

TRANSACTION COSTS AND RESAMPLING IN MEAN-VARIANCE PORTFOLIO OPTIMIZATION

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ABSTRACTS OF THESIS

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Transaction costs and resampling are two important issues that need great attention in every portfolio investment planning. In practice costs are incurred to rebalance a portfolio. Every investor tries to find a way of avoiding high transaction cost as much as possible. In this thesis, we investigated how transaction costs and resampling affect portfolio investment.

We modified the basic mean-variance optimization problem to include rebalancing costs we incur on transacting securities in the portfolio. We also reduce trading as much as possible by applying the resampling approach any time we rebalance our portfolio. Transaction costs are assumed to be a percentage of the amount of securities transacted. We applied the resampling approach and tracked the performance of portfolios over time, assuming transaction costs and then no transaction costs are incurred. We compared how the portfolio is affected when we incorporated the two issues outlined above to that of the basic mean-variance optimization.

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Chapter 1

Introduction

Mathematical methods of optimization have been successfully used in financial portfolio management. The Mean-Variance method initiated by Harry Markowitz is one of the most widely used techniques of portfolio selection. The risk of the portfolio is modeled by the quadratic variance of its daily returns. The Mean-Variance method seeks to minimize this quadratic objective function subject to constraints concerning the permissible structure of the portfolio and a requirement that the expected returns exceeds a pre-specified level. This leads to a constrained quadratic optimization problem which has been extensively studied in the mathematical literature. Efficient computational algorithms have been developed and coded to solve even very large scale quadratic optimization problems.

As market conditions change, portfolios need to be rebalanced to keep them optimal. One known practical shortcoming of the Mean-Variance approach is that it requires frequent significant rebalancing of the portfolio. To make things worse, these rebalancing not only cut into profits through the inherent transaction costs, but are also not improving portfolio returns significantly. The cause of the apparent need for rebalancing is not the discovery of more promising investment opportunities, but statistical instability of some underlying estimates.

The key quantities in the Mean-Variance portfolio model are the covariance matrix and the mean vector of the daily returns of the securities in the portfolio. Portfolio theory assumes these quantities to be know. But in real life, they have to be estimated from observed market data. To get good quality statistical estimates for the large covariance and mean return structures one needed a large sample of historical returns data. But since market change over the time, only a limited set of recent historical data are relevant for the purposes of planning for a future investment horizon. The estimated covariance matrix and the mean return vectors are the key inputs of the optimization program. Since

they are large data structures based on a relatively small sized samples, their estimates are of poor statistical quality. This statistical instability is even more accentuated by the non-linear mapping represented by the optimization procedure. The result is the unstable composition of the mean-square optimized portfolio and the apparent need for frequent rebalancing.

The objective of the present thesis is to investigate two issues related to the statistical instability of the mean-variance portfolio optimization approach. The first is the effect of transaction costs. We will investigate on a concrete example, how transaction cost affects the returns on a mean-square optimized portfolio. The second objective is to explore the effectiveness of one particular methodology that has been proposed to remedy the statistical instability issues outlined above. We investigate how resampling (bootstrap) combined with optimization can reduce the extent of rebalancing needs and the resulting transaction costs.

We performed our investigation on 12 years of historical data on 20 securities representing all major sectors of the financial markets: US stocks, fixed income securities, international stocks, commodities and cash. We implemented the estimation and optimization procedures required to construct optimal portfolios. We build optimized portfolios based on 3-month sections of historical data and investigate their performance on market data from the quarter that follows the 3 months period used in the modeling. At the end of each quarter we rebalance the portfolio according to the prescriptions of the optimization algorithm and deduct the resulting transaction costs. This is the same methodology that a portfolio manager would use in real time, but allows us to assess the performance of the resulting portfolios on a long period covering various market conditions.

Chapter 2

Statement of Problem

A. Objective of present study.

One standard formulation of the basic mean-variance (MV) portfolio optimization problem is to minimize quadratic risk objective function, specifying the minimum expected returns on the portfolio subject to a set of linear constraints. Returns on the portfolio are usually measured daily, weekly or monthly. In this research work returns are all measured daily.

