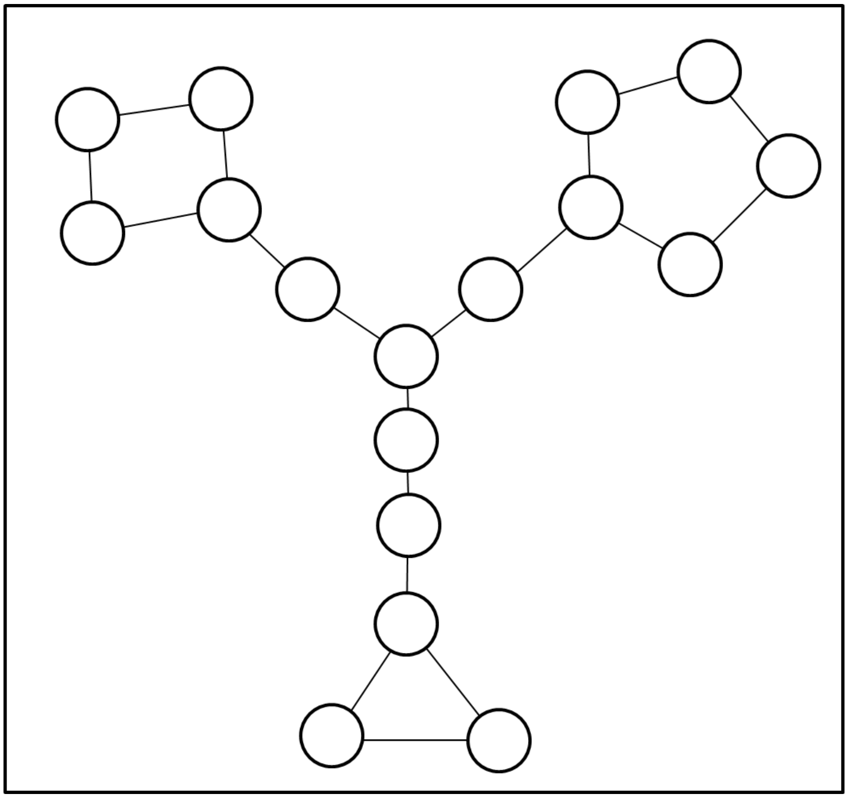


Figure 15 - Tri-cyclic extension of lollipop and dumbbell graphs

This graph consists of three cycles, each coalesced with paths that meet at some center point. Further research could be done to test the spectrum of these classes of graphs, to possibly determine if these graphs, or perhaps a certain subset of these graphs, are determined by their spectra.

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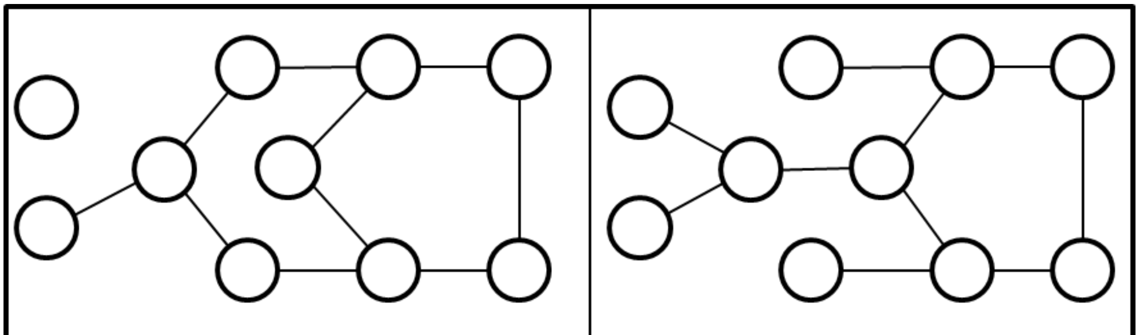
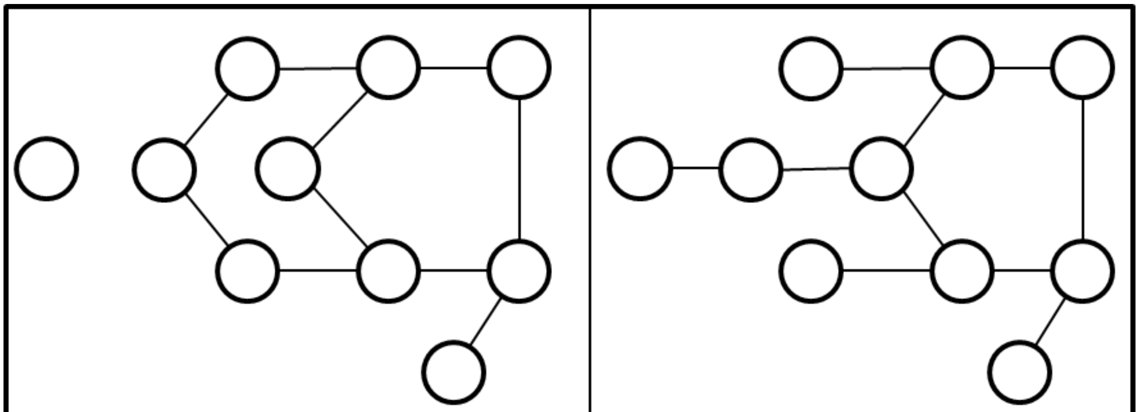
Appendix A - Spectra of Constructed Cospectral Graphs

