Comparative Analysis of Ledoit's Covariance Matrix and Comparative Adjustment Liability Management (CALM) Model
Within the Markowitz Framework

by

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Abstract

Estimation of the covariance matrix of asset returns is a key component of portfolio optimization. Inherent in any estimation technique is the capacity to inaccurately reflect current market conditions. Typical of Markowitz portfolio optimization theory, which we use as the basis for our analysis, is to assume that asset returns are stationary. This assumption inevitably causes an optimized portfolio to fail during a market crash since estimates of covariance matrices of asset returns no longer reflect current conditions. We use the market crash of 2008 to exemplify this fact. A current industry-standard benchmark for estimation is the Ledoit covariance matrix, which attempts to adjust a portfolio's aggressiveness during varying market conditions. We test this technique against the CALM (Covariance Adjustment for Liability Management Method), which incorporates forward-looking signals for market volatility to reduce portfolio variance, and assess under certain criteria how well each model performs during recent market crash. We show that CALM should be preferred against the sample convariance matrix and Ledoit covariance matrix under some reasonable weight constraints.

Key Words: covariance matrix estimation, Ledoit's model, shrinkage parameter, CALM, forward looking signal

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1 Introduction

This work is a continuation of the Summer 2012 WPI Center for Industrial Mathematics & Statistics NSF funded Research Experience for Undergraduates (REU) Financial Mathematics project sponsored by Wellington Management. The project remained under the direction of WPI Professor Marcel Blais and includes work from Masters candidates Yafei Zhang and Gregory McArthur.

A well-known problem with Markowitz portfolio theory is the assumption of stationarity of asset returns [12]. In other words, the joint distribution of asset returns does not change over time. The covariance matrix of asset returns is used to determine, in a Markowitz setting, how much an investor should choose to hold in the context of diversification. This calculation is used to create a mean-variance portfolio, which determines how much risk we will have to incur for an expected return. Accurately estimating a covariance matrix is important to the study of portfolio optimization and risk management.

Markowitz originally proposed using the sample covariance matrix; however, we have come to realize that this is certainly not the best technique. Essentially when the number of stocks, N, is large relative to the historical data, estimation error occurs. Also since Markowitz portfolio theory assumes the stationarity of asset returns, a sample covariance matrix tells us nothing about how to invest given a variety of possible market changes.

Designing a covariance matrix estimate that can work around this issue has been an important study for many years. Currently, the Ledoit [1] covariance matrix is one of the industry-standard benchmarks. In order to reflect market reality, Ledoit developed a shrinkage parameter, which can adjust the aggressiveness of a portfolio automatically according to different market conditions. We show in this paper that Ledoit is certainly a good choice, but that under certain mathematical conditions, other models may be preferred. In our case, we test against the CALM (Covariance Adjustment for Liability Management) model.

CALM originated from the Summer 2012 REU Financial Mathematics project at WPI. The aim of CALM is to use shrinkage based information on forward-looking signals to create a covariance matrix that better reflects market conditions. Essentially the goal is to try and improve upon the Ledoit covariance matrix. For more information on this topic, we refer to *Incorporat-*

ing Forward-Looking Signals Into Covariance Matrix Estimation for Portfolio Optimization [2] .

We study the mathematics behind the two models and compare their performance in last financial crisis from 2007 to 2009.

2 Background

This section attempts to explain the mathematics behind the shrinkage estimator of the covariance matrix introduced by Ledoit.

2.1 Ledoit's Model

2.1.1 Statistical Model

We first focus on understanding the Ledoit technique [1] for estimating the covariance matrix of stock returns, as it is currently an industry-standard.

Let $\mathbf{X}$ denote an $N \times T$ matrix of T observations on a system of N random variables representing T returns on a universe of N stocks.

Assumption 1. *Stock returns are independent and indentically dstributed (iid) through time and are not assumed to be normally distributed.*

This assumption implies that the time-series representing stock returns are stationary. We note that actual stock returns do not verify this assumption.

Assumption 2. *The number of stocks N is fixed and finite, while the number of observations T goes to infinity.*

Assumption 3. *Stock returns have a finite fourth moment:*

$$\forall \ i, j, k, l = 1, \ldots, n \qquad \forall \ t = 1, \ldots, T \qquad E\left[|x_{it}x_{jt}x_{kt}x_{lt}|\right] < \infty.$$

A fourth moment is a measure of the *peak* of a distribution. A finite fourth moment implies that a peak is not infinite, and hence we have a finite variance (and covariance) and can apply the central limit theorem.

2.1.2 Sample Covariance Matrix

We define the sample mean vector m and the sample covariance matrix S by:

$$m = \frac{1}{T}\mathbf{X}\mathbf{1},$$

and

$$S = \frac{1}{T}\mathbf{X}\left(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}'\right)\mathbf{X}'. \tag{1}$$

Here, $\mathbf{1}$ is a conformable vector of 1's, $\mathbf{1}'$ represents its transpose. S represents a $T \times 1$ vector of ones, $\mathbf{I}$ is a $T \times T$ identity matrix.

2.1.3 Single-Index Covariance Matrix Estimator

Sharpe's single-index model assumes that stock returns are generated by:

$$x_{it} = \alpha_i + \beta_i x_{0t} + \varepsilon_{it}. \tag{2}$$

Here the residuals $Var\epsilon_{it}$ are uncorrelated to market returns x_{0t} and to one another. We also have that $Var\left(\epsilon_{it}\right) = \delta_{ii}$, which gives that the variance between stocks is constant. We can see this by taking $Var\left(x_{it}\right), \forall t$.

The covariance matrix implied by this model is:

$$\Phi = \sigma_{00}^2 \beta\beta' + \Delta. \tag{3}$$

Here σ_{00}^2 is the $N \times N$ covariance matrix of market returns, β is the $N \times 1$ vector of slopes, and Δ is the $N \times N$ diagonal matrix containing residual variances δ_{ii}. We denote ϕ_{ij} by the (i,j)-th entry of Φ.

We note that this model can be estimated by running a regression of stock $i's$ returns on the market. Call b_i the slope estimate and d_{ii} the residual variance estimate, then the single-index model yields the following estimator for the covariance matrix of stock returns:

$$\mathbf{F} = s_{00}^2 \mathbf{b}\mathbf{b}' + \mathbf{D}. \tag{4}$$

Here, s_{00}^2 is the sample variance of market returns, $\mathbf{b}$ is the vector of slope estimates, and $\mathbf{D}$ is the diagonal matrix containing residual variance estimates d_{ii}. Call f_{ij} the (i,j)-th entry of $\mathbf{F}$.

Assumption 4. $\Phi \neq \Sigma$, where Σ is the sample covariance matrix.

Assumption 5. The returns of market portfolio has positive variance, that is, $\sigma_{00}^2 > 0$.

2.1.4 Formula for the Optimal Shrinkage Intensity

In order to understand the Ledoit covariance matrix, we need to understand *shrinkage*.

When considering a large number of stocks, the estimated sample covariance matrix tends to have a large error. The error implies that extreme coefficients tend to not be representative of the true covariance matrix. This, in turn, causes a mean-variance portfolio optimizer to place its biggest bets on those coefficients which are extremely unreliable.

The main idea behind shrinkage is that coefficients with positive error need to be compensated for by pulling them downward and the reverse for coefficients with negative error. Essentially we are shrinking the error towards the center.

We need to question what it is we are shrinking and to what intensity. Consider the model:

$$\mathbf{F}\delta + (1 - \delta)\,\mathbf{S}. \tag{5}$$

$\mathbf{F}$ is defined by (4), $\mathbf{S}$ by (1), and δ is our shrinkage estimate to be found. We consider a quadratic loss function defined by:

$$L(\delta) = \|\delta F + (1 - \delta)S - \Sigma\|_F^2. \tag{6}$$

Notice how we are calculating the distance between our shrinkage model and the sample covariance matrix of stock returns, Σ. This is a quadratic measure of distance between the true and the estimated covariance matrices based on the Frobenius norm.

Now consider

$$\mathrm{R}(\delta) = \mathrm{E}\left[L(\delta)\right]. \tag{7}$$

We can rewrite (7) in summation form which considers the components of the matrices. This in conjunction with (6) yields

$$R\left(\delta\right) = E\left(L\left(\delta\right)\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} E(f_{ij}\delta + (1-\delta)\,s_{ij} - \sigma_{ij})^2$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} Var\left(f_{ij}\delta + (1-\delta)\,s_{ij}\right) + \left[E\left(\delta f_{ij} + (1-\delta)\,s_{ij} - \sigma\right)\right]^2$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \delta^2 Var(f_{ij}) + (1-\delta)^2 Var(s_{ij}) + 2\delta(1-\delta)Cov(f_{ij}, s_{ij}) + \delta^2(\phi_{ij} - \sigma_{ij})^2.$$

We achieve these equations using the properties of variance.