MV models found in literature often assume the following;

- No short selling allowed
- Fully invested portfolio
- Dividends are paid at the end of each rebalancing period
- No transaction cost are incurred to rebalance or set up the portfolio

In reality, it may cost some money to rebalance your portfolio as the need arises when data is updated with time. The model in the present thesis is the same as the MV analysis model but, it also takes into consideration the following real-life situations:

- Dividends are not paid out after each rebalancing period. All yields on the portfolio after rebalancing are reinvested during the next period.
- Transaction costs occur when portfolio is rebalanced.
- The maximum allocation to any single security is bounded. This makes sense in real life because, portfolio managers usually put a limit on the amount that he or she is willing to invest in a single security in his or her portfolio.

The objective of this present study is to progressively rebalance portfolios to keep them optimal and investigate the effects of the following factors:

- Return expectations
- Transaction cost
- Uncertainty inherited in statistical estimates of expected returns on the securities in the portfolio under study and their covariance.

Chapter 3

The Mathematical Model and Methods

3.1 Model and notations:

x_i = Weight of security i in portfolio

μ_i = Expected return of security i in the portfolio

E = Required expected returns on portfolio

Q = Covariance matrix of the securities in the portfolio

e = Vector of all ones

n = Total number of securities in portfolio

The basic model under investigation is to minimize an objective quadratic risk function

$$\min \frac{1}{2} x^T Q x$$

Subject to some given linear constraints;

$$\mu^T x \geq E \tag{1}$$

$$e^T x = 1$$

$$0 \leq a \leq x \leq b$$

The first constraint requires that the expected return on the portfolio exceed the required amount E . The second assumption requires that the portfolio is fully invested. The third constraint prohibits short selling and limits the maximum allocation to the individual constituents' securities.

3.2 Portfolio Rebalancing

We introduce the following notations:

\bar{x} = weights of securities in existing portfolio before rebalancing

x = weights of securities in new portfolio after rebalancing

u = weights of securities bought to rebalance old portfolio

v = weights of securities sold to rebalance old portfolio

Using these notations, we can express the weights of securities in new portfolio as;

$$x = \bar{x} + u - v \quad (2)$$

The main assumption we made is that transaction costs are proportional to the value of the securities bought or sold. If we denote by

C_B = transaction cost vector for buying one unit of a security

C_S = transaction cost vector for selling one unit of a security

where, $0 \leq C_B, C_S \leq e$ and $C_B + C_S \geq 0$

If we denote the total transaction cost spent to rebalance the portfolio by x_o , then

$$x_o = C_B^T u + C_S^T v \quad (3)$$

Let P be the total weight of the portfolio before rebalancing. Since part of this is spent to rebalance the portfolio, the actual weight left to be allocated to the securities is less than P . The total weight left for re-investing is $P - x_o$. Thus, if we assume full investment then the following equations hold;

$$e^T x = P - x_o$$

$$e^T x = P - C_B^T u - C_S^T v \quad (4)$$

We can now replace $e^T x = 1$ from the constraints in the above model labeled (1) with equation (4).

Mitchell and Braun stated that the appropriate objective function when we consider transaction cost is fractional and it is given by;

$$\frac{\frac{1}{2} x^T Q x}{(P - x_o)^2} \quad (5)$$

The denominator is just the square of equation (4). They explained how it makes sense to include the fractional quadratic objective function. For a fixed initial portfolio \bar{x} , when transaction cost is increased, you have smaller amounts of principal weight available for investment. However, in order to get the same returns on the portfolio with this smaller principal weight, we will need to reach for higher returns. And this can be achieved with higher levels of risk. This fractional objective function measures this risk for these types of transaction cost exhausted resource portfolios.

Our revised model now has a fractional objective function given by

$$\min \frac{\frac{1}{2} x^T Q x}{(P - C_B^T u - C_S^T v)^2}$$

subject to the following constraints;

$$\mu^T x \geq E \quad (6)$$

$$x = \bar{x} + u - v$$

$$e^T x = P - C_B^T u - C_S^T v$$

$$x \leq zP \quad \leftarrow \text{upper bounds}$$

$$x, u, v \geq 0 \quad \leftarrow \text{lower bounds}$$

Note that z is a scalar and it represents the maximum fraction of the total weight that can be assigned to a security in the portfolio.

3.3 Transforming Fractional Quadratic Program (FQP) into a Quadratic Program (QP)

We made used an extension of the technique of Charnes and Cooper for this transformation.

Let the variable

$$t = \frac{1}{(P - C_B^T u - C_S^T v)^2} \quad (7)$$

Note: The denominator of the variable t is strictly greater than zero. This actually makes sense in reality because no one will be willing to spend all his/ her principal weight on transaction cost to balance his/ her portfolio.