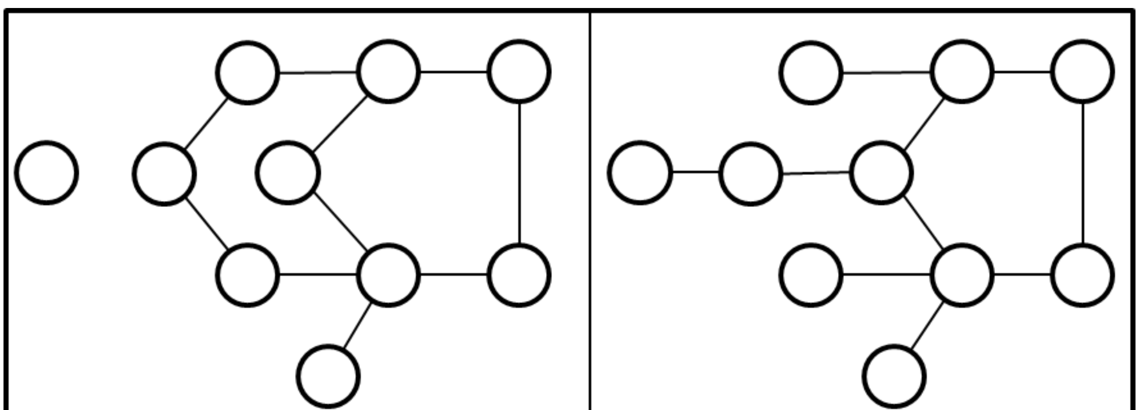
Constructed Cospectral Graphs with 10 Vertices:



Characteristic Polynomial: $7x^2 + 4x^3 - 33x^4 - 2x^5 + 31x^6 - 10x^8 + x^{10}$
Spectrum: $-2.1906,\ -1.8019,\ -1.3582,\ -.44504,\ 0,\ 0,\ .64428,\ 1.2470,\ 1.5528, 2.3517$

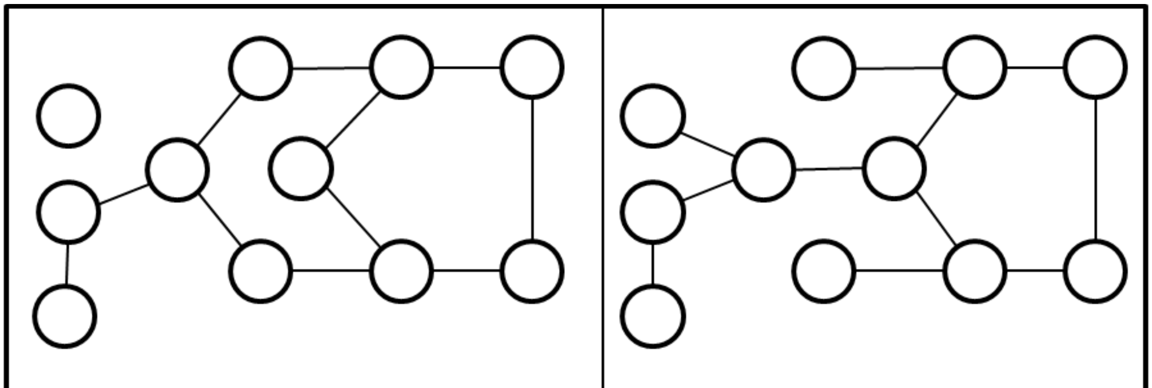


Characteristic Polynomial: $11x^2 + 2x^3 - 34x^4 - 2x^5 + 31x^6 - 10x^8 + x^{10}$
Spectrum: $-2.1365, -1.9551, -1.0903, -.69636, 0, 0, .80566, 1.2572, 1.4439, 2.3717$