We want to now minimize the risk of R (δ) with respect to δ. To do this we calculate the first two derivatives of R (δ). We have

$$\mathrm{R}'(\delta) = 2\sum_{i=1}^{N}\sum_{j=1}^{N} \delta Var(f_{ij}) - (1-\delta)\,Var(s_{ij}) + (1-2\delta)\,Cov(f_{ij}, s_{ij}) + \delta\,(\phi_{ij} - \sigma_{ij})^2,$$

$$\mathrm{R}''(\delta) = 2\sum_{i=1}^{N}\sum_{j=1}^{N} Var(f_{ij} - s_{ij}) + (\phi_{ij} - \sigma_{ij})^2.$$

We set $\mathrm{R}'(\delta) = 0$, and we find that

$$\delta^* = \frac{\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} Var(s_{ij}) - Cov(f_{ij}, s_{ij})}{\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} Var(f_{ij} - s_{ij}) + (\phi_{ij} - \sigma_{ij})^2}.$$

We note that since $\mathrm{R}''(\delta)$ is positive everywhere, this solution minimizes the risk function.

Let $\hat{\theta}$ denote the an estimator for parameter vector θ, and $\sqrt{n}\left(\hat{\theta} - \theta\right) \to n\left(0, V\right)$ in distribution, then $\hat{\theta} \to n\left(\theta, \frac{1}{n}V\right)$ asymptotically. The term $\frac{1}{n}V$ is called the *'asymptotic variance or covariance'*. [6]

Let π denote the sum of asymptotic variances of the entries of the sample covariance matrix scaled by $\sqrt{T}$: $\pi = \sum_{i=1}^{N}\sum_{j=1}^{N} AsyVar\left[\sqrt{T}s_{ij}\right]$. Similarly let ρ denote the sum of the asymptotic covariances of the entries of the single-

index covariance matrix with the entries of the sample covariance matrix scaled by $\sqrt{T}$: $\rho = \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} \text{AsyCov}\left[\sqrt{T}f_{ij}, \sqrt{T}s_{ij}\right]$. Finally let γ measure the misspecification of the single-index model: $\gamma = \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} (\phi_{ij} - \sigma_{ij})^2$. Then the optimal shrinkage δ^* satisfies: [1]

$$\delta^* = \frac{1}{T}\frac{\pi - \rho}{\gamma} + \mathcal{O}\left(\frac{1}{T^2}\right). \tag{8}$$

From equation (8), we have that:

$$T\delta^* = \frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} Var(\sqrt{T}s_{ij}) - Cov(\sqrt{T}f_{ij}, \sqrt{T}s_{ij})}{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} Var(f_{ij} - s_{ij}) + (\phi_{ij} - \sigma_{ij})^2}.$$

Using the assumptions 1 and 3, that the data is *iid* and from finite fourth moments, we have that:

$$\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} Var(\sqrt{T}s_{ij}) \to \pi,$$

$$\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} Cov(\sqrt{T}f_{ij}, \sqrt{T}s_{ij}) \to \rho,$$

$$\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} Var(f_{ij} - s_{ij}) = \mathcal{O}(\frac{1}{T}).$$

Hence the optimal shrinkage is constant $k = \frac{\pi - \rho}{\gamma}$ [1] .

Finally, using this notation, the shrinkage estimator for the stock return covariance matrix that Ledoit recommend is:

$$S = \frac{k}{T}\mathbf{F} + (1 - \frac{k}{T})\mathbf{S}. \tag{9}$$

Since the shrinkage estimator captures the current market status, it contains forward looking signal that will make the covariance matrix estimate more responsive to the changes in market.

2.2 CALM Model

Covariance Adjustment for Liability Management (CALM) is a new model that incorporates signals for market volatility to minimize portfolio variance. This model originated from the REU aforementioned in the Introduction section of this paper.[2].

We wish to construct a covariance matrix such that it accurately reflects a stressed market. Stressed market regimes are commonly observed to have higher correlations between stocks [3]. We shall try to account for this property by incorporating high correlations into a stressed covariance matrix, $\mathbf{H}$. We construct a covariance matrix with constant high correlation with the method used by Bollerslev [4].

Let $\mathbf{C}$ be a high correlation matrix with a constant high correlation. Then $\mathbf{C}_{i,i} = 1$ and $\mathbf{C}_{i,j} = \rho_{high}$. Let $\mathbf{V}$ be the diagonal volatility matrix. Then,

$$\mathbf{H} = \mathbf{VCV} \tag{10}$$

We call $\mathbf{H}$ the *stressed covariance matrix* and expect that when the market is in turmoil, ρ_{high} approximates the stock correlations and $\mathbf{H}$ approximates Σ. In addition we continue to use the shrinkage parameter defined in Ledoit's Model to balance the weights between the sample covariance matrix $\mathbf{S}$ and highly structured matrix $\mathbf{H}$.

2.2.1 The Choice of Constant $\rho_{\mathbf{high}}$

A qualified highly structured correlation matrix should also be invertible. We derive the range of ρ_{high} in which the correlation matrix is positive definite and thus invertible.

Let p denote the constant value of ρ_{high} and $\mathbf{C}$ denote the $N \times N$ highly

structured correlation matrix,

$$\mathbf{C} = \begin{bmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ p & p & 1 & \cdots & p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & p & p & \cdots & 1 \end{bmatrix}.$$

We calculate the 1st to 3rd principle minor of $\mathbf{C}$ below:

$$\Delta_1 = 1,$$
$$\Delta_2 = 1 - p^2,$$
$$\Delta_3 = 2p^3 - 3p^2 + 1 = (1 - p)^2 (1 + 2p).$$

Further $\mathbf{C}'s$ 4th principle minor is:

$$\Delta_4 = (1 - p)^3 (1 + 3p). \tag{11}$$

The Nth principle minor is

$$\Delta_n = (1 - p)^{n-1} [1 + (n - 1) p].$$

We calculate the determinant of $\mathbf{C}$ using Gaussian elimination

$$\begin{vmatrix} 1 & p & p & \cdots & p \\ p & 1 & p & \cdots & p \\ p & p & 1 & \cdots & p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & p & p & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1 + (n-1)p & 1 + (n-1)p & 1 + (n-1)p & \cdots & 1 + (n-1)p \\ p & 1 & p & \cdots & p \\ p & p & 1 & \cdots & p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & p & p & \cdots & 1 \end{vmatrix}$$

$$= [1 + (n-1)p] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ p & 1 & p & \cdots & p \\ p & p & 1 & \cdots & p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & p & p & \cdots & 1 \end{vmatrix} = [1 + (n-1)p] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1-p & 0 & \cdots & 0 \\ 0 & 0 & 1-p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-p \end{vmatrix}$$

$$= [1 + (n-1)p] (1 - p)^{n-1} = \Delta_n.$$

In order to be positive definite, each kth principle minor of $\mathbf{C}$ has to be

positive for every k.

$$\Delta_1 = 1 > 0,$$
$$\Delta_2 = 1 - p^2 = (1 - p)(1 + p) > 0 \Rightarrow p \in (-1, 1),$$
$$\Delta_3 = (1 - p)^2 (1 + 2p) > 0 \Rightarrow p \in \left(-\tfrac{1}{2}, 1\right) \bigcup (1, +\infty),$$
$$\Delta_4 = (1 - p)^3 (1 + 3p) > 0 \Rightarrow p \in \left(-\tfrac{1}{3}, 1\right),$$
$$\vdots$$
$$\Delta_n = (1 - p)^{n-1} [1 + (n - 1)p] > 0 \Rightarrow p \in \left(-\tfrac{1}{n-1}, 1\right).$$

Thus as long as $p \in \left(-\tfrac{1}{n-1}, 1\right)$, $\mathbf{C}$ is positive definite and invertible for each n. As n gets large, the range of p converges to $(0, 1)$.

We use a value of 0.7 and 0.9 for p, which came empirically from observing the average correlation as implied market volatility rises. This concept was support by Engles in a recent article regarding the use of a constant for high correlation [5].

3 Empirical Results

In order to test the portfolio performance under different covariance matrix estimates, we use Markowitz optimization theory [7] to calculate tangency weights on a standard Markowitz model, and also Ledoit's model and the CALM model [2] with constant correlation 0.7 and 0.9.

Markowitz portfolio theory is based on the assumption that the past market behavior is consistent with future behavior. We relax this assumption here, assuming that market behavior changes dramatically during a crisis. We examine how the covariance matrix estimates change after implementing the forward-looking signals from a crisis across the four different models. We choose weekly data from 2007 to 2009 as our holding period to perform the back-testing. This time interval allows us to test under a pre-crisis bubble, the actual crash, and the steady market recovery.