Multiplying the (FQP) given in (6) by the variable t , we have

$$\min \frac{1}{2} x^T Q x$$

subject to the following constraints

$$\mu^T x t \geq E t$$

$$x t = \bar{x} t + u t - v t \quad (8)$$

$$e^T x t = P t - C_B^T u t - C_S^T v t$$

$$\begin{aligned}
xt &\leq zPt && \leftarrow \text{upper bounds} \\
xt, ut, vt &\geq 0 && \leftarrow \text{lower bounds}
\end{aligned}$$

We redefine the decision variables as follows:

$$\hat{u} = ut \quad \hat{v} = vt \quad \hat{x} = xt \quad \text{and} \quad t \succ 0$$

Substituting the above variables into the (FQP) equation (8), our model becomes a quadratic program (QP). Our new covariance matrix becomes C. Thus we minimize the quadratic problem

$$\min \frac{1}{2} \hat{x}^T C \hat{x}$$

subject to the following constraints:

$$\begin{aligned}
-\mu^T \hat{x} + Et &\leq 0 \\
\hat{x} - \hat{u} + \hat{v} - \bar{x}t &= 0 && (9) \\
e^T \hat{x} + C_B^T \hat{u} + C_S^T \hat{v} - Pt &= 0 \\
\hat{x} - zPt &\leq 0 && \leftarrow \text{upper bounds} \\
\hat{x}, \hat{u}, \hat{v}, t &\geq 0 && \leftarrow \text{lower bounds}
\end{aligned}$$

If $(\hat{x}^*, \hat{u}^*, \hat{v}^*, t^*)$ is the optimal solution to the above QP, then we can find its corresponding solution to the FQP (x^*, u^*, v^*) by rescaling $(\hat{x}^*, \hat{u}^*, \hat{v}^*)$ so that

$$x^* = \frac{1}{t^*} \hat{x}^* ; \quad u^* = \frac{1}{t^*} \hat{u}^* ; \quad v^* = \frac{1}{t^*} \hat{v}^* \quad (10)$$

3.4 Efficient Frontier

The efficient frontier is the set of portfolios with

- Expected return greater than any other portfolio with the same or lesser risk
- Lesser risk than any other portfolio with the same or greater return

The efficient frontier (EF) can be obtained by optimizing the above (QP) for various values of the expected returns on the portfolio, E .

Procedure for generating the efficient frontier:

- Calculate the daily returns from three months (one quarter) data for n securities.
- Compute the expected daily returns and covariance estimates for these n securities
- Run the minimization program (QP) for varying expected returns, E .
- Plot various expected returns, E versus their corresponding minimum risk value obtained from the QP to generate the efficient frontier.

The question that comes up is which of these portfolios on the efficient frontier should we choose for our investment? Two different portfolios on the EF will be considered under this study. The first is the portfolio with the maximum reward to risk ratio and the second is the lowest risk fully invested portfolio on the EF. These two portfolios are rebalanced progressively at the end of every business quarter to keep them optimal. The main objective is to track how they will perform on out of sample data. The performance of each the two portfolios is compared to a benchmark portfolio of equal weights. At each time when we rebalance the portfolio, the following steps will be performed to find the new portfolio.

- Pick the point on the efficient frontier that corresponds to the best reward to risk ratio (or the lowest risk fully invested portfolio) on the EF.
- Find the expected return and risk corresponding to this point.

- Compute the actual portfolio composition and the corresponding risk after rebalancing.

3.5 Determination of the maximum reward to risk and lowest risk fully invested portfolios.

Computation of actual portfolio value is given by

$$P_j = \left\{ \sum_{i=1}^n \bar{x}_i^{(j-1)} \prod_{k=1}^m (1 + r(i, k)) \right\} \quad (11)$$

And its corresponding risk is

$$(\mathbf{x}^{(j-1)})^T \mathbf{C} \mathbf{x}^{(j-1)} \quad (12)$$

Where,

P_j = Portfolio value at the end of quarter j

$\bar{x}^{(j-1)}$ = Optimization solution at the end of quarter $(j - 1)$

$r(i, k)$ = returns of security i on trading day k

n = total number of securities in the portfolio

m = total number of trading days for quarter j

3.6 Critique of the validity of appropriate theory and research literature.

The above method is a power tool for portfolio optimization. However, one of the main critiques behind this rigorous theory is that it turns to maximize error and invest in non-relevant portfolios. Markowitz assumes that the inputs of the MV portfolio optimization are 100 percent certain. However, these inputs; expected returns on the securities and their corresponding covariance matrix are measured from historical data and fed into the model as if they were know perfectly. Actually, these inputs are subject to significant statistical and specification errors. They are estimated on the basis of a statistical sample of limited size of historical data and then applied to a different set of future data. The MV analysis does not take into account the uncertainty incorporated in the input parameters.