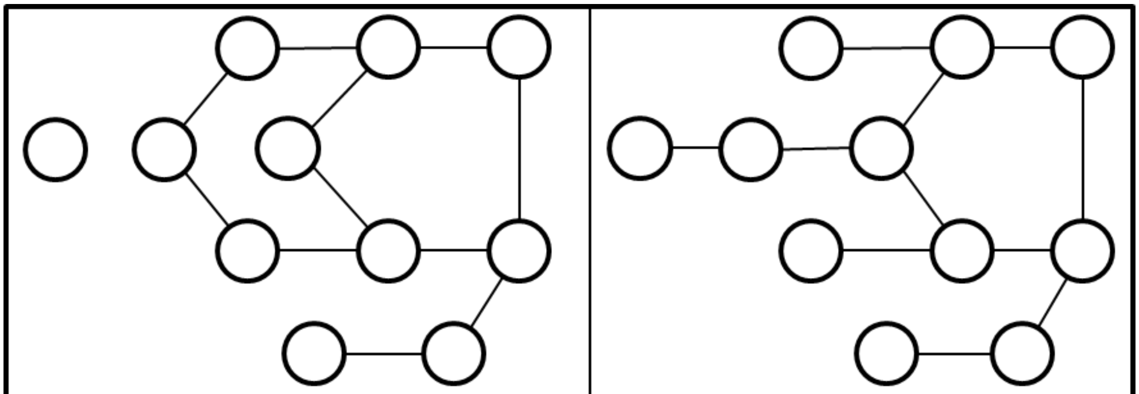


Characteristic Polynomial: $7x^2 + 2x^3 - 330 - 2x^5 + 30x^6 - 10x^8 + x^{10}$
Spectrum: $-2.2370, -1.8749, -1.1263, -.52775, 0, 0, .62844, 1.2852, 1.4429, 2.4093$

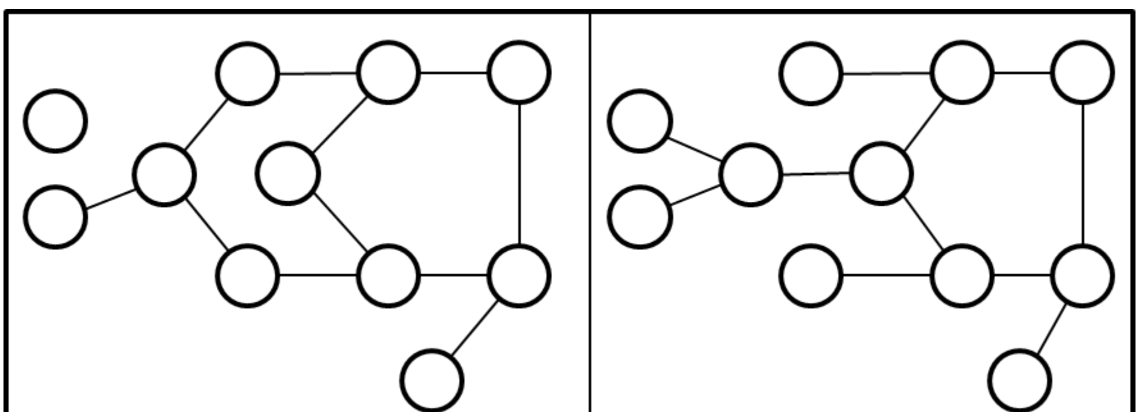
Constructed Cospectral Graphs with 11 Vertices:



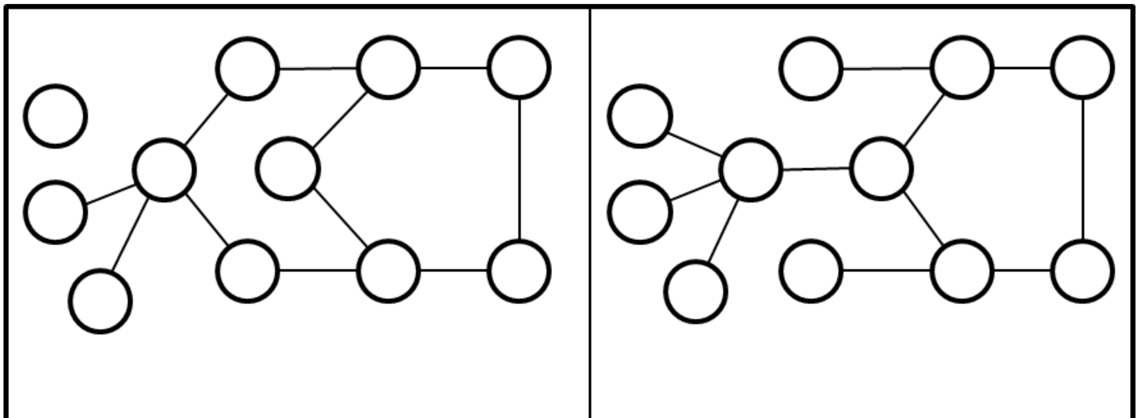
Characteristic Polynomial: $4x + 2x^2 - 28x^3 - 6x^4 + 57x^5 + 2x^6 - 40x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2143, -1.8019, -1.5085, -.67964, -.44504, 0, .53919, .82578, 1.2470, 1.6751, 2.3623$