The key method we used to obtain the historical data is from a *sliding window* observation. A sliding window observation considers data from the past n days and is applied each day starting from the $n+1$ day. Our holding period is designed to be 3 years. We use 1 year as our length of sliding window. In order to simplify our project, we used a *rectangular window*, in which every past net return has a weight of 1.

A popular approach to manage risk is through diversification. Consider our portfolio as a linear combination of N risky assets; investing in a diversified portfolio can help to reduce the risk from market changes and lower the portfolio volatility as long as the assets correlation coefficient is less than 1.

3.1 Assets Overview

Our portfolio is composed by 29 stocks traded on the NYSE. The construction of our portfolio is based on following principles:

1. Diversification: Markowitz portfolio theory is especially useful when the portfolio contains a significant number of assets. Since market indices are generally well diversified, investing in a market index is a reasonable choice. The Dow Jones Industrial Average only contains 30 stocks and we decided to construct our portfolio by investing in every component of the the Dow. Since our back-test starts at the beginning of 2007, we use the historical components of the Dow as of Nov 21, 2005.

One of the major historical components of the Dow in 2005 was General Motor Corporation (GM). GM Corporation filed for bankruptcy on July 8 2009, making its historical prices unattainable for the late holding period. So we exclude GM Corporation, and for that reason there are only 29 stocks in our portfolio, rather than 30.

2. FULLY INVESTED: All of our available capital is invested in the risky assets.

3.2 Test for Properties of a Covariance Matrix

Before using these estimators of a covariance matrix to calculate our tangency weights, we first need to make sure that these estimators contain some essential necessary properties of a covariance matrix.

A covariance matrix should have the following property:

- The covariance matrix must be a positive semi-definite matrix.

Standard Markowitz Model: Our covariance matrix estimator is the sample covariance matrix, S.

First, Let $\mathbf{X}$ denote an $N \times T$ matrix of T observations on a vector of N random variables representing T returns on a universe of N stocks. Let $\mathbf{1}$ denote a conformable vector of ones and $\mathbf{I}$ denote a conformable identity matrix. We assume that N is a finite number while T goes to infinity. We note that $\left(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}'\right)$ is a $T \times T$ matrix, $\mathbf{X}'$ is a $T \times N$ matrix.

$$S = \frac{1}{T}\mathbf{X}\left(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}'\right)\mathbf{X}'. \tag{12}$$

Given two matrices, $\mathbf{A}$ and $\mathbf{B}$, we know that the rank of the product $\mathbf{AB}$ is less than or equal to the minimum of the ranks of $\mathbf{A}$ and $\mathbf{B}$ i.e.,

$$\text{rank}\left(AB\right) \leq \min\left\{\text{rank}\left(A\right), \text{rank}\left(B\right)\right\}. \tag{13}$$

Applying (12) to property (13), we get:

$$\text{rank}\left(S\right) = \text{rank}\left(\frac{1}{T}\mathbf{X}\left(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}'\right)\mathbf{X}'\right) \leq \min\left\{\text{rank}\left(\mathbf{X}\right), \text{rank}\left(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}'\right)\right\},$$

where $\mathbf{X}$ is assumed to have full rank since we can take $\mathbf{X}$ inverse. Thus $\left(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}'\right)$ has smaller rank than $\mathbf{X}$. As a result

$$\text{rank}\,(\mathrm{S}) \leq \text{rank}\left(\mathbf{I} - \frac{1}{\mathrm{T}}\mathbf{1}\mathbf{1}'\right).$$

Using Gaussian elimination, we can show that

$$\text{rank}\left(\mathbf{I} - \frac{1}{\mathrm{T}}\mathbf{1}\mathbf{1}'\right) = \mathrm{T} - 1.$$

Thus

$$\text{rank}\,(\mathrm{S}) \leq \mathrm{T} - 1.$$

As an $N \times N$ matrix, as long as $T > N$, S has full rank and is invertible. In our portfolio, $N = 29$ and $T = 3 \times 52 = 156$. Notice that T is larger than N. Hence, S is invertible.

Since the sample covariance matrix S is also a covariance matrix, it is positive semi-definite. Thus the estimator S does not lose this necessart property.

Ledoit Model: Our covariance matrix estimator is $\frac{k}{T}\mathbf{F} + (1 - \frac{k}{T}\mathbf{S})$.

As stated before, $\mathbf{F}$ is the covariance matrix implied by the single factor model. Since $\mathbf{F}$ is also a covariance matrix, it is positive semi-definite as well.

We know that if $\mathbf{M}$ is positive semi-definite and $r > 0$ is a real number, then $r\mathbf{M}$ is positive semi-definite. If $\mathbf{M}$ and $\mathbf{N}$ are positive semi-definite, then $\mathbf{M} + \mathbf{N}$ is also positive semi-definite. Since our shrinkage parameter $\frac{k}{T}$ is between 0 and 1, $\frac{k}{T}\mathbf{H} + (1 - \frac{k}{T}\mathbf{S})$ is also positive semi-definite. Ledoit's estimator, implemented in the Markowitz portfolio framework, maintains this necessary property of a covariance matrix.

CALM Model: Our covariance matrix estimator is $\frac{k}{T}\mathbf{H} + (1 - \frac{k}{T}\mathbf{S})$, where $\mathbf{H}$ is as defined in (10).

Here $\mathbf{S}$ is the same sample covariance matrix used within the standard Markowitz model, so it is positive semi-definite. As we previously proved, as long as the off-diagonal constant number $p \in (0,1)$, all k^{th} principle minors of $\mathbf{H}$ are positive and hence $\mathbf{H}$ is also positive definite, which implies $\mathbf{H}$ must be positive semi-definite. For the same reason, since shrinkage parameter $\frac{k}{T}$ is between 0 and 1, $\frac{k}{T}\mathbf{H} + (1 - \frac{k}{T}\mathbf{S})$ is positive semi-definite.

We can conclude that the CALM estimator also has the desired property of a covariance matrix.

3.3 Tangency Weight Programming

As stated before, the length of our sliding window is 1 year. US Treasury bills are issued by the United States government, which can be considered free of risk. To match the window length, we choose the return of the 1 year Treasury bill as our risk-free rate. The data was obtained from the Federal Reserve's website [11].

The unconstrained tangency portfolio is useful theoretically, but has several problems in practice. The method requires the inverse of the covariance matrix; however, numerical error can occur when the covariance matrix is nearly singular. In addition, elements of w (the portfolio weights) can be negative, which represents shorting assets. This involves borrowing stock on margin, which is a form of leveraging and easily can trigger margin calls. Moreover, the unconstrained method for obtaining a minimum variance portfolio does not limit portfolio turnover. The weight vector can change substantially without restriction between time periods, which means that asset turnover has the potential to be high, and we may need to long or short large amounts of stock on each rebalance date. In practice this causes transaction fees to cut into profit. In order to avoid large negative weights and margin calls, we used constrained quadratic programming. Quadratic programming is used to minimize a quadratic objective function subject to linear constraints [12]. In this project we try the following two constraints.

1. We permit shorting stocks but limit the maximum shorting weight to be -0.2 and the maximum longing weight to be 0.2.

2. We prohibit short selling, limiting all tangency weights to be between 0 and 1.

We calculate unconstrained weights and two different constrained weights for the four models and compared their return distributions. Hence we have 12 models to analyze. We used MATLAB to calculate the initial tangency portfolio weights and formed our portfolio beginning on Jan 1, 2007. Our principle amount was $1 million. During the holding period we used weekly returns and rebalanced the portfolio monthly.

3.4 Comparison Across Models

3.4.1 Holding period return

Comparing across different strategies, as we can see from Table 1, the constraint [-0.2 0.2] gave the best returns while the unconstrained models gave the worst of all our tests. The highly structured correlation matrix with constant 0.9 and constraint [-0.2 0.2] has the highest holding period return, while the unconstrained highly structured matrices have the greatest loss.

Table 1: Holding Period Net Return

	Unconstrained	[-0.2 0.2]	[0 1]
Markowitz	-0.70	0.61	-0.08
Ledoit	-0.71	0.38	0.03
$p = 0.7$	-1.21	0.42	-0.10
$p = 0.9$	-1.21	0.64	-0.07

Unconstrained Weights: Unconstrained weights imply significant liability and hence high margin requirements. This result indicates that during a market catastrophe, the theory that one should continue to take risk to garner large gains does not hold. Moreover, since unconstrained weights performed the worst, they are both unrealistic and unprofitable for this portfolio - no matter which model is used.