In order to control estimation error in our MV optimization inputs, we will apply a powerful statistical procedure called re-sampling. The re-sampling method defines a new efficient frontier that is consistent with most applications of MV efficiency. One advantage of the re-sampling is that it reduces the need to trade. This saves money on transaction cost when rebalancing the portfolio.

3.7 Resampling procedure

We will apply the bootstrap method for our re-sampling. The actual historical daily returns are used as a psuedo-population or as an estimate of the true population.

1. Run the minimization program only once for $E = E_{sh}$ or $E = E_f$ where E comes from the first non-bootstrap stage for either the maximum reward to risk ratio or the first fully invested portfolio.
2. Resample from m daily returns to get a new bootstrap sample of size m trading days.
3. Use the bootstrap sample to recalculate the new mean μ_i^b and covariance C_{ij}^b of the n securities.
4. Re-run the minimization program again with the following:

$$\min \left\{ (x^b)^T C^b x^b : \mu^b x^b \geq E_{sh} \text{ or } E_f \right\} \quad (14)$$

5. The bootstrap mean vector and bootstrap covariance matrix of the daily returns are now our inputs for the optimization program.
6. Repeat steps 2 through 4 several (B) times and compute the average of these re-sampled portfolios.

$$x^{avg} = \frac{1}{B} \sum_{b=1}^B x^b \quad (15)$$

7. Use x^{avg} as the re-sampled portfolio for the next period.

Chapter 4

Computation Implementation

All the computational work underlying the present thesis has been implemented in MATLAB. First, we will illustrate the interconnection among the MATLAB code modules in a form of a flow chart.