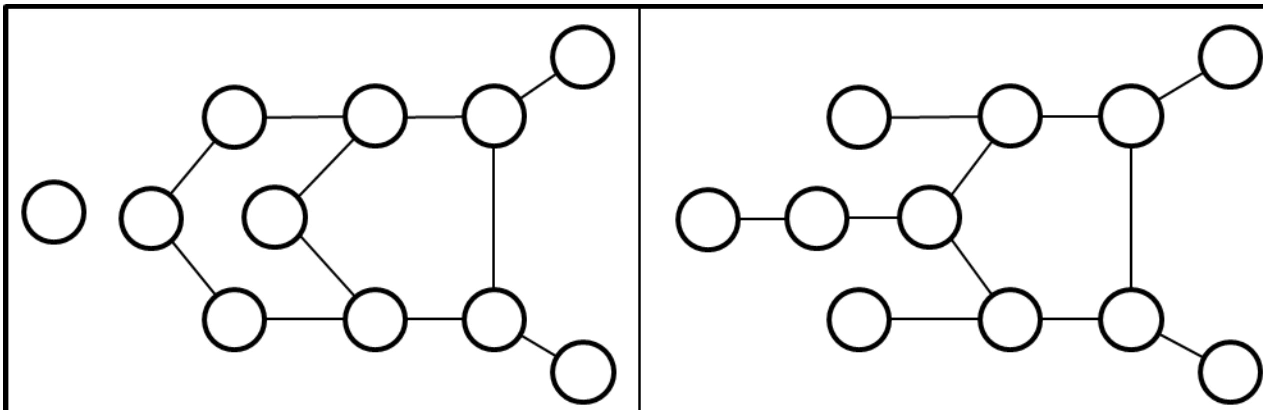
Characteristic Polynomial: $4x + 2x^2 - 32x^3 - 4x^4 + 58x^5 + 2x^6 - 40x^7 + 11x^9 - x^{11}$
Spectrum: $-2.1493, -2, -1.1455, -1, -.37400, 0, .46351, 1, 1.2584, 1.5603, 2.3865$



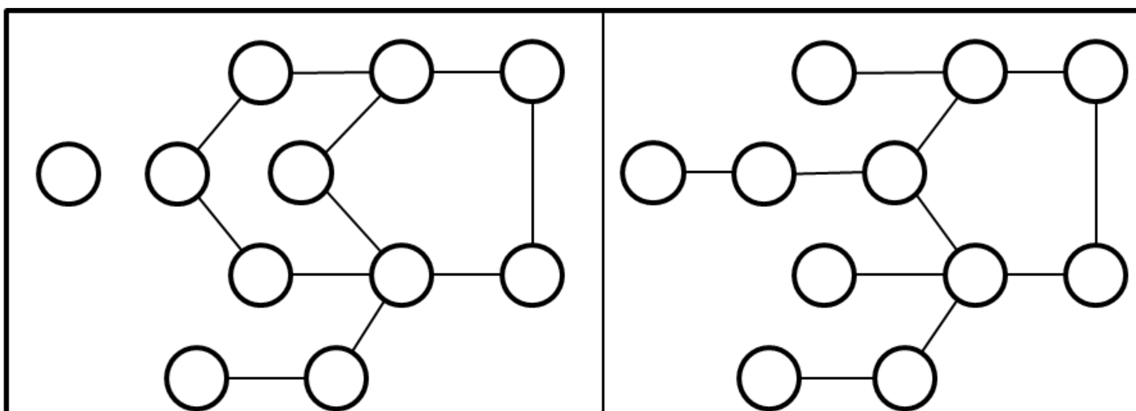
Characteristic Polynomial: $-19x^3 - 4x^4 + 51x^5 + 2x^6 - 39x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2069, -1.9607, -1.3697, -.70080, 0, 0, 0, .90032, 1.2927, 1.6401, 2.4009$