Constrained Weights [-0.2 0.2]: This allows short selling but limits the percentage of the short position for each stock and properly balances the risks and rewards. Portfolios under this constraint have the capacity to garner positive return in a bear or bull market. For this reason we believe that this model performed best out of the 4 possible models.

Constrained Weights [0 1]: Short selling is prohibited under this constraint. For the conservative and cautious investor, these weights are ideal since these constraints imply zero liability and no risk of being margin called. However, since this long-only constraint takes less risk by not short selling any stock, the portfolio is not able to earn a return on stocks for which prices are dropping. Hence this constraint makes less money than a portfolio that allows short selling during a normal market and has zero profit (or a loss) when market is in catastrophe. This is the reason why 3 of the 4 models have negative returns under this constraint for our specific portfolio.

3.4.2 Leverage Ratio

Table 2: Leverage Ratio

	Unconstrained	[-0.2 0.2]	[0 1]
Markowitz	27.4	3.4	1.0
Ledoit	5.8	2.6	1.0
$p = 0.7$	22.5	3.1	1.0
$p = 0.9$	22.5	3.4	1.0

$$Leverage = \frac{\sum\limits_{i=1}^{n} |w_i|}{\sum\limits_{i=1}^{n} w_i} \tag{14}$$

The Leverage ratio is defined as the sum of absolute value of each tangency weight divided by the sum of each tangency weight. Unconstrained portfolio optimization models may introduce significant short sell positions. Therefore, as we can see from the Table 2, their leverage ratios are also very high. However, portfolios with a large leverage ratio are not only very risky (as stated before, they have unlimited potential liability) but also have to meet large margin requirements. Rarely is a portfolio manager is willing to construct his or her portfolio using unconstrained weights. Since unconstrained models are neither realistic nor profitable, we consider the unconstrained cases only as basic case general models but not as practical models. The rest of this report only analyzes the performances of 8 constrained models.

3.4.3 Yearly Return Comparison Across Models

As stated before, we use weekly data to calculate the expected returns and covariance matrix estimates and rebalance the portfolio on the first day of each month according to the new tangency weights. During each month we hold the portfolio and do not make any changes. We only observe and track the portfolios' weekly performances based on the monthly initial weights. At the end of holding period we collect 156 returns for each model. We plot the weekly returns in MATLAB using a BOXPLOT function and group by year and model. We can see this in Fig. 1.
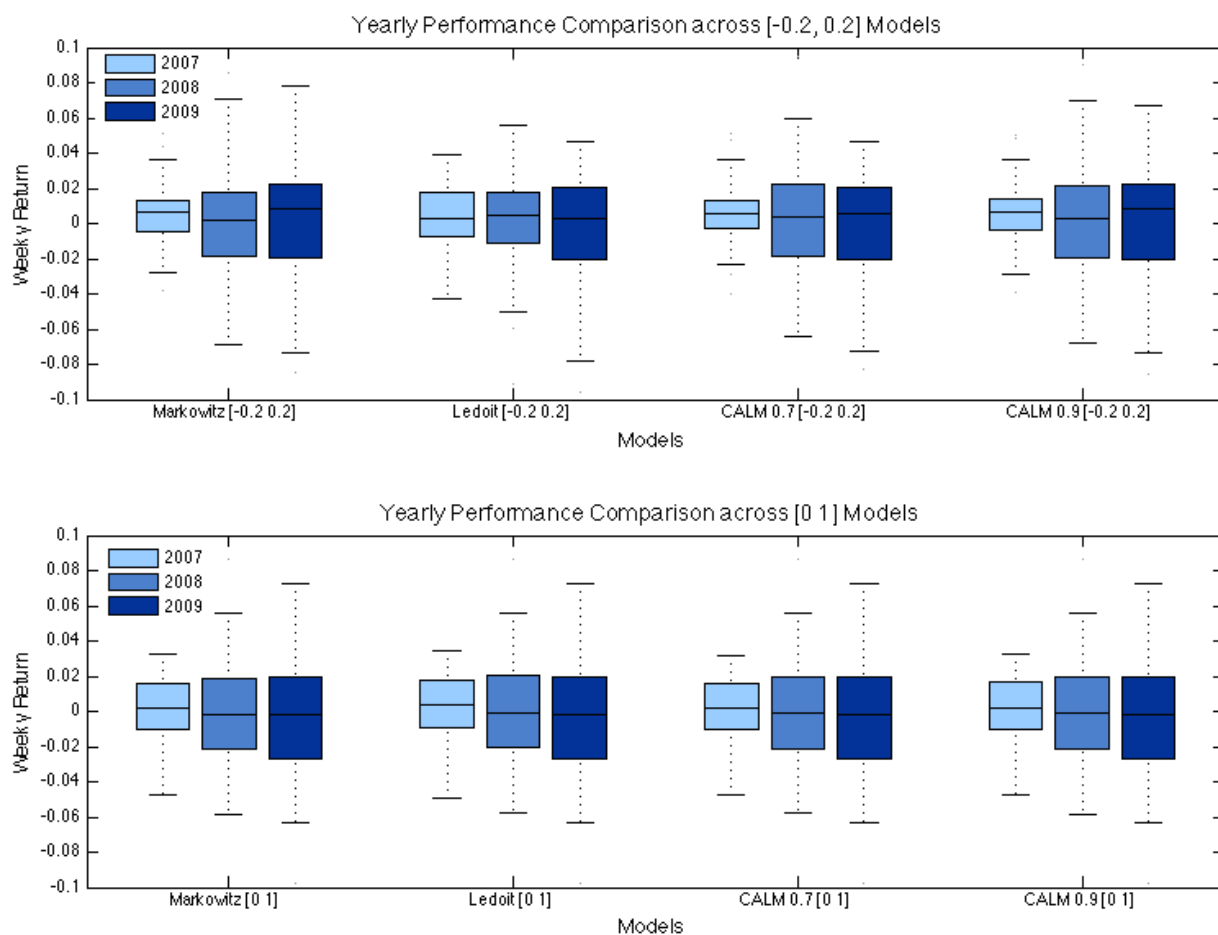
Figure 1: Yearly Performance Comparison Across Models

BOXPLOT provides a clear comparison of model performance for each year. On each box, the central mark is the median, the edges of the box are the 25th and 75th percentiles and the whiskers extend to the most extreme data points are considered as outliers.

If we divide the 8 models into two groups by their weight constraints, we get four models with [-0.2, 0.2] constraints and four models with [0 1] constraints. Generally, the [-0.2, 0.2] constrained models perform better than [0 1] constrained models.

Table 3: Yearly Performance Comparison Across Models

	Marko [-0.2, 0.2]			Ledoit [-0.2, 0.2]		
	2007	2008	2009	2007	2008	2009
Mean	0.0049	-0.0013	-0.0008	0.0035	0.0015	-0.0039
Std. deviation	0.0182	0.0473	0.0340	0.0174	0.0379	0.0333
	CALM 0.7 [-0.2, 0.2]			CALM 0.9 [-0.2, 0.2]		
	2007	2008	2009	2007	2008	2009
Mean	0.005	0.0011	-0.0022	0.005	0.0001	-0.0011
Std. deviation	0.0179	0.0435	0.0321	0.0180	0.0468	0.0338
	Marko [0 1]			Ledoit [0 1]		
	2007	2008	2009	2007	2008	2009
Mean	0.0017	-0.0010	0.0015	0.0021	-0.0008	0.0015
Std. deviation	0.0190	0.0351	0.0489	0.0185	0.0350	0.0487
	CALM 0.7 [0 1]			CALM 0.9 [0 1]		
	2007	2008	2009	2007	2008	2009
median	0.0019	-0.0010	0.0015	0.0018	-0.001	0.0016
Std. deviation	0.0189	0.0354	0.0489	0.0189	0.0353	0.0489

[-0.2, 0.2] Constraint: Before the 2008 crisis, Markowitz, CALM 0.7 and CALM 0.9 outperformed Ledoit. They either have lower risk or higher return. Except for the Ledoit model, the other 3 models experienced return deteriorations in 2008, but rapidly recovered from it once market conditions improved. This means the shrinkage parameter takes effect under the [-0.2 0.2] constraint by lifting the median return despite lifting risk as well. The Ledoit model performed best among the 4 models during the crisis. If we use the return standard deviation as the measure of risk, it had the highest return and the lowest risk and should be considered the best model for use in 2008. Despite having the lowest return of 2009, its risk was the lowest.

[0 1] Constraint: We observe that every model with [0 1] constraint experienced return deterioration from 2007 to 2009. Their returns kept decreasing while risk increased. We consider the [0 1] an inferior constraint compared to the [-0.2, 0.2] constraint.