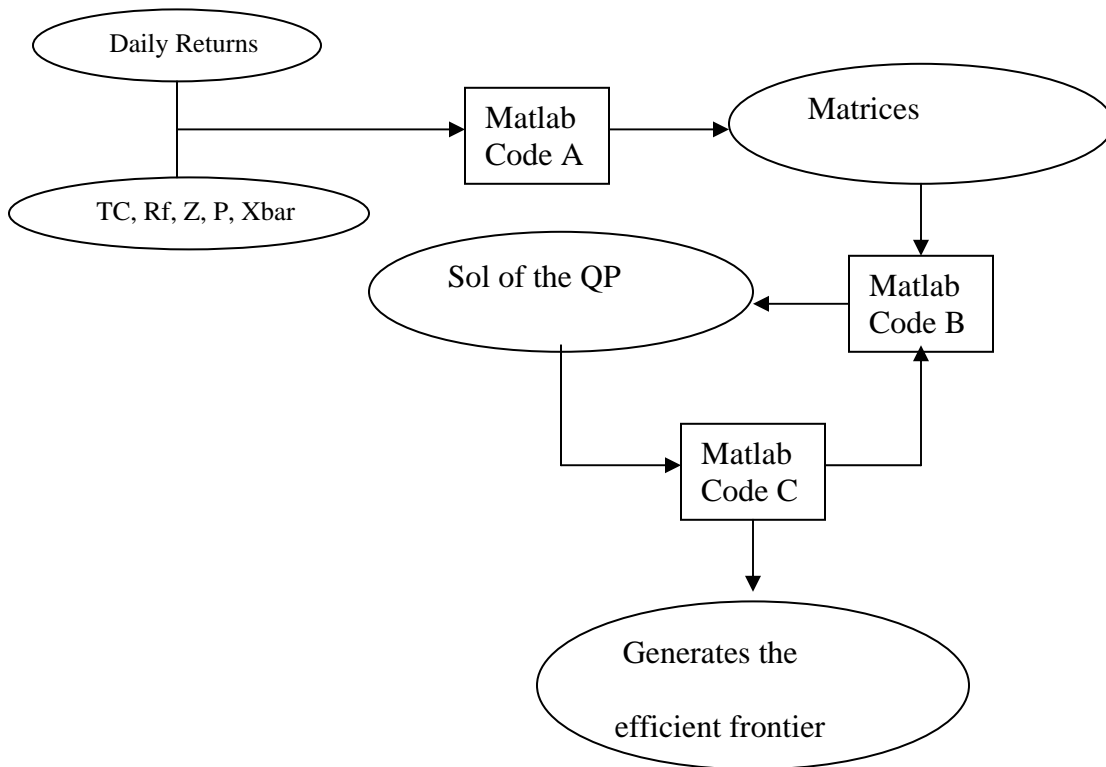


Figure 1. Computational Flowchart 1

- The daily closing prices of the $n=20$ securities involved in this portfolio have been downloaded from yahoo finance. The daily returns are calculated for each of the trading days in the given quarter.

$$\text{Return}(\text{day } i) = \frac{\text{Closing Price}(\text{day } i) - \text{Closing Price}(\text{day } i - 1)}{\text{Closing Price}(\text{day } i - 1)}$$

- 3-months risk-free rates (Rf) are downloaded from the Missouri Federal Reserve Bank. The need for these risk free rates comes into play when we use the maximum reward to risk option of selecting the portfolio on the efficient frontier. We should note that in this option, cash is invested in T-Bills.
- Matlab Code A calculates the estimated expected returns and covariance matrices from the historical security returns from Item 1 above.
- Matlab Code B implements the solution to the QP problem
- Matlab Code B passes the solution to Matlab Code C. Code C increases the expected daily return on the portfolio and passes it back to Code B. The process continues in a loop until the maximum expected return is reached.
- Matlab Code C then generates the efficient frontier.

The actual portfolio weight P measures the growth of the portfolio. Because dividends are not paid out, all yields on the portfolio from the prior quarter are re-invested in the next quarter. Thus the starting portfolio weight for any given quarter is the ending portfolio weight prior to that particular quarter. Moreover, we assumed a portfolio weight of one at the beginning of the investment.

The $xbar$ vector represents the weight of each of the n -securities in the portfolio we want to rebalance. This is a zero vector at the time we build our initial portfolio. Transaction cost, TC and the maximum weight on each security Z are already explained at the beginning of Chapter 2.

These inputs discussed above are then passed onto the Matlab Code A, which formulates the matrices of the quadratic minimization problem in model (9). These matrices are then passed onto program Code B, which calls the QUADPROG from the optimization toolbox in matlab. This solves the quadratic risk minimization problem discussed in model (9). Program Code C stores the above solution and varies the expected daily return, E on the portfolio. It then passes this new expected daily return to program code

B, which solves the quadratic risk minimization problem again with this updated return. This process is repeated for several E's and their corresponding risk are store in Code C. The efficient frontier is obtained by plotting the expected return E with the corresponding minimum risk from the QP model. The question is, which of these portfolios on the frontier should we choose for our investment?