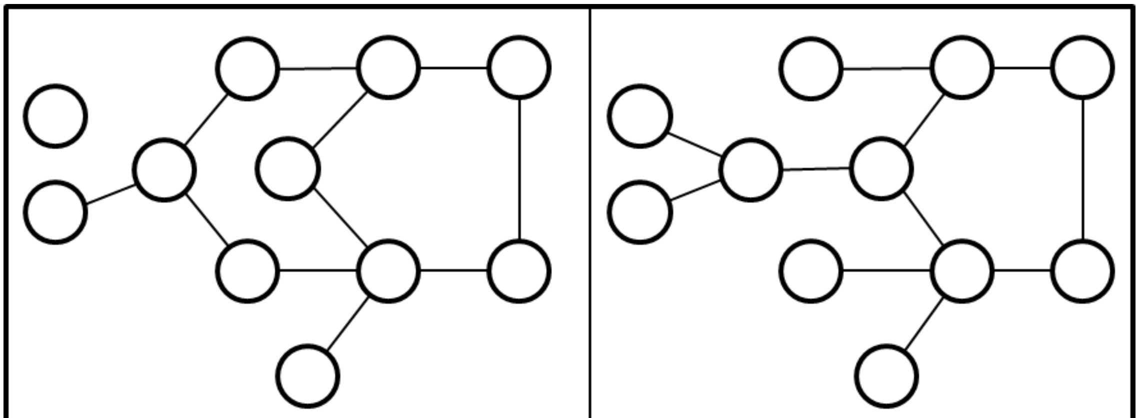
Characteristic Polynomial: $-10x^3 - 6x^4 + 45x^5 + 2x^6 - 38x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2958, -1.8019, -1.5412, -.44504, 0, 0, 0, .66601, 1.2470, 1.7601, 2.4109$



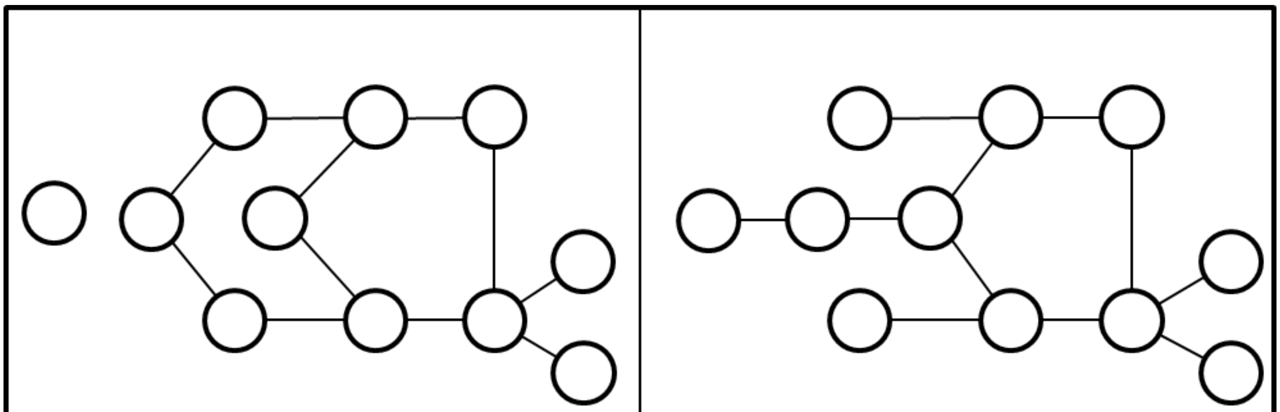
Characteristic Polynomial: $4\mathrm{x} - 27x^3 - 2x^4 + 53x^5 + 2x^6 - 39x^7 + 11x^9 - x^{11}$
Spectrum: $-2.1394, -2.0953, -1.1178, -.73764, -.53289, 0, .47726, .86230, 1.3557, 1.4986, 2.4291$