A good portfolio should have either higher returns given a specific level of risk or lower risks given a specific level of return. These 4 [0 1] constrained

models do not meet this criteria well. Since the U.S. equity market started to recover in 2009, the [0 1] constraints weakened the model's ability to capture this market characteristic and adjust shrinkage parameters in time.

Each model under [-0.2, 0.2] constraint has its own strengths; we will focus on these 4 models in the following analysis.

3.4.4 Performance Ratio

It is not comprehensive to focus only on the final holding period returns. In order to compare the weekly performance over these 4 models, we calculate their Sharpe Ratio, Treynor Ratio and Information Ratio using the DJIA as a benchmark.

Sharpe's Ratio

Sharpe's ratio (SR) is the industry standard for measuring risk-adjusted return. Sharpe's ratio is what reward an investor could expect on average for investing in a risky asset versus a risk-free asset. The numerator of the ratio is the expected portfolio return R_p less the risk-free rate R_f, and the denominator is the portfolio return's volatility or standard deviation of returns σ_p (less that of the risk-free asset's standard deviation, which is zero). The resulting ratio isolates the expected excess return that the portfolio could be expected to generate per unit of portfolio return variability. Sharpe ratio uses actual instead of expected returns and is calculated as: [12]

$$Sharpe's Ratio = \frac{R_P - R_f}{\sigma_p}.$$

Table 4: Sharpe's Ratio from 2007 to 2009

	Unconstrained	[-0.2 0.2]	[0 1]
Markowitz	-0.1714	0.2818	0.0142
Ledoit	-0.0291	0.2173	0.0419
$p = 0.7$	-0.0635	0.2964	0.0169
$p = 0.9$	-0.0635	0.2933	0.0170

Sharpe's ratio informs an investor what portion of a portfolio's performance is associated with risk taking. It measures a portfolio's added value relative to its total risk. Table 4 shows the Sharpe's Ratio for the 12 com-

binations of correlations and constraints. The risk-free rate is chosen to be the 4-week treasury bill rate.

Generally Sharpe's ratio is useful in pratice but it has its own set of limitations to consider. It is based on Markowitz portfolio theory, which proposes that a portfolio can be described by just two measures: its mean return and its variance of returns. Sharpe's ratio measures only one dimension of risk, the variance. Sharpe's ratio is designed to be applied to investment strategies that have normal expected return distributions; it is not suitable for measuring investments that are expected to have asymmetric returns. The study on whether our portfolio has an asymmetric return will be performed by fitting a normal and a student t-distribution according to the parameters we estimated from the available data. Details will be discussed later.

There are two obvious downfalls in using Sharpe's Ratio [13], even in the framework of normally distributed returns:

- Sharpe's ratio cannot tell an investor whether a high standard deviation is due to large upside deviations or downside deviations; the Sharpe ratio penalizes both equally.

- Negative Sharpe ratios, such as those arising during portfolio under performance (which often occurs during bear markets) are also uninformative.

For the CALM model with constant 0.9 and constraint [-0.2, 0.2], its Sharpe ratio is slightly smaller than the one with constant 0.7. However, if we look at their normal distribution fitting results, the mean for CALM 0.9 is higher than CALM 0.7, along with their standard deviations, respectively. The normal fitting did not tell us whether this high standard deviation was due to large upside or downside deviations. Sharpe's ratio does not distinguish between them.

Treynor's Ratio

Let σ_{MP} denote the return covariance between market portfolio and our portfolio, and σ_M^2 denote the market portfolio's return variance, Treynor's ratio is defined as the following: [12]

$$Treynor'sRatio = \frac{R_P - R_f}{\beta_p},$$

where
$$\beta_p = \frac{\sigma_{MP}}{\sigma_M^2}.$$

Unlike Sharpe's ratio, Treynor's ratio (TR) uses beta in the denominator instead of the standard deviation. The beta measures only the portfolio's sensitivity to the market movement, while the standard deviation is a measure of the total volatility (upside as well as downside). The Treynor ratio relates excess return over the risk-free rate to the additional risk taken; however, systematic risk is used instead of total risk.[12] The Treynor ratio is interpreted as excess returns per unit of systematic risk. As our portfolio contains 29 stocks and can be considered well diversified, we can say that the effect of unsystematic risk is very small. Table 5 summarizes the Treynor's Ratio of the 12 models.

Table 5: Treynor's Ratio from 2007 to 2009

	Unconstrained	[-0.2 0.2]	[0 1]
Markowitz	0.1102	-0.0909	0.0114
Ledoit	0.0088	-0.0445	0.0342
$p = 0.7$	0.0108	-0.0753	0.0142
$p = 0.9$	0.0108	-0.0842	0.0137

Among our 12 combinations of models and constraints, only long-only portfolios have a positive beta. This means only long-only portfolios are positively correlated to market. Many negative TR appeared, which are uninformative. The unconstrained Markowitz model had the highest TR. However, its distribution fitting result shows that it may not be a good model because of the large negative return and standard deviation.

Information Ratio

The information ratio (IR) is often referred to as a variation or generalized version of the Sharpe ratio. It evolved as users of the Sharpe ratio began substituting passive benchmarks for the risk-free rate. The information ratio tells an investor how much excess return is generated from the amount of excess risk taken relative to the benchmark. The information ratio is calculated by dividing the portfolio's mean excess return relative to its benchmark by the variability of that excess return. The portfolio's excess return is also known as its active return, and the variability of the excess return is also referred to as active risk.[12]

$$InformationRatio = \frac{\overline{R_P} - \overline{R_B}}{\sigma_{P-B}}.$$

Table 6: Information Ratio from 2007 to 2009

	unconstrained	[-0.2 0.2]	[0 1]
Markowitz	-0.1054	0.2430	0.0753
Ledoit	-0.0155	0.1889	0.0962
const 0.7	-0.0484	0.2416	0.0772
const 0.9	-0.0484	0.2481	0.0773

Table 6 shows the CALM model with constant 0.9 and constraint [-0.2, 0.2] had the highest IR using the DJIA as the benchmark. Other models with weight constraint [-0.2, 0.2] also have comparative IR's, while models with other weight constrainst have worse IR's.

Generally when evaluating the information ratio we consider the higher IR to be the better. However, if we move into more detailed criterion, evaluating the information ratio for portfolio is more challenging [13]. Grinold and Kahn [8] contended that top-quartile active equity managers generally have information ratios of 0.50 or higher. In another work, Grinold and Kahn [9] rated an information ratio of 1.0 as "exceptional", 0.75 as "very good", and 0.50 as "good". Goodwin [10] measured IR's over a 10-year period and found that even among consistently outperforming long-only managers, very few are able to sustain an IR of 0.50 or higher, suggesting that the ranking criteria may be too high. Goodwin further suggested that IR's are most useful when comparing managers within their own style universe rather than among styles. A general consensus among the investment profession is that an IR of 0.20 or 0.30 is superior. Since our IR is 0.248, we can conclude that our investment performance is within good standing, especially during the period of financial crisis.

3.4.5 Normal Distribution Fitting

We fit our portfolio's weekly return to a normal distribution and performed a hypothesis test to determine if the returns distribution is consistent with the fitted distribution. Moreover, we use 2-D and 3-D figures to demonstrate how their distributions change over time.

The hypothesis tests of Chi square goodness of fit test that we used to determine whether or not our data fit a normal distribution is as follows:

- H_0: Portfolio daily returns were taken from a normal distribution.

- H_1: Portfolio daily returns were not taken from a normal distribution.

Figure 2, 3, 4 and 5 are the histograms and normal distribution fitting results over time for 4 [-0.2, 0.2] constrained models.

Unlike the [0 1] models, the post-crisis 2-D fitted distributions have more peak than those during the crisis, indicating a post-crisis portfolio is less risky. This is evidence that these [-0.2, 0.2] constrained models are more effective in reducing risk than [0 1] constrained models. (In order to avoid tediousness, we do not put all the normal fitting for the [0 1] constrained models here. Instead, we include them in the appendix.) Moreover, pre-crisis and post-crisis portfolio returns cannot reject the null hypothesis at a 5% significant level while returns during crisis reject the null hypothesis. Since the crisis, the returns distribution of each stock in the market is no longer normal because they are severely negatively skewed. No matter what weights we use, the portfolio return distribution cannot be normal.

Also from both 2-D and 3-D plot, it is well demonstrated that the time-series of portfolio returns are not stationary during the holding period.

3.4.6   Tests for Normality

If we want to test the normality of the returns distribution considering the entire holding period, we can use a Q-Q plot. The Figure 6 is the Q-Q plots for 4 [-0.2, 0.2] constrained models.