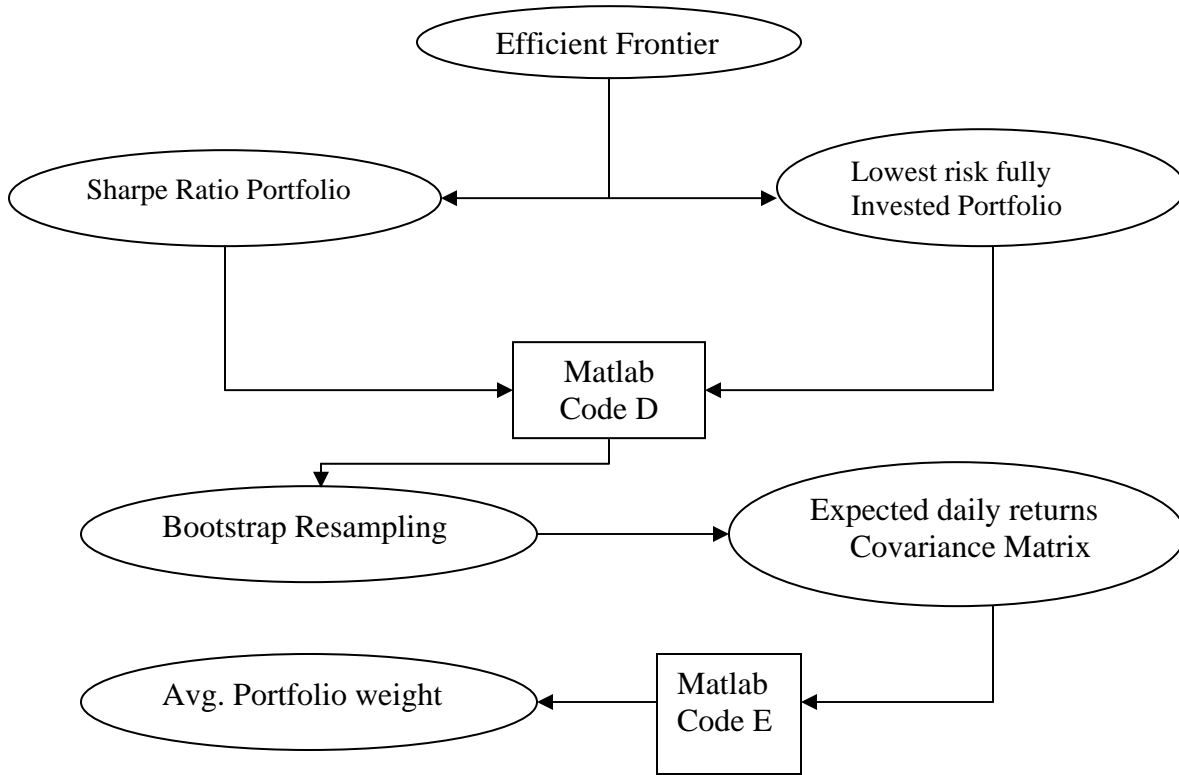


Figure 2. Computational Flowchart 2

Two main portfolios on the EF were considered in this research work; the Sharpe ratio (best reward to risk portfolio) and the lowest risk fully invested portfolio. The expected daily returns on these portfolios are passed to Matlab Code D. This code also performs a bootstrap re-sampling by treating the original daily returns as the pseudo-population. It then computes the mean and covariance of the re-sampled daily returns and these serve as an input to Matlab Code E, which solves the quadratic risk optimization model (9). Note that we use the expected daily returns obtained from the Sharpe ratio or the first fully invested portfolios in place of the value of E in model (9). The whole resampling process is repeated several times and its corresponding solutions (optimal portfolio weights)

recorded. The average of these optimal portfolio weights was computed and this served as our investment portfolio for the next quarter. Note that rebalancing of the portfolio was done quarterly throughout this research.

Chapter 5

Findings

We used the same data set consisting of 20 securities to arrive at our findings. The efficient frontier was generated by solving the QP model with various expected daily returns, E , and plotting these returns with their corresponding minimum risk obtain from solving the model.

5.1 Uncertainty in the estimates of our inputs in MV optimization.

We investigated the uncertainty in Mean-Variance optimization model when used on out of sample data. Our results are based on the 20-securities historical data. We achieved our goal by solving the QP minimization model with these 20-securities data and tracking how our investment grows with time when the portfolio is rebalanced quarterly.

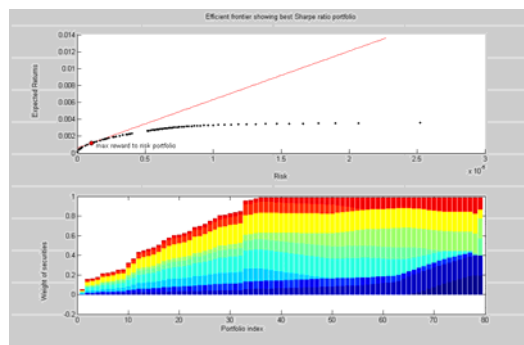


Figure 5-1. Efficient Frontier showing Sharpe ratio portfolio and security compositions

Figure 5-1 above shows the results obtained from solving our QP minimization problem using the 20-securities. The upper part of the figure shows the efficient frontier displaying the maximum reward to risk portfolio. Refer to Sharpe on how this portfolio can be obtained. The lower part of figure 5-1 shows the distribution of weights of each of the portfolios on the frontier. For this particular figure above, we relaxed the fully invested constraint and found out that the maximum reward to risk portfolio was not fully invested. Some money was invested in the best risk free T-Bills available. Michaud, stated that in order to correct this uncertainty in the QP optimization portfolio selection model, there is a need to do resampling. In this research paper, we performed a bootstrap resampling quarterly on the daily returns. We tracked the performances of two portfolios on the efficient frontier under this method.

- The best reward to risk ratio portfolio and
- Minimum risk fully invested portfolio

Below are some findings from our investigation. Figure 5-2 below shows the best reward to risk and minimum risk fully invested portfolios with some resampled portfolios.