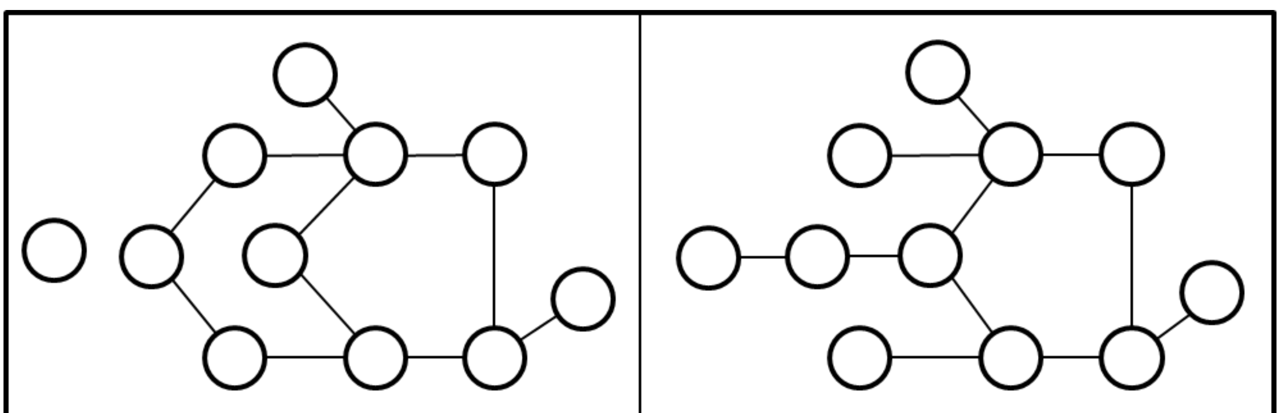
Characteristic Polynomial: $4\mathrm{x} + 2x^2 - 28x^3 - 4x^4 + 54x^5 + 2x^6 - 39x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2727, -1.8918, -1.1909, -.85942, -.41696, 0, .55725, .81642, 1.2855, 1.5456, 2.4315$



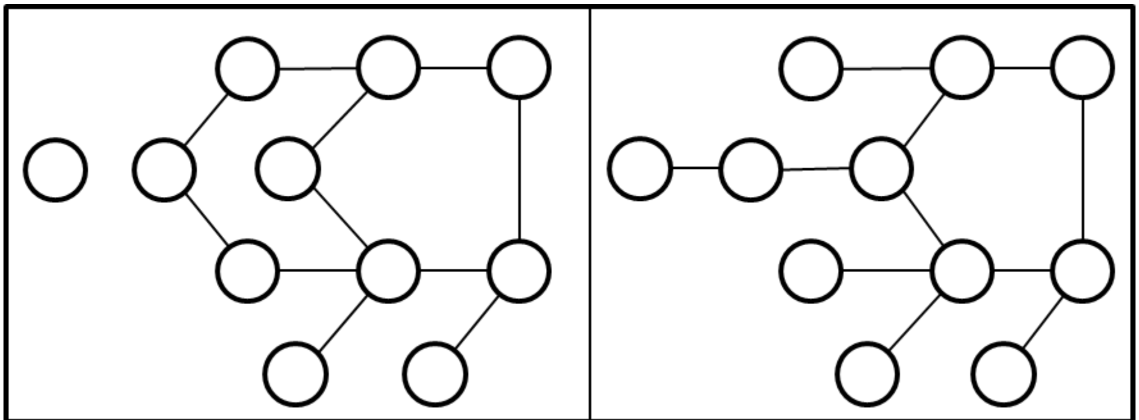
Characteristic Polynomial: $-12x^3 - 4x^4 + 46x^5 + 2x^6 - 38x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2855, -1.8882, -1.4142, -.53055, 0, 0, 0, .67576, 1.4142, 1.5908, 2.4377$



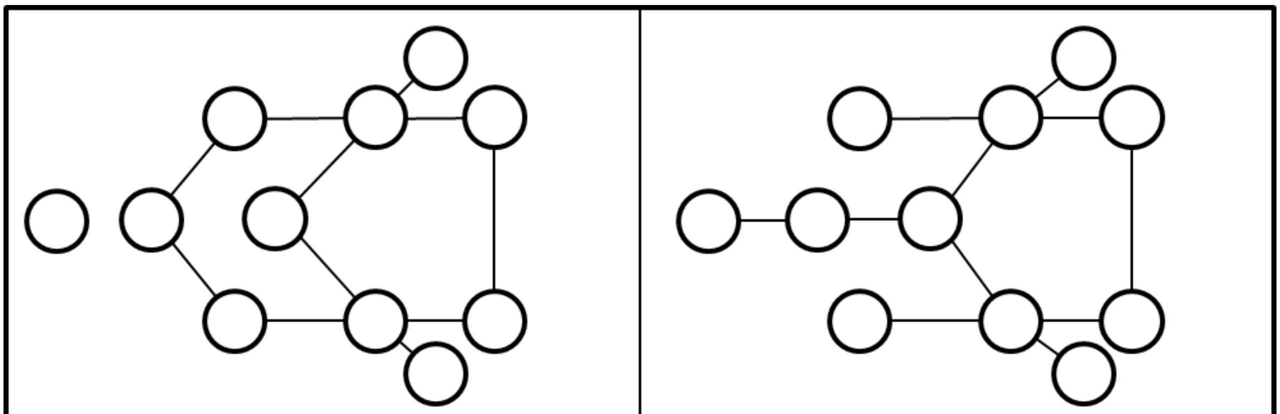
Characteristic Polynomial: $-18x^3 - 2x^4 + 47x^5 + 2x^6 - 38x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2203, -2.0569, -1.1084, -.81294, 0, 0, 0, .88301, 1.2582, 1.6066, 2.4508$