There is not much difference in normality for the 4 models. If the two distributions being compared are similar, the points in the Q-Q plot will approximately lie on the line $y = x$. The linearity of the points suggests that the data is normally distributed. Only the middle part of the portfolio returns lie on the $y = x$ line, large deviation from $y = x$ happens on the upper and lower tails. The extreme outliers may be caused by market catastrophe. The Ledoit model improves the normality of the upper tail, however, its severely deviated lower tail still shows that the entire distribution is not normal.
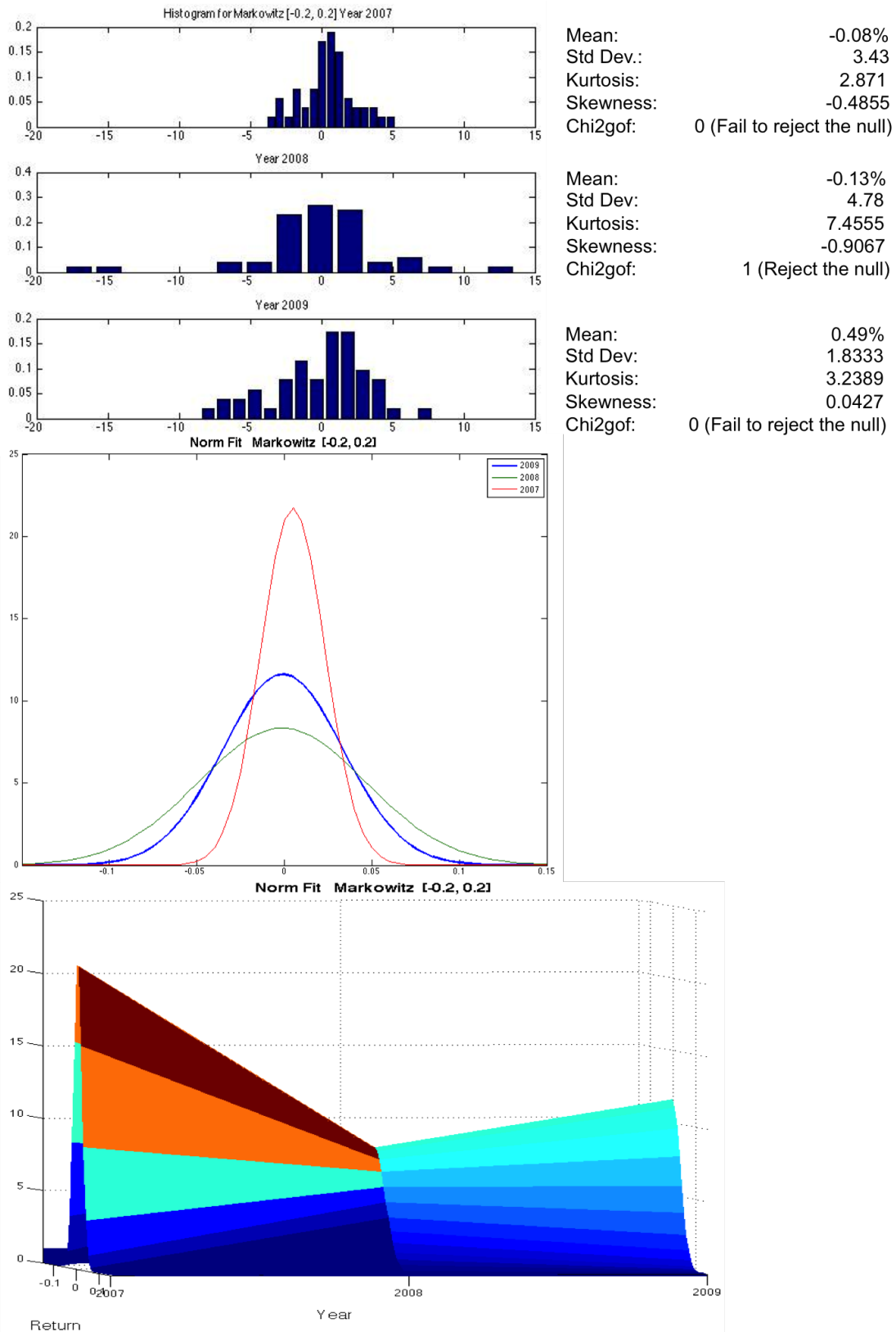
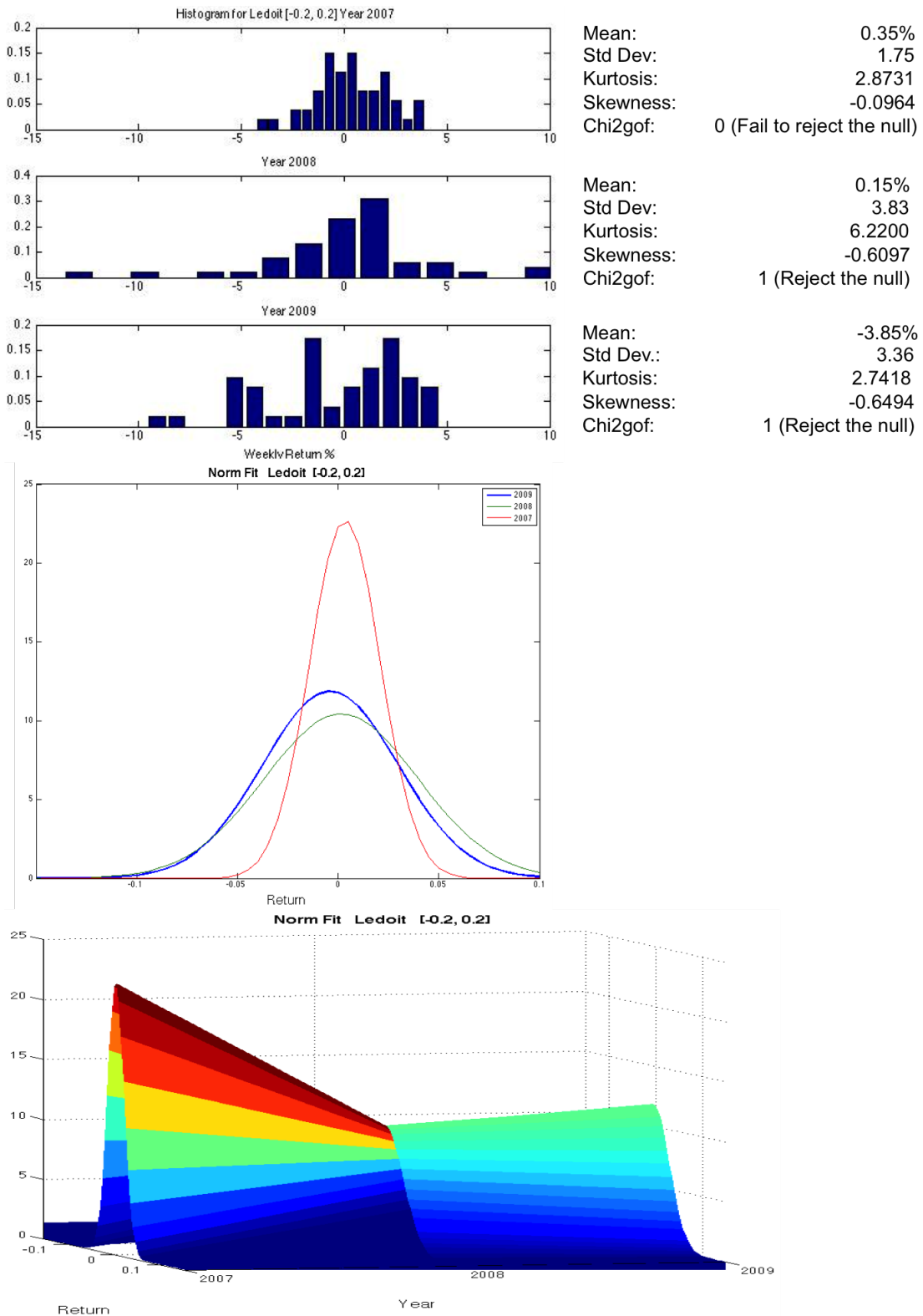Figure 2: Yearly Normal Distribution Fitting Result for Markowitz model with constraint [-0.2, 0.2]