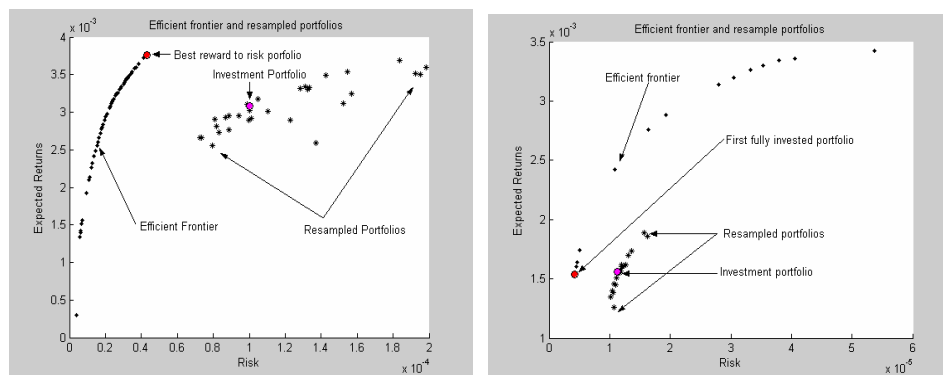


Figure 5-2. Efficient frontier and resampled portfolios

Please refer to the previous chapter for how the resampled portfolios were obtained. The average of the resampled portfolio served as a very important portfolio for our investment. We used this portfolio as our investment portfolio for the coming quarter. We investigated how our model performed on an out sample data by tracking the

performance of this portfolio over a 3-year period. Though best reward to risk portfolio has a higher expected return than the investment portfolio, the investment portfolio yields better return when applied on real data. The compositions of the securities in the portfolio over a 3-year period are shown below.

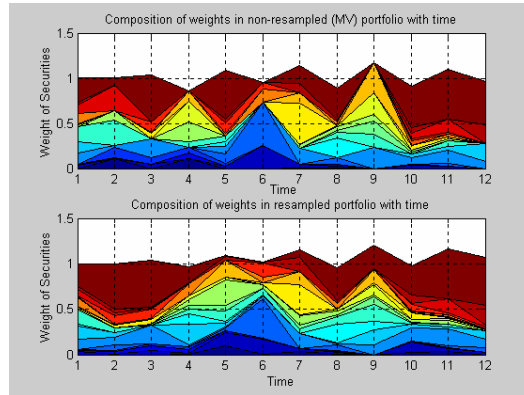


Figure 5-3. Composition of weights in portfolio with time

From figure 5-3 above, we observed the effect of resampling on our investment portfolio with time. Each color above represents a particular type of security in the portfolio. The composition of securities in the resampled portfolio has less trading effect as compared to that of the non-resampled portfolio. When we incorporate transaction costs are in the model, then the non-resampled portfolio will incur more cost during rebalancing periods because of the often transacting of securities. Figure 5-4 also shows the performance of our investment with time when we apply resampling to obtain our investment portfolio. The average resampled portfolio yielded better returns than the usual mean-variance portfolio obtained with resampling. It also shows how the risk on portfolio for these two cases varies with time. One interesting observation is that the risk of the resampled portfolio is more stable with time.

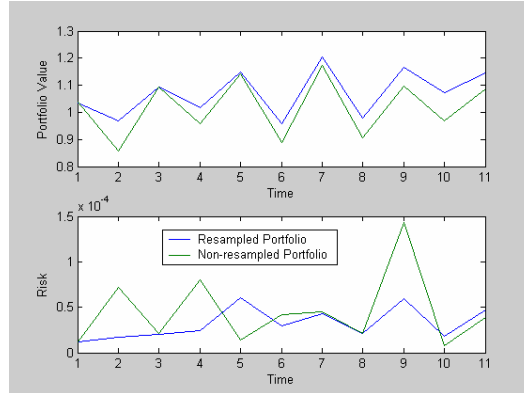


Figure 5-4. Portfolio value and risk with time

We compared resampled portfolio with a benchmark portfolio with equally weighted securities over this 3-year period. We found that for this particular portfolio, when we assign equal weights to the 20 securities in the portfolio, the investment returns were slightly better when we assign mean-optimized weights. We should be careful not to conclude that the mean-optimized portfolio is not the optimal portfolio for the data set it was obtained. Of course it is the optimal portfolio for the data set it was obtained, but it served as an investment portfolio for out-of-data sample. We assumed that the daily returns of the 20 securities for the investment period would not be that different from the returns from the previous quarter, which we used for the computation of the mean-optimized portfolio. The return from the minimum risk portfolio was slightly better than maximum reward to risk portfolio returns.