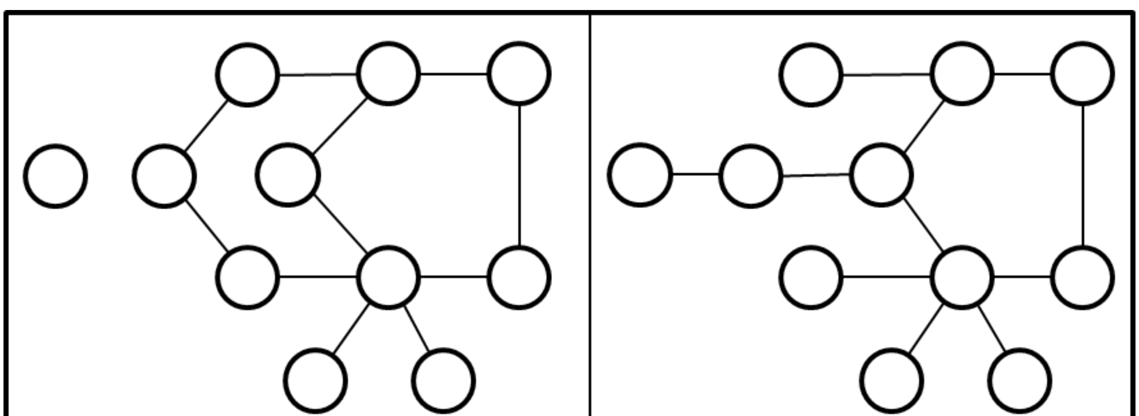
Characteristic Polynomial: $-17x^3 - 2x^4 + 47x^5 + 2x^6 - 38x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2378, -2.0271, -1.1686, -.75370, 0, 0, 0, .81862, 1.3813, 1.5339, 2.4533$



Characteristic Polynomial: $-19x^3 - 2x^4 + 48x^5 + 2x^6 - 38x^7 + 11x^9 - x^{11}$
Spectrum: $-2.2751, -1.9765, -1.1263, -.83559, 0, 0, 0, .89842, 1.3583, 1.4935, 2.4633$



Characteristic Polynomial: $-12x^3 - 2x^4 + 42x^5 + 2x^6 - 37x^7 + 11x^9 - x^{11}$
Spectrum: $-2.3028, -2, -1.1701, -.61803, 0, 0, 0, .68889,, 1.3028, 1.6180, 2.4812$

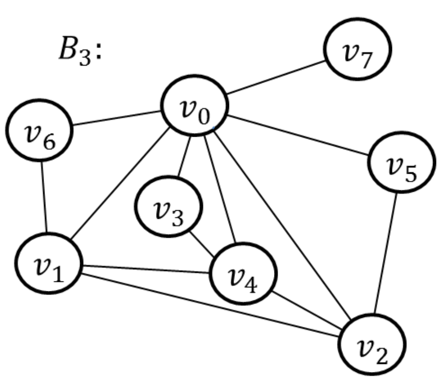


Characteristic Polynomial: $-10x^3 - 2x^4 + 39x^5 + 2x^6 - 36x^7 + 11x^9 - x^{11}$
Spectrum: $-2.3950, -1.9101, -1.1606, -.56969, 0, 0, 0, .64484, 1.2854, 1.5785, 2.5266$

Appendix B - Pascal’s Triangle Mod 3

In their paper “Binomial Graphs and Their Spectra”, Christopher and Kennedy [5] examined the properties of binomial graphs as they relate to spectral graph theory. A binomial graph is a type of graph constructed by translating the entries of Pascal’s triangle, modulo 2, into a matrix, and then considering this as the adjacency matrix of some graph. They identified a self-similarity in the matrix of binomial graphs, such that the matrices could be described in terms of a Kronecker product of smaller matrices. They used this fact to formulaically describe the spectra of binomial graphs. We examined a similar type of graph, whose adjacency matrix consists of the entries of Pascal’s triangle taken modulo 3, to see if a similar argument could be made. However, since this type of graph could not be represented as a Kronecker product, no formula for the spectrum of such a graph could be found.

A binomial graph B_n is defined to have vertex set $V_n = \{v_j \,:\, j = 0, 1, \dots, 2^n - 1)$ and edge set $E_n = \left\{\{v_i, v_j\} \,:\, \binom{i+j}{j} \equiv 1 \ (mod\ 2)\right\}$. With such a graph, the adjacency matrix $A(B_n)$ bears a close similarity to Pascal’s triangle, modulo 2. The binomial graph B_3 and its adjacency matrix $A(B_3)$ are shown below in Figure 16.