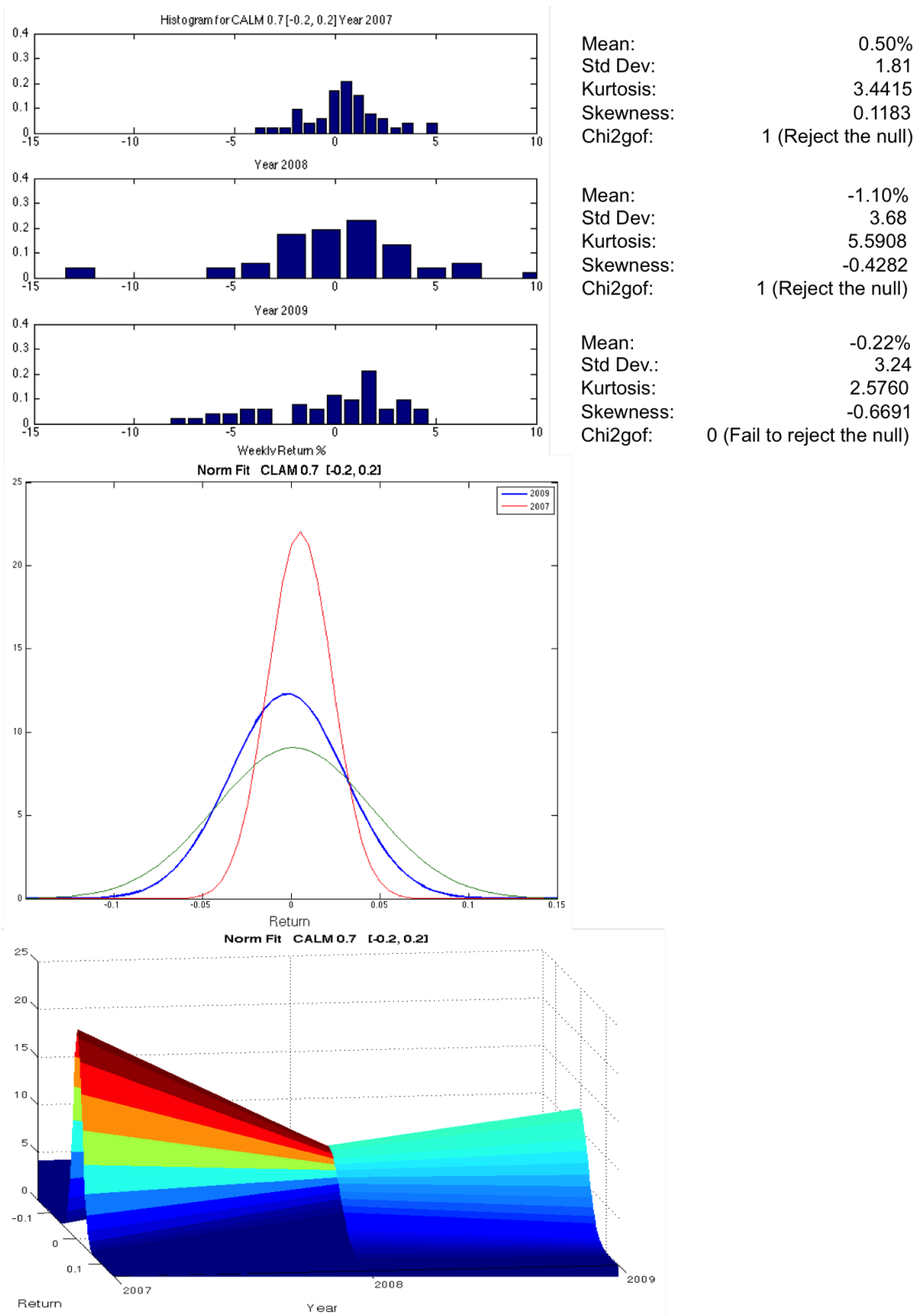Figure 3: Yearly Normal Distribution Fitting Result for Ledoit's model with constraint [-0.2, 0.2]

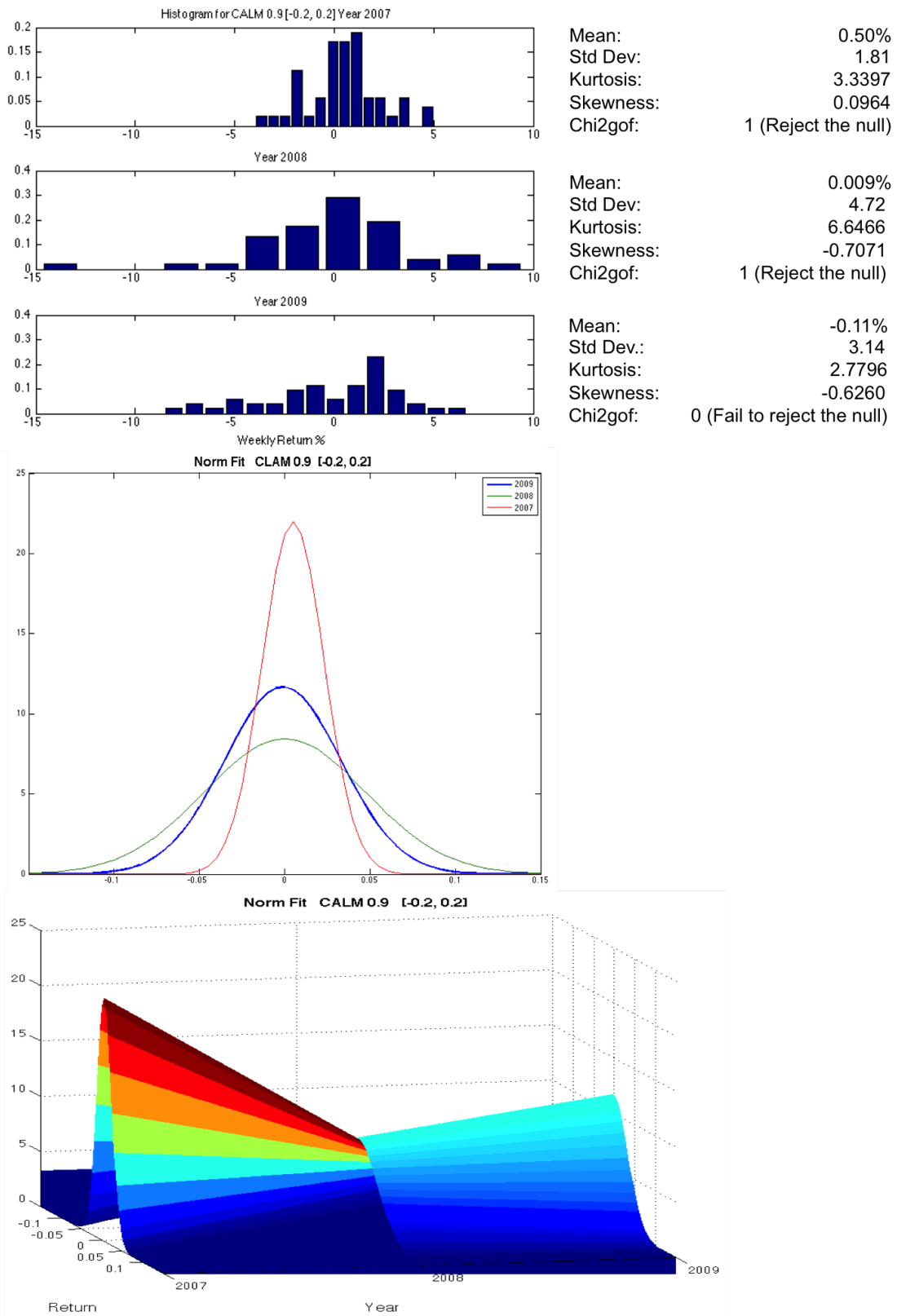Figure 4: Yearly Normal Distribution Fitting Result for CALM 0.7 model with constraint [-0.2, 0.2]