5.2 Effect of transaction cost on portfolio investment

We examined the effect of transaction cost on efficient frontier when underlying securities weights are unchanged. We varied transaction cost from (0%, 10%, 20%, to 30%) and generated the efficient frontiers for these varying transaction costs. Our results are displayed in the figure 5.6 below.

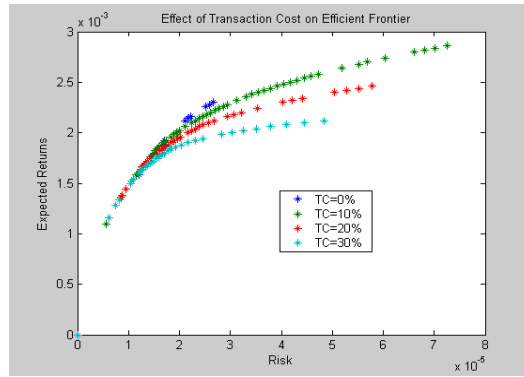


Figure 5-5. Effect transaction cost on efficient frontier

At a fixed level of risk, say $2.5e^{-5}$, the corresponding expected daily returns increases as we reduce transaction cost. Also for a fixed expected return E_o , a portfolio with a higher transaction cost can attain this return with a higher risk level. Increasing transaction cost pushes our efficient frontier inwardly from that of a zero transaction cost efficient frontier. What does this tells us in reality? Increasing transaction cost has a decreasing effect on our feasible set of the QP model, hence, reducing the choice of portfolios for investment.

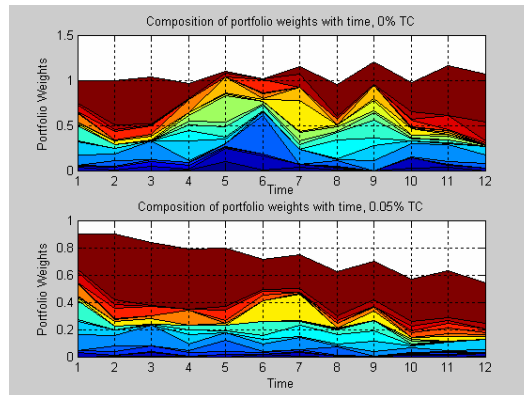


Figure 5-6. Composition of portfolio weights with time

Incorporation of transaction cost in the basic mean-variance optimization makes we loose money fast even with low transaction cost. This involves a lot trading during resampling.

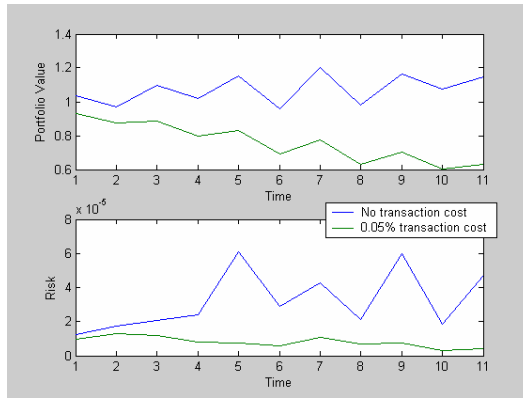


Figure 5-7. Portfolio value and risk with time

Chapter 6

Conclusion

Portfolio optimization is directly opposite to the traditional stock picking. We explored how resampling diversify our resulting mean-optimized portfolio after rebalancing to withstand the uncertainty in the mean-variance method. Resampling works in real returns. Though it may have a lower expected return than that of the non-resample portfolio, it provides higher real returns at lower risk. Resampling also reduces trading.

In real life, transaction costs are incurred to rebalance our portfolio to keep it optimal. This makes intuitive sense because we increasing costs make one loses money fast with optimization strategy. Transaction cost optimization strategies should be left for large investment firms for whom transaction costs are low. On the other hand, transaction cost may reduce trading and risk.

From our findings, simple strategies such as equal weighting of securities in a portfolio may beat the mean-variance resampling approach when used on an out-of-sample data. An equally weighted securities portfolio from a cleverly chosen asset universe may often outperform the mean-variance approach during investment.

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