$A(B_3)$:

	j = 0	1	2	3	4	5	6	7
i = 0	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0
2	1	1	0	0	1	1	0	0
3	1	0	0	0	1	0	0	0
4	1	1	1	1	0	0	0	0
5	1	0	1	0	0	0	0	0
6	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0

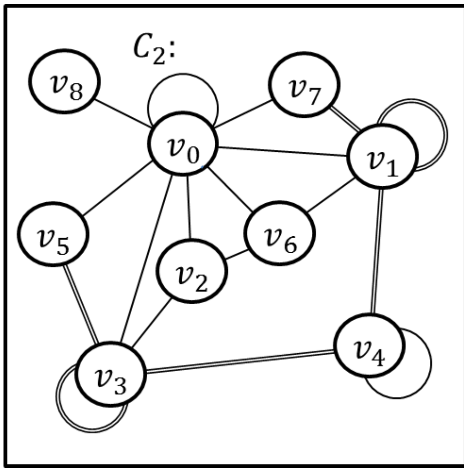
Figure 16 - Binomial Graph B_3 and its adjacency matrix $A(B_3)$.

The adjacency matrix $A(B_n)$ thus contains the first n rows of Pascal's triangle, modulo 2, in the top left corner, with zeroes below and to the right. It can also be seen that the adjacency matrix shows a self-similarity which allows it to be expressed as a Kronecker product of smaller binomial graphs. The Kronecker product of two matrices is defined as follows: If $A = [a_{ij}]$ is an $m \times n$ matrix, and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix, $A \otimes B = [a_{ij}B]$. By using this operator, we can define B_n as follows.

$$A(B_n) = \begin{bmatrix} A(B_{n-1}) & A(B_{n-1}) \\ A(B_{n-1}) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(B_{n-1}) = A(B_1) \otimes A(B_{n-1})$$

Furthermore, if a graph can be represented as the Kronecker product of two smaller graphs, then it is possible to easily calculate the spectrum of the larger graph from the spectra of the smaller graphs. If A is a $n \times n$ matrix with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and B is a $m \times m$ matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$, then $A \otimes B$ has nm eigenvalues of the form $\alpha_i\beta_j$ for each $i = 1, 2, \ldots, n$ and each $j = 1, 2, \ldots, m$. Thus, since $A(B_n)$ can be expressed recursively as the Kronecker product of $A(B_n)$, a 2x2 matrix, it is not difficult to calculate the spectrum of any binomial graph.

Here, we investigate whether this process can be extended by constructing a graph from Pascal's triangle, modulo 3. We define a graph C_n to have vertex set $V_n = \{v_j : j = 0, 1, \ldots, 3^n - 1)$ and to have an edge set such that if $\binom{i+j}{j} \equiv 1 \ (mod\ 3)$, then there is one edge between v_i and v_j, and if $\binom{i+j}{j} \equiv 2 \ (mod\ 3)$, then there are two edges between v_i and v_j. With this definition, the adjacency matrix $A(C_n)$ contains the first 3^n rows of Pascal's triangle, modulo 3, in its top left corner, and zeroes below and to the right of it. The graph C_2 and its adjacency matrix $A(C_2)$ are shown below in Figure 17.



$A(C_2)$:

	j = 0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
i = 0	1	1	1	1	1	1	1	1	1
1	1	2	0	1	2	0	1	2	0
2	1	0	0	1	0	0	1	0	0
3	1	1	1	2	2	2	0	0	0
4	1	2	0	2	1	0	0	0	0
5	1	0	0	2	0	0	0	0	0
6	1	1	1	0	0	0	0	0	0
7	1	2	0	0	0	0	0	0	0
8	1	0	0	0	0	0	0	0	0

Figure 17 - Graph C_2 and its adjacency matrix $A(C_2)$.

Looking at the adjacency matrix $A(C_2)$, we can see that it has close to the same self-similarity as was seen with the binomial graph above. However, this self-similarity is not perfect, and this prevents the matrix from being represented as a Kronecker product. The matrix $A(C_2)$ is identical to the Kronecker product $C_1 \otimes C_1$ in every cell except for cell (4,4). In the Kronecker product, this cell has the value 4, however in $A(C_2)$ this value is only one due to the matrix being taken modulo 3. This discrepancy means that the adjacency matrix cannot be represented as a Kronecker product, and thus that there is no easy method for calculating the spectrum of the graph C_n.