Figure 5: Yearly Normal Distribution Fitting Result for CALM 0.9 model with constraint [-0.2, 0.2]
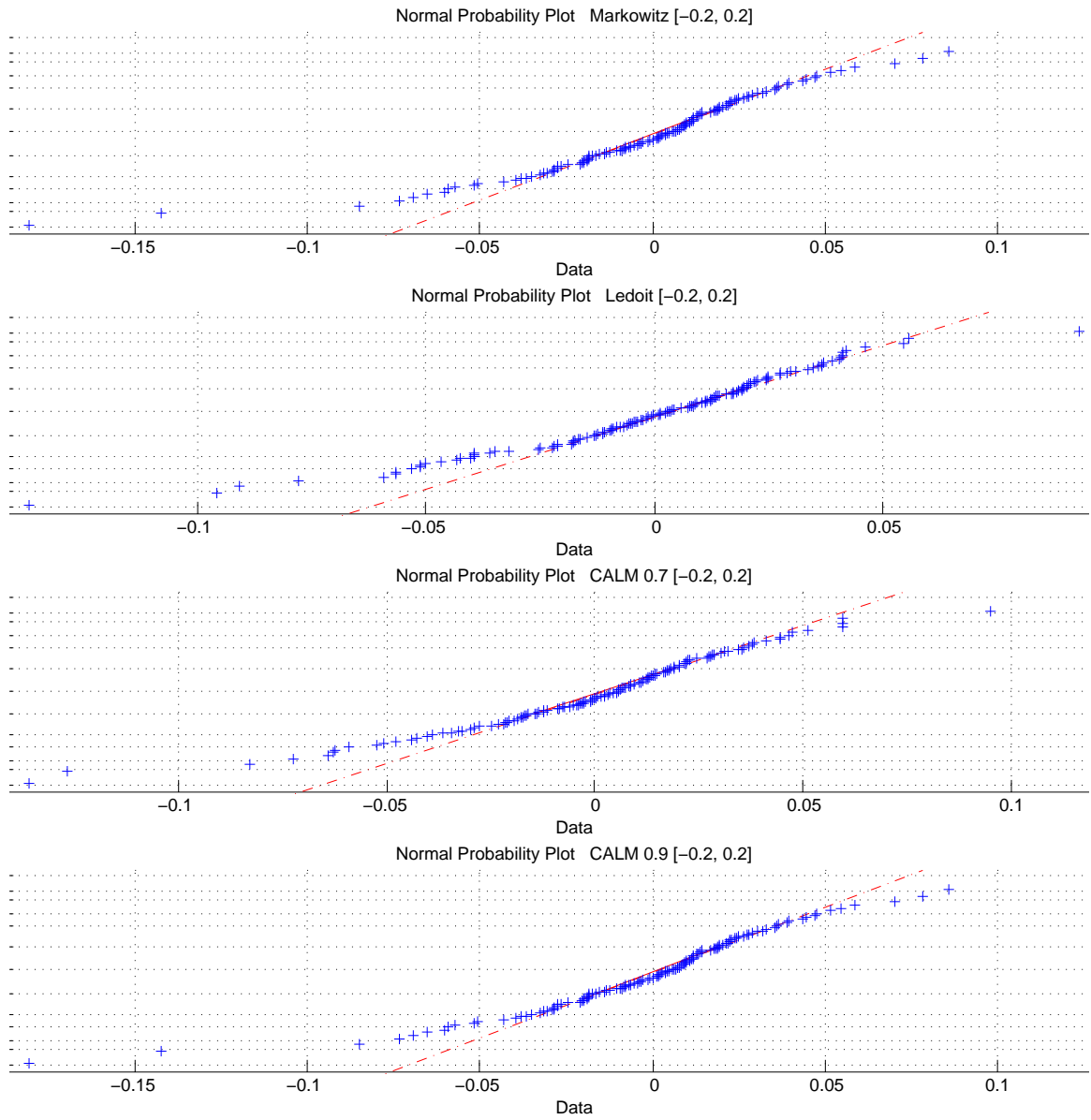
Figure 6: Q-Q Plot for Four [-0.2, 0.2] Constrained Models

4   Conclusion

Our empirical analysis based on the portfolio performance from the financial crisis from 2007 to 2009 indicates that incorporating forward-looking signals into covariance matrix estimation is an effective method to improve return and reduce risk in market catastrophes.

Among the 12 combinations of models and constraints, Ledoit's model does not perform as well as the other 3 models as measured by Sharpe's Ratio, holding period return, and weekly return and risk. CALM with constant 0.7 and 0.9 performed similarly, indicating that the value of the constant number in a highly structured correlation matrix may not have a large impact on a portfolio's performance. The standard Markowitz model performed moderately well among these 4 models.

We found that [-0.2, 0.2] constrained models outperformed unconstrained and [0 1] constrained models. Although their leverage ratios are higher and take more risk than the [0 1] models, they obtain enough rewards, indicating that the excess risk is worth taking.

When the market is in a pre-crisis state, the standard Markowitz model performed the greatest. Since the market is calm before a crisis, the assumption that market behavior in the future is consistent with the past holds. If we include a highly structured correlation matrix, the covariance matrix overestimates the market risk and thus the model instructs us to invest conservatively and take less risk. As a result, the return is lower. However, during a post-crisis, the Ledoit model and CALM overcome the weaknesses of the Markowitz model and fairly measure the market risk. Thus they obtain better returns.

Should we decide to change investment models over time, Markowitz' model with constraint [-0.2, 0.2] works well during periods of relative ease. When market conditions worsen, we should switch to CALM 0.7 or CALM 0.9 in order to achieve better results.

A Appendix

A.1 Matlab Code

The following is the core code computing the tangency weights for constrained Ledoit's Model and CALM.

```
[t,n]=size(data);
meanasset=mean(data);
data=data-meanasset(ones(t,1),:);

% compute sample covariance matrix
sample=(1/t).*(data'*data);

% compute prior
var=diag(sample);
sqrtvar=sqrt(var);
rBar=(sum(sum(sample./(sqrtvar(:,ones(n,1)).*sqrtvar(:,ones(n,1))'))))-n)...
     /(n*(n-1));
prior=rBar*sqrtvar(:,ones(n,1)).*sqrtvar(:,ones(n,1))';
prior(logical(eye(n)))=var;

%compute prior for CALM
%constant = 0.9;
%prior = constant*ones(n,n);


% what we call phi-hat
y=data.^2;
phiMat=y'*y/t - 2*(data'*data).*sample/t + sample.^2;
phi=sum(sum(phiMat));

% what we call rho-hat
term1=((data.^3)'*data)/t;
help = data'*data/t;
helpDiag=diag(help);
term2=helpDiag(:,ones(n,1)).*sample;
term3=help.*var(:,ones(n,1));
```

```
term4=var(:,ones(n,1)).*sample;
thetaMat=term1-term2-term3+term4;
thetaMat(logical(eye(n)))=zeros(n,1);
rho=sum(diag(phiMat))+rBar*sum(sum(((1./sqrtvar)*sqrtvar').*thetaMat));

% what we call gamma-hat
gamma=norm(sample-prior,'fro')^2;

% compute shrinkage constant
kappa=(phi-rho)/gamma;
shrinkage=max(0,min(1,kappa/t));


% compute the estimator
sigma=shrinkage*prior+(1-shrinkage)*sample

bOmega = sigma;
bmu = mean(asset);
ngrid = 50;
muP= linspace(rf,max(bmu),ngrid);

weights = zeros(29,ngrid);
sigmaP = muP;

LB = 0*ones(29,1);
UB = 1*ones(29,1);
Aeq = [ones(1,29);bmu];
f = zeros(29,1);

for i = 1:1:ngrid
    beq = [1;muP(i)];
    w = quadprog(bOmega,f,'','',Aeq,beq,LB,UB);
    weights(:,i) = w;
    sigmaP(i) = sqrt(w'*bOmega*w);
end

Imin = find(sigmaP == min(sigmaP));
Ieff = (muP >= muP(Imin));
```

```
sharperatio = (muP-rf)./sigmaP;
Itangency = find(sharperatio == max(sharperatio));

weightsT=weights(:,Itangency)
ExpReturnT=bmu*weightsT;
VarT=weightsT'*bOmega*weightsT;
```

The following code fit the data with normal and run chi2gof test to test whether the data were taken from a specific distribution.

```
netreturn7 = 100*netreturn7;

% fit the distribution of original data with normal dist.
pd_norm = fitdist(netreturn7,'Normal')
[h,p] = chi2gof(netreturn7,'cdf',pd_norm)
[f,x] = hist(netreturn7,15);
subplot(3,1,1)

% plot the percentage histogram
bar(x,f/sum(f));
title('Histogram for CALM 0.9 [-0.2, 0.2] Year 2007');
xlim([-15,10])
```

This code gives us the 2D and 3D surface plot of portfolio return.

```
xrange = 0.8;
x = [-xrange:.005:xrange];

% normal distribution fitting
return09 = xlsread('DATA_DOW','weighted return','AF3:AF54');
pd_norm09 = fitdist(return09,'Normal');
norm09 = normpdf(x,pd_norm09.mu,pd_norm09.sigma);

return08 = xlsread('DATA_DOW','weighted return','AF56:AF107');
pd_norm08 = fitdist(return08,'Normal');
norm08 = normpdf(x,pd_norm08.mu,pd_norm08.sigma);

return07 = xlsread('DATA_DOW','weighted return','AF109:AF161');
pd_norm07 = fitdist(return07,'Normal');
```

```
norm07 = normpdf(x,pd_norm07.mu,pd_norm07.sigma);

% 2D plot
plot(x,norm09,x,norm08,x,norm07)
legend('2009','2008','2007')

%3D surface plot
norm = [norm07; norm08; norm09;];
xx = x(ones(3,1),:);
t = [2007:1:2009]';
tt = t(:,ones(length(x),1));
z = norm;
mesh(x,t,z)
set(gca,'YTick',[2007 2008 2009])
xlim([-xrange,xrange])
```

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